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FINITE RANK TRANSFORMATION AND WEAK CLOSURE THEOREM

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ABSTRACT. – We introduce a new class of cocycles which provides examples of measure preserving dynamical systems $(X, \mathcal{B}, \mu, T)$, such that given positive integers $r \geqslant 2$ and $m \geqslant 1$, possibly infinite, with $(r, m) \neq (\infty, \infty)$, the rank is r and the order of the quotient group in the measure-theoretic centralizer, $\#\frac{C(T)}{\mathrm{wcl}\{T^n;\ n\in\mathbb{Z}\}}$, is m. Moreover, $\mathrm{wcl}\{T^n;\ n \in \mathbb{Z}\}$ is uncountable. For the case $(r, m) = (\infty, \infty)$, we produce a mixing T. This completes the weak closure theorem of Jonathan King. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous introduisons une nouvelle classe de cocycles qui permet d'obtenir des exemples de flots $(X, \mathcal{B}, \mu, T)$, tels qu'étant donnés deux entiers $r \geqslant 2$ et $m \geqslant 1$, éventuellement infinis, avec $(r, m) \neq (\infty, \infty)$, le rang soit r et l'ordre du groupe quotient dans le centralisateur, $\#\frac{C(T)}{\mathrm{wcl}\{T^n;\ n\in\mathbb{Z}\}}$, soit m. En outre ces exemples sont tels que $\mathrm{wcl}\{T^n;\ n \in \mathbb{Z}\}$ est non dénombrable. Pour $(r, m) = (\infty, \infty)$, nous construisons un exemple avec T mélangeant. Ceci en particulier complète le Théorème de Clôture Faible de Jonathan King. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and let $C(T)$ be the measure-theoretic centralizer of T. The Weak Closure Theorem [10] asserts that $C(T)$ coincides with the weak closure of the set of powers of T, denoted $\mathrm{wcl}\{T^n,\ n \in \mathbb{Z}\}$, whenever $r(T) = 1$, where $r(T)$ is the rank of T.

Hence the question of the existence of a relationship between $r(T)$ and the cardinality $q(T)$ of the quotient group $\frac{C(T)}{\mathrm{wcl}\{T^n;\ n\in\mathbb{Z}\}}$ in the general case naturally arises.

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For mixing T it follows from [11] that $q(T) \leqslant r(T)$. It is not difficult to show that the same inequality holds for T's defined in [2] – generalized Morse flows, for which $q(T) = 2$ and $r(T)$ can be arbitrarily large.

Each automorphism T in [2] has a partially discrete spectrum whence the two kinds of automorphisms previously described are rather far from each other.

However for both cases $\mathrm{wcl}\{T^n,\ n \in \mathbb{Z}\} = \{T^n,\ n \in Z\}$. Therefore an interesting additional feature is to construct arbitrary pairs $(q(T), r(T))$ with an uncountable $\mathrm{wcl}\{T^n,\ n \in Z\}$.

In this paper we shall introduce some new classes of cocycles, which define flows that are ergodic group extensions of rank 1 systems, and are tractable enough to allow an exact computation of both $r(T)$ and $q(T)$.

We shall pick within these classes, for each possible pair (q, r), examples of ergodic automorphisms T such that $(q(T), r(T)) = (q, r)$. Moreover, for $(q, r) \neq (\infty, \infty)$, our examples produce an uncountable $\mathrm{wcl}\{T^n,\ n \in Z\}$.

The difficulty lies both in the proposition of a good candidate, and in the computation of the rank and the order of the quotient group. For $(r, m) \neq (\infty, \infty)$, our examples lie in the class of group extensions determined by r-Toeplitz sequences. For the (∞, ∞) case, the example is mixing: it is a weakly mixing extension of a rank 1 mixing transformation [1].

From these examples, it now follows that in its generality, the weak closure theorem is the only one for limitations concerning the coexistence of the measure-theoretic invariants $q(T)$ and $r(T)$.

The investigations of ergodicity and that of the measure-theoretic centralizer both rely on Newton's functional equation [21] and are carried out partially on a measure-theoretic group extension representation of the system.

Investigating the rank (and partly the centralizer too) we use a shift representation of those extensions.

2. Preliminaries

2.1. Notations and definitions

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and T a measure-preserving invertible ergodic transformation of $(X, \mathcal{B}, \mu)$. By the centralizer (measure-theoretic) of T we mean the set of all measure-preserving automorphisms of $(X, \mathcal{B}, \mu)$ which commute with T and we denote it by $C(T)$. Then $C(T)$ is a topological group with the standard operation of composition of transformations and with a topology (called the weak topology) defined as follows: $\{S_n\}_{n\in\mathbb{N}} \in C(T)$ converges to $S \in C(T)$ if for every $A \in \mathcal{B}$

$$\mu(S_n A \triangle SA) \longrightarrow 0.$$

We shall indicate this convergence by $S_n \rightharpoonup S$. With this topology, $C(T)$ is metric, complete. By $\mathrm{wcl}\{T^n,\ n \in \mathbb{Z}\}$ we mean the weak closure of the powers of T in $C(T)$.

We say that a sequence of sets $A_1, \ldots, A_k \in \mathcal{B}$ is a T-stack if these sets are pairwise disjoint and $TA_i = A_{i+1}$, $i = 1, \ldots, k-1$.

If we are given a collection of r measurable subsets F_i of X, and r positive integers h_i, such that $\mathcal{C} := \{T^l F_i\colon 1 \leqslant i \leqslant r,\ 0 \leqslant l < h_i\}$ is a collection of disjoint sets (a union of r disjoint T-stacks), setting $Y = X \setminus (\bigcup_{C \in \mathcal{C}} C)$, this union of r disjoint T-stacks defines a partition $\tilde{\mathcal{C}} := \mathcal{C} \cup \{Y\}$, and a σ-algebra $\sigma(\tilde{\mathcal{C}})$.

The rank of T is the smallest integer r such that given $\varepsilon > 0$, there exists a union of r disjoint T-stacks $\mathcal{C}$, such that for any measurable $A \in \mathcal{B}$, there exists $B \in \sigma(\tilde{\mathcal{C}})$ with $\mu(A \Delta B) < \varepsilon$. If such a positive integer does not exist then we say that $r(T) = \infty$.

We shall give a symbolic version for the definition of the rank in 2.2. and 4., which are shown to be equivalent to the one above in [3] and [20]. We reffer the interested reader to [5–7] for more on rank and partitions.

Suppose now that G is a compact metric abelian group and $\varphi : X \longrightarrow G$ is a measurable function which we will call a cocycle. The G-extension of $(X, \mathcal{B}, \mu, T)$ given by the cocycle φ is the dynamical system $\mathcal{X}_\varphi = (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times \nu, T_\varphi)$, where $\mathcal{B}_G$ is the Borel σ-algebra in G, ν is the normalized Haar measure on G and

$$T_\varphi(x, g) = \bigl(Tx, g + \varphi(x)\bigr)$$

for $x \in X,\ g \in G$. It is well known [22] that for ergodic $(X, \mathcal{B}, \mu, T)$ the following theorem is true.

THEOREM A. – *T_φ is ergodic iff the functional equation*

$$\frac{f(Tx)}{f(x)} = \gamma\bigl(\varphi(x)\bigr) \tag{1}$$

has no measurable solutions $f : X \longrightarrow K$ for any nontrivial character γ of G (K is the unit complex circle).

It is known (see [21] for the definition) that if $(X, \mathcal{B}, \mu, T)$ is a canonical factor of T_φ (for example if T is with discrete spectrum) then, assuming that T_φ is ergodic, $C(T_\varphi)$ is given by the triples (S, f, τ), where $S \in C(T),\ f : X \to G$ is measurable and τ is a group automorphism of G such that

$$f(Tx) - f(x) = \varphi(Sx) - \tau\bigl(\varphi(x)\bigr). \tag{2}$$

This means that every element $R \in C(T_\varphi)$ is of a form

$$R(x, g) = \bigl(Sx, \tau(g) + f(x)\bigr). \tag{3}$$

In such a case we write $R \sim (S, f, \tau)$. The following property is proved in [17] and [18], using Theorem A.

THEOREM B. – *If $R_n, R \in C(T_\varphi)$ and $R_n \sim (S_n, f_n, \mathrm{id})$, $R \sim (S, f, \mathrm{id})$ then $R_n \rightharpoonup R$ iff $S_n \rightharpoonup S$ and $f_n \longrightarrow f$ in measure μ.*

Let $\sigma_a : X \times G \longrightarrow X \times G$ be given by the formula

$$\sigma_a(x, g) = (x, g + a), \quad a \in G. \tag{4}$$

Then $\sigma_a \in C(T_\varphi)$, $\sigma_a \sim (\mathrm{id}, a, \mathrm{id})$. For every integer n, $(T_\varphi)^n$ is given by the formula

$$(T_\varphi)^n(x,g) = \big(T^n x, g + \varphi^{(n)}(x)\big), \tag{5}$$

where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \cdots + \varphi(T^{n-1}x), & \text{if } n \geqslant 0, \\ -\varphi(T^{-1}x) - \cdots - \varphi(T^n x), & \text{if } n < 0. \end{cases} \tag{6}$$

Then it follows from Theorem B that

COROLLARY 1. – *$(T_\varphi)^{n_k} \rightharpoonup \sigma_a$ in $C(T_\varphi)$ iff $T^{n_k} \rightharpoonup \mathrm{id}$ in $C(T)$ and $\varphi^{(n_k)} \longrightarrow a$ in measure.*

2.2. Sequences and blocks

A finite sequence $B = (B[0], \ldots, B[k-1])$, $B[i] \in G$, $0 \leqslant i \leqslant k-1$, $k \geqslant 1$, is called a block over G. The number k is called the length of B and is denoted by $|B|$. If $C = (C[0], \ldots, C[n-1])$ is another block then the concatenation of B and C is the block

$$BC = \big(B[0], \ldots, B[k-1], C[0], \ldots, C[n-1]\big).$$

Inductively we define the concatenation of an arbitrary number of blocks. By B_g, $g \in G$, we will denote the block

$$B_g = \big(B[0] + g, \ldots, B[k-1] + g\big)$$

and by $B[i,s] (0 \leqslant i \leqslant s \leqslant k-1)$ the block

$$B[i,s] = \big(B[i], \ldots, B[s]\big).$$

Assume that

$$B = B(0) \ldots B(r-1)$$

is a concatenation of r blocks $B(0), \ldots, B(r-1)$ having the same lengths and

$$C = C[0] \ldots C[rm-1]$$

for some $m \geqslant 1$. We define the product $B \overset{r}{\times} C$ of B and C as follows:

$$\begin{aligned} B \overset{r}{\times} C = {} & B_{C[0]}(0) \ldots B_{C[r-1]}(r-1) B_{C[r]}(0) \ldots \\ & B_{C[2r-1]}(r-1) B_{C[r(m-1)]}(0) \ldots B_{C[rm-1]}(r-1). \end{aligned} \tag{7}$$

Then

$$\big|B \overset{r}{\times} C\big| = \frac{|B||C|}{r} = \big|B(i)\big| rm, \quad \text{for every } i = 0, \ldots, r-1.$$

Let Ω by the space of all bi-infinite sequences over G. If $\omega \in \Omega$ or ω is a one-sided infinite sequence over G then $\omega[i,s]$, $i \leqslant s$, denotes the block $(\omega[i], \ldots, \omega[s])$.

A block B is said to occur at place i in ω (resp. in a block C, $|C| = n$, if $|B| \leqslant n$) if $\omega[i, i + |B| - 1] = B$ (resp. $C[i, i + |B| - 1] = B$). The frequencies of B in C or ω are the numbers

$$\mathrm{fr}(B, C) = |C|^{-1}\#\{0 \leqslant i \leqslant |C| - |B|;\ B \text{ occurs at place } i \text{ in } C\},$$

$$\mathrm{fr}(B, \omega) = \lim_{s\to\infty} \mathrm{fr}\big(B, \omega[0, s-1]\big),$$

if this limit exists.

For an infinite subsequence of ω, $E = \{\omega[n],\ n \in I \subset \mathbb{Z}\}$ (resp. $E = \{\omega[n],\ n \in I \subset \mathbb{N}\}$), we call the density of E the density of the set I in $\mathbb{Z}$ (resp. in $\mathbb{N}$), whenever it exists. Let $\delta > 0$. We say that B δ-occurs at place i in C (resp. in ω) if

$$d\big(B, C[i, i + |B| - 1]\big) < \delta \quad \big(\text{resp. } d\big(B, \omega[i, i + |B| - 1]\big) < \delta\big),$$

where

$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = n^{-1}\#\{i;\ x_i \neq y_i\}$$

(d is called the normalized Hamming distance or d-bar distance between sequences). We will say also that $B\delta$-occurs on the fragment $\omega[i, i + |B| - 1]$ of ω.

We will use the following elementary properties of the distance d;

$$d\big(B \overset{r}{\times} C, B \overset{r}{\times} D\big) = d(C, D) \quad (\text{see (7)}), \tag{8}$$

$$d(B_g, C_g) = d(B, C), \tag{9}$$

$$d(A_1A_2, B_1B_2) = \frac{|A_1|}{|A_1| + |A_2|} d(A_1, B_1) + \frac{|A_2|}{|A_1| + |A_2|} d(A_2, B_2), \tag{10}$$

where $|A_1| = |B_1|,\ |A_2| = |B_2|$.

If $D_1 \subset D (D_1$ is a subblock of $D)$ and $C_1 \subset C$, $|D_1| = |C_1|$, both appearing in the corresponding same positions, then

$$d(D, C) \geqslant \frac{|D_1|}{|D|} d(D_1, C_1). \tag{11}$$

$$d(A_1A_2 \ldots A_s, B_1B_2 \ldots B_s) = \frac{1}{s}\sum_{i=1}^{s} d(A_i, B_i) \tag{12}$$

if $|A_1| = |A_2| = \cdots = |A_s| = |B_1| = \cdots = |B_s|$.

By T_σ we denote the left shift homeomorphism of Ω. If $\omega \in \Omega$ then $O(\omega)$ denotes the T_σ-orbit of ω and Ω_ω the T_σ-orbit closure of ω in Ω. The T_σ-orbit closure Ω_ω is well-defined if ω is a one-sided sequence. Namely, we first let $\diamondsuit \notin G$ be an additional symbol. Then we let $\omega^\diamondsuit$ denote the bi-infinite sequence which agrees with ω at positive coordinates and has only squares appearing at the negative ones. Then we say that a bi-infinite y belongs to Ω_ω if there exists $n_i \to +\infty$ such that $T_\sigma^{n_i}\omega \to y$ in Ω (the convergence is for all coordinates of y, and the limiting element y does not contain any more squares). The topological flow $(\Omega_\omega, T_\sigma)$ is called minimal if there is no non trivial closed and T_σ-invariant subset of Ω_ω. We say that $(\Omega_\omega, T_\sigma)$ is uniquely ergodic if there is a unique borelian normalized T_σ-invariant measure μ_ω on Ω_ω. Then $(\Omega_\omega, T_\sigma)$ is said to be strictly ergodic if it is minimal and uniquely ergodic. Suppose $(\Omega_\omega, T_\sigma)$ is strictly ergodic. The unique T_σ-invariant measure μ_ω is determined by the condition

$$\mu_\omega(B) = \mathrm{fr}(B, \omega)$$

for each block B. In the case of a discrete group G, the definition of the rank has the following symbolic transcription.

The system $(\Omega_\omega, T_\sigma, \mu_\omega)$ is of rank at most r if for any $\delta > 0$ and every n, there exist r blocks $B_1, \ldots, B_r$, $|B_i| \geqslant n$, such that for all N large enough, for any $s \in \mathbb{N}$, the fragment $\omega[s, s+N-1]$ has a form

$$\omega[s, s+N-1] = \varepsilon_1 W_1 \varepsilon_2 W_2 \cdots \varepsilon_k W_k \varepsilon_{k+1},$$

where $|\varepsilon_1| + \cdots + |\varepsilon_k| + |\varepsilon_{k+1}| < \delta N$ and the distance d between W_j and some B_m, $j = 1, \ldots, k$, $1 \leqslant m \leqslant r$, is less than δ. The system $(\Omega_\omega, T_\sigma, \mu_\omega)$ is of rank r if it is of rank at most r and not of rank at most $r-1$.

2.3. Adding machines and r-Toeplitz cocycles

Now, let $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be a $\{p_t\}$-adic adding machine i.e.

$$p_{t+1} = \lambda_{t+1} p_t, \quad \lambda_0 = p_0, \quad \lambda_t \geqslant 2 \text{ for } t \geqslant 0,$$

$$X = \left\{ x = \sum_{t=0}^{\infty} q_t p_{t-1};\ 0 \leqslant q_t \leqslant \lambda_t - 1,\ p_{-1} = 1 \right\}$$

is the group of $\{p_t\}$-adic integers and $Tx = x + \hat{1}$, where

$$\hat{1} = 1 + 0p_1 + 0p_2 + \cdots.$$

The space X has a standard sequence $\{\xi_t\}_{t \geqslant 0}$ of T-stacks. Namely

$$\xi_t = \left(D_0^t, \ldots, D_{p_t - 1}^t\right),$$

where

$$D_0^t = \{x \in X; q_0 = \cdots = q_t = 0\}, \qquad D_s^t = T^s\left(D_0^t\right)$$

for $s = 1, \ldots, p_t - 1$. We have

$$X = \bigcup_{i=0}^{p_t - 1} D_i^t.$$

Then ξ_{t+1} refines ξ_t and the sequence of partitions $\{\xi_t\}_{t \geqslant 0}$ converges to the point partition.

We will define a special class of cocycles $\varphi : X \longrightarrow G$ that are determined by Toeplitz sequences over G.

Let $r \geqslant 2$ be an integer, and assume that $b^0, b^1, \dots$ are finite blocks over G with $|b^t| = \lambda_t r$, $\lambda_t \geqslant 2$, such that

$$b^t[0, r-1] = (\underbrace{0, \dots, 0}_{r \text{ times}}).$$

We shall introduce a particular sequence (p_t), and some new blocks (B^t).

We can write

$$b^t = b^t(0) \dots b^t(r-1), \quad \left|b^t(i)\right| = \lambda_t, \ i = 0, \dots, r-1. \tag{13}$$

Define another sequence of blocks $\{B^t\}$ letting

$$B^0 = b^0, \quad B^{t+1} = B^t \overset{r}{\times} b^{t+1}, \quad t \geqslant 0. \tag{14}$$

Then we have

$$\left|B^t\right| = r m_t = p_t, \quad m_t = \lambda_0 \cdots \lambda_t, \tag{15}$$

and we can represent B^t as

$$B^t = B^t(0) \cdots B^t(r-1), \quad \left|B^t(i)\right| = m_t, \ i = 0, \dots, r-1. \tag{16}$$

Moreover

$$B^{t+1}[0, p_t - 1] = B^t. \tag{17}$$

Now we can define a cocycle φ by

$$\varphi(x) = B^t[i+1] - B^t[i] \tag{18}$$

if $x \in D_i^t$ except of $i = m_t - 1, 2m_t - 1, \dots, p_t - 1$. Let us observe that φ is well defined. Such a cocycle is called r-Toeplitz cocycle. For every $t \geqslant 0$, φ is constant on the levels of ξ_t except of r levels.

The sequence $\{B^t\}_{t \geqslant 0}$ determines a one-sided sequence ω as follows:

$$\omega[0, p_t - 1] = B^t, \quad t = 0, 1, \dots. \tag{19}$$

The condition (17) guarantees that ω is well defined.

It is not hard to show that the condition

$$\mathrm{fr}\big(g, b^t\big) \geqslant \rho > 0 \quad (\text{if } G \text{ is finite}) \tag{20}$$

for every $g \in G$ and $t = 0, 1 \dots,$ implies that the system $(\Omega_\omega, T_\sigma)$ is strictly ergodic. Then using (19), (20), and arguments as in [16], we deduce that the dynamical systems $(\Omega_\omega, T_\sigma, \mu_\omega)$ and $(X \times G, T_\varphi, \mu \times \nu)$ are measure-theoretically isomorphic when T_φ is ergodic.

The group extensions defined by r-Toeplitz cocycles shall be called r-Toeplitz extensions.

In the sequel we will write

$$\omega = b^0 \overset{r}{\times} b^1 \overset{r}{\times} b^2 \overset{r}{\times} \cdots.$$

Except of ω we need the sequences $\omega_t,\ t \geqslant 0$, defined by

$$\omega_t = b^t \overset{r}{\times} b^{t+1} \overset{r}{\times} \cdots. \tag{21}$$

3. Examples of r-Toeplitz extensions

In this part, given $r \geqslant 2$ and $m \geqslant 1$, we define r-Toeplitz group extensions having cardinality of the quotient group $C(T_\varphi)/\mathrm{wcl}\{T_\varphi^n;\ n \in \mathbb{Z}\}$ equal to m.

3.1. The case $r \geqslant 2, m \geqslant 2$

Let $G = \mathbb{Z}/m\mathbb{Z} = \{0, \ldots, m-1\}$. Define

$$F^{(i)} = \overbrace{00\ldots0}^{r(2^{i+2}-1)}\overbrace{0\ldots0\underbrace{\ 1\ }_{i+1}0\ldots0}^{r}, \quad i = 0, \ldots, r-1;$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \ldots F_{m-1}^{(i)}.$$

We have $|H^{(i)}| = mr2^{i+2}$. Next define

$$b^t(0) = \underbrace{H^{(0)} H^{(0)} \ldots H^{(0)}}_{x_0 \text{ times}}$$

$$b^t(1) = \underbrace{H^{(1)} H^{(1)} \ldots H^{(1)}}_{x_1 \text{ times}}$$

$$\vdots$$

$$b^t(r-1) = \underbrace{H^{(r-1)} H^{(r-1)} \ldots H^{(r-1)}}_{x_{r-1} \text{ times}}$$

where

$$x_i = 2^{t+r-1-i}, \quad 0 \leqslant i \leqslant r-1,$$

and

$$b^t = b^t(0) \ldots b^t(r-1), \quad t \geqslant 0.$$

Then we have

$$\lambda_t = \left|b^t(i)\right| = mr2^{t+r+1}, \quad \text{for } i = 0, \ldots, r-1 \quad \text{(see (13))}$$

and

$$\left|b^t\right| = mr^2 2^{t+r+1}.$$

Now define the blocks B^t, $t \geqslant 0$, by (14) and the cocycle φ by (18). Then from (15)

$$p_t = \left|B^t\right| = m^{t+1} r^{2t} 2^{r+1}\left(2^{t+1}-1\right), \quad t \geqslant 0.$$

3.2. The case $r \geqslant 2$, $m = 1$

Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$, $n \geqslant 4$. Then define

$$F^{(i)} = \overbrace{00\ldots0}^{3r}\overbrace{0\ldots\underbrace{1}_{i+1}0\ldots0}^{r},$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \ldots F_{n-1}^{(i)},$$

and

$$b^t(i) = \underbrace{H^{(i)} H^{(i)} \ldots H^{(i)}}_{x \text{ times}}, \quad x = 2^t.$$

Next set

$$b^t = b^t(0) \ldots b^t(r-1),$$

$$B^t = b^0 \overset{r}{\times} b^1 \overset{r}{\times} \cdots \overset{r}{\times} b^t, \quad t \geqslant 0,$$

and define φ by (18). In this case we have

$$\lambda_t = rn2^{t+2} = \left|b^t(i)\right|, \quad |b^t| = r^2 n 2^{t+2}, \text{ for } i = 0, 1, \ldots, r-1 \text{ and } t \geqslant 0.$$

3.3. Ergodicity and the measure-theoretic centralizer

THEOREM 1. – *T_φ is ergodic.*

Proof. – We will prove ergodicity of T_φ in both cases 3.1 and 3.2. Assume that there exists a measurable function $f : X \longrightarrow K$ satisfying (1). Then (see (5), (6))

$$\frac{f(T^n x)}{f(x)} = \gamma\left(\varphi^{(n)}(x)\right) \tag{22}$$

for μ-a.e. $x \in X$ and every $n \in \mathbb{Z}$.

In particular (22) holds for $n = p_t$, $t = 0, 1, \ldots$. The measurability of f and the fact that $\xi_t \longrightarrow \varepsilon$ (the partition into points) in X imply

$$\gamma\left(\varphi^{(p_t)}(x)\right) = 1 \tag{23}$$

except of a subset of measure ε_t and $\varepsilon_t \longrightarrow 0$.

Let $x \in D_j^{t+1}$, $0 \leqslant j \leqslant p_{t+1} - 1$. We can represent j as

$$j = up_t + vm_t + \rho, \tag{24}$$

where $0 \leqslant u \leqslant \lambda_{t+1} - 1$, $0 \leqslant v \leqslant r-1$, $0 \leqslant \rho \leqslant m_t - 1$ (see (15)).

It follows from (18) (with $t := t + 1$) that

$$\varphi^{(p_t)}(x) = B^{t+1}[j + p_t] - B^{t+1}[j] \tag{25}$$

except j for which $u = u_1 = \frac{\lambda}{r} - 1, \ldots, u = u_r = \frac{r\lambda}{r} - 1 = \lambda - 1, \lambda = \lambda_{t+1}$. At the same time we have

$$B^{t+1}[j] = b[ur + v] + B^t(v)[\rho], \quad b = b^{t+1} \quad \text{(see (14), (16)).}$$

Then (25) can be rewritten as

$$\varphi^{(p_t)}(x) = b\big[(u+1)r + v\big] - b[ur + v], \quad u \neq u_1, \ldots, u_r. \tag{26}$$

The last equality and (23) imply that

$$\gamma\big(c[q]\big) = 1 \quad (q = ur + v) \tag{27}$$

for $q \in V_t \subset \{0, 1, \ldots, r\lambda_{t+1} - 1\}$, $\frac{\#V_t}{r\lambda_{t+1}} \geqslant 1 - \varepsilon_t - \frac{2}{\lambda_{t+1}}$, where $c = c^t$ is given by

$$c[q] := b[q + r] - b[q], \quad q = 0, \ldots, r\lambda - r - 1.$$

Further the blocks $c = c^t$ have the following forms:

$$c = \overbrace{E^{(0)} \ldots E^{(0)}}^{(mx_0-1)\ \text{times}} L^{(0)} \overbrace{E^{(1)} \ldots E^{(1)}}^{(mx_1-1)\ \text{times}} L^{(1)} \ldots \overbrace{E^{(r-1)} \ldots E^{(r-1)}}^{(mx_{r-1}-1)\ \text{times}} \tag{28}$$

where

$$E^{(0)} = \overbrace{0\ldots0}^{2r}\overbrace{10\ldots0}^{r}\overbrace{01\ldots1}^{r}, \quad \big|L^0\big| = r,$$

$$E^{(1)} = \overbrace{0\ldots0}^{6r}\overbrace{010\ldots0}^{r}\overbrace{101\ldots1}^{r}, \quad \big|L^1\big| = r,$$

$$\vdots$$

$$E^{(r-1)} = \overbrace{0\ldots0}^{(2^{r+1}-2)r}\ \overbrace{0\ldots01}^{r}\overbrace{1\ldots10}^{r}, \quad \big|L^{(r-2)}\big| = r,$$

in the case 3.1. In the case 3.2 we have

$$c = \overbrace{E^{(0)} \ldots E^{(0)}}^{(nx-1)\ \text{times}} L^{(0)} \overbrace{E^{(1)} \ldots E^{(1)}}^{(nx-1)\ \text{times}} L^{(1)} \ldots \overbrace{E^{(r-1)} \ldots E^{(r-1)}}^{(nx-1)\ \text{times}} \tag{29}$$

where

$$E^{(0)} = \overbrace{0\ldots0}^{2r}\overbrace{10\ldots0}^{r}\overbrace{01\ldots1}^{r}, \quad \big|L^0\big| = r,$$

$$E^{(1)} = \overbrace{0\ldots0}^{2r}\overbrace{010\ldots0}^{r}\overbrace{101\ldots1}^{r}, \quad \big|L^1\big| = r,$$

$$\vdots$$

$$E^{(r-1)} = \overbrace{0\ldots0}^{2r}\overbrace{0\ldots01}^{r}\overbrace{1\ldots10}^{r}, \quad |L^{(r-2)}| = r.$$

In both cases 1 appears in c with frequency $> \frac{1}{r2^{r+2}}$ for each $t \geqslant 0$. Then (27) implies $\gamma(1) = 1$ so γ is trivial. We have proved that T_φ is ergodic. □

3.4. The centralizer of T_φ

The p_t-adic adding machine $(X, \mathcal{B}, \mu, T)$ is a canonical factor of the group extension $(X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times \nu, T_\varphi)$. Then $C(T_\varphi)$ is described in 2.1. We can distinguish the following subgroups of $C(T_\varphi)$:

$$C_1 = \mathrm{wcl}\{T_\varphi^n;\ n \in \mathbb{Z}\},$$
$$C_2 = \{\sigma_a \circ \widetilde{S};\ \widetilde{S} \in C_1 \text{ and } a \in G\},$$
$$C_3 = \{R \sim (S, f, \tau);\ \tau = \mathrm{id}\}.$$

Of course C_1, C_2, C_3 are closed subgroups of $C(T_\varphi)$ and

$$C_1 \subset C_2 \subset C_3 \subset C(T_\varphi).$$

We prove in Lemmas 1 and 2 that $C(T_\varphi)$ reduces to C_2 when φ is the r-Toeplitz cocycle defined in 3.1 or in 3.2.

In the sequel n means the same n as the one defined in 3.2 if this case is considered, and $n := m$ if the case 3.1 is considered.

LEMMA 1. – $C(T_\varphi) = C_3$.

Proof. – Take R as in (3). Then the triple (S, f, τ) satisfies (2). Putting $x := Tx, \ldots, T^{p_t-1}x$ in (2) and summing we obtain

$$f(T^{p_t}x) - f(x) = \varphi^{(p_t)}(Sx) - \tau(\varphi^{(p_t)}(x)) \tag{30}$$

for μ-a.e. $x \in X$ and each $t \geqslant 0$. Using the same arguments as in the proof of Theorem 1 we get from (30)

$$\varphi^{(p_t)}(Sx) - \tau(\varphi^{(p_t)}(x)) = 0 \tag{31}$$

for $x \in X_t$ and $\mu(X_t) \longrightarrow 1$.

Further we know [21] that there exists $q_0 \in X$ such that

$$S(x) = x + g_0, \quad x \in X.$$

Let

$$g_0 = \sum_{t=0}^{\infty} u_t p_{t-1}, \quad 0 \leqslant u_t \leqslant \lambda_t - 1, t \geqslant 1 \text{ and } 0 \leqslant u_0 \leqslant \lambda_0 r - 1.$$

Fix t and consider (31) on the stack ξ_{t+1}. Let

$$j_t = \sum_{j=0}^{t} u_j p_{j-1}.$$

Then (see (24))

$$j_t = v_0 m_t + \rho_0, \qquad j_{t+1} = u_0 p_t + v_0 m_t + \rho_0, \qquad u_0 = u_{t+1}.$$

If $x \in D_j^{t+1}$, $0 \leqslant j \leqslant p_{t+1} - 1$, then $Sx \in D_{j+j_{t+1}}^{t+1}$, where $j + j_{t+1}$ is taken mod p_{t+1}. We can write

$$j + j_{t+1} = \bar{u} p_t + \bar{v} m_t + \bar{\rho}, \quad 0 \leqslant \bar{u} \leqslant \lambda - 1,\ 0 \leqslant \bar{v} \leqslant r - 1,\ 0 \leqslant \bar{\rho} \leqslant m_t - 1.$$

Let us denote (use (24) for j)

$$q_0 = \begin{cases} u_0 r + v_0 & \text{if } \rho = 0, \dots, m_t - \rho_0 - 1, \\ u_0 r + v_0 + 1 & \text{if } \rho = m_t - \rho_0, \dots, m_t - 1, \end{cases}$$

and $q = ur + v$, $\bar{q} = \bar{u} r + \bar{v}$. Then $\bar{q} = q + q_0 (\operatorname{mod} r\lambda_{t+1})$. Thus (26) and (31) give

$$c[q + q_0] = \tau\big(c[q]\big) \quad \text{if } q \in V_t \subset \{0, 1, \dots, r\lambda_{t+1} - 1\} \tag{32}$$

and $\frac{1}{\lambda_{t+1}} \# V_t \longrightarrow 1$. Analysing the sequences (28) and (29) it is easy to observe that they do not satisfy (32) with any q_0 whenever $\tau \neq \mathrm{id}$ (i.e., $\tau(1) \neq 1$). The lemma is proved. □

LEMMA 2. – $C(T_\varphi) = C_2$.

Proof. – Let $R \sim (S, f, \mathrm{id}) \in C_3$. Then (32) means

$$c[q + q_0] = c[q], \quad q \in V_t.$$

The last condition implies

$$q_0(t) = q_0 = 2^{r+1} r m w, \quad w = w_t, \tag{33}$$

in the case 3.1 and

$$q_0(t) = q_0 = 4 r n w, \quad w = w_t, \tag{34}$$

in the case 3.2, where $0 \leqslant w \leqslant r 2^{t+1} - 1$ (see again (28) and (29)). Moreover

$$\min\left(\frac{q_0(t)}{\lambda_{t+1}}, 1 - \frac{q_0(t)}{\lambda_{t+1}}\right) \longrightarrow 0.$$

The above condition implies

$$\min\left(\frac{j_t}{p_t}, 1 - \frac{j_t}{p_t}\right) \longrightarrow 0.$$

Assume that $j_t/p_t \longrightarrow 0$ along some subsequence of t. It follows from the definition of the p_t-adic adding machine that

$$T^{j_t} \rightharpoonup S. \tag{35}$$

Now we will prove that there exists $a \in G$ such that

$$\varphi^{(j_t)} \longrightarrow f + a \tag{36}$$

in measure μ.

The function f satisfies the condition (see (2) with $\tau = \mathrm{id}$)

$$f(Tx) - f(x) = \varphi(Sx) - \varphi(x).$$

The measurability of f and $\xi_t \longrightarrow \varepsilon$ imply that there exists $a_t \in G$ such that the functions f_t defined by

$$f_t(y) = a_t + \varphi^{(i)}(Sx) - \varphi^{(i)}(x), \quad y \in D_i^t,\ y = T^i x,\ x \in D_0^t, \tag{37}$$
$$i = 0, \ldots, p_t - 1,$$

satisfy the condition

$$f_t \longrightarrow f \text{ in measure } \mu.$$

We can assume that $a_t = b$. We can rewrite (37) as

$$f_t(y) = b + \varphi^{(i)}(Sx) - \varphi^{(i)}\big(T^{j_t}x\big) + \varphi^{(i)}\big(T^{j_t}x\big) - \varphi^{(i)}(x).$$

Further we have (see (6))

$$\varphi^{(i)}\big(T^{j_t}x\big) - \varphi^{(i)}(x) = \varphi^{(j_t)}\big(T^i x\big) - \varphi^{(j_t)}(x). \tag{38}$$

Because of $j_t < m_t$ then $\varphi^{(j_t)}(x) = b_t$ for all $x \in D_0^t$. Assuming again $b_t = b_1$ we can write (38) as

$$\varphi^{(i)}\big(T^{j_t}x\big) - \varphi^{(i)}(x) = \varphi^{(j_t)}(y) - b_1$$

and (37) as

$$f_t(y) = b_2 + \varphi^{(j_t)}(y) + \varphi^{(i)}(Sx) - \varphi^{(i)}\big(T^{j_t}x\big). \tag{39}$$

Assume that

$$x \in D_{up_t}^{t+1}, \quad 0 \leqslant u \leqslant \lambda_{t+1} - 1.$$

Then

$$T^{j_t}x \in D_{up_t+j_t}^{t+1} \quad \text{and} \quad Sx \in D_{(u+u_0)p_t+j_t}^{t+1},$$

where $u_0 = q_0/r$.

For $i \leqslant p_t - j_t - 1$, $i = vm_t + \rho$ and $u \neq u_1, \ldots, u_r$ we have

$$\varphi^{(i)}\big(T^{j_t}x\big) = B^{t+1}[up_t + j_t + i] - B^{t+1}[up_t + j_t] = b^{t+1}[ur + v] - b^{t+1}[ur]$$

and

$$\begin{aligned}\varphi^{(i)}(Sx) &= B^{t+1}\big[(u+u_0)p_t + j_t + i\big] - B^{t+1}\big[(u+u_0)p_t + j_t\big]\\ &= b^{t+1}\big[(u+u_0)r + v\big] - b^{t+1}\big[(u+u_0)r\big].\end{aligned}$$

Thus

$$\varphi^{(i)}(Sx) - \varphi^{(i)}\big(T^{j_t}x\big) = \big(b[q+q_0] - b[q]\big) - \big(b[ur+q_0] - b[ur]\big), \quad q = ur + v.$$

Then (33) and (34) imply

$$\varphi^{(i)}(Sx) - \varphi^{(i)}\big(T^{j_t}x\big) = 0 \tag{40}$$

except of a set of measure $\leqslant (r/\lambda_t) + (j_t/p_t)$.

Now (39) and (40) imply (36) with $a = -b_2$. Notice that (35) and (36) and Theorem B imply

$$T_\varphi^{j_t} \rightharpoonup R \circ \sigma_a.$$

This proves the lemma. □

To prove that

$$\#\frac{C(T_\varphi)}{\mathrm{wcl}\{T_\varphi^n;\ n \in \mathbb{Z}\}} = m$$

in case 3.1 it is sufficient to show that $\sigma_a \notin C_1$ whenever $a \in \mathbb{Z}_m, a \neq 0$. In the case 3.2 we will prove that $\sigma_a \in C_1$ for every $a \in \mathbb{Z}_n$ what implies

$$\#\frac{C(T_\varphi)}{\mathrm{wcl}\{T_\varphi^n;\ n \in \mathbb{Z}\}} = 1.$$

To do this we need estimations of the d-distance between blocks occurring in ω and $\omega_t, t \geqslant 0$.

3.5. $\bar{d}$-bar distance between blocks

The sequence $\omega = b^0 \overset{r}{\times} b^1 \overset{r}{\times} \cdots$ is a concatenation of the blocks of the form

$$E_k(t) = B^t \overset{r}{\times} \bar{e}_k, \quad E_k^{(s)}(t) = B^t \overset{r}{\times} \bar{e}_k^{(s)}, \quad k \in \mathbb{Z}_n,\ s = 0, \ldots, r-1,$$

where

$$\bar{e}_k = (\overbrace{k, \ldots, k}^{r}), \quad \bar{e}_k^{(s)} = (\overbrace{k, \ldots, k,\ \underbrace{k+1}_{s\text{th place}}, k, \ldots, k}^{r}).$$

The sequence $\omega_t = b^t \overset{r}{\times} b^{t+1} \overset{r}{\times} \cdots$ is a concatenation of the blocks of the form

$$e_k(t) = b^t \overset{r}{\times} \bar{e}_k, \qquad e_k^{(s)}(t) = b^t \overset{r}{\times} \bar{e}_k^{(s)}.$$

The blocks $E_k = E_k(t)$, $E_k^{(s)} = E_k^{(s)}(t)$ are called t-symbols and the blocks $e_k = e_k(t)$, $e_k^{(s)} = e_k^{(s)}(t)$ are called "small" t-symbols. Each fragment $\omega[kp_t, (k+1)p_t - 1]$ of ω, $k \in \mathbb{Z}$, is a t-symbol, and $\omega_t[k\lambda_t r, (k+1)\lambda_t r - 1]$ is a "small" t-symbol. The positions $[kp_t, (k+1)p_t - 1]$ and $[k\lambda_t r, (k+1)\lambda_t r - 1]$ will be called the natural positions in ω and ω_t respectively.

We will examine d-bar distance between the blocks mentioned above or between their special fragments. In particular, we will examine the pairs

$$b_k(i)b_k(i+1),\ b_k(i)b_{k+1}(i+1),\ b_{k+1}(i)b_k(i+1),$$

for $i = 0, \ldots, r-2$ and $k \in \mathbb{Z}_n$ and

$$b_k(r-1)b_k(0), b_k(r-1)b_{k+1}(0).$$

PROPOSITION 1. – *Let*

$$\begin{cases} I = b_0^t(i)[0, \lambda_t - j - 1], & j \leqslant \frac{1}{2}\lambda_t, \\ II = b_k^t(i')[j, \lambda_t - 1], & k \in \mathbb{Z}_n,\ i, i' = 0, \ldots, r-1,\ t \geqslant 0. \end{cases} \tag{41}$$

If

$$d(I, II) < \frac{1}{r2^{r+2}} \tag{42}$$

then $i' = i$ and

$$j = (n-k)r2^{i+2} + anr2^{i+2}, \quad a \geqslant 0, \quad \textit{if } 3.1 \textit{ holds}, \tag{43}$$

$$j = (n-k)r4 + anr4, \quad a \geqslant 0, \quad \textit{if } 3.2 \textit{ holds}. \tag{44}$$

Proof. – It is easy to observe that if $i' \neq i$ or $i' = i$ and (43) (or (44) in the case 3.2) does not hold then every subblock $F_k^{(i)}$ of I differs from the corresponding fragment in II at least in one position. Since $j \leqslant \frac{1}{2\lambda_t}$, this would imply the converse inequality in (42). □

In Propositions 2–6 the blocks $b_k^t(i) = b_k(i)$, $k \in \mathbb{Z}_n$, $0 \leqslant i < r$, are those defined in 3.1.

PROPOSITION 2. – *Let*

$$I = b_0(0) \ldots b_0(r-1)[0, r\lambda_t - j - 1],$$

$$II = b_k(0) \ldots b_k(r-1)[j, r\lambda_t - 1], \quad j \leqslant \frac{1}{2}r\lambda_t, \quad k \in \mathbb{Z}_n.$$

If

$$d(I, II) < \frac{1}{r^2 2^{r+3}} \tag{45}$$

then $j \leqslant \frac{1}{2}\lambda_t$, $k = 0$, and

$$j \equiv 0 \big(\operatorname{mod} nr2^{r+1}\big). \tag{46}$$

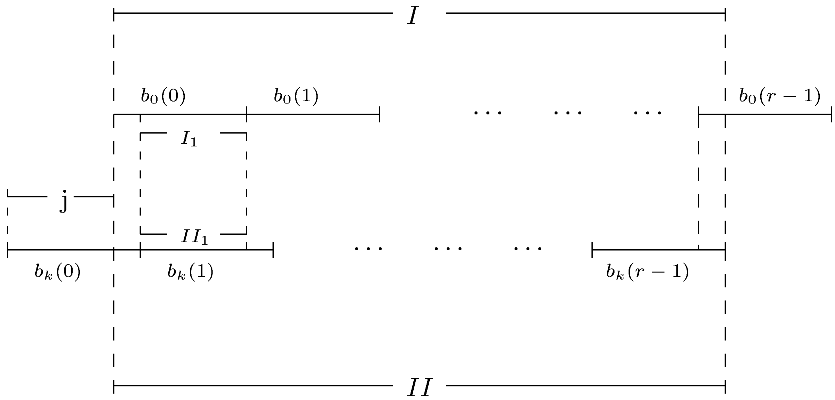

Fig. 1.

Proof. – If $j > \frac{1}{2}\lambda_t$ then we can find subblocks I_1 of I and II_1 of II such that II_1 is under I_1 (see Fig. 1) having the form (41) with different $j's$ and with $i' \neq i$.

It follows from Proposition 1 that $d(I_1, II_1) \geqslant \frac{1}{r2^{r+2}}$ and using (11) we obtain

$$d(I, II) \geqslant \frac{\frac{1}{2}\lambda_t}{r\lambda_t} d(I_1.II_1) \geqslant \frac{1}{r^2 2^{r+3}}.$$

in spite of (45). Therefore $j \leqslant \frac{1}{2}\lambda_t$.

It follows from (11) and (45) that

$$d(I_i, II_i) < \frac{1}{r2^{r+2}} \quad \text{for } i = 0, \ldots, r-1, \tag{47}$$

where

$$I_i = b_0(i)[0, \lambda_t - j - 1], \quad II_i = b_k(i)[j, \lambda_t - 1].$$

Then (47) implies (43) to hold for each $i = 0, \ldots, r-1$. In particular taking $i = 0, 1$ we get

$$-kr4 + 2kr4 = a_1 nr4.$$

Thus $k = 0$ in $\mathbb{Z}_n$. The proposition is proved. □

PROPOSITION 3. – *Let*

$$I = b_k(i)b_{k+1}(i+1)[0, 2\lambda - j - 1], \quad j \leqslant \frac{1}{2}\lambda;\ \lambda = \lambda_t,$$

$$II = b_{k_1}(i)b_{k_2}(i+1)[j, 2\lambda - 1], \quad i = 0, \ldots, r-2,\ k, k_1, k_2 \in \mathbb{Z}_n$$

and $k_2 = k_1 + 1$ *or* $k_2 = k_1 - 1$.

If

$$d(I, II) < \frac{1}{r2^{r+4}} \tag{48}$$

then

$$(k_1k_2) = (k, k+1) \quad or \quad (k_1k_2) = (k+4, k+3) \quad if\, n \geqslant 3 \tag{49}$$

and

$$(k_1k_2) = (k, k+1) \quad if\, n = 2. \tag{50}$$

Proof. – It follows from (48) and (11) that

$$d(I_1, II_1) < \frac{1}{r2^{r+2}}$$

and

$$d(I_2, II_2) < \frac{1}{r2^{r+2}},$$

where

$$I_1 = b_k(i)[0, \lambda - j - 1], \quad II_1 = b_{k_1}(i)[j, \lambda - 1],$$

$$I_2 = b_{k+1}(i+1)[0, \lambda - j - 1], \quad II_2 = b_{k_2}(i+1)[j, \lambda - 1].$$

Now, we apply Proposition 1. It follows from (43) that

$$k - k_1 = 2(k + 1 - k_2) \ (\mathrm{mod}\ n).$$

The above condition implies (49) and (50). □

PROPOSITION 4. – *Let*

$$I_k = b_k(r-1)b_k(0)[0, 2\lambda - j - 1] \quad or \quad I'_k = b_k(r-1)b_{k+1}(0)[0, 2\lambda - j - 1],$$

$$II = b_{k_1}(r-1)b_{k_2}(0)[j, 2\lambda - 1], \quad k, k_1, k_2 \in \mathbb{Z}_n,\ j \leqslant \frac{1}{2}\lambda_t,$$

and

$$k_2 = k_1 \quad or \quad k_2 = k_1 + 1. \tag{51}$$

If

$$d(I, II) < \frac{1}{r2^{r+4}}, \quad I = I_k \ or\ I'_k,$$

then

$$k_1 = k_2 = k \quad if\, I = I_k \qquad and \qquad k_1 = k, k_2 = k+1 \quad if\, I = I'_k \tag{52}$$

whenever

$$\left(2^{r-1} - 1, n\right) > 1, \tag{53}$$

and there is a unique $l \in \mathbb{Z}_n$ such that

$$\begin{cases} (k_1k_2) = (kk) \quad \textit{or} \quad (k_1k_2) = (l, l+1) \textit{ and } l \textit{ satisfies} \\ l(2^{r-1}-1) = (2^{r-1}-1)k + 1 \textit{ in } \mathbb{Z}_n \quad \textit{if } I = I_k, \\ \textit{and} \\ (k_1k_2) = (k, k+1) \quad \textit{or} \quad (k_1k_2) = (ll) \textit{ and } l \textit{ satisfies} \\ l(2^{r-1}-1) = (2^{r-1}-1)k - 1 \textit{ in } \mathbb{Z}_n \quad \textit{if } I = I'_k, \end{cases} \tag{54}$$

whenever

$$\left(2^{r-1}-1, n\right) = 1. \tag{55}$$

Proof. – Using the same arguments as in the proof of Proposition 3 we obtain from (43)

$$(k_1 - k)2^{r-1} = k - k_2 \pmod n \quad \text{if } I = I_k$$

and

$$(k_1 - k)2^{r-1} = k - k_2 + 1 \pmod n \quad \text{if } I = I'_k.$$

The above, (51), (53) and (55) imply (52) and (54) respectively. □

The next proposition is an easy consequence of (9) and the definition of the blocks $b(0), \ldots, b(r-1)$.

PROPOSITION 5. – *Let*

$$I_l = b_l(i)[0, \lambda_t - j - 1], \quad II_k = b_k(i)[j, \lambda_t - 1],$$
$$j \leqslant \frac{1}{2}\lambda_t, \ k, l \in \mathbb{Z}_n, \ 0 \leqslant i \leqslant r-1.$$

If $j \equiv 0 \pmod{nr2^{r+1}}$ *and* $k \neq l$ *then*

$$d(I_l, II_k) = 1.$$

PROPOSITION 6. – *Let*

$$I = b^t \overset{r}{\times} C, \quad II = b^t \overset{r}{\times} D[j, j + \lambda_t|D| - 1], \quad 0 \leqslant j \leqslant r\lambda_t - 1,$$

where $|C| \geqslant 3r, \ |D| = |C| + r, \ C, D \subset \omega_{t+1}$ (*see* (21)) *and* $C = \omega_{t+1}[pr, pr + |C| - 1], \ D = \omega_{t+1}[qr, qr + |D| - 1]$. *If*

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}}, \tag{56}$$

then either

$$j < \delta r 2^{r+1}\lambda_t \quad \textit{and} \quad d(C, D_1) < \delta \tag{57}$$

or

$$r\lambda_t - \delta r 2^{r+1}\lambda_t < j \leqslant r\lambda_t \quad \textit{and} \quad d(C, D_1) < \delta, \tag{58}$$

where

$$D_1 = D[0, |D| - r - 1] \quad \textit{if } j \leqslant \frac{1}{2} r\lambda_t,$$
$$D_1 = D[r, |D| - 1] \quad \textit{if } j > \frac{1}{2} r\lambda_t.$$

Proof. – We can represent C and D as

$$C = C_1 C_2 \dots C_s, \qquad D = D_1 D_2 \dots D_s D_{s+1},$$

where

$$|C_1| = \dots = |C_s| = |D_1| = \dots = |D_s| = |D_{s+1}| = r,\ s \geqslant 3,$$

and every $C_1, \dots, C_s, D_1, \dots, D_{s+1}$ is equal to one of the blocks $\bar{e}_k, \bar{e}_k^{(v)}, k \in \mathbb{Z}_n, v = 0, \dots, r-1$ (see 3.5). Assume that $j \leqslant \frac{1}{2} r\lambda_t$. Using (12) we get

$$d(I, II) = \frac{1}{s} \sum_{p=1}^{s} \bigl(b \overset{r}{\times} C_p, A_p\bigr), \tag{59}$$

where

$$A_p = \bigl(b \overset{r}{\times} D_p\bigr)\bigl(b \overset{r}{\times} D_{p+1}\bigr)[j, j + r\lambda_t - 1].$$

Then (56) implies that

$$d\bigl(b \overset{r}{\times} C_p, A_p\bigr) < \frac{1}{3r^2 2^{r+3}}$$

for at least one p. Using the same arguments as in the proof of Proposition 2 we obtain $j \leqslant \frac{1}{2}\lambda_t$.

Let

$$Q = \{1 \leqslant p \leqslant s,\ C_p \text{ and } D_p \text{ are equal } \bar{e}_k, \bar{e}_l \text{ for some } k, l \in \mathbb{Z}_n\}.$$

It follows from the definitions of ω, ω_t and b^t's that

$$\#Q \geqslant \frac{1}{3} s.$$

This inequality, (56), and (59), imply

$$\frac{1}{|Q|} \sum_{p \in Q} d\bigl(b \overset{r}{\times} C_p, A_p\bigr) < \frac{1}{r^2 2^{r+3}}.$$

Now we conclude that there is at least one $p \in Q$ such that

$$d\bigl(b \overset{r}{\times} C_p, A_p\bigr) < \frac{1}{r^2 2^{r+3}}.$$

It follows from Proposition 2 that $j \equiv 0 \pmod{nr2^{r+1}}$.

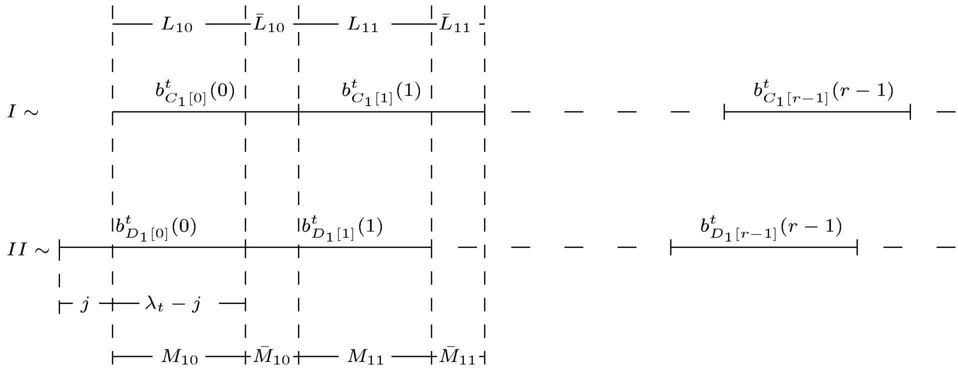

Fig. 2.

Now, using (10) and (12) again we get (see Fig. 2)

$$d(I,II) = \frac{1}{r}\sum_{i=0}^{r-1}\frac{1}{s}\left(\left(1-\frac{j}{\lambda_t}\right)\sum_{u=1}^{s} d(L_{ui},M_{ui}) + \frac{j}{\lambda_t}\sum_{u=1}^{s} d\left(\bar{L}_{ui},\bar{M}_{ui}\right)\right), \tag{60}$$

where

$$\begin{aligned} L_{ui} &= b^t_{C_u[i]}(i)[0,\lambda_t - j - 1], \quad & M_{ui} = b^t_{D_u[i]}(i)[j,\lambda_t - 1],\\ \bar{L}_{ui} &= b^t_{C_u[i]}(i)[\lambda_t - j,\lambda_t - 1], \quad & \bar{M}_{ui} = b^t_{D_u[i]}(i+1)[0,j-1]. \end{aligned}$$

It is not hard to remark that if $j \neq 0$

$$d\left(\bar{L}_{ui},\bar{M}_{ui}\right) \geqslant \frac{1}{r2^{r+1}} \tag{61}$$

for every u and i, $1 \leqslant u \leqslant s$, $0 \leqslant i \leqslant r-1$. Let

$$a = \#\left\{0 \leqslant k \leqslant |C|-1, C[k] \neq D[k]\right\}.$$

Then using Proposition 5, (60) and (61) we get

$$\delta > d(I,II) \geqslant \frac{a}{|C|}\left(1-\frac{j}{\lambda_t}+\frac{j}{\lambda_t}\frac{1}{r2^{r+1}}\right). \tag{62}$$

The above gives

$$\delta > \frac{a}{|C|}\left(1-\frac{j}{\lambda_t}\right) \geqslant \frac{a}{|C|}\frac{1}{2}$$

and then $\frac{a}{|C|} < 2\delta$. This inequality, (56) and (62) imply

$$\delta > \frac{a}{|C|} + \frac{j}{\lambda_t}\left(\frac{1}{r2^{r+1}} - \frac{a}{|C|}\right) > \frac{a}{|C|} + \frac{j}{\lambda_t}\left(\frac{1}{r2^{r+1}} - 2\delta\right) > \frac{a}{|C|} = d(C,D_1).$$

We have obtained the second inequality of (57). To get the first inequality of (57) we use (62) to obtain

$$\delta > \frac{j}{\lambda_t}\frac{1}{r2^{r+1}}.$$

This implies (57). We have proved the proposition if $j \leqslant \frac{1}{2}r\lambda_t$. The case $\frac{1}{2}r\lambda_t < j < r\lambda_t$ leads to (58) in a similar way. The proposition is proved. □

PROPOSITION 7. – *Let*

$$I = B^t \overset{r}{\times} C, \quad II = B^t \overset{r}{\times} D[j, j + m_t|D| - 1], \quad 0 \leqslant j \leqslant p_t - 1,$$

where C and D satisfy the same conditions as in Proposition 6. *If*

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

then either

$$j < \delta r 2^{r+} p_t \quad \text{and} \quad d(C, D_1) < \delta$$

or

$$p_t - \delta r 2^{r+1} p_t < j < p_t \quad \text{and} \quad d(C, D_1) < \delta,$$

where

$$D_1 = D[0, |D| - r - 1] \quad \text{if } j \leqslant \frac{1}{2}p_t,$$

and

$$D_1 = D[r, |D| - 1] \quad \text{if } r > \frac{1}{2}p_t.$$

Proof. – We use an induction argument and can repeat the proof of Lemma 3 from [8, p. 198], using (8), (9), and also using Proposition 6 instead of using a Lemma 2 as in [8, p. 196]. □

3.6. d-bar distance between blocks – the case 3.2

Using the same methods as in 3.5 we can estimate the distance between blocks $b_k^t(i)$ and $B_k^t(i)$, $i = 0, \ldots, r-1$, $k \in \mathbb{Z}_n$, $t \geqslant 0$, defined in the case 3.2.

As an easy consequence of Proposition 1 we get

PROPOSITION 8. – *Let*

$$I_l = b_{l_0}(0) \ldots b_{l_{r-1}}(r-1)[0, r\lambda_t - j - 1],$$
$$II_k = b_{k_0}(0) \ldots b_{k_{r-1}}(r-1)[j, r\lambda_t - 1],$$

$j \leqslant \frac{1}{2}r\lambda_t$, *where $(l_0, \ldots, l_{r-1})$ (resp. $(k_0, \ldots, k_{r-1})$) is of the form $\bar{e}_l$ or $\bar{e}_l^{(v)}$ (resp. $\bar{e}_k$ or $\bar{e}_k^{(v')}$), $k, l \in \mathbb{Z}_n$ and $v, v' = 0, \ldots, r-1$. If*

$$d(I_l, II_k) < \frac{1}{r^2 2^{r+3}}$$

then $j \leqslant \frac{1}{2}\lambda_t$ *and there is a unique* $s \in \mathbb{Z}_n$, $s = s(t)$, *such that* $l_i = k_i + s$ *for every* $i = 0, \ldots, r-1$. *Moreover* j *has a form*

$$j = (n-s)r4 + anr4, \quad a \geqslant 0.$$

As an analogue of Proposition 5 we obtain

PROPOSITION 9. – *Let* I_l, II_k *be as in Proposition* 5,

$$j \leqslant \frac{1}{2}\lambda_t \quad \textit{and} \quad j \equiv (n-s) \pmod{4rn}$$

for some $s \in \mathbb{Z}_n$. *Then*

$$d(I_l, II_k) = 1 \quad \textit{whenever } k - l \neq s.$$

Then using Propositions 8 and 9 we have

PROPOSITION 10. – *Let* I *and* II *be as in Proposition* 6 *and*

$$|C| \geqslant r, \quad |D| = |C| + r, \quad C, D \subset \omega_{t+1},$$
$$C = \omega_{t+1}[pr, pr + |C| - 1], \quad D = \omega_{t+1}[qr, qr + |D| - 1].$$

If

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

then there is an unique $s \in \mathbb{Z}_n$, $s = s(t)$, *such that*

$$j < \delta r 2^{r+1}\lambda_t \quad \textit{and} \quad d(C, D_1) < \delta$$

or

$$r\lambda_t - \delta r 2^{r+1}\lambda_t < j \leqslant r\lambda_t \quad \textit{and} \quad d(C, D_1) < \delta,$$

where $D_1 = D[0, |D| - r - 1] = C + s$ *if* $j \leqslant \frac{1}{2}\lambda_t r$, *and* $D_1 = D[r, |D| - 1] = C + s$ *if* $j > \frac{1}{2} r\lambda_t$.

Using arguments as in Lemma 3 in [8] and Proposition 10 we get

PROPOSITION 11. – *Let* I *and* II *be as in the Proposition* 7 *and* C, D *satisfy the same conditions as in Proposition* 10.

If

$$d(I, II) < \delta, \quad \delta < \frac{1}{3r^2 2^{r+3}},$$

there exists an unique $s \in \mathbb{Z}_n$, $s = s(t)$, *such that either*

$$j < \delta r 2^{r+1} p_t \quad \textit{and} \quad d(C, D_1) < \delta$$

or

$$p_t \delta r 2^{r+1} p_t < j < p_t \quad \textit{and} \quad d(C, D_1) < \delta,$$

where

$$D_1 = D[0, |D| - r - 1] + s \quad \textit{if } j \leqslant \frac{1}{2} p_t$$

and

$$D_1 = D[r, |D| - 1] + s \quad \textit{if } j > \frac{1}{2} p_t.$$

3.7. The centralizer of T_φ (continuation)

In 3.4 we have proved that $C(T_\varphi)$ consists of the elements $R \circ \sigma_a$, where R is a limit of powers of T_φ and σ_a is defined by (4), $a \in \mathbb{Z}_n$. Now we are in a position to show that

$$\# \frac{C(T_\varphi)}{\mathrm{wcl}\{T_\varphi^n;\ n \in \mathbb{Z}\}} = \begin{cases} n & \text{in the case 3.1,} \\ 1 & \text{in the case 3.2.} \end{cases}$$

LEMMA 3. – *If the case* 3.1 *holds and* $\sigma_a \in C_1$ *then* $a = 0$.

Proof. – Let us suppose that $T_\varphi^{n_s} \rightharpoonup \sigma_a,\ a \in \mathbb{Z}_n$. Then Corollary 1 says that $\varphi^{(n_s)} \longrightarrow a$ in measure. Let

$$\varepsilon_s = \mu\{x \in X;\ \varphi^{(n_s)}(x) \neq a\}. \tag{63}$$

We have $\varepsilon_s \longrightarrow 0$. Now for every s find t_s such that

$$\frac{n_s}{p_{t_s}} < \frac{\varepsilon_s}{r}. \tag{64}$$

To shorten notation we let $t := t_s + 1, \bar{t} := t_s$. Take $x \in D_j^t$. Then using (18) we get

$$\varphi^{(n_s)}(x) = B^t[j + n_s] - B^t[j] \tag{65}$$

except of j's satisfying $m_t - 1 - n_s \leqslant j \leqslant m_t - 1,\ 2m_t - 1 - n_s \leqslant j \leqslant 2m_t - 1,\ \ldots,\ p_t - 1 - n_s \leqslant j \leqslant p_t - 1$. Then (63) and (64) imply

$$\frac{1}{p_t} \#\{0 \leqslant j \leqslant p_t - 1, B^t[j + n_s] - B^t[j] \neq a\} < \varepsilon_s + \varepsilon_s = 2\varepsilon_s.$$

This means that

$$d\big(B^t[0, p_t - n_s - 1], B^t_{-a}[n_s, p_t - 1]\big) < 2\varepsilon_s.$$

We can write

$$B^t = B^{\bar{t}} \overset{r}{\times} b^t, \qquad B^t_{-a} = B^{\bar{t}} \overset{r}{\times} b^t_{-a}.$$

If $\varepsilon_s < \frac{1}{6r^2 2^{r+3}}$ then we apply Proposition 7 to the blocks $I = B^{\bar{t}} \overset{r}{\times} b^t$ and $II = B^{\bar{t}} \overset{r}{\times} b^t_{-a}$. As a consequence we obtain

$$d\big(b^t, b^t_{-a}\big) < 2\varepsilon_s.$$

This equality implies (Proposition 2) $a = 0$. The lemma is proved. □

From Lemmas 2 and 3 we obtain

THEOREM 2. –

$$\#\frac{C(T_\varphi)}{\mathrm{wcl}\{T_\varphi^n,\ n\in\mathbb{Z}\}}=n$$

if the case 3.1 *holds.*

Now, we examine the case 3.2. It follows from the definition of the blocks $b_0(i)=b_0^t(i), i=0,\dots,r-1, a\in\mathbb{Z}_n$ that

$$b(i)[(n-a)4r,\lambda-1]=b_a(i)[0,\lambda-(n-a)4r-1], \tag{66}$$

for every $i=0,\dots,r-1$.

Set $n_t=(n-a)4rp_{t-1}$. Then (66) implies

$$B^t(i)[j+n_t]-B^t(i)[j]=a$$

for $j=0,\dots,p_t-n_t-1$, and $i=0,\dots,r-1$. (65) and the above imply $\varphi^{(n_t)}(x)=a$ except of a set of measure $<r\frac{n_t}{p_t}\leqslant\frac{4r^2n}{\lambda_t}$.

Hence $\varphi^{(n_t)}\longrightarrow a$ in measure which implies that $T_\varphi^{n_t}\rightharpoonup\sigma_a,\ a\in\mathbb{Z}_n$. We have shown that $\sigma_a\in C_1$ for every $a\in\mathbb{Z}_n$ and as a consequence of Lemma 2 we get

THEOREM 3. –

$$\#\frac{C(T_\varphi)}{\mathrm{wcl}\{T_\varphi^n;\ n\in\mathbb{Z}\}}=1$$

if the case 3.2 *holds.*

THEOREM 3′. – $\mathrm{wcl}\{T_\varphi^n,\ n\in Z\}$ *is uncountable.*

Proof. – Let

$$g_0=\sum_0^\infty u_tp_{t-1},\qquad u_t=w_t\big(rm2^{r+1}\big)$$

in the case (3.1) and $u_t=w_t(4rn)$ in the case (3.2) $0\leqslant u_t\leqslant r\lambda_t-1$ and assume that

$$\sum_{t=0}^\infty\min\left(\frac{w_t}{r2^t},1-\frac{w_t}{r2^t}\right)<\infty.$$

Repeating the same arguments as in Lemma 4 of [9] we can construct a measurable function $f:X\longrightarrow G$ such that

$$f(Tx)-f(x)=\varphi(Sx)-\varphi(x),\qquad\text{for a.e. } x\in X.$$

Thus the triple $R=(S,f,\mathrm{id})\in C(T_\varphi)$. Of course, there is a continuum of g_0's in X satisfying the above conditions. Hence $C(T_\varphi)$ is uncountable. Then Theorem 2 and 3 imply $\mathrm{wcl}\{T_\varphi^n,n\in Z\}$ is uncountable. □

4. Rank of T_φ is r

In this section we use the shift representation $(\Omega_\omega, T_\sigma)$ of $(X \times \mathbb{Z}_n, T\varphi)$ (see 2.3) and the definition of rank given at the end of 2.2.

We will also require the notion of δ-cover: let $\mathcal{A}$ be a (finite) family of blocks and B a block such that $|B| \in \{|A|\colon\ A \in \mathcal{A}\}$, we let

$$d(B, \mathcal{A}) = \min\bigl\{d(B, A)\colon\ A \in \mathcal{A},\ |A| = |B|\bigr\}.$$

If $\mathcal{A} = \{A_1, \ldots, A_k\}$, C is a block, and $\delta > 0$, we define

$$t_\delta(\mathcal{A}, C) = t_\delta(A_1, \ldots, A_k, C) = \max\left\{\frac{|C_1| + \cdots + |C_p|}{|C|}\right\},$$

where the maximum is taken over all concatenations of the form

$$C = \epsilon_1 C_1 \epsilon_2 \ldots \epsilon_p C_p \epsilon_{p+1}$$

for which $d(C_i, \mathcal{A}) < \delta$, $1 \leqslant i \leqslant p$. Then we define, for a strictly ergodic one-sided sequence ω,

$$t_\delta(\mathcal{A}, \omega) = \liminf_{N\to\infty} t_\delta\bigl(\mathcal{A}, \omega[0, N]\bigr) \ \bigl(= \lim_{N\to\infty} t_\delta\bigl(\mathcal{A}, \omega[0, N]\bigr)\bigr).$$

In particular, $t_\delta(A, \omega)$ is defined for a single block A, or if $\omega = C$ is finite.

It is known ([3,20]) that in the case under consideration the *rank of* $(\Omega_\omega, S, \mu_\omega)$ *is at most* r if for any $\delta > 0$ and any $N \in \mathbb{N}$, there exists $\mathcal{A}$ of cardinality r such that $|A| \geqslant N$, $A \in \mathcal{A}$, and

$$t_\delta(\mathcal{A}, \omega) \geqslant 1 - \delta.$$

This definition agrees with that of sub-section 2.2.

Given a one-sided η, some $\delta > 0$, and a family $\mathcal{A}$ of blocks, we will say that *the subsequence* $\tilde{\eta}$ *of* η (*finite or infinite*) *is* δ*-covered by* $\mathcal{A}$ if $t_\delta(\tilde{\eta}, \mathcal{A}) \geqslant 1 - \delta$.

4.1. The frequencies of t-symbols and an estimation of the rank

Let $\mathrm{Fr}(E, \omega)$ be the average frequency of a t-symbol E (see 3.5) appearing in ω at natural positions. Similarly, let $\mathrm{Fr}(e, \omega_t)$ denote the average frequency of a "small" t-symbol e appearing in ω_t at natural positions. It is easy to get the following equalities;

$$\begin{cases} \mathrm{Fr}(E_k, \omega) = \mathrm{Fr}(e_k, \omega_t) = \dfrac{1}{rn}\displaystyle\sum_{i=0}^{r-1}\left(1 - \frac{1}{2^{i+2}}\right) = \frac{1}{n}\left[1 - \frac{1}{r}\sum_{i=0}^{r-1}\frac{1}{2^{i+2}}\right] \\ \text{and} \\ \mathrm{Fr}\bigl(E_k^{(s)}, \omega\bigr) = \mathrm{Fr}\bigl(e_k^{(s)}, \omega_t\bigr) = \dfrac{1}{rn2^{s+2}}, \quad s = 0, \ldots, r-1,\ k \in \mathbb{Z}_n, \end{cases} \tag{67}$$

if the case 3.1 holds. In the case 3.2 we have

$$\begin{cases} \operatorname{Fr}(E_k, \omega) = \operatorname{Fr}(e_k, \omega_t) = \dfrac{3}{4n}, \\ \operatorname{Fr}\bigl(E_k^{(s)}, \omega\bigr) = \operatorname{Fr}\bigl(e_k^{(s)}, \omega_t\bigr) = \dfrac{1}{4nr}, \quad k \in \mathbb{Z}_n, s = 0, \ldots, r-1. \end{cases} \tag{68}$$

PROPOSITION 12. – $r(T_\varphi) \leqslant r$.

Proof. – Consider the blocks

$$L_k^{(s)} = L_k^{(s)}(t) = B^t \overset{r}{\times} b_k^{t+1}(s), \quad s = 0, \ldots, r-1,\ t \geqslant 0,\ k \in \mathbb{Z}_n.$$

We have

$$E_k = L_k^{(0)} \ldots L_k^{(r-1)}, \qquad E_k^{(s)} = L_k^{(0)} \ldots L_k^{(s-1)} L_{k+1}^{(s)} L_k^{(s+1)} \ldots L_k^{(r-1)}$$

for every $k \in \mathbb{Z}_n$ and $s = 0, \ldots, r-1$.

Because the blocks $E_k, E_k^{(s)}$ cover completely the sequence ω then the blocks $L_k^{(0)} \ldots L_k^{(r-1)}, k \in \mathbb{Z}_n$, also cover ω.

We know that

$$\begin{gathered} b^{t+1}(s)\bigl[0, \lambda_{t+1} - knr2^{r+1}\bigr] = b_{-k}^{t+1}(0)\bigl[knr2^{r+1}, \lambda_{t+1} - 1\bigr], \\ k \in \mathbb{Z}_n, \quad s = 0, \ldots, r-1, \quad \text{if 3.1 holds,} \end{gathered}$$

and

$$\begin{gathered} b^{t+1}(s)[0, \lambda_{t+1} - knr4] = b_{-k}^{t+1}(0)[knr4, \lambda_{t+1} - 1], \\ k \in \mathbb{Z}_n, \quad s = 0, \ldots, r-1, \quad \text{if 3.2 holds.} \end{gathered}$$

The last equalities imply that the block $L_0^{(s)}$ cover each block $L_k^{(s)}, k \in \mathbb{Z}_n$, except of a part with the length $\leqslant n^2 2^{r+1} p_t$ in the case 3.1 and $\leqslant n^2 4 p_t$ in the case 3.2, for $s = 0, \ldots, r-1$. Thus the blocks $L_0^{(0)}, \ldots, L_0^{(r-1)}$ cover the sequence ω except of a part with the density $\leqslant n^2 2^{r+1}/\lambda_{t+1}$ if 3.1 holds and $\leqslant n^2 4/\lambda_{t+1}$ if 3.2 holds. Simultaneously $|L_0^{(s)}(t)| \xrightarrow{t\to\infty} \infty$. According to the definition of the rank (see 2.2) we have $r(T_\varphi) \leqslant r$. □

4.2. Special subblocks of ω_t

Fix $t \geqslant 0$. We distinguish special subblocks C of ω_t of the form $b^t \overset{r}{\times} \bar{C}$, where $\bar{C}$ is a strict subblock of one of the following blocks (cf. 3.5)

$$\begin{cases} e_k e_k, e_k e_k^{(s)}, e_k^{(s)} e_{k+1}, \quad k \in \mathbb{Z}_n, s = 0, \ldots, r-1, \\ \quad \text{where } e_k = e_k(t+1), e_k^{(s)} = e_k^{(s)}(t+1), \\ \quad \text{if the case 3.2 is considered,} \end{cases} \tag{69}$$

or

$$\begin{cases} e_k e_k e_k e_k, e_k e_k e_k e_k^{(s)}, e_k e_k e_k^{(s)} e_{k+1}, e_k e_k^{(s)} e_{k+1} e_{k+1}, e_k^{(s)} e_{k+1} e_{k+1} e_{k+1}, \\ \quad k \in \mathbb{Z}_n, s = 0, \ldots, r-1, \\ \quad \text{if the case 3.1 is considered.} \end{cases} \tag{70}$$

Notice that blocks (69) are all pairs of “small” $(t+1)$-symbols appearing in ω_{t+1}, as well as the blocks (70) are all possible quadruples of “small” $(t+1)$-symbols appearing in ω_{t+1}. Let us list the different cases we shall deal with afterwards:

(A) $\bar{C} \subset b_{k_0}^{t+1}(i_0)$ for some $k_0 \in \mathbb{Z}_n$ and $i_0 = 0, \ldots, r-1$ (cases 3.1 or 3.2);

(B) (the case 3.2) $\bar{C} = b_{k_{i_0}}(i_0) \ldots b_{k_{r-1}}(r-1) \mid b_{l_0} \ldots b_{l_{i_1}}(i_1)$ where $b(i) = b^{t+1}(i)$, $i_0 > 0, i_1 < r-1$.

$E := (k_{i_0} \ldots k_{r-1} l_0 \ldots l_{i_1})$ is contained in one of the following blocks;

$$\bar{e}_k \bar{e}_k,\ \bar{e}_k \bar{e}_k^{(s)},\ \bar{e}_k^{(s)} \bar{e}_{k+1}, \quad k \in \mathbb{Z}_n, s = 0, \ldots, r-1, \tag{71}$$

and $2 \leqslant |E| < 2r$;

(B′) (the case 3.1) $\bar{C} = b_{k_{i_0}}(i_0)..b_{k_{r-1}}(r-1) \mid b_{u_0}(0)..b_{u_{r-1}}(r-1) \mid b_{v_0}(0)..b_{v_{r-1}}(r-1) \mid b_{l_0}(0)..b_{l_{i_1}}(i_1)$ and $E = (k_{i_0} \ldots k_{r-1} \mid u_0 \ldots u_{r-1} \mid v_0 \ldots v_{r-1} \mid l_0 \ldots l_{i_1})$, $2 \leqslant |E| < 4r$, $i_0 > 0$, $i_1 < r-1$, is contained in one of the blocks

$$\bar{e}_k \bar{e}_k \bar{e}_k \bar{e}_k, \bar{e}_k \bar{e}_k \bar{e}_k \bar{e}_k^{(s)},\ \bar{e}_k \bar{e}_k \bar{e}_k^{(s)} \bar{e}_{k+1},\ \bar{e}_k \bar{e}_k^{(s)} \bar{e}_{k+1} \bar{e}_{k+1},\ \bar{e}_k^{(s)} \bar{e}_{k+1} \bar{e}_{k+1} \bar{e}_{k+1}. \tag{72}$$

In general we can write

$$\bar{C} = \bar{C}_1 \bar{C}_2 \bar{C}_3 \tag{73}$$

where $\bar{C}_2$ is as in (A) or as in (B) (the case 3.2) or (B′) (the case 3.1), and

$$\begin{cases} \bar{C}_1 = b_{k'}^{t+1}(i_0 - 1)[l_1 r, \lambda - 1], \quad \bar{C}_3 = b_{k''}^{t+1}(i_1 + 1)[0,\ l_2 r - 1], \\ 0 < l_1 \leqslant \lambda - 1,\ 0 < l_2 \leqslant \lambda - 1,\ \lambda = \lambda_{t+1}, \end{cases} \tag{74}$$

and $k' E k''$ is contained in one of the blocks (71) or (72) respectively (E is defined by $\bar{C}_2$).

Then we can distinguish the next special kinds of blocks (73) for given $\delta > 0$:

(G1) $|\bar{C}_1|/|\bar{C}| > \delta$ and $|\bar{C}_3|/|\bar{C}| > \delta$,

(G2) $|\bar{C}_1|/|\bar{C}| > \delta$ and $|\bar{C}_3|/|\bar{C}| \leqslant \delta$,

(G3) $|\bar{C}_1|/|\bar{C}| \leqslant \delta$ and $|\bar{C}_3|/|\bar{C}| > \delta$,

(G4) $|\bar{C}_1|/|\bar{C}| \leqslant \delta$ and $|\bar{C}_3|/|\bar{C}| \leqslant \delta$.

4.3. $r(T_\varphi) = r$: the case 3.2

Take $0 < \delta^2 < 1/(r^2 2^{2r+3})$.

PROPOSITION 13. – *Assume that $\bar{C}$ is as in* (B) *and let $d(C, D) < \delta^2$, $D \subset \omega_t$. Then D has a form*

$$D = \left(b^t \overset{r}{\times} \bar{D}\right)[j, j + |D| - 1], \quad \textit{where } \bar{D} \subset \omega_{t+1} \tag{75}$$

and

$$\begin{cases} \bar{D} = b_{k'_{i_0}}^{t+1}(i_0) \ldots b_{k'_{r-1}}^{t+1}(r-1) \mid b_{l'_0}^{t+1}(0) \ldots b_{l'_{i_1}}^{t+1}(i_1) b_{l'_{i_1+1}}^{t+1}(i_1 + 1), \\ \quad \textit{and } j < \delta^2 r 2^{r+1} \lambda_{t+1},\ l'_{i_1+1} \in \mathbb{Z}_n \end{cases} \tag{76}$$

or

$$\begin{cases} \bar{D} \text{ is as in } (76) \text{ and} \\ j > r\lambda_{t+1} - \delta^2 r 2^{r+1}\lambda_{t+1}. \end{cases} \tag{77}$$

Moreover, there is a unique $s_0 \in \mathbb{Z}_n$ *such that*

$$(k'_0 \dots k'_{r-1} \mid l'_0 \dots l'_{i_1}) = (k_0 \dots k_{r-1} \mid l_0 \dots l_{i_1}) + s_0$$

if (76) *holds and*

$$(k'_1 \dots k'_{r-1} \mid l'_0 \dots l'_{i_1+1}) = (k_0 \dots k_{r-1} \mid l_0 \dots l_{i_1}) + s_0$$

if (77) *holds.*

Proof. – The proposition is an easy consequence of the Proposition 10 where t is taken instead of $t+1$ ($\delta^2 < 1/(r^2 2^{2r+3}) < 1/(3r^2 2^{r+3})$). □

Given a block $A \subset \omega$ or ω_t, $A = \omega[l, l+|A|-1]$ we define $A(\delta)$ as $A(\delta) = \omega[l - \delta|A|, l+|A| + \delta|A| - 1]$, $\delta > 0$. The next proposition says that if C is as in (G1), (G2), (G3), or (G4), there is a block $C' = b^t \overset{r}{\times} \widetilde{C}$ such that $\widetilde{C}$ is as in (B) and either $\widetilde{C}$ contains $\bar{C}$ or $\bar{C}$ is contained in $\widetilde{C}(\delta_1)$, where $\delta_1 < \delta^2 r 2^{r+1}$.

PROPOSITION 14. – *Let* $C = b^t \overset{r}{\times} \bar{C}$ *and let* $\bar{C}$ *be as in* (G1), (G2), (G3) *or* (G4). *Assume that*

$$d\big(C, \omega_t[l, l+|C|-1]\big) < \frac{\delta^2}{3}. \tag{78}$$

Then

$$d\big(C', \omega_t[l', l'+|C'|-1]\big) < \delta^2$$

where $C' = b^t \overset{r}{\times} \widetilde{C}$, $\widetilde{C} \subset \omega_{t+1}$ *and*

(g1) $\widetilde{C} = b^{t+1}_{k'}(i_0 - 1)\bar{C}_2 b^{t+1}_{k''}(i_1+1)$, $l' = l - l_1 r$ (*cf.* (73), (74)), *if* (G1) *holds,*
(g2) $\widetilde{C} = b^{t+1}_{k'}(i_0 - 1)\bar{C}_2$, $l' = l - l_1 r$, *if* (G2) *holds,*
(g3) $\widetilde{C} = \bar{C}_2 b^{t+1}_{k''}(i_1+1)$, $l' = l$, *if* (G3) *holds,*
(g4) $\widetilde{C} = \bar{C}_2$, $l' = l$, *if* (G4) *holds.*

Proof. – Consider the case (G2). Then (11) and (78) imply ($C_2 = b^t \overset{r}{\times} \bar{C}_2$)

$$d\big(b^t \overset{r}{\times} \bar{C}_2, \omega_t[\bar{l_2}, \bar{l_2} + |C_2| - 1]\big) < \delta^2$$

where $\bar{l_2} = l + |b^t \overset{r}{\times} \bar{C}_1|$.

It follows from Proposition 13 that $\omega_t[\bar{l_2}, \bar{l_2} + |C_2| - 1]$ is of the form (75). Assume that the case (76) holds. Set

$$\widetilde{C_1} = \bar{C}_1[0, |\bar{C}_1| - j - 1],$$

$$\widetilde{D_1} = \omega_{t+1}\left[\frac{1}{\lambda_t}(l-j), \frac{1}{\lambda_t}(l-j) + |\widetilde{C_1}| - 1\right] \quad \text{(see Fig. 3).}$$

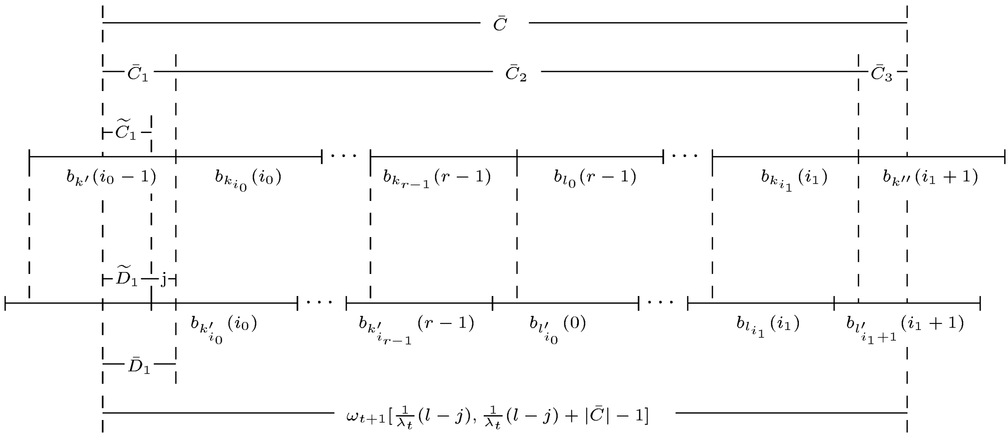

Fig. 3.

If follows from Proposition 8 that

$$j \equiv (n-s_0)r4 \ (\mathrm{mod}\ 4nr). \tag{79}$$

The fragment of ω_{t+1} from the left side of $b^{t+1}_{k'_0}(i_0)$ having the length λ_{t+1} is of a form $b^{t+1}_u(i_0-1)$ and either $u = k'+s_0$ or $u = k'+s_0+1$. Assume that $u = k'+s_0+1$. Then Proposition 9 implies

$$d(\widetilde{C_1}, \widetilde{D_1}) = 1. \tag{80}$$

Let $\bar{D}_1$ denote the block $\omega_{t+1}[\frac{1}{\lambda_t}(l-j), \frac{1}{\lambda_t}(l-j)+|\bar{C}_1|-1]$ (see Fig. 3). Obviously we have

$$\frac{|\bar{C}_1|}{|C|} d(\bar{C}_1, \bar{D}_1) \overset{(11),(8)}{\leqslant} d\bigl(C, \omega_t[l, l+|C|-1]\bigr) < \delta^2.$$

Further

$$\delta^2 > \frac{|\bar{C}_1|}{|\bar{C}|} d(\bar{C}_1, \bar{D}_1) > \delta d(\bar{C}_1, \bar{D}_1) \overset{(11)}{\geqslant} \frac{|\widetilde{C_1}|}{|\bar{C}_1|} \delta d(\widetilde{C_1}, \widetilde{D_1}) \overset{(80)}{=} \frac{|\bar{C}_1|-j}{|\bar{C}_1|}\delta$$
$$= \delta\left(1-\frac{j}{|\bar{C}_1|}\right) \overset{(G2)}{\geqslant} \delta\left(1-\frac{j}{\delta|\bar{C}|}\right) \overset{(76)}{\geqslant} \delta\left(1-\frac{\delta^2 r 2^{r+1}\lambda_{t+1}}{\delta|\bar{C}|}\right) \geqslant \delta\bigl(1-\delta r 2^{r+1}\bigr),$$

because $|\bar{C}| \geqslant \lambda_{t+1}$.

Thus

$$1-\delta r 2^{r+1} < \delta$$

which is in contradiction with the inequality $\delta^2 < 1/(r^2 2^{2r+3})$. We have shown $u-k' = s_0 = k'_0 - k_0$.

Now, using (79) and the definition of $b_{k'}(i_0-1)$ and $b_u(i_0-1)$ we obtain $C[v] = \omega_t[l'+v]$ for each $v = 0, \ldots, |\bar{C}_1|-1$, $l' = l - l_1 r$ (see (74)). This last equality implies (g2). The proofs of the remaining cases are similar. $\square$

PROPOSITION 15. – *Assume that* $\mathcal{F} = \{C_1, \ldots, C_d\}, d \leqslant r - 1$, *is a family of subblocks of* ω_t *such that*

$$C_j = b^t \overset{r}{\times} \bar{C}_j \quad \textit{and} \quad \textit{each } \bar{C}_j \textit{ is as in (B).} \tag{81}$$

Let $\omega_t(\mathcal{F})$ *be the maximal subsequence of* ω_t *that can be* δ^2*-covered by the family* $\mathcal{F}$ *in a disjoint way,* $\delta^2 < 1/(r^2 2^{2r+3})$, *and let* $\bar{\omega}_t(\mathcal{F})$ *be the complementary part of* ω_t*. Then it is an union of at least* $(r-d)$ *blocks* $b^t \overset{r}{\times} b^{t+1}(i_j)$, $j = 1, \ldots, r-d$.

Proof. – Denote by $\mathcal{F}_i$ the set of all blocks $C \in \mathcal{F}$ such that $\bar{C}\delta^2$-covers a subblock of ω_{t+1} containing one of the form

$$b_1^{t+1}(i) b^{t+1}(i+1), \quad i = 0, \ldots, r-2,$$

and by $\mathcal{F}_{r-1}$ those C for which $\bar{C}$ δ^2-covers a block containing $b^{t+1}(r-1)b^{t+1}(0)$. We show that $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ whenever $i \neq j$. Take $C \in \mathcal{F}_i$, $D \in \mathcal{F}_j$ and let $\bar{C}$, $\bar{D}$ be the blocks defined by (81), $\bar{C}$ as in (B) and

$$\bar{D} = b^{t+1}_{k'_{i'_0}}(i'_0) \ldots b^{t+1}_{k'_{r-1}}(r-1) \mid b^{t+1}_{l'_0}(0) \ldots b^{t+1}_{l'_{i'_1}}(i'_1).$$

If $(i_0 \ldots (r-1) \mid 0 \ldots i_1) \neq (i'_0 \ldots (r-1) \mid 0 \ldots i'_1)$ then $C \neq D$. If $(i_0 \ldots (r-1) \mid 0 \ldots i_1) = (i'_0 \ldots (r-1) \mid 0 \ldots i'_1)$ then using Proposition 13 we obtain

$$(k_{i_0} \ldots k_{r-1} \mid l_0 \ldots l_{i_1}) = (k'_{i_0} \ldots k'_{r-1} \mid l'_0 \ldots l'_{i_1}) + s_0$$

for some $s_0 \in \mathbb{Z}_n$. The last condition is impossible since $i \neq j$. The proposition follows because $\#\{\mathcal{F}_i;\ 0 \leqslant i < r\} = r$. $\square$

THEOREM 4. – $r(T_\varphi) = r$.

Proof. – According to Proposition 12 it remains to show that $r(T_\varphi) > r - 1$. Let

$$\frac{\delta^2}{9} < \frac{1}{r^2 2^{2r+3}}$$

and let $A_1, \ldots, A_x$ be blocks occurring in ω, $|A_i| \geqslant p_{t_0}$ and t_0 satisfies $r/\lambda_t < \delta^2 r 2^{r+1}$, if $t \geqslant t_0$, $x \leqslant r - 1$. For each $u = 1, \ldots, x$ there exists an unique $t = t(u)$ such that A_u contains at least one t-symbol and does not contain any $(t+1)$-symbol. Then A_u has a form

$$A_u = \widetilde{E_1}\big(B^{t-1} \overset{r}{\times} C_u\big)\widetilde{E_2}, \tag{82}$$

where $C_u \subset \omega_t$ is as in 4.2, $|C_u| = qr$, $q = q(u) \geqslant 1$, E_1 is a right-side part of a t-symbol and E_2 is a left-side part of a t-symbol. We divide the set $\{t(1), \ldots, t(x)\}$ by arithmetic order. More precisely, we put

$$\tau_1 = \max\big\{t(1), \ldots, t(x)\big\}, \qquad T_1 = \big\{u; t(u) = \tau_1\big\}, \qquad d_1 = \#T_1.$$

Next we define

$$\tau_2 = \max\{t(u);\ u \notin T_1\}, \qquad T_2 = \{u;\ t(u) = \tau_2\}, \qquad d_2 = \#T_2.$$

Similarly we define sets $T_3, \ldots, T_v$, numbers $\tau_3, \ldots, \tau_v$ and $d_3, \ldots, d_v$. We have

$$\tau_1 > \cdots > \tau_v, \qquad d_1 + \cdots + d_v = x.$$

Let

$$\mathcal{A}_p = \{A_u;\ u \in T_p\}, \quad p = 1, \ldots, v.$$

The families $\mathcal{A}_1, \ldots, \mathcal{A}_v$ are pairwise disjoint and $\bigcup_{p=1}^{v} \mathcal{A}_p = \{A_1, \ldots, A_x\}$.

Consider the family $\mathcal{A}_1$. Assume that

$$\mathcal{A}_1 = \{A_1, \ldots, A_{d_1}\}.$$

Then

$$C_u = b^t \overset{r}{\times} \bar{C}_u$$

and

$$\bar{C}_u \subset \omega_{t+1}, \quad u \in T_1, t = \tau_1.$$

If $d(A_u, \omega[\widetilde{l}, \widetilde{l} + |A_u| - 1]) < \delta^2/9$ then by (11), (8),

$$d\big(B^{t-1} \overset{r}{\times} C_u, \omega[l, l + m_{t-1}|C_u| - 1]\big) < \frac{\delta^2}{3} \tag{83}$$

where $l = \widetilde{l} + |\widetilde{E_1}|$.

According to Proposition 11

$$d\big(C_u, \omega_t[l', l' + |C_u| - 1]\big) < \frac{\delta^2}{3} \tag{84}$$

for some $l' \in \mathbb{Z}$ and

$$|l - p_t l'| < \frac{1}{3}\delta^2 r 2^{r+1} p_t. \tag{85}$$

We can write

$$\bar{C}_u = \bar{C}_u^{(1)} \bar{C}_u^{(2)} \bar{C}_u^{(3)}$$

according to (73).

We distinguish among the blocks $A_1, \ldots, A_{d_1}$ three types $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, as follows;

$A_u \in \mathcal{F}_1$ if C_u is as in (A) or (G4),
$A_u \in \mathcal{F}_2$ if C_u is as in (G1), (G2), or (G3),
$A_u \in \mathcal{F}_3$ if C_u is as in (B).

Let $d_{11} = \#\mathcal{F}_1, d_{12} = \#\mathcal{F}_2, d_{13} = \#\mathcal{F}_3$. We have

$$d_{11} + d_{12} + d_{13} = d_1.$$

Let $\omega(A_1, \ldots, A_{d_1})$ be a subsequence of ω that is $\frac{\delta^2}{9}$-covered by the blocks $A_1, \ldots, A_{d_1}$ in a disjoint way. By $\omega(\mathcal{F}_i), i = 1, 2, 3,$ we denote the subsequence of $\omega \frac{\delta^2}{9}$-covered in a disjoint way by the families $\mathcal{F}_i$. Of course, $\omega(A_1, \ldots, A_{d_1}) \subset \omega(\mathcal{F}_1) \cup \omega(\mathcal{F}_2) \cup \omega(\mathcal{F}_3)$. Denoting by $\bar{\omega}(A_1, \ldots, A_{d_1}), \bar{\omega}(\mathcal{F}_i)$ the complementary parts of $\omega(A_1, \ldots, A_{d_1}), \omega(\mathcal{F}_i)$, $i = 1, 2, 3,$ respectively, we have

$$\bar{\omega}(A_1, \ldots, A_{d_1}) \supset \bar{\omega}(\mathcal{F}_1) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_3).$$

According to (83)–(85) and Proposition 15 we have that $\bar{\omega}(\mathcal{F}_3)$ is an union of at least

$$(r - d_{13}) \text{ blocks } E(\delta_1), \tag{86}$$

where

$$\begin{cases} E = B^t \overset{r}{\times} b^{t+1}(i_j), \quad j = 1, \ldots, r - d_{13}, \quad \text{and} \\ \delta_1 \leqslant 2\delta^2 r 2^{r+1}, \end{cases} \tag{87}$$

because of

$$\frac{|\widetilde{E_1}|}{|A_u|} \overset{(84)}{\leqslant} \frac{p_t}{m_{t+1}} = \frac{r}{\lambda_{t+1}} < \frac{1}{2}\delta_1, \quad \text{and} \quad \frac{|\widetilde{E_2}|}{|A_u|} < \frac{1}{2}\delta_1.$$

Consider the family $\mathcal{F}_2$. Let $A_u \in \mathcal{F}_2$. If $A_u \frac{\delta^2}{9}$-covers a fragment I_u of ω then (83) and (84) imply that $\bar{C}_u \frac{\delta^2}{3}$-covers a fragment $I_u = I_u(t)$ of ω_{t+1} and (85) implies

$$I_u \subset \left(B^t \overset{r}{\times} I_u(t)\right)(\delta_1).$$

It follows from Proposition 14 that there is $A_{\bar{u}}$ of a form as in $\mathcal{F}_3$ such that $\widetilde{C}_{\bar{u}} \frac{\delta^2}{3}$-covers another fragment $I_{\bar{u}}(t)$ of ω_{t+1} such that

$$I_u(t) \subset I_{\bar{u}}(t)(\delta).$$

Applying Proposition 15 to the family $\{A_{\bar{u}}\}$ we obtain that $\bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2)$ is an union of at least $(r - d_{13} - d_{12})$ blocks $E(\delta_2)$, E is as (87) and $\delta_2 = \max(\delta, \delta_1)$.

Each block $E(\delta_2) \in \bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2)$ is an union of at least $(r - d_{13} - d_{12})$ blocks of the form $B^t \overset{r}{\times} e_k^{(s)}, k \in \mathbb{Z}_n, s \in S, \#S = r - d_{13} - d_{12}$.

Using the same arguments as before we get that

$$\begin{cases} \bar{\omega}(\mathcal{F}_3) \cap \bar{\omega}(\mathcal{F}_2) \cap \bar{\omega}(\mathcal{F}_1) \text{ is a union} \\ \text{of at least } (r - d_{13} - d_{12} - d_{11}) \text{ blocks of the form } B^{t-1} \overset{r}{\times} e_k^{(s)}, \\ \quad s \in S_1, \#S_1 = r - d_{13} - d_{12} - d_{11}. \end{cases} \tag{88}$$

Denoting $P(\omega_1, \omega)$ the density of a subsequence ω_1 in ω and using (69), (86), (88) we have

$$\begin{aligned}P\left(\bar{\omega}(A_1,\ldots,A_{d_1}),\omega\right) &\geqslant P\left(\bar{\omega}(\mathcal{F}_3)\cap\bar{\omega}(\mathcal{F}_2)\cap\bar{\omega}(\mathcal{F}_1),\omega\right)\\ &\geqslant\left(1-\frac{d_{13}+d_{12}}{r}\right)\left(1-\frac{d_{11}}{r}\right)\left(\frac{1}{4nr}\right)^2(1-\delta_2)\\ &\geqslant\left(1-\frac{1}{r}\right)^2\left(\frac{1}{4nr}\right)^2(1-\delta_2)\geqslant\left(1-\frac{1}{r}\right)^2\left(\frac{1}{4nr}\right)^2\frac{1}{2}.\end{aligned}$$

If $T_1\neq\{1,\ldots,x\}$ then we repeat the above reasoning to the subsequence $\bar{\omega}(\mathcal{F}_3)\cap\bar{\omega}(\mathcal{F}_2)\cap\bar{\omega}(\mathcal{F}_1)$ and $t=\tau_2$, and so on. As a consequence we get

$$P\left(\bar{\omega}(A_1,\ldots,A_x),\omega\right)\geqslant\left(1-\frac{1}{r}\right)^{2r}\frac{1}{2^r}\left(\frac{1}{4nr}\right)^{2r}.$$

This implies $r(T_\varphi)>r-1$. Thus we have shown $r(T_\varphi)=r$. □

4.4. $r(T_\varphi)=r$: the case 3.1

To prove that $r(T_\varphi)=r$ in the case 3.1 we can repeat the same arguments an in 4.3. Similarly as in the Theorem 4 we consider blocks $A_u, u=1,\ldots,x, x\leqslant r-1$, and A_u are as in (82), $C_u=b^t\overset{r}{\times}\bar{C}_u$ but $\bar{C}_u$ are as in (A), (B′) and (G1), (G2), (G3), (G4).

As an analogue of Propositions 13–15 and Theorem 4 we obtain

PROPOSITION 13′. – *Assume that C is as in* (B′) *and let $d(C,D)<\delta^2$, $D\subset\omega_t$. Then D has a form* (75), *and*

$$\bar{D}=b_{k'_{i_0}}(i_0)..b_{k'_{r-1}}(r-1)\mid b_{u'_0}(0)..b_{u'_{r-1}}(r-1)\mid b'_{v_0}(0)..b_{v'_{r-1}}(r-1)\mid b_{l'_0}(0)..b_{l'_{i_1}}(i_1),$$

$b_k(i)=b_k^{t+1}(i)$, *and j satisfies either* (76) *or* (77).

PROPOSITION 14′. – *Let C be as in Proposition* 14, *$\bar{C}$ is as in* (73) *and $\bar{C}_2$ is as in* (B′). *Then we get* (g1), (g2), (g3) *or* (g4).

The proofs of Propositions 13′ and 14′ are similar to the proofs of Propositions 13 and 14.

PROPOSITION 15′. – *Let $\mathcal{F}=\{C_1,\ldots,C_d\}$, $d\leqslant r-1$, $C_j=b^t\overset{r}{\times}\bar{C}$, and C_j are as in* (B′). *Then we have the same thesis as in Proposition* 15.

Proof. – Let $\mathcal{F}_{i,k}, i=0,\ldots,r-2, k\in\mathbb{Z}_n$, be the set of all blocks $C\in\mathcal{F}$ such that $\bar{C}(C=b^t\overset{r}{\times}\bar{C})\delta^2$-covers a subblock of ω_{t+1} containing one of the form $b_k^{t+1}(i)b_{k+1}^{t+1}(i+1)$. By $\mathcal{F}^{(1)}_{r-1,k},\mathcal{F}^{(2)}_{r-1,k}$ we denote those $C\in\mathcal{F}$ such that $\bar{C}$ does so for the pairs $b_k^{t+1}(r-1)b_k^{t+1}(0)$ or $b_k^{t+1}(r-1)b_{k+1}^{t+1}(0)$ respectively.

Using Propositions 3 and 7 we get that

$$\begin{cases} \text{if } C \in \mathcal{F}_{i,k} \text{ then } \bar{C}\delta^2\text{-covers (up to } \delta r^2 2^{r+3}\lambda_{t+1}) \text{ only those} \\ \text{fragments of } \omega_{t+1} \text{ containing blocks of the form} \\ (89')\ b^{t+1} \overset{r}{\times} \bar{e}_k^{(i+1)} \text{ or } b^{t+1} \overset{r}{\times} \bar{e}_{k+4}^{(i)},\ \text{if } n \geqslant 3, \\ \text{and} \\ (89'')\ b^{t+1} \overset{r}{\times} \bar{e}_k^{(i+1)} \text{ if } n = 2, \end{cases} \tag{89}$$

whenever $i = 0, \ldots, r-2$, $k \in \mathbb{Z}_n$. Using Propositions 4 and 7 we get that

$$\begin{cases} \text{if } C \in \mathcal{F}_{r-1,k}^{(1)} \text{ then } \bar{C}\delta^2\text{-covers only those fragments} \\ \text{of } \omega_{t+1} \text{ containing blocks of the form} \\ (90')\ b_k^{t+1}(r-1)b_k^{t+1}(0) \text{ or } b_l^{t+1}(r-1)b_{l+1}^{t+1}(0), \\ l \text{ satisfies (54),} \end{cases} \tag{90}$$

and

$$\begin{cases} \text{if } C \in \mathcal{F}_{r-1,k}^{(2)} \text{ then } \bar{C}\delta^2\text{-covers only those fragments} \\ \text{of } \omega_{t+1} \text{ containing blocks of the form} \\ (91')\ b_k^{t+1}(r-1)b_{k+1}^{t+1}(0) \text{ or } b_l^{t+1}(r-1)b_l^{t+1}(0), \\ l \text{ satisfies (54).} \end{cases} \tag{91}$$

Now notice that each two blocks $b^{t+1} \overset{r}{\times} e_k^{(i)}$ and $b^{t+1} \overset{r}{\times} e_{k'}^{(i)}$, $k' \in \mathbb{Z}_n$, $k \neq k'$, appearing in ω_{t+1} are separated by at least three blocks of the form $b^{t+1} \overset{r}{\times} e_{k+1}$. This, (89) and the condition $|E| < 4r$ (see (B$'$)) imply that $\mathcal{F}_{i,k} \cap \mathcal{F}_{i,k'} = \emptyset$, if $k \neq k'$, $i = 0, \ldots, r-2$. Similarly $\mathcal{F}_{r-1,k}^{(1)} \cap \mathcal{F}_{r-1,k'}^{(1)} = \emptyset$ and $\mathcal{F}_{r-1,k}^{(2)} \cap \mathcal{F}_{r-1,k'}^{(2)} = \emptyset$, if $k \neq k'$.

Further (89) implies that if $C \in \mathcal{F}_{i,k} \cap \mathcal{F}_{i',k'}$ then $i' = i+1, k' = k+4$ if $n \geqslant 3$ ((89$'$)) and $i' = i, k' = k$ if $n = 2$ ((89$''$)), $i = 0, \ldots, r-2$. (90) implies that if $C \in \mathcal{F}_{r-1,k}^{(1)} \cap \mathcal{F}_{r-1,k'}^{(2)}$ then $k' = l, l$ satisfying (54). Combining the above arguments we get that there is at least $\frac{rn}{2} - d$ fragments of ω_{t+1} of the form (89$'$) and (90) or (91) that are not covered by the family $\mathcal{F}$. The Proposition follows because $\frac{rn}{2} \geqslant r$. $\square$

THEOREM 4$'$. – $r(T_\varphi) = r$.

Proof. – We repeat the same reasoning as in the proof of Theorem 4 using blocks $A_1, \ldots, A_x$ of the form (82) with $q \geqslant 3$. We use Proposition 7 instead of Proposition 11 and the Propositions 14$'$ and 15$'$ instead of Propositions 14 and 15. Then using (67) instead of (68) we get

$$P\big(\bar{\omega}(A_1, \ldots, A_x), \omega\big) \geqslant \left(1 - \frac{1}{r}\right)^{2r} \frac{1}{2^r} \left(\frac{1}{rn2^{r+1}}\right)^{2r},$$

what implies $r(T_\varphi) > r-1$ and by Proposition 12 we have $r(T_\varphi) = r$. $\square$

5. Pairs (r, ∞) or (∞, m)

In this part we construct group extensions $(X \times G, T_\varphi)$ such that $r(T_\varphi) = r, q(T_\varphi) = \infty, 2 \leqslant r < \infty$ or $r(T_\varphi) = \infty, q(T_\varphi) = m, 1 \leqslant m < \infty$.

5.1. The case (r, ∞)

Take a sequence $\{s_t\}_{t=0}^{\infty}, s_{t+1} = \mu_{t+1}s_t, s_0 = \mu_0,\ \mu_t \geqslant 2$ for $t \geqslant 0$ and let G be the group of $\{s_t\}$-adic integers. Let $e = 1 + 0s_1 + 0p_2 + \cdots$. The set of all $\{s_t\}$-adic rational integers of G coincides with the set $\{e_n,\ n \in Z\}$, where $e_n = ne$. Similarly as in the case 3.1 we define an adding machine $(X, \mathcal{B}, \mu, T)$ and a cocycle $\varphi : X \longrightarrow G$. To do this we define blocks $F^{(0)}, F^{(1)}, \ldots, F^{(r-1)}$ ($r \geqslant 2$ is given) over G.
Put

$$F^{(i)}(t) = F^{(i)} = \overbrace{0\ldots0}^{r(2^{i+1}-1)}\ \overbrace{0\ldots0e0\ldots0}^{r}, \quad i = 0, \ldots, r-1,$$
$$H^{(i)} = F^{(i)} F_e^{(i)} \ldots F_{(s_t-1)e}^{(i)}.$$

Then $|H^{(i)}| = s_t r 2^{t+1}$. Next define $b^t(0), \ldots, b^t(r-1)$ as in 3.1 and $b^t = b^t(0)\ldots b^t(r-1), t \geqslant 0$.
We have

$$\lambda_t = \left|b^t(i)\right| = s_t r 2^{r+t+1}, \quad i = 0, \ldots, r-1$$

and

$$\left|b^t\right| = s_t r^2 2^{r+t+1}.$$

Then we define the blocks $B^t,\ t \geqslant 0$, by (14). We have $p_t = |B^t| = s_0 \ldots s_t r^{2t} 2^{r+1}(2^{t+1} - 1)$. Let $(X, \mathcal{B}, \mu, T)$ be the $\{p_t\}$-adic adding machine and define a cocycle $\varphi : X \longrightarrow G$ by (18).

THEOREM 5. – *$r(T\varphi) = r$ and $q(T_\varphi) = \infty$*

Proof. – Let $\Pi_t : G \longrightarrow Z/s_t Z$ be the natural group homomophism. We can define cocycles $\varphi_t : X \longrightarrow Z/s_t Z$ by $\varphi_t = \varphi \circ \Pi_t$. It is evident that φ_t is a r-Toeplitz cocycle as in 3.1 defined by the blocks $\Pi_t(B_k), u \geqslant 0$. According to Theorems 2 and 4 we have $r(T_{\varphi_t}) = r$ and $q(T_{\varphi_t}) = s_t$. It follows from the definitions of φ and φ_t that the dynamical system $(X \times G, T_\varphi)$ is the inverse limit of the systems $(X \times Z/s_t Z, T_{\varphi_t})$. Then from the definition of the rank we obtain $r(T_\varphi) = r$. It is proved in Theorem 2 that $\sigma_{je} \notin \mathrm{wcl}\{T_{\varphi_t}^n,\ n \in Z\}$ if $j = 0, \ldots, s_t - 1,\ t \geqslant 0$. This means that $\sigma_{je} \notin \mathrm{wcl}\{T_\varphi^n,\ n \in Z\}$ for every $j \in Z, j \neq 0$ which implies $q(T_\varphi) = \infty$. □

5.2. The case (∞, m)

First consider the case $m \geqslant 2$. Let $r_t = 2^{t+1},\ t \geqslant 0$, and define blocks $F^{(i)} = F^{(i)}(t)$ over $G = Z/mZ,\ i = 0, \ldots, r_{r+1} - 1$, as follows:

$$vF^{(i)} = \overbrace{0\ldots0}^{2^{i+1}r_t}\overbrace{0\ldots0\underbrace{\ 1\ }_{i+1}0\ldots0}^{r_{t+1}},$$

$$H^{(i)} = F_0^{(i)} F_1^{(i)} \dots F_{m-1}^{(i)}, \quad i = 0, \dots, r_{t+1} - 1.$$

We have $|H^{(i)}| = mr_t 2^{i+3}$. Next define $b^t(0), \dots, b^t(r_{t+1}-1), b^t, B^t$ by putting

$$b^t(i) = \overbrace{H^{(i)} H^{(i)} \dots H^{(i)}}^{x}, \quad x = 2^{t+r_{t+1}-i-1}$$
$$b^t = b^t(0) b^t(1) \dots b^t(r_{t+1}-1), \quad \text{and}$$
$$B^t = b^0 \overset{r_0}{\times} b^1 \overset{r_1}{\times} \cdots \overset{r_{i-1}}{\times} b^t.$$

Then $\lambda_t = |b^t(i)| = m2^{2t+\rho+2}$, $\rho = r_{t+1}$ and $p_t = m_t r_{t+1}, m_t = \lambda_0 \cdots \lambda_t$. We define a cocycle $\varphi : X \longrightarrow G$ by

$$\varphi(x) = B^t[j+1] - b^t[j]$$

if $x \in D_j^t$ except if $j = m_t - 1, \dots, p_t - 1$. The cocycle φ is constant on the levels D_j^t except of r_{t+1} consecutive levels.

In a similar way we construct a cocycle φ if $m = 1$. Take n as in the case 3.2 and define

$$F^{(i)}(t) = F^{(i)} = \overbrace{0 \dots 0}^{2r_t} \overbrace{0 \dots 0 \underbrace{\ 1\ }_{i+1} 0 \dots 0}^{2r_t},$$
$$H^{(i)} = F_0^{(i)} F_1^{(i)} \dots F_{n-1}^{(i)}, \quad i = 0, 1, \dots, r_{t+1} - 1.$$

The next steps of the definition φ are the same as in the case $m \geqslant 2$.

THEOREM 6. – *$r(T_\varphi) = \infty, q(T_\varphi) = m$ and $\mathrm{wcl}\{T_\varphi^n,\ n \in z\}$ is uncountable.*

Proof. – For the dynamical system $(X \times G, T_\varphi)$ we can use the same arguments as in the parts 3 and 4 taking r_t instead of r. Theorems 2, 3 and 3′ are valid. To estimate the rank of T_φ we use the shift representations $(\Omega\omega, T_\sigma)$ of $(X \times G, T_\varphi)$ where $\omega = b^0 \overset{r_0}{\times} b^1 \overset{r_1}{\times} \cdots$. Repeating the proof of Theorem 4 and 4′ we get $r(T_\varphi) > r_t - 1$ for every $t \geqslant 0$. Thus $r(T_\varphi) = \infty$. □

6. The pair (∞, ∞)

This case is easy to handle: first let (Y, S, ν) be the rank 1 mixing staircase transformation [1]. Then let G denote the group of dyadic integers, and let m_G denote its normalized Haar measure.

Then consider a *Morse cocycle* [23] $\phi : Y \to G$, that is a measurable map which is constant on the levels of the stacks defining the rank 1 S, except the top level, at each step.

To select a ϕ such that the system $(Y \times G, S_\phi, \nu \otimes m_G)$ is mixing, where

$$S_\phi(y, g) = \big(Sy, g + \phi(y)\big),$$

we proceed as follows.

The system is the inverse limit of the sequence of systems $(Y \times Z/2^t Z, S_{\phi_t}, \nu \otimes m_t)$ where m_t is Haar measure on $Z/2^t Z$ and $\phi_t = \pi_t \circ \phi$.

Therefore [13] enough is to make sure that ϕ is such that each S_{ϕ_t} is mixing. Using [24], sufficient is that each ϕ_t is a weakly-mixing cocycle. This in turn is easy to ensure using [14, Theorems 3, 4].

So we take a ϕ such that S_ϕ is mixing. Now because S_{ϕ_t} is a factor of S_ϕ, we have the inequality $r(S_{\phi_t}) \leqslant r(S_\phi)$.

But since S_{ϕ_t} is mixing, using [12], it follows that $r(S_{\phi_t}) = 2^t$. Whence $r(S_\phi) = \infty$.

Now S_ϕ is mixing therefore $\{S_\phi^n\colon n \in Z\} = \mathrm{wcl}\{S_\phi^n\colon n \in Z\}$. Else for each $g \in G$, $\sigma_g \in C(S_\phi)$, and G is uncountable.

We deduce that $q(S_\phi) = \infty$.

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