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Dilation of a class of quantum dynamical semigroups with unbounded generators on UHF algebras

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Dedicated to the memory of Professor Paul-André Meyer

Abstract

Evans–Hudson flows are constructed for a class of quantum dynamical semigroups with unbounded generator on UHF algebras, which appeared in [Rev. Math. Phys. 5 (3) (1993) 587–600]. It is shown that these flows are unital and covariant. Ergodicity of the flows for the semigroups associated with partial states is also discussed.

Résumé

Les flots d'Evans–Hudson sont construits pour une classe de semi-groupes dynamiques quantiques à générateur non borné sur une algèbre UHF, définie dans la référence [Rev. Math. Phys. 5 (3) (1993) 587–600]. On montre que ces flots préservent l'unité et sont covariants. L'ergodicité des flots associés à des états partiels est également discutée.

1. Introduction

Quantum dynamical semigroups, to be abbreviated as QDS, constitute a natural generalization of classical Markov semigroups arising as expectation semigroups of Markov processes. A QDS $\{T_t\colon t \geqslant 0\}$ on a C^*-algebra $\mathcal{A}$ is a C_0-semigroup of completely positive maps T_t on $\mathcal{A}$. Given such a QDS, it is interesting and important to look for a dilation in the sense of Evans–Hudson, i.e. a family of $*$-homomorphisms $\eta_t : \mathcal{A} \to A'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0)))$

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where $\mathbf{k}_0$ is some separable Hilbert space and $\Gamma(\cdot)$ denotes the symmetric Fock space, satisfying a suitable quantum stochastic differential equation. This problem has been completely solved for QDS with bounded generators by Goswami, Sinha and Pal [2,4], where a canonical Evans–Hudson flow for an arbitrary QDS with bounded generator has been constructed. However, only partial success has been achieved for QDS with unbounded generator. It is perhaps too much to expect a complete general theory for an arbitrary QDS. It may be wiser to look for Evans–Hudson flow for special classes of QDS. In [3] for example, the authors gave a general theory of dilation for QDS on a C^*-algebra $\mathcal{A}$, which is covariant with respect to an action of a Lie group and also symmetric with respect to a given faithful semifinite trace. However, in the present article, we shall try to construct an Evans–Hudson flow for another class of QDS on a UHF C^*-algebra, studied by T. Matsui in [6]. This construction has some similarity with the earlier one, but the action of the discrete group $\mathbb{Z}^d$ instead of a Lie group action as in [3] makes the present model somewhat different from that of [3]. We have not only proved the existence of a dilation in Section 3, we are also able to prove in Section 4 that the Evans–Hudson (EH) flow is indeed covariant with respect to the $\mathbb{Z}^d$ action. Some ergodicity properties of the flows are also discussed briefly in Section 5.

2. Notation and preliminaries

T. Matsui [6] constructed a class of conservative QDS on the UHF C^*-algebra $\mathcal{A}$ generated as the C^*-completion of infinite tensor product $\bigotimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, where N and d are two fixed positive integers. This C^*-algebra can also be described as the inductive limit of full matrix algebras $\{M_{N^n}(\mathbb{C}),\ n\geqslant 1\}$ with respect to the imbedding $M_{N^n}\subseteq M_{N^{n+1}}$ by sending a to $a\otimes 1$. The unique normalized trace tr on $\mathcal{A}$ is given by $\operatorname{tr}(x)=\frac{1}{N^n}\operatorname{Tr}(x)$, for $x\in M_{N^n}(\mathbb{C})$, where Tr denotes the ordinary trace on $M_{N^n}(\mathbb{C})$. For $x\in M_N(\mathbb{C})$ and $j\in\mathbb{Z}^d$, let $x^{(j)}$ denote an element in $\mathcal{A}$ whose jth component is x and rest are identity of $M_N(\mathbb{C})$. For a simple tensor element $a\in\mathcal{A}$, let $a_{(j)}$ be the jth component of a. The support of a, denoted by supp(a) is defined to be the set $\{j\in\mathbb{Z}^d\colon\ a_{(j)}\neq 1\}$. For a general element $a\in\mathcal{A}$ such that $a=\sum_{n=1}^\infty c_n a_n$ with a_n's simple tensor elements in $\mathcal{A}$ and c_n's complex coefficients, we define $\operatorname{supp}(a):=\bigcup_{n\in\mathbb{N}}\operatorname{supp}(a_n)$ and we set $|a|=$ cardinality of supp(a). For any $\Lambda\subseteq\mathbb{Z}^d$, let $\mathcal{A}_\Lambda$ denote the $*$-subalgebra generated by elements of $\mathcal{A}$ with support Λ. When $\Lambda=\{k\}$, we write $\mathcal{A}_k$ instead of $\mathcal{A}_{\{k\}}$. Let $\mathcal{A}_{\mathrm{loc}}$ be the $*$-subalgebra of $\mathcal{A}$ generated by elements $a\in\mathcal{A}$ of finite support or equivalently by $\{x^{(j)}\colon\ x\in M_N(\mathbb{C}),\ j\in\mathbb{Z}^d\}$. Clearly $\mathcal{A}_{\mathrm{loc}}$ is dense in $\mathcal{A}$. For $k\in\mathbb{Z}^d$, the translation τ_k on $\mathcal{A}$ is an automorphism determined by $\tau_k(x^{(j)}):=x^{(j+k)}\ \forall x\in M_N(\mathbb{C})$ and $j\in\mathbb{Z}^d$. Thus, we get an action τ of the infinite discrete group $\mathbb{Z}^d$ on $\mathcal{A}$. For $x\in\mathcal{A}$ we denote $\tau_k(x)$ by x_k. The algebra $\mathcal{A}$ is naturally sitting inside $\mathbf{h}_0=L^2(\mathcal{A},\operatorname{tr})$, the GNS Hilbert space for $(\mathcal{A},\operatorname{tr})$. It is easy to see that τ_k extends to a unitary on $\mathbf{h}_0$, to be denoted by the same symbol τ_k, giving rise to a unitary representation τ of the group $\mathbb{Z}^d$ on $\mathbf{h}_0$, which implements the action τ. It is also clear that this action extends as an action of $\mathbb{Z}^d$ by normal automorphisms on the von Neumann algebra $\mathcal{A}''$.

We also need another dense subset of $\mathcal{A}$, which is in a sense like the first Sobolev space in $\mathcal{A}$. For this, we need to note that $M_N(\mathbb{C})$ is spanned by a pair of noncommutative representatives $\{U,V\}$ of $\mathbb{Z}_N=\{0,1,\ldots,N-1\}$ such that $U^N=V^N=1\in M_N(\mathbb{C})$ and $UV=wVU$, where $w\in\mathbb{C}$ is the primitive Nth root of unity. These U,V can be chosen to be the $N\times N$ circulant matrices. In particular for $N=2$, a possible choice is given by $U=\sigma_x$ and $V=\sigma_z$, where σ_x and σ_z denote the Pauli-spin matrices. For $j\in\mathbb{Z}^d$ and $(\alpha,\beta)\in G\equiv\mathbb{Z}_N\times\mathbb{Z}_N$, we set $\sigma_{j;\alpha,\beta}(x)=[U^{(j)^\alpha}V^{(j)^\beta},x]\ \forall x\in\mathcal{A}$, $\|x\|_1=\sum_{j;\alpha,\beta}\|\sigma_{j;\alpha,\beta}(x)\|$ and $\mathcal{C}^1(\mathcal{A})=\{x\in\mathcal{A}\colon\ \|x\|_1<\infty\}$. It is easy to see that $\|x^*\|_1=\|\tau_j(x)\|_1=\|x\|_1$ and since $\mathcal{C}^1(\mathcal{A})$ contains the dense $*$-subalgebra $\mathcal{A}_{\mathrm{loc}}$, $\mathcal{C}^1(\mathcal{A})$ is a dense τ invariant $*$-subalgebra of $\mathcal{A}$. Let $\mathcal{G}:=\prod_{j\in\mathbb{Z}^d}G$ be the infinite direct product of the finite group G at each lattice site. Thus each $g\in\mathcal{G}$ has jth component $g_{(j)}=(\alpha_j,\beta_j)$ with $\alpha_j,\beta_j\in\mathbb{Z}_N$. For $g\in\mathcal{G}$ we define its support by $\operatorname{supp}(g)=\{j\in\mathbb{Z}^d\colon\ g_{(j)}\neq(0,0)\}$ and $|g|=$ cardinality of supp(g). Let us consider the projective unitary representation of $\mathcal{G}$, given by $\mathcal{G}\ni g\mapsto U_g=\prod_{j\in\mathbb{Z}^d}U^{(j)^{\alpha_j}}V^{(j)^{\beta_j}}\in\mathcal{A}$. For a given completely positive map T on $\mathcal{A}$, we formally define the Linbladian

$$\mathcal{L} = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k,$$
$$\text{where } \mathcal{L}_k x = \tau_k \mathcal{L}_0(\tau_{-k}x),\ \forall x \in \mathcal{A},$$
$$\text{with } \mathcal{L}_0(x) = -\frac{1}{2}\{T(1), x\} + T(x), \tag{2.1}$$

and $\{A, B\} := AB + BA$.

In particular we consider the Lindbladian $\mathcal{L}$ for the completely positive map T,

$$Tx := \sum_{n=0}^{\infty} a_n^* x a_n, \quad \forall x \in \mathcal{A},$$

associated with a sequence of elements $\{a_n\}_{n\geqslant 0}$ in $\mathcal{A}$, with $a_n = \sum_{g\in\mathcal{G}} c_{n,g} U_g$ such that $\sum_{n=0}^{\infty} \sum_{g\in\mathcal{G}} |c_{n,g}||g|^2 < \infty$. Matsui has proven the following in the paper referred earlier [6].

Theorem 2.1. (i) *The map $\mathcal{L}$ formally define above is well defined on $\mathcal{C}^1(\mathcal{A})$ and the closure $\hat{\mathcal{L}}$ of $\mathcal{L}/_{\mathcal{C}^1(\mathcal{A})}$ is the generator of a conservative QDS $\{P_t\colon t \geqslant 0\}$ on $\mathcal{A}$,*

(ii) *The semigroup $\{P_t\}$ leaves $\mathcal{C}^1(\mathcal{A})$ invariant.*

The semigroup P_t satisfies

$$P_t(x) = x + \int_0^t P_s\big(\hat{\mathcal{L}}(x)\big)\,\mathrm{d}s, \quad \forall x \in \mathrm{Dom}(\hat{\mathcal{L}}).$$

Since $1 \in \mathcal{C}^1(\mathcal{A})$ and $\hat{\mathcal{L}}(1) = \mathcal{L}(1) = 0$, it follows that $P_t(1) = 1, \forall t \geqslant 0$.

Following [6], we say that P_t is ergodic if there exists an invariant state ψ satisfying

$$\big\| P_t(x) - \psi(x)1 \big\| \to 0 \quad \text{as } t \to \infty,\ \forall x \in \mathcal{A}. \tag{2.2}$$

In [6], the author has discussed some criteria for ergodicity of the QDS P_t. Some examples of such semigroups associated with partial states on the UHF algebra and their perturbation are given.

For a state ϕ on $M_N(\mathbb{C})$ and $k \in \mathbb{Z}^d$, the partial state ϕ_k on $\mathcal{A}$ is determined by $\phi_k(x) = \phi(x_{(k)})x_{\{k\}^c}$, for $x = x_{(k)}x_{\{k\}^c}$, where $x_{(k)} \in \mathcal{A}_k$ and $x_{\{k\}^c} \in \mathcal{A}_{\{k\}^c}$. We can find a natural number N' and elements $\{L^{(m)}\colon m = 1, 2, \ldots, N'\}$ in $M_N(\mathbb{C})$ such that

$$\phi(x) = \sum_{m=1}^{N'} L^{(m)*} x L^{(m)} \quad \forall x \in M_N(\mathbb{C}) \quad \text{and} \quad \sum_{m=1}^{N'} L^{(m)*} L^{(m)} = 1.$$

For $m = 1, \ldots, N'$, let us consider the element $L_0^{(m)} \in \mathcal{A}_0$ with the zeroth component being $L^{(m)}$. Now for $k \in \mathbb{Z}^d$ and $m = 1, \ldots, N'$, writing $L_k^{(m)} = \tau_k(L_0^{(m)})$, the partial state ϕ_k is given by,

$$\phi_k(x) = \sum_{m=1}^{N'} L_k^{(m)*} x L_k^{(m)} \quad \forall x \in \mathcal{A}.$$

By (2.1), the Linbladian $\mathcal{L}^\phi$ corresponding to the partial state ϕ_0 is formally given by

$$\mathcal{L}^\phi(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{L}_k^\phi(x),$$

where

$$\mathcal{L}_k^{\phi}(x)=\phi_k(x)-x=\frac{1}{2}\sum_{m=1}^{N'}\big[L_k^{(m)^*},x\big]L_k^{(m)}+L_k^{(m)^*}\big[x,L_k^{(m)}\big].$$

It follows from Theorem 2.1 that $\mathcal{L}^{\phi}$ is defined on $\mathcal{C}^1(\mathcal{A})$. Moreover, the closure $\hat{\mathcal{L}}^{\phi}$ of $\mathcal{L}^{\phi}/\mathcal{C}^1(\mathcal{A})$ generates a conservative QDS P_t^{ϕ} on $\mathcal{A}$ given by

$$P_t^{\phi}\Big(\prod_{k\in\Lambda}x_{(k)}\Big)=\prod_{k\in\Lambda}\big\{\phi(x_{(k)})+\mathrm{e}^{-t}\big(x_{(k)}-\phi(x_{(k)})\big)\big\}.$$

We note that the map Φ defined by,

$$\Phi\Big(\prod_{k\in\Lambda}x_{(k)}\Big)=\lim_{t\to\infty}P_t^{\phi}\Big(\prod_{k\in\Lambda}x_{(k)}\Big)=\prod_{k\in\Lambda}\phi(x_{(k)})$$

extends as a state on $\mathcal{A}$ which is the unique invariant state for the ergodic QDS P_t^{ϕ}. For any real number c, we consider the perturbation

$$\mathcal{L}^{(c)}(x)=\mathcal{L}^{\phi}(x)+c\mathcal{L}(x),\quad \forall x\in\mathcal{C}^1(\mathcal{A}).$$

It is clear that $\mathcal{L}^{(c)}$ is the Linbladian associated with the completely positive map

$$T(x)=\sum_{m=1}^{N'}L_k^{(m)^*}xL_k^{(m)}+c\sum_{l=0}^{\infty}a_l^*xa_l,\quad \forall x\in\mathcal{A}$$

and by Theorem 2.1 it follows that the closure $\hat{\mathcal{L}}^{(c)}$ of $\mathcal{L}^{(c)}/\mathcal{C}^1(\mathcal{A})$ generate a QDS $P_t^{(c)}$. Moreover, one has

Theorem 2.2 [6]. *There exists a constant c_0 such that for $0\leqslant c\leqslant c_0$ the above QDS $P_t^{(c)}$ is ergodic with the invariant state $\Phi^{(c)}$ satisfying*

$$\begin{aligned}&\big\|P_t^{(c)}(x)\big\|_1\leqslant 2\mathrm{e}^{-(1-c/c_0)t}\|x\|_1,\\ &\big\|P_t^{(c)}(x)-\Phi^{(c)}(x)1\big\|\leqslant\frac{4}{N^2}\mathrm{e}^{-(1-c/c_0)t}\|x\|_1,\quad \forall x\in\mathcal{C}^1(\mathcal{A}).\end{aligned}\tag{2.3}$$

Remark 2.3. The invariant state $\Phi^{(c)}$ corresponding to the ergodic QDS $P_t^{(c)}$ is given by

$$\Phi^{(c)}(x)=\Phi(x)+c\int_0^{\infty}\Phi\big(\mathcal{L}\big(P_t^{(c)}(x)\big)\big)\,\mathrm{d}t,\quad \forall x\in\mathcal{C}^1(\mathcal{A}).$$

Let us conclude the present section with a brief discussion on the fundamental integrator processes of quantum stochastic calculus, introduced by Hudson and Parthasarathy [5]. Let $\mathbf{k}=L^2(\mathbb{R}_+,\mathbf{k}_0)$ where $\mathbf{k}_0=l^2(\mathbb{Z}^d)$ with the canonical orthonormal basis $\{e_j\colon j\in\mathbb{Z}^d\}$ and $\Gamma=\Gamma_{\mathrm{sym}}(\mathbf{k})$, the symmetric Fock space over $\mathbf{k}$. For $f\in\mathbf{k}$, we denote by $\mathbf{e}(f)$ the exponential vector in Γ associated with f:

$$\mathbf{e}(f)=\bigoplus_{n\geqslant 0}\frac{1}{\sqrt{n!}}f^{(n)},$$

where $f^{(n)}=\underbrace{f\otimes f\otimes\cdots\otimes f}_{n\text{-copies}}$ for $n>0$ and by convention $f^{(0)}=1$. For $f=0$, $\mathbf{e}(f)$ is called the vacuum vector in Γ. Let $\mathcal{C}$ be the space of all bounded continuous functions from $\mathbb{R}_+$ to $\mathbf{k}_0$, so that $\mathcal{E}(\mathcal{C})\equiv\{\mathbf{e}(f)\colon f\in\mathcal{C}\}$ is total in $\Gamma(\mathbf{k})$. Any $f \in L^2(\mathbb{R}_+, \mathbf{k}_0)$ decomposes as $f = \sum_{k\in\mathbb{Z}^d} f_k e_k$ with $f_k \in L^2(\mathbb{R}_+)$. We take the freedom to use the same symbol f_k to denote the function in $L^2(\mathbb{R}_+, \mathbf{k}_0)$ as well, whenever it is clear from the context. The fundamental processes $\{\Lambda_i^j\colon i, j \in \mathbb{Z}^d\}$ associated with the orthonormal basis $\{e_j\colon j \in \mathbb{Z}^d\}$ are given by

$$\begin{aligned}
\Lambda_j^i(t) &= a_{\chi_{[0,t]}\otimes e_i} \quad \text{for } i \neq 0,\ j = 0\\
&= a^\dagger_{\chi_{[0,t]}\otimes e_j} \quad \text{for } i = 0,\ j \neq 0\\
&= \Lambda_{M_{\chi_{[0,t]}}\otimes |e_j\rangle\langle e_i|} \quad \text{for } i, j \neq 0\\
&= t1 \quad \text{for } i = j = 0,
\end{aligned}$$

where $M_{\chi_{[0,t]}}$ is the multiplication operator on $L^2(\mathbb{R}_+)$ by characteristic function of the interval $[0,t]$. For details the reader is referred to [10] and [7].

3. Evans–Hudson type dilation

In this section we investigate the possibility of constructing EH flows for the QDS on UHF C^*-algebra, discussed in the previous section. Although the question is not answered in full generality, EH flows for a class of QDS are constructed.

Let $r = \sum_{g\in\mathcal{G}} c_g U_g \in \mathcal{A}$ such that $\sum_{g\in\mathcal{G}} |c_g||g|^2 < \infty$. The Lindbladian $\mathcal{L}$ associated with the element r, i.e. associated with the CP map T, $T(x) = r^* x r$, $\forall x \in \mathcal{A}$, takes the form

$$\mathcal{L}(x) = \sum_{k\in\mathbb{Z}^d} \delta_k^\dagger(x) r_k + r_k^* \delta_k(x), \tag{3.1}$$

where $r_k := \tau_k(r)$ and $\delta_k, \delta_k^\dagger$ are bounded derivation on $\mathcal{A}$ given by

$$\delta_k(x) = [x, r_k] \quad \text{and} \quad \delta_k^\dagger(x) := \big(\delta_k(x^*)\big)^* = [r_k^*, x], \quad \forall x \in \mathcal{A}. \tag{3.2}$$

It follows from [6] that the closure $\hat{\mathcal{L}}$ of $\mathcal{L}/\mathcal{C}^1(\mathcal{A})$ is the generator of a contractive QDS P_t on $\mathcal{A}$. In order to construct an EH flow for the QDS P_t, we would like to solve the following QSDE in $\mathcal{B}(L^2(\mathcal{A}, \mathrm{tr})) \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0)))$:

$$\begin{aligned}
\mathrm{d}j_t(x) &= \sum_{j\in\mathbb{Z}^d} j_t\big(\delta_j^\dagger(x)\big)\,\mathrm{d}a_j(t) + \sum_{j\in\mathbb{Z}^d} j_t\big(\delta_j(x)\big)\,\mathrm{d}a_j^\dagger(t) + j_t\big(\hat{\mathcal{L}}(x)\big)\,\mathrm{d}t,\\
j_0(x) &= x \otimes 1_\Gamma, \quad x \in \mathcal{A}_{\mathrm{loc}}.
\end{aligned} \tag{3.3}$$

Let us first look at the corresponding Hudson–Parthasarathy equation in $L^2(\mathcal{A}, \mathrm{tr}) \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0))$, given by

$$\begin{aligned}
\mathrm{d}U_t &= \Big\{\sum_{j\in\mathbb{Z}^d}\big[r_j^*\,\mathrm{d}a_j(t) - r_j\,\mathrm{d}a_j^\dagger(t)\big] - \frac{1}{2}\sum_{j\in\mathbb{Z}^d} r_j^* r_j\,\mathrm{d}t\Big\} U_t,\\
U_0(x) &= 1_{L^2\otimes\Gamma}.
\end{aligned} \tag{3.4}$$

However, though each $r_j \in \mathcal{A}$ and hence is in $\mathcal{B}(L^2(\mathcal{A}, \mathrm{tr}))$, Eq. (3.4) does not in general admit a solution since

$$\Big\langle u, \sum_{j\in\mathbb{Z}^d} r_j^* r_j u\Big\rangle = \sum_{j\in\mathbb{Z}^d} \|r_j u\|^2 \quad \forall u \in L^2(\mathcal{A}, \mathrm{tr}),$$

is not convergent in general and hence $\sum_{j\in\mathbb{Z}^d} r_j \otimes e_j$ does not define an element in $\mathcal{A}\otimes\mathbf{k}_0$. For example, let r be the single-supported unitary element $U^{(k)}\in\mathcal{A}$ for some $k\in\mathbb{Z}^d$ so that $r_j = U^{(k+j)}$ is a unitary for each $j\in\mathbb{Z}^d$ and hence

$$\sum_{j\in\mathbb{Z}^d} \|r_j u\|^2 = \sum_{j\in\mathbb{Z}^d} \|u\|^2 = \infty.$$

However, as we shall see, in many situation there exist Evans–Hudson flows, even though the corresponding Hudson–Parthasarathy equation (3.4) does not admit a solution.

Remark 3.1. There are some cases when an Evans–Hudson flow can be seen to be implemented by a solution of a Hudson–Parthasarathy equation. For example, given a self adjoint $r\in\mathcal{A}$

$$\mathrm{d}V_t = V_t\left\{\sum_{k\in\mathbb{Z}^d}\left(S_k^*\,\mathrm{d}a_k(t) - S_k\,\mathrm{d}a_k^\dagger(t)\right) - \frac{1}{2}\sum_{k\in\mathbb{Z}^d} S_k^* S_k\,\mathrm{d}t\right\},\quad V_0 = 1,$$

where S_k is defined by $S_k(x) = [r_k, x]$ for $x\in\mathcal{A}\subseteq L^2(\mathcal{A},\mathrm{tr})$, admits a unique unitary solution and

$$x\mapsto V_t(x\otimes 1)V_t^*$$

gives an Evans–Hudson dilation for P_t [8,9].

Let $a, b\in\mathbb{Z}_N$ be fixed and $W = U^a V^b\in\mathcal{M}_N(\mathbb{C})$. We consider the following representation of the infinite product group $\mathcal{G}' := \prod_{j\in\mathbb{Z}^d}\mathbb{Z}_N$, given by

$$\mathcal{G}'\ni g\mapsto W_g = \prod_{j\in\mathbb{Z}^d} W^{(j)^{\alpha_j}},\quad \text{where } g = (\alpha_j).$$

For any $y\in\mathcal{A}$, $y = \sum_{g\in\mathcal{G}} c_g U_g$ and for $n\geqslant 1$ we define

$$\vartheta_n(y) = \sum_{g\in\mathcal{G}} |c_g||g|^n.$$

Now we consider $r\in\mathcal{A},\ r = \sum_{g\in\mathcal{G}'} c_g W_g$ such that $\sum_{g\in\mathcal{G}'}|c_g||g|^2 < \infty$. It is clear that $\vartheta_1(r) = \sum_{g\in\mathcal{G}'}|c_g||g| < \infty$. We note that any $x\in\mathcal{A}_{\mathrm{loc}}$ can be written as $x = \sum_{h\in\mathcal{G}} c_h U_h$, with complex coefficients c_h satisfying $c_h = 0$ for all h such that $\mathrm{supp}(h)\cap\mathrm{supp}(x)$ is empty. So

$$\vartheta_n(x) = \sum_{h\in\mathcal{G}} |c_h||h|^n < \infty\quad \text{for } n\geqslant 1,$$

and it is clear that

$$\vartheta_n(x)\leqslant |x|^n\sum_{h\in\mathcal{G}}|c_h|\leqslant c_x^n$$

where $c_x = |x|(1+\sum_{h\in\mathcal{G}}|c_h|)$. Let us consider the formal Lindbladian $\mathcal{L}$ associated with the element r,

$$\mathcal{L} = \sum_{k\in\mathbb{Z}^d}\mathcal{L}_k,$$

where $\mathcal{L}_k(x) = \frac{1}{2}\delta_k^\dagger(x)r_k + r_k^*\delta_k(x)$.

For $n\geqslant 1$, we denote the set of integers $\{1,2,\ldots,n\}$ by I_n and for $1\leqslant p\leqslant n,\ P = \{l_1, l_2,\ldots,l_p\}\subseteq I_n$ with $l_1 < l_2 < \cdots < l_p$, we define a map from the n-fold Cartesian product of $\mathbb{Z}^d$ to that of p copies of $\mathbb{Z}^d$ by

$$\bar{k}(I_n) = (k_1, k_2,\ldots,k_n)\mapsto \bar{k}(P) := (k_{l_1}, k_{l_2},\ldots,k_{l_p})$$

and similarly, $\bar{\varepsilon}(P) := (\varepsilon_{l_1}, \varepsilon_{l_2}, , \ldots, \varepsilon_{l_p})$ for a vector $\bar{\varepsilon}(I_n) = (\varepsilon_1, \varepsilon_2, , \ldots, \varepsilon_n)$ in the n-fold Cartesian product of $\{-1, 0, 1\}$.

For brevity of notations, we write $\bar{\varepsilon}(P) \equiv c \in \{-1, 0, 1\}$ to mean that all $\varepsilon_{l_i} = c$ and denote $\bar{k}(I_n)$ and $\bar{\varepsilon}(I_n)$ by $\bar{k}(n)$ and $\bar{\varepsilon}(n)$ respectively. Setting $\delta_k^\varepsilon = \delta_k^\dagger, \mathcal{L}_k$ and δ_k depending upon $\varepsilon = -1, 0$ and 1 respectively, we write $R(\bar{k}) = r_{k_1} r_{k_2} \cdots r_{k_p}$ and $\delta(\bar{k}, \bar{\varepsilon}) = \delta_{k_p}^{\varepsilon_p} \cdots \delta_{k_1}^{\varepsilon_1}$ for any $\bar{k} = (k_1, k_2, \ldots, k_p)$ and $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$. Now we have the following useful lemma,

Lemma 3.2. *Let r, x and constant c_x be as above. Then*

(i) *For any $n \geqslant 1$,*

$$\sum_{\bar{k}(n)} \big\| \delta\big(\bar{k}(n), \bar{\varepsilon}(n)\big)(x) \big\| \leqslant \big(2\vartheta_1(r) c_x\big)^n \quad \forall x \in \mathcal{A}_{\mathrm{loc}},$$

where $\bar{\varepsilon}(n)$ is such that $\varepsilon_l \neq 0, \ \forall l \in I_n$.

(ii) *For any $n \geqslant 1$ and $\bar{k}(n)$,*

$$\mathcal{L}_{k_n} \cdots \mathcal{L}_{k_1}(x) = \frac{1}{2^n} \sum_{p=0,1,\ldots,n} \ \sum_{P \subseteq I_n:\ |P| = p} R\big(\bar{k}(P^c)\big)^* \delta\big(\bar{k}(n), \bar{\varepsilon}_{(P)}(n)\big)(x) R\big(\bar{k}(P)\big),$$

where $\bar{\varepsilon}_{(P)}(n)$ is such that $\bar{\varepsilon}_{(P)}(P) \equiv -1$ and $\bar{\varepsilon}_{(P)}(P^c) \equiv 1$.

(iii) *For any $n \geqslant 1$, $p \leqslant n$, $P \subseteq I_n$ and $\bar{\varepsilon}(n)$ such that $\bar{\varepsilon}(P)$ contains all those components equal to* 0*, we have,*

$$\sum_{\bar{k}(n)} \big\| \delta\big(\bar{k}(n), \bar{\varepsilon}(n)\big)(x) \big\| \leqslant \|r\|^p \big(2\vartheta_1(r) c_x\big)^n \leqslant \big(1 + \|r\|\big)^n \big(2\vartheta_1(r) c_x\big)^n.$$

(iv) *Let $m_1, m_2 \geqslant 1$; $x, y \in \mathcal{A}_{\mathrm{loc}}$ and $\bar{\varepsilon}'(m_1), \bar{\varepsilon}''(m_2)$ be two fixed tuples. Then for $n \geqslant 1$ and $\bar{\varepsilon}(n)$ as in* (iii)*, we have,*

$$\sum_{\bar{k}(n), \bar{k}'(m_1), \bar{k}''(m_2)} \big\| \delta\big(\bar{k}(n), \bar{\varepsilon}(n)\big) \big\{ \delta\big(\bar{k}'(m_1), \bar{\varepsilon}'(m_1)\big)(x) \cdot \delta\big(\bar{k}''(m_2), \bar{\varepsilon}''(m_2)\big)(y) \big\} \big\|$$
$$\leqslant 2^n \big(1 + \|r\|\big)^{2n + m_1 + m_2} \big(2\vartheta_1(r) c_{x,y}\big)^{n + m_1 + m_2},$$

where $c_{x,y} = \max\{c_x, c_y\}$.

Proof. (i) As r^* is again of the same form as r, it is enough to observe the following:

$$\sum_{k_n, \ldots, k_1} \big\| [r_{k_n}, \cdots [r_{k_1}, x]] \cdots] \big\| \leqslant \big(2\vartheta_1(r) c_x\big)^n \quad \forall x \in \mathcal{A}_{\mathrm{loc}}.$$

In order to prove this let us consider

$$LHS = \sum_{k_n, \ldots, k_1} \ \sum_{g_n, \ldots, g_1 \in \mathcal{G}';\ h \in \mathcal{G}} |c_{g_n}| \cdots |c_{g_1}| |c_h| \big\| [\tau_{k_n} W_{g_n}, \cdots [\tau_{k_1} W_{g_1}, U_h]] \cdots] \big\|.$$

We note that for any two commuting elements A, B in $\mathcal{A}$, $[A,\ [B, x]] = [B,\ [A, x]]$. Thus, for the commutator $[\tau_{k_n} W_{g_n}, \cdots [\tau_{k_1} W_{g_1}, U_h]] \cdots]$ to be nonzero, it is necessary to have $(\mathrm{supp}(g_i) + k_i) \cap \mathrm{supp}(h) \neq \phi$ for each $i = 1, 2, \ldots, n$. Clearly the number of choices of such $k_i \in \mathbb{Z}^d$ is at most $|g_i| \cdot |h|$. Thus we get,

$$\sum_{k_n, \ldots, k_1} \big\| [r_{k_n}, \cdots [r_{k_1}, x]] \cdots] \big\| \leqslant \sum_{g_n, \ldots, g_1 \in \mathcal{G}';\ h \in \mathcal{G}} |c_{g_n}| \cdots |c_{g_1}| |c_h| |g_n| \cdots |g_1| |h|^n 2^n \leqslant \big(2\vartheta_1(r) c_x\big)^n.$$

(ii) The proof is by induction. For any $k \in \mathbb{Z}^d$ we have,

$$\mathcal{L}_k(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \delta_k^\dagger(x) r_k + r_k^* \delta_k(x),$$

so it is trivially true for $n = 1$. Let us assume it to be true for some $m > 1$ and for any $k_{m+1} \in \mathbb{Z}^d$ consider $\mathcal{L}_{k_{m+1}} \mathcal{L}_{k_m} \cdots \mathcal{L}_{k_1}(x)$. By applying the statement for $n = m$ we get,

$$\mathcal{L}_{k_{m+1}} \mathcal{L}_{k_m} \cdots \mathcal{L}_{k_1}(x) = \frac{1}{2^{m+1}} \sum_{p=0,1,\ldots,m} \sum_{P \subseteq I_m\colon |P|=p} \bigl[\delta_{k_{m+1}}^* \bigl\{R\bigl(\bar{k}(P^c)\bigr)^* \delta\bigl(\bar{k}(m), \bar{\varepsilon}_{(P)}(m)\bigr)(x) R\bigl(\bar{k}(P)\bigr)\bigr\} r_{k_{m+1}}$$
$$+ r_{k_{m+1}}^* \delta_{k_{m+1}} \bigl\{R\bigl(\bar{k}(P^c)\bigr)^* \delta\bigl(\bar{k}(m), \bar{\varepsilon}_{(P)}(m)\bigr)(x) R\bigl(\bar{k}(P)\bigr)\bigr\}\bigr].$$

Since r_k's are commuting with each other, the above expression becomes

$$\frac{1}{2^{m+1}} \sum_{p=0,1,\ldots,m} \sum_{P \subseteq I_m\colon |P|=p} \bigl[R\bigl(\bar{k}(P^c)\bigr)^* \delta_{k_{m+1}}^* \delta\bigl(\bar{k}(m), \bar{\varepsilon}_{(P)}(m)\bigr)(x) R\bigl(\bar{k}(P)\bigr) r_{k_{m+1}}$$
$$+ r_{k_{m+1}}^* R\bigl(\bar{k}(P^c)\bigr)^* \delta_{k_{m+1}} \delta\bigl(\bar{k}(m), \bar{\varepsilon}_{(P)}(m)\bigr)(x) R\bigl(\bar{k}(P)\bigr)\bigr]$$
$$= \frac{1}{2^{m+1}} \sum_{p=0,1,\ldots,m+1} \sum_{P \subseteq I_{m+1}\colon |P|=p} R\bigl(\bar{k}(P^c)\bigr)^* \delta\bigl(\bar{k}(m+1), \bar{\varepsilon}_{(P)}(m+1)\bigr)(x) R\bigl(\bar{k}(P)\bigr).$$

(iii) By simple application of (ii),

$$\delta\bigl(\bar{k}(n), \bar{\varepsilon}(n)\bigr)(x) = \frac{1}{2^p} \sum_{q=0,1,\ldots,p} \sum_{Q \subseteq P\colon |Q|=q} R\bigl(\bar{k}(P \setminus Q)\bigr)^* \delta\bigl(\bar{k}(n), \bar{\varepsilon}_{(Q,P)}(n)\bigr)(x) R\bigl(\bar{k}(Q)\bigr), \tag{3.5}$$

where $\bar{\varepsilon}_{(Q,P)}(n)$ is defined to be the map from the n-fold Cartesian product of $\{-1, 0, 1\}$ to itself, given by $\bar{\varepsilon}(n) \mapsto \bar{\varepsilon}_{(Q,P)}(n)$ such that $\bar{\varepsilon}_{(Q,P)}(Q) \equiv -1$, $\bar{\varepsilon}_{(Q,P)}(P \setminus Q) \equiv 1$ and $\bar{\varepsilon}_{(Q,P)}(I_n \setminus P) = \bar{\varepsilon}(I_n \setminus P)$. Now (iii) follows from (i).

(iv) By (3.5) we have,

$$LHS = \frac{1}{2^p} \sum_{\bar{k}(n), \bar{k}'(m_1), \bar{k}''(m_2)} \sum_{q=0,1,\ldots,p} \sum_{Q \subseteq P\colon |Q|=q} \bigl\| R\bigl(\bar{k}(P \setminus Q)\bigr)^*$$
$$\times \delta\bigl(\bar{k}(n), \bar{\varepsilon}_{(Q,P)}(n)\bigr)\bigl[\delta\bigl(\bar{k}'(m_1), \bar{\varepsilon}'(m_1)\bigr)(x) \cdot \delta\bigl(\bar{k}''(m_2), \bar{\varepsilon}''(m_2)\bigr)(y)\bigr] R\bigl(\bar{k}(Q)\bigr)\bigr\|.$$

Now applying the Leibnitz rule, it can be seen to be less than or equal to

$$\frac{\|r\|^p}{2^p} \sum_{\bar{k}(n), \bar{k}'(m_1), \bar{k}''(m_2)} \sum_{q=0,1,\ldots,p} \sum_{Q \subseteq P\colon |Q|=q} \sum_{l=0,1,\ldots,n} \sum_{L \subseteq I_n\colon |L|=l} \bigl\|\delta\bigl(\bar{k}(L), \bar{\varepsilon}_{(Q,P)}(L)\bigr) \delta\bigl(\bar{k}'(m_1), \bar{\varepsilon}'(m_1)\bigr)(x)\bigr\|$$
$$\times \bigl\|\delta\bigl(\bar{k}(L^c), \bar{\varepsilon}_{(Q,P)}(L^c)\bigr)\bigl[\delta\bigl(\bar{k}''(m_2), \bar{\varepsilon}''(m_2)\bigr)(y)\bigr]\bigr\|.$$

Using (iii), we obtain,

$$LHS \leqslant \frac{(1+\|r\|)^n}{2^p} \sum_{q=0,1,\ldots,p} \frac{p!}{(p-q)!\,q!} \sum_{l=0,1,\ldots,n} \frac{n!}{(n-l)!\,l!} \bigl(1+\|r\|\bigr)^{l+m_1} \bigl(2\vartheta_1(r) c_x\bigr)^{l+m_1}$$
$$\times \bigl(1+\|r\|\bigr)^{n-l+m_2} \bigl(2\vartheta_1(r) c_y\bigr)^{n-l+m_2}$$
$$\leqslant 2^n \bigl(1+\|r\|\bigr)^{2n+m_1+m_2} \bigl(2\vartheta_1(r) c_{x,y}\bigr)^{n+m_1+m_2}. \qquad \square$$

Now we are in a position to prove the following result about existence of an Evans–Hudson flow for QDS P_t associated with the element $r \in \mathcal{A}$ discussed above.

Theorem 3.3. (a) *For $t \geqslant 0$, there exists a unique solution j_t of the QSDE,*

$$\mathrm{d}j_t(x) = \sum_{j\in\mathbb{Z}^d} j_t(\delta_j^\dagger x)\,\mathrm{d}a_j(t) + \sum_{j\in\mathbb{Z}^d} j_t(\delta_j x)\,\mathrm{d}a_j^\dagger(t) + j_t(\hat{\mathcal{L}}x)\,\mathrm{d}t, \tag{3.6}$$

$$j_0(x) = x\otimes 1_\Gamma, \quad \forall x\in\mathcal{A}_{\mathrm{loc}},$$

such that $j_t(1)=1,\ \forall t\geqslant 0$.

(b) *For $x, y\in\mathcal{A}_{\mathrm{loc}}$ and $u, v\in\mathbf{h}_0$, $f, g\in\mathcal{C}$,*

$$\langle u\mathbf{e}(f), j_t(xy)v\mathbf{e}(g)\rangle = \langle j_t(x^*)u\mathbf{e}(f), j_t(y)v\mathbf{e}(g)\rangle. \tag{3.7}$$

(c) *j_t extends uniquely to a unital C^*-homomorphism from $\mathcal{A}$ into $\mathcal{A}''\otimes\mathcal{B}(\Gamma)$.*

Proof. We note first that $\mathcal{A}_{\mathrm{loc}}$ is a dense $*$-subalgebra of $\mathcal{A}$.

(a) As usual, we solve the QSDE by iteration. For $t_0\geqslant 0$, $t\leqslant t_0$ and $x\in\mathcal{A}_{\mathrm{loc}}$, we set

$$j_t^{(0)}(x) = x\otimes 1_\Gamma \quad \text{and}$$
$$j_t^{(n)}(x) = x\otimes 1_\Gamma + \int_0^t \sum_{j\in\mathbb{Z}^d} j_s^{(n-1)}\big(\delta_j^\dagger(x)\big)\,\mathrm{d}a_j(s) + \sum_{j\in\mathbb{Z}^d} j_s^{(n-1)}\big(\delta_j(x)\big)\,\mathrm{d}a_j^\dagger(s) + j_s^{(n-1)}\big(\hat{\mathcal{L}}(x)\big)\,\mathrm{d}s. \tag{3.8}$$

Then for $u\in\mathbf{h}_0$ and $f\in\mathcal{C}$, we can show by induction, that

$$\big\|\big\{j_t^{(n)}(x) - j_t^{(n-1)}(x)\big\}u\mathbf{e}(f)\big\| \leqslant \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}}\big\|u\mathbf{e}(f)\big\| \sum_{\bar{k}(n)}\sum_{\bar{\varepsilon}(n)} \big\|\delta\big(\bar{k}(n),\bar{\varepsilon}(n)\big)(x)\big\|, \tag{3.9}$$

where $c_f = 2\mathrm{e}^{\gamma_f(t_0)}(1+\|f\|_\infty^2)$, with $\gamma_f(t_0) = \int_0^{t_0}(1+\|f(s)\|^2)\,\mathrm{d}s$. For $n=1$, by the basic estimate of quantum stochastic integral [10,7],

$$\begin{aligned}
&\big\|\big\{j_t^{(1)}(x) - j_t^{(0)}(x)\big\}u\mathbf{e}(f)\big\|^2\\
&\quad = \left\|\left\{\int_0^t \sum_{j\in\mathbb{Z}^d}\delta_j^\dagger(x)\,\mathrm{d}a_j(s) + \sum_{j\in\mathbb{Z}^d}\delta_j(x)\,\mathrm{d}a_j^\dagger(s) + \hat{\mathcal{L}}(x)\,\mathrm{d}s\right\}u\mathbf{e}(f)\right\|^2\\
&\quad \leqslant 2\mathrm{e}^{\gamma_f(t_0)}\big\|\mathbf{e}(f)\big\|^2\int_0^t\left\{\sum_{j\in\mathbb{Z}^d}\big\|\delta_j^\dagger(x)u\big\|^2 + \sum_{j\in\mathbb{Z}^d}\big\|\delta_j(x)u\big\|^2 + \big\|\hat{\mathcal{L}}(x)u\big\|^2\right\}\big(1+\big\|f(s)\big\|\big)^2\,\mathrm{d}s\\
&\quad \leqslant c_f t_0\big\|\mathbf{e}(f)\big\|^2\left\{\sum_{j\in\mathbb{Z}^d}\big\|\delta_j^\dagger(x)u\big\| + \big\|\delta_j(x)u\big\| + \big\|\mathcal{L}_j(x)u\big\|\right\}^2.
\end{aligned}$$

Thus (3.9) is true for $n=1$. Inductively assuming the estimate for some $m>1$, we have by the same argument as above,

$$\begin{aligned}
&\big\|\big\{j_t^{(m+1)}(x) - j_t^{(m)}(x)\big\}u\mathbf{e}(f)\big\|^2\\
&\quad = \left\|\left\{\int_0^t \sum_{j\in\mathbb{Z}^d}\big[j_{s_m}^{(m)}\big(\delta_j^\dagger(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j^\dagger(x)\big)\big]\,\mathrm{d}a_j(s_m) + \sum_{j\in\mathbb{Z}^d}\big[j_{s_m}^{(m)}\big(\delta_j(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j(x)\big)\big]\,\mathrm{d}a_j^\dagger(s_m)\right.\right.\\
&\qquad \left.\left. + \big[j_{s_m}^{(m)}\big(\hat{\mathcal{L}}(x)\big) - j_{s_m}^{(m-1)}\big(\hat{\mathcal{L}}(x)\big)\big]\,\mathrm{d}s_m\right\}u\mathbf{e}(f)\right\|^2
\end{aligned}$$

$$\leqslant 2\mathrm{e}^{\gamma_f(t_0)} \int_0^t \Big\{ \sum_{j\in\mathbb{Z}^d} \big\| \big[j_{s_m}^{(m)}\big(\delta_j^\dagger(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j^\dagger(x)\big)\big] u\mathbf{e}(f)\big\|^2$$

$$+ \sum_{j\in\mathbb{Z}^d} \big\| \big[j_{s_m}^{(m)}\big(\delta_j(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j(x)\big)\big] u\mathbf{e}(f)\big\|^2$$

$$+ \big\| \big[j_{s_m}^{(m)}\big(\hat{\mathcal{L}}(x)\big) - j_{s_m}^{(m-1)}\big(\hat{\mathcal{L}}(x)\big)\big] u\mathbf{e}(f)\big\|^2 \Big\} \big(1 + \|f(s_m)\|^2\big)\,\mathrm{d}s_m$$

$$\leqslant c_f \int_0^t \Big[\sum_{j\in\mathbb{Z}^d} \Big\{ \big\| \big[j_{s_m}^{(m)}\big(\delta_j^\dagger(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j^\dagger(x)\big)\big] u\mathbf{e}(f)\big\|$$

$$+ \sum_{j\in\mathbb{Z}^d} \big\| \big[j_{s_m}^{(m)}\big(\delta_j(x)\big) - j_{s_m}^{(m-1)}\big(\delta_j(x)\big)\big] u\mathbf{e}(f)\big\| + \big\| \big[j_{s_m}^{(m)}\big(\hat{\mathcal{L}}(x)\big) - j_{s_m}^{(m-1)}\big(\hat{\mathcal{L}}(x)\big)\big] u\mathbf{e}(f)\big\| \Big\} \Big]^2 \mathrm{d}s_m.$$

Now applying (3.9) for $n = m$, we get the required estimate for $n = m + 1$ and furthermore by the estimate of Lemma 3.2(iii),

$$\big\| \big\{ j_t^{(n)}(x) - j_t^{(n-1)}(x) \big\} u\mathbf{e}(f) \big\| \leqslant 3^n \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}} \|u\mathbf{e}(f)\| \big(1 + \|r\|\big)^n \big(1 + 2\vartheta_1(r)c_x\big)^n.$$

Thus it follows that the sequence $\{j_t^{(n)}(x)u\mathbf{e}(f)\}$ is Cauchy. We define $j_t(x)u\mathbf{e}(f)$ to be $\lim_{n\to\infty} j_t{}^{(n)} u\mathbf{e}(f)$, that is

$$j_t(x)u\mathbf{e}(f) = xu \otimes \mathbf{e}(f) + \sum_{n\geqslant 1} \big\{ j_t^{(n)}(x) - j_t^{(n-1)}(x) \big\} u\mathbf{e}(f) \tag{3.10}$$

and one has

$$\big\| j_t(x)u\mathbf{e}(f) \big\| \leqslant \big\| u\mathbf{e}(f) \big\| \Big[\|x\| + \sum_{n\geqslant 1} 3^n \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}} \big(1 + \|r\|\big)^n \big(1 + 2\vartheta_1(r)c_x\big)^n \Big]. \tag{3.11}$$

Uniqueness follows by setting,

$$q_t(x) = j_t(x) - j_t'(x)$$

and observing

$$\mathrm{d}q_t(x) = \sum_{j\in\mathbb{Z}^d} q_t\big(\delta_j^\dagger(x)\big)\,\mathrm{d}a_j(t) + \sum_{j\in\mathbb{Z}^d} q_t\big(\delta_j(x)\big)\,\mathrm{d}a_j^\dagger(t) + q_t\big(\mathcal{L}(x)\big)\,\mathrm{d}t, \quad q_0(x) = 0.$$

Exactly similar estimate as above shows that, for all $n \geqslant 1$,

$$\big\| q_t(x)u\mathbf{e}(f) \big\| \leqslant \frac{(t_0 c_f)^{n/2}}{\sqrt{n!}} \big\| u\mathbf{e}(f) \big\| \sum_{\bar{k}(n)} \sum_{\bar{\varepsilon}(n)} \big\| \delta\big(\bar{k}(n), \bar{\varepsilon}(n)\big)(x) \big\|.$$

Since by Lemma 3.2(iii) the sum grows as nth power, $q_t(x) = 0\ \forall x \in \mathcal{A}_{\mathrm{loc}}$, showing the uniqueness of the solution. As $1 \in \mathcal{A}_{\mathrm{loc}}$ with $\mathcal{L}_k(1) = \delta_k^\dagger(1) = \delta_k(1) = 0$ it follows from the QSDE (3.6) that $j_t(1) = 1$.

(b) For $u\mathbf{e}(f), v\mathbf{e}(g) \in h \otimes \mathcal{E}(\mathcal{C})$ and $x, y \in \mathcal{A}_{\mathrm{loc}}$, we have, by induction,

$$\big\langle j_t^{(n)}(x^*)u\mathbf{e}(f), v\mathbf{e}(g) \big\rangle = \big\langle u\mathbf{e}(f), j_t^{(n)}(x)v\mathbf{e}(g) \big\rangle.$$

Now as n tends to ∞, we get

$$\big\langle j_t(x^*)u\mathbf{e}(f), v\mathbf{e}(g) \big\rangle = \big\langle u\mathbf{e}(f), j_t(x)v\mathbf{e}(g) \big\rangle.$$

We define

$$\Phi_t(x,y) = \langle u\mathbf{e}(f), j_t(xy)v\mathbf{e}(g)\rangle - \langle j_t(x^*)u\mathbf{e}(f), j_t(y)v\mathbf{e}(g)\rangle.$$

Setting $(\zeta_k(l), \eta_k(l)) = (\delta_k, \mathrm{id})$, (id, δ_k), $(\delta_k^\dagger, \mathrm{id})$, $(\mathrm{id}, \delta_k^\dagger)$, $(\mathcal{L}_k, \mathrm{id})$, $(\mathrm{id}, \mathcal{L}_k)$ and $(\delta_k^\dagger, \delta_k)$ for $l = 1, 2, \ldots, 7$ respectively, one has

$$\begin{aligned} &\left|\Phi_t(x,y)\right| \\ &\quad\leqslant c_{f,g}^n \sum_{l_n,\ldots,l_1} \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_1} \sum_{k_n,\ldots,k_1} \left|\Phi_{s_1}\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x, \eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y\big)\right| \mathrm{d}s_0\cdots \mathrm{d}s_{n-1} \\ &\qquad \forall n \geqslant 1, \end{aligned} \tag{3.12}$$

where $c_{f,g} = (1 + {t_0}^{1/2})(\|f\|_\infty + \|g\|_\infty)$. By the quantum Ito formula and cocycle properties of structure operators, i.e. $\hat{\mathcal{L}}(xy) = x\hat{\mathcal{L}}(y) + \hat{\mathcal{L}}(x)y + \sum_{k\in\mathbb{Z}^d} \delta_k^\dagger(x)\delta_k(y)$, we have,

$$\begin{aligned} \Phi_t(x,y) &= \int_0^t \sum_k \big\{\Phi_s\big(\delta_k(x), y\big) + \Phi_s\big(x, \delta_k(y)\big)\big\} f_k(s)\,\mathrm{d}s + \int_0^t \sum_k \big\{\Phi_s\big(\delta_k^\dagger(x), y\big) + \Phi_s\big(x, \delta_k^\dagger(y)\big)\big\}\bar{g}_k(s)\,\mathrm{d}s \\ &\quad + \int_0^t \sum_k \big\{\Phi_s\big(\mathcal{L}_k(x), y\big) + \Phi_s\big(x, \mathcal{L}_k(y)\big) + \Phi_s\big(\delta_k^\dagger(x), \delta_k(y)\big)\big\}\,\mathrm{d}s, \end{aligned}$$

which gives the estimate for $n = 1$:

$$\left|\Phi_t(x,y)\right| \leqslant c_{f,g} \sum_{l=1,\ldots,7} \int_0^t \sum_k \left|\Phi_s\big(\zeta_k(l)(x), \eta_k(l)(y)\big)\right| \mathrm{d}s. \tag{3.13}$$

If we now assume (3.12) for some $m > 1$, an application of (3.13) gives the required estimate for $n = m + 1$.

At this point we note the following, which can be verified easily by (3.10), (3.11) and Lemma 3.2(iv).

(1) For any n-tuple $(l_1, l_2, \ldots, l_n)$ in $\{1, 2, \ldots, 7\}$

$$\begin{aligned} &\sum_{k_n,\ldots,k_1} \left\| j_s\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)(x)\cdot\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)(y)\big)v\mathbf{e}(g)\right\| \\ &\quad\leqslant C_{g,x,y}\big\{\big(1 + \|r\|\big)\big(1 + 2\vartheta_1(r)c_{x,y}\big)\big\}^{2n}\left\|v\mathbf{e}(g)\right\|, \end{aligned} \tag{3.14}$$

where for any $g \in \mathcal{C}$

$$C_{g,x,y} = 1 + \sum_{m\geqslant 1} 3^m \frac{(t_0 c_g)^{m/2}}{\sqrt{m!}}\big\{\big(1 + \|r\|\big)\big(1 + 2\vartheta_1(r)c_{x,y}\big)\big\}^{2m}.$$

(2) For any $s \leqslant t_0$, $p \leqslant n$ and $\bar{\varepsilon}(p)$,

$$\sum_{\bar{k}(p)} \left\| j_s\big\{\delta\big(\bar{k}(p), \bar{\varepsilon}(p)\big)(y)\big\}v\mathbf{e}(g)\right\| \leqslant C_{g,x,y}\big\{\big(1 + \|r\|\big)\big(1 + 2\vartheta_1(r)c_{x,y}\big)\big\}^n \left\|v\mathbf{e}(g)\right\|. \tag{3.15}$$

(3) Since $\vartheta_p(x) = \vartheta_p(x^*)$ and $\{\delta(\bar{k}(p), \bar{\varepsilon}(p))(x)\}^*$ can also be written as $\delta(\bar{k}(p), \bar{\varepsilon}'(p))(x^*)$ for some $\bar{\varepsilon}'(p)$, we have

$$\sum_{\bar{k}(p)} \left\| j_s\big\{\delta\big(\bar{k}(p), \bar{\varepsilon}(p)\big)(x)\big\}^* u\mathbf{e}(f)\right\| \leqslant C_{f,x,y}\big\{\big(1 + \|r\|\big)\big(1 + 2\vartheta_1(r)c_{x,y}\big)\big\}^n \left\|u\mathbf{e}(f)\right\|. \tag{3.16}$$

For any fixed n-tuple $(l_1,\dots,l_n)$, it is easy to observe from the definition of Φ_s that

$$\begin{aligned}\sum_{\bar{k}(n)}&\big|\Phi_s\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x,\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y\big)\big|\\&\leqslant\sum_{k_n,\dots,k_1}\big\|u\mathbf{e}(f)\big\|\cdot\big\|j_s\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x\cdot\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y\big)v\mathbf{e}(g)\big\|\\&\quad+\big\|j_s\big\{\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)(x)\big)^*\big\}u\mathbf{e}(f)\big\|\cdot\big\|j_s\big(\eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)(y)\big)v\mathbf{e}(g)\big\|.\end{aligned}$$

The estimates (3.14), (3.15) and (3.16) yield:

$$\begin{aligned}\sum_{\bar{k}(n)}&\big|\Phi_s\big(\zeta_{k_n}(l_n)\cdots\zeta_{k_1}(l_1)x,\ \eta_{k_n}(l_n)\cdots\eta_{k_1}(l_1)y\big)\big|\\&\leqslant\big\{\big(1+\|r\|\big)\big(1+2\vartheta_1(r)c_{x,y}\big)\big\}^{2n}\big\|u\mathbf{e}(f)\big\|\cdot\big\|v\mathbf{e}(g)\big\|(C_{g,x,y}+C_{f,x,y}C_{g,x,y})\\&=C\big\{\big(1+\|r\|\big)\big(1+2\vartheta_1(r)c_{x,y}\big)\big\}^{2n},\end{aligned}$$

with $C=\|u\mathbf{e}(f)\|\cdot\|v\mathbf{e}(g)\|(C_{g,x,y}+C_{f,x,y}C_{g,x,y})$.

Now by (3.12),

$$\big|\Phi_t(x,y)\big|\leqslant C\frac{(7t_0c_{f,g})^n}{n!}\big\{\big(1+\|r\|\big)\big(1+2\vartheta_1(r)c_{x,y}\big)\big\}^{2n},\quad\forall n\geqslant 1,$$

which implies $\Phi_t(x,y)=0$.

(c) Let $\xi=\sum c_ju_j\mathbf{e}(f_j)$ be a vector in the algebraic tensor product of $\mathbf{h}_0$ and $\mathcal{E}(\mathcal{C})$. If $y\in\mathcal{A}_{\mathrm{loc}}^+$, y is actually an $N^{|y|}\times N^{|y|}$-dim positive matrix and hence it admits a unique square root $\sqrt{y}\in\mathcal{A}_{\mathrm{loc}}^+$. For any $x\in\mathcal{A}_{\mathrm{loc}}^+$, setting $y=\sqrt{\|x\|1-x}$ so that $y\in\mathcal{A}_{\mathrm{loc}}^+$, we get

$$\begin{aligned}\big\|j_t(y)\xi\big\|^2=\big\langle j_t(y)\xi,\,j_t(y)\xi\big\rangle&=\sum\bar{c}_ic_j\big\langle j_t(y)u_i\mathbf{e}(f_i),\,j_t(y)u_j\mathbf{e}(f_j)\big\rangle\\&=\sum\bar{c}_ic_j\big\langle u_i\mathbf{e}(f_i),\,j_t\big(\|x\|1-x\big)u_j\mathbf{e}(f_j)\big\rangle\quad\text{(by (b))}\\&=\|x\|\cdot\|\xi\|^2-\big\langle\xi,\,j_t(x)\xi\big\rangle,\end{aligned}$$

where we have used the fact that $1\in\mathcal{A}_{\mathrm{loc}}$ and $j_t(1)=1$. Now let $x\in\mathcal{A}_{\mathrm{loc}}$ be arbitrary and applying the above for x^*x as well as (b) we get,

$$\begin{aligned}\big\|j_t(x)\xi\big\|^2=\big\langle j_t(x)\xi,\,j_t(x)\xi\big\rangle&=\sum\bar{c}_ic_j\big\langle j_t(x)u_i\mathbf{e}(f_i),\,j_t(x)u_j\mathbf{e}(f_j)\big\rangle\\&=\sum\bar{c}_ic_j\big\langle u_i\mathbf{e}(f_i),\,j_t(x^*x)u_j\mathbf{e}(f_j)\big\rangle=\big\langle\xi,\,j_t(x^*x)\xi\big\rangle\leqslant\|x^*x\|\cdot\|\xi\|^2=\|x\|^2\cdot\|\xi\|^2\end{aligned}$$

or

$$\big\|j_t(x)\xi\big\|\leqslant\|x\|\cdot\|\xi\|.$$

This inequality obviously extends to all $\xi\in\mathbf{h}_0\otimes\Gamma$. Noting that $j_t(1)=1$, $\forall t$, we get

$$\big\|j_t(x)\big\|\leqslant\|x\|\quad\text{and}\quad\|j_t\|=1.$$

Thus j_t extends uniquely to a unital C^*-homomorphism satisfying the QSDE (3.6) and hence is an Evans–Hudson flow on $\mathcal{A}$ with P_t as its expectation semigroup. That the range of j_t is in $\mathcal{A}''\otimes\mathcal{B}(\Gamma)$ is clear from the construction of j_t. $\square$

We have also obtained an Evans–Hudson type dilation for the QDS P_t^{ϕ} associated with the partial state ϕ_0. It may be noted that the generator $\hat{\mathcal{L}}^{\phi}$ of P_t^{ϕ} satisfies

$$\hat{\mathcal{L}}^{\phi}(x) = \sum_{k\in\mathbb{Z}^d} \frac{1}{2} \sum_{m=1}^{N'} \big[L_k^{(m)^*}, x\big]L_k^{(m)} + L_k^{(m)^*}\big[x, L_k^{(m)}\big], \quad \forall x \in \mathcal{A}_{\mathrm{loc}}.$$

Now we have the following,

Theorem 3.4. *Let $\hat{\mathcal{L}}^{\phi}$ and P_t^{ϕ} be as discussed earlier. Then*:

(a) *For each $k \in \mathbb{Z}^d$ and $t \geqslant 0$ there exists a unique solution $\eta_t^{(k)}$ for the QSDE,*

$$\begin{aligned} \mathrm{d}\eta_t^{(k)}(x) &= \eta_t^{(k)}\left(\sum_{m=1}^{N'}\big[L_k^{(m)^*}, x_{(k)}\big]\right)\mathrm{d}a_k(t) + \eta_t^{(k)}\left(\sum_{m=1}^{N'}\big[x_{(k)}, L_k^{(m)}\big]\right)\mathrm{d}a_k^{\dagger}(t) + \eta_t^{(k)}\big(\mathcal{L}_k^{\phi}x_{(k)}\big)\,\mathrm{d}t,\\ j_0(x_{(k)}) &= x_{(k)} \otimes 1_{\Gamma}, \quad \forall x_{(k)} \in \mathcal{A}_k, \end{aligned} \tag{3.17}$$

as a unital $*$-homomorphism from $\mathcal{A}_k$ into $\mathcal{A}_k \otimes \mathcal{B}(\Gamma)$. Moreover, for different k and k', $\eta_t^{(k)}$ and $\eta_t^{(k')}$ commute in the sense that, $\eta_t^{(k)}(x_{(k)})$ and $\eta_t^{(k')}(x_{k'})$ commute for every $x_{(k)} \in \mathcal{A}_k$ and $x_{k'} \in \mathcal{A}_{k'}$;

(b) *There exists a unique unital $*$-homomorphism η_t from $\mathcal{A}_{\mathrm{loc}}$ into $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$ such that it coincide with $\eta_t^{(k)}$ on $\mathcal{A}_k$;*

(c) *η_t extends uniquely as a unital C^*-homomorphism from $\mathcal{A}$ into $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$.*

Proof. (a) For any $k \in \mathbb{Z}^d$ and $t \geqslant 0$ let us consider the QSDE (3.17). Here we have only finitely many nontrivial structure maps on the finite dimensional unital C^*-algebra $\mathcal{A}_k$, satisfying the structure equation. So there exists a unique solution $\eta_t^{(k)}$ as a unital $*$-homomorphism from $\mathcal{A}_k$ into $\mathcal{A}_k \otimes \mathcal{B}(\Gamma)$. Since for different k and k' the associated structure maps commute and for any $x_{(k)} \in \mathcal{A}_k$ and $x_{(k')} \in \mathcal{A}_{k'}$, Ito term absent in $d(\eta_t^{(k)}(x_{(k)})\eta_t^{(k')}(x_{(k')}))$, it follows that $\eta_t^{(k)}(x_{(k)})$ and $\eta_t^{(k')}(x_{(k')})$ commute.

(b) For any finite $\Lambda \subseteq \mathbb{Z}^d$, $t \geqslant 0$ and simple tensor element $x_\Lambda = \prod_{k\in\Lambda} x_{(k)} \in \mathcal{A}_\Lambda$, the map $\eta_t^{(\Lambda)}$ given by

$$\eta_t^{(\Lambda)}(x_\Lambda) := \prod_{k\in\Lambda} \eta_t^{(k)}(x_{(k)})$$

is well defined from $\mathcal{A}_\Lambda$ to $\mathcal{A}_\Lambda \otimes \mathcal{B}(\Gamma)$ as $\eta_t^{(k)}$'s commute. Differentiating $\eta_t^{(\Lambda)}(x_\Lambda)$ with respect to t, it follows that $\eta_t^{(\Lambda)}(x_\Lambda)$ satisfies the QSDE,

$$\begin{aligned} \mathrm{d}\eta_t^{(\Lambda)}(x_\Lambda) &= \sum_{k\in\Lambda} \eta_t^{(\Lambda)}\left(\sum_{m=1}^{N'}\big[L_k^{(m)^*}, x_\Lambda\big]\right)\mathrm{d}a_k(t) + \sum_{k\in\Lambda} \eta_t^{(\Lambda)}\left(\sum_{m=1}^{N'}\big[x_\Lambda, L_k^{(m)}\big]\right)\mathrm{d}a_k^{\dagger}(t) + \eta_t^{(\Lambda)}\big(\mathcal{L}_k^{\phi}x_\Lambda\big)\,\mathrm{d}t,\\ \eta_0^{(\Lambda)}(x_\Lambda) &= x_\Lambda \otimes 1_{\Gamma}. \end{aligned} \tag{3.18}$$

We now want to show

$$\eta_t^{(\Lambda)}(xy) = \eta_t^{(\Lambda)}(x)\cdot\eta_t^{(\Lambda)}(y), \quad \text{for simple tensor elements } x, y \in \mathcal{A}_{\mathrm{loc}}. \tag{3.19}$$

Since each $\eta_t^{(k)}$ is unital and $\eta_t^{(\Lambda')}$ agrees with $\eta_t^{(\Lambda)}$ for simple tensor elements in $\mathcal{A}_\Lambda$ whenever Λ is a finite subset of Λ', it is suffices to show (3.19) for $x, y \in \mathcal{A}_\Lambda$, where $\Lambda \subseteq \mathbb{Z}^d$ is a finite set. For $x = \prod_{k\in\Lambda} x_{(k)}$ and $y = \prod_{k\in\Lambda} y_{(k)} \in \mathcal{A}_\Lambda$ we have,

$$\begin{aligned}\eta_t^{(\Lambda)}(xy) &= \eta_t^{(\Lambda)}\prod_{k\in\Lambda}(x_{(k)}y_{(k)}) = \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)}y_{(k)}) \\ &= \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)})\eta_t^{(k)}(y_{(k)}) = \prod_{k\in\Lambda}\eta_t^{(k)}(x_{(k)})\prod_{k\in\Lambda}\eta_t^{(k)}(y_{(k)}).\end{aligned}$$

Similarly

$$\eta_t^{(\Lambda)}(x^*) = \big(\eta_t^{(\Lambda)}(x)\big)^*. \tag{3.20}$$

Noting that any element $x \in \mathcal{A}_{\mathrm{loc}}$ can be written as a linear combination of simple tensor elements $\{U_g\colon g \in \mathcal{G}\}$, say $x = \sum_{g\in\mathcal{G}} c_g U_g$ with $c_g = 0$ when $\mathrm{supp}(g)$ is outside $\mathrm{supp}(x) = \Lambda$, we define

$$\eta_t(x) = \sum_{g\in\mathcal{G}} c_g \eta_t^{(\Lambda)}(U_g).$$

For x and $y \in \mathcal{A}_{\mathrm{loc}}$, with $x = \sum_{g\in\mathcal{G}} c_g U_g$ and $y = \sum_{h\in\mathcal{G}} c_h U_h$, such that $\mathrm{supp}(x) = \mathrm{supp}(y) = \Lambda$,

$$\begin{aligned}\eta_t(xy) &= \eta_t\Bigg(\sum_{g,h\in\mathcal{G}} c_g c_h U_g U_h\Bigg) \\ &= \sum_{g,h\in\mathcal{G}} c_g c_h \eta_t^{(\Lambda)}(U_g U_h) = \sum_{g,h\in\mathcal{G}} c_g c_h \eta_t^{(\Lambda)}(U_g)\eta_t^{(\Lambda)}(U_h) \quad \text{(by (3.19))} \\ &= \eta_t\Bigg(\sum_{g\in\mathcal{G}} c_g U_g\Bigg)\eta_t\Bigg(\sum_{h\in\mathcal{G}} c_h U_h\Bigg) = \eta_t(x)\eta_t(y).\end{aligned}$$

It follows from (3.20) that $\eta_t(x^*) = (\eta_t(x))^*$ $\forall x \in \mathcal{A}_{\mathrm{loc}}$. Thus η_t is a unital $*$-homomorphism from $\mathcal{A}_{\mathrm{loc}}$ into $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$.

(c) We recall that $\mathcal{A}^+_{\mathrm{loc}}$ is closed under taking square root, as already noted in the proof of Theorem 3.3(c). Thus for $x \in \mathcal{A}_{\mathrm{loc}}$, $\sqrt{\|x\|^2 1 - x^*x} \in \mathcal{A}^+_{\mathrm{loc}}$. Since η_t is a unital $*$-homomorphism on $\mathcal{A}_{\mathrm{loc}}$,

$$\eta_t\big(\|x\|^2 1 - x^*x\big) \geqslant 0 \Rightarrow \eta_t(x^*x) \leqslant \|x\|^2 1 \Rightarrow \big\|\eta_t(x^*x)\big\| \leqslant \|x\|^2 \Rightarrow \big\|\eta_t(x)\big\| \leqslant \|x\|.$$

So η_t extends uniquely as a unital C^*-homomorphism from $\mathcal{A}$ into $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$. $\square$

4. Covariance of the Evans–Hudson flows

In this section we shall prove that the Evans–Hudson flows constructed in the last section are covariant. Let $\mathcal{B}$ be a C^* (or von Neumann) algebra, G be a locally compact group with an action α on $\mathcal{B}$. Let $\{T_t\colon t > 0\}$ be a covariant QDS on $\mathcal{B}$ with respect to α, i.e.

$$\alpha_g \circ T_t(x) = T_t \circ \alpha_g(x), \quad \forall t \geqslant 0,\ g \in G,\ x \in \mathcal{B}.$$

Then a natural question arises whether there exists a covariant Evans–Hudson dilation for $\{T_t\}$. The question is discussed in [1] for uniformly continuous QDS. There is no such general result for QDS with unbounded generators.

We shall show that the Evans–Hudson flows $\{j_t\}$ and $\{\eta_t\}$ constructed in the previous section are covariant with respect to the actions τ and λ of $\mathbb{Z}^d$, where λ will be introduced later in this section.

It can be easily observed that

$$\delta_k \tau_j = \tau_j \delta_{k-j} \quad \text{and} \quad \delta_k^\dagger \tau_j = \tau_j \delta_{k-j}^\dagger,\ \forall j, k \in \mathbb{Z}^d, \tag{4.1}$$

and we have the following lemma,

Lemma 4.1.

(i) $\hat{\mathcal{L}}\tau_j(x) = \tau_j\hat{\mathcal{L}}(x)\ \forall x \in \mathrm{Dom}(\hat{\mathcal{L}})$,
(ii) $P_t\tau_j = \tau_j P_t$, *i.e.* P_t *is covariant.*

Proof. (i) We note that $\mathcal{C}^1(\mathcal{A})$ is invariant under τ and thus for $x \in \mathcal{C}^1(\mathcal{A})$,

$$\begin{aligned}\mathcal{L}\big(\tau_j(x)\big) &= \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\delta_k^\dagger\big(\tau_j(x)\big)r_k + r_k^*\delta_k\big(\tau_j(x)\big)\\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}^d}\tau_j\delta_{k-j}^\dagger(x)r_k + r_k{}^*\tau_j\delta_{k-j}(x) \quad \text{(by (4.1))}\\ &= \frac{1}{2}\tau_j\left\{\sum_{k\in\mathbb{Z}^d}\delta_{k-j}^\dagger(x)r_{k-j} + r_{k-j}^*\delta_{k-j}(x)\right\} = \tau_j\big(\mathcal{L}(x)\big).\end{aligned}$$

For $x \in \mathrm{Dom}(\hat{\mathcal{L}})$, we choose a sequence $\{x_n\}$ in $\mathcal{C}^1(\mathcal{A})$ and an element $y \in \mathcal{A}$ such that $y = \hat{\mathcal{L}}(x),\ x_n$ converge to x and $\mathcal{L}(x_n)$ converge to y. As τ_j is an automorphism for any $j \in \mathbb{Z}^d,\ \tau_j(x_n)$ and $\tau_j\mathcal{L}(x_n)$ converge to $\tau_j(x)$ and $\tau_j(y)$ respectively. Since $x_n \in \mathcal{C}^1(\mathcal{A})$ and $\mathcal{L}(\tau_j(x_n)) = \tau_j\mathcal{L}(x_n)$, we get

$$\tau_j(x) \in \mathrm{Dom}(\hat{\mathcal{L}}) \quad \text{and} \quad \hat{\mathcal{L}}\tau_j(x) = \tau_j\hat{\mathcal{L}}(x).$$

(ii) By (i), for $x \in \mathrm{Dom}(\hat{\mathcal{L}})$ and $0 \leqslant s \leqslant t$ we have,

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s \circ \tau_j \circ P_{t-s}(x) = P_s \circ \hat{\mathcal{L}} \circ \tau_j \circ P_{t-s}(x) - P_s \circ \tau_j \circ \hat{\mathcal{L}} \circ P_{t-s}(x) = 0.$$

This implies that $P_s \circ \tau_j \circ P_{t-s}(x)$ is independent of s for every j and $0 \leqslant s \leqslant t$. Setting $s = 0$ and t respectively and using the fact that P_t is bounded we get $P_t\tau_j = \tau_j P_t$. □

We note that $j_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k}_0)))$, where $\mathbf{k}_0 = l^2(\mathbb{Z}^d)$ with a canonical basis $\{e_k\}$, as mentioned earlier. We define the canonical bilateral shift s by $s_j e_k = e_{k+j},\ \forall j, k \in \mathbb{Z}^d$ and let $\gamma_j = \Gamma(1 \otimes s_j)$ be the second quantization of $1 \otimes s_j$, i.e. $\gamma_j\mathbf{e}(\sum f_l(\cdot)e_l) = \mathbf{e}(\sum f_l(\cdot)e_{l+j})$. This defines a unitary representation of $\mathbb{Z}^d$ in Γ. We set an action $\sigma = \tau \otimes \lambda$ of $\mathbb{Z}^d$ on $\mathcal{A}'' \otimes \mathcal{B}(\Gamma)$, where $\lambda_j(y) = \gamma_j y \gamma_{-j}\ \forall y \in \mathcal{B}(\Gamma)$.

By definition of fundamental processes $a_k(t)$ given by $a_k(t)\mathbf{e}(g) = \int_0^t g_k(s)\,\mathrm{d}s\,\mathbf{e}(g)$, it can be observed that

$$\begin{aligned}\lambda_j a_k(t)\mathbf{e}(g) &= \gamma_j a_k(t)\gamma_{-j}\mathbf{e}(g) = \gamma_j a_k(t)\mathbf{e}\Big(\sum\langle g, e_{l+j}\rangle(\cdot)e_l\Big)\\ &= \int_0^t\langle g, e_{k+j}\rangle(s)\,\mathrm{d}s\,\gamma_j\Big(\mathbf{e}\Big(\sum\langle g, e_{l+j}\rangle(\cdot)e_l\Big)\Big)\\ &= \int_0^t\langle g, e_{k+j}\rangle(s)\,\mathrm{d}s\,\mathbf{e}\Big(\sum\langle g, e_{l+j}\rangle(\cdot)e_{l+j}\Big)\\ &= a_{k+j}(t)\mathbf{e}(g).\end{aligned}$$

Since $\langle\mathbf{e}(f), \lambda_j a_k(t)\mathbf{e}(g)\rangle = \langle\lambda_j a_k^\dagger(t)\mathbf{e}(f), \mathbf{e}(g)\rangle$, it follows that

$$\lambda_j a_k(t) = a_{k+j}(t) \quad \text{and} \quad \lambda_j a_k^\dagger(t) = a_{k+j}^\dagger(t). \tag{4.2}$$

Theorem 4.2. *The Evans–Hudson flow j_t of the QDS P_t is covariant with respect to the actions τ and σ, i.e.*

$$\sigma_j j_t \tau_{-j}(x) = j_t(x) \quad \forall x \in \mathcal{A},\ t \geqslant 0 \text{ and } k \in \mathbb{Z}^d.$$

Proof. For a fixed $j \in \mathbb{Z}^d$ we set $j_t' = \sigma_j j_t \tau_{-j},\ \forall t \geqslant 0$. Using the QSDE (3.6) and Lemma 4.1, (4.1), (4.2) we have for $x \in \mathcal{A}_{\mathrm{loc}}$,

$$\begin{aligned}
&j_t'(x) - j_0'(x) \\
&\quad= \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s\big(\delta_k^\dagger\big(\tau_{-j}(x)\big)\big)\,\mathrm{d}a_k(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s\big(\delta_k\big(\tau_{-j}(x)\big)\big)\,\mathrm{d}a_k^\dagger(s) + \int_0^t \sigma_j j_s\big(\hat{\mathcal{L}}\big(\tau_{-j}(x)\big)\big)\,\mathrm{d}s \\
&\quad= \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s \tau_{-j}\big(\delta_{k+j}^\dagger(x)\big)\,\mathrm{d}a_{k+j}(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} \sigma_j j_s \tau_{-j}\big(\delta_{k+j}(x)\big)\,\mathrm{d}a_{k+j}^\dagger(s) + \int_0^t \sigma_j j_s \tau_{-j}\big(\hat{\mathcal{L}}(x)\big)\,\mathrm{d}s \\
&\quad= \int_0^t \sum_{k\in\mathbb{Z}^d} j_s'\big(\delta_k^\dagger(x)\big)\,\mathrm{d}a_k(s) + \int_0^t \sum_{k\in\mathbb{Z}^d} j_s'\big(\delta_k(x)\big)\,\mathrm{d}a_k^\dagger(s) + \int_0^t j_s'(\hat{\mathcal{L}}x)\,\mathrm{d}s.
\end{aligned}$$

Since $j_0'(x) = \sigma_j j_0 \tau_{-j}(x) = \sigma_j(\tau_{-j}(x) \otimes 1_\Gamma) = x \otimes 1_\Gamma = j_0(x)$, it follows from the uniqueness of solution of the QSDE (3.6) that $j_t'(x) = j_t(x)$ for all $t \geqslant 0$ and $x \in \mathcal{A}_{\mathrm{loc}}$. As both j_t' and j_t are bounded maps, we have $j_t' = j_t$. $\square$

Remark 4.3. By similar arguments as above, the Evans–Hudson flow for the QDS P_t^ϕ associated with partial state ϕ_0 can be seen to be covariant with respect to the same actions.

5. Ergodicity of the Evans–Hudson flows

Let us recall the ergodic QDS P_t^ϕ associated with the partial state ϕ_0, for which we have constructed an Evans–Hudson flow η_t in Section 3. It may be noted that P_t^ϕ has the unique invariant state Φ. We have the following result on ergodicity of η_t with respect to the weak operator topology.

Theorem 5.1. *The Evans–Hudson flow η_t of the ergodic QDS P_t^ϕ is ergodic with respect to the unique invariant state Φ, in the sense that*

$$\eta_t(x) \to \Phi(x) \otimes 1_\Gamma \quad \textit{weakly}\ \forall x \in \mathcal{A}.$$

Proof. Since η_t and P_t^ϕ are norm contractive, $\mathcal{A}_{\mathrm{loc}}$ is norm-dense in $\mathcal{A}$, and $P_t^\phi(x)$ converges to $\Phi(x)1$ for all $x \in \mathcal{A}$, it is enough to show that $\eta_t(x) - P_t^\phi(x) \otimes 1_\Gamma \to 0$ weakly as $t \to \infty$. Furthermore, it suffices to show that $\langle \xi_1, (\eta_t(x) - P_t^\phi(x) \otimes 1_\Gamma)\xi_2\rangle \to 0$ as $t \to \infty$, where ξ_1, ξ_2 vary over the linear span of vectors of the form $ve(f)$, with $f = \sum_{|k|\leqslant n} f_k \otimes e_k$ for some n and f_k's are in $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$.

For notational simplicity denoting the bounded derivations on $\mathcal{A}$,

$$x \mapsto \sum_{m=1}^{N'} \big[x, L_k^{(m)}\big] \quad \text{and} \quad x \mapsto \sum_{m=1}^{N'} \big[L_k^{(m)*}, x\big]$$

by ρ_k and $\rho_k^\dagger$ respectively. We note that η_t satisfies the QSDE

$$\begin{aligned} \mathrm{d}\eta_t(x) &= \sum_{k\in\mathbb{Z}^d} \eta_t\big(\rho_k^\dagger(x)\big)\,\mathrm{d}a_k(t) + \sum_{k\in\mathbb{Z}^d} \eta_t\big(\rho_k(x)\big)\,\mathrm{d}a_k^\dagger(t) + \sum_{k\in\mathbb{Z}^d} \eta_t\big(\mathcal{L}_k^\phi(x)\big)\,\mathrm{d}t, \\ \eta_0(x) &= x\otimes 1_\Gamma, \quad \forall x\in\mathcal{A}_{\mathrm{loc}}. \end{aligned} \tag{5.1}$$

For $t \geqslant 0$, $u, v \in \mathbf{h}_0$ and $f, g \in L^2(\mathbb{R}_+, \mathbf{k}_0) \cap L^1(\mathbb{R}_+, K_0)$ such that $f = \sum_{|k|\leqslant n} f_k \otimes e_k$ and $g = \sum_{|k|\leqslant n} g_k \otimes e_k$ and $x \in \mathcal{A}_{\mathrm{loc}}$, we consider the following,

$$\begin{aligned} &\big|\big\langle u\mathbf{e}(f), \big[\eta_t(x) - P_t^\phi(x)\otimes 1_\Gamma\big] v\mathbf{e}(g)\big\rangle\big| \\ &\quad = \left|\left\langle u\mathbf{e}(f), \left[\int_0^t \sum_{k\in\mathbb{Z}^d} \eta_q\big\{\rho_k\big(P_{t-q}^\phi(x)\big)\big\}\,\mathrm{d}a_k^\dagger(q) + \eta_q\big\{\rho_k^\dagger\big(P_{t-q}^\phi(x)\big)\big\}\,\mathrm{d}a_k(q)\right] v\mathbf{e}(g)\right\rangle\right| \\ &\quad \leqslant \sum_{|k|\leqslant n} \int_0^t \big|\big\langle u\mathbf{e}(f), \eta_q\big\{\rho_k\big(P_{t-q}^\phi(x)\big)\big\} v\mathbf{e}(g)\big\rangle\big|\,\big\|g(q)\big\|\,\mathrm{d}q \\ &\qquad + \sum_{|k|\leqslant n} \int_0^t \big|\big\langle u\mathbf{e}(f), \eta_q\big\{\rho_k^\dagger\big(P_{t-q}^\phi(x)\big)\big\} v\mathbf{e}(g)\big\rangle\big|\,\big\|f(q)\big\|\,\mathrm{d}q. \end{aligned}$$

As η_t, P_t^ϕ are contractive, $P_t^\phi(x)$ tends to $\Phi(x)1$ as t tends to ∞ and ρ_k, $\rho_k^\dagger$ are uniformly bounded with $\rho_k(1) = \rho_k^\dagger(1) = 0$ for all $k \in \mathbb{Z}^d$, we have,

$$\big|\big\langle u\mathbf{e}(f), \eta_q\big\{\rho_k\big(P_{t-q}^\phi(x)\big)\big\} v\mathbf{e}(g)\big\rangle\big| \quad \text{and} \quad \big|\big\langle u\mathbf{e}(f), \eta_q\big\{\rho_k^\dagger\big(P_{t-q}^\phi(x)\big)\big\} v\mathbf{e}(g)\big\rangle\big| \leqslant M,$$

for some constant M independent of t and q. The fact that $f, g \in L^1(\mathbb{R}_+, K_0)$ allows us to conclude that both the terms of the above expression tend to 0 as t tends to ∞. This completes the proof. □

Remark 5.2. $\eta_t(x)$ does not converge strongly, for if it did, then $x \mapsto \Phi(x) \otimes 1_\Gamma$ would be a homomorphism, i.e. Φ would be a multiplicative nonzero functional on the UHF algebra $\mathcal{A}$, contradictory to the fact that $\mathcal{A}$ does not have any such functional.

Remark 5.3. If we look at the perturbation of the ergodic QDS P_t^ϕ by the QDS associated with some single-supported $r \in \mathcal{A}_0$, then by the same arguments used in the construction of the Evans–Hudson flow for the unperturbed semigroup one can obtain an Evans–Hudson flow for the perturbed one. For small perturbation parameter $c \geqslant 0$ for which $P_t^{(c)}$ is ergodic, the associated Evans–Hudson flow is also ergodic with respect to the same invariant state in the above sense.

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