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Local behaviour of local times of super-Brownian motion

Comportement local des temps locaux du super-mouvement brownien

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Abstract

For $x_0 \in \mathbb{R}^d \setminus \{0\}$, $d \leqslant 3$, we study the local behaviour near 0 of the local times (L_t^y) of a super-Brownian motion X initially in δ_{x_0}. More precisely, if $\psi(c)$ is a suitable normalization, our main result implies that the process $(\psi(c)(L_t^{x/c} - L_t^0),\ x \in \mathbb{R}^d,\ t \geqslant 0)$ converges in distribution to a non-degenerate limit as $c \to \infty$. This allows us to study the local behaviour of the occupation measure of X, then to recover and to generalise a result of Lee concerning the occupation measure of three-dimensional super-Brownian motion conditioned to hit a distant ball.

Résumé

Pour $x_0 \in \mathbb{R}^d \setminus \{0\}$, $d \leqslant 3$, on étudie le comportement local au voisinage de 0 des temps locaux (L_t^y) du super-mouvement brownien X de valeur initiale δ_{x_0}. Plus précisément, si on note $\psi(c)$ la normalisation adéquate, notre résultat principal implique que le processus $(\psi(c)(L_t^{x/c} - L_t^0),\ x \in \mathbb{R}^d,\ t \geqslant 0)$ converge en loi lorsque $c \to \infty$ vers une limite non dégénérée. Ce résultat nous permettra d'étudier le comportement local de la mesure d'occupation de X, puis de redémontrer et généraliser un résultat de Lee concernant la mesure d'occupation d'un super-mouvement brownien tri-dimensionel conditionné à toucher une boule lointaine.

Keywords: Super-Brownian motion; Local time; Occupation measure

1. Introduction and statement of results

1.1. Introduction

The main goal of this work is to study the local behaviour of the occupation measure of super-Brownian motion in dimension $d \leqslant 3$.

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Super-Brownian motion is a model for spatial populations undergoing a continuous branching phenomenon, which arises in a variety of different contexts. It was introduced independently by Watanabe (69) and Dawson (75) as the weak limit of branching particle systems. Connections between the wider class of superprocesses and partial differential equations have helped derive some of the basic properties of super-Brownian motion and have also allowed to prove analytic results using probabilistic arguments. More recently, it has been shown that super-Brownian motion appears in scaling limit of a wide range of lattice systems such as the voter model, the contact process, lattice trees or oriented percolation. Therefore, in a way similar to standard Brownian motion, super-Brownian motion can be thought of as a universal object providing information on the asymptotics of many interacting particle systems or statistical mechanics models.

Local times of superprocesses have been studied by many authors (cf. Sugitani [9], Adler and Lewin [1], Krone [4]). Our main result Theorem 1 gives precise information about the local behaviour of local times of super-Brownian motion, in dimension $d = 2$ or 3. Let us be more precise. Let $x_0 \in \mathbb{R}^d \setminus \{0\}$, X a super-Brownian motion initially in δ_{x_0}, and L_t^x the local time of X at time t and point x. As a direct consequence of Theorem 1, if $x \in \mathbb{R}^d$ and if we set $\psi(c) = \sqrt{c}$ when $d = 3$, $\psi(c) = c(\ln(c))^{-1/2}$ when $d = 2$, we will obtain that $(\psi(c)(L_t^{x/c} - L_t^0))_{t \geqslant 0}$ converges in distribution as $c \to \infty$ to $(\beta_{a(x)L_t^0})_{t \geqslant 0}$ where β is a one-dimensional Brownian motion which is independent of X and $a(x)$ is a constant depending only on x. Theorem 1 in fact gives a more general statement involving finitely many different values of x. This allows us to study the local behaviour of the occupation measure of X (Proposition 2).

Our results are related to a recent paper of Lee [5]. Lee considers a super-Brownian motion started at δ_{cx_0} with c large and conditioned on hitting the unit ball. Using analytic methods, he obtains limit theorems for the occupation measure of this ball by super-Brownian motion (see Proposition 1 below). We will show how to recover and to generalise Lee's results from our main theorem. To do this, we will need to study the local behaviour of the occupation measure of X under its canonical measure $\mathbb{N}_{x_0}$ (Proposition 5).

Our results on the local behaviour of local times of super-Brownian motion are analogous to the ones obtained by Yor for local times of standard Brownian motion. Let B be a linear Brownian motion started at the origin and let ℓ_t^x denote the local time of B at level x and time t. Then Yor [11], proved that $(\sqrt{c}(\ell_t^{x/c} - \ell_t^0))_{t \geqslant 0,\, x \geqslant 0}$ converges in distribution to a Brownian sheet as $c \to \infty$. This should be compared with Theorem 1 below.

After introducing the basic notation, Section 1 first summarises known results which motivate our study, mainly found in [5,9], and [7], then states our results. It also provides an outline of the proof of our main result (Theorem 1) about the local behaviour of local times. Section 2 is devoted to the proof of Theorem 1. In Section 3, we apply Theorem 1 to prove a non-conditional equivalent form of Lee's result (Proposition 2), then we recover and extend Lee's result (Proposition 1). Section 4 is devoted to the one-dimensional case.

1.2. Notation

Let $M_F(\mathbb{R}^d)$ be the space of finite measures on $\mathbb{R}^d$. For $\mu \in M_F(\mathbb{R}^d)$, $f : \mathbb{R}^d \to \mathbb{R}$, $\langle \mu, f \rangle$ is shorthand for $\int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x)$.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra in $\mathbb{R}^d$. The notation $B(x, r)$ stands for the open ball of radius r centered at $x \in \mathbb{R}^d$.

We denote by $C_b(\mathbb{R}^d)$ the space of bounded continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$, and by $C_b(\mathbb{R}^d)_+$ the space of non-negative functions in $C_b(\mathbb{R}^d)$. If K is a compact subset of $\mathbb{R}^d$, $C_K(\mathbb{R}^d)$ is the subset of $C_b(\mathbb{R}^d)$ consisting of functions supported on K. If $n \in \mathbb{N}$, we let $C_b^n(\mathbb{R}^d)$ be the set of all n times continuously differentiable functions on $\mathbb{R}^d$ with bounded derivatives of order less than n, and $C_b^\infty(\mathbb{R}^d) := \bigcap_{n \geqslant 0} C_b^n(\mathbb{R}^d)$.

We denote by p_t the transition density of d-dimensional Brownian motion, that is for $t \geqslant 0$, $z \in \mathbb{R}^d$,

$$p_t(z) = (2\pi t)^{-d/2} \exp\left(-\frac{|z|^2}{2t}\right).$$

Set $q_t(x) = \int_0^t p_s(x)\,\mathrm{d}s$, and if $\mu \in M_F(\mathbb{R}^d)$,

$$\mu q_t(z) = \int_{\mathbb{R}^d} q_t(z - y)\mu(\mathrm{d}y).$$

We consider a super-Brownian motion $(X_t,\ t \geqslant 0)$ in the space $M_F(\mathbb{R}^d)$. We will write P_μ for the probability measure under which $X_0 = \mu$. When there is no confusion we will write P for P_μ, E for E_μ. We denote by $(\mathcal{F}_t^X)_{t\geqslant 0}$ the right-continuous filtration generated by X.

1.3. Basic properties of super-Brownian motion and its local times

In this section we consider a super-Brownian motion X under P_μ, for $\mu \in M_F(\mathbb{R}^d)$.

It is well known (see for example [7], Theorem II.5.9) that for any function $\phi \in C_b(\mathbb{R}^d)_+$ we can express $E_\mu[\exp(-\int_0^t \langle X_s, \phi\rangle\, \mathrm{d}s)]$ in terms of the solution of a partial differential equation:

$$E_\mu\left(\exp\left\{-\int_0^t \langle X_s, \phi\rangle\, \mathrm{d}s\right\}\right) = \exp\left(-\int v(t,x,\phi)\mu(\mathrm{d}x)\right), \tag{1}$$

where $(t,x) \to v(t,x,\phi)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^d$ and solves the partial differential equation

$$\frac{\partial v(s,x)}{\partial s} = \frac{1}{2}\Delta v(s,x) - \big(v(s,x)\big)^2 + \phi, \tag{2}$$

on $]0,\infty[\, \times \mathbb{R}^d$, with initial value $v(0,x) = 0$.

(1) is a particular case of the *Laplace equation* for the super-Brownian motion X. Note that formula (1) remains valid if $\phi \in C_K(\mathbb{R}^d)$ is not necessarily non-negative, but only then if t is less than a certain explosion time $t^* > 0$. More precisely, for a fixed $T > 0$, (1) will remain valid for $t \leqslant T$ for any compactly supported function that is bounded from below by a constant depending on T.

As shown in Chapter IV of [6] by the Brownian snake approach (see also Section II.7 of [7]), for any $x \in \mathbb{R}^d$, there exists a σ-finite measure $\mathbb{N}_x$ on $\mathcal{C}(\mathbb{R}_+, M_F(\mathbb{R}^d))$ called the excursion measure of super-Brownian motion such that the law of $(X_t)_{t>0}$ under P_μ is the same as the law of $(\sum X_t^i)_{t>0}$, where $\sum \delta_{X^i}$ is a Poisson point process with intensity $\int \mathbb{N}_x\, \mathrm{d}\mu(x)$. This fact is called the *canonical decomposition of super-Brownian motion*.

Intuitively, if one thinks of super-Brownian motion as the scaling limit of critical branching random walks as it is introduced in [7], the measure X_t under $\mathbb{N}_x$ for $t > 0$ represents the contribution to the population of the descendants at time t of one single individual alive at time 0 at the point x. The canonical decomposition expresses that the super-Brownian motion at time t is obtained by superimposing a Poisson number of such contributions.

It is also well known that the process X solves a *martingale problem* (see [7], Theorem II.5.1). For any $\phi \in C_b^2(\mathbb{R}^d)$,

$$\langle X_t, \phi\rangle = \langle X_0, \phi\rangle + M_t(\phi) + \frac{1}{2}\int_0^t \langle X_s, \Delta\phi\rangle\, \mathrm{d}s, \tag{3}$$

where $M_t(\phi)$ is an $\mathcal{F}_t^X$-martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$\big\langle M(\phi)\big\rangle_t = \int_0^t \big\langle X_s, \phi^2\big\rangle \mathrm{d}s.$$

Sugitani [9] proved that for $d \leqslant 3$, there exists a random continuous function $(t,x) \to L_t^x$ from $(0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}_+$ such that for any $\Psi \in C_b(\mathbb{R}^d)$,

$$\int_0^t \langle X_s, \Psi\rangle\, \mathrm{d}s = \int_{\mathbb{R}^d} \Psi(x) L_t^x\, \mathrm{d}x. \tag{4}$$

L_t^x is called the *local time* of X at point $x \in \mathbb{R}^d$ and time $t > 0$.

Take $L_0^x = 0$ for every $x \in \mathbb{R}^d$. The function $(L_t^x)_{t\geqslant 0,\ x\in\mathbb{R}^d}$ needs not being continuous at points of the form $(0,x)$, $x \in \mathbb{R}^d$. However Sugitani established for $d \geqslant 2$ that L_t^x is continuous in the pair (x,t) on the set of continuity points for $\mu q_t(x)$. For example, note that this set contains $\mathbb{R}_+ \times B(0,r)$ whenever $\mu(B(0,r)) = 0$.

Sugitani also proved that for $d=1$, L_t^x is always continuous in the pair (x,t) and even differentiable with respect to the space variable at points $x\in\mathbb{R}^d$ where $\mu(\{x\})=0$.

Under the assumption that $(t,x)\to\mu q_t(x)$ is continuous in $\mathbb{R}_+\times\mathbb{R}^d$, Sugitani [9] obtained the existence of *exponential moments for local times*: for any $T>0$, there exists a constant $\mathcal{K}(\mu,T)>0$ such that

$$E_\mu\left[\exp\left(rL_t^x\right)\right]<\infty \tag{5}$$

holds for any $t\leqslant T$, $x\in\mathbb{R}^d$ and $r<\mathcal{K}(\mu,T)$. We shall denote by $G_\mu^{t,x}:\{z\in\mathbb{C},\ |z|<\mathcal{K}(\mu,t)\}\to\mathbb{C}$ the function such that for any $z\in\mathbb{C}$, $|z|<\mathcal{K}(\mu,t)$,

$$E_\mu\left[\exp\left(zL_t^x\right)\right]=\exp\left(G_\mu^{t,x}(z)\right). \tag{6}$$

For convenience we will write $G_{\delta_{x_0}}^{t,x}=:G_{x_0}^{t,x}$.

If K is a compact set in $\mathbb{R}^d$, and if we only know that $(t,x)\to\mu q_t(x)$ is continuous in $\mathbb{R}_+\times K$ (for example if $\mu(K)=0$), it is easy to adapt the proof of (5) in [9] to obtain for any $T>0$ the existence of constants $\mathcal{K}(\mu,T,K)>0$, $\mathcal{C}(\mu,T,K)>0$ such that

$$E_\mu\left[\exp\left(rL_t^x\right)\right]\leqslant\mathcal{C}(\mu,T,K) \tag{7}$$

holds for any $t\leqslant T$, $x\in K$ and $r<\mathcal{K}(\mu,T,K)$. In this case, this clearly allows us to define the functions $G_\mu^{t,x}$ for $t\geqslant 0$, $x\in K$ in a way such that (6) remains valid for $z\in\mathbb{C}$, $|z|<\mathcal{K}(\mu,t,K)$.

Let us now discuss the *scaling properties of super-Brownian motion*. Let $c>0$, and if $f\in C_b(\mathbb{R}^d)_+$ let f_c be the function $f_c(x)=f(cx)$. From (2) we get

$$v(t,x,f_c)=c^2v\left(c^2t,cx,c^{-4}f\right).$$

The scaling properties of super-Brownian motion easily follow from (1) and that observation. Let $\mu\in M(\mathbb{R}^d)$. For any $A\in\mathcal{B}(\mathbb{R}^d)$, $t\geqslant 0$, let

$$X_t^{(c)}(A)=c^{-2}X_{c^2t}(cA),\qquad \mu^{(c)}(A)=c^2\mu\left(c^{-1}A\right).$$

Then, the law of the process $(X_t^{(c)},\ t\geqslant 0)$ under $P_{\mu^{(c)}}$ is equal to the law of the process $(X_t,\ t\geqslant 0)$ under P_μ.

Consequently, for any $\phi\in C_K(\mathbb{R}^d)$, the law of the process

$$\left(c^{-4}\int_0^{c^2t}\langle X_u,\phi\rangle\,\mathrm{d}u,\ t\geqslant 0\right)$$

under $P_{\mu^{(c)}}$ is the same as the law of the process

$$\left(\int_0^{t}\langle X_u,\phi_c\rangle\,\mathrm{d}u,\ t\geqslant 0\right)$$

under P_μ.

We are now in position to state our main result concerning the local behaviour of local times of super-Brownian motion.

1.4. The main result

Theorem 1. *Assume $d=2$ or 3. Let*

$$\psi(c)=\begin{cases}\sqrt{c} & \text{if } d=3,\\ \frac{c}{\sqrt{\ln(c)}} & \text{if } d=2.\end{cases}$$

Let $\nu\in M_F(\mathbb{R}^d)$ such that $\nu(B(0,\rho))=0$ for some $\rho>0$, and let X be a super-Brownian motion in $\mathbb{R}^d$ with initial value ν. If $x_1,\ldots,x_k$ are fixed points in $\mathbb{R}^d\setminus\{0\}$, the process

$$\left(X_t,\psi(c)\left(L_t^{x_1/c}-L_t^0\right),\ldots,\psi(c)\left(L_t^{x_k/c}-L_t^0\right)\right)_{t\geqslant 0}$$

converges as $c \to \infty$ in the sense of weak convergence in the space $C(\mathbb{R}^+, M(\mathbb{R}^d) \times \mathbb{R}^k)$ to a limiting process which can be written in the form

$$\left(X_t, \beta^{x_1}_{L^0_t}, \dots, \beta^{x_k}_{L^0_t}\right)_{t\geqslant 0}.$$

Here, $(\beta^x_t)_{t\geqslant 0, x\in\mathbb{R}^d}$ is a centered Gaussian process independent of X such that

$$\operatorname{cov}\left(\beta^x_t, \beta^y_s\right) = a(x,y)(t\wedge s),$$

and $a(x,y)$ is given by

$$a(x,y) = \begin{cases} \frac{1}{4\pi^2}\int_{\mathbb{R}^3} \mathrm{d}z\left(\frac{1}{|z-x|} - \frac{1}{|z|}\right) \times \left(\frac{1}{|z-y|} - \frac{1}{|z|}\right) & \text{if } d = 3, \\ \frac{1}{\pi} x\cdot y & \text{if } d = 2. \end{cases}$$

For convenience we will write $a(x,x) =: a(x)$.

Part of the motivation for Theorem 1 came from a recent paper of Lee [5] dealing with asymptotics for the occupation measure of super-Brownian motion.

1.5. Lee's result and extensions

Lee [5] only considers the three-dimensional case, but we will extend his result to the case of dimension 2.

Let $d = 2$ or 3, X a super-Brownian motion in $\mathbb{R}^d$, with initial value $\mu = \delta_{cx_0}$, where $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$ and $c > 0$.

Let K be a compact set in $\mathbb{R}^d$, and let ϕ, ξ be integrable functions on $\mathbb{R}^d$ supported on K satisfying $\int_K \xi(y)\,\mathrm{d}y = 0$, $\int_K \phi(y)\,\mathrm{d}y \neq 0$. We also define ϕ_c, ξ_c the functions such that for any $x\in\mathbb{R}^d$,

$$\phi_c(x) = \phi(cx), \qquad \xi_c(x) = \xi(cx).$$

Note that ϕ_c, ξ_c are supported on $K^{(c)} := c^{-1}K = \{y\colon cy \in K\}$.

One may think of the particular example where $K = \overline{B}(0,1)$, and where we consider the functions

$$\phi^0 = \mathbf{1}_{\overline{B}(0,1)}, \qquad \xi^0 = \mathbf{1}_{\overline{B}(0,1)\cap\{x\colon x_d\geqslant 0\}} - \mathbf{1}_{\overline{B}(0,1)\cap\{x\colon x_d< 0\}}.$$

For $T > 0$, let us introduce the quantities

$$D_{\phi,T} = \int_0^T \langle X_t, \phi\rangle\,\mathrm{d}t, \quad D_{\xi,T} = \int_0^T \langle X_t, \xi\rangle\,\mathrm{d}t, \quad T\geqslant 0.$$

We also set

$$a_\xi := \int_K\int_K \xi(y)\xi(z)a(y,z)\,\mathrm{d}y\,\mathrm{d}z.$$

The following result, is proved by Lee in [5] (for the case $d = 3$), by analysing the behaviour of $v(t, x_0, ac^3\phi_c + bc^{7/2}\xi_c)$ as $c\to\infty$ (cf. (2)).

Proposition 1. *Let $d = 2$ or 3, $t > 0$, $x_0 \in \mathbb{R}^d\setminus\{0\}$ and let K be a compact subset of $\mathbb{R}^d$. Let ϕ and ξ be integrable functions on $\mathbb{R}^d$ supported on K satisfying $\int_K \phi(y)\,\mathrm{d}y \neq 0$, $\int_K \xi(y)\,\mathrm{d}y = 0$. Under $P_{\delta_{cx_0}}(\cdot|X$ hits $K)$ the random vector*

$$\left(c^{d-4}D_{\phi,c^2t}, c^{d-4}\psi(c)D_{\xi,c^2t}\right)$$

converges in distribution as $c\to\infty$ to a non-degenerate limit (D_1, D_2). Furthermore, if $|a|$ and $|b|$ are small enough the Fourier transform of the limit is

$$E\left[\exp(\mathrm{i}aD_1 + \mathrm{i}bD_2)\right] = 1 + \frac{2|x_0|^2}{4-d}G^{T,0}_{x_0}\left(\mathrm{i}a\int_K \phi(y)\,\mathrm{d}y - \frac{b^2a_\xi}{2}\right),$$

where the function $G^{T,0}_{x_0}$ is determined by (6).

The case $d = 3$ stands as Corollary 1.2 in [5]. The form of the Fourier transform of (D_1, D_2) implies that the conditional law of D_2 knowing D_1 is Gaussian with mean 0 and variance $a_\xi D_1(\int \phi(y)\,\mathrm{d}y)^{-1}$.

In the example $\phi = \phi^0$, $\xi = \xi^0$, Proposition 1 has the following interpretation: conditionally on the event that a super-Brownian motion X started at δ_{cx_0} hits the unit ball, the occupation time of the unit ball up to time c^2t (that is D_{ϕ^0,c^2t}) is of order c^{4-d}, while the difference between the occupation time of the top half of the unit ball and the occupation time of the bottom half (that is D_{ξ^0,c^2t}) is of order $\sqrt{c}$ when $d = 3$, and $c(\ln(c))^{1/2}$ when $d = 2$.

In Section 3.2, we will derive Proposition 1 from the following proposition, which is itself a consequence of Theorem 1, as we will see in Section 3.1.

Proposition 2. *Let $d = 2$ or 3. Let t, K, ϕ, ξ be as in Proposition 1, and let ν be as in Theorem 1. Under P_ν, the random vector*

$$\left(c^d D_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t}\right)$$

converges in distribution as $c \to \infty$ to

$$\left(L_t^0 \int_K \phi(y)\,\mathrm{d}y, U_t\right),$$

where conditionally on L_t^0, U_t is Gaussian with variance $a_\xi L_t^0$.

As we will explain below, the first component $c^d D_{\phi_c,t}$ indeed converges almost surely. Also note that, from (6), the Fourier transform of $(\int \phi(y)\,\mathrm{d}y L_t^0, U_t)$ is simply

$$E_\nu\left[\exp\left(\mathrm{i}aL_t^0 \int_K \phi(y)\,\mathrm{d}y + \mathrm{i}bU_t\right)\right] = \exp\left(G_\nu^{T,0}\left(\mathrm{i}a\int_K \phi(y)\,\mathrm{d}y - \frac{b^2a_\xi}{2}\right)\right).$$

Let us explain how Proposition 2 follows from Theorem 1. The following approach is also valid in the one-dimensional case, and so we consider the general case $d \leqslant 3$. Let us first give a simple argument showing that $c^d D_{\phi_c,t}$ converges P_ν-almost surely as $c \to \infty$. By (4), and our assumption on the support of ϕ,

$$c^d D_{\phi_c,t} = c^d \int_{\mathbb{R}^d} \phi(cx)L_t^x\,\mathrm{d}x = \int_K \phi(y)L_t^{y/c}\,\mathrm{d}y.$$

Since $(t, x) \to L_t^x$ is continuous on $\mathbb{R}_+ \times B(0, \rho)$,

$$c^d D_{\phi_c,t} \underset{c\to\infty}{\longrightarrow} L_t^0 \int_K \phi(y)\,\mathrm{d}y$$

almost surely under P_ν by dominated convergence. See Theorem 5 in [9] for a related statement ([9] only states the convergence in distribution of $c^d D_{\phi_c,t}$).

If ϕ is replaced by ξ, the preceding argument is not sufficient. Indeed, since $\int \xi(x)\,\mathrm{d}x = 0$, the last step yields $c^d D_{\xi_c,t} \to 0$ as $c \to \infty$. Still, we can write

$$\int_{\mathbb{R}^d} \xi(y)L_t^{y/c}\,\mathrm{d}y = \int_{\mathbb{R}^d} \xi(y)\left(L_t^{y/c} - L_t^0\right)\mathrm{d}y,$$

which suggests to focus on the convergence of $\psi(c)(L_t^{y/c} - L_t^0)$, where $\psi(c)$ is a suitable normalization. This approach is similar to the work of Stroock–Varadhan–Papanicolaou [10] and Yor [11] concerning limit theorems for additive functionals of standard Brownian motion.

The one-dimensional case follows almost immediately from [9] and will be treated briefly in Section 4. *Until then we set $d = 2$ or 3.*

1.6. Outline of the proof of Theorem 1

We will first consider the case $k=1$. *We fix* $x \in \mathbb{R}^d \setminus \{0\}$ and a measure ν satisfying the assumption of Theorem 1. For $\alpha \geqslant 0$ ($\alpha > 0$ if $d=2$) we set $g_\alpha(z) = \int_0^\infty \exp(-\alpha t) p_t(z)\,\mathrm{d}t$, and for $y \in \mathbb{R}^d$, $g_\alpha^y(z) = g_\alpha(z-y)$. If $d=3$, note that we have $g_0(z) = 1/(2\pi|z|)$.

The key idea is to use the Tanaka formula for local times of super-Brownian motion in dimension less than 3 (see [2], Theorem 6.1). Let $y \in \mathbb{R}^d$, $d \leqslant 3$, $\mu \in M_F(\mathbb{R}^d)$, and let $\alpha \geqslant 0$ if $d=3$ or $\alpha > 0$ if $d=2$. Under the assumption $\langle \mu, g_\alpha^y \rangle < \infty$, we have P_μ-almost surely,

$$\langle X_t, g_\alpha^y \rangle = \langle \mu, g_\alpha^y \rangle + M_t(g_\alpha^y) + \alpha \int_0^t \langle X_s, g_\alpha^y \rangle \mathrm{d}s - L_t^y, \tag{8}$$

where $M_t(g_\alpha^y)$ is an $\mathcal{F}_t^X$-martingale which is defined in terms of the martingale measure associated with super-Brownian motion. In particular, $M_0(g_\alpha^y) = 0$ and $M_t(g_\alpha^y)$ has quadratic variation

$$\langle M(g_\alpha^y) \rangle_t = \int_0^t \langle X_s, (g_\alpha^y)^2 \rangle \mathrm{d}s.$$

If c is large enough so that $x/c \in B(0,\rho)$, the conditions $\langle \nu, g_\alpha^0 \rangle < \infty$, $\langle \nu, g_\alpha^{x/c} \rangle < \infty$, will clearly be satisfied. Thus we can use (8) for $y = x/c$ and $y=0$ to obtain, P_ν-almost surely,

$$L_t^{x/c} - L_t^0 = \langle X_0 - X_t, g_\alpha^{x/c} - g_\alpha^0 \rangle + M_t(g_\alpha^{x/c}) - M_t(g_\alpha^0) + \alpha \int_0^t \langle X_s, g_\alpha^{x/c} - g_\alpha^0 \rangle \mathrm{d}s. \tag{9}$$

In what follows we will take $\alpha = 0$ when $d=3$. Note that the last term in the right-hand side of the previous formula then vanishes.

In Section 2.1 we will prove the following lemmas:

Lemma 1. *Let $d=3$, $T>0$. Then P_ν-almost surely*

$$\sup_{t \leqslant T} \left| \sqrt{c} \langle X_0 - X_t, g_0^{x/c} - g_0^0 \rangle \right| \underset{c\to\infty}{\longrightarrow} 0.$$

Lemma 2. *Let $d=2$, $T>0$. Then P_ν-almost surely*

$$\text{(a)} \quad \sup_{t \leqslant T} \left| \left\langle X_0 - X_t, \frac{c}{\sqrt{\ln(c)}} \left(g_\alpha^{x/c} - g_\alpha^0\right) \right\rangle \right| \underset{c\to\infty}{\longrightarrow} 0,$$

$$\text{(b)} \quad \sup_{t \leqslant T} \left| \int_0^t \left\langle X_s, \frac{c}{\sqrt{\ln(c)}} \left(g_\alpha^{x/c} - g_\alpha^0\right) \right\rangle \mathrm{d}s \right| \underset{c\to\infty}{\longrightarrow} 0.$$

From (9), Lemmas 1 and 2 we see that the convergence of $(\psi(c)(L_t^{x/c} - L_t^0))_{t\geqslant 0}$ follows from that of $(\psi(c) M_t(g_\alpha^{x/c} - g_\alpha^0))_{t \geqslant 0}$.

All that remains to do is thus to study the convergence of the martingales

$$M_t^{x,c} := \begin{cases} \sqrt{c} M_t((g_0^{x/c} - g_0^0)) & \text{if } d=3, \\ \frac{c}{\sqrt{\ln(c)}} M_t((g_\alpha^{x/c} - g_\alpha^0)) & \text{if } d=2. \end{cases}$$

Lemma 3. *P_ν-almost surely,*

$$\langle M^{x,c} \rangle_t \underset{c\to\infty}{\longrightarrow} a(x) L_t^0.$$

This lemma will be proven in Section 2.2.

We then have to discuss the convergence in distribution of the martingale $M_t^{x,c}$. Using the Dubins–Schwarz theorem (see [8], Theorem V.1.6), we can write

$$M_t^{x,c} = \beta^{x,c}_{\langle M^{x,c}\rangle_t}, \tag{10}$$

where $\beta_t^{x,c}$ is a standard Brownian motion. We may and will assume that for $s \geqslant u \geqslant \langle M^{x,c}\rangle_\infty$, $\beta_s^{x,c} - \beta_u^{x,c} = \gamma_s^x - \gamma_u^x$, where γ^x is a one-dimensional Brownian motion independent of X. The collection of the laws of the family $(X, \beta^{x,c})_{c>0}$ is clearly tight.

In Section 2.3 we will prove

Lemma 4. *Suppose that along a subsequence $c_n \nearrow \infty$ we have*

$$\left(X, \beta^{x,c_n}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \left(X, \beta^x\right).$$

Then β^x is independent of X.

From the tightness of the laws of $(X, \beta^{x,c})$ and Lemma 4, it follows that

$$\left(X, \beta^{x,c}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \left(X, \beta^x\right) \tag{11}$$

with a Brownian motion β^x independent of X. We know from Lemma 3 that $\langle M^{x,c}\rangle_t$ converges P_ν-almost surely to $a(x)L_t^0$, and the convergence is uniform on every compact time interval by Dini's theorem. It then follows from (10) and (11) that $(X_t, M_t^{x,c})_{t\geqslant 0}$ converges in distribution to $(X_t, \beta^x_{a(x)L_t^0})_{t\geqslant 0}$.

The case $k = 1$ of Theorem 1 now follows from this fact and Lemmas 1 and 2. We will then extend this argument to the general case in Section 2.4.

2. Proof of Theorem 1

2.1. Preliminary reduction

In this section we will prove Lemmas 1 and 2. Recall that the point $x \neq 0$ is fixed, and that we have also fixed a measure ν satisfying the assumption of Theorem 1: there exists $\rho > 0$ such that $\nu(B(0,\rho)) = 0$.

We start with a preliminary result providing a uniform bound for the measure of small balls. From [2], Corollary 4.8, we know the following

Proposition 3. *Let $\delta > 0$ be fixed. If $d = 2$ or 3 then, for any $\mu \in M_F(\mathbb{R}^d)$, P_μ-almost surely,*

$$\limsup_{r\searrow 0}\sup_{t\geqslant\delta}\sup_{y\in\mathbb{R}^d} X_t\big(B(y,r)\big)\Phi(r)^{-1} \leqslant \kappa_\mu,$$

where $\Phi(r) = r^2(\ln(1/r))^{4-d}$ and κ_μ is a constant depending on μ.

An easy consequence is the following

Corollary 1. *P_ν-almost surely,*

$$\limsup_{r\searrow 0}\sup_{t\geqslant 0}\sup_{y\in B(0,\rho/2)} X_t\big(B(y,r)\big)r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4} \leqslant \kappa_\nu.$$

Proof of Corollary 1. Since $\nu(B(0,\rho)) = 0$, it is well known that P_ν-almost surely there exists $n_0(\omega) \in \mathbb{N}$ such that $\mathrm{supp}(X_t)$ does not intersect $B(0, \frac{3\rho}{4})$ for any $t \in [0, 2^{-n_0}]$. Provided $r \leqslant \rho/4$, we then have

$$\sup_{t\leqslant 2^{-n_0(\omega)}}\sup_{y\in B(0,\rho/2)} X_t\big(B(y,r)\big)r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4} = 0,$$

and thanks to Proposition 3, for P_ν-almost all ω,

$$\limsup_{r\searrow 0}\sup_{t\geqslant 2^{-n_0(\omega)}}\sup_{y\in\mathbb{R}^d} X_t\big(B(y,r)\big)r^{-2}\left(\ln\left(\frac{1}{r}\right)\right)^{d-4}\leqslant \kappa_\nu.$$

Corollary 1 follows. □

In the following we write $\kappa_0=\kappa_0(x)$, $\kappa_1=\kappa_1(x),\dots$ for constants that depend on the point x which is fixed.

2.1.1. The case $d=3$

For convenience we will use the notation

$$h^{x,c}(z):=\psi(c)\big(g_0^{x/c}(z)-g_0^0(z)\big)=\frac{\sqrt{c}}{2\pi}\left(\frac{1}{|z-x/c|}-\frac{1}{|z|}\right),\quad \text{for } z\neq 0, x/c.$$

We will use the following easy estimates on $h^{x,c}$:

(A_1) If $|z|\leqslant\frac{|x|}{2c}$, then $|h^{x,c}(z)|\leqslant\frac{\sqrt{c}}{|z|}$.

(A_2) Let $r\geqslant\frac{2|x|}{c}$. Then, the maximum of $|h^{x,c}|$ outside the ball $B(0,r)$ is attained at the point $\frac{rx}{|x|}$ and its value is $(2\pi\sqrt{c}\,r(\frac{r}{|x|}-\frac{1}{c}))^{-1}$.

(A_3) If $|z|\wedge|z-x/c|\geqslant\frac{|x|}{2c}$ and $|z|\leqslant\frac{4|x|}{c}$, then $|h^{x,c}(z)|\leqslant\frac{2c\sqrt{c}}{\pi|x|}$.

Proof of Lemma 1. If c is sufficiently large and $z\notin B(0,\rho)$, using (A_2),

$$\big|h^{x,c}(z)\big|\leqslant\frac{1}{2\pi\sqrt{c}\rho(\rho/|x|-1/c)}\underset{c\to\infty}{\longrightarrow}0.$$

Thus $\langle X_0,h^{x,c}\rangle=\langle\nu,h^{x,c}\rangle$ clearly goes to 0 as $c\to\infty$.

Hence, to get Lemma 1, it is enough to prove that P_ν-almost surely,

$$\sup_{t\leqslant T}\big|\langle X_t,h^{x,c}\rangle\big|=\sup_{t\leqslant T}\left|\int_{\mathbb{R}^3}h^{x,c}(z)X_t(\mathrm{d}z)\right|\underset{c\to\infty}{\longrightarrow}0.\tag{12}$$

If $t>0$ is fixed we know that 0 does not belong to $\operatorname{supp}X_t$, P_ν-almost surely. Thus the same argument as when $t=0$ gives $\langle X_t,h^{x,c}\rangle\to 0$ as $c\to\infty$. The point is that we need this convergence uniformly in $t\leqslant T$, and we know that there may exist exceptional times $t\leqslant T$ such that $0\in\operatorname{supp}X_t$.

It will be convenient to cut the domain $\mathbb{R}^3$ of the integral in (12) into different areas where we will be able to estimate the integrand. First of all, if $r>0$ is fixed, if $z\notin B(0,r)$ and c is large enough, using again (A_2) we have $|h^{x,c}(z)|\leqslant\kappa_0/\sqrt{c}$ so that

$$\left|\int_{\mathbb{R}^3\setminus B(0,r)}h^{x,c}(z)X_t(\mathrm{d}z)\right|\leqslant\frac{\kappa_0}{\sqrt{c}}\langle X_t,1\rangle.$$

Since $\sup_{t\geqslant 0}\langle X_t,1\rangle$ is P_ν-almost surely bounded, we get

$$\sup_{t\leqslant T}\left|\int_{\mathbb{R}^3\setminus B(0,r)}h^{x,c}(z)X_t(\mathrm{d}z)\right|\underset{c\to\infty}{\longrightarrow}0.$$

Thus, to get (12) it only remains to prove that for some $r_1>0$,

$$\sup_{t\leqslant T}\left|\int_{B(0,r_1)}h^{x,c}(z)X_t(\mathrm{d}z)\right|\underset{c\to\infty}{\longrightarrow}0.\tag{13}$$

From Corollary 1 it follows that P_ν-almost surely, there exists $r_1>0$ such that, for any $r\leqslant r_1$,

$$\sup_{t\geqslant 0}\sup_{y\in B(0,\rho/2)}X_t\big(B(y,r)\big)\leqslant 2\kappa_\nu r^2\ln\left(\frac{1}{r}\right).\tag{14}$$

Clearly we can assume $r_1 \leqslant \rho/4$. Let c be large enough so that $B(0, 4|x|/c) \subset B(0, r_1)$. To get rid of the singularities of $h^{x,c}$, we will first deal with the integrals in neighbourhoods of 0 and x/c. First, we have

$$\left| \int_{B(0,\frac{|x|}{2c})} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant \sum_{p \geqslant 1} \left| \int_{B(0,\frac{|x|}{2^p c}) \setminus B(0,\frac{|x|}{2^{p+1}c})} h^{x,c}(z) X_t(\mathrm{d}z) \right|.$$

Using (A_1) and (14) we see that P_ν-almost surely

$$\sup_{t \leqslant T} \left| \int_{B(0,\frac{|x|}{2^p c}) \setminus B(0,\frac{|x|}{2^{p+1}c})} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant 2\kappa_\nu \frac{2^{p+1} c\sqrt{c}}{|x|} \left(\frac{|x|}{2^p c} \right)^2 \ln\left(\frac{2^p c}{|x|} \right).$$

It is now clear that P_ν-almost surely

$$\sup_{t \leqslant T} \left| \int_{B(0,\frac{|x|}{2c})} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant \frac{4\kappa_\nu |x|}{c^{1/2}} \sum_{p \geqslant 1} \big(p\ln(2) + \ln(c) - \ln(|x|) \big) 2^{-p} \leqslant \kappa_1 \kappa_\nu c^{-1/2} \ln(c).$$

Clearly we can obtain an analogous bound for the quantity

$$\sup_{t \leqslant T} \left| \int_{B(\frac{x}{c},\frac{|x|}{2c})} h^{x,c}(z) X_t(\mathrm{d}z) \right|.$$

Now using (A_3) and (14) we get

$$\sup_{t \leqslant T} \left| \int_{B(0,\frac{4|x|}{c}) \setminus (B(0,\frac{|x|}{2c}) \cup B(\frac{x}{c},\frac{|x|}{2c}))} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant 2\kappa_\nu \frac{2c\sqrt{c}}{\pi |x|} \left(\frac{4|x|}{c} \right)^2 \ln\left(\frac{c}{4|x|} \right),$$

so we finally obtain that

$$\sup_{t \leqslant T} \left| \int_{B(0,\frac{4|x|}{c})} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant \kappa_\nu \kappa_2 \frac{\ln(c)}{\sqrt{c}}. \tag{15}$$

Let us now consider the integral on $B(0, r_1) \setminus B(0, \frac{4|x|}{c})$. Let N be such that

$$\frac{r_1}{2^N} \leqslant \frac{4|x|}{c} \leqslant \frac{r_1}{2^{N-1}}.$$

Note that, for $1 \leqslant p \leqslant N$, $r_1/2^p \geqslant 2^{N-p+1}|x|/c$ while $r_1/2^{p-1} \leqslant 2^{N-p+3}|x|/c$. Once again, using ($\mathrm{A}_2$) and (14), we obtain for $1 \leqslant p \leqslant N$

$$\sup_{t \leqslant T} \left| \int_{B(0,\frac{r_1}{2^{p-1}}) \setminus B(0,\frac{r_1}{2^p})} h^{x,c}(z) X_t(\mathrm{d}z) \right|$$
$$\leqslant \frac{2\kappa_\nu c^{3/2}}{|x|(2^{N-p+1}-1)^2} \left(\frac{2^{N-p+3}|x|}{c} \right)^2 \ln\left(\frac{c}{2^{N-p+3}|x|} \right) \leqslant 2\kappa_\nu \frac{2^6 |x|}{c^{1/2}} \ln\left(\frac{c}{|x|} \right).$$

Therefore,

$$\sup_{t \leqslant T} \left| \int_{B(0,r_1) \setminus B(0,\frac{4|x|}{c})} h^{x,c}(z) X_t(\mathrm{d}z) \right| \leqslant N \times \frac{2^7 \kappa_\nu |x|}{c^{1/2}} \ln\left(\frac{c}{|x|} \right) \leqslant \kappa_\nu \kappa_3 \frac{(\ln(c))^2}{c^{1/2}}.$$

The limit (13) now follows from the above and (15), which completes the proof of Lemma 1. □

2.1.2. The case d=2

Now we consider $\alpha > 0$ and the function

$$h^{x,c}(z) := \frac{c}{\sqrt{\ln(c)}}\big(g_\alpha^{x/c} - g_\alpha^0\big)(z), \quad \text{for } z \neq 0, x/c.$$

We will need the following estimates on $h^{x,c}$ which shall be proven in Appendix A. When c is large enough,

(B_1) If $z \in B(0, \frac{|x|}{2c})$,

$$\big|h^{x,c}(z)\big| \leqslant \kappa_4 \frac{c}{\sqrt{\ln(c)}} \ln^+\left(\frac{1}{|z|}\right).$$

If $z \in B(\frac{x}{c}, \frac{|x|}{2c})$,

$$\big|h^{x,c}(z)\big| \leqslant \kappa_4 \frac{c}{\sqrt{\ln(c)}} \ln^+\left(\frac{1}{|z - x/c|}\right).$$

(B_2) If $z \in B(0, c^{-3/4}) \setminus (B(0, \frac{|x|}{2c}) \cup B(\frac{x}{c}, \frac{|x|}{2c}))$,

$$\big|h^{x,c}(z)\big| \leqslant \kappa_5 c\sqrt{\ln(c)}.$$

(B_3) Let $r > c^{-7/8}$. Then if $z \notin B(0, r)$,

$$\big|h^{x,c}(z)\big| \leqslant \kappa_6 \frac{1}{r\sqrt{\ln(c)}}.$$

Proof of Lemma 2. We will use a similar method as for proving Lemma 1.

Fix $T > 0$. To obtain Lemma 2(a), it is enough to establish that P_ν-almost surely,

$$\sup_{t \leqslant T} \big|\langle X_t, h^{x,c}\rangle\big| \underset{c\to\infty}{\longrightarrow} 0. \tag{16}$$

If $r > 0$ is fixed and c is sufficiently large we can use (B_3) to get

$$\sup_{t \leqslant T} \left| \int_{\mathbb{R}^3 \setminus B(0,r)} h^{x,c}(z) X_t(\mathrm{d}z)\right| \leqslant \frac{\kappa_6}{r\sqrt{\ln(c)}} \sup_{t \leqslant T} \langle X_t, 1\rangle.$$

Since $\sup_{t \geqslant 0} |\langle X_t, 1\rangle|$ is finite P_ν-almost surely, we have

$$\sup_{t \leqslant T} \left| \int_{\mathbb{R}^2 \setminus B(0,r)} h^{x,c}(z) X_t(\mathrm{d}z)\right| \underset{c\to\infty}{\longrightarrow} 0. \tag{17}$$

Now, from Corollary 1 we know that P_ν-almost surely, there exists $r_2 > 0$ such that for any $r \leqslant r_2$,

$$\sup_{t \geqslant 0} \sup_{y \in B(0,\rho/2)} X_t\big(B(y, r)\big) \leqslant 2\kappa_\nu r^2 \left(\ln\left(\frac{1}{r}\right)\right)^2. \tag{18}$$

We can choose $r_2 \leqslant \rho/4$ and c large enough so that $B(0, c^{-3/4}) \subset B(0, r_2)$. We will first deal with the neighbourhood of singularities 0 and x/c. Using (18) and the estimate (B_1) we get for every $p \geqslant 1$,

$$\sup_{t \leqslant T} \left| \int_{B(0, \frac{|x|}{2^p c}) \setminus B(0, \frac{|x|}{2^{p+1} c})} h^{x,c}(z) X_t(\mathrm{d}z)\right| \leqslant \kappa_4 \frac{2\kappa_\nu c}{\sqrt{\ln(c)}} \left(\ln\left(\frac{2^{p+1}c}{|x|}\right)\right)^3 \left(\frac{|x|}{2^p c}\right)^2,$$

and thus

$$\sup_{t \leqslant T} \left| \int_{B(0, \frac{|x|}{2c})} h^{x,c}(z) X_t(\mathrm{d}z)\right| \leqslant \sum_{p \geqslant 1} \frac{2\kappa_4\kappa_\nu c}{\sqrt{\ln(c)}} \left(\ln\left(\frac{2^{p+1}c}{|x|}\right)\right)^3 \left(\frac{|x|}{2^p c}\right)^2 \leqslant \kappa_7 \frac{(\ln(c))^{5/2}}{c} \underset{c\to\infty}{\longrightarrow} 0.$$

We can bound the integral on $B(\frac{x}{c}, \frac{|x|}{2c})$ by the same quantity. Furthermore using (B_2) and (18) we have

$$\left| \int\limits_{B(0,c^{-3/4})\setminus(B(0,\frac{|x|}{2c})\cup B(\frac{x}{2c},\frac{|x|}{2c}))} h^{x,c}(z) X_t(\mathrm{d}z)\right| \leqslant 2\kappa_\nu\kappa_5(\ln(c))^{5/2}c^{-1/2} \underset{c\to\infty}{\longrightarrow} 0.$$

Let us now consider the integral on $D := B(0, r_2) \setminus B(0, c^{-3/4})$. Let $N \in \mathbb{N}$ such that

$$\frac{r_2}{2^N} \leqslant c^{-3/4} \leqslant \frac{r_2}{2^{N-1}}.$$

We may assume that c is large enough so that $c^{-7/8} < 2^{-N} r_2$. The domain D is contained in the union of the sets $B(0, \frac{r_2}{2^{p-1}}) \setminus B(0, \frac{r_2}{2^p})$ for $1 \leqslant p \leqslant N$. Since $c^{-7/8} \leqslant r_2/2^N$ we can use (B_3). Together with (18) this leads to:

$$\begin{aligned}\left| \int\limits_{B(0,r_2)\setminus B(0,c^{-3/4})} h^{x,c} X_t(\mathrm{d}z)\right| &\leqslant \sum_{p=1}^{N} \frac{2^{p+1}\kappa_\nu\kappa_6}{r_2\sqrt{\ln(c)}}\left(\frac{r_2}{2^{p-1}}\right)^2\left(\ln\left(\frac{2^{p-1}}{r_2}\right)\right)^2 \\ &\leqslant \sum_{p=1}^{N} \frac{8\kappa_\nu\kappa_6}{\sqrt{\ln(c)}} 2^{-p}\left(\ln\left(\frac{2^p}{r_2}\right)\right)^2 \leqslant \frac{\kappa_7}{\sqrt{\ln(c)}} \underset{c\to\infty}{\longrightarrow} 0.\end{aligned}$$

By considering the preceding estimates, we get

$$\sup_{t\leqslant T}\left| \int\limits_{B(0,r_2)} h^{x,c}(z) X_t(\mathrm{d}z)\right| \underset{c\to\infty}{\longrightarrow} 0.$$

Together with (17) this proves (16) and thus Lemma 2(a).

Lemma 2(b) is also a consequence of (16). Simply notice that if $t \leqslant T$,

$$\left|\int_0^t \langle X_s, h^{x,c}\rangle \,\mathrm{d}s\right| \leqslant t \times \sup_{s\leqslant T}\left|\langle X_s, h^{x,c}\rangle\right|,$$

which goes to 0 as $c \to \infty$ by (16). □

2.2. *Convergence of the quadratic variation of $M_t^{x,c}$*

From the martingale problem for X (see (3)) we know that $M_t^{x,c}$ is a local martingale whose quadratic variation is $\psi(c)^2 \int_0^t \langle X_s, (g_\alpha^{x/c} - g_\alpha^0)^2\rangle \,\mathrm{d}s$.

Let us now prove that this quantity converges to a non-degenerate limit as $c \to \infty$.

2.2.1. *Proof of Lemma 3, the case $d = 3$*

Recall we set $\alpha = 0$ for $d = 3$. We simply have

$$\langle M^{x,c}\rangle_t = \int_0^t c\langle X_s, (g^{x/c} - g^0)^2\rangle \mathrm{d}s = \frac{c}{4\pi^2}\int\limits_{\mathbb{R}^3} \mathrm{d}z\, L_t^z\left(\frac{1}{|z - x/c|} - \frac{1}{|z|}\right)^2 = \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^3} \mathrm{d}z\, L_t^{z/c}\left(\frac{1}{|z-x|} - \frac{1}{|z|}\right)^2.$$

Note that the function $z \to (1/|z - x| - 1/|z|)^2$ is integrable over $\mathbb{R}^3$. From Sugitani [9] we know that the function $x \to L_t^x$ is continuous with compact support. Hence, by dominated convergence, the above quantity goes P_ν-almost surely to $a(x)L_t^0$ where

$$a(x) = \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^3} \mathrm{d}z\left(\frac{1}{|z-x|} - \frac{1}{|z|}\right)^2.$$

2.2.2. Proof of Lemma 3, the case $d=2$

We now have

$$\langle M^{x,c}\rangle_t = \int_0^t \langle X_s, (h^{x,c})^2\rangle \mathrm{d}s = \int_{\mathbb{R}^2} L_t^z\big(h^{x,c}(z)\big)^2\,\mathrm{d}z. \tag{19}$$

Changing into polar coordinates (r,θ) and then setting $r=c^\delta$ leads to

$$\langle M^{x,c}\rangle_t = \int_{-\infty}^{\infty}\int_0^{2\pi} c^{2\delta}\ln(c)L_t^{(c^\delta,\theta)}\big(h^{x,c}(c^\delta,\theta)\big)^2\,\mathrm{d}\theta\,\mathrm{d}\delta,$$

where $L_t^{(r,\theta)}$ refers to the local time at time t and at the point with polar coordinates (r,θ).

We will then need some sharper estimates on $h^{x,c}(z)$. These will also be proven in Appendix A.

For two positive functions f and g we will write $f(c)=\mathrm{o}(g(c))$, if for any $\epsilon>0$ there is c_ϵ such that $f(c)\leqslant \epsilon g(c)$ for any $c\geqslant c_\epsilon$.

In the following estimates, z and δ are linked by the relation $|z|=c^\delta$, and c is supposed to be large enough.

(C_1) If $z\in B(0,\frac{1}{c(\ln(c))^{1/8}})$ or equivalently if $\delta\leqslant -1-\frac{\ln(\ln(c))}{8\ln(c)}$,

$$\big|h^{x,c}(z)\big|\leqslant \kappa_5|1+\delta|c\sqrt{\ln(c)}.$$

(C_2) If $\frac{1}{c(\ln(c))^{1/8}}\leqslant |z|\leqslant \frac{(\ln(c))^{1/8}}{c}$ and $z\notin B(\frac{x}{c},\frac{1}{c(\ln(c))^{1/8}})$,

$$\big|h^{x,c}(z)\big|\leqslant \kappa_8\frac{c}{(\ln(c))^{1/4}}.$$

(C_3) If $\frac{(\ln(c))^{1/8}}{c}\leqslant |z|\leqslant (\ln(c))^{-1/4}$ or equivalently if $-1+\frac{\ln(\ln(c))}{8\ln(c)}\leqslant\delta\leqslant -\frac{\ln(\ln(c))}{4\ln(c)}$,

$$\big|h^{x,c}(z)\big| = \frac{|z\cdot x|}{\pi\sqrt{\ln(c)}}c^{-2\delta}+\mathrm{o}\big(c^{-\delta}(\ln(c))^{-1/2}\big),$$

where $z\cdot x$ denotes the usual scalar product.

(C_4) If $(\ln(c))^{-1/4}\leqslant |z|$ or equivalently if $-\frac{\ln(\ln(c))}{4\ln(c)}\leqslant\delta$,

$$\big|h^{x,c}(z)\big|\leqslant \kappa_9\big(\ln(c)\big)^{-1/4}.$$

Now, in order to use these estimates, we will split $\mathbb{R}^2$ into the following five sets:

$$D_0^{(c)} = B\left(0,\frac{1}{c(\ln(c))^{1/8}}\right),$$
$$D_1^{(c)} = B\left(\frac{x}{c},\frac{1}{c(\ln(c))^{1/8}}\right),$$
$$D_2^{(c)} = B\left(0,\frac{(\ln(c))^{1/8}}{c}\right)\Big\backslash\big(D_0^{(c)}\cup D_1^{(c)}\big),$$
$$D_3^{(c)} = B\left(0,\frac{1}{(\ln(c))^{1/4}}\right)\Big\backslash\big(D_0^{(c)}\cup D_1^{(c)}\cup D_2^{(c)}\big),$$
$$D_4^{(c)} = \mathbb{R}^2\setminus\big(D_0^{(c)}\cup D_1^{(c)}\cup D_2^{(c)}\cup D_3^{(c)}\big).$$

We suppose c is large enough so that the sets $D_0^{(c)}$ and $D_1^{(c)}$ do not intersect, so that we can write

$$\langle M^{x,c}\rangle_t = \sum_{i=0}^4\int_{D_i^{(c)}} L_t^z\big(h^{x,c}(z)\big)^2\,\mathrm{d}z.$$

Fix $T>0$ and consider $t\in[0,T]$. First notice that P_ν-almost surely,

$$L_t^x \leqslant \sup_{y\in\mathbb{R}^d} L_T^y := L_T^* < \infty. \tag{20}$$

Using (C_4) on the domain $D_4^{(c)}$ we obtain

$$\int_{D_4^{(c)}} L_t^z\big(h^{x,c}(z)\big)^2\,\mathrm{d}z \leqslant \frac{(\kappa_9)^2}{\sqrt{\ln(c)}}\int_{D_4^{(c)}} L_t^z\,\mathrm{d}z = \frac{(\kappa_9)^2}{\sqrt{\ln(c)}}\int_0^T \langle X_s,1\rangle\,\mathrm{d}s \leqslant \frac{(\kappa_9)^2}{\sqrt{\ln(c)}}T\sup_{0\leqslant s\leqslant T}\langle X_s,1\rangle,$$

which goes P_ν-almost surely to 0 as $c\to\infty$.

The integrals over $D_0^{(c)}$, $D_1^{(c)}$ are treated in a symmetric way. We have

$$\begin{aligned}\int_{D_0^{(c)}} L_t^z\big(h^{x,c}(z)\big)^2\,\mathrm{d}z &= \int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}}\int_0^{2\pi} \ln(c)c^{2\delta}L_t^{(c^\delta,\theta)}\big(h^{x,c}\big(c^\delta,\theta\big)\big)^2\,\mathrm{d}\theta\,\mathrm{d}\delta\\ &\leqslant 2\pi L_T^*(\kappa_5)^2\big(\ln(c)\big)^2\int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}}(1+\delta)^2c^{2+2\delta}\,\mathrm{d}\delta,\end{aligned}$$

where we used (20) and the estimate (C_1). Then,

$$\big(\ln(c)\big)^2\int_{-\infty}^{-1-\frac{\ln(\ln(c))}{8\ln(c)}}(1+\delta)^2c^{2+2\delta}\,\mathrm{d}\delta \leqslant \kappa_{10}\frac{(\ln(\ln(c)))^2}{(\ln(c))^{5/4}}$$

so that $\int_{D_0^{(c)}} L_t^z(h^{x,c}(z))^2\,\mathrm{d}z$ goes almost surely to 0 as $c\to\infty$, and so does $\int_{D_1^{(c)}} L_t^z(h^{x,c}(z))^2\,\mathrm{d}z$.

Consider now the integral over $D_2^{(c)}$. Using (20) and the estimate (C_2) we get

$$\int_{D_2^{(c)}}\mathrm{d}z\,L_t^z\big(h^{x,c}(z)\big)^2 \leqslant 2\pi L_T^*\kappa_8^2\frac{c^2}{\sqrt{\ln(c)}}\int_{-1-\frac{\ln(\ln(c))}{8\ln(c)}}^{-1+\frac{\ln(\ln(c))}{8\ln(c)}} c^{2\delta}\ln(c)\,\mathrm{d}\delta \leqslant \pi L_T^*\kappa_8^2\big(\ln(c)\big)^{-1/4}$$

which goes almost surely to 0 as $c\to\infty$.

It remains to compute the integral over $D_3^{(c)}$ which is the preponderant part. We use (C_3), and the fact that P_ν-almost surely, $\sup\{|L_t^z-L_t^0|:\ z\in D_3^{(c)}\}$ tends to 0 as $c\to\infty$ to obtain

$$\begin{aligned}\int_{D_3^{(c)}}\mathrm{d}z\,L_t^z\big(h^{x,c}\big)^2(z) &= \int_{-1+\frac{\ln(\ln(c))}{8\ln(c)}}^{-\frac{\ln(\ln(c))}{4\ln(c)}}\int_0^{2\pi} c^{2\delta}\ln(c)L_t^{(c^\delta,\theta)}\big(h^{x,c}\big)^2\big(c^\delta,\theta\big)\,\mathrm{d}\theta\,\mathrm{d}\delta\\ &= \int_{-1+\frac{\ln(\ln(c))}{8\ln(c)}}^{-\frac{\ln(\ln(c))}{4\ln(c)}}\int_0^{2\pi}\frac{1}{\pi^2}\big(x_1^2\cos^2(\theta)+x_2^2\sin^2(\theta)\big)L_t^{(c^\delta,\theta)}\,\mathrm{d}\theta\,\mathrm{d}\delta + \mathrm{o}(1)\xrightarrow[c\to\infty]{}\frac{|x|^2}{\pi}L_t^0.\end{aligned}$$

This completes the proof of Lemma 3. □

As we explained in Section 1.6, to get the convergence of $(X, M^{x,c})$, and thus the case $k=1$ in Theorem 1, it only remains to prove Lemma 4.

2.3. *Independence of β^x and X*

Proof of Lemma 4. In what follows, when there is no ambiguity we will omit the x exponent in the notation $M^{x,c}$, $\beta^{x,c}$, β^x, γ^x.

By assumption, along a subsequence $c_n \nearrow \infty$,

$$\big(X, \beta^{c_n}\big) \xrightarrow[c\to\infty]{(\mathrm{d})} (X, \beta). \tag{21}$$

Recall from (3) the definition of the martingales $M_t(\phi)$. The formula for the quadratic variation of $M_t(\phi)$ shows that the collection $(M_t(\phi))_{t\geqslant 0,\ \phi\in\mathcal{C}_b^2(\mathbb{R}^d)}$ generates $\mathcal{F}^X$. Hence it is enough to check that β is independent of $(M_t(\phi))_{t\geqslant 0,\phi\in\mathcal{F}}$, where $\mathcal{F}$ is dense in $\mathcal{C}_b^2(E)$ for the topology $\mathcal{T}$ induced by the norm

$$\|f\| = \max\Big(\|f\|_\infty, \max_{i\in\{1,\dots,d\}}\big\|\partial^i f\big\|_\infty, \max_{i,j\in\{1,\dots,d\}}\big\|\partial^{i,j} f\big\|_\infty\Big)$$

for $f\in\mathcal{C}_b^2(\mathbb{R}^d)$. For instance, we let $\mathcal{F}$ be the space of all functions $\phi\in C_b^\infty(\mathbb{R}^d)$ such that there exists $A>0$ such that for all $n\in\mathbb{N}$, for all $\alpha=(\alpha_1,\dots,\alpha_n)\in\{1,\dots,d\}^n$,

$$\sup_{x\in\mathbb{R}^d}\big|\partial^\alpha\phi(x)\big| \leqslant A^{n+1}.$$

We will use the following notation: Let $0\leqslant t_1\leqslant\cdots\leqslant t_p$, $0=s_0\leqslant s_1\leqslant\cdots\leqslant s_q$, $\phi_1,\dots,\phi_p\in\mathcal{F}$, $\lambda_1,\dots,\lambda_p\in\mathbb{R}$ and $\mu_1,\dots,\mu_q\in\mathbb{R}$. We set

$$N_t = \sum_{j=1}^p \lambda_j M_{t\wedge t_j}(\phi_j),$$

and

$$f = \sum_{j=1}^q \mu_j \mathbf{1}_{(s_{j-1},s_j]}.$$

We let $\Lambda := \max_{1\leqslant i\leqslant p}|\lambda_i|$, and $K := \max_{1\leqslant i\leqslant p}\|\phi_i\|_\infty$. If W is a standard Brownian motion we also set

$$W_t(f) = \int_0^t f(s)\,\mathrm{d}W_s = \sum_{j=1}^q \mu_j(W_{t\wedge s_j} - W_{t\wedge s_{j-1}}).$$

We finally set $B = \exp(\mathrm{i}\beta_\infty(f)) = \exp(\mathrm{i}\int_0^\infty f(s)\,\mathrm{d}\beta_s)$.

In order to prove Lemma 4, it is enough to establish the following statement

Lemma 5. *For any choice of $(\phi_1,\dots,\phi_p)\in\mathcal{F}^p$, for any $0\leqslant t_1\leqslant\cdots\leqslant t_p$,*

$$E\big[BM_{t_1}(\phi_1)\cdots M_{t_p}(\phi_p)\big] = E[B]E\big[M_{t_1}(\phi_1)\cdots M_{t_p}(\phi_p)\big]. \tag{22}$$

The proof of Lemma 5 is based on

Lemma 6. *If $\Lambda = \max_{1\leqslant i\leqslant p}|\lambda_i|$ is small enough,*

$$E\left[B\exp\left(\mathrm{i}\sum_{j=1}^p\lambda_j M_{t_j}(\phi_j) + \frac{1}{2}\sum_{j,k=1}^p \lambda_j\lambda_k\big\langle M(\phi_j), M(\phi_k)\big\rangle_{t_j\wedge t_k}\right)\right] = E[B]. \tag{23}$$

Proof of Lemma 6. From the definition of the Brownian motion β^c (see (10)) we have

$$\beta_\infty^c(f) = \sum_{j=1}^q \mu_j\big(\beta_{s_j}^c - \beta_{s_{j-1}}^c\big) = \int_0^\infty f\big(\langle M^c\rangle_s\big)\,\mathrm{d}M_s^c + \int_{\langle M^c\rangle_\infty}^\infty f(s)\,\mathrm{d}\gamma_s. \tag{24}$$

Since γ is independent of X we also have

$$E\left[\exp\left(\mathrm{i}\int_{\langle M^c\rangle_\infty}^{\infty} f(s)\,\mathrm{d}\gamma_s+\frac{1}{2}\int_{\langle M^c\rangle_\infty}^{\infty} f^2(s)\,\mathrm{d}s\right)\middle|X\right]=1. \tag{25}$$

We use the notation $\mathcal{E}(M)_t=\exp(M_t-1/2<M\rangle_t)$ for the exponential martingale of the martingale M.

Lemma 7. *If Λ is small enough, the exponential martingale*

$$\mathcal{E}\left(\mathrm{i}\left(\int_0^t f\left(\langle M^c\rangle_s\right)\mathrm{d}M_s^c+N_t\right)\right)$$

is uniformly integrable.

Proof of Lemma 7. It suffices to check that

$$E\left[\exp\left(\int_0^\infty f^2\left(\langle M^c\rangle_s\right)\mathrm{d}\langle M^c\rangle_s+\langle N\rangle_\infty\right)\right]<\infty.$$

Since

$$\int_0^\infty f^2\left(\langle M^c\rangle_s\right)\mathrm{d}\langle M^c\rangle_s\leqslant\int_0^\infty f^2(s)\,\mathrm{d}s<\infty,$$

we only have to prove that

$$E\left[\exp\left(\langle N\rangle_\infty\right)\right]<\infty. \tag{26}$$

Note that

$$\langle N\rangle_\infty=\langle N\rangle_{t_p}=\sum_{i=1}^p\sum_{j=1}^p\lambda_i\lambda_j\int_0^{t_p}\langle X_s,\phi_i\phi_j\rangle\,\mathrm{d}s.$$

Recall the notation $K=\max_{1\leqslant i\leqslant p}\|\phi_i\|_\infty$. We have

$$E\left[\exp\left(\langle N\rangle_{t_p}\right)\right]\leqslant E\left[\exp\left(p^2\Lambda^2K^2\int_0^{t_p}\langle X_s,1\rangle\,\mathrm{d}s\right)\right]\leqslant\frac{1}{t_p}\int_0^{t_p}E\left[\exp\left(p^2\Lambda^2K^2t_p\langle X_s,1\rangle\right)\right]\mathrm{d}s,$$

where in the previous line we used Jensen's inequality. We know (see for example [7], p. 32) that

$$E\left[\exp\left(\lambda\langle X_s,1\rangle\right)\right]=\exp\left(\frac{\langle\nu,1\rangle\lambda}{1-\lambda s}\right)\quad\text{for }\lambda<\frac{1}{s}. \tag{27}$$

It follows that $\int_0^{t_p}E[\exp(p^2\Lambda^2K^2t_p\langle X_s,1\rangle)]\,\mathrm{d}s$ is finite as soon as we have $p^2\Lambda^2K^2t_p<(2t_p)^{-1}$, which is equivalent to $\Lambda<(\sqrt{2}Kpt_p)^{-1}$. Under this condition, (26) holds and the exponential martingale $\mathcal{E}(\mathrm{i}\int_0^t f(\langle M^c\rangle_s)\,\mathrm{d}M_s^c+N_t)$ is uniformly integrable. □

Let us now get back to the proof of Lemma 6. Using Lemma 7, we now have for every $t\in[0,\infty]$,

$$E\left[\exp\left\{\mathrm{i}\left(\int_0^t f\left(\langle M^c\rangle_s\right)\mathrm{d}M_s^c+N_t\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(\int_0^t f^2\left(\langle M^c\rangle_s\right)\mathrm{d}\langle M^c\rangle_s+\langle N\rangle_t+2\int_0^t f\left(\langle M^c\rangle_s\right)\mathrm{d}\langle M^c,N\rangle_s\right)\right\}\right]=1. \tag{28}$$

Define $H_t^c := \int_0^t f(\langle M^c\rangle_s)\,\mathrm{d}\langle M^c, N\rangle_s$.

Lemma 8. $\sup_{t\in[0,\infty]} |H_t^c| \to 0$ *almost surely as* $c\to\infty$.

Proof of Lemma 8. From the definition of the martingales M^c and N we have

$$H_t^c = \sum_{j=1}^p \int_0^{t\wedge t_j} \lambda_j f\big(\langle M^c\rangle_s\big)\langle X_s, h^{x,c}\phi_j\rangle\,\mathrm{d}s.$$

Thus, if $\mu = \max_{1\leqslant j\leqslant p}|\mu_j|$, for any $t\in[0,\infty]$,

$$\big|H_t^c\big| \leqslant p t_p \Lambda\mu \sup_{s\leqslant t_p}\big|\langle X_s, h^{x,c}\phi_j\rangle\big|.$$

Since the functions ϕ_j are bounded, the same arguments as in the proofs of Lemmas 1 and 2 show that P_ν-almost surely the right-hand side of the above display goes to 0 as $c\to\infty$, which completes the proof of Lemma 8. □

Let us now complete the proof of Lemma 6. We use (28) with $t=\infty$ and (25), together with (24) and the fact that

$$\int_0^\infty f\big(\langle M^c\rangle_s\big)\,\mathrm{d}\langle M^c\rangle_s = \int_0^{\langle M^c\rangle_\infty} f^2(s)\,\mathrm{d}s$$

to get

$$E_\nu\left[\exp\left\{\mathrm{i}\big(\beta_\infty^c(f)+N_\infty\big)+\frac{1}{2}\left(\int_0^\infty f^2(s)\,\mathrm{d}s + \langle N\rangle_\infty + H_\infty^c\right)\right\}\right]=1. \tag{29}$$

By the Kunita–Watanabe inequality (see for example [8], Corollary IV.1.16), for any $c>0$, P_ν-almost surely

$$\begin{aligned}\big|H_\infty^c\big| &\leqslant \left(\int_0^\infty f^2\big(\langle M^c\rangle_s\big)\,\mathrm{d}\langle M^c\rangle_s\right)^{1/2}\left(\int_0^\infty \mathrm{d}\langle N\rangle_s\right)^{1/2} \leqslant \left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)^{1/2}\big(\langle N\rangle_\infty\big)^{1/2}\\ &\leqslant \frac{1}{2}\int_0^\infty f^2(s)\,\mathrm{d}s + \frac{1}{2}\langle N\rangle_\infty,\end{aligned}$$

so that

$$\frac{1}{2}\left[\int_0^\infty f^2(s)\,\mathrm{d}s + \langle N\rangle_\infty + 2\big|H_\infty^c\big|\right] \leqslant \int_0^\infty f^2(s)\,\mathrm{d}s + \langle N\rangle_\infty. \tag{30}$$

From (21), and the fact that both $M_t(\phi_j)$ and $\langle M(\phi_j), M(\phi_k)\rangle_t$ are continuous functions of X (cf. (3)), we see that

$$\begin{aligned}&\Big(\beta_\infty^c(f), \big(M_{t_j}(\phi_j)\big)_{1\leqslant j\leqslant p}, \big(\langle M(\phi_k), M(\phi_l)\rangle_{t_k\wedge t_l}\big)_{1\leqslant k,l\leqslant p}\Big)\\ &\quad\xrightarrow[n\to\infty]{(\mathrm{d})}\Big(\beta_\infty(f), \big(M_{t_j}(\phi_j)\big)_{1\leqslant j\leqslant p}, \big(\langle M(\phi_k), M(\phi_l)\rangle_{t_k\wedge t_l}\big)_{1\leqslant k,l\leqslant p}\Big).\end{aligned} \tag{31}$$

Since $N_\infty = \sum_{j=1}^p \lambda_j M_{t_j}(\phi_j)$, $\langle N\rangle_\infty = \sum_{j,k=1}^p \lambda_j\lambda_k\langle M(\phi_j), M(\phi_k)\rangle_{t_j\wedge t_k}$, we can use (31) and Lemma 8 to pass to the limit $c\to\infty$ in the left-hand side of (29). Note that (30) and (26) provide the domination required to justify the passage to the limit. In this way we get

$$E\left[\exp\left\{\mathrm{i}\left(\beta_\infty(f)+\sum_{j=1}^p \lambda_j M_{t_j}(\phi_j)\right)+\frac{1}{2}\left(\int_0^\infty f^2(s)\,\mathrm{d}s + \sum_{j,k=1}^p \lambda_j\lambda_k\langle M(\phi_j), M(\phi_k)\rangle_{t_j\wedge t_k}\right)\right\}\right]=1,$$

where $\beta_\infty(f)=\sum_{j=1}^q \mu_j(\beta_{s_j}-\beta_{s_{j-1}})$. Since $E[B]=\exp(-\frac{1}{2}\int_0^\infty f^2(s)\,\mathrm{d}s)$, we get Lemma 6. □

Proof of Lemma 5. For any $(y_1,\dots,y_p)\in\mathbb{R}^p$, $(z_{j,k})_{1\leqslant j,k\leqslant p}\in\mathbb{R}^{p^2}$, let us write

$$\exp\left(\mathrm{i}\sum_{j=1}^p\lambda_j y_j+\frac{1}{2}\sum_{j,k=1}^p\lambda_j\lambda_k z_{j,k}\right)-1=\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}\lambda_1^{n_1}\cdots\lambda_p^{n_p}Q_{n_1,\dots,n_p}\big((y_j),(z_{j,k})\big),$$

where the series converges absolutely and for every choice of $n_1,\dots,n_p$, $Q_{n_1,\dots,n_p}((y_j),(z_{j,k}))$ is a polynomial of the $p+p^2$ variables $(y_j),(z_{j,k})$. Furthermore, the highest degree term in $Q_{1,\dots,1}$ is clearly $\mathrm{i}^p y_1\cdots y_p$. Thus (23) can be rewritten as

$$E\left[B\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}\lambda_1^{n_1}\cdots\lambda_p^{n_p}Q_{n_1,\dots,n_p}\big(\big(M_{t_j}(\phi_j)\big),\big(\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big)\big)\right]=0. \tag{32}$$

We now observe that for Λ small enough,

$$\begin{aligned}&E\left[B\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}\lambda_1^{n_1}\cdots\lambda_p^{n_p}Q_{n_1,\dots,n_p}\big(\big(M_{t_j}(\phi_j)\big),\big(\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big)\big)\right]\\&\quad=\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}\lambda_1^{n_1}\cdots\lambda_p^{n_p}E\big[BQ_{n_1,\dots,n_p}\big(\big(M_{t_j}(\phi_j)\big),\big(\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big)\big)\big].\end{aligned}\tag{33}$$

To justify the interchange of summation and expectation it is enough to verify that

$$E\left[|B|\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}|\lambda_1|^{n_1}\cdots|\lambda_p|^{n_p}\big|Q_{n_1,\dots,n_p}\big(\big(M_{t_j}(\phi_j)\big),\big(\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big)\big)\big|\right]$$

is finite. Let us define new polynomials $\widehat{Q}_{n_1,\dots,n_p}$ by

$$\exp\left(\sum_{j=1}^p\lambda_j|y_j|+\frac{1}{2}\sum_{j,k=1}^p\lambda_j\lambda_k|z_{j,k}|\right)-1=\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}\lambda_1^{n_1}\cdots\lambda_p^{n_p}\widehat{Q}_{n_1,\dots,n_p}\big(\big(|y_j|\big),\big(|z_{j,k}|\big)\big),$$

and observe that we always have

$$\big|Q_{n_1,\dots,n_p}\big((y_j),(z_{j,k})\big)\big|\leqslant\widehat{Q}_{n_1,\dots,n_p}\big(\big(|y_j|\big),\big(|z_{j,k}|\big)\big).$$

Since $|B|=1$, it is then enough to prove that

$$E\left[\sum_{\substack{n_1,\dots,n_p\in\mathbb{N}\\ n_1+\dots+n_p\geqslant 1}}|\lambda_1|^{n_1}\cdots|\lambda_p|^{n_p}\widehat{Q}_{n_1,\dots,n_p}\big(\big(\big|M_{t_j}(\phi_j)\big|\big),\big(\big|\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big|\big)\big)\right]$$

is finite, which from the definition of $\widehat{Q}$ holds if

$$E\left[\exp\left\{\sum_{j=1}^p\big|\lambda_jM_{t_j}(\phi_j)\big|+\frac{1}{2}\sum_{j,k=1}^p\big|\lambda_j\lambda_k\langle M(\phi_j),M(\phi_k)\rangle_{t_j\wedge t_k}\big|\right\}\right]<\infty.$$

By the Cauchy–Schwarz inequality, it is enough to check the finiteness of

$$A_p(\lambda_1,\dots,\lambda_p)=E\left[\exp\left(2\sum_{j=1}^p\big|\lambda_jM_{t_j}(\phi_j)\big|\right)\right]$$

and

$$B_p(\lambda_1, \dots, \lambda_p) = E\left[\exp\left(\sum_{j,k=1}^{p} \left|\lambda_j \lambda_k \langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right|\right)\right],$$

provided Λ is small enough. The fact that both $A_p(\lambda_1, \dots, \lambda_p)$ and $B_p(\lambda_1, \dots, \lambda_p)$ are finite when Λ is sufficiently small follows from (27) by arguments similar to the proof of Lemma 7. Thus, the interchange of summation and expectation in (33) is justified.

From (32) we now get

$$\sum_{\substack{n_1,\dots,n_p \in \mathbb{N} \\ n_1+\dots+n_p \geqslant 1}} \lambda_1^{n_1} \cdots \lambda_p^{n_p} E\left[B Q_{n_1,\dots,n_p}\left(\left(M_{t_j}(\phi_j)\right), \left(\langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right)\right] = 0.$$

Since this is true for any $(\lambda_1, \dots, \lambda_p)$ such that Λ is sufficiently small we obtain that for any $n_1, \dots, n_p \in \mathbb{N}$ such that $n_1 + \dots + n_p \geqslant 1$,

$$E\left[B Q_{n_1,\dots,n_p}\left(\left(M_{t_j}(\phi_j)\right), \left(\langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right)\right] = 0.$$

Specialising to the case $f = 0$ we have also, for any $n_1, \dots, n_p \in \mathbb{N}$ such that $n_1 + \dots + n_p \geqslant 1$,

$$E\left[Q_{n_1,\dots,n_p}\left(\left(M_{t_j}(\phi_j)\right), \left(\langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right)\right] = 0.$$

We have finally proven that

$$\begin{aligned} 0 &= E\left[B Q_{n_1,\dots,n_p}\left(\left(M_{t_j}(\phi_j)\right), \left(\langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right)\right] \\ &= E[B] E\left[Q_{n_1,\dots,n_p}\left(\left(M_{t_j}(\phi_j)\right), \left(\langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right)\right]. \end{aligned} \tag{34}$$

We are now in a position to finish the proof of Lemma 5. We prove (22) by induction on p. For $p = 1$, we simply use (34) with $p = 1, n_1 = 1$ and observe that $Q_1(y) = \mathrm{i}y$ to get

$$E\left[B M_{t_1}(\phi_1)\right] = E[B] E\left[M_{t_1}(\phi_1)\right].$$

Now let $p \geqslant 2$ and let us assume that (22) holds up to the order $p - 1$. Observe first that we can write

$$Q_{1,\dots,1}\left((y_j), (z_{j,k})\right) = \mathrm{i}^p y_1 \cdots y_p + \sum_{\substack{J \subset \{1,\dots,p\} \\ K \subset \{1,\dots,p\}^2}} \alpha_{J,K} \left(\prod_{i \in J} y_i\right) \left(\prod_{(j,k) \in K} z_{j,k}\right)$$

where the constants $\alpha_{J,K}$ may be non-zero only if $\operatorname{Card} J + \operatorname{Card} K < p$.

Using (34) with $n_1 = \dots = n_p = 1$, we see that (22) will follow if we can prove that for any choice of (J, K) such that $\operatorname{Card} J + \operatorname{Card} K < p$,

$$\begin{aligned} &E\left[B \left(\prod_{i \in J} M_{t_i}(\phi_i)\right) \left(\prod_{(j,k) \in K} \langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right] \\ &\quad = E[B] E\left[\left(\prod_{i \in J} M_{t_i}(\phi_i)\right) \left(\prod_{(j,k) \in K} \langle M(\phi_j), M(\phi_k)\rangle_{t_j \wedge t_k}\right)\right]. \end{aligned} \tag{35}$$

To get rid of the quadratic variation terms we write

$$\langle M(\phi_j), M(\phi_k)\rangle_t = \int_0^t \langle X_s, \phi_j \phi_k\rangle \, \mathrm{d}s,$$

and from the martingale problem for X, using an easy induction on n,

$$
\begin{aligned}
\langle X_s, \phi_j\phi_k\rangle &= \langle X_0, \phi_j\phi_k\rangle + M_s(\phi_j\phi_k) + \frac{1}{2}\int_0^s \mathrm{d}u\big\langle X_u, \Delta(\phi_j\phi_k)\big\rangle \\
&= \langle X_0, \phi_j\phi_k\rangle + \frac{s}{2}\big\langle X_0, \Delta(\phi_j\phi_k)\big\rangle + M_s(\phi_j\phi_k) \\
&\quad + \frac{1}{2}\int_0^s \mathrm{d}u M_u\big(\Delta(\phi_j\phi_k)\big) + \frac{1}{4}\int_0^s \mathrm{d}u\int_0^u \mathrm{d}r\big\langle X_r, \Delta^2(\phi_j\phi_k)\big\rangle \\
&= X_s^n(\phi_j\phi_k) + R_s^n(\phi_j\phi_k),
\end{aligned}
$$

where

$$
\begin{aligned}
X_s^n(\phi_j\phi_k) &= \langle X_0, \phi_j\phi_k\rangle + \frac{s}{2}\big\langle X_0, \Delta(\phi_j\phi_k)\big\rangle + \cdots + \frac{s^{n-1}}{2^{n-1}(n-1)!}\big\langle X_0, \Delta^{n-1}(\phi_j\phi_k)\big\rangle \\
&\quad + M_s(\phi_j\phi_k) + \frac{1}{2}\int_0^s \mathrm{d}u M_u\big(\Delta(\phi_j\phi_k)\big) \\
&\quad + \cdots + \frac{1}{2^{n-1}}\int_0^s \mathrm{d}u_1\int_0^{u_1} \mathrm{d}u_2\cdots\int_0^{u_{n-2}} \mathrm{d}u_{n-1} M_{u_{n-1}}\big(\Delta^{n-1}(\phi_j\phi_k)\big),
\end{aligned}
$$

and

$$
R_s^n(\phi_j\phi_k) = \frac{1}{2^n}\int_0^s \mathrm{d}u_1\int_0^{u_1} \mathrm{d}u_2\cdots\int_0^{u_{n-1}} \mathrm{d}u_n\big\langle X_{u_n}, \Delta^n(\phi_j\phi_k)\big\rangle.
$$

By assumption both ϕ_j and ϕ_k belong to $\mathcal{F}$, and it easily follows that $\phi_j\phi_k$ is also in $\mathcal{F}$. Let A be the constant associated with $\phi_j\phi_k$ in the definition of $\mathcal{F}$. From the formula for $R_s^n(\phi_j\phi_k)$ one easily gets

$$
E_\nu\big[\big(R_s^n(\phi_j\phi_k)\big)^p\big] \leqslant \frac{d^{pn}A^{p(2n+1)}s^{pn}}{(n!)^p} E_\nu\Big[\Big(\sup_{u\in[0,s]}\langle X_u, 1\rangle\Big)^p\Big].
$$

Recall that $\langle X_t, 1\rangle$ is an $\mathcal{F}_t^X$-martingale (cf. (3) with $\phi = 1$). Furthermore we know from (27) that this quantity has finite moments of any order. If $q > 0$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, using Doob's inequality, we see that for any $s \leqslant t$,

$$
E_\nu\big[\big(R_s^n(\phi_j\phi_k)\big)^p\big] \leqslant \frac{d^{pn}A^{p(2n+1)}t^{pn}}{(n!)^p} q^p E_\nu\big[\big(\langle X_t, 1\rangle\big)^p\big],
$$

so that $R_s^n(\phi_j\phi_k) \to 0$ as $n \to \infty$, in L^p for every $p < \infty$, uniformly on $[0, t]$.

From the expression of $X_s^n(\phi_j\phi_k)$, we see that we can use the induction hypothesis (recall $\mathrm{Card}\, J + \mathrm{Card}\, K < p$) to get

$$
\begin{aligned}
&E\Bigg[B\bigg(\prod_{i\in J} M_{t_i}(\phi_i)\bigg)\bigg(\prod_{(j,k)\in K}\int_0^{t_j\wedge t_k} \mathrm{d}s X_s^n(\phi_j\phi_k)\bigg)\Bigg] \\
&\quad = E[B]E\Bigg[\bigg(\prod_{i\in J} M_{t_i}(\phi_i)\bigg)\bigg(\prod_{(j,k)\in K}\int_0^{t_j\wedge t_k} \mathrm{d}s X_s^n(\phi_j\phi_k)\bigg)\Bigg].
\end{aligned}
$$

and letting $n \to \infty$ leads to (35), which completes the proof of Lemma 5, and thus the one of Lemma 4. □

We have proven that for every $x \neq 0$,

$$
\big(X, M_t^{x,c}\big) \xrightarrow[n\to\infty]{(\mathrm{d})} \big(X, \beta^x_{a(x)L_t^0}\big),
$$

with β^x a standard Brownian motion independent of X. As it was explained in Section 1.6, this gives the case $k=1$ of Theorem 1.

We now address the problem of the convergence in distribution of

$$\left(X, M_t^{x_1,c}, M_t^{x_2,c}, \dots, M_t^{x_k,c}\right)$$

as $c \to \infty$, where $\mathcal{X} = (x_1, \dots, x_k)$ is a k-tuple of non-zero points in $\mathbb{R}^d$.

2.4. Space dependence for the limit

We denote by $\overline{M}_t^{\mathcal{X},c}$ the k-tuple of martingales $(M_t^{x_1,c}, \dots, M_t^{x_k,c})$, and by $\langle \overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c}\rangle_t$ the matrix $(\langle M^{x_i,c}, M^{x_j,c}\rangle)_{1\leqslant i,j\leqslant k}$. Recall that for a fixed $x \neq 0$, $\langle M^{x,c}\rangle_t \to a(x)L_t^0$ as $c \to \infty$. By adapting the argument used in Section 2.2 to prove this convergence, it is easy to see that P_ν-almost surely,

$$\left\langle \overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c}\right\rangle_t \underset{n\to\infty}{\longrightarrow} \left(a(x_i,x_j)L_t^0\right)_{1\leqslant i,j\leqslant k}. \tag{36}$$

By the Dini theorem, the convergence of the diagonal terms in (36) is uniform in $t \geqslant 0$. Then, using the Kunita–Watanabe inequality, it is not hard to see that the convergence of the non-diagonal terms is also uniform in $t \geqslant 0$.

Let A be the matrix $(a(x_i,x_j))_{1\leqslant i,j\leqslant k}$. Since A is symmetric, there exists an orthogonal matrix O such that $D := OA\,{}^tO$ is a diagonal matrix.

Now let

$$\overline{N}_t^c = \left(N_t^{1,c}, \dots, N_t^{k,c}\right) := O\overline{M}_t^{\mathcal{X},c}$$

and $\langle \overline{N}^c, \overline{N}^c\rangle_t$ be the matrix $(\langle N^{i,c}, N^{j,c}\rangle_t)_{1\leqslant i,j\leqslant k}$.

We clearly have $\langle \overline{N}^c, \overline{N}^c\rangle_t = O\langle \overline{M}^{\mathcal{X},c}, \overline{M}^{\mathcal{X},c}\rangle_t\,{}^tO$, so that P_ν-almost surely,

$$\left\langle \overline{N}^c, \overline{N}^c\right\rangle_t \underset{n\to\infty}{\longrightarrow} L_t^0 D \tag{37}$$

uniformly in $t \geqslant 0$.

If for $s \geqslant 0$ we let $\tau_s^{j,c} := \inf\{t \geqslant 0\colon\ \langle N^{j,c}\rangle_t \geqslant s\}$, we thus have that for any $j \neq k$, for any $s \geqslant 0$, both $\langle N^{j,c}, N^{k,c}\rangle_{\tau_s^{j,c}}$, and $\langle N^{j,c}, N^{k,c}\rangle_{\tau_s^{k,c}}$ go to 0 as $c \to \infty$.

Using the Dubins–Schwarz theorem, for every $i \in \{0, \dots, k\}$, we have

$$N_t^{i,c} = B^{i,c}_{\langle N^{i,c}\rangle_t},$$

where $B^{i,c}$ is a linear Brownian motion. For every $i \in \{1, \dots, k\}$, we may and will assume that for $s \geqslant u \geqslant \langle N^{i,c}\rangle_\infty$, we have $B_s^{i,c} - B_u^{i,c} = \gamma_s^{i,c} - \gamma_u^{i,c}$, where $(\gamma^{1,c}, \dots, \gamma^{k,c})_{c>0}$ is a family of independent k-dimensional Brownian motions, independent of X.

By an evident adaptation of Theorem 2.3 of [8], Chapter XIII, the convergence to 0 of $\langle N^{j,c}, N^{k,c}\rangle_{\tau_s^{j,c}}$, and $\langle N^{j,c}, N^{k,c}\rangle_{\tau_s^{k,c}}$ implies that

$$\left(B^{1,c}, \dots, B^{k,c}\right) \underset{n\to\infty}{\overset{\text{(d)}}{\longrightarrow}} \left(B^1, \dots, B^k\right),$$

where $(B^1, \dots, B^k)$ is a k-dimensional Brownian motion. By adapting the arguments of Section 2.3, we can also verify that

$$\left(X, B^{1,c}, \dots, B^{k,c}\right) \underset{n\to\infty}{\overset{\text{(d)}}{\longrightarrow}} \left(X, B^1, \dots, B^k\right),$$

where the k-dimensional Brownian motion $B = (B^1, \dots, B^k)$ is independent of X.

It follows that

$$\left(X_t, N_t^{1,c}, \dots, N_t^{k,c}\right)_{t\geqslant 0} \underset{n\to\infty}{\longrightarrow} \left(X_t, B^1_{D_{11}L_t^0}, \dots, B^k_{D_{kk}L_t^0}\right)_{t\geqslant 0}.$$

Now recall that $\overline{M}^c = O^{-1}\overline{N}^c$ so that

$$\big(X_t, \overline{M}_t^{\mathcal{X},c}\big)_{t\geqslant 0} \xrightarrow[n\to\infty]{\text{(d)}} \big(X_t, \beta^{\mathcal{X},1}_{L_t^0}, \ldots, \beta^{\mathcal{X},k}_{L_t^0}\big), \tag{38}$$

where

$$\beta_t^X = \big(\beta_t^{\mathcal{X},1}, \ldots, \beta_t^{\mathcal{X},k}\big) := O^{-1}\big(B^1_{D_{11}t}, \ldots, B^k_{D_{kk}}\big)$$

is a centered k-dimensional Gaussian process satisfying

$$\operatorname{cov}\big(\beta_t^{\mathcal{X}}, \beta_s^{\mathcal{X}}\big) = (t\wedge s)O^{-1}D\,{}^tO^{-1} = (t\wedge s)A.$$

It is immediate that $\beta^{\mathcal{X}} \overset{\text{(d)}}{=} (\beta^{x_1}, \ldots, \beta^{x_k})$, where the collection $(\beta^x)_{x\in\mathbb{R}^d}$ is as in Theorem 1. The same arguments as in the case $k=1$, using Lemmas 1 and 2, show that the general case of Theorem 1 follows from (38). □

3. Applications of Theorem 1

3.1. A non-conditioned result

We now turn to the convergence of the vector $\Xi_c = (c^d D_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t})$ under P_ν, ν being as in Theorem 1.

Proof of Proposition 2. Let $t>0$. Recall from Section 1.6 that

$$\Xi_c = \bigg(\int_{\mathbb{R}^d} L_t^{y/c}\phi(y)\,\mathrm{d}y, \psi(c)\int_{\mathbb{R}^d} \xi(y)\big(L_t^{y/c} - L_t^0\big)\,\mathrm{d}y\bigg). \tag{39}$$

We already noticed that P_ν-almost surely,

$$\lim_{c\to\infty}\int_{\mathbb{R}^d} L_t^{y/c}\phi(y)\,\mathrm{d}y = \bigg(\int_{\mathbb{R}^d}\phi(y)\,\mathrm{d}y\bigg)L_t^0. \tag{40}$$

Furthermore it is easy to check that

$$\lim_{c\to\infty} E_\nu\bigg[\bigg|\psi(c)\int_{\mathbb{R}^d}\xi(y)\big(L_t^{y/c}-L_t^0\big)\,\mathrm{d}y - \int_{\mathbb{R}^d}\xi(y)M_t^{y,c}\,\mathrm{d}y\bigg|\bigg] = 0. \tag{41}$$

Indeed the Tanaka formula (8) shows that for every $y\in K$ and c sufficiently large,

$$\psi(c)\big(L_t^{y/c}-L_t^0\big) - M_t^{y,c} = \big\langle X_0 - X_t, h^{y,c}\big\rangle + \alpha\int_0^t \big\langle X_s, h^{y,c}\big\rangle\,\mathrm{d}s.$$

Hence (41) follows from the convergence

$$\lim_{c\to\infty}\Big(\sup_{y\in K}\sup_{s\in[0,t]} E_\nu\big[\big\langle X_s, \big|h^{y,c}\big|\big\rangle\big]\Big) = 0,$$

which is itself an easy consequence of the first moment formula for X (see Proposition 2.10 in [6]). By (39), (40) and (41), Proposition 2 reduces to verifying that

$$\bigg(L_t^0, \int_{\mathbb{R}^d}\xi(y)M_t^{y,c}\,\mathrm{d}y\bigg) \xrightarrow[n\to\infty]{\text{(d)}} \big(L_t^0, U_t\big), \tag{42}$$

where, conditionally given L_t^0, U_t is centered Gaussian with variance $a_\xi L_t^0$.

Lemma 9. *We can find $c_0>0$ such that for every integer $p\geqslant 1$,*

$$\sup_{y\in K}\sup_{c\geqslant c_0} E_\nu\big[\big|M_t^{y,c}\big|^p\big] < \infty. \tag{43}$$

Let us postpone the proof of Lemma 9 and complete that of Proposition 2. From (7), the limiting law in (42) is determined by its moments. Hence to get the convergence (42) it is enough to prove that, for every integers k and $p \geqslant 1$,

$$\lim_{c\to\infty} E\left[\left(L_t^0\right)^k\left(\int \xi(y)M_t^{y,c}\,\mathrm{d}y\right)^p\right] = E\left[\left(L_t^0\right)^k(U_t)^p\right]. \tag{44}$$

Note that

$$\left(L_t^0, U_t\right) \overset{(\mathrm{d})}{=} \left(L_t^0, \int \xi(y)\beta^y_{L_t^0}\right)$$

with the notation of Theorem 1. By the Fubini theorem, (44) follows from the fact that for every $y_1, \ldots, y_p \in K$,

$$\lim_{c\to\infty} E\left[\left(L_t^0\right)^k M_t^{y_1,c}\cdots M_t^{y_p,c}\right] = E\left[\left(L_t^0\right)^k \beta^{y_1}_{L_t^0}\cdots\beta^{y_p}_{L_t^0}\right]. \tag{45}$$

However we proved (Theorem 1) that the $(p+1)$-tuple $(L_t^0, M_t^{y_1,c}, \ldots, M_t^{y_k,c})$ converges in distribution to $(L_t^0, \beta^{x_1}_{L_t^0}, \ldots, \beta^{x_k}_{L_t^0})$, and the bound of Lemma 9 allows us to derive (45) from this convergence in distribution. This completes the proof of Proposition 2.

Proof of Lemma 9. We will only give the proof in the three-dimensional case. In the two-dimensional case, there are a few technical differences as can be guessed by looking at Sections 2.1 and 2.2, but the ideas remain very similar, and we leave this case to the reader.

Let $d = 3$, $a := \sup\{|y|,\ y \in K\}$, $c_0 := 4a\rho^{-1}$, $t > 0$, $p \geqslant 1$, and $c \geqslant c_0$. For $y \in K$, we first use the Burkholder–Davis–Gundy inequality to obtain

$$E\left[\left(\left|M_t\left(h^{y,c}\right)\right|\right)^p\right] \leqslant c_p E\left[\left\langle M\left(h^{y,c}\right)\right\rangle_t^{p/2}\right], \tag{46}$$

where c_p is a constant. Let $\eta^{y,c} := E[\langle M(h^{y,c})\rangle_t^{p/2}]$.

From (3), we have

$$\eta^{y,c} = E\left[\left(\int_0^t \left\langle X_s, \left(h^{y,c}\right)^2\right\rangle \mathrm{d}s\right)^{p/2}\right].$$

From the fact that $c \geqslant 4a\rho^{-1}$, we can split $\mathbb{R}^d$ into the domains $B(0, 2ac^{-1})$, $B(0,\rho/2)\setminus B(0,2ac^{-1})$, $\mathbb{R}^d \setminus B(0,\rho/2)$ and introduce

$$h_a^{y,c} := h^{y,c}\mathbf{1}_{B(0,\frac{2a}{c})}, \qquad \eta_a^{y,c} := E\left[\left(\int_0^t \left\langle X_s, \left(h_a^{y,c}\right)^2\right\rangle \mathrm{d}s\right)^{p/2}\right],$$

$$h_\rho^{y,c} := h^{y,c}\mathbf{1}_{B(0,\frac{\rho}{2})\setminus B(0,\frac{2a}{c})}, \qquad \eta_\rho^{y,c} := E\left[\left(\int_0^t \left\langle X_s, \left(h_\rho^{y,c}\right)^2\right\rangle \mathrm{d}s\right)^{p/2}\right],$$

$$\bar h^{y,c} := h^{y,c}\mathbf{1}_{\mathbb{R}^3\setminus B(0,\frac{\rho}{2})}, \qquad \bar\eta^{y,c} := E\left[\left(\int_0^t \left\langle X_s, \left(\bar h_\rho^{y,c}\right)^2\right\rangle \mathrm{d}s\right)^{p/2}\right].$$

We have to verify that the quantities

$$\sup_{c\geqslant c_0,\ y\in K} \bar\eta^{y,c}, \qquad \sup_{c\geqslant c_0,\ y\in K} \eta_\rho^{y,c}, \qquad \sup_{c\geqslant c_0,\ y\in K} \eta_a^{y,c}$$

are finite.

From (A_2), we get that $\bar h^{y,c}$ is bounded from above by a constant $\tilde\kappa$ neither depending on y nor on c. Thus using (27), we have

$$\sup_{c\geqslant c_0}\sup_{y\in K}\bar{\eta}_\rho^{y,c}\leqslant\tilde{\kappa}\,E\left[\left(\int_0^t\langle X_s,1\rangle\,\mathrm{d}s\right)^{p/2}\right]<\infty.$$

Let us turn to $\eta_\rho^{y,c}$. From (7), introducing for convenience the notation

$$\mathcal{C}_\rho:=\mathcal{C}\left(\nu,t,\overline{B}\left(0,\frac{\rho}{2}\right)\right),\qquad \mathcal{K}_\rho:=\mathcal{K}\left(\nu,t,\overline{B}\left(0,\frac{\rho}{2}\right)\right),$$

we have for any integer $n\geqslant 0$

$$\sup_{z\in B(0,\frac{c\rho}{2})}E\big[\big(L_t^{z/c}\big)^n\big]\leqslant\frac{n!}{(\mathcal{K}_\rho)^n}\mathcal{C}_\rho. \tag{47}$$

Using the trivial inequality $a^{p/2}\leqslant 1+a^p$ for $a\geqslant 0$, we see that

$$\eta_\rho^{y,c}\leqslant 1+E\left[\left(\int_0^t\big\langle X_s,\big(h_\rho^{y,c}\big)^2\big\rangle\,\mathrm{d}s\right)^p\right]\leqslant 1+E\left[\left(\frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}\mathrm{d}z\left(\frac{1}{|z-y|}-\frac{1}{|z|}\right)^2L_t^{z/c}\right)^p\right]. \tag{48}$$

Set

$$\hat{\kappa}(y,c)=\frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}\mathrm{d}z\left(\frac{1}{|z-y|}-\frac{1}{|z|}\right)^2.$$

We clearly have $\sup_{c\geqslant c_0}\sup_{y\in K}\hat{\kappa}(y,c)<\infty$. Using the Jensen inequality in (48) we obtain

$$\eta_\rho^{y,c}\leqslant 1+\hat{\kappa}(y,c)^{p-1}E\left[\frac{1}{4\pi^2}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}\mathrm{d}z\left(\frac{1}{|z-y|}-\frac{1}{|z|}\right)^2\big(L_t^{z/c}\big)^p\right].$$

Thus, using (47), we get

$$\eta_\rho^{y,c}\leqslant 1+\hat{\kappa}(y,c)\frac{p!\mathcal{C}_\rho}{(\mathcal{K}_\rho)^p}\int_{B(0,\frac{c\rho}{2})\setminus B(0,2a)}\left(\frac{1}{|z|}-\frac{1}{|z-y|}\right)^{2p},$$

so that $\sup_{c\geqslant c_0,\ y\in K}\eta_\rho^{y,c}$ is also finite.

It remains to bound $\eta_a^{y,c}$. Using the trivial inequality $(a+b)^{p/2}\leqslant 1+2^p(a^p+b^p)$ for $a,b\geqslant 0$, we obtain

$$\eta_a^{y,c}\leqslant\frac{1}{2\pi^2}+\frac{2^p}{2\pi^2}\left(E\left[\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z|^2}L_t^{z/c}\right)^p+\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z-y|^2}L_t^{z/c}\right)^p\right]\right).$$

From the Hölder inequality with conjugate exponents $\frac{5}{4}$ and 5, we then get

$$E\left[\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z|^2}L_t^{z/c}\right)^p\right]\leqslant E\left[\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z|^{5/2}}\right)^{4p/5}\left(\int_{B(0,2a)}\mathrm{d}z\big(L_t^{z/c}\big)^5\right)^{p/5}\right]$$

and also,

$$E\left[\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z-y|^2}L_t^{z/c}\right)^p\right]\leqslant E\left[\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z|^{5/2}}\right)^{4p/5}\left(\int_{B(0,2a)}\mathrm{d}z\big(L_t^{z/c}\big)^5\right)^{p/5}\right].$$

Using the inequality $a^{p/5}\leqslant 1+a^p$ for $a\geqslant 0$, and then the Jensen inequality, we then obtain

$$\eta_a^{y,c}\leqslant\frac{1}{2\pi^2}+\frac{2^p}{\pi^2}\left(\int_{B(0,2a)}\frac{\mathrm{d}z}{|z|^{5/2}}\right)^{4p/5}\left(1+\left(\frac{32\pi a^3}{3}\right)^{p/5-1}E\left[\int_{B(0,2a)}\big(L_t^{z/c}\big)^{5p}\right]\right).$$

Using (47) we now get that $\sup_{c\geqslant c_0,\ y\in K}\eta_a^{y,c}<\infty$. We thus have proven that $\sup_{c\geqslant c_0,\ y\in K}\eta^{y,c}$ is finite, and (43) now follows from (46).

We thus have finished the proof of Lemma 9 and of the non-conditioned result Proposition 2. We now get back to Lee's result.

3.2. *Back to Lee's result*

In this section we prove Proposition 1 with the help of Proposition 2.

Proof of Proposition 1. We know from the scaling properties of super-Brownian motion that the law of $c^{d-4}D_{\phi,c^2t}$ under $P_{\nu^{(c)}}$ is the same as that of $c^dD_{\phi_c,t}$ under P_ν. Proposition 1 is thus equivalent to the following statement

Proposition 4. *Consider t, x_0, K, ϕ, ξ as in Proposition* 1.

Under $P_{c^{-2}\delta_{x_0}}(\cdot|X$ hits $K^{(c)})$, the random vector $(c^dD_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t})$ converges in distribution as $c\to\infty$ to (D_1,D_2).

Rather than proving Proposition 4 immediately, we will first establish an analogous statement under the excursion measure $\mathbb{N}_{x_0}$ of super-Brownian motion. Let us give an informal explanation for this intermediate step.

Let $q_c := P_{c^{-2}\delta_{x_0}}(X \text{ hits } K^{(c)})$. From [3] or [5],[1]

$$q_c \underset{c\to\infty}{\sim} \frac{4-d}{2c^2|x_0|^2}.$$

Whenever a super-Brownian motion started at $c^{-2}\delta_{x_0}$ hits $K^{(c)}$, the probability that this is done by a single excursion goes to one as $c\to\infty$. At least informally it follows that, when c is large enough, $P_{c^{-2}\delta_{x_0}}(\cdot|X \text{ hits } K^{(c)})$ is in some sense close to $\mathbb{N}_{x_0}(\cdot|X \text{ hits } \{0\})$. This idea will be used below to reduce the proof of Proposition 4 to the following statement.

Proposition 5. *Let t, x_0, K, ϕ, ξ be as in Proposition* 1. *Let l_t^0 denotes the local time of X under $\mathbb{N}_{x_0}(\cdot|X$ hits $\{0\})$ at level* 0 *and time t.*

Under $\mathbb{N}_{x_0}(\cdot|X$ hits $\{0\})$, the vector $\Xi_c := (c^dD_{\phi_c,t}, c^d\psi(c)D_{\xi_c,t})$ converges in distribution as $c\to\infty$ to

$$\left(l_t^0\int_K\phi(y)\,\mathrm{d}y,\ \widetilde{U}_t\right),$$

where conditionally on l_t^0, $\widetilde{U}_t$ is centered Gaussian with variance $a_\xi l_t^0$.

We now prove Proposition 5 as a consequence of Proposition 2.

Proof of Proposition 5. Fix $t>0$ and let $\rho_0 := 2^{-1}|x_0|$. Using the notation of (7), we also set $r_0 := \mathcal{K}(\delta_{x_0}, t, \overline{B}(0,\rho_0))$. Note from the discussion following (6) and (7) that the function $G_{x_0}^{t,0}$ is well defined on $\{z\in\mathbb{C}, |z|<r_0\}$. Let us introduce the function $g_{x_0}^{t,0}$ such that for any $z\in\mathbb{C}$, $|z|<r_0$,

$$g_{x_0}^{t,0}(z) = \mathbb{N}_{x_0}\big(\exp\big(zl_t^0\big)|X \text{ hits } \{0\}\big).$$

The canonical decomposition of super-Brownian motion ensures that $g_{x_0}^{t,0}$ is also well defined for $|z|<r_0$. It is proven in Chapter VI of [6] that $\mathbb{N}_{x_0}(X \text{ hits } \{0\}) = (2-\frac{d}{2})|x_0|^{-2}$, so using the canonical decomposition under $P_{\delta_{x_0}}$ we can write $P_{\delta_{x_0}}$-almost surely:

$$L_t^0 = \sum_{i=1}^N l_t^{0,(i)},$$

[1] in [5] as in [2], q_c is twice bigger. In [5] this comes from the non-standard underlying Brownian motion, whereas in [2] it comes from the branching rate being 2 instead of 1.

where the random variables $l_t^{0,(i)}$ are independent and distributed as l_t^0 under $\mathbb{N}_{x_0}(\cdot|X \text{ hits } \{0\})$, and N is an independent Poisson variable with parameter $(2-\frac{d}{2})|x_0|^{-2}$.

We thus have for $z \in \mathbb{C}$, $|z| \leqslant r_0$,

$$\exp\bigl(G_{x_0}^{t,0}(z)\bigr) = E_{\delta_{x_0}}\bigl[\exp\bigl(zL_t^0\bigr)\bigr] = E_{\delta_{x_0}}\bigl[E_{\delta_{x_0}}\bigl[\exp\bigl(zL_t^0\bigr)\big|N\bigr]\bigr] = \sum_{k=0}^{\infty}\left(\frac{4-d}{2|x_0|^2}\right)^k \frac{1}{k!}\exp\left(-\frac{4-d}{2|x_0|^2}\right) g_{x_0}^{t,0}(z)^k$$
$$= \exp\left\{\frac{4-d}{2|x_0|^2}\bigl(g_{x_0}^{t,0}(z)-1\bigr)\right\}.$$

From the fact that $G_{x_0}^{t,0}(0)=0$ and $g_{x_0}^{t,0}(0)=1$ and the continuity of the functions $g_{x_0}^{t,0}$, $G_{x_0}^{t,0}$ we then deduce that for any $z \in \mathbb{C}$, $|z| \leqslant r_0$,

$$g_{x_0}^{t,0}(z) = 1 + \frac{2|x_0|^2}{4-d} G_{x_0}^{t,0}(z). \tag{49}$$

Let a and b two real numbers and $Z_{a,b}^c := ac^d D_{\phi_c,t} + bc^d\psi(c) D_{\xi_c,t}$. Let $H_c(a,b)$, respectively $h_c(a,b)$ be the Fourier transform of Ξ_c with respect to the measure $P_{\delta_{x_0}}$, respectively $\mathbb{N}_{x_0}(\cdot|X \text{ hits } \{0\})$, that is

$$H_c(a,b) = E_{\delta_{x_0}}\bigl[\exp\bigl(\mathrm{i}Z_{a,b}^c\bigr)\bigr],$$
$$h_c(a,b) = \mathbb{N}_{x_0}\bigl[\exp\bigl(\mathrm{i}Z_{a,b}^c\bigr)\big|X \text{ hits } \{0\}\bigr].$$

Recall that $\mathbb{N}_{x_0}(X \text{ hits } \{0\}) = (2-\frac{d}{2})|x_0|^{-2}$. Using the canonical decomposition of super-Brownian motion we obtain that $Z_{a,b}^c$ is distributed under $P_{\delta_{x_0}}$ as $Z_{a,b}^{c,(1)} + Z_{a,b}^{c,(2)} + \cdots + Z_{a,b}^{c,(N)} + R_{a,b}^c$ where $Z_{a,b}^{c,(1)}, Z_{a,b}^{c,(2)}, \ldots$ are independent and distributed as $Z_{a,b}^c$ under $\mathbb{N}_{x_0}(\cdot|X \text{ hits } \{0\})$, and N is Poisson with parameter $(2-\frac{d}{2})|x_0|^{-2}$. Also, $R_{a,b}^c$ represents the contribution to $Z_{a,b}^c$ under $P_{\delta_{x_0}}$ of excursions that hit $K^{(c)}$ but do not hit 0. Since the compact sets $K^{(c)}$ converge to $\{0\}$, it is easy to verify that $P_{\delta_{x_0}}(R_{a,b}^c = 0) \underset{c\to\infty}{\longrightarrow} 1$. It follows that, uniformly in $(a,b) \in \mathbb{R}^2$,

$$\bigl|E_{\delta_{x_0}}\bigl[\exp\bigl(\mathrm{i}Z_{a,b}^c\bigr)\bigr] - E\bigl[\exp\bigl(\mathrm{i}\lambda\bigl(Z_{a,b}^{c,(1)} + \cdots + Z_{a,b}^{c,(N)}\bigr)\bigr)\bigr]\bigr| \underset{c\to\infty}{\longrightarrow} 0,$$

that is

$$\left|H_c(a,b) - \exp\left[\frac{4-d}{2|x_0|^2}\bigl[h_c(a,b)-1\bigr]\right]\right| \underset{c\to\infty}{\longrightarrow} 0, \quad \text{uniformly in } (a,b) \in \mathbb{R}^2. \tag{50}$$

On the other hand we know from Proposition 2

$$H_c(a,b) \underset{c\to\infty}{\longrightarrow} H(a,b) := E\left[\exp\left\{\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right)L_t^0 + \mathrm{i}bU_t\right\}\right] \tag{51}$$

uniformly when (a,b) varies over a compact subset of $\mathbb{R}^2$. We also have

$$H(a,b) = E\left[E\left[\exp\left\{\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right)L_t^0 + \mathrm{i}bU_t\right\}\middle|L_t^0\right]\right]$$
$$= E\left[\exp\left(\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right)L_t^0\right)\exp\left(-\frac{L_t^0 a_\xi b^2}{2}\right)\right]$$
$$= \exp\left\{G_{x_0}^{t,0}\left(\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right) - \frac{a_\xi b^2}{2}\right)\right\},$$

assuming that (a,b) belongs to a sufficiently small neighbourhood $\mathcal{V}$ of $(0,0)$ (see the discussion following (7)). We then get by (49)

$$H(a,b) = \exp\left\{\frac{4-d}{2|x_0|^2}\left(g_{x_0}^{t,0}\left(\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right) - \frac{a_\xi b^2}{2}\right) - 1\right)\right\}.$$

If we let

$$h(a,b) := g_{x_0}^{t,0}\left(\mathrm{i}a\left(\int \phi(x)\,\mathrm{d}x\right) - \frac{a_\xi b^2}{2}\right),$$

we thus have from (50) and (51)

$$\exp\left[\frac{4-d}{2|x_0|^2}\left[h_c(a,b)-1\right]\right] \underset{c\to\infty}{\longrightarrow} \exp\left\{\frac{4-d}{2|x_0|^2}\big(h(a,b)-1\big)\right\} \tag{52}$$

uniformly in $\mathcal{V}$. From the fact that for any (a,b), the function $c \to h_c(a,b)$ is continuous, it follows from (52) that for any $(a,b) \in \mathcal{V}$, there exists an integer $k(a,b)$ such that

$$\frac{4-d}{2|x_0|^2} h_c(a,b) \underset{c\to\infty}{\longrightarrow} \frac{4-d}{2|x_0|^2} h(a,b) + 2\mathrm{i}k(a,b)\pi.$$

Now from the continuity of both functions $(a,b) \to h_c(a,b)$ and $(a,b) \to h(a,b)$ and the uniformity of the convergence in (52), it follows that $k(a,b)$ does not depend on a nor b. Since $h_c(0,0) \to h(0,0)$ as $c \to \infty$, $k(a,b) = 0$ for every $(a,b) \in \mathcal{V}$. We have thus proved that for $(a,b) \in \mathcal{V}$,

$$h_c(a,b) = \mathbb{N}_{x_0}\big(\exp\big(\mathrm{i}Z^c_{a,b}\big)\big| X \text{ hits } \{0\}\big) \underset{c\to\infty}{\longrightarrow} h(a,b),$$

and $h(a,b)$ can be interpreted as the Fourier transform at 1 of $al^0_t \int \phi(x)\,\mathrm{d}x + b\widetilde{U}_t$. The statement of Proposition 5 follows. □

Proof of Proposition 4. From the canonical decomposition of super-Brownian motion, the law under $P_{c^{-2}\delta_{x_0}}$ of Ξ^c coincides with the law of $\sum_{i=1}^{N_c}(U_i^c, V_i^c)$ where the variables (U_i^c, V_i^c) are independent and distributed as Ξ^c under $\mathbb{N}_{x_0}(\cdot | X \text{ hits } K^{(c)})$, and N_c is an independent Poisson variable with parameter $c^{-2}\mathbb{N}_{x_0}(X \text{ hits } K^{(c)})$. Clearly $\{N_c \geqslant 1\} = \{X \text{ hits } K^{(c)}\}$, and moreover

$$P_{c^{-2}\delta_{x_0}}(N_c = 1) \underset{c\to\infty}{\sim} \frac{4-d}{2c^2|x_0|^2}.$$

It is also clear that $P(N_c \geqslant 2) \leqslant \kappa(x_0)c^4$ where $\kappa(x_0)$ is a constant depending on x_0. Since the laws of Ξ under $E_{c^{-2}\delta_{x_0}}(\cdot | N_c = 1)$ and $\mathbb{N}_{x_0}(\cdot | X \text{ hits } K^{(c)})$ coincide, we have

$$\begin{aligned}\big|E_{c^{-2}\delta_{x_0}}\big(\exp\big(\mathrm{i}Z^c_{a,b}\big)\big| X \text{ hits } K^{(c)}\big) - \mathbb{N}_{x_0}\big(\exp\big(\mathrm{i}Z^c_{a,b}\big)\big| X \text{ hits } K^{(c)}\big)\big| &\leqslant \frac{|E_{c^{-2}\delta_{x_0}}(\exp(\mathrm{i}Z^c_{a,b}), N_c \geqslant 2)|}{P_{c^{-2}\delta_{x_0}}(X \text{ hits } K^{(c)})} \\ &\leqslant 2|x_0|^2\kappa(x_0)c^{-2}.\end{aligned}$$

From the fact that $\mathbb{N}_{x_0}(X \text{ hits } K^{(c)}) \to \mathbb{N}_{x_0}(X \text{ hits } \{0\})$ as $c \to \infty$ we now have

$$\big|E_{c^{-2}\delta_{x_0}}\big(\exp\big(\mathrm{i}Z^c_{a,b}\big)\big| X \text{ hits } K^{(c)}\big) - h_c(a,b)\big| \underset{c\to\infty}{\longrightarrow} 0.$$

From the proof of Proposition 5 we know that $h_c(a,b) \to h(a,b)$ as $c \to \infty$, and from (49)

$$h(a,b) = \frac{2|x_0|^2}{4-d} G^{t,0}_{x_0}\left(\mathrm{i}a \int_K \phi(y)\,\mathrm{d}y - \frac{a_\xi b^2}{2}\right) + 1$$

is the Fourier transform of (D_1, D_2). This finishes the proof of Proposition 4, and thus of its rescaled equivalent form Proposition 1. □

4. The case $d = 1$

Sugitani showed in [9] (Theorems 1 and 4) that under the condition that ν does not charge points in $\mathbb{R}$, P_ν-almost surely, L^x_t is continuously differentiable with respect to both time and space variables on $(0,\infty) \times \mathbb{R}$. We will denote $D_x L^y_t$ its continuous derivative with respect to the space variable taken at point (t, y). It is not hard to deduce from [9] the following extension of Theorem 1 to the one-dimensional case

Proposition 6. *Suppose $\nu \in M_F(\mathbb{R})$ is atomless in a certain neighbourhood of* 0. *Let $T > 0$, $K > 0$ be fixed. Then P_ν-almost surely, uniformly in $y \in [-K, K]$, $t \in [0, T]$, the random variable $c(L^{y/c}_t - L^0_t)$ converges to $yD_x L^0_t$, where $D_x L^0_t$ denotes the derivative of L^x_t with respect to the x-variable taken at point $(t, 0)$.*

Also, it is evident that uniformly in $y \in [-K, K]$, $t \in [0, T]$, $L_t^{y/c}$ converges P_ν-almost surely to L_t^0. As a direct consequence of these results we obtain a statement analogous to Proposition 2 in the one-dimensional case.

Proposition 7. *Fix $t > 0$ and let ν be as in Proposition 6, and let ϕ, ξ be integrable function with compact support on $\mathbb{R}$ such that $\int \phi(y)\,\mathrm{d}y \neq 0$, $\int \xi(y)\,\mathrm{d}y = 0$. Then, for every $t > 0$, we have P_ν-almost surely*

$$\left(cD_{\phi_c,t}, c^2 D_{\xi_c,t}\right) \underset{c\to\infty}{\longrightarrow} \left(\left(\int_{\mathbb{R}} \phi(y)\,\mathrm{d}y\right) L_t^0, \left(\int_{\mathbb{R}} y\,\xi(y)\,\mathrm{d}y\right) D_x L_t^0\right).$$

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Appendix A. Proof of the estimates on $h^{x,c}$

Recall

$$h^{x,c}(z) = \frac{c}{\sqrt{\ln(c)}} \int_0^\infty (2\pi t)^{-1}\,\mathrm{e}^{-\alpha t}\left(\mathrm{e}^{-|z-x/c|^2/2t} - \mathrm{e}^{-|z|^2/2t}\right)\mathrm{d}t. \tag{53}$$

We will still use the notation $c^\delta = |z|$, and introduce δ' so that $c^{\delta'} = |z - x/c|$. By a symmetry argument, without loss of generality we may and will always assume $\delta \leqslant \delta'$. Note that we then have for every $t > 0$

$$\left|\mathrm{e}^{-|z-x/c|^2/2t} - \mathrm{e}^{-|z|^2/2t}\right| \leqslant \mathrm{e}^{-|z|^2/2t}. \tag{54}$$

Proof of ($\mathbf{B_1}$), ($\mathbf{B_2}$). Notice that the maximum of the function $t \to t^{-1}\,\mathrm{e}^{-|z|^2/2t}$ is attained at $t = |z|^2/2$. Its value at this point is $2\mathrm{e}^{-1}/|z|^2$, so that

$$\left|\int_0^\infty (2\pi t)^{-1}\,\mathrm{e}^{-\alpha t}\,\mathrm{e}^{-|z|^2/(2t)}\right| \leqslant \mathrm{e}^{-1} + \left|\int_{|z|^2/2}^\infty (2\pi t)^{-1}\,\mathrm{e}^{-\alpha t}\right| \leqslant \kappa_{11}\left(1 + \ln^+\left(1/|z|\right)\right).$$

The above and (54) imply (B_1), (B_2). □

Let us now prove the remaining estimates. The change of variable $t = c^\beta$ in the integral (53) leads to

$$h^{x,c}(z) = \frac{c}{2\pi\sqrt{\ln(c)}} \int_{-\infty}^\infty \mathrm{e}^{-\alpha c^\beta}\left(\mathrm{e}^{-\frac{1}{2}c^{2\delta'-\beta}} - \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\right)\ln(c)\,\mathrm{d}\beta.$$

For convenience let us define

$$F(u) = \int_{-\infty}^u \mathrm{e}^{-\alpha c^\beta}\left(\mathrm{e}^{-\frac{1}{2}c^{2\delta'-\beta}} - \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\right)\ln(c)\,\mathrm{d}\beta,$$

and

$$\widetilde{F}(u) = \int_u^\infty \mathrm{e}^{-\alpha c^\beta}\left(\mathrm{e}^{-\frac{1}{2}c^{2\delta'-\beta}} - \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\right)\ln(c)\,\mathrm{d}\beta,$$

so that for any real x, $h^{x,c}(z) = c(2\pi\sqrt{\ln(c)})^{-1}(F(u) + \widetilde{F}(u))$. Note that $F(u)$ and $\widetilde{F}(u)$ still depend on c, x and z, even though this is not apparent in our notation.

Proof of (C_2). On the domain of (C_2), $-1-\frac{\ln(\ln(c))}{8\ln(c)} \leqslant \delta \leqslant -1+\frac{\ln(\ln(c))}{8\ln(c)}$. Using (54) we first obtain

$$\left|F(2\delta)\right| \leqslant \int_{-\infty}^{2\delta} \mathrm{e}^{-\alpha c^\beta}\,\mathrm{e}^{-\frac{|z|^2}{2c^\beta}}\ln(c)\,\mathrm{d}\beta \leqslant \int_{-\infty}^{2\delta} c^{2\delta-\beta}\,\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\ln(c)\,\mathrm{d}\beta \leqslant 2\left[\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\right]_{-\infty}^{2\delta} \leqslant 2.$$

Using a Taylor expansion and the fact $\mathrm{e}^{-\alpha c^\beta} \leqslant 1$, we also have

$$\left|\widetilde{F}(2\delta)\right| \leqslant \left|\int_{2\delta}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n! c^{\beta n}}\left(c^{2\delta' n}-c^{2\delta n}\right)\ln(c)\,\mathrm{d}\beta\right)\right| \leqslant \left|\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n n!}\left(c^{2(\delta'-\delta)n}-1\right)\right|.$$

Since the Taylor expansion of $\int_0^u \frac{\mathrm{e}^{-y}-1}{y}\,\mathrm{d}y$ is $\sum_{n\geqslant 1}\frac{(-1)^n u^n}{n n!}$, this last quantity is equal to

$$\left|\int_{\frac{1}{2}}^{\frac{1}{2}|z-x/c|^2 c^{-2\delta}} \frac{\mathrm{e}^{-y}-1}{y}\,\mathrm{d}y\right| \leqslant \left(\frac{|z\cdot x|}{c}c^{-2\delta}+\frac{x^2}{2c^2}c^{-2\delta}\right)\sup_{y\in[1/2,\infty)}\left|\frac{\mathrm{e}^{-y}-1}{y}\right|.$$

Using the fact that $\delta \geqslant -1-\frac{\ln(\ln(c))}{8\ln(c)}$ we thus obtain that $|\widetilde{F}(2\delta)| \leqslant \kappa_{12}(\ln(c))^{1/4}$. Combining the above inequalities for $|F(2\delta)|$, $|\widetilde{F}(2\delta)|$, we obtain (C_2). □

Proof of (C_1). Here $\delta \leqslant -1-\frac{\ln(\ln(c))}{8\ln(c)}$. As in the proof of ($C_2$) we first have $|F(2\delta)| \leqslant 2$. Furthermore, using (54) we get

$$\left|\widetilde{F}(-4-2\delta)-\widetilde{F}(2\delta)\right| \leqslant \int_{2\delta}^{-2+(-2-2\delta)} \mathrm{e}^{-\alpha c^\beta}\,\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\ln(c)\,\mathrm{d}\beta \leqslant \frac{-4-4\delta}{2\pi}\ln(c) \leqslant |1+\delta|\ln(c).$$

Using the same Taylor expansion technique as in the proof of (C_2) we also obtain

$$\left|\widetilde{F}(-4-2\delta)\right| \leqslant \left|\int_{\frac{c^{4+4\delta}}{2}}^{\frac{c^{4+4\delta}}{2}-(z\cdot x)c^{3+2\delta}+|x|^2\frac{c^{2+2\delta}}{2}} \frac{\mathrm{e}^{-y}-1}{y}\,\mathrm{d}y\right| \leqslant \kappa_{13}\left(\ln(c)\right)^{-1/4}.$$

Combining the above inequalities leads to (C_1). □

Proof of (C_3), (C_4), (B_3). Here $\delta \geqslant -1+\frac{\ln(\ln(c))}{8\ln(c)}$. In particular $\frac{3}{2}\delta-\frac{1}{2} \leqslant 2\delta$. Using (54) we first obtain

$$\left|F\left(\frac{3}{2}\delta-\frac{1}{2}\right)\right| \leqslant \left|\int_{-\infty}^{\frac{3}{2}\delta-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\ln(c)\,\mathrm{d}\beta\right| \leqslant \mathrm{e}^{-c^{1/2\delta+1/2}} \tag{55}$$

which is $o(c^{-\delta-1})$ since $\delta \geqslant -1+\frac{\ln(\ln(c))}{8\ln(c)}$. Furthermore, when $\beta \geqslant \frac{3}{2}\delta-\frac{1}{2}$, we have

$$\left|\frac{|z|^2-|z-x/c|^2}{2c^\beta}\right| = \left|\left(\frac{z\cdot x}{c}-\frac{|x|^2}{2c^2}\right)c^{-\beta}\right| \underset{c\to\infty}{\longrightarrow} 0,$$

so that

$$\mathrm{e}^{-\frac{1}{2}c^{2\delta'-\beta}}-\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \underset{c\to\infty}{\sim} \frac{z\cdot x}{c^{\beta+1}}\,\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}.$$

By dominated convergence, we thus have

$$\widetilde{F}\left(\frac{3}{2}\delta-\frac{1}{2}\right) \underset{c\to\infty}{\sim} \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\infty} \mathrm{e}^{-\alpha c^\beta}\,\frac{z\cdot x}{c^{\beta+1}}\mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}}\ln(c)\,\mathrm{d}\beta. \tag{56}$$

Let us first prove (C_3), for which $\delta \leqslant -\frac{\ln(\ln(c))}{\ln(c)}$, in particular $\delta < 0$. Let us split the integral in the right-hand side of (56) into two parts:

$$\int_{\frac{3}{2}\delta-\frac{1}{2}}^{\delta} \mathrm{e}^{-\alpha c^\beta} \frac{z \cdot x}{c^{\beta+1}} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c)\,\mathrm{d}\beta \underset{c\to\infty}{\sim} \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\delta} \frac{z \cdot x}{2c^{\beta+1}} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c)\,\mathrm{d}\beta \underset{c\to\infty}{\sim} 2(z\cdot x)c^{-2\delta-1}, \tag{57}$$

and

$$\int_{\delta}^{\infty} \mathrm{e}^{-\alpha c^\beta} \frac{z \cdot x}{2c^{\beta+1}} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c)\,\mathrm{d}\beta \leqslant (z\cdot x)c^{-1-2\delta} \int_{\delta}^{\infty} c^{2\delta-\beta} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c)\,\mathrm{d}\beta,$$

which, since $\delta < 0$ is $\mathrm{o}(c^{-\delta-1})$ as $c \to \infty$. This fact, (55) and (57) give us the estimate (C_3).

Let us now prove (C_4), for which $\delta \geqslant \frac{\ln(\ln(c))}{4\ln(c)}$. Using (56), we obtain

$$\left|\widetilde{F}\left(\frac{3}{2}\delta - \frac{1}{2}\right)\right| \leqslant \left| \int_{\frac{3}{2}\delta-\frac{1}{2}}^{\infty} \frac{z \cdot x}{c^{\beta+1}} \mathrm{e}^{-\frac{1}{2}c^{2\delta-\beta}} \ln(c)\,\mathrm{d}\beta \right| \leqslant \kappa_{14} c^{-\delta-1}. \tag{58}$$

The above and (55) give us (C_4).

Finally (B_3) is obtained as a combination of (55), (58) and (C_3). □

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