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CUT TIMES FOR RANDOM WALKS ON THE DISCRETE HEISENBERG GROUP

TEMPS DE COUPURE POUR LES MARCHES ALÉATOIRES SUR LE GROUPE DE HEISENBERG DISCRET

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ABSTRACT. – This article studies the occurrence of cut times along the path of a random walk (with finite support) on the discrete Heisenberg group. We establish the existence of an infinite number of cut times almost surely, using sharp estimates of the Green function and its gradient. This example (up to some extensions) happens to be the last unsolved case in the study of cut times for random walks with finite support, on finitely generated groups.

Keywords: Random walks on groups; Green function; Cut times; Heisenberg group

RÉSUMÉ. – Cet article étudie l’occurrence des temps de coupure le long du chemin d’une marche aléatoire (à support fini) sur le groupe de Heisenberg discret. Nous établissons l’existence d’un nombre infini de temps de coupure, presque sûrement, à l’aide d’estimations précises de la fonction de Green et de son gradient. Cet exemple (à quelques extensions près) apparaît comme le dernier cas non résolu dans l’étude des temps de coupure pour les marches aléatoires à support fini sur les groupes finiment engendrés.

1. Introduction

Let start with the definition of cut times for random walks on discrete, finitely generated groups. Then, we will give an overview of the known results about the occurence of cut times along the path of such random walks.

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As we will not deal only with the Heisenberg group (see Definition 2.4), but also with its finite extensions and finite extensions of $\mathbb{Z}^3$ and $\mathbb{Z}^4$, we present the definitions for a random walk $S(k)$ on a general group Γ having polynomial growth of degree D (see Definition 2.2). Then, we study the occurrence of cut times on the path of this random walk.

DEFINITION 1.1. – *A time n is a* cut time *if*

$$S([0,n]) \cap S([n+1,\infty[) = \emptyset,$$

where $S([a,b]) = \{S(k):\ a \leqslant k \leqslant b\}$.

We first remark that every recurrent random walk has no cut times, with probability 1. Indeed, the existence of an infinite number of returns to the starting point implies that this point belongs to $S([0,n]) \cap S([n+1,\infty[)$ for all n, almost surely.

Lawler [13] studied the case of the simple random walk on $\mathbb{Z}^d$. For $d \leqslant 2$, the simple random walk is recurrent, so the work deals with the case $d \geqslant 3$. He builds a two-sided walk (denoted $\widetilde{S}$) as an extension of the random walk to $k \in \mathbb{Z}$: for k negative,

$$\mathbf{P}\{\widetilde{S}(k-1) = y \mid \widetilde{S}(k) = x\} = p(y,x).$$

Then, each positive cut time for $\widetilde{S}$ is a cut time for S. For $\widetilde{S}$, the probability that k is a cut time does not depend on k. Thus, taking $k = 0$, by symmetry, the problem is reduced to the study of the non-intersection probability of two simple random walks with the same starting point. This quantity has also been studied by Lawler [13]. Then, Lawler proved that, for $d \geqslant 5$, simple random walk has an infinite number of cut times, with probability 1. He also proved the same result for the simple random walk on $\mathbb{Z}^4$ with more complex methods (see [13]). Finally, James and Peres [11] solved the 3-dimensional case and extended the result:

THEOREM 1.2. – *Every transient random walk on $\mathbb{Z}^d$ with finite range has an infinite number of cut times, almost surely.*

The case where the random walks are not centered is a simple consequence of the ergodic theorem. For centered random walks, they defined an infinite sequence of spheres with an infinite subsequence of *cut spheres*. It means that after the first hitting time of these spheres, with probability 1, the random walk does not come back within the sphere. Hence, as each hitting time of a cut sphere is a cut time, they get the result.

On groups having polynomial growth of degree $D \leqslant 2$, every centered (see Definition 3.1) random walk is recurrent. When $D \geqslant 5$ (and for discrete groups with super-polynomial growth), James and Peres [11] adapt the method used by Lawler on $\mathbb{Z}^D$ to get also the existence, with probability 1, of an infinite number of cut times for symmetric irreducible random walk with finite range. Actually, this method needs only to have a decay of the type

$$\mathbf{P}^e\{S(k) = e\} = \mathrm{O}(k^{-5/2}).$$

Alexopoulos [1] proved that this occurs even for centered random walk with $D \geqslant 5$ and for non-centered ones, with any degree (still with finite support). So we are left with the case of centered random walks on groups of degree 3 or 4.

When $D = 3$, the group has $\mathbb{Z}^3$ as a sub-group of finite index and when $D = 4$, the group has either $\mathbb{Z}^4$ or the *Heisenberg group* $\mathbb{H}_3$ as a sub-group of finite index (Proposition 2.5). As every nilpotent, torsion-free group, $\mathbb{H}_3$ is isomorphic to a discrete lattice in a nilpotent simply connected Lie group. On this Lie group, we have an exact formula for the *Green function* (see [6]). Then we can compare its gradient with the one computed on a finite extension of the discrete group $\mathbb{H}_3$ (Corollary 3.3) thanks to the work of Alexopoulos [1]. Then, we build spheres corresponding to level lines of the Green function. On a portion of such a sphere, the gradient of the Green function has a lower bound (Proposition 3.5) similar to the one on $\mathbb{Z}^D$. Thus, the techniques developed by James and Peres can be applied to these portions of spheres, which appears to be sufficient to prove the existence of an infinite number of cut times almost surely (Theorem 4.1). We also note that the same techniques work on finite extensions of $\mathbb{Z}^3$ or $\mathbb{Z}^4$, which completes the study of cut times for random walks, with finite support, on finitely generated groups (Corollary 4.8).

Finally, note that all these results deal with random walks on which transience coincide with the existence, with probability 1, of an infinite number of cut times. However, James [10] gave an example of a graph where the simple random walk is transient but get only a finite number of cut times, with probability 1. Thus, this coincidence cannot be a general property.

2. Discrete groups having polynomial growth

We give some definitions from geometric group theory.

DEFINITION 2.1. – *Let Γ be a finitely generated discrete group and V be a symmetric finite generating set. We call* word distance (*corresponding to V*) *between two elements x, y of Γ the minimal number of generators we need to go from e* (*identity of Γ*) *to $x^{-1}y$ by right multiplication. We denote this distance $|x^{-1}y|$.*

DEFINITION 2.2. – *Let Γ be a finitely generated discrete group and V be a symmetric finite generating set. We denote* $\mathrm{Vol}(n) = \#B(e,n)$ *the cardinality of the ball of radius n for the word distance associated to V. We say that Γ has a* polynomial growth *of degree D if there is a constant C such that for all n,*

$$C^{-1}n^D \leqslant \mathrm{Vol}(n) \leqslant Cn^D.$$

By left invariance, this property remains true if we center the balls at any $x \in \Gamma$. The original definition of the polynomial growth needs only the upper bound of the volume. But since the famous article by Gromov [7], these two definitions are known to be equivalent. Moreover, these groups are exactly the finitely generated virtually nilpotent groups.

DEFINITION 2.3. – *A group Γ is called* virtually nilpotent *if Γ has a nilpotent sub-group Γ^N of finite index.*

Let Γ^N be a r-step nilpotent group. Then, its lower central series is

$$\Gamma^N = \Gamma_1^N \supseteq \Gamma_2^N \supseteq \cdots \supseteq \Gamma_{r+1}^N = e,$$

where each $\Gamma_i^N / \Gamma_{i+1}^N$ is a finitely generated Abelian group. Bass [2] gives a formula for the degree of growth $D(\Gamma^N)$ in term of the (torsion-free) rank of the quotients $\Gamma_i^N / \Gamma_{i+1}^N$. Namely,

$$D(\Gamma^N) = \sum_{i=1}^{r} i \cdot \mathrm{rk}(\Gamma_i^N / \Gamma_{i+1}^N). \tag{1}$$

By [8, Theorem 7.8], as Γ^N is nilpotent, it has a torsion-free subgroup of finite index. Then, in any virtually nilpotent group Γ, there is a nilpotent torsion-free subgroup Γ' of Γ, of finite index.

Outside the Abelian case, a typical example of such a group is the Heisenberg group:

DEFINITION 2.4. – *The Heisenberg group $\mathbb{H}_3$ is the group of the upper triangular integer-valued* (3×3) *matrices with* 1*'s on the diagonal*:

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad (x, y, z) \in \mathbb{Z}^3.$$

It is easy to check that $\mathbb{H}_3$ has polynomial growth of degree 4.

As the subgroup Γ' of Γ defined above, is torsion-free and nilpotent, it is isomorphic to a cocompact lattice in a simply connected nilpotent Lie group N (see [14]). Then, by a classification, due to Dixmier [4], of these Lie groups for low dimension, we get the following result (see [3] for a detailed proof).

PROPOSITION 2.5. – *Let Γ be a discrete group having polynomial growth of degree D. If $D \leqslant 3$, Γ has a subgroup Γ' of finite index, which is isomorphic to $\mathbb{Z}^D$. If $D = 4$, Γ has a subgroup Γ' of finite index, which is isomorphic to $\mathbb{Z}^4$ or $\mathbb{H}_3$.*

Using this proposition, we will answer the question of the existence of cut times for random walks, with finite support, on finitely generated groups, only with a study on $\mathbb{H}_3$ (and some remarks on finite extensions of $\mathbb{H}_3$, $\mathbb{Z}^3$ and $\mathbb{Z}^4$).

The method we use relies on sharp estimates of the Green function and its gradient. These estimates are obtained in Section 3, using exact computation of the Green function on the Heisenberg Lie group due to Folland [5] and Gaveau [6], and error estimates with the discrete setting, due to Alexopoulos [1]. In Section 4, we adapt the method by James and Peres [11] to prove the existence of infinitely many cut times for random walks on $\mathbb{H}_3$, with probability 1.

3. Discrete gradient of the Green function

Let Γ be a discrete group having polynomial growth. Let Γ' be a torsion-free nilpotent subgroup of Γ of finite index. We denote by N the simply connected nilpotent Lie group in which we can embed Γ'.

The group $\Gamma/[\Gamma, \Gamma]$ is an Abelian subgroup of Γ. Hence it can be written as $\mathbb{Z}^k \times I$, with I finite.

DEFINITION 3.1. – *We say that a probability measure on Γ is* centered *if its canonical projection on $\mathbb{Z}^k$ is centered.*

Let μ be such a measure, with a finite support U, such that U generates Γ. We will associate to μ a left-invariant sub-Laplacian L^μ on N. We denote p_t^μ the heat kernel on N associated to L^μ (i.e. the fundamental solution of $(\frac{\partial}{\partial t} + L^\mu)f = 0$). We define the two *Green functions*:

$$\text{for all } (h_1, h_1') \text{ in } \Gamma, \quad G^\mu(h_1, h_1') = \sum_{n \geqslant 0} \mu^n(h_1^{-1} h_1')$$

and

$$\text{for all } (h_2, h_2') \text{ in } N, \quad \mathcal{G}^\mu(h_2, h_2') = \int_0^\infty p_t^\mu(h_2, h_2')\, dt,$$

where $\mu^n = \mu^{*n}$ the nth convolution power of μ and $\mu f(x) = \sum_{y \in \Gamma} \mu(x^{-1}y) f(y)$. By translation invariance, we only need to consider

$$p_t^\mu(h) = p_t^\mu(e, h); \quad \mathcal{G}^\mu(h) = \mathcal{G}^\mu(e, h); \quad G^\mu(h) = G^\mu(e, h).$$

3.1. Construction of L^μ

We follow the construction of [1]. The Lie algebra $\mathcal{N}$ of N can be identified with a vector space

$$\mathcal{N}_\infty = \bigoplus_{i=1}^{r} \mathcal{N}_i / \mathcal{N}_{i+1}$$

with the following Lie bracket:

$$\text{for } h \in \mathcal{N}_i \text{ and } h' \in \mathcal{N}_j, \quad [\overline{h}, \overline{h'}\,] = \overline{[h, h']},$$

where $[h, h']$ is the Lie bracket on $\mathcal{N}$.

Let $X = \{X_1, \ldots, X_k\}$ be a basis of $\mathcal{N}_\infty$ such that $\{X_{n_{i-1}+1}, \ldots, X_{n_i}\}$ is a basis of $\mathcal{N}_i/\mathcal{N}_{i+1}$. Here, $\{n_i\colon\ i = 0 \text{ to } r\}$ is a finite increasing sequence with $n_0 = 0$ and, for $i \geqslant 1$,

$$n_i = \dim \bigoplus_{j=1}^{i} \mathcal{N}_j / \mathcal{N}_{j+1}.$$

Fix $\{g_l\colon\ l = 0 \text{ to } p\}$ a coset representative of Γ/Γ', with $g_0 = e$. If $h \in \Gamma'$, we can identify h with an element of $\mathcal{N}$. Then, we can define $P_i(h)$ as the ith coordinate of h in the basis X. We extend P_i to Γ by saying $P_i(hg_l) = P_i(h)$. We define the following coefficients:

$$\begin{aligned}
b_i(g_l) &= \sum_{h \in \Gamma} P_i(g_l h)\mu(h), \quad 1 \leqslant i \leqslant n_2;\\
a_{ij}(g_l) &= \sum_{h \in \Gamma} P_i(g_l h) P_j(g_l h)\mu(h), \quad 1 \leqslant i, j \leqslant n_1;\\
a_i(g_l) &= b_i(g_l), \quad 1 \leqslant i \leqslant n_1.
\end{aligned}$$

Now, we define the first order correctors ψ^j (see [1]) by

$$\psi^j(hg_l) = \sum_{n\geqslant 0} \mu^n a_j(g_l), \quad 1 \leqslant j \leqslant n_1.$$

We write

$$b_{ij}(g_l) = \sum_{h\in\Gamma} \psi^j(g_l h) P_i(g_l h)\mu(h), \quad 1 \leqslant i, j \leqslant n_1.$$

Finally, we define the coefficients

$$q_{ij} = \langle (1/2)a_{ij} + b_{ij} \rangle, \quad 1 \leqslant i, j \leqslant n_1,$$

and

$$q_i = \langle b_i \rangle, \quad n_1 < i \leqslant n_2,$$

where $\langle f \rangle = (1/(p+1)) \sum_{l=0}^{p} f(g_l)$. Then, we define the sub-Laplacian as

$$L^\mu = -\sum_{1\leqslant i,j\leqslant n_1} q_{ij} X_i X_j - \sum_{n_1<i\leqslant n_2} q_i X_i. \tag{2}$$

It is constructed in such a way that we can compare the asymptotic behavior of μ^n and p_n^μ. It is called the *homogenized sub-Laplacian* associated to μ (see [1]).

When Γ is nilpotent and torsion-free, we take the same definition with $\Gamma = \Gamma'$. We remark that $\psi^j \equiv 0$ in this case.

3.2. Error estimates

We define the discrete gradient of a function f in the direction $w \in U$, by

$$\nabla_w f(h) = \left| f(hw) - f(h) \right|.$$

And,

$$\nabla f(h) = \max_{w\in U} \nabla_w f(h).$$

Recall that any element h in Γ' can be identified with an element of N. From [1], we have

THEOREM 3.2. – *Let Γ be a discrete group having polynomial growth of degree D, then for all $\varepsilon \in (0,1)$, there exists a constant C such that, for all $w \in U$, $h \in \Gamma'$, and $g_l \in \Gamma/\Gamma'$*

$$\left| \nabla_w \mu^n(hg_l) - \nabla_w p_n^\mu(h) - \sum_{j\leqslant n_1} \left(\nabla_w \psi^j(e)\right) X_j p_n^\mu(h) \right| \leqslant Cn^{-(D+1+\varepsilon)/2} \exp\left(-\frac{|hg_l|^2}{Cn} \right).$$

From this, we deduce error estimates for the gradient of the Green function.

COROLLARY 3.3. – *If Γ is a discrete group having polynomial growth of degree D, then for all $\varepsilon \in (0,1)$, there exists a constant C such that, for all $w \in U$, $h \in \Gamma'$, and $g_l \in \Gamma/\Gamma'$*

$$\left|\nabla_w G^\mu(hg_l) - \nabla_w \mathcal{G}^\mu(h) - \sum_{j\leqslant n_1} \left(\nabla_w \psi^j(e)\right) \int_0^\infty X_j p_t^\mu(h)\,dt\right| \leqslant C|hg_l|^{1-D-\varepsilon}.$$

Proof. – We integrate the result of Theorem 3.2.

$$\begin{aligned}
&\left|\nabla_w G^\mu(hg_l) - \nabla_w \mathcal{G}^\mu(h) - \sum_{j\leqslant n_1} \left(\nabla_w \psi^j(e)\right) \int_0^\infty X_j p_t^\mu(h)\,dt\right| \\
&\quad\leqslant \sum_{n\geqslant 0}\left|\nabla_w \mu^n(hg_l) - \nabla_w p_n^\mu(h) - \sum_{j\leqslant n_1}\left(\nabla_w \psi^j(e)\right) X_j p_n^\mu(h)\right| \\
&\qquad+ \sum_{n\geqslant 0}\left|\nabla_w p_n^\mu(h) - \int_n^{n+1} \nabla_w p_t^\mu(h)\,dt\right| \\
&\qquad+ \sum_{j\leqslant n_1}\left(\nabla_w \psi^j(e)\right)\sum_{n\geqslant 0}\left|X_j p_n^\mu(h) - \int_n^{n+1} X_j p_t^\mu(h)\,dt\right|.
\end{aligned}$$

By Theorem 3.2, the first term of the right hand side can be easily bounded by $c|hg_l|^{1-D-\varepsilon}$. For the second term, using [15, Theorem VIII 2.7], we get

$$\begin{aligned}
&\sum_{n\geqslant 0}\left|\nabla_w p_n^\mu(h) - \int_n^{n+1} \nabla_w p_t^\mu(h)\,dt\right| \\
&\quad= \sum_{n\geqslant 0}\left|\int_n^{n+1}\int_n^t \left(-\nabla_w \frac{\partial}{\partial s} p_s^\mu(h)\right) ds\,dt\right| \\
&\quad\leqslant c\sum_{n\geqslant 0}\left|\int_n^{n+1}\int_n^t s^{-(D+3)/2}\exp\left(-\frac{|h|^2}{Cs}\right) ds\,dt\right| \\
&\quad\leqslant c'|h|^{-D-1}.
\end{aligned}$$

Likewise for the third term, we get

$$\begin{aligned}
&\sum_{n\geqslant 0}\left|X_j p_n^\mu(h) - \int_n^{n+1} X_j p_t^\mu(h)\,dt\right| \\
&\quad\leqslant c\sum_{n\geqslant 0}\left|\int_n^{n+1}\int_n^t s^{-(D+3)/2}\exp\left(-\frac{|h|^2}{Cs}\right) ds\,dt\right| \\
&\quad\leqslant c'|h|^{-D-1}.
\end{aligned}$$

This completes the proof. □

Remark 3.4. – When Γ is nilpotent and torsion-free, we can remove the term depending on the ψ^j's in the previous results.

Now, we consider the case where Γ is finite extension of $\mathbb{H}_3$, so $D = 4$. The associated Lie group N is $\mathbb{R}^3$ with the following product:

$$(x, y, z)(x', y', z') = \big(x + x', y + y', z + z' + (1/2)(xy' - x'y)\big).$$

Let $\widetilde{N}$ be also $\mathbb{R}^3$, but with the product:

$$(x, y, z)(x', y', z') = \big(x + x', y + y', z + z' - 2(xy' - x'y)\big).$$

We give this other definition of the Heisenberg group in order to use results from Folland [5] and Gaveau [6].

As $\Gamma/[\Gamma, \Gamma]$ is finitely generated and Abelian, $\Gamma/[\Gamma, \Gamma] \simeq \mathbb{Z}^2 \times I$, where I is finite. Let μ_1 be the canonical projection of μ on $\mathbb{Z}^2$. By assumption, μ_1 is centered with finite support U_1, which generates $\mathbb{Z}^2$. Let Q_1 be the symmetric positive definite matrix whose entries are the coefficients (q_{ij}) defined above. We define an isomorphism $\phi : N \to \widetilde{N}$:

$$\phi(x, y, z) = (ax + by, bx + cy, dz) \tag{3}$$

with $a = |Q_1|^{1/2} + q_{22}$, $b = -q_{12}$, $c = |Q_1|^{1/2} + q_{11}$, and $d = -4[2|Q_1| + |Q_1|^{1/2}(q_{11} + q_{22})]$. Let

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad \text{and} \quad X_3 = \frac{\partial}{\partial z}. \tag{4}$$

The set $X = \{X_1, X_2, X_3\}$ is a set of left-invariant vector fields, basis of the Lie algebra associated to N. We write $p_t^\mu(x, y, z)$ the fundamental solution in N of $(\frac{\partial}{\partial t} + L^\mu)f = 0$. The expression (2) of L^μ is

$$L^\mu = -q_{11}X_1^2 - q_{22}X_2^2 - q_{12}(X_1X_2 + X_2X_1) - q_3X_3.$$

Then, we check that $p_t^\mu \circ \phi^{-1}$ is the fundamental solution in $\widetilde{N}$ of the equation

$$\left(\frac{\partial}{\partial t} - \frac{\beta}{2}\Delta_K - \beta\delta\frac{\partial}{\partial z}\right)f = 0,$$

where

$$\beta = 4|Q_1|^{3/2} + 2|Q_1|(q_{11} + q_{22}) \quad \text{and} \quad \delta = -2q_3|Q_1|^{-1/2}$$

and Δ_K is the *Kohn Laplacian*, defined by

$$\Delta_K = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4y\frac{\partial^2}{\partial x\partial z} - 4x\frac{\partial^2}{\partial y\partial z} + 4\big(x^2 + y^2\big)\frac{\partial^2}{\partial z^2}.$$

Folland [5] and later Gaveau [6] have computed the explicit form of this solution, and then the expression of the associated Green function. We get

$$\begin{aligned}\widetilde{\mathcal{G}}^\mu(x,y,z) &= \mathcal{G}^\mu \circ \phi^{-1}(x,y,z) \\ &= \frac{[(x^2+y^2)^2+z^2]^{-1/2}}{\beta(1+\exp(-\delta\pi/2))} \exp\left(\frac{\delta}{2}\arctan\left(\frac{x^2+y^2}{z}\right)\right). \end{aligned}\tag{5}$$

Let $h=(h_1,h_2,h_3)$. We define a distance from the origin $|\cdot|_K$ on $\mathbb{H}_3$, called the *Koranyi distance*:

$$|h|_K = \left[\left(h_1^2+h_2^2\right)^2+h_3^2\right]^{1/4}. \tag{6}$$

This distance can be compared to the word distance on Γ (roughly-isometric to $\mathbb{H}_3$) since, by [9, Theorem 5.1] and (5), there exists a constant C such that, for all h in $\mathbb{H}_3$ and l,

$$C^{-1}|hg_l| \leqslant |h|_K \leqslant C|hg_l|,$$

where hg_l is the unique decomposition defined above. For positive real numbers α, M, we define also the subsets $\mathcal{S}(\alpha,M)$ of Γ by

$$\mathcal{S}(\alpha,M) = \left\{g = hg_l \in \Gamma\colon\ h_1^2+h_2^2 \geqslant \alpha|h_3|,\ c_1h_1+c_2h_2 < -M,\ \text{and}\ c_3h_1+c_4h_2 < -M\right\},$$

where the constants c_1, c_2, c_3, c_4 will be fixed below.

PROPOSITION 3.5. – *There is a positive constant $c = c(\alpha)$, such that for all $g \in \mathcal{S}(\alpha,0)$,*

$$\nabla G^\mu(g) \geqslant c|g|^{-3}.$$

Proof. – We write $\phi(h) = H = (H_1,H_2,H_3)$. Then

$$\begin{aligned}&\left(c^2+b^2\right)\frac{|Q_1|}{\beta^2}H_1^2 + \left(a^2+b^2\right)\frac{|Q_1|}{\beta^2}H_2^2 - 2b(a+c)\frac{|Q_1|}{\beta^2}H_1H_2 \\ &\quad = h_1^2+h_2^2 \geqslant \alpha|h_3| = \frac{\alpha}{|d|}|H_3|.\end{aligned}$$

It implies that there is a positive constant γ, which depends only on μ_1, such that

$$H_1^2+H_2^2 \geqslant \gamma\alpha|H_3|. \tag{7}$$

We write $\phi(w) = W = (W_1,W_2,W_3)$, and so $\phi(w)\phi(h) = WH = (H_1+W_1, H_2+W_2, H_3+W_3-2(W_1H_2-W_2H_1))$. We need to bound from below

$$\nabla_w\mathcal{G}^\mu(h) = \left|\widetilde{\mathcal{G}}^\mu\left(\phi(w)\phi(h)\right) - \widetilde{\mathcal{G}}^\mu\left(\phi(h)\right)\right|.$$

Using (7) and the boundedness of $\phi(U)$, we get

$$|H|_K^4 - |WH|_K^4 = 4\left(\left(H_1^2+H_2^2\right)(W_1H_1+W_2H_2) + H_3(W_2H_1-W_1H_2)\right) + \mathrm{O}\left(|H|_K^2\right),$$

and

$$\left(|H|_K^2+|WH|_K^2\right)|H|_K^2|WH|_K^2 = 2|H|_K^6\left(1+\mathrm{o}(1)\right),$$

where the o(1) tends to 0 when $|H|_K$ goes to infinity.

Let $u = (H_1, H_2)$ and $u' = (-H_2, H_1)$ be two vectors in $\mathbb{R}^2$. Let $\widetilde{W}_1$ and $\widetilde{W}_2$ be the coordinates of (W_1, W_2) in the basis $(u/\|u\|, u'/\|u'\|)$. Here, $\|.\|$ is the Euclidean norm on $\mathbb{R}^2$. Let $v = ((H_1^2 + H_2^2)/|H|_K^2, H_3/|H|_K^2)$ in the basis $(u/\|u\|, u'/\|u'\|)$. We get

$$|H|_K^{-2} - |WH|_K^{-2} = 2\big(H_1^2 + H_2^2\big)^{1/2}|H|_K^{-4}\langle\widetilde{W}, v\rangle\big(1 + \mathrm{o}(1)\big). \tag{8}$$

Likewise we get

$$\begin{aligned}&\arctan\left(\frac{H_1^2 + H_2^2}{H_3}\right) - \arctan\left(\frac{(H_1 + W_1)^2 + (H_2 + W_2)^2}{H_3 + W_3 - 2(W_1H_2 - W_2H_1)}\right)\\ &\quad = 2\big(H_1^2 + H_2^2\big)^{1/2}|H|_K^{-4}\big\langle\widetilde{W}, (\beta/2)v'\big\rangle\big(1 + \mathrm{o}(1)\big),\end{aligned} \tag{9}$$

with $v' = (-H_3/|H|_K^2, (H_1^2 + H_2^2)/|H|_K^2)$ in the basis $(u/\|u\|, u'/\|u'\|)$. So, (7), (8) and (9) yields

$$\nabla_w\mathcal{G}^\mu(h) \geqslant c\big(H_1^2 + H_2^2\big)^{1/2}|H|_K^{-4}\big|\big\langle\widetilde{W}, \big(v + (\beta/2)v'\big)\big(1 + \beta^2/4\big)^{-1/2}\big\rangle\big|. \tag{10}$$

At this point, we need a technical lemma. The set $\phi(U_1)$ is bounded and it generates $\mathbb{Z}^2$.

LEMMA 3.6. – *Let $K = \max_{W\in\phi(U_1)}\|W\|$. There is a strictly positive constant $C(K)$ such that for all W and $W' \neq 0$ in $\phi(U_1)$, non collinear, and for all $v \in V = \{v \in \mathbb{R}^2\colon \|v\| = 1\}$,*

$$\max\big\{\big|\langle W, v\rangle\big|; \big|\langle W', v\rangle\big|\big\} \geqslant C(K).$$

Proof. – We write $W = (W_1, W_2)$ and $W' = (W_1', W_2')$. We can take $\|W\| \geqslant \|W'\|$. As W and W' have integer coordinates, $\|W\| \geqslant 1$. Then

$$\max\big\{\big|\langle W, v\rangle\big|; \big|\langle W', v\rangle\big|\big\} \geqslant \max\big\{\big|\cos(\widehat{v, W})\big|; \big|\cos(\widehat{v, W'})\big|\big\}.$$

The infimum of the right term, over $v \in V$, is attained for

$$v = \frac{W/\|W\| - W'/\|W'\|}{\|(W/\|W\| - W'/\|W'\|)\|}.$$

So,

$$\begin{aligned}\big\{\max\big\{\big|\langle W, v\rangle\big|; \big|\langle W', v\rangle\big|\big\}\big\} &\geqslant \sin\big(\big|(\widehat{W, W'})\big|/2\big) \geqslant \frac{1}{2}\sin\big|(\widehat{W, W'})\big|\\ &\geqslant \frac{1}{2K^2}|W_1W_2' - W_1'W_2|.\end{aligned}$$

If $W_1 = 0$, then $W_1'W_2 \neq 0$, and if $W_1' = 0$, then $W_1W_2' \neq 0$. So, $|W_1W_2' - W_1'W_2| \geqslant 1$ is true in both cases. Now, suppose $W_1W_1' \neq 0$. As there is a finite number of vectors in $\phi(U_1)$, we get, under the hypothesis of the lemma,

$$\min_{W_1W_1'\neq 0}|W_2/W_1 - W_2'/W_1'| \geqslant c(K).$$

The lemma follows. □

As $\phi(U_1)$ contains more than two elements, by Lemma 3.6 and (10), we get for $|h|_K$ large enough,

$$\nabla\mathcal{G}^\mu(h) = \max_{W\in\phi(U_1)} \nabla_w\mathcal{G}^\mu(h) \geqslant c\big[\big(H_1^2+H_2^2\big)^2+H_3^2\big]^{-3/4} \geqslant c'|h|_K^{-3}. \tag{11}$$

Now, we compute $\int_0^\infty X_j p_t^\mu(h)\,dt$ for $j=1,2$. As $p_t^\mu(h)$ and $X_j p_t^\mu(h)$ are continuous in t and h (see [6] for an explicit formula of $p_t^\mu(h)$),

$$\int_0^\infty X_j p_t^\mu(h)\,dt = X_j\mathcal{G}^\mu(h).$$

We denote

$$A(h) = \frac{[(H_1^2+H_2^2)^2+H_3^2]^{-3/2}}{\beta(1+\exp(-\delta\pi/2))}\exp\left(\frac{\delta}{2}\arctan\left(\frac{H_1^2+H_2^2}{H_3}\right)\right)\big(H_1^2+H_2^2\big).$$

For $j=1$, using (3), (4), (5) and the definition of $\mathcal{S}(\alpha,M)$,

$$\begin{aligned}X_1\mathcal{G}^\mu(h) > -2A(h)\big[&\big(a^2+b^2\big)h_1+\big(b(a+c)-\delta/8\big)h_2\\ &+c(\alpha)\big|\big(a^2+b^2\big)\delta h_1+\big(b(a+c)\delta+1/4\big)h_2\big|\big],\end{aligned}$$

and

$$\begin{aligned}X_2\mathcal{G}^\mu(h) = -2A(h)\big[&\big(b(a+c)+\delta/8\big)h_1+\big(b^2+c^2\big)h_2\\ &+c(\alpha)\big|\big(b(a+c)\delta-1/4\big)h_1+\big(b^2+c^2\big)\delta h_2\big|\big],\end{aligned}$$

where the constant $c(\alpha)$ goes to 0 when α tends to infinity. So, $X_1\mathcal{G}^\mu(h)>0$ and $X_2\mathcal{G}^\mu(h)>0$ under the hypothesis

$$\begin{aligned}&\big(a^2+b^2\big)h_1+\big(b(a+c)-\delta/8\big)h_2\\ &\quad+c(\alpha)\big|\big(a^2+b^2\big)\delta h_1+\big(b(a+c)\delta+1/4\big)h_2\big|<0\end{aligned}$$

and

$$\begin{aligned}&\big(b(a+c)+\delta/8\big)h_1+\big(b^2+c^2\big)h_2\\ &\quad+c(\alpha)\big|\big(b(a+c)\delta-1/4\big)h_1+\big(b^2+c^2\big)\delta h_2\big|<0.\end{aligned}$$

The two lines of $\mathbb{R}^2$ defined by $(a^2+b^2)h_1+(b(a+c)-\delta/8)h_2=0$ and $(b(a+c)+\delta/8)h_1+(b^2+c^2)h_2=0$ are not parallel because $(a^2+b^2)(b^2+c^2)-b^2(a+c)^2+\delta^2/64=|Q_1|+\delta^2/64>0$. So, for α large enough, there exist constants c_1,c_2,c_3,c_4 such that, for $|h|_K$ large enough, the above hypothesis is satisfied when

$$c_1h_1+c_2h_2<0,\quad\text{and}\quad c_3h_1+c_4h_2<0.$$

Finally, as $\nabla_w\psi^j(e)>0$ for all w and j (by definition of ∇_w), we get

$$\nabla_w\psi^1(e)X_1\mathcal{G}^\mu(h)+\nabla_w\psi^2(e)X_2\mathcal{G}^\mu(h)>0.$$

Then the result follows from Corollary 3.3 and (11). □

When Γ is a finite extension of $\mathbb{Z}^d$, then the associated Lie group is $\mathbb{R}^d$ with the usual additive law. So, up to an isomorphism, with $X_j = \partial_j$, L^μ is the usual Laplacian and then $p_t^\mu(h)$ is the fundamental solution of the usual d-dimensional heat equation. If we write h in an appropriated basis depending on the covariance matrix Q_1 associated to μ, then, as soon as $h_j < 0$, we get

$$\int_0^\infty X_j p_t^\mu(h) > 0.$$

Using Corollary 3.3 and the well-known estimate $\nabla \mathcal{G}^\mu(h) \geqslant c\|h\|^{1-d}$ (see [11] for instance), we get

$$\nabla \mathcal{G}^\mu(g) \geqslant c\|g\|^{1-d}$$

as soon as $g_j < 0$ for all j. Here $\|\cdot\|$ denotes the Euclidean norm.

4. Cut times for random walks on $\mathbb{H}_3$

Following the introduction, we study the occurrence of cut times for random walks with finite support, which generates $\mathbb{H}_3$. Our result is the following

THEOREM 4.1. – *Let $S(k)$ be a random walk on Γ, a finite extension of $\mathbb{H}_3$ such that the transition probability $\mu(x)$ has a finite support and generates Γ. Then $S(k)$ has infinitely many cut times, almost surely.*

We adapt the proof given in [11] for $\mathbb{Z}^4$. For simplicity, we remove the exponent μ for the Green function.

By translation invariance, we can take $S(0) = e$, the identity of Γ. We denote $G(x) = G(e, x)$ the Green function on Γ associated to μ, and U is the finite support of μ, which is also a finite generating set of Γ. We define a sequence of spheres $\partial B(n)$ which can be seen as "level lines" of $G(x)$ by setting

$$B(n) = \{x \in \Gamma \colon\ G(x) \geqslant (\delta/n)^2\}, \tag{12}$$
$$\partial B(n) = \{x \in B(n) \colon\ xU \text{ contains some } y \notin B(n)\}.$$

Here, δ is a suitable constant such that the following lemma is true. For completeness, we write its short proof taken from [11].

LEMMA 4.2 [11, Lemma 2.3]. – *Let $r(n) = (\delta/n)^2$ and $R(n) = (1 + 1/n)(\delta/n)^2$. Then for all $n \geqslant 1$ and $x \in \partial B(n)$, the Green function satisfies $r(n) \leqslant G(x) \leqslant R(n)$.*

Proof. – Since $x \in B(n)$, the definition of $B(n)$ gives the lower bound. As $x \in \partial B(n)$, it has a neighbor $y \notin B(n)$. So, [9, Theorem 5.1] and (12) yield $(c_1|y|)^{-2} \leqslant G(y) < (\delta/n)^2$, hence $|y| > n/(c_1\delta)$. Again by [9, Theorem 5.1], we get $\nabla G(y) \leqslant (\delta c_1 c_2/n)^3$. Finally,

$$G(x) \leqslant G(y) + \nabla G(y) \leqslant \left(1 + \frac{\delta(c_1c_2)^3}{n}\right)\left(\frac{\delta}{n}\right)^2 \leqslant R(n),$$

as soon as $\delta \leqslant (c_1c_2)^{-3}$. $\square$

The aim is to prove that $\partial B(n)$ is a cut sphere for infinitely many n, almost surely. That implies the existence of infinitely many cut times, almost surely.

DEFINITION 4.3. – *The sphere $\partial B(n)$ is a* cut sphere *for $S(k)$ if*

$$S\big([\tau(n)+1,\infty)\big)\cap B(n)=\emptyset,$$

where $\tau(n)=\inf\{k\geqslant 1\colon\ S(k)\in\partial B(n)\}$.

In [11], James and Peres need to be on $\mathbb{Z}^4$ to use a lower bound for the gradient of the Green function of the type

$$\nabla G(x)\geqslant C d(x)^{-3}, \tag{13}$$

where C is some positive constant and $d(x)$ the smallest radius of a ball $B(n)$ containing x. Such an inequality cannot hold in general. For instance on $\mathbb{H}_3$, taking $x=(0,0,z)$ with $z>0$, leads to $d(x)\leqslant c\sqrt{z}$ by looking at the Koranyi distance. With (5), we can compute $\nabla\mathcal{G}(x)$ as in the proof of Proposition 3.5 and get, for z large enough, $\nabla\mathcal{G}(x)\leqslant c'z^{-2}\leqslant c''d(x)^{-4}$. Therefore, by Corollary 3.3 and Remark 3.4, for all $\varepsilon\in(0,1)$, there exists a constant c such that

$$\nabla G(x)\leqslant c d(x)^{-4+\varepsilon}.$$

On Γ, by [9, Theorem 5.1] and Proposition 3.5, the inequality (13) is satisfied for $x\in\mathcal{S}(\alpha,0)$ with a constant C depending on α. It will appear to be sufficient to prove Theorem 4.1.

Now, we use (13) to build a path of bounded length from a point in $\partial B(n)\cap\mathcal{S}(2\alpha,M)$ (with M large enough) to $\partial B(n+1)\cap\mathcal{S}(\alpha,0)$.

LEMMA 4.4. – *There exist three constants J, N_0 and M_0 such that for every $n\geqslant N_0$, $M\geqslant M_0$, and for any $x\in\partial B(n)\cap\mathcal{S}(2\alpha,M)$, there is a finite path $x_0=x,x_1,\ldots,x_j$ of length $j\leqslant J$ that ends in $\partial B(n+1)\cap\mathcal{S}(\alpha,0)$ and such that $x_i\in x_{i-1}U$ and $x_i\notin B(n)$ for $i\geqslant 1$.*

Proof. – By the definition of $\partial B(n)$, there exists $x_1\in xU$ with $x_1\notin B(n)$. Once x_i is defined, we choose $x_{i+1}\in x_iU$ such that $G(x_{i+1})=\min_{z\in x_iU}G(z)$. Since G is harmonic on $\Gamma\backslash\{e\}$, the sequence $\{G(x_i)\}$ is strictly decreasing. As G vanishes at infinity, we can choose j such that $G(x_j)\geqslant r(n+1)>G(x_{j+1})$. We have

$$j\cdot\min_{i<j}\big[G(x_i)-G(x_{i+1})\big]\leqslant G(x_1)-G(x_j).$$

By Lemma 4.2,

$$G(x_1)-G(x_j)\leqslant R(n)-r(n+1)\leqslant C_1 n^{-3}$$

for some constant C_1. Moreover, by definition of ∇G and harmonicity of G, we get that there is a constant C_0 such that

$$G(x_i)-G(x_{i+1})\geqslant C_0\nabla G(x_i).$$

By (13), for all $x \in \mathcal{S}(\alpha, 0) \cap (B^c(n) \cup \partial B(n))$, there exists a constant C_2 such that

$$\nabla G(x) \geqslant C_2 n^{-3}.$$

We write $J = C_0C_2/C_1$. At a bounded distance of $x = (a_1, a_2, a_3)$, the third coordinate cannot change of more than a $\mathrm{O}(\sqrt{a_1^2 + a_2^2})$. In the same time, $a_1^2 + a_2^2$ changes of a bounded value. As $x \in \mathcal{S}(2\alpha, M)$, for n and M large enough, we have $x_i \in \mathcal{S}(\alpha, 0)$ for all $i \leqslant J$, and the result follows. □

From this lemma and the fact that U is bounded, we get that, for all $x \in \partial B(n) \cap \mathcal{S}(2\alpha, M)$, with n large enough,

$$\mathbf{P}^x\{S(k) \text{ visits } \partial B(n+1) \text{ before } \partial B(n)\} \geqslant p_{\min}^J$$

where $p_{\min} = \min_{y \in U} \mathbf{P}\{S(1) = y\}$.

From [11], we get

LEMMA 4.5 [11, Lemma 2.4]. – *There exist a real C and an integer N_0 such that for all n, m $(N_0 \leqslant n < m)$ and all starting point $x \in \partial B(n+1)$,*

$$\frac{m}{Cn(m-n)} \leqslant \mathbf{P}^x\{S(k) \textit{ visits } \partial B(m) \textit{ before } \partial B(n)\} \leqslant \frac{Cm}{n(m-n)}$$

and consequently for all $x \in \partial B(n+1)$,

$$\frac{1}{Cn} \leqslant \mathbf{P}^x\{S(k) \textit{ never visits } \partial B(n)\} \leqslant \frac{C}{n}.$$

We define new spheres $\partial' B(n) \subset \partial B(n)$ as the "internal part" of $\partial B(n)$. Namely,

$$\partial' B(n) = \{x \in \partial B(n)\colon\ xU \text{contains some } y \in B(n) \backslash \partial B(n)\}.$$

We need a lower bound for the harmonic measure:

LEMMA 4.6. – *There exists a constant K such that for every set $\mathcal{D} \in \partial' B(n) \cap \mathcal{S}(2\alpha, M)$*

$$h(\partial' B(n), \mathcal{D}) \geqslant K(\#\mathcal{D})n^{-3},$$

where $h(\partial' B(n), .)$ is the hitting probability of $\partial' B(n)$.

Proof. – We only need to prove the statement for $\mathcal{D} = \{z\}$ and conclude for a general $\mathcal{D}$ by summing over all its elements. We denote $\sigma(n) = \tau(n) \wedge \tau_e$, where τ_e is the first hitting time of e. We define the *stopped Green function*:

$$\forall x, y \in B(n), \quad G_n(x, y) = \sum_{k \geqslant 0} \mathbf{P}^x\{S(k) = y,\ k \leqslant \tau(n)\}.$$

By the same argument used in Lemma 4.4, there is a path of length at most J from any point in $\partial B(n) \cap \mathcal{S}(2\alpha, M)$ to $\partial B(n-1) \cap \mathcal{S}(\alpha, 0)$. So, if we denote $\widetilde{S}$ the reversed random walk,

$$\begin{aligned} h\big(\partial' B(n), \{z\}\big) &= \mathbf{P}\big\{S\big(\tau(n)\big) = z\big\} \\ &\geqslant \mathbf{P}\big\{S\big(\sigma(n)\big) = z\big\} = \mathbf{P}^z\big\{\widetilde{S}\big(\sigma(n)\big) = e\big\} \\ &\geqslant p_{\min}^J \min_{y \in \partial B(n-1) \cap \mathcal{S}(\alpha,0)} \mathbf{P}^y\big\{\widetilde{S}\big(\sigma(n)\big) = e\big\}. \end{aligned}$$

But

$$\widetilde{G}_n(y, e) = \widetilde{G}_n(e, e)\mathbf{P}^y\big\{\widetilde{S}\big(\sigma(n)\big) = e\big\}.$$

And, for $y \in \partial B(n-1)$, by Lemma 4.2

$$\begin{aligned} \widetilde{G}_n(y, e) &= \widetilde{G}(y, e) - \mathbb{E}^y\big\{\widetilde{G}\big(\widetilde{S}\big(\tau(n)\big), e\big)\big\} \\ &= G(e, y) - \mathbb{E}^y\big\{G\big(e, \widetilde{S}\big(\tau(n)\big)\big)\big\} \\ &\geqslant \left(\frac{\delta}{n-1}\right)^2 - \left(1 + \frac{1}{n}\right)\left(\frac{\delta}{n}\right)^2 \geqslant \left(\frac{\delta}{n}\right)^3. \end{aligned}$$

Hence, as $\widetilde{G}_n(e, e) \leqslant \widetilde{G}(e, e)$, we get for some constant K

$$h\big(\partial' B(n), \{z\}\big) \geqslant K n^{-3}. \qquad \square$$

LEMMA 4.7. – *There exists constants c_1 and $\gamma > 1$ such that for any p large enough,*

$$\sum_{n=\gamma^p}^{\gamma^{p+1}} \mathrm{Vol}\big(\partial' B(n) \cap \mathcal{S}(2\alpha, M)\big) \geqslant c_1 \gamma^{4p}. \tag{14}$$

Proof. – Let $\widetilde{B}(n)$ be the balls of $\widetilde{N}$ (the Lie group associated to Γ, defined in Section 3) defined like $B(n)$ in the discrete setting. We also define the balls $\widetilde{B}_\Gamma(n)$ on Γ as follows:

$$\widetilde{B}_\Gamma(n) = \big\{g = hg_l \in \Gamma \colon\ h \in \widetilde{B}(n)\big\}.$$

Here, we identify the elements of $\mathbb{H}_3$ and $\widetilde{N}$. As Γ is a finite extension of $\mathbb{H}_3$ and $\nabla G(g) \leqslant c|g|^{-3}$ (see [9]), there is a constant C such that for all h and l,

$$C^{-1} G(h) \leqslant G(hg_l) \leqslant C G(h).$$

Moreover, the two Green functions, in Γ and in $\widetilde{N}$, are of order $|h|_K^{-2}$ by [9, Theorem 5.1] and (5). So, there is a constant $\gamma > 1$ such that for n large enough,

$$B(n) \subset \widetilde{B}_\Gamma\big((\gamma + 2)n/3\big) \subset \widetilde{B}_\Gamma\big((2\gamma + 1)n/3\big) \subset B(\gamma n).$$

As we know the exact expression (5) of the Green function $\widetilde{\mathcal{G}}$ in $\widetilde{N}$, we deduce that

$$\begin{aligned} &\mathrm{Vol}\big(\big[B(\gamma n) \backslash B(n)\big] \cap \mathcal{S}(4\alpha, 2M)\big) \\ &\quad \geqslant c\, \mathrm{Vol}\big(\big[\widetilde{B}\big((\gamma + 2)n/3\big) \backslash \widetilde{B}\big((\gamma + 1)n/3\big)\big] \cap \mathcal{S}(4\alpha, 2M)\big) \geqslant c' n^4, \end{aligned}$$

for some constants c and c'. The same argument used in Lemma 4.4 gives that, for all n large enough and for each point y in $[B(n+1) \backslash B(n)] \cap \mathcal{S}(4\alpha, 2M)$, there is a path from

y to $\partial' B(n) \cap \mathcal{S}(2\alpha, M)$ of length at most J. And, as the random walk has finite support, there exists a constant ν such that, each point has at most ν accessible points. So,

$$\begin{aligned}\mathrm{Vol}\big(\big[B(n+1)\backslash B(n)\big] \cap \mathcal{S}(4\alpha, 2M)\big) &\leqslant \#\big\{y\colon \exists x \in \partial' B(n) \cap \mathcal{S}(2\alpha, M),\ \big|x^{-1}y\big| \leqslant J\big\} \\ &\leqslant \nu^J \,\mathrm{Vol}\big(\partial' B(n) \cap \mathcal{S}(2\alpha, M)\big).\end{aligned}$$

So, if we denote $c_J = c'/\nu^J$, for any p large enough,

$$\sum_{n=\gamma^p}^{\gamma^{p+1}} \mathrm{Vol}\big(\partial' B(n) \cap \mathcal{S}(2\alpha, M)\big) \geqslant c_1 \gamma^{4p}. \qquad \square$$

We will now complete the proof of Theorem 4.1. We define the event Cut_n by

$$\mathrm{Cut}_n = \big\{\partial B(n) \text{ is a cut sphere and } S\big(\tau(n)\big) \in \mathcal{S}(2\alpha, M)\big\}.$$

By Lemma 4.5, we get

$$\begin{aligned}\mathbf{P}\{\mathrm{Cut}_n\} &\leqslant \mathbf{P}\big\{\partial B(n) \text{ is a cut sphere}\big\} \\ &\leqslant \max_{x \in \partial B(n+1)} \mathbf{P}^x\big\{S(k) \text{ never visits } \partial B(n)\big\} \leqslant \frac{C}{n}.\end{aligned}$$

Then, for $N_0 \leqslant n < m$,

$$\begin{aligned}\mathbf{P}\{\mathrm{Cut}_n \cap \mathrm{Cut}_m\} &\leqslant \max_{x \in \partial B(n+1)} \mathbf{P}^x\big\{S(k) \text{ visits } \partial B(m) \text{ before } \partial B(n)\big\} \\ &\quad \times \max_{y \in \partial B(n+1)} \mathbf{P}^y\big\{S(k) \text{ never visits } B(m)\big\} \\ &\leqslant \frac{Cm}{n(m-n)} \frac{C}{m} \leqslant \frac{C^2}{n(m-n)}.\end{aligned}$$

So,

$$\sum_{n,m=N_0}^{N} \mathbf{P}\{\mathrm{Cut}_n \cap \mathrm{Cut}_m\} \leqslant 2 \sum_{N_0 \leqslant n < m \leqslant N} \frac{C^2}{n(m-n)} + \sum_{n=N_0}^{N} \frac{C}{n} \leqslant C' \ln N,$$

for some constant C'. By Lemmata 4.4 and 4.5, we get

$$\mathbf{P}\big\{\mathrm{Cut}_n \mid S\big(\tau(n)\big) \in \mathcal{S}(2\alpha, M)\big\} \geqslant \frac{p_{\min}^J}{n}.$$

So, Lemma 4.6 implies

$$\begin{aligned}\mathbf{P}\{\mathrm{Cut}_n\} &= \mathbf{P}\big\{\mathrm{Cut}_n \mid S\big(\tau(n)\big) \in \mathcal{S}(2\alpha, M)\big\}\mathbf{P}\big\{S\big(\tau(n)\big) \in \mathcal{S}(2\alpha, M)\big\} \\ &\geqslant c\frac{\mathrm{Vol}(\partial' B(n) \cap \mathcal{S}(2\alpha, M))}{n^4}.\end{aligned}$$

Moreover,

$$\sum_{n=N_0}^{N} \mathbf{P}\{\mathrm{Cut}_n\} \geqslant \sum_{k=K_0}^{K} \sum_{n=\gamma^k}^{\gamma^{k+1}} \mathbf{P}\{\mathrm{Cut}_n\},$$

with $K_0 = [\log_\gamma N_0] + 1$ and $K = [\log_\gamma N] - 1$. Hence, by Lemma 4.7,

$$\begin{aligned}\sum_{n=N_0}^{N} \mathbf{P}\{\mathrm{Cut}_n\} &\geqslant c \sum_{k=K_0}^{K} \sum_{n=\gamma^k}^{\gamma^{k+1}} \frac{\mathrm{Vol}(\partial' B(n) \cap \mathcal{S}(2\alpha, M))}{n^4} \\ &\geqslant c \sum_{k=K_0}^{K} \gamma^{4(k+1)} \sum_{n=I^k}^{I^{k+1}} \mathrm{Vol}\big(\partial' B(n) \cap \mathcal{S}(2\alpha, M)\big) \\ &\geqslant C'' \ln N,\end{aligned}$$

for some constant C''. By the Kochen–Stone Lemma [12], there exists a strictly positive constant ε such that,

$$\mathbf{P}\{\mathrm{Cut}_n \text{ i.o.}\} \geqslant \limsup_{N\to\infty} \frac{(\sum_{n=1}^{N} \mathbf{P}\{\mathrm{Cut}_n\})^2}{\sum_{n,m=1}^{N} \mathbf{P}\{\mathrm{Cut}_n \cap \mathrm{Cut}_m\}} \geqslant \varepsilon > 0.$$

Now, if we fix the first k steps $(e, x_1, \ldots, x_k)$ of the random walk, we can choose N_0 large enough so that we still have

$$\mathbf{P}\big\{\mathrm{Cut}_n \text{ i.o.} \mid S(0) = e, \ldots, S(k) = x_k\big\} \geqslant \varepsilon > 0.$$

But, when n tends to infinity, the left hand side tends to $\mathbb{1}\{\mathrm{Cut}_n \text{ i.o.}\}$. Therefore we obtain $\mathbf{P}\{\mathrm{Cut}_n \text{ i.o.}\} = 1$, which implies the result.

The same argument works when Γ is a finite extension of $\mathbb{Z}^d$ using

$$\mathcal{S}(M) = \{g = hg_l \in \Gamma \colon \forall j, h_j < -M\},$$

where (h_j) are the coordinates of h in a basis depending on the covariance matrix Q_1 associated to μ. The use of the constant M appears, as for the extensions of $\mathbb{H}_3$, in the proof of Lemma 4.4.

By Proposition 2.5, when $D = 3$, Γ is isomorphic to a finite extension of $\mathbb{Z}^3$. When $D = 4$, Γ is isomorphic to a finite extension of $\mathbb{Z}^4$ or $\mathbb{H}_3$. Thus, Theorem 4.1 together with [11] and the remark in the introduction about non-centered and centered (but non-symmetric) random walks, leads to the following corollary.

COROLLARY 4.8. – *Every transient random walk, with finite support on a finitely generated group, has infinitely many cut times, almost surely.*

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