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HOW A CENTRED RANDOM WALK ON THE AFFINE GROUP GOES TO INFINITY

COMMENT UNE MARCHE ALÉATOIRE CENTRÉE SUR LE GROUPE AFFINE TEND VERS L'INFINI

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ABSTRACT. – We consider the processes obtained by (left and right) products of random i.i.d. affine transformations of the Euclidean space $\mathbb{R}^d$. Our main goal is to describe the geometrical behavior at infinity of the trajectories of these processes in the critical case when the dilatation of the random affinities is centred. Then we derive a proof of the uniqueness of the invariant Radon measure for the Markov chain induced on $\mathbb{R}^d$ by the left random walk and prove a stronger property of divergence for the discrete time process on $\mathbb{R}^d$ induced by the right random walk.

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RÉSUMÉ. – On considère le processus obtenu par produit (à droite et à gauche) de transformations affines aléatoires de l'espace Euclidien $\mathbb{R}^d$, indépendantes et de même loi. Notre fin principale est de décrire le comportement geometrique à l'infini des trajectoires de ces processus, lors que la dilatation des affinités alátoires est centrée. On en déduit une nouvelle démonstration de l'unicité de la mesure de Radon invariante pour la chaîne de Markov induite sur $\mathbb{R}^d$ par la marche aléaroire gauche et on démontre une plus forte propriété de divergence pour le processus à temps discret induit par la marche aléatoire droite.

We consider the group $\mathrm{Aff}(\mathbb{R}^d)$ of affine transformations of the space $\mathbb{R}^d$:

$$(a, b) : x \mapsto ax + b$$

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where a is a positive real number and b a vector of $\mathbb{R}^d$. Let $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ be a sequence of random independent and identically distributed affine transformations. We are interested in the behavior of their composition products, that is in the behavior of the right and left random walks on $\mathrm{Aff}(\mathbb{R}^d)$:

$$R_n = (A_1, B_1)\cdots(A_n, B_n) \quad \text{and} \quad L_n = (A_n, B_n)\cdots(A_1, B_1).$$

We will identify the group $\mathrm{Aff}(\mathbb{R}^d)$ with the half-space $\mathbb{H} = \mathbb{R}_*^+ \times \mathbb{R}^d$. General results on random walks ensure that random walks on the affine group are transient, so that accumulation points of the trajectories are on the boundary $\partial\mathbb{H} = \mathbb{R}^d \cup \{\infty\}$ of the geometrical compactification of $\mathbb{H}$. In particular, it is known and easy to show that if the mean of the logarithm of the component on $\mathbb{R}_+^*$ is positive, then the random walks converge to ∞, while, if this mean is negative, the right random walk converges to a random element of the boundary different from ∞. Although random walks on this group are well studied (e.g. Kesten [9], Grincevicius [7], Élie [3], Le Page and Peigné [10], Goldie and Maller [6]), it was not yet known what happens in the so-called “critical case” when the projection on $\mathbb{R}_+^*$ is recurrent. We will prove, under a weak moment hypothesis, but without supposing that the step distribution has a density or is spread out, that a centred right random walk converges to the point ∞. The argument is inspired by an analogous result on the affine group of a tree obtained by Cartwright, Kaimanovich and Woess [2].

We will then apply this result to the study of the Markov chain induced by the left random walk on $\mathbb{R}^d$, that is the process $\{Y_n^y\}_n$ defined recursively by the sequence of i.i.d. random variables $\{(A_n, B_n)\}_n$ as:

$$Y_{n+1}^y = A_{n+1} Y_n^y + B_{n+1}, \quad Y_0^y = y.$$

Besides its intrinsic interest, this process, also known as first order random coefficient auto-regressive model, has various applications (especially in economy and biology, see for instance Engle and Bollerslev [4], Nicholls and Quinn [11], Goldie [5]). M. Babillot, Ph. Bougerol and L. Elie have already shown in [1] that even when the coefficients A_n are centred, that is $\mathbb{E}[\log(A_n)] = 0$, the trajectories of this process satisfy a property that may be seen as a global stability at finite distance or as a local contraction, that is

$$|Y_n^x - Y_n^y| \mathbf{1}_K(Y_n^x) \to 0 \quad \text{as } n \to +\infty$$

almost surely for all compact subset K of $\mathbb{R}^d$. As they had noticed, this property is related to the uniqueness of an invariant Radon measure. We will show that the local contraction property is a straightforward geometrical consequence of the convergence of the right random walk to ∞, and we will give a proof of the uniqueness of the Radon invariant measure for the chain $\{Y_n\}_n$ via the Chacon–Ornstein theorem, thus correcting an error in [1].

In the last section we will look at the projection of the right random walk on $\mathbb{R}^d$, that is the series

$$Z_n^{(a,b)} = b + a\sum_{k=1}^{n} A_1\cdots A_{k-1}B_k. \tag{1}$$

Using a stronger moment hypothesis and a density condition for the marginal on $\mathbb{R}^d$, we will prove that Z_n^g is transient, in the sense that almost surely $\lim_n |Z_n^g| = +\infty$. Although Z_n^g is not Markov, it is of some interest for various problems. For instance if we consider the continuous time process $\widetilde{Z}_t = \int_0^t \mathrm{e}^{W_s}\, dB_s$, where W_t and B_t are two independent Brownian motions, and we look at it at integer times we obtain a series of the type (1). $\widetilde{Z}$ is a well known economic model (cf. for instance [13] and [14]) and it is easy to show that $\underline{\lim}_{t\to\infty} \widetilde{Z}_t = -\infty$ and $\overline{\lim}_{t\to\infty} \widetilde{Z}_t = +\infty$ so that, being continuous, $\widetilde{Z}$ has to visit infinitely often every open set of $\mathbb{R}$. The result of the last section implies that every discretization of time leads to a transient process and shows therefore that the recurrence of this model is not very robust.

We can remark that the results of Section 2 and Section 3 are still valid, and can be proved exactly with the same techniques, for a random walk on the group of affine conformal transformations, that is when the variables A_n are not real positive numbers but, more generally, matrices that live in a group that is direct product of $\mathbb{R}_+^*$ and a compact subgroup of $GL(\mathbb{R}^d)$.

1. Notation and hypotheses

We will denote by $\mathrm{Aff}(\mathbb{R}^d)$ the group of affine transformations of the Euclidean space $\mathbb{R}^d$, that is transformations of the form $x \mapsto ax + b$ with a a positive number and b a vector in $\mathbb{R}^d$; thus $\mathrm{Aff}(\mathbb{R}^d)$ may be identified with the hyperbolic half-space $\mathbb{H} = \mathbb{R}_*^+ \times \mathbb{R}^d$. We will denote by a and b the projections of $\mathrm{Aff}(\mathbb{R}^d)$ on $\mathbb{R}_+^*$ and $\mathbb{R}^d$ respectively, so that $g = (a(g), b(g))$ for each $g \in \mathrm{Aff}(\mathbb{R}^d)$.

Adding a sphere at infinity leads to the geometrical compactification of $\mathbb{H}$, where the boundary of $\mathbb{H}$ is $\partial\mathbb{H} = \mathbb{R}^d \cup \{\infty\}$. The group of hyperbolic isometries of $\mathbb{H}$ that fix the point ∞ is nothing but the group of affine conformal transformations which contains $\mathrm{Aff}(\mathbb{R}^d)$ as a subgroup. The action of $\mathrm{Aff}(\mathbb{R}^d)$ on $\partial\mathbb{H} - \{\infty\}$ is then the canonical action on $\mathbb{R}^d$ and will be denoted by

$$g \cdot x = a(g)x + b(g).$$

We will sometimes use the fact that the action and the projection on $\mathbb{R}^d$ coincide, in the sense that $g \cdot x = b(gh)$ for every $h = (a, x) \in \mathrm{Aff}(\mathbb{R}^d)$.

For the composition of two affinities we have the identity

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$$

so that, from an algebraic point of view, $\mathrm{Aff}(\mathbb{R}^d)$ is a semi-direct product of $\mathbb{R}_+^*$ and $\mathbb{R}^d$.

We consider a sequence $X_n = (A_n, B_n)$ of random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathrm{Aff}(\mathbb{R}^d)$, that we suppose independent and identically distributed with distribution μ. The right and left random walks with law μ are the Markov chains on $\mathrm{Aff}(\mathbb{R}^d)$ defined respectively by

$$R_{n+1} = R_n X_{n+1}, \quad R_0 = \mathbf{1}$$

and

$$L_{n+1} = X_{n+1} L_n, \quad L_0 = \mathbf{1}.$$

For a fixed n, R_n and L_n are both distributed as $\mu^{(n)}$, the n-th convolution power of μ. The expected number of visits of these random walks in a Borel set $B \subset \mathbb{H}$ is given by

$$U(B) = \sum_{n=0}^{+\infty} \mathbb{P}[R_n \in B] = \sum_{n=0}^{+\infty} \mathbb{P}[L_n \in B]$$

that is by the potential measure $U = \sum_{n=0}^{+\infty} \mu^{(n)}$. The potential kernels of the right and left random walk give the expected numbers of visits when the random walks start from a generic point $g \in \mathrm{Aff}(\mathbb{R}^d)$, and are respectively given by $U^r \mathbf{1}_B(g) = \delta_g * U(B)$ and $U^l \mathbf{1}_B(g) = U * \delta_g(B)$.

We observe that

$$a(R_n) = a(L_n) = A_1 \cdots A_n$$

is a classical multiplicative random walk on $\mathbb{R}_+^*$.

We will suppose that the random walk is non-degenerate in the sense that there exists no $y \in \mathbb{R}^d$ fixed by X_1 and that we are really dealing with affinities and not just with translations, i.e.

$$\forall y \in \mathbb{R}^d: \quad \mathbb{P}[X_1 \cdot y = y] < 1 \quad \text{and} \quad \mathbb{P}\big[a(X_1) = 1\big] < 1. \tag{H1}$$

Under this hypothesis the closed group generated by the support of μ is non-unimodular, thus the random walks are transient and the expected number of visits, $U(K)$, in every compact set K is finite (cf. [8]).

We will not need, at least in the first part, any density hypothesis but only a weak moment condition that is

$$\mathbb{E}\big[\big|\log\big(a(X_1)\big)\big|\big] < +\infty \quad \text{and} \quad \mathbb{E}\big[\log^+\big|b(X_1)\big|\big] < +\infty. \tag{H2}$$

As announced, we will only consider the most critical case for which the random walk projected on $\mathbb{R}_+^*$ is recurrent, that is

$$\mathbb{E}\big[\log\big(a(X_1)\big)\big] = 0. \tag{H3}$$

2. Convergence to infinity

In this section we shall prove:

THEOREM 1. – *Under the hypotheses* (H1), (H2) *and* (H3), *almost surely for every* $g \in \mathrm{Aff}(\mathbb{R}^d)$:

$$\lim_{n \to +\infty} g R_n = \infty \in \partial \mathbb{H}$$

in the hyperbolic topology of $\mathbb{H}$.

To prove that gR_n converges to ∞ is equivalent to show that the sequence gR_n is definitely (i.e. for any sufficiently large n) in every neighborhood of ∞, or equivalently that every set of the form $C_{s,t} = \{g \in \mathrm{Aff}(\mathbb{R}^d)\colon\ a(g) < s \text{ and } |b(g)| < t\}$, with s and t real positive numbers, is transient. On the other hand we will see that the left random walk visits this set infinitely often, so that the potential measure of $C_{s,t}$ is infinite. Therefore the number of visits of gR_n to $C_{s,t}$ is almost surely finite, but has infinite expectation. The transience of the set $C_{s,t}$ for the right random walk is thus a quite subtle phenomenon, and instead to prove it directly we will show, following the ideas of [2], that the right random walk can cross the border between $C_{s,t}$ and its complement only a finite numbers of times.

PROPOSITION 1. – *Suppose that the hypotheses* (H1) *and* (H2) *are verified. Then if* $C = \{g \in \mathrm{Aff}(\mathbb{R}^d)\colon\ a(g) < 1 \text{ and } |b(g)| < 1\}$

$$\mathbb{P}[gR_{n+1} \in C,\ gR_n \notin C \text{ infinitely often}] = 0 \tag{2}$$

for almost all $g \in \mathrm{Aff}(\mathbb{R}^d)$ *with respect to the Haar measure.*

This proposition cannot be proven as in [2] because in our case the group does not act on a discrete space, such as the tree. So we have to find a different way and we will need the following lemma that estimates the potential of integrable functions on a locally compact group.

LEMMA 1. – *Let* U^r *be the potential kernel of a transient right random walk on a locally compact second countable group* G *and* dg *the right Haar measure of* G. *Then for every non-negative function* $f \in L^1(dg)$, $U^r f(g)$ *is* dg*-almost surely finite.*

Proof. – We will show that $U^r f$ is locally dg-integrable and thus necessarily dg-almost surely finite. If μ is the distribution of the random walk, we recall that potential measure is $U = \sum_{n=0}^{+\infty} \mu^{(n)}$. Let K be a compact set of G. Then by the right invariance of dg:

$$\begin{aligned}
\int_G U^r f(g_1)\mathbf{1}_K(g_1)\,dg_1 &= \int_G \int_G f(g_1 g_2)\mathbf{1}_K(g_1)U(dg_2)\,dg_1 \\
&= \int_G \int_G f(g_1)\mathbf{1}_K\bigl(g_1 g_2^{-1}\bigr)\,dg_1 U(dg_2) \\
&= \int_G f(g_1)\check{U}^r\mathbf{1}_K(g_1)\,dg_1 \\
&\leqslant \sup_{g\in G} \check{U}^r\mathbf{1}_K(g) \int_G f(g_1)\,dg_1
\end{aligned}$$

where $\check{U}^r$ is the potential kernel of the right random walk whose law $\check{\mu}$ is the image of μ under the map $g \mapsto g^{-1}$. The transience of the random walk with law $\check{\mu}$ follows from the transience of random walk with law μ by duality. Therefore $\check{U}^r\mathbf{1}_K(g)$ is finite and thus uniformly bounded by the maximum principle (cf. [12], Corollary 3.6). Since f is integrable $\int_G U^r f(g_1)\mathbf{1}_K(g_1)\,dg_1$ has to be finite. □

Proof of Proposition 1. – By the Borel–Cantelli Lemma, in order to prove (2), it is sufficient to show that

$$\sum_{n=0}^{\infty} \mathbb{P}[gR_{n+1} \in C,\ gR_n \notin C] < +\infty.$$

On the other hand we can write

$$\mathbb{P}[gR_{n+1} \in C,\ gR_n \notin C] = \mathbb{E}\big[\mathbb{P}[gR_n X_{n+1} \in C \mid R_n]\,\mathbf{1}_{C^c}(gR_n)\big] = \mathbb{E}\big[\phi(gR_n)\big],$$

where

$$\phi(g) = \mathbb{P}[gX_1 \in C]\mathbf{1}_{C^c}(g)$$

and therefore

$$\sum_{n=0}^{\infty} \mathbb{P}[gR_{n+1} \in C,\ gR_n \notin C] = U^r\phi(g).$$

So, if $U^r\phi$ is almost everywhere finite, (2) will hold for almost all g. Using the previous lemma and [8] which insures that R_n is transient, we just need to show that ϕ is integrable with respect to the right Haar measure $dg = \frac{da\,db}{a}$ of $\mathrm{Aff}(\mathbb{R}^d)$. As

$$C^c = \big\{g \in \mathrm{Aff}\big(\mathbb{R}^d\big):\ a(g) \geqslant 1\big\} \cup \big\{g \in \mathrm{Aff}\big(\mathbb{R}^d\big):\ a(g) < 1,\ \big|b(g)\big| \geqslant 1\big\}$$

we can split the integral of $\int_{\mathrm{Aff}(\mathbb{R}^d)} \phi(g)\,dg$ into two parts. On the first set we have

$$\begin{aligned}\iint \phi(a,b)\mathbf{1}_{[a\geqslant 1]}\frac{db\,da}{a} &= \mathbb{E}\left[\int_1^{+\infty}\int_{\mathbb{R}^d} \mathbf{1}_C\big((a,b)(A_1,B_1)\big)\frac{db\,da}{a}\right]\\ &= \mathbb{E}\left[\int_1^{+\infty}\int_{\mathbb{R}^d} \mathbf{1}_{[aA_1<1]}\mathbf{1}_{[|aB_1+b|<1]}\frac{db\,da}{a}\right]\\ &= \mathbb{E}\left[v_d \log\left(\frac{1}{A_1} \vee 1\right)\right] = v_d\mathbb{E}\big[\log^-(A_1)\big],\end{aligned}$$

where v_d is the volume of the ball of radius 1 in $\mathbb{R}^d$. On the second set we have that

$$\iint \phi(a,b)\mathbf{1}_{[a<1,|b|\geqslant 1]}\frac{db\,da}{a} \leqslant \mathbb{E}\left[\int_0^1 \left(\int_{\mathbb{R}^d} \mathbf{1}_{[|b|\geqslant 1]}\mathbf{1}_{[|aB_1+b|<1]}\,db\right)\frac{da}{a}\right].$$

An easy computation shows that $\int_{\mathbb{R}^d} \mathbf{1}_{[|x+b|<1\leqslant |b|]}\,db \leqslant \min\{v_d, \pi v_{d-1}|x|/2\}$, whence

$$\begin{aligned}\iint \phi(a,b)\mathbf{1}_{[a<1,|b|\geqslant 1]}\frac{db\,da}{a} &\leqslant \pi v_{d-1}\mathbb{E}\left[\int_0^1 \min\{1, |aB_1|\}\frac{da}{a}\right]\\ &\leqslant \pi v_{d-1}\big(1 + \mathbb{E}\big[\log^+(|B_1|)\big]\big). \qquad \square\end{aligned}$$

We are now able to prove Theorem 1.

Proof of Theorem 1. – A direct consequence of Proposition 1 is that, for almost all $g \in \mathrm{Aff}(\mathbb{R}^d)$, gR_n is either definitely in C or definitely in C^c. As we assumed that $\mathbb{E}[\log(a(X_1))] = 0$, $\log(a(gR_n))$ is a recurrent random walk on the real line; hence gR_n visits the set $[1, +\infty] \times \mathbb{R} \subset C^c$ infinitely often. Therefore we can conclude that for almost all g, almost surely, gR_n is definitely in C^c, or equivalently that for almost all g, R_n is definitely in gC^c.

Combining the fact that $\{gC^c\}_{g\in\mathrm{Aff}(\mathbb{R}^d)}$ is a base of open neighborhoods of ∞ and the fact that for every fixed g_0 the set of g such that $gC^c \subset g_0C^c$ has positive Haar measure, it is possible to choose a sequence g_k such that almost surely R_n is definitely in every g_kC^c and such that $\{g_kC^c\}_k$ remains a base of open neighborhoods of ∞. Hence almost surely $\lim_{n\to\infty} R_n = \infty$.

To conclude that almost surely for every g in $\mathrm{Aff}(\mathbb{R}^d)$, the random walk gR_n converges to ∞ we only need to notice that the action of g on $\mathbb{H} \cup \partial\mathbb{H}$ is continuous, so that $\lim_{n\to\infty} gR_n = g \cdot \infty = \infty$. □

3. Local contraction

Let y_0 be a random vector independent of the $\{X_i\}_i$. The left random walk induces a Markov chain on $\mathbb{R}^d$

$$Y_n^{y_0} = L_n \cdot y_0$$

for every $n \geqslant 0$. Since $Y_n = X_n \cdot Y_{n-1}$, this process satisfies the random difference equation

$$Y_n = a(X_n)Y_{n-1} + b(X_n).$$

From the geometrical viewpoint that we have adopted, this process may be seen as the projection on $\mathbb{R}^d$ of the left random walk starting from a point whose $\mathbb{R}^d$ component is y_0, that is

$$Y_n^{y_0} = L_n \cdot y_0 = b\big(L_n(a, y_0)\big)$$

for every a in $\mathbb{R}_+^*$.

A consequence of the previous theorem concerns the dependence on the starting point of the process Y_n^y. Let us consider the distance $|Y_n^x - Y_n^y| = a(L_n)|x - y|$ between two trajectories starting from two different points x and y of $\mathbb{R}^d$; since we assumed that $\mathbb{E}[\log(A_1))] = 0$, we have that

$$\overline{\lim}\big|Y_n^x - Y_n^y\big| = +\infty \quad \text{while} \quad \underline{\lim}\big|Y_n^x - Y_n^y\big| = 0.$$

Thus it is not possible to globally control this distance. However M. Babillot, Ph. Bougerol and L. Elie have noticed that if we look at the Markov chain only at the times it visits a compact subset K, the process Y_n becomes contractive, in the sense that almost surely for every x and y

$$\lim_{n\to 0}\big|Y_n^x - Y_n^y\big|\mathbf{1}_K\big(Y_n^y\big) = 0.$$

This property was obtained in [1] using an asymptotic estimate of the potential. The proof we propose here is more geometrical and relies on the convergence of the right random walk to ∞.

THEOREM 2. – *Under the hypotheses* (H1)–(H3)*, almost surely for every compact set* $K \subset \mathbb{R}^d$ *and every* $x, y \in \mathbb{R}^d$

$$\lim_{n\to+\infty} |L_n \cdot x - L_n \cdot y| \mathbf{1}_K(L_n \cdot y) = \lim_{n\to+\infty} a(L_n) \mathbf{1}_K(L_n \cdot y) = 0. \tag{3}$$

Proof. – We first observe that

$$L_n = \left(X_1^{-1} \cdots X_n^{-1}\right)^{-1} = \check{R}_n^{-1}$$

where $\check{R}_n$ is the right random walk with law $\check{\mu}$, obtained from μ by composing with the inversion on the group; therefore, as $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$, we have

$$b(L_n) = -\frac{b(\check{R}_n)}{a(\check{R}_n)} = -a(L_n) b(\check{R}_n). \tag{4}$$

Let k be a real positive number such that K is contained in the disc centred at the origin and of radius k; then $L_n \cdot y = a(L_n)y + b(L_n) \in K$ implies that $|b(L_n)| \leqslant k + a(L_n)|y|$. Using the equality (4) we have $|b(\check{R}_n)| \leqslant \frac{k}{a(L_n)} + |y|$ so that:

$$L_n \cdot y \in K \quad \Rightarrow \quad \max\left\{a(\check{R}_n), b(\check{R}_n)\right\} \leqslant (k \vee 1) \frac{1}{a(L_n)} + |y|.$$

As the right random walk $\check{R}_n$ satisfies the hypotheses of Theorem 1, we have that $\max\{a(\check{R}_n), b(\check{R}_n)\}$ converges to $+\infty$ and we conclude. □

Let us consider the attractor set $A(\omega, y) \subset \mathbb{R}^d$ of each trajectory, that is the set of accumulation points of $\{L_n(\omega) \cdot y\}_n$. It is well known (cf. [3], Lemma 5.49) that if we add to the hypotheses (H1)–(H3) a little stronger moment condition, that is

$$\mathbb{E}\left[\left|\log\left(a(X_1)\right)\right|^2\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\left(\log^+\left|b(X_1)\right|\right)^{2+\eta}\right] < +\infty \tag{H4}$$

for some $\eta > 0$, then the Markov chain $L_n \cdot y$ is recurrent in the sense that the attractor sets $A(\omega, y)$ are almost surely non-empty. A direct consequence of the local contraction property is that the set $A(\omega, y)$ does not depend on y, and then on ω by the 0–1 law; thus there exists a set $A \subseteq \mathbb{R}^d$ such that $A = A(\omega, y)$ $\mathbb{P}(d\omega)$-almost surely for all $y \in \mathbb{R}^d$.

Although in the centred case, $L_n \cdot y$ is not positive recurrent and does not have an invariant probability measure, M. Babillot, Ph. Bougerol and L. Elie have constructed in [1] an invariant Radon measure for this process. We will see in the next theorem that the invariant Radon measure is unique and that its support is A.

THEOREM 3. – *Under the hypotheses* (H1)–(H4)*, the Markov chain* $L_n \cdot y$ *has a unique invariant Radon measure on* $\mathbb{R}^d$ *up to a multiplicative constant.*

Furthermore for every couple f and h of continuous functions with compact support such that h is non-negative and not identically zero on A, there exists a constant $c_{f,h}$ such that almost surely for every y

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n} f(L_k \cdot y)}{\sum_{k=0}^{n} h(L_k \cdot y)} = c_{f,h}. \tag{5}$$

Proof. – Let m be an invariant Radon measure on $\mathbb{R}^d$. Its existence is guaranteed by [1]. The (infinite) measure $\mathbb{P}_m$ on the space $(\mathbb{R}^d)^{\mathbb{N}}$ of trajectories of the Markov chain Y_n, obtained as the image of the measure $m \times \mathbb{P}$ on $\mathbb{R}^d \times (\mathrm{Aff}(\mathbb{R}^d))^{\mathbb{N}}$ by the mapping $(y_0, (x_1, x_2, \ldots)) \mapsto (y_0, x_1 \cdot y_0, x_2 x_1 \cdot y_0, \ldots)$, is invariant by the shift θ on $(\mathbb{R}^d)^{\mathbb{N}}$. Therefore the linear transformation induced by the shift on $L^1(\mathbb{P}_m)$ is a positive contraction. Moreover the recurrence and the local contraction imply that this linear transformation is also conservative. Indeed, if we consider the non-negative integrable function $\mathbf{1}_D$, where D is an open and relatively compact set such that $A \cap D \neq \emptyset$, we have

$$\mathbb{P}_m\left[\left\{\mathbf{y} \in \left(\mathbb{R}^d\right)^{\mathbb{N}}\colon \sum_{k=0}^{\infty} \mathbf{1}_D\left(\theta^k \mathbf{y}\right) = \sum_{k=0}^{\infty} \mathbf{1}_D(y_k) < +\infty\right\}\right] = 0.$$

We can then apply the Chacon–Ornstein theorem and we have that for every non-negative function f and h in $L^1(m)$ on the set $\{(y,\omega) \mid \sum_{k=0}^{n} h(L_k(\omega)\cdot y) > 0\}$:

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n} f(L_k \cdot y)}{\sum_{k=0}^{n} h(L_k \cdot y)} = \frac{\mathbb{E}_m[f(Y_0)|\mathcal{I}]}{\mathbb{E}_m[h(Y_0)|\mathcal{I}]} \quad dm(y) \times \mathbb{P}\text{-almost surely} \tag{6}$$

where $\mathcal{I}$ is the σ-algebra of invariant sets for the shift on $(\mathbb{R}^d)^{\mathbb{N}}$.

We now prove that the ratio limit in (6) does not depend on the starting point. We will denote $\sum_{k=0}^{n} f(L_k \cdot y)$ by $S_n f(y)$.

Let f and h two continuous functions with compact support, such that h is non-negative and not identically zero on A, and let K be a compact set which contains their support. For every $\delta > 0$, let $K_\delta = \{z \in \mathbb{R}^d \mid \mathrm{dist}(z, K) \leqslant \delta\}$. Since f is uniformly continuous, using (3), almost surely for every y and x in $\mathbb{R}^d$ and for every positive number ε there exists a random $N \in \mathbb{N}$ such that if $k \geqslant N$:

$$\left|f(L_k \cdot y) - f(L_k \cdot x)\right| \leqslant \varepsilon \max\left\{\mathbf{1}_K(L_k \cdot y), \mathbf{1}_K(L_k \cdot x)\right\} \leqslant \varepsilon \mathbf{1}_{K_\delta}(L_k \cdot y).$$

As h is a non-negative function and not identically zero on A, it is possible, using (6), to choose y such that $\mathbb{P}$-almost surely $\frac{S_n \mathbf{1}_{K_\delta}(y)}{S_n h(y)}$, converges. Therefore:

$$\overline{\lim_{n\to\infty}} = \left|\frac{S_n f(y) - S_n f(x)}{S_n h(y)}\right| \leqslant \varepsilon \lim_{n\to+\infty} \frac{S_n \mathbf{1}_{K_\delta}(y)}{S_n h(y)}.$$

As ε was chosen arbitrarily, we have that almost surely, for all x,

$$\lim_{n\to\infty}\left|\frac{S_n f(y) - S_n f(x)}{S_n h(y)}\right| = 0.$$

If f/h is bounded, we have:

$$\lim_{n\to+\infty}\left|\frac{S_nf(y)}{S_nh(y)}-\frac{S_nf(x)}{S_nh(x)}\right|$$
$$\leqslant \lim_{n\to+\infty}\left(\left|\frac{S_nf(y)-S_nf(x)}{S_nh(y)}\right|+\left|\frac{S_nf(x)}{S_nh(x)}\right|\left|\frac{S_nh(y)-S_nh(x)}{S_nh(y)}\right|\right)=0$$

because $\frac{S_nf(x)}{S_nh(x)}$ is bounded. So if we had chosen y such that $\frac{S_nf(y)}{S_nh(y)}$ converges to a random variable $c_{f,h}$ then almost surely, for every x,

$$\lim_{n\to+\infty}\frac{S_nf(x)}{S_nh(x)}=\lim_{n\to+\infty}\frac{S_nf(y)}{S_nh(y)}=c_{f,h}. \tag{7}$$

To suppose that f/h is bounded, is not a restrictive hypothesis, as one can always obtain the behaviour of S_nf/S_nh as ratio of $S_nf/S_n(h+|f|)$ and $S_nh/S_n(h+|f|)$.

The fact that the limit $c_{f,h}$ is constant is due to the 0–1 law. In fact for every $i\in\mathbb{N}$

$$\lim_{n\to+\infty}\frac{\sum_{k=i}^n f(X_k\cdots X_{i+1}\cdot y)}{\sum_{k=i}^n h(X_k\cdots X_{i+1}\cdot y)}=\lim_{n\to+\infty}\frac{\sum_{k=i}^n f(L_kL_i^{-1}\cdot y)}{\sum_{k=i}^n h(L_kL_i^{-1}\cdot y)}$$
$$=\lim_{n\to+\infty}\frac{S_nf(L_i^{-1}\cdot y)}{S_nh(L_i^{-1}\cdot y)}=\lim_{n\to+\infty}\frac{S_nf(y)}{S_nh(y)}=c_{f,h}$$

because of (7), so that $c_{f,h}$ is measurable with respect to the σ-algebra of the $\{X_k\}_{k>i}$, for each $i\in\mathbb{N}$.

We now can easily deduce the uniqueness of m. Because of (7) and (6), we have then

$$\frac{\mathbb{E}_m[f(Y_0)|\mathcal{I}]}{\mathbb{E}_m[h(Y_0)|\mathcal{I}]}=c_{f,h}\quad dm(y)\times\mathbb{P}\text{-almost surely},$$

and this implies that for every invariant measure m we have

$$m(f)=\mathbb{E}_m\big[\mathbb{E}_m\big[f(Y_0)\mid\mathcal{I}\big]\big]=\mathbb{E}_m\big[c_{f,h}\mathbb{E}_m\big[h(Y_0)\mid\mathcal{I}\big]\big]=c_{f,h}m(g).$$

Therefore m is unique up to a constant. □

4. Divergence of the right projection

In this section we will reinforce the result of Theorem 1 by showing that, under some density hypotheses, not only the right random walk goes to infinity but its projection onto $\mathbb{R}^d$ do the same, in other words the process

$$Z_n^g=b(gR_n)=b(g)+a(g)\sum_{k=1}^n A_1\cdots A_{k-1}B_k$$

is transient. We can remark that it was known (cf. [13]) that in the centred case both process $|Z_n|$ and $|Y_n|$, that have the same law for a fixed n, converge in probability to $+\infty$; but while Y_n is a recurrent process, we will prove that $|Z_n|$ converges almost surely to $+\infty$.

THEOREM 4. – *Suppose that the hypotheses* (H1)–(H3) *are satisfied, that the marginal of* μ *on* $\mathbb{R}^d$*, that is the law of* B_1*, has a bounded density and that* $\mathbb{E}[|B_1|^\rho]$ *is finite for some* $\rho > 1$*. Then almost surely for every* g *in* $\mathrm{Aff}(\mathbb{R}^d)$

$$\lim_{n\to\infty} \left|b(gR_n)\right| = +\infty.$$

Proof. – We first observe that the d-dimensional affine group $\mathrm{Aff}(\mathbb{R}^d)$ may be projected on a one-dimensional affine group just taking the first coordinate $b_1(g)$ of the vector $b(g)$. As obviously when the first coordinate diverges also the vector diverge, we can restrict to the case $d = 1$ without loss of generality.

We will proceed as in the proof of Theorem 1 and we will start to show that if $S = \{g \in \mathrm{Aff}(\mathbb{R}) | |b(g)| \leqslant 1\}$, gR_n does not cross the border of S but a finite number of times. As we have seen in Proposition 1, we only need to show that the potential of the function

$$\psi(g) = \mathbb{P}[gX_1 \in S]\mathbf{1}_{S^c}(g)$$

is finite. We will split the function ψ in two parts and study their potentials with two different techniques. Let

$$S_1 = \left\{g \in \mathrm{Aff}(\mathbb{R}) \mid \left|b(g)\right| > 1, a(g) \leqslant 1\right\}$$

and

$$S_2 = \left\{g \in \mathrm{Aff}(\mathbb{R}) \mid \left|b(g)\right| > 1, a(g) > 1\right\}$$

so that:

$$\psi = \psi\mathbf{1}_{S_1} + \psi\mathbf{1}_{S_2} = \psi_1 + \psi_2.$$

The integral of ψ_1 with respect with the right Haar measure was already calculated in the proof of the Proposition 1 where we proved that

$$\iint \psi_1(a,b)\frac{db\,da}{a} = \mathbb{E}\left[\int_0^1 \int_{\mathbb{R}} \mathbf{1}_{[|aB_1+b|<1\leqslant|b|]}\,db\frac{da}{a}\right] \leqslant c\left(1 + \mathbb{E}\left[\log^+\left(|B_1|\right)\right]\right)$$

so that $U^r\psi_1(g)$ is finite for almost all g.

It is easily checked that ψ_2 is not integrable for the right Haar measure so, to prove that its potential is finite, we will need to use a more specific method. Let

$$F_1 = \left\{g \in \mathrm{Aff}(\mathbb{R}){:}\ 1 < a(g) \leqslant 2,\ 0 < b(g) \leqslant 1\right\}$$

and for every $k \in \mathrm{Aff}(\mathbb{R})$

$$F_k = kF_1 = \left\{g \in \mathrm{Aff}(\mathbb{R}){:}\ a(k) < a(g) \leqslant 2a(k),\ b(k) < b(g) \leqslant a(k) + b(k)\right\}.$$

As the random walk is transient and the sets F_k are relatively compact, their potential is bounded and we have for every k

$$\left\|U^r\mathbf{1}_{\overline{F}_k}\right\|_\infty = \sup_{g\in\mathrm{Aff}(\mathbb{R})} \left|\delta_g * U(k\overline{F}_1)\right| = \left\|U^r\mathbf{1}_{\overline{F}_1}\right\|_\infty.$$

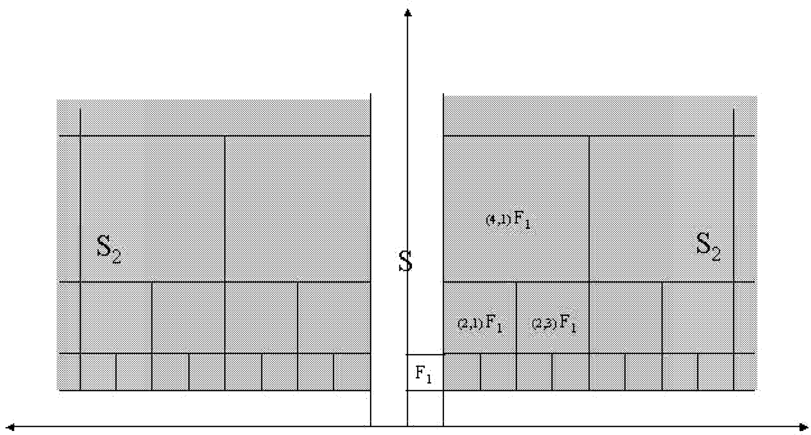

Fig. 1. Partition of S_2 into "equipotential" squares.

We denote $-F_{(2^m,2^m n+1)}$ the image of the set $F_{(2^m,2^m n+1)}$ under the mapping $(a,b)\mapsto(a,-b)$ and we observe that $-\overline{F}_{(2^m,2^m n+1)}=\overline{F}_{(2^m,-2^m(n+1)-1)}$; thus the family $\{F_{(2^m,2^m n+1)}\}_{n\in\mathbb{N},m\in\mathbb{N}}\cup\{-F_{(2^m,2^m n+1)}\}_{n\in\mathbb{N},m\in\mathbb{N}}$ is a partition of S_2 into "equipotential" squares. We observe that

$$\begin{aligned}
U^r\psi_2 &= \sum_{\substack{k=(2^m,2^m n+1)\\ n,m\in\mathbb{N}}} \bigl(U^r(\psi_2\mathbf{1}_{F_k})+U^r(\psi_2\mathbf{1}_{-F_k})\bigr)\\
&\leqslant \sum_{\substack{k=(2^m,2^m n+1)\\ n,m\in\mathbb{N}}} \bigl(\bigl\|U^r\mathbf{1}_{F_k}\bigr\|_\infty\|\psi_2\mathbf{1}_{F_k}\|_\infty+\bigl\|U^r\mathbf{1}_{-F_k}\bigr\|_\infty\|\psi_2\mathbf{1}_{-F_k}\|_\infty\bigr)\\
&\leqslant \bigl\|U^r\mathbf{1}_{\overline{F}_1}\bigr\|_\infty \sum_{\substack{k=(2^m,2^m n+1)\\ n,m\in\mathbb{N}}} \bigl(\|\psi_2\mathbf{1}_{F_k}\|_\infty+\|\psi_2\mathbf{1}_{-F_k}\|_\infty\bigr)
\end{aligned}\tag{8}$$

so that to prove that the potential is bounded we need to estimate the function ψ_2 on the sets F_k.

It may be worth observing that this approach allows us to compare the right potential kernel with something that roughly looks like a left invariant measure, in the sense that we sum the maximum of the function over a collection of sets obtained by left translation (and ψ_2 is integrable for the left Haar measure). The problem is that the sizes of the squares over which we sum are fixed and we need the function ψ_2 to be smooth enough on them. Now if f is the bounded density of the law of B_1, we have:

$$\mathbb{P}\bigl[(a,b)X_1\in S\bigr]=\mathbb{P}\left[\frac{-1-b}{a}<B_1<\frac{1-b}{a}\right]\leqslant\|f\|_\infty\frac{2}{a}.$$

To control ψ_2 when $|b|$ is big we observe that for every $p>1$ and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$:

$$\mathbb{P}\bigl[(a,b)X_1\in S\bigr]\leqslant\left(\frac{|b|-1}{a}\right)^{-\rho/p}\mathbb{E}\bigl[|B_1|^{\rho/p}\mathbf{1}_{[\frac{-1-b}{a},\frac{1-b}{a}]}(B_1)\bigr]$$

$$\leqslant \left(\frac{|b|-1}{a}\right)^{-\rho/p} \mathbb{E}\left[|B_1|^\rho\right]^{1/p} \left(\int\limits_{\frac{-1-b}{a}}^{\frac{1-b}{a}} f(x)\,dx \right)^{1/q}$$

$$\leqslant 2^{1/q} \|f\|_\infty^{1/q} \mathbb{E}\left[|B_1|^\rho\right]^{1/p} \frac{a^{\rho/p-1/q}}{(|b|-1)^{\rho/p}}$$

whenever $|b| \geqslant 1$. Therefore there exist $0 < \alpha < \beta$ and $\beta > 1$ and a suitable constant C such that, for every $(a,b) \in S^c$, we have

$$\psi(a,b) \leqslant C \min\left\{a^\alpha \left(|b|-1\right)^{-\beta}, a^{-1}\right\}.$$

Then for every $g \in F_{(2^m,2^m n+1)} \cup -F_{(2^m,2^m n+1)}$

$$\psi(g) \leqslant C \min\left\{2^{\alpha(m+1)} \left(2^m n\right)^{-\beta}, 2^{-m}\right\} \leqslant C \min\left\{2^{(\alpha-\beta)(m+1)} n^{-\beta}, 2^{-m}\right\}$$

and using the decomposition (8) we have

$$U^r \psi_2 \leqslant \left\|U^r \mathbf{1}_{F_1}\right\|_\infty 2C \sum_{m,n\in\mathbb{N}} \min\left\{2^{(\alpha-\beta)(m+1)} n^{-\beta}, 2^{-m}\right\} < +\infty.$$

We proved that, for almost all $g \in \mathrm{Aff}(\mathbb{R})$, gR_n can cross the border of S only a finite number of times.

Combining the recurrence of the real random walk $a(gR_n)$ and the transience of gR_n, we see that gR_n visits an infinite number of times the set $([1,2] \times \mathbb{R}) \cap S^c$ so that, for almost all g, gR_n is definitely in S^c. As for every $s > 0$ the set of g such that

$$gS^c \subset \left\{(a,b) \in \mathrm{Aff}(\mathbb{R}):\ |b| > s\right\}$$

has positive Haar measure, we may conclude that for every $s > 0$ almost surely $|b(R_n)|$ is definitely greater than s, so that, letting s go to $+\infty$ on a countable sequence, we obtain that, almost surely, for every $g \in \mathrm{Aff}(\mathbb{R})$

$$\lim_{n\to+\infty} \left|b(gR_n)\right| = \lim_{n\to+\infty} \left|a(g)b(R_n) + b(g)\right| = +\infty. \qquad \square$$

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