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INTERNAL DLA IN A RANDOM ENVIRONMENT

AGRÉGATION LIMITÉE PAR DIFFUSION INTERNE DANS UN ENVIRONEMENT ALÉATOIRE

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ABSTRACT. – In this article, Internal DLA is studied with a random, homogeneous, distribution of traps. Particles are injected at the origin of a d-dimensional Euclidean lattice and perform independent random walks until they hit an unsaturated trap, at which time the particle dies and the trap becomes saturated. It is proved that the large scale effect of the randomness of the traps on the speed of growth of the set of saturated traps depends of the strength of the injection, and separates into several regimes. In the subcritical regime, the set of saturated traps is asymptotically an Euclidean ball whose radius is determined in a trivial way from the trap density. In the critical regime, there is a nontrivial interplay between the density of traps and the rate of growth of the ball. The supercritical regime is studied using order statistics for free random walks. This restricts us to $d = 1$. In the supercritical, subexponential regime, there is an overall effect of the traps, but their density does not affect the growth rate. Finally, in the supercritical, superexponential regime, the traps have no effect at all, and the asymptotics is governed by that of free random walks on the lattice.

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RÉSUMÉ. – Dans cet article on étudie le modèle d'Agrégation Limitée par Diffusion Interne sous l'hypothèse d'une distribution aléatoire homogène de pièges. Ce modèle correspond à l'injection de particules à l'origine d'un réseau euclidien d-dimensionnel; chaque particule évoluant ensuite, de façon indépendante, selon une marche aléatoire jusqu'à l'instant oú elle tombe dans un piège non saturé. A ce moment la particule meurt et le piège devient saturé. Nous prouvons que l'effet à grand échelle de l'aspect aléatoire de pièges sur la vitesse de croissance de l'ensemble de pièges saturés dépend du taux d'injection, ce qui definit plusiers régimes d'injection. Dans le régime sous-critique, l'ensemble de pièges saturés est asymptotiquement égal à une boule euclidienne dont le rayon dépend trivialement de la densité de pièges. Dans le régime critique, il y a un rapport non trivial entre la densité de pièges et le taux de croissance de la boule. Le régime sur-critique est étudié à l'aide des techniques de statistiques d'ordre pour des marches aléatoires libres. Pour ce faire on se restreint au cas $d = 1$. Dans le régime sur-critique et sous-exponentiel, on trouve un effet global des pièges, mais leur densité n'affecte pas le taux de croissance. Finalement, dans le régime sur-critique et sous-exponentiel, les pièges n'ont aucun effet, et le comportement asymptotique est régi par celui des marches aléatoires libres dans le réseau.

0. Introduction

Internal DLA is a stochastic particle system in which traps are distributed on a d-dimensional integer lattice, and particles are produced at the origin and move as independent random walks until they hit a trap, at which time the particle stops, and the trap becomes saturated.

The model arises in several applied problems, for example, nuclear waste management: Radioactive waste is placed in a container and buried, but, since even the best containers have some leakage, particles leave the container and as a rough approximation, perform independent random walks. In order to contain them, chemical traps are distributed around the area. When a radioactive particle hits a trap it is destroyed and the trap becomes saturated. Internal DLA also serves as a model for the behaviour of the chemical reaction $A + B \to$Inert, in the special case where there is a source of A particles, and the B particles are fixed. It has also been used to model erosion processes, as well as melting processes, among others.

A basic problem in such a model is to understand the asymptotic shape and growth rate for the set of saturated traps. Fixing the rate of the walks, and the field of traps, we have as free parameter the rate of injection of particles at the origin. The way in which the growth rate is affected by the presence of the trap field depends on the rate of injection of particles at the origin. We identify four different regimes: Subcritical, critical, supercritical with subexponential injection, and supercritical with superexponential injection. The critical case is when $N(t)$, the number of particles injected up to time t, is asymptotic to $t^{d/2}$, and the exponential case is when $N(t) \sim \exp\{ct\}$.

In the subcritical case, the asymptotic shape is a ball in any dimension. Asymptotically there is a zero density of live particles, and therefore the size of the ball can be determined easily: It has to contain as many traps as particles which have been injected, up to a negligible error. By the law of large numbers the volume of the ball is clearly $N(t)/m$ where m is the mean number of traps per site, and the radius can be read off directly. Because the density of live particles is negligible, the model can be coupled to a discrete time version, as was done in [6] (see also [3], where a discrete time asymmetric version of the model is studied, and [5] where refinements of [6] are obtained). The shape theorem follows from the analogous shape theorem for the discrete time case, following the methods of [6]. This was proved in [1] (see also [2]).

In the critical case, the shape is still a ball in any dimension, but now there is a nontrivial density of live particles inside the ball. The boundary, and density of particles evolve together asymptotically according to a one-phase Stefan problem. This can be solved explicitly, yielding the shape and the rate of growth, which is a nontrivial function of the density of traps. This is proved here by an appropriate adaptation of the hydrodynamic limit method introduced in [4] (see also [8]).

We also study here the supercritical case using order statistics. This restricts us to one dimension. Two regimes are identified. In the subexponential regime, there is still a net effect of the traps, but the density of traps plays no role. In the superexponential regime, the traps play no role at all and the speed of propagation is controlled by the range of free random walks. In dimensions greater than one, in the supercritical case, one expects similar results. However, here the shape will not be a sphere, but a certain level set of the rate function for large deviations of random walks on the lattice. This reflects the fact that as the strength of the injection is increased, the range of random walks becomes the dominant factor in the asymptotic shape and size of the saturated set. We do not prove this fact here, as we decided to stress the application of order statistics: We study the $n(t)$th rightmost particle at time t in a system of random walks on the integer lattice, starting at the origin, at given times. Under quite general conditions it is shown that the asymptotics of this particle is governed by an appropriate transform of the large deviation rate function for a random walk on the lattice. This is then used to prove the asymptotics for the internal DLA model in the one dimensional supercritical case.

Section 1 contains a more precise description of the model, and the main results. We have stated these in the simplest cases in order to highlight the main point of the article which is the identification of the different regimes. In Section 2 we explain how the methods of [4] need to be adapted in order to handle the random trap field in the critical case. In Section 3 we state the main results about order statistics of random walks and from this obtain the asymptotics for internal DLA in the supercritical case in one dimension. Section 4 contains the technical proofs of the order statistics results.

1. Model and main results

We start with a field ζ of traps on $\mathbf{Z}^d$. For each $x \in \mathbf{Z}^d$, $\zeta_x \in \{0, 1\}$ denotes the absence or presence of a trap initially at x.

Particles are injected at the origin at times $t_1, t_2, \ldots$ and then perform independent continuous time random walks at rate 1. The rule is that the first particle which hits a trap stops moving.

The position at time t of the particle produced at time t_n will be denoted $X_n(t)$. Sometimes it is convenient to use the convention that $X_n(t) = 0$ for $0 \leqslant t \leqslant t_n$. Another way to describe the model is through the occupation number $\eta_x(t) = \sum_n \mathbf{1}_{X_n(t)=x}$, the number of particles at site x at time t. This includes the particles which have stopped moving, and clearly $\xi_x(t) = (\eta_x(t) - \zeta_x)_+$ is the number of 'live' particles at x at time t, and

$$A_t = \left\{x \in \mathbf{Z}^d\colon\ \zeta_x = 1, \eta_x(t) \geqslant 1\right\}$$

is the set of saturated traps. We will make a convention that the occupation number at 0 will not include particles which are not yet born, so at 0 we modify the definition to be $\eta_0(t) = \sum_n \mathbf{1}_{X_n(t)=0,\ t \geqslant t_n}$.

Let N_t denote the number of particles created up to time t. We will call such an N_t an *injection*. For simplicity, in this article we will always take N_t to be deterministic, though it is certainly not necessary. For each field ζ of traps, and each injection N we have a process $Y(t) = \{X_1(t), X_2(t), \ldots\}$ which also has a reduced description $\eta_x(t), x \in \mathbf{Z}^d$. We will denote the distribution of this process by P_ζ.

In all cases we will assume that there is a positive density of traps – for simplicity we assume that $\zeta_x, x \in \mathbf{Z}^d$ are independent, with $P(\zeta_x = 1) = m$ and $P(\zeta_x = 0) = 1 - m$ for some fixed $m \in (0, 1]$.

Our main question is how the distribution of the traps affects the growth rate of the random set of saturated traps. We distinguish three cases, based on the strength of the injection: If $t^{-d/2}N_t$ has a nontrivial limit as $t \to \infty$ we say the model is *critical* and use the notation $N_t \sim t^{d/2}$, if $t^{-d/2}N_t \to 0$ as $t \to \infty$ we say the model is *subcritical* and use the notation $N_t \ll t^{d/2}$ while if $t^{-d/2}N_t \to \infty$ as $t \to \infty$ we say the model is *supercritical* and use the notation $N_t \gg t^{d/2}$. Three regimes are displayed in Table 1.

In any dimension d, we denote by $B(\mathbf{x}, r)$ the Euclidean ball of radius r centered at $\mathbf{x}$. Note that $B(0, v^{1/d}a_d)$ has volume v, where $a_d = d\Gamma(d/2)/2)^{1/d}/\sqrt{\pi}$. Here $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,dx$ is the Gamma function. We use the notation $a_t \ll b_t$ when $a_t/b_t \to 0$ as $t \to \infty$ and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to $x \in \mathbf{R}$. We also will define the constant K as the unique solution of the equation,

$$\Gamma(d/2)\exp\left\{-K^2/4\right\} = m\pi^{d/2}K^d. \tag{1.1}$$

Table 1
Injection regimes

Subcritical	$N_t \ll t^{d/2}$
Critical	$N_t \sim t^{d/2}$
Supercritical	$N_t \gg t^{d/2}$

The large deviation rate function of a continuous time simple symmetric total jump rate one random walk will appear in some asymptotics. So let

$$I(x)=\sup_{\lambda\in\mathbf{R}}\bigl\{\lambda x-(\cosh\lambda-1)\bigr\}=x\sinh^{-1}x-\sqrt{1+x^2}+1. \tag{1.2}$$

Note that $I:[0,\infty)\to[0,\infty)$ is one to one and therefore has an inverse function $I^{-1}:[0,\infty)\to[0,\infty)$. If n_t is an increasing function define

$$w_{n.}(t)=\sup_{0\leqslant y\leqslant t}(t-y)I^{-1}\left(\frac{1}{t-y}\log\frac{N_y}{n_t}\right), \tag{1.3}$$

with the convention that $I^{-1}(x)=0$ for $x<0$. Finally, given two set U,V, we denote by $U\triangle V$ their symmetric difference, and $\#[U]$ the cardinality of U.

We can now state our main results.

THEOREM 1.1. – (1) Subcritical case. *In any dimension $d\geqslant 1$ we have that,*

$$A_t\sim B\bigl(0,(N_t/m)^{1/d}a_d\bigr)$$

in the sense that for any $\delta>0$, for almost every ζ, with P_ζ probability one, for sufficiently large t,

$$B\bigl(0,(1-\delta)(N_t/m)^{1/d}a_d\bigr)\cap\{\zeta_x=1\}\subset A_t\subset B\bigl(0,(1+\delta)(N_t/m)^{1/d}a_d\bigr)\cap\{\zeta_x=1\}.$$

(2) Critical case. *In any dimension $d\geqslant 1$, suppose that $N_t=\lfloor t^{d/2}\rfloor$. Then,*

$$A_t\sim B(0,K\sqrt{t}\,),$$

in the sense that for almost every realization ζ of the trap field,

$$t^{-d/2}\#\bigl[A_t\triangle B(0,K\sqrt{t}\,a_d)\cap\{\zeta_x=1\}\bigr]\to 0, \tag{1.4}$$

in P_ζ-probability.

(3) Supercritical case. *In dimension $d=1$, suppose that either $\log N_t\gg\log t$ or that $N_t=\lfloor t^\alpha\rfloor$ for some $\alpha>1/2$. Furthermore, assume that there is a $\delta>0$ and a function f_t such that $1\ll f_t\ll t$ and $N(f_t)\gg(\log N_t)^{1+\delta}$. Then*

$$A_t\sim B\bigl(0,w_{\sqrt{\cdot}}(t)\bigr)$$

in the sense that if r_t and ℓ_t are the rightmost and leftmost particles, then for almost every realization ζ of the trap field,

$$\lim_{t\to\infty}r_t/w_{\sqrt{\cdot}}(t)=1\quad\textit{and}\quad\lim_{t\to\infty}\ell_t/w_{\sqrt{\cdot}}(t)=-1 \tag{1.5}$$

in P_ζ-probability.

Remarks. – (1) Note that part (3) of Theorem 1.1 does not cover the whole supercritical case. In fact, by technical reasons which somehow simplify the proofs, we have included the additional hypothesis that either $\log N_t \gg \log t$ or that $N_t = \lfloor t^\alpha \rfloor$ for some $\alpha > 1/2$.

(2) The main conclusion is that the effect of the traps depends on the strength of the injection.

In case (1), where the injection is subcritical, the occupied set is approximately a ball of volume $m^{-1}N_t$, which by the law of large numbers contains approximately N_t traps. Essentially all the particles at time t have been trapped. The influence of the random trap field on the speed of growth is a fairly trivial averaging.

In case (2), where the injection is critical, the randomness in the trap field enters only through the mean on a large scale but has a nontrivial effect (1.1) on the speed of growth.

In case (3), where the injection is supercritical, the density $m \in (0, 1]$ of traps does not enter at all. However from the asymptotics of $w_{\sqrt{t}}(t)$ one finds a transition at exponential injections. If $\log N_t \ll t$ then for almost every trap configuration ζ, in P_ζ-probability,

$$\lim_{t\to\infty} \frac{r_t}{\sup_{0\leqslant y\leqslant t} \sqrt{2(t-y)\log(N_y/t^{1/2})}} = 1. \tag{1.6}$$

If $t \ll \log N_t$ then for almost every trap configuration ζ in P_ζ-probability,

$$\lim_{t\to\infty} \frac{r_t}{\sup_{0\leqslant y\leqslant t} \log N_y/\log(\frac{\log N_y}{t-y})} = 1. \tag{1.7}$$

One can check that the latter corresponds to the rightmost particle for free random walks with the same injection, but the former does not (the final denominator $t^{1/2}$ would be absent). Hence for subexponential injections, the traps slow down the growth rate in a way which does not depend on the density. For superexponential injections, there is no slowdown effect at all.

(3) The transition at exponential injections is also seen in the effect of the lattice. We can replace the random walks in our model by Brownian motions. The traps live at the integers, as before, and the first Brownian motion at a trap stops there forever. One can check that in that case the asymptotic in case (3) is always as in (1.6). Hence for injections much weaker than exponential large scale lattice effects are not seen, but for stronger than exponential injection rates large scale lattice effects correspond to an increase in the speed of growth with respect to the Brownian motion version of Internal DLA just described. In other words, for given N_t stronger than exponential, the rate of growth of Internal DLA is larger than the rate of growth of the corresponding model of Brownian motions with traps at the integers. The simple point is that as the rate of injection becomes larger, one has to look farther into the tails of the distribution of the particles for the main contribution to the asymptotics.

(4) The difference in the formulation of the shape results in the three different regimes reflects the different techniques used. The subcritical case (1) is proved in [1] using the methods of [6]. In this article we prove the critical and supercritical cases. The critical case is proved in Section 3 using the method of [4] where the theorem was proved

before in the special case $\zeta_x = 1$ for all $x \in \mathbf{Z}^d$. The supercritical case is proved in Section 4 using order statistics. This complements the proof in [4] in the special case of one dimension with injection $N_t = t$ and $\zeta_x = 1$ for all x.

(5) The conclusion of case (3) (in particular concerning $r(t)$) is valid for any distribution of traps satisfying $\liminf_{n\to\infty} \sum_{x=0}^{n} \zeta_x = m > 0$ and does not depend on the randomness of the trap distribution.

(6) Using the methods of [4] one can study a variant of the model where live particles have a zero-range interaction, in the critical and subcritical case (see [4,8]). Analogous results hold.

(7) If more than one trap is allowed at each site the same results hold, with analogous proofs, with $m = E[\zeta_x]$. It would be interesting to know what happens in the case $m = \infty$.

2. Critical case

Recall the reduced description $\eta_x(t) = \sum_n \mathbf{1}_{X_n(t)=x}$ of our process. Since the field of traps ζ_x is fixed throughout, the variable

$$\xi_x(t) = \eta_x(t) - \zeta_x$$

together with the initial condition $\xi_x(0) = -\zeta_x$ gives a full description of our Markov process. For any local function f,

$$f\big(\xi(t)\big) - \int_0^t Lf\big(\xi(s)\big)\,ds - \sum_{\{i:\ t_i \leqslant t\}} \big(f\big(\xi^{0,+}(t_i)\big) - f\big(\xi(t_i)\big)\big), \tag{2.1}$$

is a martingale, where the Markov generator is

$$Lf(\xi) = \sum_{x,e} (\xi_x)_+ \big(f(\xi^{x,x+e}) - f(\xi)\big),$$

where e are unit vectors in the lattice, x are sites in $\mathbf{Z}^d$, $\xi^{x,x+e}$ denotes the configuration obtained from ξ by moving one particle from site x to site $x+e$, and $\xi^{0,+}$ denotes the configuration obtained from ξ by adding one particle at 0,

$$\xi^{x,x+e} = \xi - \delta_x + \delta_{x+e}, \qquad \xi^{0,+} = \xi + \delta_0.$$

For any real number x we use $(x)_+$ or x_+ to denote $\max(x,0)$. The last term of (2.1) corresponds to the deterministic injection of particles, and $0 \leqslant t_1 < t_2 < \cdots$ are the times when particles are added; the jumps of $\lfloor t^{d/2} \rfloor$.

Let ε be a small parameter and introduce macroscopic space and time variables $\mathbf{x} = \varepsilon x$ and $\mathbf{t} = \varepsilon^2 t$ in $\mathbf{R}^d$ and $[0,\infty)$. The main result of this section is

THEOREM 2.1. – *For almost every realization ζ of the trap field, as $\varepsilon \to 0$,*

$$\big[\xi_{\lfloor \varepsilon^{-1}\mathbf{x}\rfloor}\big(\varepsilon^{-2}\mathbf{t}\big)\big]_+ \rightharpoonup \rho(\mathbf{x},\mathbf{t}) \tag{2.2}$$

and

$$\mathbf{1}_{\zeta_{\lfloor \varepsilon^{-1}\mathbf{x}\rfloor}>0,\xi_{\lfloor \varepsilon^{-1}\mathbf{x}\rfloor}(\varepsilon^{-2}\mathbf{t})\geqslant 0} \rightharpoonup m\mathbf{1}_{s(\mathbf{x})\leqslant \mathbf{t}} \tag{2.3}$$

weakly in P_ζ probability, where $\rho(\mathbf{x},\mathbf{t})\geqslant 0$ and $s(\mathbf{x})$ are the unique solutions of the one-phase Stefan problem

$$\begin{cases} \frac{\partial\rho}{\partial t} = \Delta\rho + t^{(d-2)/2}\delta_0 & s(\mathbf{x}) < \mathbf{t}, \\ \rho = 0 & s(\mathbf{x}) \geqslant \mathbf{t}, \\ \nabla_0\rho\cdot\nabla s = -m & s(\mathbf{x}) = \mathbf{t}. \end{cases} \tag{2.4}$$

Remarks. – (1) The Stefan problem says that the expansion of the boundary is in the normal direction with velocity proportional to the density gradient $\nabla_0\rho$ taken from inside the region. The only large scale effect of the randomness of the traps is that this expansion is slowed down by a factor $m = E[\zeta_x]$.

(2) The solution of (2.4) is given explicitly by $\rho(\mathbf{x},\mathbf{t}) = \frac{\Gamma(d/2)}{2\pi^{d/2}}\int_{|\mathbf{x}|/\sqrt{\mathbf{t}}}^{K} s^{1-d}\mathrm{e}^{-s^2/4}\,ds$ and $s(\mathbf{x}) = K^{-2}|\mathbf{x}|^2$ where $\Gamma(d/2)\mathrm{e}^{-K^2/4} = m\pi^{d/2}K^d$. Case (2) of Theorem 1.1 follows.

Theorem 2.1 is proved by suitably modifying the method of [4]. We indicate only the main steps and differences from the earlier proof and refer the reader to [4] when the proofs only require straightforward modifications.

2.1. Invariant measures

We consider the system without creation, i.e., with Markov generator L.

LEMMA 2.2. – *Let μ be any invariant measure for L. Then*

$$\mu\big(\exists\, x, y\in \mathbf{Z}^d, \xi_x > 0, \xi_y < 0\big) = 0.$$

Proof. – It suffices to show that for arbitrary sites x and y, $\mu(\xi_x > 0, \xi_y < 0) = 0$. We will prove it by induction on n, the lattice distance between x and y.

To start the induction let us take x and y to be nearest neighbour sites. Consider the function $f = \mathbf{1}_{\xi_y<0}$. Since f is a bounded local function and μ is an invariant measure we have $E_\mu[Lf] = 0$. Now $Lf = -\sum_e(\xi_{y+e})_+\mathbf{1}_{\xi_y<0}$. Since each term in the sum is non-negative we have $E_\mu[(\xi_x)_+\mathbf{1}_{\xi_y<0}] = 0$ which we rewrite as $0 = \sum_{k=1}^\infty k_+\mu(\xi_x = k, \xi_y < 0) = 0$. This proves that $\mu(\xi_x > 0, \xi_y < 0) = 0$.

Now suppose the statement holds for sites at distance n and let x be at distance $n+1$ from y. Then there exists a site z of distance n from x and 1 from y. By the inductive hypothesis $f = \mathbf{1}_{\xi_z>0,\xi_x<0} = 0$ almost surely with respect to μ. Therefore for any lattice site u and unit e, $L_{u,u+e}f = (\xi_u)_+(f(\xi^{u,u+e}) - f(\xi)) \geqslant 0$ almost surely with respect to μ as well. However since f is a bounded local function and μ is invariant we have also $E_\mu[L_0 f] = 0$ and it follows that each $L_{u,u+e}f = 0$ almost surely with respect to μ. In particular, $E_\mu[L_{y,z}f] = 0$, or

$$0 = E_\mu\big[(\xi_y)_+\mathbf{1}_{\xi_z=0,\xi_x<0}\big] = \sum_{k=1}^{\infty} k_+\mu(\xi_y = k, \xi_z = 0, \xi_x < 0).$$

By the induction hypothesis, $\mu(\xi_y = k, \xi_z = 0, \xi_x < 0) = \mu(\xi_y = k, \xi_x < 0)$. But then we must have $\mu(\xi_y = k, \xi_x < 0) = 0$ for all k which completes the induction.

COROLLARY 2.3. – *The set of extremal invariant measures for L consists of*
(i) *the Dirac mass on any configuration ξ with $\xi_x \leqslant 0$ for all $x \in \mathbf{Z}^d$,*
(ii) *density any $\rho > 0$.*

Proof. – All the measures in group i are clearly invariant. Suppose μ is some other extremal invariant measure. Then $\xi_x > 0$ for some x, and by the previous lemma, $\xi_x \geqslant 0$ for all x, μ almost surely. Hence, L is the generator of independent random walks on the support of μ, and it therefore follows that μ must be an extremal invariant measure for independent random walks, which are known to be product Poisson measures [7].

2.2. Hydrodynamic limit

The $H_{-1,\varepsilon}$ norm is defined on functions $f : \varepsilon\mathbf{Z}^d \to \mathbf{R}$ of mean $\varepsilon^d \sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d} f_{\mathbf{x}} = 0$ by

$$\begin{aligned}\|f\|^2_{-1,\varepsilon} &= \sup_{\phi} \varepsilon^d \sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d} \left\{ 2 f_{\mathbf{x}}\phi_{\mathbf{x}} - \frac{1}{2}\varepsilon^{-2} \sum_{|\mathbf{e}|=\varepsilon} |\phi_{\mathbf{x}+\mathbf{e}} - \phi_{\mathbf{x}}|^2 \right\} \\ &= \varepsilon^{2d} \sum_{\mathbf{x},\mathbf{y}\in\varepsilon\mathbf{Z}^d} g^{\varepsilon}_{\mathbf{y}-\mathbf{x}} f_{\mathbf{x}} f_{\mathbf{y}},\end{aligned} \tag{2.5}$$

where $g^{\varepsilon}_{\mathbf{x}} = \varepsilon^2 \sum_{n=0}^{\infty} p^n_{\mathbf{x}}$ in $d \geqslant 3$ and $g^{\varepsilon}_{\mathbf{x}} = \lim_{N\to\infty} \varepsilon^2 \sum_{n=0}^{N} p^n_{\mathbf{x}} - p^n_0$ in $d = 1$ or 2. Here p^n are the n step transition probabilities of a symmetric nearest neighbour discrete time random walk on $\varepsilon\mathbf{Z}^d$. Note that in [4] the factors ε^2 are missing in the definition of g^{ε}.

We can observe our system on the lattice $\varepsilon\mathbf{Z}^d$ by defining

$$\xi^{\varepsilon}_{\mathbf{x}}(\mathbf{t}) = \xi_{\lfloor \varepsilon^{-1}\mathbf{x}\rfloor}\left(\varepsilon^{-2}\mathbf{t}\right).$$

LEMMA 2.4. – *For almost every realization ζ of the traps, for each $\mathbf{t} \geqslant 0$, as $\varepsilon \to 0$, $\xi^{\varepsilon}(\mathbf{t}) - \rho^{\varepsilon}(\mathbf{t}) \rightharpoonup 0$ weakly, in probability, where $\rho^{\varepsilon}_{\mathbf{x}}(\mathbf{t})$, $\mathbf{x} \in \varepsilon\mathbf{Z}^d$, $\mathbf{t} \geqslant 0$, is the solution of the lattice Stefan problem*

$$\frac{\partial\rho^{\varepsilon}}{\partial\mathbf{t}} = \Delta_{\varepsilon}(\rho^{\varepsilon})_+ + d P^{\varepsilon}, \qquad \rho^{\varepsilon}(\mathbf{t} = 0) = -m. \tag{2.6}$$

Here $\Delta_{\varepsilon}\phi_{\mathbf{x}} = \varepsilon^{-2}\sum_{|\mathbf{e}|=\varepsilon} \phi_{\mathbf{x}+\mathbf{e}} - \phi_{\mathbf{x}}$ is the lattice Laplacian and $\varepsilon^2 P^{\varepsilon}(\mathbf{t})$ is the number of particles created in the microscopic system up to time $\mathbf{t}$.

In Sections 2 and 4 of [4] it is explained in detail how part (2) of Theorem 1.1 follows from this lemma. The weak convergence (2.2) of the density field follows rather easily because the solution of (2.6) converges to the solution of (2.4) away from the creation points. However there is a fair amount of work to do to obtain the weak convergence (2.3) of the saturated set, as well as (1.4). This is done in [4].

Proof of lemma. – *Step* 1. Fix a large time $\mathbf{T}$. There is a finite B such that if the initial condition in (2.6) are replaced by $\rho^{\varepsilon}_{\mathbf{x}}(\mathbf{t} = 0) = -\zeta_x$ if $|\mathbf{x}| > B\mathbf{T}$ and $\rho^{\varepsilon}_{\mathbf{x}}(\mathbf{t} = 0) = -m$ if $|\mathbf{x}| \leqslant B\mathbf{T}$ then the solution remains the same up to time T (finite propagation speed). In the following we work with the modified initial conditions for ρ^{ε}.

Let q^ε be the solution of (2.6) with the initial condition changed to $q^\varepsilon(\mathbf{t}=0)=-\zeta$ everywhere. Note that $\varepsilon^d\sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d}[q^\varepsilon_{\mathbf{x}}-\rho^\varepsilon_{\mathbf{x}}]$ is constant in time and given by $Z^\varepsilon=\varepsilon^d\sum_{|x|\leqslant\varepsilon^{-1}B\mathbf{T}}(m-\zeta_x)$. Note that $Z^\varepsilon=\mathrm{O}(\varepsilon^{d/2})$ and vanishes for almost every realization ζ of the traps, by the law of large numbers. Now the $H_{-1,\varepsilon}$ norm of $q^\varepsilon-\rho^\varepsilon-Z^\varepsilon$ makes sense and it is straightforward to check that

$$\left\|q^\varepsilon-\rho^\varepsilon-Z^\varepsilon\right\|^2_{-1,\varepsilon}\Big|_{\mathbf{t}=0}^{\mathbf{t}=\mathbf{T}}=-2\int_0^{\mathbf{T}}\sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d}\big((q^\varepsilon_{\mathbf{x}})_+-(\rho^\varepsilon_{\mathbf{x}})_+\big)(q^\varepsilon_{\mathbf{x}}-\rho^\varepsilon_{\mathbf{x}})\,dt$$
$$+2Z^\varepsilon\int_0^{\mathbf{T}}\sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d}\big((q^\varepsilon_{\mathbf{x}})_+-(\rho^\varepsilon_{\mathbf{x}})_+\big)\,dt.$$

The last integral is bounded uniformly in ε, for fixed $\mathbf{T}$. Since $Z^\varepsilon\to0$, the last term goes to zero and we see that $q^\varepsilon-\rho^\varepsilon$ tends weakly to 0. As in [4] it can be shown from this that q^ε_+ converges strongly to the solution ρ of (2.4).

Step 2. By direct computation one shows that

$$\left\|\xi^\varepsilon-q^\varepsilon\right\|^2_{-1,\varepsilon}\Big|_{\mathbf{t}=0}^{\mathbf{t}=\mathbf{T}}=2\int_0^{\mathbf{T}}\varepsilon^d\sum_{\mathbf{x}\in\varepsilon\mathbf{Z}^d}V\big(\xi^\varepsilon_{\mathbf{x}}(\mathbf{t}),q^\varepsilon_{\mathbf{x}}(\mathbf{t})\big)\,d\mathbf{t}+M^\varepsilon(\mathbf{T})$$

where $M^\varepsilon(\mathbf{T})$ is a martingale and

$$V(\xi,q)=-(\xi-q)(\xi_+-q_+)+\xi_+.$$

Step 3. We cut off a small region around the creation site, as well as large values of V, and perform some time averaging using the strong convergence of ρ^ε to ρ, and the apriori smoothness of ρ away from 0. The result is that

$$E\big[\left\|\xi^\varepsilon-q^\varepsilon\right\|^2_{-1,\varepsilon}\big]\Big|_{\mathbf{t}=0}^{\mathbf{t}=\mathbf{T}}\leqslant2\int_0^{\mathbf{T}}\int_{|\mathbf{x}|\geqslant\delta}E_{\bar\mu^{\varepsilon,\sigma}_{\mathbf{x},\mathbf{t}}}\big[V_\ell(\xi_0,\rho(\mathbf{x},\mathbf{t}))\big]\,d\mathbf{t}\,d\mathbf{x}+\Omega(\varepsilon,\ell,\sigma,\delta),$$

where $\bar\mu^{\varepsilon,\sigma}_{\mathbf{x},\mathbf{t}}$ denotes the average over $B_\sigma(\mathbf{t})=\{,\mathbf{s}\in[0,\mathbf{T}]\colon|\mathbf{s}-\mathbf{t}|\leqslant\sigma\}$ of $\tau_{\mathbf{x}}\mu_{\mathbf{s}}$ where $\mu_{\mathbf{s}}$ is the distribution of $\xi(\mathbf{s})$ and

$$\limsup_{\delta\downarrow0}\limsup_{\sigma\downarrow0}\limsup_{\ell\uparrow\infty}\limsup_{\varepsilon\downarrow0}\Omega(\varepsilon,\ell,\sigma,\delta)=0.$$

Here

$$V_\ell(\xi,\rho)=-\phi_\ell\big((\xi-\rho)(\xi_+-\rho_+)\big)+\phi_\ell(\xi_+)$$

where $\phi_\ell(x)=x$ if $x\leqslant\ell$ and ℓ otherwise. Finally one shows that the family $\bar\mu^{\varepsilon,\sigma}_{\mathbf{x},\mathbf{t}}$, $\varepsilon>0$ is tight.

Step 4. Let $\bar\mu^{\sigma}_{\mathbf{x},\mathbf{t}}$ be any weak limit of $\bar\mu^{\varepsilon,\sigma}_{\mathbf{x},\mathbf{t}}$ as $\varepsilon\to0$. Let f be any local function and $|\mathbf{x}|\geqslant\delta$. Recall that L is the generator of the dynamics without creation, and σ is much smaller than δ. We have

$$E_{\bar{\mu}^{\sigma}_{\mathbf{x},\mathbf{t}}}[Lf] = \lim_{\varepsilon\to 0} Av_{|\mathbf{s}-\mathbf{t}|\leqslant\sigma} E\big[Lf(\tau_x\xi_{\mathbf{s}})\big].$$

The latter term can be written as the limit as $\varepsilon \to 0$ of

$$\big|[\mathbf{t}-\sigma,\mathbf{t}+\sigma]\cap[0,\mathbf{T}]\big|^{-1} \int\limits_{[\mathbf{t}-\sigma,\mathbf{t}+\sigma]\cap[0,\mathbf{T}]} E\big[Lf(\tau_x\xi_{\mathbf{s}})\big].$$

By the definition of the generator this becomes,

$$\varepsilon^2\big|[\mathbf{t}-\sigma,\mathbf{t}+\sigma]\cap[0,\mathbf{T}]\big|^{-1} E\big[Av_{|y-\mathbf{x}|\leqslant\sigma\varepsilon^{-1}} f(\tau_y\xi_{\mathbf{s}})\big]\big|_{\mathbf{s}=(\mathbf{t}-\sigma)\wedge 0}^{\mathbf{s}=(\mathbf{t}+\sigma)\vee\mathbf{T}}.$$

Note that $\tau_y f$ never depends on ξ_0, and therefore the creation part of the ξ dynamics does not appear in the last expression. Taking $\varepsilon \to 0$ we obtain $E_{\bar{\mu}^{\sigma}_{\mathbf{x},\mathbf{t}}}[Lf] = 0$ for any bounded local f and therefore $\bar{\mu}^{\sigma}_{\mathbf{x},\mathbf{t}}$ is invariant for L.

Step 5. We arrive at

$$\int\limits_{|\mathbf{x}|\geqslant\delta}\int\limits_0^{\mathbf{T}}\int\limits_{\mathcal{B}} E_{\mu_\beta}\big[V_\ell(\xi_0,\rho(\mathbf{x},\mathbf{t})\big]\Psi_{\mathbf{x},\mathbf{t}}(d\beta)\,d\mathbf{t}\,d\mathbf{x}$$

where each $\Psi_{\mathbf{x},\mathbf{t}}$ is a probability measure on the parameter space $\mathcal{B}$ parametrizing the extremal invariant measures L. We let $\ell \to \infty$ and use the monotone convergence theorem to remove the cutoff ℓ on V. Since

$$E_{\mu_\beta}\big[V(\xi_0,\rho)\big] \leqslant 0$$

for any such μ_β, and any ρ, we have shown that $E[\|\xi^\varepsilon(\mathbf{t}) - q^\varepsilon(\mathbf{t})\|^2_{-1,\varepsilon}]|_{\mathbf{t}=0}^{\mathbf{t}=\mathbf{T}}$ vanishes in the limit of small ε. Hence $\xi^\varepsilon - q^\varepsilon$ tends to zero weakly in probability. From step 1, we know that for almost every realization ζ of the traps, $q^\varepsilon - \rho^\varepsilon$ tend to zero weakly as well, and this completes the proof.

3. Asymptotics for order statistic of free random walks and supercritical IDLA in one dimension

In this section we state asymptotic estimates on the position of free random walks and use this to compute the size of supercritical IDLA in dimension $d = 1$.

We begin by defining an order statistics on a sequence of real numbers $a_1, a_2, \ldots$. Let $M \in \mathbf{N}$ and $a^M_{(1)}$ be the largest among the first M members of such sequence,

$$a^M_{(1)} = \sup_{1\leqslant n\leqslant M}\{a_n\}$$

and recursively define the kth largest $a^M_{(k)}$ among the first M members of this sequence

$$a^M_{(k)} = \sup_{1\leqslant n\leqslant M}\big\{a_n\colon a_n \neq a^M_{(j)} \text{ for } 1 \leqslant j \leqslant k-1\big\}.$$

Let $Y_1(t), Y_2(t), \ldots$ be independent continuous time simple symmetric random walks on $\mathbf{Z}$ created at times $t_1, t_2, \ldots$ and jumping after that at rate 1. Let $N_t = \max\{i\colon\ t_i \leqslant t\}$. We can add a convention that $Y_n(t) = 0$ for $0 \leqslant t \leqslant t_n$. Then, we have an order statistics on the first M born random walks at time t, given by $\{Y_{(k)}^M(t)\colon\ k \in \mathbf{N}\}$. Similarly we have an order statistics on the rightmost positions attained by each random walk between time 0 and t, and denoted by $\{\overline{Y}_{(k)}^M(t)\colon\ k \in \mathbf{N}\}$. The following asymptotics will be proved in Section 4.

THEOREM 3.1. – *Let $n_t : [0, \infty) \to \mathbf{N}$ be increasing and assume that either $\log N_t \gg \log t$ or that $N_t = \lfloor t^\alpha \rfloor$ for some $\alpha > 1/2$. Furthermore, assume that there is a $\delta > 0$ and a function f_t such that $1 \ll f_t \ll t$ and $N_{f_t} \gg (\log N_t)^{1+\delta}$ and that $n_t \leqslant C\sqrt{t}\,\log N_t$, for some constant C. Then*

(i) *In probability*

$$\lim_{t\to\infty} Y_{(n_t)}^{N_t}(t)/w_{n.}(t) \geqslant 1. \tag{3.1}$$

(ii) *If we assume in addition that there is a $\beta > 0$ such that $n_t \ll t^\beta$, then equality holds in* (3.1). *The theorem holds also if $Y_{(n_t)}^{N_t}(t)$ is replaced by $\overline{Y}_{(n_t)}^{N_t}(t)$.*

Remarks. – (1) The speed $w_{n.}(t)$ reflects an interplay on how the random walks Y_i affect the value of $Y_{n_t}^{N_t}(t)$. At time s, N_s random walks have been born which by time t have evolved at least a time $t - s$. Suppose one wants to measure the effect of these N_s random walks on $Y_{n_t}^{N_t}(t)$. For s small enough this effect should be negligible, since $N_s \to 0$ when $s \to 0$. On the other hand, if s is to close to t, there can be many random walks within the first N_s which evolved a time $t - s$, so that they do not contribute significantly to $Y_{n_t}^{N_t}(t)$. The supremum in $w_{n.}(t)$ corresponds to choosing the optimal time s.

(2) Most of the hypothesis of Theorem 3.1 are of a more technical nature. The hypothesis $\log N_t \gg \log t$ or $N_t = \lfloor t^\alpha \rfloor$ is a restricition from the set of N's such that $N_t \gg t^{1/2}$, and basically discards injections that could oscillate between some polynomial injection and something much larger than a polynomial injection. The hypothesis concerning the function f_t such that $1 \ll f_t \ll t$ and $N_{f_t} \gg \log N_t$ discards injections with a sudden big jump (for example $N_s = \mathrm{O}(\mathrm{e}^s)$ for $s \leqslant t - 1/t$ and $N_t = \mathrm{O}(\mathrm{e}^{\mathrm{e}^s})$). These assumptions could be weakened, but we decided in favour of shorter proofs over the most general statements.

Proof of the lower bound of Theorem 1.1(3). – Here we prove that for almost every realization of the trap configuration ζ, in P_ζ-probability

$$\lim_{t\to\infty} r_t/w_{\sqrt{t}}(t) \geqslant 1. \tag{3.2}$$

Note that $r_t \leqslant \overline{Y}_{(1)}(t)$. By Theorem 3.1 part (ii) applied to the order statistics of the rightmost position $\overline{Y}(t)$ in the time interval $[0, t]$ of the random walks $Y(t)$, with $n_t = 1$, for every ζ, in P_ζ-probability,

$$\limsup_{t\to\infty} r_t/w_1(t) \leqslant 1. \tag{3.3}$$

By symmetry, the same statement holds for the (reflected) leftmost particle $-l_t$. The number of stopped random walks X_i at time t is given by $\sum_{x=l(t)}^{r_t} \zeta_x$. Therefore if t is sufficiently large, less than $\sum_{x=-2w_1(t)}^{2w_1(t)} \zeta_x \leq 5w_1(t)$ random walks $X_i(t)$ have been stopped. The smallest possible value of r_t then corresponds to stopping the rightmost $5w_1(t)$ random walks and hence for each ζ, in P_ζ-probability

$$\limsup_{t\to\infty} r_t / Y_{(\lfloor 5w_1(t)\rfloor)}(t) \geqslant 1. \tag{3.4}$$

Now, by standard estimates on the function $I^{-1}(x)$ (see Proposition 4.1, Section 4) $w_1(t) \leqslant C\sqrt{t}\log N_t$, for some $C < \infty$. Therefore, (1.5) together with an application of part (i) of Theorem 3.1, this time with $n_t = \lfloor 5w_1(t)\rfloor$, shows that for every ζ, in P_ζ-probability

$$\limsup_{t\to\infty} r_t / w_{5w_1(\cdot)}(t) \geqslant 1.$$

The lower bound (3.2) now follows from the equality $\lim_{t\to\infty} w_{5w_1(\cdot)}(t)/w_{\sqrt{t}}(t) = 1$, which is verified using the concavity property of the function $I^{-1}(x)$ (see Proposition 4.1, Section 4) and considering separately the cases $N_t = \lfloor t^\alpha \rfloor$, $\alpha > 1/2$, and $\log N_t \gg \log t$.

Proof of the upper bound of Theorem 1.1(3). – First we claim that for each $k \geqslant 1$,

$$r_t \leqslant \overline{Y}_{(k)}(t) + M(k) \tag{3.5}$$

where $M(k)$ represents the number of sites between $\overline{Y}_{(k)}(t)$ and the position of the $(k-1)$th trap strictly to the right of $\overline{Y}_{(k)}(t)$.

Indeed for all j, $\overline{X}_{(j)}(t) \leqslant \overline{Y}_{(j)}(t)$. Let us fix a t and renumber the particles according to their record values in $[0, t]$. More precisely, let $n(j)$ be defined by $\overline{X}_{n(j)}(t) = \overline{X}_{(j)}(t)$. Since $X_{n(j)}(t) \leqslant \overline{X}_{n(j)}(t)$ we certainly have $X_{n(j)}(t) \leqslant \overline{Y}_{(j)}(t)$. Hence the only particles whose positions at time t could possibly be larger than $\overline{Y}_{(k)}(t)$ are $X_{n(1)}, \ldots, X_{n(k-1)}$. There are $k-1$ such particles, so if one of them is stricly to the right of the $(k-1)$th trap to the right of $\overline{Y}_{(k)}(t)$, then by the pigeonhole principle one of the traps must be empty. Since there is a particle to the right of it, this contradicts the definition of the internal DLA dynamics. Hence (1.7) holds.

By the strong law of large numbers, for almost every realization of the trap configuration we have $m = \lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^{n} \zeta_x > 0$. Therefore, for almost every realization of the trap configuration we have that $M(k) \leqslant \frac{2}{m}k$, eventually in k. Choosing $k = \lfloor\sqrt{t}\rfloor$ we can now conclude from (1.7) that for almost every realization of the trap configuration,

$$r_t \leqslant \overline{Y}_{(\lfloor\sqrt{t}\rfloor)}(t) + \frac{2}{m}\sqrt{t},$$

eventually in t. By Theorem 3.1, for almost every realization of the trap configuration in P_ζ-probability, we have the upper bound $\lim_{t\to\infty} r_t / w_{\sqrt{t}}(t) \leqslant 1$.

4. Asymptotics for order statistic of free random walks

In this section we prove Theorem 3.1. Although the methods are standard, we were unable to find relevant references in the literature. So complete proofs are included here. Our first step in Section 4.1 will be to derive precise asymptotic estimates on the tail distribution of a continuous time symmetric simple random walk. In Section 4.2 we will derive tail estimates on the order statistics of independent random walks born at the same time, and then on the right-most random walk from the set $\{Y_i(t)\colon 1 \leqslant i \leqslant N_t\}$. In Section 4.3, we first derive the lower bound of part (i) of Theorem 3.1. This is based in finding a time s for N_s independent random walks born at time s that maximizes their order statitics positions. Next, in Section 4.3 we derive the upper bound of part (ii) of Theorem 3.1. This will be an application of the estimates of Section 4.2 analyzing separately the case $\log N_t \gg \log t$ and $N_t = \lfloor t^\alpha \rfloor$, with $\alpha > 1/2$.

4.1. Asymptotics for a continuous time symmetric simple random walk

The main result of this subsection is Lemma 4.2 which gives the asymptotics for the tail distribution of a simple continuous time random walk. The result is standard in the sense that different versions of these estimates can be found in the literature, however, never in the particular form needed in this paper. Note in particular that in Lemma 4.2 the time α_t may even go to 0 as $t \to \infty$.

Before we start we collect some basic information about the rate function (1.2).

PROPOSITION 4.1. –
(i) *$I(x)$ is convex and $I^{-1}(x)$ is concave.*
(ii) $I'(x) = \sinh^{-1} x = \log(x + \sqrt{1+x^2})$.
(iii) $I^{-1}(x) \geqslant \sqrt{2x}(1 - 4\sqrt{x})\mathbf{1}_{x \leqslant I(1/2)} + \frac{x}{3\log x}\mathbf{1}_{x \geqslant I(1/2)} \geqslant \sqrt{x/2}$.
(iv) $I^{-1}(x) \leqslant \sqrt{2x}(1 + \sqrt{x})\mathbf{1}_{x \leqslant I(1/2)} + 10x\mathbf{1}_{x > I(1/2)}$.

Proof. – (i) and (ii) are clear. To prove (iii), note that $1 + x^2/2 - x^4/8 \leqslant \sqrt{1+x^2} \leqslant 1 + x^2/2$, and $\log(1+x) \leqslant x$. Therefore, $I(x) \leqslant x^2/2 + x^3/2 + x^4/8 \leqslant x^2/2 + x^4$, when $x \leqslant 1/2$. Inverting this relationship we obtain the lower bound on I^{-1} for $x \leqslant I(1/2)$. For $x \geqslant 1/2$, note that $I(x) \leqslant x\log(6x)$. But the inverse of the function $x\log(6x)$ is larger than $x/(6\log(x))$ when $x \geqslant I(1/2)$. This finishes the proof of the lower bound. To prove (iv), note that $\log(1+x) \geqslant x - x^2/2$. Therefore, $I(x) \geqslant x\log(1+x) - \sqrt{1+x^2} + 1 \geqslant x(x - x^2/2) - x^2/2 \geqslant x^2/2 - x^3/2$ if $x \leqslant 1/2$. Inverting we obtain the upper bound on $I^{-1}(x)$ for $x \leqslant I(1/2)$. The large x upper bound is similar.

LEMMA 4.2. – *Let $Z(t)$ be a continuous time symmetric simple random walk on $\mathbf{Z}$, starting at the origin at time 0 and running at rate 1. Let $\alpha_t, \beta_t : [0,\infty) \to (0,\infty)$ satisfy $\beta_t \gg 1$ and $\beta_t \geqslant C\sqrt{\alpha_t \log(\alpha_t^2 + 1)}$ for some $C > 0$. Let $a = (\alpha^2 + \beta^2)^{1/4}$. Then,*

$$P\big(Z(\alpha_t) \geqslant \beta_t\big) = \frac{\mathrm{e}^{-\alpha_t I(\beta_t/\alpha_t)}}{\sqrt{2\pi}\, a_t(1 - \mathrm{e}^{-I'(\beta_t/\alpha_t)})}[1 + R_t]$$

where $|R_t| \leqslant \frac{30}{C}(4\log a_t)^{-1/6}$.

Proof. – We fix t and drop the subindex t on α and β temporarily. Let $\nu > 0$ and $n \in \mathbf{N}$ be such that $\nu n = \alpha$. For $\lambda > 0$, let $V_1^{\lambda,\nu}, V_2^{\lambda,\nu}, \ldots,$ be independent and identically distributed with $P(V_1^{\lambda,\nu} = k) = P(Z(\nu) = k)\exp\{\lambda k - \nu(\cosh\lambda - 1)\}$ and $S_n^{\lambda,\nu} = \sum_{i=1}^n V_i^{\lambda,\nu}$ so that for $j \in \mathbf{Z}$,

$$P\big(Z(\alpha) = j\big) = P\big(S_n^{0,\nu} = j\big) = P\big(S_n^{\lambda,\nu} = j\big)\exp\big\{-\lambda j + \alpha(\cosh\lambda - 1)\big\}.$$

The supremum in (1.2) is attained at $\lambda = \sinh^{-1} x$ so by Proposition 4.1(ii),

$$P\big(Z(\alpha) = \beta + j\big) = P\big(S_n^{I',\nu} = \beta + j\big)\exp\big\{-jh - \alpha I(\beta/\alpha)\big\}. \tag{4.1}$$

The characteristic function $E[\exp\{iuS_n^{\lambda,\nu}\}]$ of $S_n^{\lambda,\nu}$ is given by

$$\begin{aligned}&\exp\big\{n\nu(\cosh(\lambda + iu) - \cosh\lambda)\big\}\\ &\quad = \exp\left\{\alpha\left(iu\sinh\lambda - \frac{1}{2}u^2\cosh\lambda - \frac{1}{6}iu^3\sinh\lambda + R_{u,\lambda}\right)\right\},\end{aligned}$$

where $R_{u,\lambda} = \frac{1}{6}\int_\gamma (z-\lambda)^3\cosh z\,dz$, the integral being taken over the contour $\gamma = \{z \in \mathbf{C}\colon\ z = \lambda + iy,\ 0 \leqslant y \leqslant u\}$. Since for $z \in \gamma$, $|\cosh z| \leqslant \cosh\lambda$, we can write $R_{u,\lambda} = (\cosh\lambda)R_1(u)$, where $|R_1(u)| \leqslant u^4/24$. By Fourier inversion, since $\alpha\cosh I' = a^2$,

$$P\big(S_n^{I',\nu} = \beta + j\big) = \int_{-\pi}^{\pi} \mathrm{e}^{-iuj}\exp\left\{-\frac{1}{2}u^2a^2 - \frac{1}{6}iu^3\beta + a^2R_1(u)\right\}\frac{du}{2\pi}.$$

After an elementary change of scale this becomes $a^{-1}\int_{-\pi a}^{\pi a}\exp\{-ivj/a - \frac{1}{2}v^2 + R_2(v)\}\frac{dv}{2\pi}$, where $R_2(v) = \frac{1}{6}iv^3\beta a^{-3} + \mathrm{O}(v^4)$, and $|\mathrm{O}(v^4)| \leqslant \frac{1}{24}v^4a^{-2}$. Substituting in Eq. (1.7), and summing over j we get that

$$P\big(Z(\alpha) \geqslant \beta\big) = \frac{1}{2\pi a}\mathrm{e}^{-\alpha I(\beta/\alpha)}\int_{-\pi a}^{\pi a}\big(1 - \exp\{-I' - iv/a\}\big)^{-1}\mathrm{e}^{-\frac{v^2}{2} + R_2(v)}\,dv. \tag{4.2}$$

Let I_1 denote the integration restricted to $|v| \leqslant (\log a)^{1/3}$ and I_2 the remainder. We now claim that if $|v| \leqslant (\log a)^{1/3}$, then

$$\left|\frac{1 - \exp\{-I'\}}{1 - \exp\{-I' - iv/a\}} - 1\right| \leqslant \frac{6}{C(\log a)^{1/6}}. \tag{4.3}$$

In order to check this note that the left hand side is always bounded by $\frac{v}{a}\frac{\exp\{-I'\}}{|1-\exp\{-I'-iv/a\}|}$. We can bound the absolute value in the denominator below by $(1 - \exp\{-1\})\min(1, I')$. So using $1 - \exp\{-1\} \geqslant 1/3$ and Proposition 4.1(ii) and dropping a few terms, using $(\alpha + \beta)^2 \geqslant \alpha^2 + \beta^2$, we see that the left hand side of (4.3) is bounded above by $3v\sqrt{\alpha}\,a^{-2}\max(1, 1/I')$.

Now $I'(\beta/\alpha) \geqslant \log(1 + \frac{\beta}{\alpha})$. Note that for $x \geqslant 1/2$, $\log(1 + x) \geqslant \log(3/2)$ and for $0 \leqslant x \leqslant 1/2$, $\log(1 + x) \geqslant x/2$. Also $1/\log(3/2) \leqslant 3$. Hence we obtain an upper bound by replacing $\max(1, 1/I')$ by $\max(3, 2\alpha/\beta)$, or, since $\beta \geqslant C\sqrt{\alpha\log(1+\alpha^2)}$, by $\max(3, 2C^{-1}\sqrt{\alpha/\log(1+\alpha^2)})$. Now

$$\sqrt{\alpha/\log(1+\alpha^2)} \leqslant \frac{(1+\alpha^2)^{1/4}}{(\log(1+\alpha^2))^{1/2}} \leqslant \frac{(\beta^2+\alpha^2)^{1/4}}{(\log(\beta^2+\alpha^2))^{1/2}}.$$

Thus, for t large enough we have $\max(3, 2C^{-1}\sqrt{\alpha/\log(1+\alpha^2)}\,) \leqslant 2\frac{(\beta^2+\alpha^2)^{1/4}}{C(\log(\beta^2+\alpha^2))^{1/2}}$. This gives (4.3).

It is also not hard to check that if $|v| \leqslant (\log a)^{1/3}$,

$$\left|\exp\{-v^2/2 + R_2(v)\} - \exp\{-v^2/2\}\right| \leqslant 4a^{-1}\log a, \tag{4.4}$$

and that

$$\left|\sqrt{2\pi} - \int_{-(\log a)^{1/3}}^{(\log a)^{1/3}} \exp\{-v^2/2\}\,dv\right| \leqslant 2\exp\left\{-\frac{1}{2}(\log a)^{2/3}\right\}. \tag{4.5}$$

Integrating (4.3), (4.4) and (4.5), we conclude that,

$$I_1 = \frac{\sqrt{2\pi}}{1-\mathrm{e}^{-I'}}\big(1 + R_3(t)\big), \qquad \big|R_3(t)\big| \leqslant 20C^{-1}(\log a)^{-1/6}. \tag{4.6}$$

On the other hand, we claim that

$$\left|\big(1-\exp\{-I'\}\big)I_2\right| \leqslant 8\exp\left\{-\frac{1}{12}(\log a)^{2/3}\right\}. \tag{4.7}$$

In fact, first note that

$$\big(1-\exp\{-I'\}\big)I_2 = \int_{(\log a)^{1/3}<|v|\leqslant\pi a} \left|\frac{1-\mathrm{e}^{-I'}}{1-\mathrm{e}^{-I'-iv}}\right| \mathrm{e}^{-v^2/2+R_2(v)}\,dv.$$

Now the term in absolute value is bounded by 1 and $|\exp R_2(v)| \leqslant \exp\{v^4a^{-2}/24\}$ so

$$\left|\big(1-\exp\{-I'\}\big)I_2\right| \leqslant \int_{(\log a)^{1/3}<|v|\leqslant\pi a} \mathrm{e}^{-v^2(\frac{1}{2}-(\frac{v}{a})^2\frac{1}{24})}\,dv.$$

But, $v/a \leqslant \pi$, so that the factor multiplying v^2 in the exponent of this bound is larger than $1/2 - \pi^2/24 \geqslant 1/12$. Hence, we get the bound

$$\left|\big(1-\exp\{-I'\}\big)I_2\right| \leqslant \int_{(\log a)^{1/3}<|v|\leqslant\infty} \mathrm{e}^{-v^2/12}\,dv,$$

from which (4.7) follows.

Combining (4.6) and (4.7) with (4.2) gives a proof of the lemma.

4.2. Order statistics: independent identically distributed random walks

As discussed earlier, a main ingredient in the proof of Theorem 2 will be to obtain asymptotic estimates on the order statistics of independent random walks born at the same time. Let $M \in \mathbf{N}$ and consider a set $Z_1^M(t), Z_2^M(t), \ldots$ of M independent continuous time random walks such that $Z_n^M(0) = 0$ for $1 \leqslant n \leqslant M$. Consider the order statistics $\{Z_{(k)}^M(t)\colon k \in \mathbf{N}\}$ on this set of random walks at time t.

PROPOSITION 4.3. – *Let $N_t, n_t, \alpha_t : [0,\infty) \to (0,\infty)$ be increasing functions. Assume that $\alpha_t \leqslant t$ and that there is a $1 \geqslant \delta > 0$ such that $N_t \gg t^{\delta+\frac{1}{2}}$ and $n_t \ll \lfloor N_t \rfloor^{1-\delta}$. Furthermore, for $-1 \leqslant \gamma \leqslant 1$ define*

$$\Phi_t^\gamma = \alpha_t I^{-1}\left(\frac{1+\gamma}{\alpha_t}\log\frac{N_t}{n_t}\right) \tag{4.8}$$

and assume that N_t, n_t and α_t are such that $\Phi_t^0 \gg 1$. Then, for every $1/2 \geqslant \varepsilon > 0$ for sufficiently large t,

(i) $P(Z_{(n_t)}^{N_t}(\alpha_t) \geqslant \Phi_t^\varepsilon) \leqslant 8\mathrm{e}^2 N_t^{-\delta\varepsilon/4}$.

(ii) $P(Z_{(n_t)}^{N_t}(\alpha_t) \leqslant \Phi_t^{-\varepsilon}) \leqslant \exp\{-\lfloor N_t \rfloor^{\delta\varepsilon/10}\}$.

Before proceeding with the proof of the proposition, we will need the following lemma, which states some properties of some expressions that will appear when applying the tail asymptotics of Lemma 4.2.

LEMMA 4.4. – *For $-1/2 \leqslant \gamma \leqslant 1/2$ let I^{-1}, α, Φ^γ, N_t and n_t be as in the previous proposition and define*

$$r_t^\gamma = \sqrt{2\pi}\left(1 - \exp\left\{-I'\left(I^{-1}(\Phi_t^\gamma)\right)\right\}\right)\left(\alpha_t^2 + [\Phi_t^\gamma]^2\right)^{1/4}.$$

Then r_γ is increasing in γ and for t large enough,

(i) $\frac{1}{6} \leqslant r_t^{-1/2} \leqslant r_t^{1/2} \leqslant 10t\log N_t$,

(ii) $(\frac{N_t}{n_t})^\gamma \frac{1}{r_t^\gamma} \gg \frac{t^{\delta/8}}{8\log t}$.

Proof. – The monotonicity can be checked directly.

(i) To prove the leftmost inequality, by Proposition 4.1,

$$r^{-1/2} = \sqrt{2\pi}\,\frac{I^{-1} + \sqrt{(I^{-1})^2+1} - 1}{I^{-1} + \sqrt{(I^{-1})^2+1}}\left(\alpha^2 + \left[\Phi^{-1/2}\right]^2\right)^{1/4},$$

where the argument of I^{-1} is $+\frac{1}{2\alpha}\log(N/n)$, and this can be bounded below by $\frac{(\alpha)^{1/2}(I^{-1})^{3/2}}{(I^{-1}+\sqrt{(I^{-1})^2+1})}$. For $I^{-1} \geqslant 1$ and t large enough this can be written as

$$(\Phi^0)^{1/2}\frac{I^{-1}}{(I^{-1} + \sqrt{(I^{-1})^2+1})} \geqslant 1\cdot\frac{1}{3},$$

where we have used the hypothesis $\Phi_t^0 \gg 1$. On the other hand, for $I^{-1} < 1$ we have the inequality

$$\frac{\alpha(I^{-1})^2}{\sqrt{\alpha I^{-1}}} \frac{1}{I^{-1} + \sqrt{(I^{-1})^2 + 1}} \geqslant \frac{1}{10}\bigl(\log(N/n)\bigr)^{1/2},$$

where we used Proposition 4.1(iii) and (iv). This proves that $\frac{1}{3} \leqslant r_t^{-1/2}$. To prove the upper bound, note that since $\alpha_t \leqslant t$, we have that $r_t^{1/2} \leqslant 6(t + \alpha I^{-1})$. Now, by Proposition 4.1(iv), we know that $I^{-1} \leqslant 10\alpha^{-1} \log N(t)$. Our upper bound now follows for t large enough.

(ii) Since $r_t^\gamma \leqslant 10t \log N_t$ for large t, if t satisfies $N_t \geqslant t^{\log t}$, the left hand side of (ii) is bounded below by $t^{\frac{\delta\gamma \log t}{4}}/10t(\log t)^2$ which certainly dominates the right hand side of (ii) for large t. So we only need to consider the case $N_t < t^{\log t}$. We divide it into two cases. If $\alpha_t \geqslant (\log t)^3$ and t is large enough then from the definition of r^γ it follows that $r_t^\gamma \leqslant 2I^{-1}(4(\log t)^2/\alpha_t)(\alpha_t^2 + [\Phi_t^\gamma]^2)^{1/4}$. By Proposition 4.1(iv) this can be bounded by

$$4\sqrt{(4(\log t)^2/\alpha_t)}\bigl(\alpha_t^2 + [\Phi_t^\gamma]^2\bigr)^{1/4} \leqslant 4\log t\bigl(1 + \bigl[I^{-1}(4/\log t)\bigr]^2\bigr)^{1/4}.$$

This is bounded by $8\log t$ when t is large enough. On the other hand, if $\alpha_t < (\log t)^3$, we have $r_t^\gamma \leqslant 2((\log t)^3 + (\log t)^4)^{1/4} \leqslant 8\log t$. This concludes the proof of the lemma.

Proof of Proposition 4.3. – (i) From the definition of the order statistics, for any $s \geqslant 0$, $x \geqslant 0$, and $M, m \in \mathbf{N}$ with $m \leqslant M$,

$$P\bigl(Z_{(m)}^M(s) \geqslant x\bigr) = \sum_{k=m}^{M} \binom{M}{k} P\bigl(Z(s) \geqslant x\bigr)^k \bigl(1 - P\bigl(Z(s) \geqslant x\bigr)\bigr)^{M-k}. \tag{4.9}$$

We take $M = N_t$, $m = n_t$, $s = \alpha_t$ and $x = \Phi_t^\varepsilon$ and apply Lemma 4.2 with $\beta_t = \Phi_t^\varepsilon$. We need to verify the hypothesis of that lemma. Since I^{-1} is increasing, $\Phi_t^\varepsilon \geqslant \Phi_t^0$ and by hypothesis $\Phi_t^0 \gg 1$, so $\Phi_t^\varepsilon \gg 1$. By Lemma 4.2, $\Phi_t^\varepsilon \geqslant \Phi_t^0 \geqslant \frac{1}{\sqrt{2}}\sqrt{\alpha_t \log(N_t/n_t)}$ and by our assumptions we have that for t large enough $\log(N/n) \geqslant \log N^{1-\delta} \geqslant \log t^{1/4}$ so $\Phi_t^0 \geqslant \sqrt{\alpha_t \log t}/4\sqrt{2} \geqslant \frac{1}{30}\sqrt{\alpha_t \log(\alpha_t^2 + 1)}$ for t large enough. So the hypothesis of Lemma 4.2 are satisfied.

Now we can apply Lemma 4.2 to (4.9). Then we use Stirling's formula and $N/[x(N - x)] \leqslant 4/N$ on the binomial coefficients to get

$$P\bigl(Z_{(n_t)}^{N_t}(\alpha_t) \geqslant \Phi_t^\varepsilon\bigr) \leqslant \frac{2\mathrm{e}}{\sqrt{N_t}} \sum_{k=n_t}^{N_t} u_k, \quad u_k = \left(\frac{a}{k}\right)^k \left(1 - \frac{k}{N_t}\right)^{k-N_t} b^{N_t-k}, \tag{4.10}$$

where $a = \frac{\lfloor n_t\rfloor^{1+\varepsilon}}{\lfloor N_t\rfloor^\varepsilon}\frac{1+R_t}{r_t^\varepsilon}$, $b = 1 - (\frac{n_t}{N_t})^{1+\varepsilon}\frac{1+R_t}{r_t^\varepsilon}$ and $|R_t| \leqslant 30^2(\log(\alpha_t^2 + [\Phi_t^0]^2))^{-1/6}$, which goes to 0 by the hypothesis $\Phi_t^0 \gg 1$. The function u is increasing for $0 < k \leqslant a$, decreasing for $a \leqslant k < N_t$, so it attains a global maximum at $k = a$. Now $a \leqslant n(n/N)^\varepsilon(2/r^\varepsilon) \leqslant 6n(n/N)^\varepsilon \ll n$, where in the second to last inequality we have applied Lemma 4.2. Since $a \ll n_t$, the largest term in the summation of the right hand side of (4.10) corresponds to $k = n_t$. We now divide the sum in (4.10) in two parts: terms from $k = n_t$ to $k = \max\{n_t, \sqrt{N_t}\}$ which we bound by $\max\{n_t, \sqrt{N_t}\}u_{n_t}$, and terms from $k = \max\{n_t, \sqrt{N_t}\}$ to $k = N_t$ which we bound by $N_t u_{\sqrt{N_t}}$. Using $(1-x/N)^{x-N} \leqslant \mathrm{e}^x$, we see that $u_n \leqslant (\mathrm{e}a/n)^n$ and $u_{\sqrt{N}} \leqslant (\mathrm{e}a/\sqrt{N})^{\sqrt{N}}$. Hence we can bound the sum in (4.10) by

$$2\mathrm{e}\frac{\max\{n_t, \sqrt{N_t}\}}{\sqrt{N_t}}\left(\left(\frac{n_t}{N_t}\right)^{\varepsilon}\frac{\mathrm{e}(1+R_t)}{r_t^{\varepsilon}}\right)^{n_t} + 2\mathrm{e}\sqrt{N_t}\left(\left(\frac{1}{\sqrt{N_t}}\right)^{\varepsilon}\frac{\mathrm{e}(1+R_t)}{r_t^{\varepsilon}}\right)^{\sqrt{N_t}}.$$

Now use $|R_t| \leqslant 1$ and absorb the prefactor $\max\{n_t, \sqrt{N_t}\}/\sqrt{N_t}$ into a factor c^{n_t} and the prefactor $\sqrt{N_t}$ into a factor $c^{\sqrt{N_t}}$ to obtain that the left hand side of (4.10) is bounded by

$$4\mathrm{e}\left(\left(\frac{n_t}{N_t}\right)^{\varepsilon}\frac{2\mathrm{e}^2}{r_t^{\varepsilon}}\right)^{n_t} + 4\mathrm{e}\left(\left(\frac{1}{\sqrt{N_t}}\right)^{\varepsilon}\frac{2\mathrm{e}^2}{r_t^{\varepsilon}}\right)^{\sqrt{N_t}}.$$

For t large, the first term dominates the second. Using the bound $r_t^{\varepsilon} \geqslant 1/6$ proved in Lemma 4.2 we obtain (i).

(ii) We apply Lemma 4.2 to the analogue of (4.9) for $P(Z^M_{(m)}(s) \leqslant x)$ as in (i) but with $x = \Phi_t^{-\varepsilon}$, $\beta_t = \Phi_t^{-\varepsilon}$. To apply the lemma we need to verify that for some $C > 0$ such that $\Phi_t^{-\varepsilon} \geqslant C\sqrt{\alpha_t \log(\alpha_t^2 + 1)}$. By the concavity of I^{-1} it follows that $\Phi_t^{-\varepsilon} \geqslant (1-\varepsilon)\Phi_t^0$. So it is enough to show that $\Phi_t^0 \geqslant C\sqrt{\alpha_t \log(\alpha_t^2 + 1)}$ which is proved in the first paragraph of the proof (i). Hence we can apply the lemma to obtain

$$P\left(Z^{N_t}_{(n_t)}(\alpha_t)\right) \leqslant \sum_{k=0}^{n_t}\binom{N_t}{k}\rho^k(1-\rho)^{N_t-k}$$

where $\rho = (\frac{n_t}{N_t})^{1-\varepsilon}\frac{1+R_t}{r_t^{-\varepsilon}}$ with $|R_t| \leqslant 50^2(2\log\Phi_t^0)^{-1/6}$. Now

$$\binom{N}{k} \leqslant 2\mathrm{e}\left(1-\frac{k}{N}\right)^{k-N}\left(\frac{N}{k}\right)^k \leqslant 2\mathrm{e}^{k+1}\left(\frac{N}{k}\right)^k.$$

Since $|R_t| \leqslant 2$ and $1/r_t^{-\varepsilon} \leqslant 6$ for t sufficiently large, $\rho \leqslant 6(n_t/N_t)^{1-\varepsilon} \leqslant 2$. Also $(1-1/x)^{-1} \leqslant \mathrm{e}^{2/x}$ whenever $x \geqslant 2$ and hence $(1-\rho)^{-n_t} \leqslant \mathrm{e}^{6n_t(n_t/N_t)^{1-\varepsilon}}$.

Now $(1-1/x) \leqslant \mathrm{e}^{-1/x}$ if $x > 0$, and therefore we also have $(1-\rho)^N \leqslant \mathrm{e}^{-n(\frac{N}{n})^{\varepsilon}\frac{1}{2r^{-\varepsilon}}}$. For $c > 0$ the function $f(x) = (\frac{c}{x})^x$ achieves its maximum at $x = c/\mathrm{e}$. And the first term in the summation in (4.10) corresponds to $f(x)$ with, $c = \mathrm{e}n_t m_t(1+R_t)$ where $m_t = (\frac{N_t}{n_t})^{\varepsilon}\frac{1}{r_t^{-\varepsilon}}$. By Lemma 4.2, $c \gg n_t$. This implies that the maximum of the first factor in the summation in (4.10) is attained at $k = n_t$. So for sufficiently large t,

$$P\left(Z^{N_t}_{(n_t)}(t) \leqslant \Phi_t^{\varepsilon}\right) \leqslant 2\mathrm{e}n_t\exp\left\{n_t\left(1 - m_t/2 + \log(2\mathrm{e}m_t)\right)\right\}.$$

(ii) Follows from this inequality using $n_t \leqslant \exp n_t$ and Lemma 4.2.

4.3. Order statistics: rightmost random walk

We now state the second ingredient in the proof of Theorem 2, a lower bound estimate on the position at time t of the rightmost random walk among the Y_i, $1 \leqslant i \leqslant N_t$. For $\gamma \in [-1, 1]$ define,

$$w_{n.}^{\gamma}(t) = \sup_{0 \leqslant y \leqslant t} (t-y) I^{-1}\left(\frac{1+\gamma}{t-y} \log \frac{N_y}{n_t}\right), \tag{4.11}$$

with the understanding that $I^{-1}(x) = 0$ if $x < 0$. Also, let $g_{n.}^{\gamma}(t) : [0, \infty) \to [0, \infty)$ be the maximizer in (4.11) and define $N_{n.}^{\gamma}(t) = N_{g_{n.}^{\gamma}(t)}$.

PROPOSITION 4.5. – *Let $N_t, n_t : \mathbf{N} \to [0, \infty)$ be increasing functions such that for some $\delta > 0$, $\lfloor N_t \rfloor^{1-\delta} \gg n_t$ and $N_t \gg t^{1/2+\delta}$. Then, for every $1 \geqslant \varepsilon > 0$, for sufficiently large t,*

$$P\left(Y_{(1)}^{N_t}(t) > w_1^{\varepsilon}(t)\right) \leqslant 4\varepsilon^{-1}(t \log N_{t/2})^{-1/4}.$$

Proof. – First note that for every $x \geqslant 0$, $P(Y_{(1)}^{N_t}(t) \leqslant x) = \prod_{i=1}^{N_t} P(Z(t-T_i) \leqslant x)$. We want to apply Lemma 4.2 to each multiplicand with $\alpha_t = t - t_i$ and $\beta_t = w_1^{\varepsilon}(t)$. We need to verify that the hypothesis are satisfied. It is trivial to verify that $w_{1,\varepsilon} \gg 1$. To show that there is a constant C such that $w_1^{\varepsilon}(t) \geqslant C\sqrt{(t-t_i)\log((t-t_i)^2+1)}$ it is enough to verify that $w_1^{\varepsilon}(t) \geqslant C\sqrt{t \log(t^2+1)}$. This is a consequence of the fact that $w_1^{\varepsilon}(t) \geqslant \frac{t}{2} I^{-1}(\frac{2}{t} \log t^{\delta/4})$ (where we have used the assumptions $\lfloor N_t \rfloor^{1-\delta} \gg n_t$ and $N_t \gg t^{1/2+\delta}$) and the lower bound $\sqrt{x/2}$ on the function $I^{-1}(x)$ given in Proposition 4.1. Therefore,

$$P\left(Y_{(1)}^{N_t}(t) \leqslant w_1^{\varepsilon}(t)\right) = \prod_{i=1}^{N_t}\left(1 - \exp\left\{-\frac{t-t_i}{v(t,t_i)} I\left(\frac{w_1^{\varepsilon}(t)}{t-t_i}\right)\right\}(1+R_{i,t})\right) \tag{4.12}$$

where

$$v(t,s) = \sqrt{2\pi}\left((t-s)^2 + \left[w_1^{\varepsilon}(t)\right]^2\right)^{1/4}\left(1 - \exp\left\{-I'\left(w_1^{\varepsilon}(t)/(t-s)\right)\right\}\right)$$

and $|R_{i,t}| \leqslant \frac{30}{C}(\log w_1^{\varepsilon}(t))^{-1/6}$. Using the definition of $w_1^{\varepsilon}(t)$, we see that if $u > 0$, $(t-u) I(\frac{w_1^{\varepsilon}(t)}{t-u}) \geqslant \log \lfloor N_u \rfloor^{1+\varepsilon}$. Using this in (4.12) and taking logarithms we get,

$$\begin{aligned}
\log P\left(Y_{(1)}^{N_t}(t) \leqslant w_1^{\varepsilon}(t)\right) &\geqslant \sum_{i=1}^{N_t} \log\left(1 - \frac{1 \wedge \lfloor N_{t_i} \rfloor^{-(1+\varepsilon)}}{v(t,t_i)}(1+R_{i,t})\right) \\
&\geqslant -\sum_{i=1}^{N_t} \frac{1+R_{i,t}}{i^{1+\varepsilon} v(t,t_i)}.
\end{aligned} \tag{4.13}$$

In the last inequality we used $N_{t_i} \geqslant i$. We want now to obtain a lower bound on the function $v(t,t_i)$, uniform on i, to show that the rightmost hand side of (4.13) goes to 0. Using Proposition 4.1(ii), we see that $v(t,s) \geqslant \frac{w_{1,\varepsilon}((t-s)^2+w_{1,\varepsilon}^2)^{1/4}}{w_{1,\varepsilon}+((t-s)^2+w_{1,\varepsilon}^2)^{1/4}}$. But for x

and y positive, the expression $\frac{xy}{x+y}$ is increasing in x. Applying this to the previous inequality with $y = w_1^\varepsilon$ and x decreasing from $((t-s)^2 + [w_1^\varepsilon]^2)^{1/4}$ to $\sqrt{w_1^\varepsilon}$, we see that $v(t,s) \geqslant \sqrt{w_1}/2$. Also note that $\sum_{i=1}^{N_t} \frac{1}{i^{1+\varepsilon}} \leqslant 1 + \int_1^\infty \frac{1}{x^{1+\varepsilon}}\,dx \leqslant \frac{2}{\varepsilon}$, where in the last inequality we have used the hypothesis $\varepsilon \leqslant 1$. Using these bounds together with the fact that $|R_{i,t}| \leqslant 1$ for t large enough uniformly in i, we can conclude that for sufficiently large t,

$$\log P\big(Y_{(1)}^{N_t}(t) \leqslant w_{1,\varepsilon}(t)\big) \geqslant -8\varepsilon^{-1}\big(w_1(t)\big)^{-1/2}. \tag{4.14}$$

Using the inequality $w_1(t) \geqslant \frac{t}{2} I^{-1}(\frac{2}{t}\log N_{t/2})$ and the lower bound $I^{-1}(x) \geqslant \sqrt{x/2}$, we see that $w_1(t) \geqslant \frac{1}{2}\sqrt{t \log N_{t/2}}$. The proposition follows from this, the inequality $1 - \mathrm{e}^{-x} \leqslant x$ for $x \geqslant 0$ and (4.14).

LEMMA 4.6. – *Let $N_t, n_t : [0,\infty)$ be increasing functions such that $\lfloor N_t \rfloor^{1-\delta} \gg n_t$, for some $\delta > 0$.*

(i) *For every function $f(t) = o(t)$ and $\kappa > 0$, we have $N_{n.}^\gamma(t) \gg \lfloor N_{f(t)} \rfloor^{1-\kappa}$.*

(ii) *Assume that $N_t = \lfloor t^\alpha \rfloor$ for some $\alpha > 1/2$. Then, $t/(\log t)^2 \ll g_{n.}^\gamma(t) \ll t(\log t)^{-1/2}$.*

Proof. – (i) First note that for any function $f(t)$ such that $0 \leqslant f(t) \leqslant t$ one has

$$(t-g)I^{-1}\left(\frac{1+\gamma}{t-g}\log\frac{N_{n.}^\gamma}{n}\right) \geqslant (t-f)I^{-1}\left(\frac{1+\gamma}{t-f}\log\frac{N_f}{n}\right),$$

where g, n, $N_{n.}^\gamma$, and f stand for $g_{n.}^\gamma(t)$, n_t, $N_{n.}^\gamma(t)$ and $f(t)$, respectively. Therefore,

$$\frac{1+\gamma}{t-g}\log\frac{N_n^\gamma}{n} \geqslant I\left(\left(1-\frac{f}{t}\right)I^{-1}\left(\frac{1+\gamma}{t-f}\log\frac{N_f}{n}\right)\right) \geqslant \frac{1+\gamma}{t-f}\log\frac{N_f}{n} - R_{3,t}, \tag{4.15}$$

where $R_{3,t} = \frac{f}{t} I^{-1}(\frac{1+\gamma}{t-f}\log\frac{N_f}{n}) I'(I^{-1}(\frac{1+\gamma}{t-f}\log\frac{N_f}{n}))$ and we have used the lower bound $I(y) \geqslant I(x) - (x-y)I'(x)$ valid for $0 \leqslant y \leqslant x$. We now claim that

$$\big|R_3(t)\big| \leqslant 2\frac{f}{t}\frac{1+\gamma}{t-f}\log\frac{N_f}{n}.$$

For this it is enough to prove that for $y > 0$, $yI'(y) \leqslant 2I(y)$ which can be checked directly. We can then conclude from (4.15) that,

$$\log\frac{N_{n.}^\gamma(t)}{n_t} \geqslant \left(1-\frac{g_{n.}^\gamma(t)}{t}\right)\left(1-\frac{f(t)}{t}\right)\log\frac{N_{f(t)}}{n_t}.$$

Therefore if $g_{n.}^\gamma(t) < f(t)$, since $f(t) = \mathrm{o}(t)$, for every $\kappa > 0$ we have that, $N_{n.}^\gamma(t)/n_t \gg (N_{f(t)}/n_t)^{1-\kappa}$. On the other hand if $g_{n.}^\gamma(t) \geqslant f(t)$ there is nothing to prove since $N_{n.}^\gamma(t) = N_{g_{n.}^\gamma(t)} \geqslant N_{f(t)}$. This completes the proof of (i).

(ii) By Proposition 4.1 we have $w_{n.}^\gamma(t) \geqslant \sqrt{\frac{t}{2}\log N_{t/2}/n_t}$ for sufficiently large t. Since $N_t = t^\alpha$, with $\alpha > 1/2$ and $n_t \ll \lfloor N_t \rfloor^{1-\delta}$, this implies that for some $c > 0$, for t large enough $w_{n(\cdot),\gamma}(t) \geqslant c\sqrt{t\log t}$. We now claim that this implies

$$\lim_{t\to\infty} \frac{\log N_{n.}^{\gamma}(t)/n_t}{t-g_{n.}^{\gamma}(t)} = 0. \tag{4.16}$$

In fact, note that by the definition of $N_{n.}^{\gamma}$ and $g_{n.}^{\gamma}$, the expression whose limit is taken in (4.16), is positive. Thus, if (4.16) is false, there is a subsequence t_m such that either $\log N_{n.}^{\gamma}(t_m)/n_{t_m} \sim C(t_m - g_{n.}^{\gamma}(t_m))$ for some $C>0$ or $\log N_{n.}^{\gamma}(t_m)/n_{t_m} \gg t_m - g_{n.}^{\gamma}(t_m)$. In this first case, this implies that for m large enough

$$w_{n.}^{\gamma}(t_m) \leqslant \frac{2I^{-1}(C)}{C}\log\frac{N_{n.}^{\gamma}(t_m)}{n_{t_m}} \leqslant \frac{2I^{-1}(C)}{C}(\log t_m)^2,$$

a contradiction. Similarly in the second case, using the upper bound $I^{-1}(x)\leqslant 10x$ for large x Proposition 4.1, we would conclude that $w_{n.}^{\gamma}(t_m) \ll (\log t_m)^2$, a contradiction. This proves (4.16).

Now, from Proposition 4.3,

$$w_{n.}^{\gamma}(t) = \sqrt{2(1+\gamma)\bigl(t-g_{n.}^{\gamma}(t)\bigr)\log\bigl(\lfloor g_{n.}^{\gamma}(t)\rfloor^{\alpha}/n\bigr)} + R_t, \tag{4.17}$$

where $|R_t| \leqslant 10(\log t)^2$ and we used the assumption $N_t = \lfloor t^{\alpha}\rfloor$. One can check that for t large enough the supremum over $y\in[0,t]$ of the function $\sqrt{(t-y)\log(\lfloor t^{\alpha}\rfloor/n)}$ is achieved at some $y = \mathrm{O}(t/\log t) + \mathrm{o}(t\log t)$ and that the supremum itself is $\mathrm{O}(\sqrt{t\log t}) \gg R_t$. Together with (4.17), this proves (ii).

4.4. Proof of Theorem 2

Proof of (i). – Step 1. Let $\varepsilon>0$. First we check that $N_{-\varepsilon,n}(t) \gg \lfloor n(t)\rfloor^{1+\delta/4}$, which will enable us to apply Proposition 4.3. Assume first that $N(t) = t^{1/2+\delta_0}$ for some $\delta_0>0$. Then, by (4.1), $N_{-\varepsilon,n}(t) \geqslant N_{t/(\log t)^2} \gg t^{1/2+\delta_0/2}$. On the other hand $n_t \leqslant \sqrt{t}\log N_t$. Therefore, $N_{-\varepsilon,n}(t) \gg n_t^{1/2+\delta_0/4}$. Choosing $\delta_0 \geqslant \delta$ we have that $N_{-\varepsilon,n}(t) \gg \lfloor n(t)\rfloor^{1+\delta/4}$. Now assume that $\log N_t \gg \log t$. Then, for t large enough, by (1.7) applied with $f(t) = t/\log t$, we have that for any $\delta_1>0$, $\lfloor N_{-\varepsilon,n}(t)\rfloor^{\delta_1} \gg \lfloor N_{t/\log t}\rfloor^{\delta_1/2} \gg t^{(1+\delta_1/4)/2}$. Now remark that by hypothesis, there is a function $1 \ll f_0(t) \ll t$ and a $\delta>0$ such that $N_{f_0(t)} \gg (\log N_t)^{1+\delta}$. By (1.7) with $f(t) = f_0(t)$, we have $N_{-\varepsilon,n}(t) \gg \lfloor N_{f_0(t)}\rfloor^{1-\delta/2} \gg (\log N_t)^{1+\delta/2}$. Thus, when $\log N_t \gg \log t$, we have that $N_{-\varepsilon,n}(t) \gg (\sqrt{t}\log N_t)^{(1+\delta/4)} \gg n_t$.

Step 2. If $H_t = \inf\{y\geqslant 0\colon\ N_y = N_{n.}^{-\varepsilon}(t)\}$ is the first time that $N_{n.}^{-\varepsilon}(t) = N_{g^{\varepsilon}(t)}$ random walks have been born then we want to show that

$$P\bigl(Y^{N_{n.}^{-\varepsilon}(t)}(t) \leqslant w_{n.}^{-\varepsilon}(t)\bigr) \leqslant P\bigl(Z^{N_{n.}^{-\varepsilon}(t)}(t-H_t) \leqslant w_{n.}^{-\varepsilon}(t)\bigr). \tag{4.18}$$

Write M and n for $N_{n.}^{-\varepsilon}(t)$ and n_t, and note that $P(Y_{(n)}^{N}(t) \leqslant w_{n.}^{-\varepsilon}(t)) = f_{M-n}(p_1,\ldots,p_M)$ and $P(Z_{(n)}^{N_{n.}^{-\varepsilon}}(t-H_t) \leqslant w_{n.}^{-\varepsilon}(t)) = f_{M-n}(q_1,\ldots,q_M)$ with $p_i = P(Y_i(t)\leqslant w_{n.}^{-\varepsilon}(t))$ and $q_i = P(Z_i(t-H_t)\leqslant w_{n.}^{-\varepsilon}(t))$ for $1\leqslant i\leqslant N$. But for $1\leqslant i\leqslant N$ the birth times t_i of the random walks Y_i have the property that $t_i\leqslant H_t$. Thus,

$$\begin{aligned}p_i &= P\big(Y_i(t) \leqslant w_{n.}^{-\varepsilon}(t)\big) = P\big(Z_i(t-t_i) \leqslant w_{n.}^{-\varepsilon}(t)\big)\\ &\leqslant P\big(Z_i(t-H_t) \leqslant w_{n.}^{-\varepsilon}(t)\big) = q_i.\end{aligned}$$

Note that f is a function of the form

$$f_M(p_1,\dots,p_N) = \sum_{n=M}^{N} \sum_{\pi\in\Pi(n)} p_{\pi_1}\cdots p_{\pi_n}(1-p_{\pi_{n+1}})\cdots(1-p_{\pi_N}) \tag{4.19}$$

where $\Pi(n)$ is the set of permutations of $\{1,\dots,n\}$. Any derivative is given by

$$\frac{\partial f_M}{\partial p_i} = \sum_{\pi\in\Pi(i,M-1)} p_{\pi_1}\cdots p_{\pi_{M-1}}(1-p_{\pi_M})\cdots(1-p_{\pi_{N-1}}) \geqslant 0,$$

where $\Pi(i, M-1)$ are the permutations of $\{1,\dots,i-1,i+1,\dots,N\}$. Hence $f_{M-n}(p_1,\dots,p_M) \leqslant f_{M-n}(q_1,\dots,q_M)$, which proves (4.18).

Step 3. Since $n_t^{1+\delta/2} \ll N_{n.}^{-\varepsilon}(t)$, we can apply Proposition 4.3 to the right hand side of (4.18), with $\alpha_t = t - g_{n.}^{\varepsilon}(t)$. This, together with the fact that $H_t \leqslant g_{n.}^{\varepsilon}(t)$ leads to the conclusion that for every $\varepsilon > 0$ for sufficiently large t, $P(Y_{(n_t)}^{N_t}(t) \leqslant w_{n.}^{-\varepsilon}(t)) \leqslant \exp\{-t^{\delta\varepsilon/20}\}$. Finally note that the concavity of the function $I^{-1}(x)$, implies that $w_{n.}^{-\varepsilon}(t) \geqslant (1-\varepsilon)w_{n.}(t)$. Thus, for every $\varepsilon > 0$ for sufficiently large t,

$$P\big(Y_{(n_t)}^{N_t}(t) \leqslant (1-\varepsilon)w_{n.}(t)\big) \leqslant \exp\big\{-t^{\varepsilon\delta/20}\big\}. \tag{4.20}$$

This completes the proof of (i).

Proof of (ii). – By (i) it is enough to prove that in probability, $\lim_{t\to\infty} Y_{(n_t)}^{N_t}/w_{n.}(t) \leqslant 1$. To prove this we show that for every $\varepsilon > 0$, for sufficiently large t,

$$P\big(Y_{(n_t)}^{N_t}(t) \geqslant w_{n.}^{\varepsilon}(t)\big) \leqslant U_t, \tag{4.21}$$

where $U_t = \frac{2}{\varepsilon}(t\log N_{t/2})^{-1/4}$ when $\log N_t \gg \log t$ and $U_t = 80/\lfloor N_t\rfloor^{\delta\varepsilon/8}$ when $N_t = \lfloor t^\alpha\rfloor$ with $\alpha > 1/2$. Note that the concavity of the function $I^{-1}(x)$ which gives us that $w_{n.}^{\varepsilon}(t) \leqslant (1+\varepsilon)w_{n.}(t)$, implies from (4.21) that for sufficiently large t,

$$P\big(Y_{(n_t)}^{N_t}(t) \geqslant (1+\varepsilon)w_{n.}(t)\big) \leqslant U_t.$$

Consider first the case $\log N_t \gg \log t$. Note that $Y_{(n_t)}^{N_t}(t) \leqslant Y_{(1)}^{N_t}(t)$. Since $n_t \leqslant t^\beta$, by (4.1), for sufficiently large t we have $w_1^{\varepsilon/2}(t) \leqslant w_{n.}^{\varepsilon}(t)$. In fact, by the definitions of g_1^ε and N_1^ε we have that

$$\begin{aligned}w_{n.}^{\varepsilon} &\geqslant (t-g_1^\varepsilon)I^{-1}\left(\frac{1+\varepsilon}{t-g_1^\varepsilon}\log\frac{N_1^\varepsilon}{n}\right)\\ &= (t-g_1^\varepsilon)I^{-1}\left(\frac{1+\varepsilon}{t-g_1^\varepsilon}\log N_1^\varepsilon\left(1-\frac{\log n}{\log N_1^\varepsilon}\right)\right).\end{aligned} \tag{4.22}$$

But by Lemma 4.6(i) with $f(t) = t/\log t$, we have that $N_1^\varepsilon(t) \gg \lfloor N_{t/\log t}\rfloor^{1/2} \gg t^{\frac{u(t/\log t)}{4}}$, where $u(t) = \log N_t/\log t \gg 1$. Therefore, since $n_t \ll t^\beta$, we have $\frac{\log n}{\log N_1^\varepsilon} = \mathrm{o}(t)$. Combining this with (4.22), for sufficiently large t we have $w_1^{\varepsilon/2}(t) \leqslant w_{n.}^\varepsilon(t)$. Using this we can now conclude that if $\log N_t \gg \log t$, for sufficiently large t, $P(Y_{(n_t)}^{N_t}(t) \geqslant w_{n.}^\varepsilon(t)) \leqslant \frac{2}{\varepsilon}(t\log N_{t/2})^{-1/4}$.

We now analyze the case $N_t = \lfloor t^\alpha\rfloor$, $\alpha > 1/2$. First note that by (4.1) for sufficiently large t, $tI^{-1}(\frac{1+\varepsilon}{t}\log\frac{N_t}{n_t}) \leqslant w_{n.}^{2\varepsilon}(t)$. Therefore by Proposition 4.3, $P(Z_{(n_t)}^{N_t}(t) \geqslant w_{n.}^\varepsilon(t)) \leqslant 80/\lfloor N_t\rfloor^{\delta\varepsilon/8}$. If $P(Y_{(n)}^N(t) \geqslant w_{n.}^\varepsilon(t)) = 1 - f_{N-n}(p_1,\dots,p_N)$ and $P(Z_{(n)}^N(t) \geqslant w_{n.}^\varepsilon(t)) = 1 - f_{N-n}(q_1,\dots,q_{N_t})$ with $p_i = P(Y_i(t) < w_{n.}^\varepsilon(t)) \geqslant q_i = P(Z_i(t) < w_{n(.}^\varepsilon(t))$ for $1 \leqslant i \leqslant N$. Thus f has the form (4.19) and hence $f_{N-n}(p_1,\dots,p_N) \geqslant f_{N-n}(q_1,\dots,q_N)$, or $P(Y_{(n)}^N(t) \geqslant w_{n.}^\varepsilon(t)) \leqslant P(Z_{(n)}^N(t) \geqslant w_{n.}^\varepsilon(t))$ which gives

$$P\big(Y_{(n_t)}^{N_t}(t) \geqslant w_{n.}^\varepsilon(t)\big) \leqslant 80\lfloor N_t\rfloor^{-\delta\varepsilon/8}.$$

This proves (4.21) and hence (ii).

To extend this to $\overline{Y}_{(n_t)}^{N_t}(t)$, note that by the reflection principle the tail estimate for $P(Z(t) \geqslant x)$, changes by a factor of 2 if we replace Z by $\overline{Z}$. Thus, all the results of Section 3.2 remain valid, and the proof of (ii) is a repetition of the above argument.

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