

Hausdorff–Besicovitch measure of fractal functional limit laws induced by Wiener process in Hölder norms

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Abstract

Let $\{W(t) : t \geq 0\}$ denote a standard Wiener process. In this paper, we first establish a de Acosta [A. de Acosta, On the functional form of Lévy's modulus of continuity for Brownian motion, *Z. Wahr. Verw. Gebiete* 69 (1985) 567–579] type strong law for a family of Hölder norms. More precisely, we obtain, for $\alpha \in (0, 1/2)$, the exact rate of convergence, as $h \downarrow 0$, of

$$T_{\alpha, f}(h) := \inf_{0 \leq t \leq 1-h} \left\| (2h \log(1/h))^{-1/2} (W(t+h\cdot) - W(t)) - f \right\|_{\alpha}$$

when $f \in \mathcal{S}$ satisfies $\int_0^1 \left\{ \frac{d}{du} f(u) \right\}^2 du < 1$, where \mathcal{S} denotes the Strassen [V. Strassen, An invariance principle for the law of the iterated logarithm, *Z. Wahr. Verw. Gebiete* 3 (1964) 211–226] set.

In a second part we give some general technical tools for evaluating the upper and the lower critical functions of the Hausdorff–Besicovitch measures respectively for limsup random sets and for random Cantor type sets. As an application we deduce the Hausdorff dimension of the random fractal constituted of exceptional points in $[0, 1]$ where the previous rate is reached.

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Résumé

Soit $\{W(t) : t \geq 0\}$ un processus de Wiener standard. Dans ce papier nous établissons, dans un premier temps, une loi forte de type de Acosta [A. de Acosta, On the functional form of Lévy's modulus of continuity for Brownian motion, *Z. Wahr. Verw. Gebiete* 69 (1985) 567–579] pour une famille de norme de Hölder. Plus exactement nous obtenons, pour $\alpha \in (0, 1/2)$, la vitesse de convergence lorsque $h \downarrow 0$, de

$$T_{\alpha, f}(h) := \inf_{0 \leq t \leq 1-h} \left\| (2h \log(1/h))^{-1/2} (W(t+h\cdot) - W(t)) - f \right\|_{\alpha}$$

quand $f \in \mathcal{S}$ vérifie $\int_0^1 \left\{ \frac{d}{du} f(u) \right\}^2 du < 1$, où \mathcal{S} désigne l'ensemble de Strassen [V. Strassen, An invariance principle for the law of the iterated logarithm, *Z. Wahr. Verw. Gebiete* 3 (1964) 211–226].

Dans une seconde partie nous proposons des outils techniques généraux servant à exhiber les fonctions critiques supérieures et inférieures qui permettent d'évaluer les mesures de Hausdorff–Besicovitch respectivement pour des ensembles aléatoires de type limsup et pour des ensembles aléatoires de type Cantor. En application nous déduisons la dimension de Hausdorff de l'ensemble fractal aléatoire constitué des points exceptionnels de $[0, 1]$ en lesquels la vitesse du résultat de la première partie est atteinte.

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1. Introduction and main results

Let $\{W(t): t \geq 0\}$ be a standard Wiener process. For each $t \geq 0$ and for each $h > 0$, consider $\xi(t, h; \cdot)$ the normalized increment function defined by

$$\xi(t, h; s) := (2h \log(1/h))^{-1/2} \Delta W(t, h; s) \quad \text{for } s \in [0, 1],$$

where

$$\Delta W(t, h; s) := W(t + hs) - W(t).$$

Let $(C_0[0, 1], \|\cdot\|_\infty)$ be the set $C_0[0, 1]$ composed with continuous functions f on $[0, 1]$ such that $f(0) = 0$, provided with the sup norm $\|f\|_\infty := \sup_{0 \leq t \leq 1} |f(t)|$, and let \mathcal{S} be the Strassen set [17] defined by

$$\mathcal{S} := \left\{ f \in C_0[0, 1]: f(t) = \int_0^t \frac{d}{du} f(u) du \text{ and } J(f) \leq 1 \right\},$$

where

$$J(f) := \int_0^1 \left\{ \frac{d}{du} f(u) \right\}^2 du$$

is the energy of the function f .

For $\alpha \in (0, 1)$ and $\lambda \in (0, 1)$, denote respectively by

$$\|f\|_\alpha := \sup_{0 < |t-s| \leq 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} \quad \text{and} \quad \omega_{\alpha, f}(\lambda) := \sup_{0 < |t-s| \leq \lambda} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

the α -Hölder norm and the α -Hölder modulus of regularity.

de Acosta [1] established that the sequence of random sets $\mathcal{F}_h := \{\xi(t, h; \cdot): t \in [0, 1-h]\}$ converges almost surely to \mathcal{S} , when $h \downarrow 0$, with respect to the Hausdorff metric set pertaining to the sup norm. This result allows to assert that for any $f \in \mathcal{S}$,

$$\lim_{h \downarrow 0} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\infty = 0 \quad \text{a.s.} \tag{1}$$

de Acosta [1] obtained the rate of convergence in (1) for $f \in \mathcal{S}$ satisfying $J(f) < 1$:

$$\lim_{h \downarrow 0} \log(1/h) \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\infty = b_f \quad \text{a.s.,} \tag{2}$$

where $b_f = \pi \{4(1 - J(f))^{1/2}\}^{-1}$.

Earlier, Csörgő and Révész [4] had established (2) for $f \equiv 0$ that is known as the rate of convergence for the modulus of non-differentiability. Lucas [14] extended (2) to the case where $f \in \mathcal{S}$ satisfies $J(f) = 1$. For such function f , denote by $|\dot{f}|_V$ the total variation of $\dot{f} := df/du$. When $|\dot{f}|_V < +\infty$, let ν be the measure defined by $\int_a^b \nu(du) = \dot{f}(b) - \dot{f}(a)$ and ν_s (resp. ν_{ac}) the singular part (resp. absolutely continuous part) of ν with respect to the Lebesgue measure. Let $\mu: \mathbb{R} \rightarrow \mathbb{R}$ be the *Csáki function* introduced by Gorn and Lifshits [9] and denote by $\mu(a)$ the smallest eigenvalue of the Sturm–Liouville system

$$\begin{cases} \frac{1}{2}y''(x) + (ax + \mu)y(x) = 0, & -1 \leq x \leq +1, \\ y(-1) = y(+1) = 0. \end{cases}$$

Then, Lucas [14] showed that for $f \in \mathcal{S}$ satisfying $J(f) = 1$ and $|\dot{f}|_V < +\infty$,

$$\lim_{h \downarrow 0} (\log(1/h))^{2/3} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\infty = z_f \quad \text{a.s.}, \tag{3}$$

where z_f is the unique positive solution of the equation (see Gorn and Lifshits [9])

$$\delta(z) := 4z^3 (|\dot{f}(1)| + |v_s|_V) - \int_0^1 \mu(4z^3 \dot{v}_{ac}(u)) \, du, \quad z > 0.$$

We refer to Kuelbs, Li and Talagrand [13] and references therein for a more general settings when $f \in \mathcal{S}$ satisfies $J(f) = 1$.

Recently, Wei [18] established a de Acosta type functional result for the Hölder topology. More precisely, he showed that for $\alpha \in (0, 1/2)$ the sequence of random sets \mathcal{F}_h converges almost surely to \mathcal{S} , when $h \downarrow 0$, for the Hausdorff metric set pertaining to the α -Hölder norm.

Our first goal is to establish, for each $\alpha \in (0, 1/2)$, the exact rate of convergence in (1) under the α -Hölder norm.

Theorem 1. *Let $\alpha \in (0, 1/2)$ and $f \in \mathcal{S}$ such that $J(f) < 1$. Then*

$$\lim_{h \downarrow 0} (\log(1/h))^{1-\alpha} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\alpha = K_\alpha(f) \quad \text{a.s.}, \tag{4}$$

where $K_\alpha(f) = 2^{-1/2} C_\alpha^{(1-2\alpha)/2} (1 - J(f))^{-(1-2\alpha)/2}$ with $0 < C_\alpha < +\infty$.

In the sequel, we say that ℓ is a Hausdorff function if ℓ is continuous on $[0, 1]$, non-decreasing and satisfying $\ell(0) = 0$. Denote by \mathbb{H} the set of all Hausdorff functions and by $|A|$ the diameter of a subset A of $[0, 1]$. Recall (see Falconer [10]) that the Hausdorff–Besicovitch measure of order $\ell \in \mathbb{H}$ of a subset A of $[0, 1]$ is defined by

$$\Lambda^\ell(A) := \lim_{\eta \downarrow 0} \Lambda_\eta^\ell(A),$$

where

$$\Lambda_\eta^\ell(A) := \inf \left\{ \sum_{i=1}^{+\infty} \ell(|A_i|) : A \subseteq \bigcup_{i=1}^{+\infty} A_i \text{ and } |A_i| \leq \eta \right\}.$$

An essential property of the Hausdorff–Besicovitch measures is the following one.

Fact 1. *For any $\ell_1, \ell_2 \in \mathbb{H}$ such that $\ell_2 = o(\ell_1)$ in the neighborhood of 0, we have*

1. $\Lambda^{\ell_2}(A) = 0$, if $\Lambda^{\ell_1}(A) < +\infty$,
2. $\Lambda^{\ell_1}(A) = +\infty$, if $\Lambda^{\ell_2}(A) > 0$.

By restricting \mathbb{H} to the class $\{\ell(s) := s^\kappa : \kappa \in (0, \infty)\}$, Fact 1 allows to introduce a dimensional index called the Hausdorff dimension. This latter is defined by

$$\dim_H A := \sup\{\kappa > 0 : \Lambda^\kappa(A) = \infty\} = \inf\{\kappa > 0 : \Lambda^\kappa(A) = 0\},$$

where $\Lambda^\kappa \equiv \Lambda^\ell$ when $\ell = s(k)$.

As a consequence of Fact 1, we get that

$$\Lambda^\kappa(A) = \begin{cases} +\infty, & \text{if } \kappa < \dim_H A, \\ 0, & \text{if } \kappa > \dim_H A. \end{cases}$$

Based on [1], Deheuvels and Mason [6] introduced, for each $f \in \mathcal{S}$, the random set

$$L(f) := \left\{ t \in [0, 1] : \liminf_{h \downarrow 0} \|\xi(t, h; \cdot) - f\|_\infty = 0 \right\}$$

composed by $t \in [0, 1]$ such that $\xi(t, h; \cdot)$ converges uniformly to f along a sequence of h values tending to 0. They proved that, for each $f \in \mathcal{S}$, almost surely $L(f)$ is a random fractal with Hausdorff dimension equal to $1 - J(f)$. Latter, Deheuvels and Lifshits [5] extended this result by showing that almost surely

$$\dim_H L_\tau(f) = 1 - J(f), \quad \text{for any } f \in \mathcal{S},$$

where $L_\tau(f)$ is the random set $L(f)$ defined above when the sup norm $\|\cdot\|_\infty$ is replaced by a consistent norm $\|\cdot\|_\tau$.

Recently, Khoshnevisan, Peres and Xiao [11] extended the result of Deheuvels and Mason [6] by showing that for any analytic set $E \subset [0, 1]$ and for any $f \in \mathcal{S}$ such that $J(f)$ is lower than the packing dimension of E (see Falconer [10]), there exists almost surely some $t \in E$ such that f can be uniformly approximated by normalized increment function $\xi(t, h; \cdot)$.

By observing (2), Deheuvels and Mason [7] considered, for each $f \in \mathcal{S}$ satisfying $J(f) < 1$ and for each $c > 1$, the random set

$$L(f, c) := \left\{ t \in [0, 1]: \liminf_{h \downarrow 0} \log(1/h) \|\xi(t, h; \cdot) - f\|_\infty \leq cb_f \right\}$$

composed by $t \in [0, 1]$ such that $\xi(t, h; \cdot)$ converges uniformly to f with the rate $(\log(1/h))^{-1}$ along a sequence of h values tending to 0. They showed that almost surely each set $L(f, c)$ is a random fractal with Hausdorff dimension equal to

$$\dim_H L(f, c) = (1 - J(f))(1 - c^{-2}). \tag{5}$$

Orey and Taylor [15] had anticipated this above result in the case $f \equiv 0$ by showing that

$$\dim_H L(0, c) = 1 - c^{-2} \quad \text{a.s.}$$

Then, by observing (3), Lucas [14] introduced, for $f \in \mathcal{S}$ satisfying $J(f) = 1$ and $|\dot{f}|_V < +\infty$ and for $c > 1$, the random set

$$D(f, c) := \left\{ t \in [0, 1]: \liminf_{h \downarrow 0} (\log(1/h))^{2/3} \|\xi(t, h; \cdot) - f\|_\infty \leq cz_f \right\}.$$

He showed that each random set $D(f, c)$ satisfies with probability 1

$$\Lambda^{\zeta_{1/3,x}}[D(f, c)] = \begin{cases} +\infty, & \text{if } x < \frac{\delta(cz_f)}{2c^2z_f^2}, \\ 0, & \text{if } x > \frac{\delta(cz_f)}{2c^2z_f^2}, \end{cases} \tag{6}$$

where the Hausdorff function $\zeta_{a,b}$, with parameters $a, b > 0$, is defined by

$$\zeta_{a,b}(s) := \begin{cases} \exp\{-b(\log(1/s))^a\}, & \text{if } s > 0, \\ 0, & \text{if } s = 0. \end{cases} \tag{7}$$

Following these previous works, it seems natural to introduce and study the random sets

$$L_\alpha(f, c) := \left\{ t \in [0, 1]: \liminf_{h \downarrow 0} (\log(1/h))^{1-\alpha} \|\xi(t, h; \cdot) - f\|_\alpha \leq cK_\alpha(f) \right\}$$

for $\alpha \in (0, 1/2)$, $f \in \mathcal{S}$ such that $J(f) < 1$ and for $c > 1$.

A classical technic for proving that the Hausdorff dimension of a limsup random set S is equal to one number r consists in two parts: first we approach S by above with an appropriate limsup random set \mathbb{L} such that $\Lambda^{r+\varepsilon}(\mathbb{L}) < +\infty$, for each $\varepsilon > 0$, second one approximate S by below with a suitable random Cantor type set \mathbb{K} such that $\Lambda^{r-\varepsilon}(\mathbb{K}) > 0$, for each $\varepsilon > 0$. In this paper, this method is extended to the Hausdorff–Besicovitch measures by following the procedure introduced by Deheuvels and Mason [7]. More precisely we give two theorems (Theorems 3 and 4 in Section 2) which allow to find the upper critical function (respectively the lower critical function) of a class of Hausdorff functions for the Hausdorff–Besicovitch measures of limsup random sets (respectively random Cantor type sets) under some conditions. Finally by using this two results we get the Hausdorff dimension of $L_\alpha(f, c)$. That constitutes our second main result.

Theorem 2. For each $\alpha \in (0, 1/2)$, each $f \in \mathcal{S}$ such that $J(f) < 1$ and each $c > 1$, we have

$$\dim_H L_\alpha(f, c) = (1 - J(f))(1 - c^{-2/(1-2\alpha)}) \quad \text{a.s.} \tag{8}$$

2. Proofs

2.1. Proof of Theorem 1

Successively, we will show that for each $\varepsilon > 0$, we have

$$\limsup_{h \downarrow 0} (\log(1/h))^{1-\alpha} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\alpha \leq (1 + \varepsilon)K_\alpha(f) \quad \text{a.s.}, \tag{9}$$

$$\liminf_{h \downarrow 0} (\log(1/h))^{1-\alpha} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\alpha \geq (1 - \varepsilon)K_\alpha(f) \quad \text{a.s.} \tag{10}$$

In the sequel, we refer to $\{h_n: n \geq 1\}$ as the sequence of positive constants defined by

$$h_n := \exp\{-n^\beta\} \tag{11}$$

for $0 < \beta < (1 - 2\alpha)/(3 - 4\alpha)$ with $\alpha \in (0, 1/2)$.

Part I. Fix $\varepsilon > 0$. For $n \geq 1$, set $t_n(i) := ih_n$ for $i = 0, 1, \dots, M_{1,n} := \lfloor 1/h_n \rfloor - 1$ be a discretization of $[0, 1]$. For $n \geq 1$, let

$$P_1(n) := \mathbb{P}\left((\log(1/h_n))^{1-\alpha} \min_{0 \leq i \leq M_{1,n}} \|\xi(t_n(i), h_n; \cdot) - f\|_\alpha > (1 + \varepsilon)K_\alpha(f)\right).$$

Lemma 1. *Let $f \in \mathcal{S}$ and $r > 0$. For each $\alpha \in (0, 1/2)$, there exists a constant $0 < C_\alpha < +\infty$ such that*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-2} \log \mathbb{P}(\|W - \lambda f\|_\alpha \leq \lambda^{-(1-2\alpha)}r) = -\frac{1}{2}J(f) - C_\alpha r^{-2/(1-2\alpha)}. \tag{12}$$

Proof. See Kuelbs and Li [12]. \square

By Lemma 1 and properties of the Wiener-process increments, we get

$$\sum_{n=1}^{+\infty} P_1(n) = O\left(\sum_{n=1}^{+\infty} \exp\{-n\}\right) < +\infty,$$

which entails by making use of the Borel–Cantelli lemma

$$\limsup_{n \rightarrow \infty} (\log(1/h_n))^{1-\alpha} \inf_{t \in [0, 1-h_n]} \|\xi(t, h_n; \cdot) - f\|_\alpha \leq (1 + \varepsilon)K_\alpha(f) \quad \text{a.s.} \tag{13}$$

Lemma 2. *Let $f \in \mathcal{S}$ and $\alpha \in (0, 1/2)$. For each $\lambda \in (0, 1)$, we have*

$$\|f(\cdot) - f(\lambda \cdot)\|_\alpha \leq 2((1 - \lambda)/\lambda)^{1/2-\alpha}. \tag{14}$$

Proof. See Baldi and Roynette [2]. \square

Now, fix $n \geq 1$ and select $h \in]h_{n+1}, h_n]$. Since $\sup_{f \in \mathcal{S}} \|f\|_\alpha \leq \sup_{f \in \mathcal{S}} (J(f))^{1/2} \leq 1$, we get

$$\begin{aligned} & (\log(1/h))^{1-\alpha} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\alpha \\ & \leq \left(\frac{\log(1/h_{n+1})}{\log(1/h_n)}\right)^{1-\alpha} (\log(1/h_n))^{1-\alpha} \inf_{t \in [0, 1-h_n]} \left\| b^{-1}(h_n) \Delta W(t, h; \cdot) - f\left(\frac{h}{h_n} \cdot\right) \right\|_\alpha \\ & \quad + (\log(1/h_{n+1}))^{1-\alpha} \left\| f\left(\frac{h}{h_n} \cdot\right) - f \right\|_\alpha + (\log(1/h_{n+1}))^{1-\alpha} \left(1 - \frac{b(h_{n+1})}{b(h_n)}\right), \end{aligned}$$

where $b(h) = (2h \log(1/h))^{1/2}$.

By combining Lemma 2 with the definition of $\{h_n; n \geq 1\}$, we show by elementary calculations that the second and the third term in RHS of the above inequality converge to 0. Making use of (13), we easily obtain (9).

Part II. Fix $\varepsilon > 0$. For $n \geq 1$, set $s_n(i) := in^{-1}h_n$ for $i = 0, 1, \dots, M_{2,n} := \lfloor n/h_n \rfloor$ a new discretization of $[0, 1]$. For $n \geq 1$, let

$$P_2(n) := \mathbb{P}\left(\left(\log(1/h_n)\right)^{1-\alpha} \min_{0 \leq i \leq M_{2,n}} \|\xi(s_n(i), h_n; \cdot) - f\|_\alpha < (1 - \varepsilon)K_\alpha(f)\right).$$

As in Part I, we get from Lemma 1 and properties of the Wiener-process increments

$$\sum_{n=1}^{+\infty} P_2(n) = O\left(\sum_{n=1}^{+\infty} \exp\{-Cn^\beta\}\right) < +\infty, \quad \text{for } C \equiv C(\varepsilon, f, \alpha) > 0,$$

which entails by using the Borel–Cantelli lemma

$$\liminf_{n \rightarrow \infty} (\log(1/h_n))^{1-\alpha} \min_{0 \leq i \leq M_{2,n}} \|\xi(s_n(i), h_n; \cdot) - f\|_\alpha \geq (1 - \varepsilon)K_\alpha(f) \quad \text{a.s.} \tag{15}$$

Lemma 3.

$$\lim_{n \rightarrow \infty} (\log(1/h_n))^{1-\alpha} \max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \|\xi(s_n(i), h_n; \cdot) - \xi(t, h_n, \cdot)\|_\alpha = 0 \quad \text{a.s.,}$$

where $I_n(i) := [s_n(i), s_n(i + 1)]$.

Proof. Berthet [3] shows that for each $\alpha \in (0, 1/2)$, we have

$$\|f\|_\alpha = \max\left\{\omega_{\alpha, f}(\lambda), \|f\|_{\alpha, \lambda} := \sup_{\lambda < |t-s| \leq 1} \frac{|f(t) - f(s)|}{|t - s|^\alpha}\right\}, \quad \forall \lambda \in (0, 1). \tag{16}$$

Let $\{\lambda_n; n \geq 1\}$ be the sequence defined by

$$\lambda_n := n^{-\gamma}, \quad \text{with } \frac{\beta(1 - \alpha)}{(1/2 - \alpha)} < \gamma < \frac{(1/2 - \beta(1 - \alpha))}{\alpha}, \quad \text{for } \beta \text{ as in (11).}$$

By using triangular inequality, we get as $n \rightarrow +\infty$

$$\max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \|\xi(s_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)\|_{\alpha, \lambda_n} = O\left(\frac{(\log(1/h_n))^{-1/2}}{\lambda_n^\alpha h_n^{1/2}} \omega_W(n^{-1}h_n)\right),$$

and

$$\max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \omega_{\alpha, \xi(s_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)}(\lambda_n) = O\left(\frac{(\log(1/h_n))^{-1/2}}{h_n^{1/2-\alpha}} \omega_{\alpha, W}(\lambda_n h_n)\right).$$

Since $\gamma < (1/2 - \beta(1 - \alpha))/\alpha$, respectively $\gamma > \beta(1 - \alpha)/(1/2 - \alpha)$, Lemma 2 joined with

$$\lim_{h \downarrow 0} \frac{h^\alpha \omega_{\alpha, W}(h)}{(2h \log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

obtained as a direct consequence of Theorem 1.1 in Wei [18], yields almost surely

$$(\log(1/h_n))^{1-\alpha} \max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \|\xi(s_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)\|_{\alpha, \lambda_n} = O(n^{1/2-\gamma\alpha+(1-\alpha)\beta}) = o(1),$$

respectively with probability 1

$$(\log(1/h_n))^{1-\alpha} \max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \omega_{\alpha, \xi(s_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)}(\lambda_n) = O(n^{\beta(1-\alpha)-\gamma(1/2-\alpha)}) = o(1).$$

The proof of Lemma 3 is achieved by combining these two last results with (16). \square

Fix $n \geq 1$ and select $h \in [h_n, h_{n-1}]$. For the same reasons as in Part I, we get

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} (\log(1/h_n))^{1-\alpha} \min_{0 \leq i \leq M_{2,n}} \|\xi(s_n(i), h_n; \cdot) - f\|_\alpha \\
 & \leq \liminf_{n \rightarrow \infty} (\log(1/h_n))^{1-\alpha} \max_{0 \leq i \leq M_{2,n}} \sup_{t \in I_n(i)} \|\xi(s_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)\|_\alpha \\
 & \quad + \liminf_{n \rightarrow \infty} \frac{b(h_{n-1})}{b(h_n)} \left(\frac{\log(1/h_n)}{\log(1/h_{n-1})} \right)^{1-\alpha} \inf_{h_n \leq h \leq h_{n-1}} (\log(1/h))^{1-\alpha} \inf_{t \in [0, 1-h]} \|\xi(t, h; \cdot) - f\|_\alpha \\
 & \quad + \liminf_{n \rightarrow \infty} \left(\frac{b(h_{n-1})}{b(h_n)} - 1 \right) + 2 \liminf_{n \rightarrow \infty} \left(\frac{h_{n-1}}{h_n} - 1 \right)^{1/2-\alpha}, \tag{17}
 \end{aligned}$$

where $I_n(i) := [s_n(i), s_n(i + 1)]$.

By making use of (15), Lemma 3 and elementary calculations, we easily obtain (10) which achieves the proof of Theorem 1. \square

2.2. Proof of Theorem 2

The proof of Theorem 2 will also be achieved in two parts. In the first part called *Upper bound*, we will prove that

$$\dim_H L_\alpha(f, c) \leq (1 - J(f))(1 - c^{-2/(1-2\alpha)}) \quad \text{a.s.} \tag{18}$$

whereas in the second part called *Lower bound*, we will prove that

$$\dim_H L_\alpha(f, c) \geq (1 - J(f))(1 - c^{-2/(1-2\alpha)}) \quad \text{a.s.} \tag{19}$$

Upper bound. Denote by $\mathcal{C}_{\mathbb{H}}^+$ the class of sequences $\{\Psi_\varepsilon: \varepsilon \in U\}$, for U an open neighborhood of 0, of Hausdorff functions defined on $[0, 1]$ and fulfilling the condition

$$(A1) \quad \forall \varepsilon_1 < \varepsilon_2, \Psi_{\varepsilon_1}(s) = o(\Psi_{\varepsilon_2}(s)), \text{ as } s \rightarrow 0.$$

Let $\{\Psi_\varepsilon: \varepsilon \in U\} \in \mathcal{C}_{\mathbb{H}}^+$ be fixed. Consider $\{\theta_n: n \geq 1\}$ be a sequence of constants such that the following assumptions are satisfied

$$(H1) \quad \theta_n \in (0, 1), \theta_n \downarrow 0 \text{ and } N_n := \theta_n^{-1} \in \mathbf{N}, \text{ for } n \geq 1,$$

$$(H2) \quad \sum_{n=1}^{+\infty} (\Psi_{-\varepsilon}(\theta_n)/\Psi_{0(1)}(\theta_n)) < +\infty, \forall \varepsilon > 0.$$

For each $n \geq 1$, consider the discretization $\{u_n(i): 1 \leq i \leq N_n\}$ defined by

$$u_n(i) := i\theta_n, \quad i = 1, \dots, N_n. \tag{20}$$

Furthermore, introduce a sequence $\{Y_{i,n}: 1 \leq i \leq N_n\}$ of Bernoulli random variables of parameters $\{p_{i,n}: 1 \leq i \leq N_n\}$ such that

$$(H3) \quad p_n := \max_{1 \leq i \leq N_n} p_{i,n} = O(\theta_n/\Psi_{0(1)}(\theta_n)), \text{ as } n \rightarrow +\infty.$$

Finally, for each $n \geq 1$, define the family $\{I_{i,n}^Y: 1 \leq i \leq N_n\}$ of closed intervals via

$$I_{i,n}^Y := \begin{cases} [u_n(i) - \theta_n; u_n(i)], & \text{if } Y_{i,n} = 1, \\ \emptyset, & \text{if } Y_{i,n} = 0. \end{cases}$$

Theorem 3. *Under assumptions (H1), (H2) and (H3), with probability 1,*

$$\Lambda^{\Psi_\varepsilon} \left(\limsup_{n \rightarrow \infty} \bigcup_{i=1}^{N_n} I_{i,n}^Y \right) = 0, \quad \forall \varepsilon > 0.$$

Remark 1. Here, the Hausdorff function Ψ_0 is considered as the upper critical function of the family $\{\Psi_\varepsilon: \varepsilon \in U\}$ to evaluate the Hausdorff–Besicovitch measure of the limsup random set $E := \limsup_{n \rightarrow \infty} \bigcup_{i=1}^{N_n} I_{i,n}^Y$. In other words,

the Hausdorff function Ψ_0 is the “best” candidate of the family to calculate the exact Hausdorff–Besicovitch measure of E . As an example, Lucas [14] showed that the function $\zeta_{a,b}$ defined in (7) is, for $a = 1/3$ and $b = \mu_\eta := \delta(cz_f(1 + \eta))/(2c^2z_f^2(1 + \eta)^2)$ with $\eta > 0$, the upper critical function of the family belongs to $\mathcal{C}_{\mathbb{H}}^+$

$$\Psi_\varepsilon(s) := \begin{cases} \exp\{-(\mu_\eta + \varepsilon)(\log(1/s))^{1/3}\}, & \text{if } s > 0, \\ 0, & \text{if } s = 0, \end{cases}$$

for the limsup random set $E_\eta \equiv E_\eta(f, c)$ given as in Theorem 3. Hence, the second part of (6) is a straightforward consequence of Theorem 3 applied with this family.

Corollary 1. *Let $\kappa \in (0, 1)$ be fixed and consider the sequence $\{\Psi_{\kappa,\varepsilon}\}_{\varepsilon \in U}$ defined by*

$$\Psi_{\kappa,\varepsilon}(s) := s^{\kappa-\varepsilon} \quad \text{for } s \in [0, 1], \quad \text{with } U = (\kappa - 1, \kappa). \tag{21}$$

Under assumptions (H1), (H2) and (H3), with probability 1,

$$\dim_H \left(\limsup_{n \rightarrow \infty} \bigcup_{i=1}^{N_n} I_{i,n}^Y \right) \leq \kappa.$$

Proof of Corollary 1. It is a straightforward consequence of Theorem 3 by observing that the specified sequence of Hausdorff function satisfies condition (A1). \square

Remark 2. The upper bound for the Hausdorff dimension of $L(f, c)$ given by Lemma 2.2 in Deheuvels and Mason [7] is a straightforward consequence of Corollary 1 with $\kappa = (1 - J(f))(1 - c^{-2}(1 - \eta)^{-2})$ for $f \in \mathcal{S}$ satisfying $J(f) < 1$, $\eta > 0$ and $c > 1$.

Proof of Theorem 3. Fix an arbitrary $\varepsilon > 0$. By combining assumption (H1), (H2) and (H3), we get that

$$\mathbb{E} \left(\sum_{n=1}^{+\infty} \sum_{i=1}^{N_n} \Psi_{-\varepsilon}(\theta_n) 1_{\{Y_{i,n}=1\}} \right) \leq \sum_{n=1}^{+\infty} N_n p_n \Psi_{-\varepsilon}(\theta_n) = O \left(\sum_{n=1}^{+\infty} \frac{\Psi_{-\varepsilon}(\theta_n)}{\Psi_{o(1)}(\theta_n)} \right) < +\infty,$$

where 1_A denotes the indicator function of a measurable set A . Finally, for each $\varepsilon \in (0, 1)$, with probability 1, $\Lambda^{\Psi_{-\varepsilon}}(E) < +\infty$. Making use of Fact 1 in combination with this above result and a sequence $\{\varepsilon_n: n \geq 1\}$ such that $\varepsilon_n \downarrow 0$, we obtain that, with probability 1, $\Lambda^{\Psi_{-\varepsilon}}(E) = 0$ for all $\varepsilon > 0$, which completes the proof of Theorem 3. \square

Now, we will make use of Corollary 1 to find the upper bound of the fractal dimension for $L_\alpha(f, c)$. Let $\alpha \in (0, 1)$, $c > 1$ and $f \in \mathcal{S}$ satisfy $J(f) < 1$. Fix an arbitrary $\eta \in (0, 1)$. Introduce the sequence $\{\theta_n: n \geq 1\}$ defined by

$$\theta_n := 1 / \left\lfloor \frac{(\log(1/h_n))^{1/\beta}}{h_n} \right\rfloor, \quad n \geq 1,$$

where $\{h_n: n \geq 1\}$ and β are defined as in (11). Here, $\lfloor u \rfloor \leq u \leq \lfloor u \rfloor + 1$ denotes the lower part of u . For each $n \geq 1$ and each $i \in \{1, \dots, N_n\}$, define the random variable

$$Y_{i,n} := \begin{cases} 1, & \text{if } (\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha \leq c(1 + \eta)K_\alpha(f), \\ 0, & \text{otherwise.} \end{cases}$$

These random variables are identically distributed following a Bernoulli distribution with parameter $p_n \in (0, 1)$, in such a way that the following property holds.

Lemma 4. *We have as $n \rightarrow +\infty$*

$$p_n = \frac{\theta_n}{\Psi_{o(1)}(\theta_n)}, \tag{22}$$

where

$$\log \Psi_\varepsilon(s) = ((1 - J(f))(1 - (c(1 + \eta))^{-2/(1-2\alpha)}) - \varepsilon) \log s.$$

Proof. For each $n \geq 1$, we get by the scaling property of the Wiener process that

$$p_n = \mathbb{P}(\|W - (2 \log(1/h_n))^{1/2} f\|_\alpha \leq 2^{1-\alpha} c(1 + \eta) K_\alpha(f) ((2 \log(1/h_n))^{1/2})^{-(1-2\alpha)})$$

Making use of Lemma 1, we get as $n \rightarrow +\infty$

$$p_n = \theta_n^{1+(1-J(f))(1-(c(1+\eta))^{-2/(1-2\alpha)})+o(1)}$$

which achieves the proof of Lemma 4. \square

By observing that assumptions (H1), (H2) and (H3) are clearly satisfied for the sequence $\{\Psi_{\kappa,\varepsilon}\}_{\varepsilon \in U}$ defined as in (21) with $\kappa = (1 - J(f))(1 - (c(1 + \eta))^{-2/(1-2\alpha)})$, we deduce from Corollary 1 that almost surely

$$\dim_H E_\eta \leq (1 - J(f))(1 - (c(1 + \eta))^{-2/(1-2\alpha)}), \tag{23}$$

where $E_\eta \equiv E_\eta(f, \alpha, c) := \limsup_{n \rightarrow \infty} \bigcup_{i=1}^{N_n} I_{i,n}^Y$.

We continue by showing that the random set $L_\alpha(f, c)$ is included for each $\eta > 0$ in the random set E_η . Let $\eta > 0$ be fixed and select an arbitrary $t \in L_\alpha(f, c)$. By construction of the sequence of discretizations $\{u_n(i) : 1 \leq i \leq N_n\}_{n \geq 1}$, there exists for each $n \geq 1$ an integer $i \in \{1, \dots, N_n\}$ such that $|t - u_n(i)| \leq \theta_n$. By triangular inequality

$$\begin{aligned} & (\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha \\ & \leq (\log(1/h_n))^{1-\alpha} \max_{1 \leq i \leq N_n} \sup_{t \in [u_n(i) - \theta_n, u_n(i)]} \|\xi(u_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)\|_\alpha \\ & \quad + \left(\frac{(\log(1/h_n))^{1-\alpha} b(h_{n-1})}{(\log(1/h_{n-1}))^{1-\alpha} b(h_n)} \right) \inf_{h_n \leq h < h_{n-1}} (\log(1/h))^{1-\alpha} \|\xi(t, h; \cdot) - f\|_\alpha \\ & \quad + 2(\log(1/h_n))^{1-\alpha} \left(\frac{h_{n-1}}{h_n} - 1 \right)^{1/2-\alpha} + (\log(1/h_n))^{1-\alpha} \left(\frac{b(h_{n-1})}{b(h_n)} - 1 \right). \end{aligned}$$

With similar arguments as in the proof of Lemma 3, we obtain that almost surely for $n \geq 1$ sufficiently large

$$(\log(1/h_n))^{1-\alpha} \max_{1 \leq i \leq N_n} \sup_{t \in [u_n(i) - \theta_n, u_n(i)]} \|\xi(u_n(i), h_n; \cdot) - \xi(t, h_n; \cdot)\|_\alpha \leq \frac{\eta}{4} c K_\alpha(f). \tag{24}$$

Since $t \in L_\alpha(f, c)$, we get that for infinitely many indices $n \geq 1$,

$$\inf_{h_n \leq h < h_{n-1}} (\log(1/h))^{1-\alpha} \|\xi(t, h; \cdot) - f\|_\alpha \leq c \left(1 + \frac{\eta}{4} \right)^{1/2} K_\alpha(f). \tag{25}$$

Finally, from (24), (25) and straightforward calculations, we obtain that for each $t \in L_\alpha(f, c)$, almost surely for infinitely many indices there exists $i \in \mathbb{N}^*$ such that simultaneously $t \in [u_n(i) - \theta_n, u_n(i)]$ and

$$(\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha \leq c(1 + \eta) K_\alpha(f). \tag{26}$$

Since this result is satisfied uniformly over t we deduce that almost surely

$$L_\alpha(f, c) \subseteq E_\eta. \tag{27}$$

The monotonicity property of the Hausdorff dimension in combination with (23) and (27) allow us to get that for each $\eta \in (0, 1)$,

$$\dim_H L_\alpha(f, c) \leq (1 - J(f))(1 - (c(1 + \eta))^{-2/(1-2\alpha)}) \quad \text{a.s.}$$

Since η can be chosen arbitrarily small we get (18) which complete the proof of upper bound.

Lower bound. Denote by $\mathcal{C}_{\mathbb{H}}^-$ the class of sequences $\{\Phi_\varepsilon : \varepsilon \in U\}$, for U an open neighborhood of 0, of Hausdorff functions defined on $[0, 1]$ and fulfilling the following conditions.

(B1) For each $\varepsilon \in U$, there exists $\kappa_\varepsilon > 0$ such that $s \leq \Phi_\varepsilon(s) \leq 1$, for $s \in (0, \kappa_\varepsilon)$.

(B2) For any $\varepsilon_1 < \varepsilon_2$, we have

$$(i) \Phi_{\varepsilon_1}(s) \leq \Phi_{\varepsilon_2}(s), \quad \text{for } s \in (0, 1), \quad \text{and} \quad (ii) \Phi_{\varepsilon_1}(3s) = o(\Phi_{\varepsilon_2}(s)), \quad \text{as } s \rightarrow 0.$$

(B3) For each $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ such that

$$(i) \frac{s}{\Phi_\varepsilon(s)} \text{ non-decreasing on } (0, \tau_\varepsilon), \quad \text{and} \quad (ii) \frac{\Phi_{\varepsilon/2}(3s)}{\Phi_\varepsilon(s)} \leq \phi_\varepsilon(s), \quad \text{for } s \in (0, \tau_\varepsilon),$$

where ϕ_ε denotes a strictly increasing continuous function such that $\phi_\varepsilon(s) \rightarrow 0$ as $s \rightarrow 0$. Denotes by ϕ_ε^{-1} the function such that $\phi_\varepsilon(\phi_\varepsilon^{-1}(s)) = s$ and assume further that $\phi_\varepsilon^{-1}(s) \leq s$ on $(0, \tau_\varepsilon)$.

(B4) There exists a function Γ defined on $(0, 1)$ such that for each $\varepsilon > 0$, we have as $s \rightarrow 0$

$$(i) \Gamma(s) \rightarrow \infty, \quad \Gamma(s)\phi_\varepsilon(s) \rightarrow 0 \quad \text{and} \quad (ii) \frac{\Phi_{o(1)}(s)}{\Phi_\varepsilon(s/\Gamma(s))} = o\left(\frac{1}{\log(1/s)}\right), \quad \text{as } s \rightarrow 0.$$

Let $\{\Phi_\varepsilon: \varepsilon \in U\} \in \mathcal{C}_{\mathbb{H}}^-$ be fixed. Consider $\{\theta_n: n \geq 1\}$ be a sequence of constants satisfying assumptions (H1) and

$$(H'2) \sum_{n=1}^{+\infty} \exp\{-a\Phi_\varepsilon(\theta_n)/\Phi_{o(1)}(\theta_n)\} < \infty, \quad \forall a > 0 \text{ and } \forall \varepsilon > 0.$$

For each $n \geq 1$, let $\{u_n(i): 1 \leq i \leq N_n\}$ the discretization defined in (20) and introduce a sequence $\{Z_{i,n}: 1 \leq i \leq N_n\}$ of independent and identically distributed (i.i.d.) random variables, with a Bernoulli distribution with parameter $p_n \in (0, 1)$ such that

$$(H'3) \theta_n/\Phi_{o(1)}(\theta_n) = O(p_n), \quad \text{as } n \rightarrow \infty.$$

Furthermore, define the sequence $\{H_n: n \geq 1\}$ by setting

$$H_n := \theta_n/\Gamma(\theta_n), \quad \text{for } n \geq 1, \tag{28}$$

where Γ appears in condition (B4), and introduce, for each $n \geq 1$, the family $\{I_{i,n}^Z: 1 \leq i \leq N_n\}$:

$$I_{i,n}^Z := \begin{cases} [u_n(i) - H_n, u_n(i)], & \text{if } Z_{i,n} = 1, \\ \emptyset, & \text{if } Z_{i,n} = 0. \end{cases}$$

Theorem 4. Under assumptions (H1), (H'2) and (H'3), for each $\varepsilon > 0$, with probability 1, there exist a sequence $1 < q_1 < q_2 < \dots < q_j < \dots$ of integers and a sequence $E_1 \supseteq E_2 \supseteq \dots \supseteq E_k \supseteq \dots$ of subintervals from $[0, 1]$ such that, for each $j \geq 1$, E_j is an union of intervals taken among intervals $\{I_{i,q_j}^Z: 1 \leq j \leq N_{q_j}\}$. Further, we have

$$\Lambda^{\Phi_\varepsilon}(\mathbb{K}_\varepsilon) > 0,$$

where $\mathbb{K}_\varepsilon = \bigcap_{j=1}^\infty E_j$.

Remark 3. Contrary to the upper bound, the Hausdorff function Φ_0 is considered here as the lower critical function of the family $\{\Phi_\varepsilon: \varepsilon \in U\}$ to evaluate the Hausdorff–Besicovitch measure of the Cantor type set \mathbb{K}_ε . In other words, the Hausdorff function Φ_0 is the “best” candidate of the family to calculate the exact Hausdorff–Besicovitch measure of \mathbb{K}_ε .

As an example, Lucas [14] showed that the function $\Psi_{a,b}$ defined in (7) is, for $a = 1/3$ and $b = v_\eta := \delta(cz_f(1 - \eta))/(2c^2z_f^2(1 - \eta)^2)$ with $\eta > 0$, the lower critical function of the family

$$\Phi_\varepsilon(s) := \begin{cases} \exp\{-(v_\eta + \varepsilon)(\log(1/s))^{1/3}\}, & \text{if } s > 0, \\ 0, & \text{if } s = 0, \end{cases}$$

for the Cantor type set $\mathbb{K}_\varepsilon \equiv \mathbb{K}_\varepsilon(f, c, \eta)$ given as in Theorem 4. Since this sequence belongs to $\mathcal{C}_{\mathbb{H}}^-$, with $\phi_\varepsilon(s) := \exp\{-\varepsilon/4(\log(1/s))^b\}$ and $\Gamma(s) := (\log(1/s)^\gamma$ for $\gamma > 0$, Theorem 4 is an extension of Theorem 3 in Lucas [14] and allows to find the second part in (6).

Corollary 2. Let $d \in (0, 1)$ be fixed and consider the sequence $\{\Phi_{d,\varepsilon}: \varepsilon \in U\}$ defined by

$$\Phi_{d,\varepsilon}(s) := s^{d-\varepsilon}, \quad \text{for } s \in [0, 1], \text{ with } U = (d - 1, d). \tag{29}$$

Under assumptions (H1), (H'2) and (H'3), for each $\varepsilon > 0$, with probability 1, there exist a sequence $1 < q_1 < q_2 < \dots < q_j < \dots$ of integers and a sequence $E_1 \supseteq E_2 \supseteq \dots \supseteq E_k \supseteq \dots$ of subintervals from $[0, 1]$ such that, for each $j \geq 1$, E_j is an union of intervals taken among intervals $\{I_{i,q_j}^Z : 1 \leq j \leq N_{q_j}\}$. Moreover, we have

$$\dim_H \mathbb{K}_\varepsilon \geq d - \varepsilon.$$

Proof. It is a straightforward consequence of Theorem 4 by observing that the sequence $\{\Phi_{d,\varepsilon} : \varepsilon \in U\} \in \mathcal{C}_{\mathbb{H}}^-$, with $\phi_\varepsilon(s) := s^{\varepsilon/4}$ and $\Gamma(s) := (\log(1/s))^\gamma$ for $\gamma > 0$. \square

Remark 4. Corollary 1 corresponds to Theorem 2.1 in Deheuvels and Mason [7]. In particular, the lower bound for the Hausdorff dimension of $L(f, c)$ given in Section 2.5 in their article is a direct consequence of this corollary with $d = (1 - J(f))(1 - c^{-2}(1 - \eta)^{-2})$ for $f \in \mathcal{S}$ satisfying $J(f) < 1$, $\eta > 0$ and $c > 1$.

Now, we will give the proof of (19) which achieves the proof of Theorem 2. Let $f \in \mathcal{S}$ satisfy $J(f) < 1$, $c > 1$ be fixed and $\eta > 0$ an arbitrary constant. Introduce the sequence $\{\theta_n : n \geq 1\}$ defined by

$$\theta_n := 1 \left/ \left\lfloor \frac{n}{h_n} \right\rfloor \right., \quad n \geq 1,$$

where $\{h_n : n \geq 1\}$ is defined as in (11). Further, consider the sequence $\{H_n : n \geq 1\}$ defined as in (28) with $\Gamma(s) = (\log(1/s))^2$. For each $i \in \{1, \dots, N_n\}$, define the random variable

$$Z_{i,n} := \begin{cases} 1, & \text{if } (\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha \leq c(1 - \eta)K_\alpha(f), \\ 0, & \text{otherwise.} \end{cases}$$

These random variables are i.i.d. following a Bernoulli distribution with parameter $p_n \in (0, 1)$ in such a way that the next property holds.

Lemma 5. We have as $n \rightarrow +\infty$

$$p_n = \frac{\theta_n}{\Phi_{O(1)}(\theta_n)}, \tag{30}$$

where

$$\log \Phi_\varepsilon(s) = ((1 - J(f))(1 - (c(1 - \eta))^{-2/(1-2\alpha)}) - \varepsilon) \log s. \tag{31}$$

Proof. See Lemma 4. \square

By observing that (H1), (H'2) and (H'3) are clearly satisfied for the sequence $\{\Phi_{d,\varepsilon}\}_{\varepsilon \in U}$ defined as in (29) with $d = (1 - J(f))(1 - (c(1 - \eta))^{-2/(1-2\alpha)})$, we deduce from Corollary 2 that, for each $\varepsilon > 0$, almost surely

$$\dim_H \mathbb{K}_\varepsilon \geq (1 - J(f))(1 - (c(1 - \eta))^{-2/(1-2\alpha)}) - \varepsilon, \tag{32}$$

where \mathbb{K}_ε is defined as in Theorem 4.

We continue by showing that the random Cantor type set \mathbb{K}_ε is included, for each $\varepsilon > 0$, in the random set $L_\alpha(f, c)$. Observe first that for all $n \geq 1$ and all $i \in \{1, \dots, N_n\}$, we have

$$\begin{aligned} (\log(1/h_n))^{1-\alpha} \|\xi(t, h_n; \cdot) - f\|_\alpha &\leq (\log(1/h_n))^{1-\alpha} \max_{1 \leq i \leq N_n} \sup_{t \in [u_n(i) - \theta_n, u_n(i)]} \|\xi(t, h_n; \cdot) - \xi(u_n(i), h_n; \cdot)\|_\alpha \\ &+ (\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha, \end{aligned}$$

uniformly in $t \in [u_n(i) - H_n, u_n(i)]$.

Using similar arguments as in the proof of Lemma 3, it is easy to see that with probability 1 the first term in the RHS of the previous inequality goes to zero as n tends to infinity. Consequently for $n \geq 1$ sufficiently large and all $i \in \{1, \dots, N_n\}$, almost surely

$$(\log(1/h_n))^{1-\alpha} \|\xi(t, h_n; \cdot) - f\|_\alpha \leq c\eta K_\alpha(f) + (\log(1/h_n))^{1-\alpha} \|\xi(u_n(i), h_n; \cdot) - f\|_\alpha \tag{33}$$

uniformly in $t \in [u_n(i) - H_n, u_n(i)]$.

Let $\varepsilon > 0$ be fixed and select an arbitrary $t \in \mathbb{K}_\varepsilon$. By definition of this Cantor type set, it is clear that for all $j \geq 1$, there exists $i \in \{1, \dots, N_{q_j}\}$ such that jointly

$$t \in [u_{q_j}(i) - H_{q_j}, u_{q_j}(i)] \tag{34}$$

and

$$(\log(1/h_{q_j}))^{1-\alpha} \|\xi(u_{q_j}(i), h_{q_j}; \cdot) - f\|_\alpha \leq c(1-\eta)K_\alpha(f). \tag{35}$$

Finally by combining (34) and (35), we obtain that almost surely, for any $t \in \mathbb{K}_\varepsilon$, there exists $j_0 < +\infty$ such that for all $j \geq j_0$,

$$(\log(1/h_{q_j}))^{1-\alpha} \|\xi(t, h_{q_j}; \cdot) - f\|_\alpha \leq cK_\alpha(f).$$

Consequently, for each $\varepsilon > 0$

$$\mathbb{K}_\varepsilon \subseteq L_\alpha(f, c) \quad \text{a.s.}$$

Since $\varepsilon > 0$ and $\eta > 0$ can be chosen arbitrarily small, it follows from (32) in combination with the property of monotonicity of the Hausdorff dimension that almost surely

$$\dim_H L_\alpha(f, c) \geq (1 - J(f))(1 - c^{-2/(1-2\alpha)}),$$

which completes the proof of the lower bound.

Proof of Theorem 4. This proof is based on the following lemma. \square

Lemma 6. Let Ψ be a Hausdorff function and let $\mathbb{K} \subseteq [0, 1]$ be such that $\mathbb{K} := \bigcap_{m \geq 1} E_m$ where $\{E_m\}_{m \geq 1}$ is a sequence of subsets of $[0, 1]$ satisfying the following conditions.

1. $\{E_m\}_{m \geq 1}$ is decreasing with respect to inclusion (i.e. $E_{m+1} \subseteq E_m, \forall m \geq 1$),
2. $\forall m \geq 1, E_m = \bigcup_{1 \leq k \leq M_m} I_{m,k}$ where $\{I_{m,k}: 1 \leq k \leq M_m\}$ is a collection of disjoint closed subintervals of $[0, 1]$, with
 - (a) $M_m \rightarrow \infty$, when $m \rightarrow \infty$,
 - (b) $\max_{1 \leq k \leq M_m} |I_{m,k}| \rightarrow 0$, when $m \rightarrow \infty$.
3. There exist two constants $\Delta > 0$ and $d > 0$ such that, for every interval $I \subseteq [0, 1]$ with $|I| \leq \Delta$ there exists a constant $m(I)$ such that for all $m \geq m(I)$,

$$M_m(I) := \#\{I_{m,k} \subseteq I: 1 \leq k \leq M_m\} \leq d\Psi(|I|)M_m. \tag{36}$$

Then we have $\Lambda^\Psi(\mathbb{K}) > 0$.

Proof. See Lemma 2.2 in Orey and Taylor [15]. \square

More precisely, we will construct for each $\varepsilon > 0$ a Cantor type set \mathbb{K}_ε such that conditions of Lemma 6 will be satisfied for $\Psi \equiv \Phi_\varepsilon$. In the sequel, to denote a random variable X with a binomial distribution with parameters n and p , we make use of the notation $X \sim \text{Bin}(n, p)$. For each $n \geq 1$ and each closed interval $I \subseteq [0, 1]$, set

$$\mathcal{N}_n(I) := \#\{I_{i,n}^Z \subseteq I: 1 \leq i \leq N_n \text{ and } Z_{i,n} = 1\}.$$

In the particular case $I = [0, 1]$, it will be convenient to set

$$\mathcal{N}_n := \mathcal{N}_n([0, 1]) = \sum_{i=1}^{N_n} Z_{i,n}.$$

Now, introduce the function (see [16])

$$\mathbf{h}(u) := \begin{cases} u \log u - u + 1 & \text{if } u > 0, \\ 1 & \text{if } u = 0. \end{cases} \tag{37}$$

Lemma 7. Let S_n be a binomial random variable with parameters n and p . Then,

1. For any $r \geq 1$,

$$\mathbb{P}(S_n \geq rnp) \leq \exp\{-nph(r)\}. \tag{38}$$

2. For any $r \leq 1$,

$$\mathbb{P}(S_n \leq rnp) \leq \exp\{-nph(r)\}. \tag{39}$$

Proof. See Lemma 3.8 in [8]. \square

Lemma 8. Under (H1), (H'2) and (H'3) we have

$$\lim_{n \rightarrow \infty} \frac{\mathcal{N}_n}{p_n N_n} = 1, \quad \text{a.s.} \tag{40}$$

Proof. Fix an arbitrary constant $\epsilon \in (0, 1)$. Since $\mathcal{N}_n \sim \text{Bin}(N_n, p_n)$, (40) is a straightforward consequence of Lemma 7 with $r = 1 \pm \epsilon$, $p = p_n$ and $n = N_n$. \square

Next, we will be interested in the random variable $\mathcal{N}_n(I)$, where $I \subseteq [0, 1]$ is a closed interval. First, for each $n \geq 1$, and for each $1 \leq k \leq N_n$, denote by $C_n(k)$ the class of all closed intervals of the form $I := [t_n(i), t_n(i + k)]$ with $0 \leq i \leq N_n - k$. Let $\eta > 0$ be an arbitrary constant. For each $n \geq 1$, define

$$k_{1,n} := \lfloor \theta_n^{-1} \Phi_{2\eta}(\theta_n) \rfloor \quad \text{and} \quad k_{2,n} := \lfloor \theta_n^{-1} \Phi_{4\eta}(\theta_n) \rfloor.$$

Lemma 9. Under (H1), (H'2) and (H'3) we have with probability 1,

$$(i) \lim_{n \rightarrow \infty} \max_{I \in C_n(k_{1,n})} \frac{\mathcal{N}_n(I)}{|I| \mathcal{N}_n} = 1 \quad \text{and} \quad (ii) \lim_{n \rightarrow \infty} \min_{I \in C_n(k_{1,n})} \frac{\mathcal{N}_n(I)}{|I| \mathcal{N}_n} = 1. \tag{41}$$

Proof. Since $|I| = k_{1,n} \theta_n$ and $\mathcal{N}_n(I) \sim \text{Bin}(k_{1,n}, p_n)$ for each $I \in C_n(k_{1,n})$, we deduce from (38) applied with $r = 1 + \epsilon$, $p = p_n$ and $n = k_{1,n}$, for $\epsilon \in (0, 1)$ be fixed, that for each $n \geq 1$

$$\begin{aligned} P_1(n) &:= \mathbb{P}\left(\max_{I \in C_n(k_{1,n})} \frac{\mathcal{N}_n(I)}{|I| p_n N_n} \geq 1 + \epsilon\right) \leq \sum_{I \in C_n(k_{1,n})} \mathbb{P}(\mathcal{N}_n(I) \geq (1 + \epsilon) p_n k_{1,n}) \\ &\leq \exp\{\log(1/\theta_n) - \mathbf{h}(1 + \epsilon) p_n k_{1,n}\}. \end{aligned} \tag{42}$$

Making use of (H'2) and the definition of $k_{1,n}$, we obtain that

$$\mathbf{h}(1 + \epsilon) \frac{\log(1/\theta_n)}{p_n k_{1,n}} = O\left(\frac{\Phi_{0(1)}(\theta_n)}{\Phi_{2\eta}(\theta_n/\Gamma(\theta_n))} \times \log(1/\theta_n)\right), \quad \text{as } n \rightarrow +\infty. \tag{43}$$

Finally, from (B4)(ii) and (H'3) in combination with (42) and (43) we deduce

$$\sum_{n=1}^{+\infty} P_1(n) = O\left(\sum_{n=1}^{+\infty} \exp\left\{-\frac{C\mathbf{h}(1 + \epsilon)}{4} \times \frac{\Phi_{2\eta}(\theta_n)}{\Phi_{0(1)}(\theta_n)}\right\}\right) < +\infty.$$

which entails by the Borel–Cantelli lemma that with probability 1,

$$\limsup_{n \rightarrow \infty} \max_{I \in C_n(k_{1,n})} \frac{\mathcal{N}_n(I)}{|I| p_n N_n} \leq 1 + \epsilon. \tag{44}$$

By similar argument, it is easy to see that with probability 1,

$$\liminf_{n \rightarrow \infty} \min_{I \in C_n(k_{1,n})} \frac{\mathcal{N}_n(I)}{|I| p_n N_n} \geq 1 - \epsilon. \tag{45}$$

Since $\epsilon \in (0, 1)$ may be chosen arbitrarily small, we obtain (41)(i) and (41)(ii) from (44), (45) and trivial inequalities. \square

Lemma 10. Assume that (H1), (H'2) and (H'3) are fulfilled. Then, for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_{2,n}} \max_{I \in C_n(k)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} = 0, \quad \text{a.s.} \tag{46}$$

Proof. Let $\varepsilon > 0$ and $\sigma \in (0, 1)$ be arbitrary constants. Making use of (B2)(ii) and the fact that for $1 \leq k \leq k_{2,n}$ be fixed $\mathcal{N}_n(I) \sim \text{Bin}(k, p_n)$ for all $I \in C_n(k)$, we deduce from (38) applied with $n = k$, $p = p_n$ and $r = \sigma(k\theta_n)^{-1}\Phi_\varepsilon(|I|)$, that for $n \geq 1$ sufficiently large

$$\begin{aligned} P_3(n) &:= \mathbb{P}\left(\max_{1 \leq k \leq k_{2,n}} \max_{I \in C_n(k)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)p_n \mathcal{N}_n} \geq \sigma\right) \leq \sum_{k=1}^{k_{2,n}} \sum_{I \in C_n(k)} \mathbb{P}(\mathcal{N}_n(I) \geq \{\sigma(k\theta_n)^{-1}\Phi_\varepsilon(|I|)\}kp_n) \\ &\leq \sum_{k=1}^{k_{2,n}} \sum_{I \in C_n(k)} \exp\{-kp_n \mathbf{h}(\sigma(k\theta_n)^{-1}\Phi_\varepsilon(|I|))\}. \end{aligned} \tag{47}$$

Next, observe that for any $a > 0$ fixed $\mathbf{h}(x) = x \log x(1 + o(1)) \geq ax$ for all sufficiently large x and $\theta_n \leq |I| = k\theta_n \leq k_{2,n}\theta_n$ for all $I \in C_n(k)$. Consequently, by easy calculations, we get from (B1), (B4)(ii), (H'2), (H'3) and (47)

$$\sum_{n=1}^{+\infty} P_3(n) = O\left(\sum_{n=1}^{+\infty} \exp\left\{-\frac{\mathcal{C}}{2} \times \frac{\Phi_\varepsilon(\theta_n)}{\Phi_{o(1)}(\theta_n)}\right\}\right) < +\infty \quad \text{with } 0 < \mathcal{C} < +\infty.$$

Consequently, the Borel–Cantelli lemma implies that with probability 1,

$$\limsup_{n \rightarrow \infty} \max_{1 \leq k \leq k_{2,n}} \max_{I \in C_n(k)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \sigma$$

which achieves the proof of Lemma 8 by noting that σ may be chosen arbitrarily small. \square

For $\varepsilon > 0$ be fixed, choose $\eta \in (0, \varepsilon/4)$ an arbitrary constant, and define for all $n \geq 1$, where $\gamma_n := \theta_n^{-1}\Phi_{3\eta}(\theta_n)$, the following classes of closed intervals

$$\begin{aligned} C_1(n) &:= \{I \subseteq [0, 1]: \gamma_n\theta_n \leq |I| < \kappa_\varepsilon \wedge \tau_\varepsilon\}, \\ C_2(n) &:= \{I \subseteq [0, 1]: \theta_n \leq |I| < \gamma_n\theta_n\}, \\ C_3(n) &:= \{I \subseteq [0, 1]: H_n \leq |I| < \theta_n\}, \\ C_4(n) &:= \{I \subseteq [0, 1]: 0 < |I| < H_n\}. \end{aligned}$$

Lemma 11. Assume that (H1), (H'2) and (H'3) are fulfilled. Then, for each $\varepsilon > 0$,

1. With probability 1,

$$\limsup_{n \rightarrow \infty} \sup_{I \in C_1(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq 1. \tag{48}$$

2. With probability 1,

$$\lim_{n \rightarrow \infty} \sup \left\{ \left| \frac{\mathcal{N}_n(I)}{|I|\mathcal{N}_n} - 1 \right| : \tau \leq |I| \right\} = 0, \quad \forall \tau \in (0, 1). \tag{49}$$

Proof. Let $\varepsilon > 0$, $\tau \in (0, 1)$ and $\sigma \in (0, 1)$ be arbitrary constants. For all $n \geq 1$, set

$$K(n) := \left\lfloor \frac{\Phi_{3\eta}(\theta_n)}{\Phi_\eta(\theta_n)} \right\rfloor.$$

From the definitions of $K(n)$ and γ_n in combination with (H1), we may choose an integer $n' < \infty$ such that for all $n \geq n'$, $K(n) \geq 3$ and the following conditions are satisfied

$$(i) \left(1 + \frac{\sigma}{2}\right) \left(1 + \frac{4}{K(n) - 2}\right) \leq 1 + \sigma \quad \text{and} \quad (ii) \left(1 - \frac{\sigma}{2}\right) \left(1 - \frac{4}{K(n) + 2}\right) \geq 1 - \sigma. \tag{50}$$

First, observe that for all $n \geq n'$,

$$K \equiv K_n(I) := \left\lfloor \frac{|I|}{k_{1,n}\theta_n} \right\rfloor \geq K(n) \geq 3, \quad \forall I \in C_1(n). \tag{51}$$

Hence, for all $n \geq n'$ and each $I \in C_1(n)$, we may find $K + 2$ distinct intervals (which will be denoted by J_1, J_2, \dots, J_{K+2}) of $C_n(k_{1,n})$ such that

$$\bigcup_{i=1}^{K-2} J_i \subseteq I \subseteq \bigcup_{i=1}^{K+2} J_i. \tag{52}$$

Since these intervals are disjoint and have the same length, (52) yields

$$(K - 2)|J_1| \leq |I| \leq (K + 2)|J_1|. \tag{53}$$

It follows from the definition of function $\mathcal{N}'_n(\cdot)$ in combination with (41)(i), (50)(i), (51), (52) and (53) that we may choose with probability 1 an integer $n_1 \geq n'$, such that for all $n \geq n_1$,

$$\sup_{I \in C_1(n)} \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} \leq \left(1 + \frac{4}{K(n) - 2}\right) \left\{ \max_{J \in C_n(k_{1,n})} \frac{\mathcal{N}'_n(J)}{|J|\mathcal{N}_n} \right\} \leq 1 + \sigma. \tag{54}$$

Consequently, with probability 1,

$$\limsup_{n \rightarrow +\infty} \sup_{I \in C_1(n)} \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} \leq 1 + \sigma. \tag{55}$$

Making use of the definition of $C_1(n)$ in combination with (B1), we get that for all $n \geq n'$, $|I| \leq \Phi_\varepsilon(|I|)$ for all $I \in C_1(n)$. Combining with (55), that gives

$$\limsup_{n \rightarrow +\infty} \sup_{I \in C_1(n)} \frac{\mathcal{N}'_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq 1 + \sigma \quad \text{a.s.}$$

Since $\sigma \in (0, 1)$ may be chosen arbitrarily small in the previous expression, we obtain (48). By following along the same lines as above, we get from (41)(ii), (50)(ii), (51), (52) and (53) that we may choose with probability 1 an integer $n_2 \geq n'$, such that for all $n \geq n_2$,

$$\inf_{I \in C_1(n)} \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} \geq \left(1 - \frac{4}{K(n) - 2}\right) \left\{ \min_{J \in C_n(k_{1,n})} \frac{\mathcal{N}'_n(J)}{|J|\mathcal{N}_n} \right\} \geq 1 - \sigma. \tag{56}$$

By combining (54) with (56), we show that with probability 1,

$$\lim_{n \rightarrow +\infty} \sup \left\{ \left| \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} - 1 \right| : I \in C_1(n) \right\} = 0.$$

Since (51) is satisfied for each $I \subseteq [0, 1]$ with $|I| \geq \gamma_n\theta_n$, we obtain that almost surely,

$$\lim_{n \rightarrow +\infty} \sup \left\{ \left| \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} - 1 \right| : \gamma_n\theta_n \leq |I| \right\} = 0. \tag{57}$$

Clearly, $\gamma_n\theta_n = \Phi_{3\eta}(\theta_n) \downarrow 0$ as $n \rightarrow +\infty$. Consequently, we get that for all $n \geq n'$,

$$\sup \left\{ \left| \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} - 1 \right| : \tau \leq |I| \right\} \leq \sup \left\{ \left| \frac{\mathcal{N}'_n(I)}{|I|\mathcal{N}_n} - 1 \right| : \gamma_n\theta_n \leq |I| \right\}.$$

By making use of this inequality with (57), (49) holds and completes the proof. \square

Lemma 12. Assume that (H1), (H'2) and (H'3) are fulfilled. Then, for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \sup_{I \in C_2(n)} \frac{\mathcal{N}'_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} = 0 \quad \text{a.s.} \tag{58}$$

Proof. Let $\varepsilon > 0$ and $\sigma \in (0, 1)$ be arbitrary constants. Choose an integer $n'' < \infty$ such that for all $n \geq n''$,

$$(i) \theta_n^{-1} \Phi_{3\eta}(\theta_n) + 3 < \theta_n^{-1} \Phi_{4\eta}(\theta_n) \quad \text{and} \quad (ii) \gamma_n \theta_n = \Phi_{3\eta}(\theta_n) < \tau_\varepsilon. \tag{59}$$

Let $n \geq n''$ be an integer. Consider an interval $I = [a, b] \in C_2(n)$. By definition of $C_2(n)$ and from (59)(i), we have

$$|I| \leq \Phi_{3\eta}(\theta_n) \leq (\theta_n^{-1} \Phi_{4\eta}(\theta_n) - 1)\theta_n - 2\theta_n \leq k_{2,n}\theta_n - 2\theta_n. \tag{60}$$

Form the closed interval $I' \equiv I'(n) := [\alpha_n^1 \theta_n, \alpha_n^2 \theta_n]$, where

$$\alpha_n^1 := \max\{i \in \mathbb{N} : i\theta_n \leq a\} \quad \text{and} \quad \alpha_n^2 := \min\{i \in \mathbb{N} : i\theta_n \geq b\}.$$

By construction, it is easy to see that on the one hand $I \subseteq I'$ whereas on the other hand, we have $|I'| \leq |I| + 2\theta_n$. From (60), and since $I \in C_2(n)$, we get $|I'| \leq 3|I|$ and $|I'| \leq k_{2,n}\theta_n$. Consequently, for all $n \geq n''$ and each $I \in C_2(n)$, there exists a closed interval $I' \in C_n(k)$ with $1 \leq k \leq k_{2,n}$ such that jointly $I \subseteq I'$ and $|I'| \leq 3|I|$. It follows from monotonicity of $\mathcal{N}_n(\cdot)$ that for all $n \geq n''$ and each $I \in C_2(n)$, we have

$$\frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \frac{\mathcal{N}_n(I')}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \frac{\Phi_{\varepsilon/2}(3|I|)}{\Phi_\varepsilon(|I|)} \left\{ \max_{1 \leq k \leq k_{2,n}} \max_{J \in C_n(k)} \frac{\mathcal{N}_n(J)}{\Phi_{\varepsilon/2}(|J|)\mathcal{N}_n} \right\}.$$

Making use of (B3)(ii) and (59)(ii), we get that for all $n \geq n''$,

$$\frac{\Phi_{\varepsilon/2}(3|I|)}{\Phi_\varepsilon(|I|)} \leq \phi_\varepsilon(|I|) \leq 1, \quad \forall I \in C_2(n).$$

Finally, we obtain that for all $n \geq n''$,

$$\sup_{I \in C_2(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \max_{1 \leq k \leq k_{2,n}} \max_{J \in C_n(k)} \frac{\mathcal{N}_n(J)}{\Phi_{\varepsilon/2}(|J|)\mathcal{N}_n}.$$

The combination of this above inequality with (46) yields (58). \square

Lemma 13. Assume that (H1), (H'2) and (H'3) are fulfilled. Then, for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \sup_{I \in C_3(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} = 0 \quad \text{a.s.} \tag{61}$$

For all $n \geq 1$,

$$\sup_{I \in C_4(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} = 0. \tag{62}$$

Proof. Let $\varepsilon > 0$ be an arbitrary constant and let $n \geq 1$ be an integer. First, for any $I \in C_3(n)$ be fixed, we have $\mathcal{N}_n(I) \leq 1$. Thus, it follows from the non-decreasing property of function Φ_ε , the definitions of $C_3(n)$, and $\{H_n : n \geq 1\}$, in combination with (40), (H'2) and (B4)(ii)

$$\sup_{I \in C_3(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \frac{1}{\Phi_\varepsilon(H_n)\mathcal{N}_n} = O\left(\frac{\Phi_{o(1)}(\theta_n)}{\Phi_\varepsilon(\theta_n/\Gamma(\theta_n))}\right) = o(1), \quad \text{a.s. (as } n \rightarrow +\infty).$$

Finally, we infer from these inequalities that (61) holds. The second result of this lemma is a direct consequence of the fact that for any $n \geq 1$, $\mathcal{N}_n(I) = 0$ for each $I \in C_4(n)$. \square

Proposition 1. Assume that (H1), (H'2) and (H'3) are fulfilled. Then, for each $\varepsilon > 0$

$$\limsup_{n \rightarrow \infty} \sup_{\{I: 0 < |I| < \kappa_\varepsilon \wedge \tau_\varepsilon\}} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq 1 \quad \text{a.s.} \tag{63}$$

Proof. Let $\varepsilon > 0$, $\eta \in (0, \varepsilon/4)$ and $\sigma \in (0, 1)$ be arbitrary constants. First, observe that for all $n \geq 1$ sufficiently large, we have: for all $I \subseteq [0, 1]$ satisfying $0 < |I| < \kappa_\varepsilon \wedge \tau_\varepsilon$, there exists $1 \leq i \equiv i(n) \leq 4$ such that $I \in C_i(n)$. Hence, by combining (48), (58), (61) and (62), we may choose almost surely a finite integer $n(\varepsilon, \sigma)$ such that for all $n \geq n(\varepsilon, \sigma)$,

$$\sup_{\{I: 0 < |I| < \kappa_\varepsilon \wedge \tau_\varepsilon\}} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \leq \max_{1 \leq i \leq 4} \left\{ \sup_{I \in C_i(n)} \frac{\mathcal{N}_n(I)}{\Phi_\varepsilon(|I|)\mathcal{N}_n} \right\} \leq 1 + \sigma, \tag{64}$$

which entails (63) by observing that $\sigma \in (0, 1)$ may be chosen arbitrarily small and that (64) is independent of the choice of η . \square

Making use of the above lemmas and proposition, we will build a Cantor type set \mathbb{K} such that assumptions of Fact 5 are satisfied. That constitutes the main object of the next paragraph.

Exhaustive construction of \mathbb{K} . Assume (H1), (H'2) and (H'3) are fulfilled. Select an arbitrary $\varepsilon > 0$. From Proposition 2, we may choose, with probability 1, an integer $m_1(\varepsilon) < \infty$ such that for all $n \geq m_1(\varepsilon)$ and for all closed interval $I \subseteq [0, 1]$ such that $0 < |I| < \kappa_\varepsilon \wedge \tau_\varepsilon$,

$$\mathcal{N}_n(I) \leq 2\Phi_{\varepsilon/2}(|I|)\mathcal{N}_n. \tag{65}$$

From (H1) and (B4), we may choose an integer $m_2(\varepsilon) < \infty$ such that for all $n \geq m_2(\varepsilon)$, we have

$$3\theta_n \leq \tau_\varepsilon \quad \text{and} \quad \Gamma(\theta_n) \geq 1. \tag{66}$$

Introduce a sequence $\{\sigma_k: k \geq 1\}$ such that

$$\sigma_k \in (0, 1) \quad \text{for all } k \geq 1 \quad \text{and} \quad \sum_{k=1}^{+\infty} \sigma_k < +\infty.$$

Observe that under the conditions above, we have $0 < \prod_{k=1}^{\infty} (1 - \sigma_k) < \prod_{k=1}^{\infty} (1 + \sigma_k) < +\infty$. From (40), we may choose, almost surely an integer $m_3(\sigma_1) < +\infty$ such that for all $n \geq m_3(\sigma_1)$,

$$(1 - \sigma_1)p_n N_n \leq \mathcal{N}_n \leq (1 + \sigma_1)p_n N_n. \tag{67}$$

Further, by definition of N_n and by (H'2), it is easy to see that $p_n N_n \uparrow +\infty$ as $n \rightarrow +\infty$. Thus, there exists an integer $m_4(\sigma_1) < +\infty$ such that for all $n \geq m_4(\sigma_1)$,

$$(1 - \sigma_1)p_n N_n \geq 1. \tag{68}$$

From (H'3), there exists an integer $m_5 < +\infty$ such that for all $n \geq m_5$,

$$p_n N_n \geq \frac{\mathcal{C}}{\Phi_{\xi(n)}(\theta_n)}, \tag{69}$$

with $\mathcal{C} > 0$, where $\xi(n) \rightarrow 0$, as $n \rightarrow +\infty$.

Without loss of generality, we can and do assume that $\xi(n) < \varepsilon/2$, for all $n \geq m_5$. Finally, we may choose with probability 1 an integer $\tilde{m} \geq \max\{m_1(\varepsilon), m_2(\varepsilon), m_3(\sigma_1), m_4(\sigma_1), m_5\}$ such that for all $n \geq \tilde{m}$, we have jointly (65)–(68) and (69). For $q_1 \geq \tilde{m}$ be fixed, set $E_1 := \bigcup\{I_{i,q_1} \subseteq [0, 1]: I_{i,q_1} \neq \emptyset \text{ and } 1 \leq i \leq N_{q_1}\}$, and $M_1 := \#\{I_{i,q_1} \subseteq [0, 1]: I_{i,q_1} \neq \emptyset \text{ and } 1 \leq i \leq N_{q_1}\} = \mathcal{N}_{q_1}$. Denote by $\{J_{i,1}: 1 \leq i \leq M_1\}$ the set of nonempty intervals of $\{I_{i,q_1}: 1 \leq i \leq N_{q_1}\}$. Then, we have $E_1 = \bigcup_{i=1}^{M_1} J_{i,1}$. Suppose that $\{q_k: k \geq 1\}$ is a strictly increasing sequence of integers. Define by induction $E_k := \bigcup\{I_{i,q_k} \subseteq [0, 1]: I_{i,q_k} \neq \emptyset, I_{i,q_k} \subseteq E_{k-1} \text{ and } 1 \leq i \leq N_{q_k}\}$, and $M_k := \#\{I_{i,q_k} \subseteq [0, 1]: I_{i,q_k} \neq \emptyset, I_{i,q_k} \subseteq E_{k-1} \text{ and } 1 \leq i \leq N_{q_k}\}$. Denote by $\{J_{i,k}: 1 \leq i \leq M_k\}$ the set of intervals of E_k . Then, we have $E_k := \bigcup_{i=1}^{M_k} J_{i,k}$.

Now, by combining (40) with (49), we obtain that for each $\beta > 0$, with probability 1 there exists, for each $\tau \in (0, 1)$ an integer $m_6(\beta, \tau) < +\infty$ such that for all $n \geq m_6(\beta, \tau)$,

$$(1 - \beta)|I|p_n N_n \leq \mathcal{N}_n(I) \leq (1 + \beta)|I|p_n N_n, \tag{70}$$

for all $I \subseteq [0, 1]$ such that $|I| \geq \tau$. Assume now that with probability 1, for $k \geq 2$, integers q_1, q_2, \dots, q_{k-1} are well defined. Select q_k large enough so that simultaneously

$$q_k > \max\{q_{k-1}, m_6(\sigma_k, H_{q_{k-1}}), m_6(1/2, \phi_\varepsilon^{-1}(H_{q_{k-1}}))\}, \tag{71}$$

$$(1 - \sigma_k)p_{q_k} N_{q_k} \geq \frac{(1 - \sigma_k)\mathcal{C}}{\Phi_{\xi(q_k)}(\theta_{q_k})} \geq H_{q_{k-1}}^{-1-\varepsilon}, \tag{72}$$

$$\theta_{q_k} < \phi_\varepsilon^{-1}(H_{q_{k-1}}), \tag{73}$$

$$\Phi_{\varepsilon/2}(\theta_{q_k}) \leq \Gamma(\theta_{q_k})\phi_\varepsilon(\theta_{q_k}) \leq \frac{1}{4} \left\{ \prod_{i=1}^{k-1} p_{q_i} N_{q_i} H_{q_i} \right\}. \tag{74}$$

Notice that this last condition is possible by (B3)(ii) and (B4)(i). The sequence $\{q_k: k \geq 1\}$ being well defined, the decreasing sequence of sets $\{E_k: k \geq 1\}$ is well defined too. We continue by showing that the sequence $\{E_k: k \geq 1\}$ satisfies the assumptions of Fact 5.

Lemma 14.

1. $\mathbb{K} = \bigcap_{k=1}^{\infty} E_k \neq \emptyset$ and $M_k \rightarrow \infty$, as $k \rightarrow \infty$.
2. For all $k \geq 1$, we have

$$M_k \geq \left\{ \prod_{i=1}^k (1 - \sigma_i) \right\} \left\{ \prod_{i=1}^k p_{q_i} N_{q_i} \right\} \left\{ \prod_{i=1}^{k-1} H_{q_i} \right\}, \tag{75}$$

where $\prod_{i \in \emptyset} (\cdot) = 1$.

Proof. First, we obtain from (67) that $M_1 = \mathcal{N}_{q_1} \geq (1 - \sigma_1) p_{q_1} N_{q_1}$. Next, by definition of M_k , we have for all $k \geq 2$,

$$M_k = \sum_{\{i: 1 \leq i \leq M_{k-1}\}} \mathcal{N}_{q_k}(J_{i,k-1}).$$

Since $|J_{i,k-1}| = H_{q_{k-1}}$, we deduce from the choice of q_k (see (70) and (71)) that $\mathcal{N}_{q_k}(J_{i,k-1}) \geq (1 - \sigma_k) p_{q_k} N_{q_k} |J_{i,k-1}|$ which implies

$$M_k \geq (1 - \sigma_k) p_{q_k} N_{q_k} H_{q_{k-1}} M_{k-1}. \tag{76}$$

By the choice of q_1 and the inequality about M_1 , we see that $M_1 \geq 1$. Assume that $M_\ell \geq 1$ for $\ell = 1, \dots, k - 1$. We will show that this inequality is fulfilled for $\ell = k$. By induction hypothesis and by (76), it is easy to see that $M_k \geq (1 - \sigma_k) p_{q_k} N_{q_k} H_{q_{k-1}}$. By combining this last inequality with (76), we get that $M_k \geq H_{q_{k-1}}^{-\varepsilon} \geq 1$. The first half part of Lemma 14 is a direct consequence of this above inequality. From (76) we get by induction that for all $k \geq 2$,

$$M_k \geq \left\{ \prod_{i=2}^k (1 - \sigma_i) \right\} \left\{ \prod_{i=2}^k p_{q_i} N_{q_i} \right\} \left\{ \prod_{i=1}^{k-1} H_{q_i} \right\} M_1. \tag{77}$$

Combining the first inequality about M_1 with (77), we readily obtain (75). \square

We next find an upper bound for $M_k(I)$. Set $\Delta := \theta_{q_1}$. Let $I = [a, b] \subseteq [0, 1]$ satisfy $|I| \leq \Delta$. Clearly, there exists a $k_0 \equiv k_0(I) \in \mathbb{N}^*$ such that $\theta_{q_{k_0+1}} < |I| \leq \theta_{q_{k_0}}$. Form the closed interval $I^* := [\alpha_1 \theta_{q_{k_0+1}}, \alpha_2 \theta_{q_{k_0+1}}]$ where α_1 and α_2 are integers defined by

$$\alpha_1 := \max\{i \in \mathbb{N}: i \theta_{q_{k_0+1}} \leq a\} \quad \text{and} \quad \alpha_2 := \min\{i \in \mathbb{N}: i \theta_{q_{k_0+1}} \geq b\}.$$

By construction of I^* , it is clear that $I \subseteq I^*$ and $|I^*| \leq 3|I|$.

Lemma 15. For all $j \geq 2$, we have

$$M_{k_0+j}(I^*) \leq \left\{ \prod_{i=2}^j (1 + \sigma_{k_0+i}) \right\} \left\{ \prod_{i=2}^j p_{q_{k_0+i}} N_{q_{k_0+i}} \right\} \left\{ \prod_{i=1}^{j-1} H_{q_{k_0+i}} \right\} M_{k_0+1}(I^*). \tag{78}$$

Proof. First, we study the case $j = 2$. By definition of I^* , we have

$$M_{k_0+2}(I^*) = \sum_{\substack{i: 1 \leq i \leq M_{k_0+1} \\ J_{i,k_0+1} \subseteq I^*}} \mathcal{N}_{q_{k_0+2}}(J_{i,k_0+1}).$$

Since $|J_{i,k_0+1}| = H_{q_{k_0+1}}$, we deduce from the choice of q_{k_0+2} (see (70) and (71)) that

$$M_{k_0+2}(I^*) \leq (1 + \sigma_{k_0+2}) p_{q_{k_0+2}} N_{q_{k_0+2}} H_{q_{k_0+1}} M_{k_0+1}(I^*),$$

which allows us to deduce that (78) is fulfilled for $j = 2$. Assume (78) is fulfilled up to $j - 1$. By the induction hypothesis, we have

$$M_{k_0+j-1}(I^*) \leq \left\{ \prod_{i=2}^{j-1} (1 + \sigma_{k_0+i}) \right\} \left\{ \prod_{i=2}^{j-1} p_{q_{k_0+i}} N_{q_{k_0+i}} \right\} \left\{ \prod_{i=1}^{j-2} H_{q_{k_0+i}} \right\} M_{k_0+1}(I^*). \tag{79}$$

Further, we have

$$M_{k_0+j}(I^*) = \sum_{\substack{i_1: 1 \leq i_1 \leq M_{k_0+1} \\ J_{i_1, k_0+1} \subseteq I^*}} \cdots \sum_{\substack{i_{j-1}: 1 \leq i_{j-1} \leq M_{k_0+j-1} \\ J_{i_{j-1}, k_0+j-1} \subseteq J_{i_{j-2}, k_0+j-2}}} \mathcal{N}_{q_{k_0+j}}(J_{i_{j-1}, k_0+j-1}). \tag{80}$$

Since $|J_{i_{j-1}, k_0+j-1}| = H_{q_{k_0+j-1}}$, we deduce from the choice of q_{k_0+j} (see (70) and (71)) that

$$M_{k_0+j}(I^*) \leq (1 + \sigma_{k_0+j}) p_{q_{k_0+j}} N_{q_{k_0+j}} H_{q_{k_0+j-1}} M_{k_0+j-1}(I^*).$$

Finally, making use of (79) and (80), we readily obtain (78), which completes the proof. \square

By construction of I^* , we have $M_{k_0+j}(I) \leq M_{k_0+j}(I^*)$ for all $j \geq 2$. Consequently, by forming the ratio of $M_{k_0+j}(I^*)$ by M_{k_0+j} , we obtain from (78) and (75) that

$$\frac{M_{k_0+j}(I)}{M_{k_0+j}} \leq \frac{M_{k_0+j}(I^*)}{M_{k_0+j}} \leq D \frac{M_{k_0+1}(I^*)}{\left\{ \prod_{i=1}^{k_0+1} p_{q_i} N_{q_i} \right\} \left\{ \prod_{i=1}^{k_0} H_{q_i} \right\}}, \quad \text{for all } j \geq 2, \tag{81}$$

where $0 < D := \prod_{i=1}^{\infty} ((1 + \sigma_i)/(1 - \sigma_i)) < +\infty$.

In order to study $M_{k_0+1}(I^*)$ we will consider two cases.

Case 1. Assume that $\theta_{q_{k_0+1}} < |I| \leq \phi_\varepsilon^{-1}(H_{q_{k_0}})$ (this is possible by (73)). By definition of I^* and by (66), we have $0 < |I^*| \leq 3|I| \leq 3\theta_{q_{k_0}} \leq 3\theta_{q_1} \leq \tau_\varepsilon$. Thus, combining (66) with (68), we get $M_{k_0+1}(I^*) \leq \mathcal{N}_{q_{k_0+1}}(I^*) \leq 4\Phi_{\varepsilon/2}(|I^*|) p_{q_{k_0+1}} N_{q_{k_0+1}}$. This inequality, (69) and (B3)(ii) in combination with (74), yields for all $j \geq 2$,

$$\begin{aligned} \frac{M_{k_0+j}(I)}{M_{k_0+j} \Phi_\varepsilon(|I|)} &\leq 4D \frac{\Phi_{\varepsilon/2}(3|I|)}{\Phi_\varepsilon(|I|)} \times \frac{p_{q_{k_0}}^{-1} \theta_{q_{k_0}} H_{q_{k_0}}^{-1}}{\left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}} \\ &\leq 4CD \phi_\varepsilon(\phi_\varepsilon^{-1}(H_{q_{k_0}})) \times \frac{H_{q_{k_0}}^{-1} \Phi_\xi(q_{k_0})(\theta_{q_{k_0}})}{\left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}} \\ &\leq 4CD \Phi_{\varepsilon/2}(\theta_{q_{k_0}}) \left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}^{-1} \leq CD. \end{aligned} \tag{82}$$

Case 2. Assume that $\phi_\varepsilon^{-1}(H_{q_{k_0}}) < |I| \leq \theta_{q_{k_0}}$. By construction of I^* , we have $I \subseteq I^*$, which entails $|I^*| \geq \phi_\varepsilon^{-1}(H_{q_{k_0}})$. Consequently, by combining (70) with (71), we get $M_{k_0+1}(I^*) \leq \mathcal{N}_{q_{k_0+1}}(I^*) \leq (4/3) p_{q_{k_0+1}} N_{q_{k_0+1}} |I^*|$. This inequality in combination with (B3)(i)–(ii), (H'3), (69) and (74), yields for all $j \geq 2$,

$$\begin{aligned} \frac{M_{k_0+j}(I)}{M_{k_0+j} \Phi_\varepsilon(|I|)} &\leq 4D \frac{|I|}{\Phi_\varepsilon(|I|)} \times \frac{p_{q_{k_0}}^{-1} \theta_{q_{k_0}} H_{q_{k_0}}^{-1}}{\left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}} \\ &\leq 4CD \theta_{q_{k_0}} H_{q_{k_0}}^{-1} \frac{\Phi_\xi(q_{k_0})(\theta_{q_{k_0}})}{\Phi_\varepsilon(\theta_{q_{k_0}})} \left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}^{-1} \\ &\leq 4CD \Gamma(\theta_{q_{k_0}}) \phi_\varepsilon(\theta_{q_{k_0}}) \left\{ \prod_{i=1}^{k_0-1} p_{q_i} N_{q_i} H_{q_i} \right\}^{-1} \leq CD. \end{aligned} \tag{83}$$

We conclude from Case 1 and Case 2, that for any $I \subseteq [0, 1]$ such that $|I| \leq \Delta$, there exists an $m_0(I) \in \mathbb{N}$ such that $M_m(I) \leq CD \Phi_\varepsilon(|I|) M_m$, for all $m \geq m_0(I)$. Consequently (36) is satisfied. This, in turn, by Lemma 6, completes the proof of Theorem 4.

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