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LARGE DEVIATIONS FOR THE RANGE OF AN INTEGER VALUED RANDOM WALK

Yuji HAMANA [a], **Harry KESTEN** [b]

[a] *Faculty of Mathematics, Kyushu University 36, Fukuoka 812-8581, Japan*
[b] *Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853, USA*

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ABSTRACT. – Let $\{S_n\}$ be a random walk on $\mathbb{Z}$ and let R_n be the number of different points among $0, S_1, \ldots, S_{n-1}$. We prove that $\psi(x) := \lim_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx\}$ exists for $x \geqslant 0$ and establish some convexity and monotonicity properties of ψ. This is a sequel to a recent paper which treats random walks on $\mathbb{Z}^d$ with $d \geqslant 2$. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Soit $\{S_n\}$ une marche aléatoire sur $\mathbb{Z}$ et soit R_n le nombre des points distincts entre $0, S_1, \ldots, S_n$. Nous démontrons que la limite $\psi(x) := \lim_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx\}$ existe pour $x \geqslant 0$ et établissons quelques propriétés de convexité et de monotonie de ψ. Ceci complète un article récent qui traite des marches aléatoires sur $\mathbb{Z}^d$ avec $d \geqslant 2$. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $X, X_1, X_2, \ldots$ be i.i.d. $\mathbb{Z}$-valued random variables such that $P\{X = 0\} < 1$. Let $S_0 = 0,\ S_k = \sum_{i=1}^k X_i$ and let $|A|$ denote the cardinality of the set A. The *range* (*at time n*) of the random walk $\{S_k\}$ is

$$R_n = \bigl|\{0, S_1, \ldots, S_{n-1}\}\bigr| = \text{number of different points among } 0, S_1, \ldots, S_{n-1}. \quad (1.1)$$

It has been known for a long time that

$$\lim_{n\to\infty} \frac{1}{n} R_n = \pi := P\{S_n \neq 0 \text{ for all } n \geqslant 1\} \quad \text{a.s.}$$

E-mail addresses: hamana@math.kyushu-u.ac.jp (Y. Hamana), kesten@math.cornell.edu (H. Kesten).

(see [8], Section 4). Throughout this article π will denote the probability on the right here (instead of half the circumference of the unit circle). Here we shall prove the following large deviation theorem:

THEOREM 1. – *Assume that* $P\{X=0\}<1$. *Then*

$$\psi(x)=\lim_{n\to\infty}\frac{-1}{n}\log P\{R_n\geqslant nx\} \quad \textit{exists} \tag{1.2}$$

for all x (*but* $\psi(x)=\infty$ *may occur*). $\psi(\cdot)$ *has the following properties*:

$$\psi(x)=0 \quad \text{for } x\leqslant\pi, \tag{1.3}$$

$$0<\psi(x)<\infty \quad \text{for } \pi<x\leqslant 1, \tag{1.4}$$

$$\psi(x)=\infty \quad \text{for } x>1, \tag{1.5}$$

$$x\mapsto\psi(x) \text{ is continuous on } [0,1] \tag{1.6}$$

and

$$x\mapsto\psi(x) \text{ is strictly increasing on } [\pi,1]. \tag{1.7}$$

In a recent paper [4] we proved this same result when X takes values in $\mathbb{Z}^d$ for $d\geqslant 2$, and we refer the reader to that paper for some brief historical remarks about the subject. In that paper we also showed that the following result follows quickly from Theorem 1:

COROLLARY 1. – *Let* μ_n *be the probability distribution of the random variable* R_n/n. *In the set-up of Theorem* 1, *we have that*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(F)\leqslant-\inf_{x\in F}\psi(x) \tag{1.8}$$

for each closed subset $F\subset[\pi,\infty)$ *and that*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu_n(G)\geqslant-\inf_{x\in G}\psi(x) \tag{1.9}$$

for each open subset $G\subset[\pi,\infty)$.

Remark 1. – For a nearest neighbor random walk ψ can be evaluated explicitly. Specifically, if $P\{X=1\}=1-P\{X=-1\}=p\geqslant 1/2$, then $\pi=2p-1$ (see [3], Section XIII.4) and for $\pi\leqslant x\leqslant 1$,

$$\psi(x)=\frac{1}{2}(1+x)\log\frac{1+x}{2p}+\frac{1}{2}(1-x)\log\frac{1-x}{2(1-p)}, \tag{1.10}$$

where $0\log 0=0$. This can be proven in the same way as formula (1.23) in [4] for the Wiener sausage. Indeed for a nearest neighbor walk, $R_n=\max_{k\leqslant n-1}S_n-\min_{k\leqslant n-1}S_n+1$. Therefore

$$P\{|S_{n-1}|\geqslant nx-1\}\leqslant P\{R_n\geqslant nx\}\leqslant\sum_{0\leqslant k,\ell\leqslant n-1}P\{|S_k-S_\ell|\geqslant nx-1\}. \tag{1.11}$$

(1.10) follows by combining this with standard large deviation estimates for the binomial distribution.

Outline of the proof of Theorem 1. – Unfortunately, the proof of [4] does not work in the one-dimensional case. In fact, the one-dimensional case seems to be harder and we have to use different methods depending on whether

$$\limsup_{n\to\infty}\left[P\{|X|\geqslant n\}\right]^{1/n}=1, \tag{1.12}$$

or

$$\limsup_{n\to\infty}\left[P\{|X|\geqslant n\}\right]^{1/n}<1. \tag{1.13}$$

In both cases, the basic idea of the proof is of course to use subadditivity arguments. However, these do not seem directly applicable to $\{R_n\geqslant nx\}$, but only to the probability of certain subevents of $\{R_n\geqslant nx\}$. We first discuss the case of (1.12) in which X does not have an exponentially bounded tail. In this case we use subadditivity for "cylinder paths". Specifically, it is easy to see that

$$\begin{aligned}&P\left\{R_{n+m}\geqslant (n+m)x,\ 0<S_i\leqslant S_{n+m},\ 1\leqslant i\leqslant n+m\right\}\\&\quad\geqslant P\left\{R_n\geqslant nx,\ 0<S_i\leqslant S_n,\ 1\leqslant i\leqslant n\right\}\\&\qquad\times P\left\{R_m\geqslant mx,\ 0<S_i\leqslant S_m,\ 1\leqslant i\leqslant m\right\}.\end{aligned} \tag{1.14}$$

Subadditivity then shows that

$$\psi^+(x):=\lim_{n\to\infty}\frac{-1}{n}\log P\left\{R_n\geqslant nx,\ 0<S_i\leqslant S_n,\ 1\leqslant i\leqslant n\right\} \tag{1.15}$$

exists. Similarly,

$$\psi^-(x):=\lim_{n\to\infty}\frac{-1}{n}\log P\left\{R_n\geqslant nx,\ 0>S_i\geqslant S_n,\ 1\leqslant i\leqslant n\right\} \tag{1.16}$$

exists. In order to obtain an upper bound for $P\{R_n\geqslant nx\}$ we now decompose a typical sample path $S_0=0,\dots,S_n$ for which $\{R_n\geqslant nx\}$ into pieces $S_{\kappa_j},\dots,S_{\kappa_{j+1}}$ for which

$$S_{\kappa_j}<S_i\leqslant S_{\kappa_{j+1}}\quad\text{for }\kappa_j<i\leqslant\kappa_{j+1} \tag{1.17}$$

or

$$S_{\kappa_j}>S_i\geqslant S_{\kappa_{j+1}}\quad\text{for }\kappa_j<i\leqslant\kappa_{j+1}. \tag{1.18}$$

The κ_j have to be chosen as certain local maxima and minima of the S_i, in order that (1.17) or (1.18) hold. We then use (1.14) to put all the pieces for which (1.17) holds together into one piece to which (1.15) applies. Similarly we combine all pieces for which (1.18) holds into another piece to which (1.16) applies. One can think of this procedure as unfolding the random walk path to pick out more or less increasing pieces and more or less decreasing pieces (for the purpose of these informal remarks we call a piece $S_k,S_{k+1},\dots,S_{k'}$ "more or less increasing" if $S_k\leqslant S_i\leqslant S_{k'}$ for $k\leqslant i\leqslant k'$ and more or less decreasing if the opposite inequalities hold for $k \leqslant i \leqslant k'$). This type of argument was first used by [5] for counting self-avoiding walks. The result is that to each path for which $R_n \geqslant nx$, one can associate a path made up of two pieces, the probabilities of which can be controled by means of (1.15) and (1.16). However, the same two pieces may be associated to many paths with $\{R_n \geqslant nx\}$. The number of such paths is bounded by the number of ways in which the κ_j can be chosen. A principal step in the proof of Lemma 3 is to show (by means of simple combinatorial arguments) that this number grows slower than exponentially in n. This yields the bound

$$\liminf_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} \geqslant \inf_{\substack{0\leqslant\alpha,y,z\leqslant 1\\ \alpha y+(1-\alpha)z=x}} \left[\alpha\psi^+(y) + (1-\alpha)\psi^-(z)\right]. \tag{1.19}$$

We then show, in Section 3, that the right hand side of (1.19) is also an upper bound for $\limsup_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx\}$. This is done by exactly the same method as used in [4] for deriving the basic subadditivity relation of Lemma 1 there. It is done by "putting together" two pieces, one more or less increasing, of length $\lceil \alpha n\rceil$ and having $R_{\lceil \alpha n\rceil} \geqslant \lceil \alpha n\rceil y$, and another more or less decreasing piece of length $\lceil (1-\alpha)n\rceil$ and $R_{\lceil (1-\alpha)n\rceil} \geqslant \lceil (1-\alpha)n\rceil z$; here α, y and z are chosen so that the infimum in the right hand side of (1.19) is taken on at these values. The two pieces have to be put together so that not too many points occur in both pieces, because such points contribute only once, rather than twice, to the range of the combination of the pieces. This is achieved by putting the initial point of the second piece not directly at the endpoint of the first piece, but at a judiciously chosen point. This will complete the proof of Theorem 1 when (1.12) prevails.

The last step which gives an upper bound for $\limsup_{n\to\infty}(-1/n)\ \log P\{R_n \geqslant nx\}$ does not work in the case of (1.13). Indeed, in this case, when X has exponentially bounded tails, the right hand side of (1.19) can be strictly smaller than the limit of $(-1/n)\log P\{R_n \geqslant nx\}$. This is so because a more or less decreasing path and a more or less increasing path will typically have many points in common when the initial point of the former is close to the endpoint of the latter (consider for instance the case of simple random walk paths). In this case we use a different decomposition of paths with $R_n \geqslant nx$. We show that any such path can be decomposed into three pieces, two of which are circuits (i.e., paths with the same final point as initial point), and these circuits are connected by a more or less decreasing or more or less increasing path. To find such circuits note that $\max_{0\leqslant i\leqslant n} S_i \geqslant 0 \geqslant \min_{0\leqslant i\leqslant n} S_i$. The first circuit is then, roughly speaking, the piece from S_0 till the last time at which the sample path jumps from $[0,\infty)$ to $(-\infty, 0)$. The difficulty is that at this time the path does not necessarily jump to 0, but jumps across 0. However, we show in Lemmas 4 and 7 that the subclass of paths which do jump to 0 at this time has a probability at least $\mathrm{e}^{\mathrm{o}(n)} P\{R_n \geqslant nx\}$. On this subclass we can use the time of this jump to 0 as the last step of the first circuit. The second circuit is found in a similar way, by interchanging the roles of the initial point S_0 and the final point S_n. Once we have the decomposition into two circuits and a more or less monotonic piece it is easy to obtain a lower bound for $\lim_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx\}$. This rests on a simple subadditivity argument which shows that $\lim_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx, S_n = 0\}$ exists, for all $x \in [0,1]$, with the possible exception of at most one value, x_0 (see Lemma 2).

Once we have the lower bound for $\lim_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx, S_n = 0\}$, it is fairly easy to show that the lower bound is also an upper bound, by combining one circuit contained in $[0,\infty)$, one circuit contained in $(-\infty,0]$ and one more or less monotonic path between them (see Lemmas 6 and 8).

2. Subadditivity arguments

Throughout we assume that the group generated by the support of X is all of $\mathbb{Z}$. This is no loss of generality, because this group is necessarily of the form $m\mathbb{Z}$, and if $m \neq 1$, then we can replace X and X_i by X/m and X_i/m, repectively, without changing R_n. The group generated by the support of X/m will then be equal to $\mathbb{Z}$.

Define the events

$$\begin{aligned} A_n^+(x) &= \{R_n \geqslant nx \text{ and } 0 < S_k \leqslant S_n, 1 \leqslant k \leqslant n\},\\ A_n^-(x) &= \{R_n \geqslant nx \text{ and } 0 > S_k \geqslant S_n, 1 \leqslant k \leqslant n\}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \widetilde{A}_n^+(x) &= \{R_n \geqslant nx \text{ and } 0 \leqslant S_k \leqslant S_n, 1 \leqslant k \leqslant n\},\\ \widetilde{A}_n^-(x) &= \{R_n \geqslant nx \text{ and } 0 \geqslant S_k \geqslant S_n, 1 \leqslant k \leqslant n\}. \end{aligned} \tag{2.2}$$

Note that the only difference between $A_n^+(x)$ and $\widetilde{A}_n^+(x)$ is that S_k has to be strictly positive in the former, while it may equal 0 in the latter.

LEMMA 1. – *For all* $x \in \mathbb{R}$,

$$\begin{aligned} &\psi^+(x) := \lim_{n\to\infty} \frac{-1}{n} \log P\{A_n^+(x)\}\\ &\textit{and} \quad \psi^-(x) := \lim_{n\to\infty} \frac{-1}{n} \log P\{A_n^-(x)\} \quad \textit{exist} \end{aligned} \tag{2.3}$$

(*but may equal* $+\infty$), *and*

$$P\{A_n^\pm(x)\} \leqslant e^{-n\psi^\pm(x)}, \quad n \geqslant 1. \tag{2.4}$$

If

$$p^+ := P\{X > 0\} > 0, \tag{2.5}$$

then in addition

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{\widetilde{A}_n^+(x)\} = \psi^+(x), \tag{2.6}$$

and $\psi^+(\cdot)$ *is convex, nondecreasing and bounded on* $[0,1]$. *Also,* $\psi^+(\cdot)$ *is continuous on* $[0,1)$. *Similarly, if*

$$p^- := P\{X < 0\} > 0, \tag{2.7}$$

then

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{\widetilde{A}_n^-(x)\} = \psi^-(x), \tag{2.8}$$

and $\psi^-(\cdot)$ *is convex, nondecreasing and bounded on* $[0,1]$. *Also,* $\psi^-(\cdot)$ *is continuous on* $[0,1)$.

Proof. – We restrict ourselves to the case corresponding to the superscript $+$. We have

$$\begin{aligned} A^+_{n+m}\left(\frac{ny+mz}{n+m}\right) \supset A_n^+(y) \ &\cap \{|\{S_n, S_{n+1}, \dots, S_{n+m-1}\}| \geqslant mz\} \\ &\cap \{S_n < S_i \leqslant S_{n+m},\ n+1 \leqslant i \leqslant n+m\}, \end{aligned} \tag{2.9}$$

because on the event in the right hand side the sets $\{0, S_1, \dots, S_{n-1}\}$ and $\{S_n, \dots, S_{n+m-1}\}$ have no points in common. It follows that

$$P\left\{A^+_{n+m}\left(\frac{ny+mz}{n+m}\right)\right\} \geqslant P\{A_n^+(y)\}P\{A_m^+(z)\}. \tag{2.10}$$

If $p^+ = P\{X > 0\} = 0$, then $P\{A_n^+(x)\} = 0$ for $n \geqslant 1$, and (2.3) and (2.4) with $\psi^+(x) = \infty$ are obvious. These relations are also obvious when $x > 1$ because necessarily $R_n \leqslant n$. We may therefore assume that (2.5) holds and that $x \leqslant 1$. Then for $x \leqslant 1$,

$$P\{A_n^+(x)\} \geqslant P\{X_i > 0, 1 \leqslant i \leqslant n\} = [p^+]^n > 0. \tag{2.11}$$

(2.3) and (2.4) now follow in the usual way from superadditivity, when we take $y = z = x$ (see [6], Problem I.98).

Now assume that (2.5) holds, so that $p^+ > 0$. Then we can extend a path of n steps which belongs to $\widetilde{A}_n^+(x)$ by inserting ℓ strictly positive steps in front. Each such step adds a point that will not be visited again by the extended path and therefore increases the range by 1. From this we see that for $\ell \geqslant 1$

$$P\{R_{n+\ell} \geqslant nx + \ell \text{ and } 0 < S_k \leqslant S_{n+\ell}, 1 \leqslant k \leqslant n+\ell\} \geqslant [p^+]^\ell P\{\widetilde{A}_n^+(x)\}. \tag{2.12}$$

In particular, for $x \leqslant 1$

$$\begin{aligned} P\{A^+_{n+1}(x)\} &\geqslant P\{R_{n+1} \geqslant nx + 1 \text{ and } 0 < S_k \leqslant S_{n+1}, 1 \leqslant k \leqslant n+1\} \\ &\geqslant p^+ P\{\widetilde{A}_n^+(x)\} \geqslant p^+ P\{A_n^+(x)\}. \end{aligned} \tag{2.13}$$

(The last inequality is trivial because $A_n^+(x) \subset \widetilde{A}_n^+(x)$.) (2.6) now follows from (2.3) when $x \leqslant 1$. Again we do not have to prove anything for $x > 1$, since $R_n \leqslant n$.

The fact that $\psi^+(x) < \infty$ for $0 \leqslant x \leqslant 1$ is immediate from (2.11). It is also clear from the definition of ψ^+ that it is nondecreasing.

The convexity of ψ^+ also follows from (2.12) and the argument for (2.10). Indeed, let $x = \alpha y + (1-\alpha)z$ and replace n by $\lceil \alpha n \rceil$ and m by $\lceil (1-\alpha)n \rceil$ in the argument for (2.10). We find that

$$
\begin{aligned}
&P\big\{R_{\lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell}\geqslant \alpha ny+(1-\alpha)nz+\ell,\\
&\quad 0<S_k\leqslant S_{\lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell},\ 1\leqslant k\leqslant \lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell\big\}\\
&\quad\geqslant [p^+]^\ell P\big\{A^+_{\lceil\alpha n\rceil}(y)\big\}P\big\{A^+_{\lceil(1-\alpha)n\rceil}(z)\big\}\\
&\quad\geqslant \exp\big[-n(\alpha\psi^+(y)+(1-\alpha)\psi^+(z))+\mathrm{o}(n)\big].
\end{aligned}\tag{2.14}
$$

If $x<1$, then we can choose ℓ such that for all $n\geqslant 1$

$$
\alpha ny+(1-\alpha)nz+\ell\geqslant (n+\ell+2)x.
$$

Indeed, this inequality always holds if $\ell\geqslant 2x/(1-x)$. For such ℓ we then have for all n that

$$
\begin{aligned}
&P\big\{R_{\lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell}\geqslant \big(\lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell\big)x,\\
&\quad 0<S_k\leqslant S_{\lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell},\ 1\leqslant k\leqslant \lceil\alpha n\rceil+\lceil(1-\alpha)n\rceil+\ell\big\}
\end{aligned}
$$

is at least as large as the left hand side of (2.14). It follows that $\psi^+(x)\leqslant\alpha\psi^+(y)+(1-\alpha)\psi^+(z)$, which is the desired convexity for $x<1$. For $x=1$ there is nothing to prove, for then $x=\alpha y+(1-\alpha)z$ with $0\leqslant\alpha, y, z\leqslant 1$ can occur only for $y=z=1$ or for $\alpha=0$ or 1.

The fact that ψ^+ is nondecreasing and bounded on [0, 1], together with the convexity shows that ψ^+ is continuous on [0, 1). □

Remark 2. – We shall find it useful to introduce the following additional events for $u\in\{1,2,\ldots\}$:

$$
\begin{aligned}
B_n^+(u)&:=\{R_n=u \text{ and } 0<S_k\leqslant S_n,\ 1\leqslant k\leqslant n\},\\
\widetilde{B}_n^+(u)&:=\{R_n=u \text{ and } 0\leqslant S_k\leqslant S_n,\ 1\leqslant k\leqslant n\}.
\end{aligned}\tag{2.15}
$$

Clearly we have

$$
A_n^+(x)=\bigcup_{u\geqslant nx}B_n^+(u)\quad\text{and}\quad \widetilde{A}_n^+(x)=\bigcup_{u\geqslant nx}\widetilde{B}_n^+(u).
$$

By imitating the proofs of (2.9) and (2.13) we further obtain

$$
\begin{aligned}
P\big\{B^+_{n+m}(u+v)\big\}&\geqslant P\big\{R[0,n-1]=u,\,R[n,n+m-1]=v,\\
&\qquad\qquad 0<S_i\leqslant S_{n+m},\ 1\leqslant i\leqslant n+m\big\}\\
&\geqslant P\big\{B_n^+(u)\big\}P\big\{B_m^+(v)\big\},
\end{aligned}\tag{2.16}
$$

and

$$
P\big\{B^+_{n+1}(u+1)\big\}\geqslant p^+P\big\{\widetilde{B}_n^+(u)\big\}\geqslant p^+P\big\{B_n^+(u)\big\}.\tag{2.17}
$$

Here $R[a,b]$ stands for $|\{S_i\colon\ a\leqslant i\leqslant b\}|$. For the case when (1.13) prevails, we shall need another subadditivity result. We define the event

$$
C_n(x)=\{R_n\geqslant nx,\,S_n=0\}.\tag{2.18}
$$

LEMMA 2. – *Let ϑ denote the period of the random walk $\{S_n\}$, that is,*

$$\vartheta = \gcd\bigl\{n \geqslant 1\colon\ P\{S_n = 0\} > 0\bigr\},$$

and define

$$x_0 = \sup\bigl\{x\colon\ P\{R_N > \lceil Nx \rceil + 2,\ S_N = 0\} > 0 \text{ for infinitely many } N\bigr\}. \tag{2.19}$$

Then for $0 \leqslant x < x_0$

$$\sigma(x) := \lim_{n\to\infty} \frac{-1}{n\vartheta} \log P\{C_{n\vartheta}(x)\} \text{ exists and is finite}, \tag{2.20}$$

and for each $x > x_0$ there exists an $n_0 < \infty$ such that

$$P\{C_n(x)\} = 0 \quad \text{for } n \geqslant n_0. \tag{2.21}$$

Moreover, for each $\eta > 0$ there exist $N = N_\eta$ and $t = t_\eta$ such that for $0 \leqslant x \leqslant x_0 - 2\eta$ it holds for all $n \geqslant 1$ that

$$P\{C_{n\vartheta}(x)\} \leqslant \frac{nN\vartheta^2}{P\{R_{N\vartheta} > t, S_{N\vartheta} = 0\}} \exp\bigl[-(n+N)\vartheta\sigma(x)\bigr]. \tag{2.22}$$

Finally, $\sigma(\cdot)$ is nondecreasing, convex and continuous on $[0, x_0)$.

Proof. – Of course if $P\{X > 0\}P\{X < 0\} = 0$, then $P\{C_n(x)\} \leqslant P\{R_n \geqslant nx, X_i = 0, 1 \leqslant i \leqslant n\} = 0$ for $nx > 1$. In this case the conclusion of the lemma with $x_0 = 0$ is obvious. We may therefore assume that

$$P\{X > 0\}P\{X < 0\} > 0. \tag{2.23}$$

Before we begin the proof proper, we show that in this case x_0 is well defined and lies in $(0, 1]$.

In view of (2.23) there exist integers $a, b \geqslant 1$ such that

$$P\{X = -a\}P\{X = b\} > 0. \tag{2.24}$$

We also have $P\{R_{p(a+b)} \geqslant pa, S_{p(a+b)} = 0\} > 0$ for all integers $p \geqslant 2$. Indeed if $X_i = b$ for $1 \leqslant i \leqslant pa$ and $X_i = -a$ for $pa + 1 \leqslant i \leqslant p(a+b)$, then we obtain a sample path with $S_{p(a+b)} = 0$ and $R_{p(a+b)} \geqslant pa$ because the numbers $S_i = ib$ for $0 \leqslant i \leqslant pa$ are all different. Thus the set in the right hand side of (2.19) contains all $x < a/(a+b)$, so that $x_0 \geqslant a/(a+b)$. By interchanging a and b we even have $x_0 \geqslant (a \vee b)/(a+b)$.

To prove (2.20) we first show that

$$P\{R_n \geqslant t,\ S_n = 0,\ S_k \geqslant 0 \text{ for } 0 \leqslant k \leqslant n\} \geqslant \frac{1}{n} P\{R_n \geqslant t,\ S_n = 0\}. \tag{2.25}$$

To this end, we introduce a map Θ from the n-step paths $S_0, S_1, \ldots, S_n$ which end at $S_n = 0$ to the subclass of these paths which in addition stay in the nonnegative halfline. Specifically, let k be the random index at which the minimum of the path $S_0, S_1, \dots, S_n$ is first reached. Then interchange the two pieces $S_0, \dots S_k$ and $S_k, \dots, S_n$, "glue them together" (i.e., identify S_0 and S_n) and shift by $-S_k$ to obtain the path $\Theta(S_0, \dots, S_n) = (S_k, \dots, S_n, S_1, \dots, S_k) - S_k$. One easily sees that the successive steps of this new path are $X_{k+1}, \dots, X_n, X_1, \dots, X_k$. These are obtained from the original steps by a random cyclical permutation, as in [8], Proof of Proposition 32.5). Obviously the probability of obtaining the successive steps $X_{k+1}, \dots, X_n, X_1, \dots, X_k$ is the same as the probability for the steps $X_1, X_2, \dots, X_n$. By construction the path $\Theta(S_0, \dots, S_n) = (S_k, \dots, S_n, S_1, \dots, S_k) - S_k$ lies in $[0, \infty)$ and has final point and initial point 0. It is also obvious that the range of the image is $|\{S_k, S_{k+1}, \dots, S_n, S_1 \dots S_{k-1}\}| = |\{S_0, S_1, \dots, S_{n-1}\}| = R_n$ (recall that $S_n = 0$). Finally the number of pre-images (under Θ) of a given path is at most n. Therefore

$$\begin{aligned} &P\{R_n \geqslant t,\ S_n = 0,\ S_k \geqslant 0 \text{ for } 0 \leqslant k \leqslant n\} \\ &\quad \geqslant P\{\Theta\{R_n \geqslant t,\ S_n = 0\}\} \geqslant \frac{1}{n} P\{R_n \geqslant t,\ S_n = 0\}, \end{aligned}$$

which proves (2.25).

We shall now "combine" two configurations in which $\{R_n \geqslant t_1, S_n = 0\}$ and a "translate" of $\{R_m \geqslant t_2, S_m = 0\}$ occur, respectively, to form a configuration in which $R_{n+m} \geqslant t_1 + t_2 - 1$. This will prove that

$$P\{R_{n+m} \geqslant t_1 + t_2 - 1, S_{n+m} = 0\} \geqslant \frac{1}{m} P\{R_n \geqslant t_1, S_n = 0\} P\{R_m \geqslant t_2, S_m = 0\}. \tag{2.26}$$

To obtain this we decompose the event $\{R_n \geqslant t_1,\ S_n = 0\}$ according to the value of the smallest index q for which

$$S_q = \max_{0 \leqslant p \leqslant n} S_p,$$

Then we see from (2.25) that the right hand side of (2.26) is not larger than

$$\begin{aligned} &\sum_{q=0}^{n} P\{R_n \geqslant t_1,\ S_n = 0,\ S_i < S_q \text{ for } 0 \leqslant i < q,\ S_j \leqslant S_q \text{ for } q < j \leqslant n\} \\ &\quad \times P\{R_m \geqslant t_2,\ S_m = 0,\ S_j \geqslant 0 \text{ for } 0 \leqslant j \leqslant m\}. \end{aligned} \tag{2.27}$$

The summand in (2.27) is equal to

$$\begin{aligned} P\{&R_n \geqslant t_1,\ S_n = 0,\ S_i < S_q \text{ for } 0 \leqslant i < q, \\ &S_j \leqslant S_q \text{ for } q < j \leqslant n,\ R[n, n+m-1] \geqslant t_2, \\ &S_{n+m} = S_n,\ S_h - S_n \geqslant 0 \text{ for } n \leqslant h \leqslant n+m\}. \end{aligned} \tag{2.28}$$

Now consider a path of $n + m$ steps with the properties listed in this probability. Let its steps be $X_1, X_2, \dots, X_{n+m}$. As in the proof of (2.25), we construct a new path by permuting the steps. More specifically, we take the piece of the path consisting of its last m steps and insert this piece right after the qth step. That is, we arrange the steps in the order $X_1, \dots, X_q, X_{n+1}, \dots, X_{n+m}, X_{q+1}, \dots, X_n$. The new path coincides with the original path up till time q. Then it follows the loop $S_n, S_{n+1}, \dots, S_{n+m} = S_n$ translated by S_q, and then ends with the piece from S_q to S_n of the original path. Because of the properties listed in (2.28), the pieces of the original path from 0 to S_q and from S_q to S_n lie in $(-\infty, S_q]$ while the loop $S_q + (S_n, S_{n+1}, \dots, S_{n+m})$ lies in $[S_q, \infty)$. Thus this last loop has only the point S_q in common with the other pieces, so that the range of the new path is at least

$$\begin{aligned}&\bigl|\{S_0, \dots, S_{n-1}\}\bigr| + \bigl|\{S_n, S_{n+1}, \dots, S_{n+m-1}\}\bigr| - 1\\ &\quad = R_n + R[n, n+m-1] - 1 = t_1 + t_2 - 1\end{aligned}$$

(here S_j is the position at time j in the original path). Because the permuted path occurs with the same probability as the original one, we see that the probability in (2.28) is at most

$$\begin{aligned}P\bigl\{&R_{n+m} \geqslant t_1 + t_2 - 1,\ S_{n+m} = 0,\\ &S_i < S_q \text{ for } 0 \leqslant i < q,\ S_h \geqslant S_q \text{ for } q \leqslant h \leqslant q+m,\\ &S_j \leqslant S_q \text{ for } q+m < j \leqslant n+m\bigr\}.\end{aligned} \tag{2.29}$$

The events in (2.29) for distinct q's are disjoint, because on the event in (2.29) q is the smallest index for which

$$\bigl|\{i\colon\ S_i > S_q\}\bigr| < m - 1 \quad \text{but } \bigl|\{i\colon\ S_i \geqslant S_q\}\bigr| \geqslant m + 1.$$

It follows that (2.27) is not larger than $P\{R_{n+m} \geqslant t_1 + t_2 - 1,\ S_{n+m} = 0\}$, which implies (2.26).

Now let $\eta > 0$ and $x \leqslant x_0 - 2\eta$. Let $N = N_\eta$ be an integer for which

$$\kappa = \kappa_\eta := P\bigl\{R_{N\vartheta} > \lceil N\vartheta(x_0 - \eta)\rceil + 2,\ S_{N\vartheta} = 0\bigr\} > 0. \tag{2.30}$$

Such an N exists by the definition of x_0. Recall that there exists some integer $\ell \geqslant 1$ such that $P\{S_{s\vartheta} = 0\} > 0$ for all $s \geqslant \ell$ (see Appendix A21 in [1]). Let $m \geqslant 1$ be a given integer. For $n \geqslant \ell$ we can express n uniquely as

$$n = r(m+N) + s \tag{2.31}$$

with integers r, s satisfying $r \geqslant 0$ and $\ell \leqslant s < \ell + m + N$. Now take $t = t_\eta = \lceil N\vartheta(x_0 - \eta)\rceil + 2$, and assume that n is so large that even

$$r[N(x_0 - \eta) - Nx] \geqslant sx,$$

for the r, s of (2.31). (Note that the left hand side here is at least $rN\eta$ by our choice of x and that the right hand side is at most $(\ell + m + N)x_0$.) We then also have

$$\begin{aligned}n\vartheta x &= [r(m+N) + s]\vartheta x\\ &\leqslant r\bigl(m\vartheta x + \lceil N\vartheta(x_0 - \eta)\rceil\bigr)\\ &= r(m\vartheta x + t_\eta - 2).\end{aligned}$$

Finally, define

$$\delta = \inf\bigl\{P\{S_{s\vartheta} = 0\}\colon\ \ell \leqslant s < \ell + m + N\bigr\}.$$

Repeated application of (2.26), with m, t_2 replaced by $m\vartheta, m\vartheta x$ or by N, t, then shows that

$$\begin{aligned}
&P\{R_{n\vartheta} \geqslant n\vartheta x,\ S_{n\vartheta} = 0\} \\
&\quad\geqslant \left[\frac{1}{m\vartheta} P\{R_{m\vartheta} \geqslant m\vartheta x,\ S_{m\vartheta} = 0\} \frac{1}{N\vartheta} P\{R_{N\vartheta} \geqslant t,\ S_{N\vartheta} = 0\}\right]^r \\
&\qquad \times P\{R_{s\vartheta} \geqslant 0,\ S_{s\vartheta} = 0\} \\
&\quad\geqslant \delta \left[\frac{1}{m\vartheta} P\{C_{m\vartheta}(x)\} \frac{\kappa}{N\vartheta}\right]^r. \qquad (2.32)
\end{aligned}$$

Noting that δ is strictly positive and independent of n, we obtain that

$$\limsup_{n\to\infty} \frac{-1}{n\vartheta} \log P\{C_{n\vartheta}(x)\} \leqslant \frac{-1}{(m+N)\vartheta} \log P\{C_{m\vartheta}(x)\} - \frac{\log\left(\kappa/(mN\vartheta^2)\right)}{(m+N)\vartheta}. \qquad (2.33)$$

Finally, if we let m go to infinity through a subsequence for which

$$\frac{-1}{m\vartheta} \log P\{C_{m\vartheta}(x)\} \to \liminf_{n\to\infty} \frac{-1}{n\vartheta} \log P\{C_{n\vartheta}(x)\},$$

then we obtain that

$$\lim_{n\to\infty} \frac{-1}{n\vartheta} \log P\{C_{n\vartheta}(x)\}$$

exists. This limit is finite by virtue of

$$P\{R_{rN\vartheta} \geqslant rN\vartheta x,\ S_{rN\vartheta} = 0\} \geqslant \left(\frac{\kappa}{N\vartheta}\right)^r$$

(which is a simplified version of (2.32) with $m = s = 0$). This proves (2.20) for any $x \leqslant x_0 - 2\eta$, and hence for all $0 \leqslant x < x_0$. Furthermore, (2.22) is contained in (2.33).

Next, if $x > x_0$, then for $x_0 < x' < x$ and large n

$$P\{C_n(x)\} \leqslant P\big\{R_n > \lceil nx' \rceil + 2,\ S_n = 0\big\} = 0,$$

by the definitions of C_n and x_0. Thus (2.21) holds.

The monotonicity, convexity and continuity of σ are proven in the same way as for $\psi^{\pm}$ in Lemma 1. □

Remark 3. – It is not hard to show that $x_0 = 1$ when the support of X is unbounded. Even if the support of X is bounded, it will be the case that $x_0 = 1$ if there exists $a, b > 1$ with $\gcd(a, b) > 1$ for which (2.24) holds. On the other hand, there are certainly examples for which $x_0 < 1$ (for instance simple random walk). We expect that $\lim_{n\to\infty} -1/(n\vartheta) \log P\{C_{n\vartheta}(x_0)\}$ also exists in these cases, but it may be infinity.

To control $C_n(x)$ for $x_0 \leqslant x \leqslant 1$, we take for $0 < \eta < x_0$, a continuous increasing function $x \mapsto r_\eta(x)$ on $[0, \eta]$ for which

$$r_\eta(0) = 0 \quad \text{and} \quad r_\eta(\eta) = \frac{1}{\eta},$$

and define

$$\sigma_\eta(x)=\begin{cases}\sigma\big(x\wedge(x_0-\eta)\big) & \text{if } 0\leqslant x\leqslant(x_0+\eta)\wedge 1,\\ \sigma(x_0-\eta)+r_\eta(x-x_0-\eta) & \text{if } x_0+\eta\leqslant x\leqslant(x_0+2\eta)\wedge 1,\\ \sigma(x_0-\eta)+1/\eta & \text{if } x_0+2\eta\leqslant x\leqslant 1.\end{cases}\tag{2.34}$$

Note that $\sigma_\eta(x)\leqslant\sigma(x_0-\eta)$ for all $x\leqslant(x_0+\eta)\wedge 1$. Moreover, $\sigma_\eta(\cdot)$ is nondecreasing, continuous and bounded on $[0,1]$. We also take M_η to be an integer such that $P\{C_n(x_0+\eta)\}=0$ for $n\vartheta\geqslant M_\eta$ (see (2.21)). By the monotonicity in x of $P\{C_n(x)\}$ and (2.21), (2.22) we then have for all $n\geqslant M_\eta$ and all $x\in[0,1]$

$$P\{C_{n\vartheta}(x)\}\leqslant\frac{nN\vartheta^2}{P\{R_{N\vartheta}>t,\,S_{N\vartheta}=0\}}\exp\big[-(n+N)\vartheta\sigma_\eta(x)\big]\tag{2.35}$$

(with $N=N_{\eta/2},t=t_{\eta/2}$).

3. An upper bound for $P\{R_n\geqslant nx\}$

Here we shall compare $\limsup\,(-1/n)\log P\{R_n\geqslant nx\}$ with

$$\psi^*(x):=\inf_{\substack{0\leqslant\alpha,y,z\leqslant 1\\ \alpha y+(1-\alpha)z=x}}\big[\alpha\psi^+(y)+(1-\alpha)\psi^-(z)\big].\tag{3.1}$$

LEMMA 3. – *If* (2.23) *holds, then for* $0\leqslant x\leqslant 1$,

$$\liminf_{n\to\infty}\frac{-1}{n}\log P\{R_n\geqslant nx\}\geqslant\psi^*(x).\tag{3.2}$$

Proof. – We shall prove this lemma by ‘unfolding’ a random walk path. To this end we define certain indices, as a function of $S_1,S_2,\dots,S_n$. Roughly speaking these are the successive times at which S_j reaches for the last or first time a maximum or minimum of a piece of the sample path till time n (see Fig. 1). This splits the path into various pieces $S_j,\dots,S_{j'}$, each of which is more or less increasing or more or less decreasing (in the terminology introduced after (1.18)). We then permute these pieces and combine the more or less increasing ones into one path, Θ_1 say, and the more or less decreasing ones into another path, Θ_2 say. The probabilities of the resulting paths Θ_1,Θ_2 are estimated by means of Lemma 1. This eventually gives the bound (3.40) for the probability of all the possible Θ_1,Θ_2. Again the permuted paths occur with the same probability as the original path, but we still need a combinatorial or numbertheoretical estimate for the number of paths which result after permutation in a particular pair of Θ_1,Θ_2. This estimate, which depends on our choice of the more or less increasing or decreasing pieces, is provided after (3.40).

We now give the somewhat tedious details. We fix $x\in[0,1]$ for the rest of this section. Define

$$\kappa_1=\max\big\{i\leqslant n\colon\ S_i=\max\{0,S_1,S_2,\dots,S_n\}\big\};\tag{3.3}$$

if $\kappa_{2j-1}<n$, then define

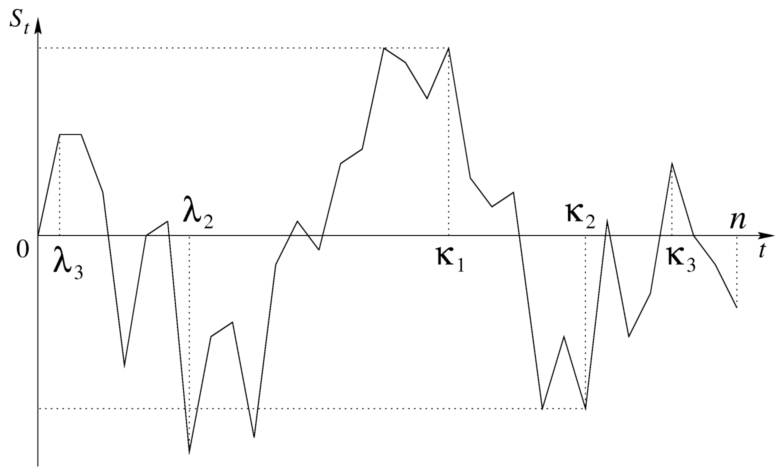

Fig. 1. Illustration of the location of $\lambda_2, \lambda_3, \kappa_1, \kappa_2$ and κ_3.

$$\kappa_{2j} = \max\{i > \kappa_{2j-1}\colon\ S_i = \min\{S_{\kappa_{2j-1}}, S_{\kappa_{2j-1}+1}, \dots, S_n\}\}; \tag{3.4}$$

and if $\kappa_{2j} < n$, then define

$$\kappa_{2j+1} = \max\{i > \kappa_{2j}\colon\ S_i = \max\{S_{\kappa_{2j}}, S_{\kappa_{2j}+1}, \dots, S_n\}\}. \tag{3.5}$$

We only define κ_ℓ in this way as long as $\kappa_{\ell-1} < n$, so that the set in the right hand sides of these definitions is nonempty. We define

$$\nu = \min\{\ell\colon\ \kappa_\ell = n\} \tag{3.6}$$

and leave κ_ℓ undefined for $\ell > \nu$. The pieces $S_{\kappa_i}, \dots, S_{\kappa_{i+1}},\ 1 \leqslant i \leqslant \nu - 1$, are some of the more or less increasing and more or less decreasing pieces into which we decompose the original path.

In order to find the remaining pieces we also define indices similar to the κ_i, but going downwards from κ_1. If $\kappa_1 > 0$, then we define

$$\lambda_2 = \min\{i < \kappa_1\colon\ S_i = \min\{S_{\kappa_1}, S_{\kappa_1-1}, \dots, S_1, 0\}\}; \tag{3.7}$$

if $\lambda_{2j} > 0$, then define

$$\lambda_{2j+1} = \min\{i < \lambda_{2j}\colon\ S_i = \max\{S_{\lambda_{2j}}, S_{\lambda_{2j}-1}, \dots, S_1, 0\}\}; \tag{3.8}$$

if $\lambda_{2j+1} > 0$, then define

$$\lambda_{2j+2} = \min\{i < \lambda_{2j+1}\colon\ S_i = \min\{S_{\lambda_{2j+1}}, S_{\lambda_{2j+1}-1}, \dots, S_1, 0\}\}. \tag{3.9}$$

These indices are defined only as long as the preceding one is strictly positive. We set

$$\mu = \min\{\ell\colon\ \lambda_\ell = 0\} \tag{3.10}$$

and leave λ_ℓ undefined for $\ell > \mu$.

Now let μ, λ and $0 = \ell_\mu < \ell_{\mu-1} < \cdots < \ell_2 < k_1 < \cdots < k_{\nu-1} < k_\nu = n$ be fixed and let us estimate the probability of

$$\{R_n \geqslant nx, \kappa_j = k_j, 1 \leqslant j \leqslant \nu, \lambda_m = \ell_m,\ 2 \leqslant m \leqslant \mu\}. \tag{3.11}$$

To this end note that on this event, by the definition of the κ's,

$$S_i < S_{k_1} \quad \text{for } k_1 < i \leqslant n, \tag{3.12}$$
$$S_i \leqslant S_{k_1} \quad \text{for } 0 \leqslant i \leqslant k_1, \tag{3.13}$$
$$S_{k_{2j-1}} > S_i \geqslant S_{k_{2j}} \quad \text{for } k_{2j-1} < i \leqslant k_{2j}, \tag{3.14}$$
$$S_i > S_{k_{2j}} \quad \text{for } k_{2j} < i \leqslant n, \tag{3.15}$$

when $1 \leqslant j \leqslant \lfloor \nu/2 \rfloor$, and

$$S_{k_{2j}} < S_i \leqslant S_{k_{2j+1}} \quad \text{for } k_{2j} < i \leqslant k_{2j+1}, \tag{3.16}$$
$$S_i < S_{k_{2j+1}} \quad \text{for } k_{2j+1} < i \leqslant n, \tag{3.17}$$

when $1 \leqslant j \leqslant \lfloor (\nu-1)/2 \rfloor$. These inequalities imply furthermore that

$$S_{k_1} - S_{k_2} > S_{k_3} - S_{k_2} > S_{k_3} - S_{k_4} > \cdots > S_{k_{\nu-1}} - S_{k_\nu} > 0 \tag{3.18}$$

if ν is even, and

$$S_{k_1} - S_{k_2} > S_{k_3} - S_{k_2} > S_{k_3} - S_{k_4} > \cdots > S_{k_\nu} - S_{k_\nu-1} > 0 \tag{3.19}$$

if ν is odd. Similarly,

$$S_{\ell_2} \leqslant S_i \leqslant S_{k_1}, \quad \ell_2 \leqslant i \leqslant k_1, \tag{3.20}$$

and

$$\begin{aligned} S_{\ell_{2j+1}} \geqslant S_i > S_{\ell_{2j}}, &\quad \ell_{2j+1} \leqslant i < \ell_{2j}, \\ S_{\ell_{2j+2}} \leqslant S_i < S_{\ell_{2j+1}}, &\quad \ell_{2j+2} \leqslant i < \ell_{2j+1}, \end{aligned} \tag{3.21}$$

provided $2j+1 \leqslant \mu$, respectively, $2j+2 \leqslant \mu$.

We introduce the following notation for the events in (3.14), (3.16), (3.20) and (3.21):

$$F_{2j-1} := \{S_{k_{2j-1}} > S_i \geqslant S_{k_{2j}} \text{ for } k_{2j-1} < i \leqslant k_{2j}\},$$
$$F_{2j} := \{S_{k_{2j}} < S_i \leqslant S_{k_{2j+1}} \text{ for } k_{2j} < i \leqslant k_{2j+1}\},$$
$$F_0 := \{S_{\ell_2} \leqslant S_i \leqslant S_{k_1} \text{ for } \ell_2 \leqslant i \leqslant k_1\},$$
$$G_{2j} := \{S_{\ell_{2j+1}} \geqslant S_i > S_{\ell_{2j}} \text{ for } \ell_{2j+1} \leqslant i < \ell_{2j}\},$$

and

$$G_{2j+1} := \{S_{\ell_{2j+2}} \leqslant S_i < S_{\ell_{2j+1}} \text{ for } \ell_{2j+2} \leqslant i < \ell_{2j+1}\}.$$

We also define

$$R[a,b] = |\{S_i\colon\ a \leqslant i \leqslant b\}|. \tag{3.22}$$

In this notation, $R_n = R[0, n-1]$ and it is easy to see that

$$R_n = R[0, n-1] \leqslant \sum_{i=2}^{\mu-1} R[\ell_{i+1}, \ell_i - 1] + R[\ell_2, k_1 - 1] + \sum_{i=1}^{\nu-1} R[k_i, k_{i+1} - 1].$$

From this one sees that the event (3.11) is contained in the union over $v_2, \dots, v_{\mu-1} \geqslant 0$, $u_0, \dots, u_{\nu-1} \geqslant 0$, with

$$n \geqslant \sum_{j=2}^{\mu-1} v_j + \sum_{i=0}^{\nu-1} u_i \geqslant nx, \tag{3.23}$$

of the events

$$\begin{aligned} \{R[\ell_2, k_1 - 1] = u_0,\ R[k_i, k_{i+1} - 1] = u_i,\ F_i,\ 0 \leqslant i \leqslant \nu - 1, \\ R[\ell_{j+1}, \ell_j - 1] = v_j,\ G_j,\ 2 \leqslant j \leqslant \mu - 1\}. \end{aligned} \tag{3.24}$$

The probability of this last event equals

$$\begin{aligned} &\left(\prod_{j=2}^{\mu-1} P\{R[\ell_{j+1}, \ell_j - 1] = v_j, G_j\}\right) P\{R[\ell_2, k_1 - 1] = u_0, F_0\} \\ &\quad \times \left(\prod_{i=1}^{\nu-1} P\{R[k_i, k_{i+1} - 1] = u_i, F_i\}\right). \end{aligned} \tag{3.25}$$

We now combine all the factors corresponding to intervals on which the last value of S exceeds the initial value, and the factor corresponding to the interval $[\ell_2, k_1]$. We combine the other factors into another product. For the sake of argument, we assume that μ and ν are even; we leave the trivial modifications for other cases to the reader. Let $\mu = 2\zeta$ and $\nu = 2\xi$. Then the factors of the first group are

$$\begin{aligned} &P\{R[\ell_{2j}, \ell_{2j-1} - 1] = v_{2j-1}, G_{2j-1}\}, \quad j = \zeta, \zeta - 1, \dots, 2, \\ &\quad P\{R[\ell_2, k_1 - 1] = u_0, F_0\} \quad \text{and} \\ &\qquad P\{R[k_{2i}, k_{2i+1} - 1] = u_{2i}, F_{2i}\}, \quad i = 1, 2, \dots, \xi - 1. \end{aligned} \tag{3.26}$$

Now, by the definition of B_n^+,

$$\begin{aligned} &P\{R[k_{2i}, k_{2i+1} - 1] = u_{2i},\ F_{2i}\} \\ &\quad = P\{R[k_{2i}, k_{2i+1} - 1] = u_{2i}, S_{k_{2i}} < S_p \leqslant S_{k_{2i+1}} \text{ for } k_{2i} < p \leqslant k_{2i+1}\} \\ &\quad = P\{B^+_{k_{2i+1} - k_{2i}}(u_{2i})\}. \end{aligned} \tag{3.27}$$

Similarly, by the definition of $\widetilde{B}_n^+$ and (2.17)

$$P\{R[\ell_2, k_1 - 1] = u_0, F_0\} = P\{\widetilde{B}^+_{k_1 - \ell_2}(u_0)\} \leqslant [p^+]^{-1} P\{B^+_{k_1 - \ell_2 + 1}(u_0 + 1)\}, \tag{3.28}$$

$$P\{R[\ell_{2j}, \ell_{2j-1} - 1] = v_{2j-1}, G_{2j-1}\} \leqslant [p^+]^{-1} P\{B^+_{\ell_{2j-1} - \ell_{2j} + 1}(v_{2j-1} + 1)\}. \tag{3.29}$$

Next we construct a new random walk path by putting together paths for which the events in the right hand sides of (3.27)–(3.29) occur. More precisely, we define m_p recursively by

$$m_0 = 0, m_{p+1} - m_p = \ell_{2\zeta-2p-1} - \ell_{2\zeta-2p} + 1 \quad \text{for } p = 0, 1, \ldots, \zeta - 2,$$

$$m_\zeta - m_{\zeta-1} = k_1 - \ell_2 + 1,$$

and finally

$$m_{\zeta+p} - m_{\zeta+p-1} = k_{2p+1} - k_{2p} \quad \text{for } p = 1, \ldots, \xi - 1.$$

Then the right hand sides of (3.27)–(3.29) equal

$$P\{B^+_{m_{\zeta+i}-m_{\zeta+i-1}}(u_{2i})\},\ [p^+]^{-1} P\{B^+_{m_\zeta - m_{\zeta-1}}(u_0+1)\},$$

and

$$[p^+]^{-1} P\{B^+_{m_{\zeta-j+1}-m_{\zeta-j}}(v_{2j-1}+1)\},$$

respectively. Define $N, \alpha \in [0,1]$ and $y \geqslant 0$ by

$$N = n + 2\zeta - 1,$$

$$\alpha N = m_{\zeta+\xi-1} = \sum_{j=2}^{\zeta} (\ell_{2j-1} - \ell_{2j} + 1) + (k_1 - \ell_2 + 1) + \sum_{i=1}^{\xi-1} (k_{2i+1} - k_{2i}),$$

and

$$\alpha N y = \sum_{j=2}^{\zeta} (v_{2j-1} + 1) + u_0 + 1 + \sum_{i=1}^{\xi-1} u_{2i}. \tag{3.30}$$

Note that automatically $y \geqslant 0$. Repeated application of (2.16) then shows that the product of the probabilities in (3.26) is at most

$$\begin{aligned}[p^+]^{-\zeta} P\{&R[m_{\zeta-j}, m_{\zeta-j-1} - 1] = v_{2j-1} + 1,\ 2 \leqslant j \leqslant \zeta,\\ &R[m_{\zeta-1}, m_\zeta - 1] = u_0 + 1,\\ &R[m_{\zeta+i-1}, m_{\zeta+i} - 1] = u_{2i},\ 1 \leqslant i \leqslant \xi - 1,\\ &0 < S_p \leqslant S_{\alpha N},\ 1 \leqslant p \leqslant \alpha N\}.\end{aligned} \tag{3.31}$$

The events in the probability here for two distinct choices of the sequences $v_{2j-1}, 2 \leqslant j \leqslant \zeta, u_{2i}, 0 \leqslant i \leqslant \xi - 1$, are disjoint. Therefore, for any fixed y, the sum of (3.31) over all v_{2j-1}, u_{2i} which satisfy (3.30) is at most

$$\begin{aligned}&[p^+]^{-\zeta} P\{R[0, \alpha N - 1] = \alpha N y,\ 0 < S_p \leqslant S_{\alpha N},\ 1 \leqslant p \leqslant \alpha N\}\\ &\quad = [p^+]^{-\zeta} P\{B^+_{\alpha N}(\alpha N y)\} \leqslant [p^+]^{-\zeta} P\{A^+_{\alpha N}(\alpha N y)\}\\ &\quad \leqslant [p^+]^{-\zeta} \exp[-\alpha N \psi^+(y)] \quad \text{(by (2.4))}.\end{aligned} \tag{3.32}$$

The remaining factors in (3.25) correspond to intervals on which the last value of S is less than the initial value. These are the factors

$$\begin{aligned}&P\{R[\ell_{2j+1},\ell_{2j}-1]=v_{2j},G_{2j}\},\quad j=1,\dots,\zeta-1,\\&P\{R[k_{2i-1},k_{2i}-1]=u_{2i-1},F_{2i-1}\},\quad i=1,\dots,\xi.\end{aligned}\tag{3.33}$$

The definitions of N and α imply that

$$(1-\alpha)N=\sum_{j=1}^{\zeta-1}(\ell_{2j}-\ell_{2j+1}+1)+\sum_{i=1}^{\xi}(k_{2i}-k_{2i-1}).$$

In analogy with (3.30) we define z by

$$(1-\alpha)Nz=\sum_{j=1}^{\zeta-1}(v_{2j}+1)+\sum_{i=1}^{\xi}u_{2i-1}.\tag{3.34}$$

In particular this forces $z\geqslant 0$. By virtue of (3.23) we only have to consider y and z which satisfy

$$\begin{aligned}\alpha Ny+(1-\alpha)Nz&=\sum_{j=2}^{2\zeta-1}(v_j+1)+(u_0+1)+\sum_{i=1}^{2\xi-1}u_i\\&\geqslant nx+2\zeta-1\geqslant Nx\end{aligned}\tag{3.35}$$

(recall $x\leqslant 1$), and hence

$$\alpha y+(1-\alpha)z\geqslant x.\tag{3.36}$$

Exactly as in (3.32) we now find that the sum over v_{2j},u_{2i-1} satisfying (3.34) for some fixed z of the product of the factors in (3.33) is at most

$$\begin{aligned}&[p^-]^{-\zeta+1}P\{R[0,(1-\alpha)N-1]=(1-\alpha)Nz,\\&\qquad 0>S_p\geqslant S_{(1-\alpha)N},\ 1\leqslant p\leqslant(1-\alpha)N\}\\&\quad\leqslant[p^-]^{-\zeta+1}P\{A^-_{(1-\alpha)N}(z)\}\\&\quad\leqslant[p^-]^{-\zeta+1}\exp[-(1-\alpha)N\psi^-(z)]\end{aligned}\tag{3.37}$$

(recall that $p^-=P\{X<0\}$; see (2.7)). Combining this with the estimate in (3.32) we find that the sum of the probabilities of the events in (3.24) over u's and v's satisfying (3.30) and (3.34) is at most

$$\begin{aligned}&[p^-]^{-\zeta+1}[p^+]^{-\zeta}\exp[-N\alpha\psi^+(y)-N(1-\alpha)\psi^-(z)]\\&\quad\leqslant p^{-2\zeta+1}\exp\Big[-n\inf_{\substack{0\leqslant\alpha\leqslant1,y,z\geqslant0\\\alpha y+(1-\alpha)z=x}}[\alpha\psi^+(y)+(1-\alpha)\psi^-(z)]\Big].\end{aligned}\tag{3.38}$$

Here p denotes $\min\{p^+, p^-\}$. In the last inequality we used (3.36) and the fact that

$$\begin{aligned}
&\inf_{\substack{0\leqslant\alpha\leqslant 1, y,z\geqslant 0\\ \alpha y+(1-\alpha)z\geqslant x}} \left[\alpha\psi^+(y) + (1-\alpha)\psi^-(z)\right]\\
&\quad= \inf_{\substack{0\leqslant\alpha\leqslant 1, y,z\geqslant 0\\ \alpha y+(1-\alpha)z= x}} \left[\alpha\psi^+(y) + (1-\alpha)\psi^-(z)\right],
\end{aligned}\tag{3.39}$$

which follows from the monotonicity of $\psi^{\pm}$. Indeed, if $\alpha y + (1-\alpha)z = x' > x$, then we have that

$$\alpha\psi^+(y) + (1-\alpha)\psi^-(z) \geqslant \alpha\psi^+\left(\frac{x}{x'}y\right) + (1-\alpha)\psi^-\left(\frac{x}{x'}z\right),$$

which is not less than the right hand side of (3.39). We may add the restriction $y, z \leqslant 1$ in the infimum here, because, as we already explained, $\psi^{\pm}(y) = \infty$ for $y > 1$. Thus the restriction $y, z \leqslant 1$ has no influence on the infimum. From the left hand inequality in (3.23) we see that we only have to consider values $\alpha N y \leqslant n + \zeta$ and $(1-\alpha)Nz \leqslant n + \zeta$. Thus the probability of the event in (3.11) is at most

$$(n+\zeta+1)^2 p^{-2\zeta+1} \exp[-n\psi^*(x)] \leqslant 4n^2 p^{-\mu+1} \exp[-n\psi^*(x)].\tag{3.40}$$

In order to complete the proof of this lemma we must now estimate how many choices there are for the μ, ν, ℓ_m and k_j in (3.23). In fact we shall only estimate this for the subclass of the sample paths which satisfy

$$\Gamma_n := \left|\{i \leqslant n\colon |X_i| > \sqrt{n}\}\right| \leqslant \gamma(n)n,\tag{3.41}$$

where we can take for $\gamma(\cdot)$ any function which satisfies

$$\gamma(n)n \geqslant 1, \gamma(n)\downarrow 0 \quad \text{but } \gamma(n)\log\frac{\gamma(n)}{P\{|X| > \sqrt{n}\}} \to \infty\tag{3.42}$$

as $n \uparrow \infty$. For instance, $\gamma(n) = [-\log P\{|X| > \sqrt{n}\}]^{-1/2} \vee n^{-1}$ will do. We may restrict ourselves to such paths, because the probability that (3.41) fails is at most

$$\binom{n}{\lceil\gamma(n)n\rceil}\left[P\{|X| > \sqrt{n}\}\right]^{\lceil\gamma(n)n\rceil} = \mathrm{O}\left(\frac{n\mathrm{e}P\{|X| > \sqrt{n}\}}{n\gamma(n)}\right)^{\gamma(n)n}.$$

Under (3.42), we therefore have

$$\limsup_{n\to\infty}\frac{1}{n}\log P\{\Gamma_n > \gamma(n)n\} = -\infty.\tag{3.43}$$

Hence, we may and shall consider only sample sequences which satisfy (3.41) for the proof of (3.2).

We claim that under (3.41) it must be the case that (for large n)

$$\nu \leqslant \gamma(n)n + 2n^{3/4}. \tag{3.44}$$

To see this, let J be the set of $j \in [1, \nu - 1]$ for which there exists an $i \in [k_j + 1, k_{j+1}]$ with $|X_i| > \sqrt{n}$. Then

$$|J| \leqslant \gamma(n)n.$$

Moreover,

$$\sum_{\substack{1 \leqslant j < \nu, \\ j \notin J}} |S_{k_{j+1}} - S_{k_j}| \leqslant \sum_{\substack{1 \leqslant i \leqslant n, \\ |X_i| \leqslant \sqrt{n}}} |X_i| \leqslant n^{3/2},$$

and the integers $|S_{k_{j+1}} - S_{k_j}|,\ 1 \leqslant j \leqslant \nu - 1,\ j \notin J$, are strictly decreasing. In fact the whole sequence of the $|S_{k_{j+1}} - S_{k_j}|$ is strictly decreasing by virtue of (3.18) and (3.19). Now it is easy to see that if $r_1, r_2, \ldots, r_q$ are distinct positive integers with $\sum_{t=1}^{q} r_t \leqslant A$, then $q(q+1)/2 \leqslant A$ since $\sum_{t=1}^{q} t \leqslant \sum_{t=1}^{q} r_t$. Thus in such a situation we must have $q \leqslant \sqrt{2A}$. If we apply this with the r_t the successive values of $|S_{k_{j+1}} - S_{k_j}|,\ 1 \leqslant j \leqslant \nu - 1,\ j \notin J$, then we see that there can be at most $\sqrt{2}n^{3/4}$ such indices j. Thus the total number of $|S_{k_{j+1}} - S_{k_j}|$ is at most $|J| + \sqrt{2}n^{3/4} + 1$, and (3.44) holds.

The bound (3.44) also holds for $\mu - 1$. Since the k_i and ℓ_j, as well as μ and ν have to take values in $[0, n]$ we find that the total number of choices for μ, ν and the k_i, ℓ_j is for large n at most

$$n^2 \binom{n+1}{\lceil 2\gamma(n)n + 4n^{3/4} \rceil} = \mathrm{e}^{\mathrm{o}(n)}.$$

For each such choice of the k_i, ℓ_j, μ and ν, the probability of the event in (3.24) is bounded by the right hand side of (3.40). We have finally proven that

$$P\{R_n \geqslant nx\} \leqslant P\{\Gamma_n > \gamma(n)n\} + 4n^2 p^{-\gamma(n)n - 2n^{3/4}} \exp\left[-n\psi^*(x) + \mathrm{o}(n)\right],$$

which, together with (3.43), implies (3.2). $\square$

4. A lower bound for $P\{R_n \geqslant nx\}$ in the case (1.12)

In this section we show that $\lim(-1/n)\log P\{R_n \geqslant nx\}$ is given by the right hand side of (3.2) when (1.12) prevails. In order to make use of (1.12), the following lemma will be helpful; (1.12) gives us that χ^+ in (4.1) (or its analogue on the negative side, χ^- equals 0).

LEMMA 4. – *Assume that* (2.23) *holds. Let*

$$\chi^+ = \liminf_{n \to \infty} \frac{-1}{n} \log P\{X = n\}. \tag{4.1}$$

If $\chi^+ < \infty$*, then there exists a constant* c_1 *and a function* $g^+ : \{1, 2, \dots\} \to \{1, 2, \dots\}$ *such that*

$$g^+(n) = \mathrm{o}(n), \tag{4.2}$$

and

$$P\big\{S_{g^+(n)} = n, -c_1 \leqslant S_i \leqslant n + c_1, 0 \leqslant i \leqslant g^+(n)\big\} \geqslant \mathrm{e}^{-\chi^+ n + \mathrm{o}(n)}. \tag{4.3}$$

Moreover, for $u \geqslant 0$

$$\begin{aligned} P\{R_n \geqslant nx, S_n = -u\} &\leqslant \mathrm{e}^{\chi^+ u + \mathrm{o}(u)} P\{R_{n+g^+(u)} \geqslant nx, S_{n+g^+(u)} = 0\} \\ &= \mathrm{e}^{\chi^+ u + \mathrm{o}(u)} P\left\{C_{n+g^+(u)}\left(\frac{nx}{n + g^+(u)}\right)\right\}. \end{aligned} \tag{4.4}$$

(*See* (2.18) *for* C_n.)

Proof. – Recall the we assumed that the group generated by the support of X equals $\mathbb{Z}$. This means that there exist integers a, b and positive integers r, s such that $P\{S_r = a\} > 0$, $P\{S_s = b\} > 0$ and $a - b = 1$. By virtue of (2.23) we may take $a, b > 0$, because we can replace a, b by $a + mc, b + mc$ if $P\{S_t = c\} > 0$ for some t. As in [1], Appendix A21, this implies that for $k \geqslant$ some k_0 there exists an $h(k) \geqslant 1$ for which $P\{S_{h(k)} = k\} > 0$. But then there also exist m and u such that

$$P\{S_{h(1+mu)+mu} = 1\} \geqslant P\{S_{h(1+mu)} = 1 + mu\}[P\{X = -u\}]^m > 0.$$

The same argument holds with positive and negative interchanged, so that there exist integers m^+, m^- and a constant $c_2 > 0$ such that

$$P\{S_{m^+} = 1\} > c_2 \quad \text{and} \quad P\{S_{m^-} = -1\} > c_2. \tag{4.5}$$

Now let $k_i \geqslant 1,\ i = 1, 2, \dots$ be such that

$$\begin{gathered} k_{i+1} \geqslant 2k_i,\ i \geqslant 0, \quad P\{X = k_i\} > 0 \\ \text{and} \quad \lim_{i\to\infty} \frac{-1}{k_i} \log P\{X = k_i\} = \chi^+ < \infty. \end{gathered} \tag{4.6}$$

For any $n \geqslant k_1$ we can then find an $n' \in [n - k_1 + 1, n]$ such that

$$n' = \sum_{i=1}^{i_0} t_i k_i$$

for some non-negative integers i_0, t_i which satisfy

$$k_{i_0} \leqslant n < k_{i_0+1}, \quad t_i < \frac{k_{i+1}}{k_i}.$$

Indeed, after finding i_0 so that the first relation holds, one merely has to take $t_{i_0} = \lfloor n/k_{i_0} \rfloor$ and then for $i = i_0 - 1, \dots, 1$,

$$t_i = \left\lfloor \frac{1}{k_i}\left(n - \sum_{j=i+1}^{i_0} t_j k_j\right)\right\rfloor.$$

Let i_1 be the unique index with $k_{i_1} \leqslant \sqrt{n} < k_{i_1+1}$. Then

$$\begin{aligned}
\sum_{i=1}^{i_0} t_i &\leqslant \sum_{i=1}^{i_0} \frac{k_{i+1} \wedge n}{k_i} \\
&\leqslant \frac{1}{k_{i_1}}\left[n + \sum_{i=i_1+1}^{i_0} k_i\right] + k_{i_1} \sum_{i=1}^{i_1-1} \frac{1}{k_i} \\
&\leqslant \frac{1}{k_{i_1}}\left[n + \sum_{i=i_1+1}^{i_0} k_{i_0} 2^{-(i_0-i)}\right] + k_{i_1} \sum_{i=1}^{i_1-1} \frac{1}{k_1} 2^{-i+1} \\
&\leqslant \frac{3n}{k_{i_1}} + 2k_{i_1} \leqslant \frac{3n}{k_{i_1}} + 2\sqrt{n} = \mathrm{o}(n).
\end{aligned}$$

Moreover, by virtue of (4.5), there exists some $\ell = \ell(n - n')$ such that $P\{S_\ell = n - n'\} > 0$. In fact the only possibilities for $n - n'$ are the integers $0, 1, \dots, k_1 - 1$, so that we can take

$$\ell(n - n') = m^+(n - n') \leqslant m^+ k_1 \quad \text{and} \quad P\{S_{\ell(n-n')} = n - n'\} \geqslant [c_2]^{n-n'} \geqslant [c_2]^{k_1} =: 2c_3,$$

uniformly in n. After that we can choose $c_1 > 0$ so that

$$\begin{aligned}
&P\{S_\ell = n - n', |S_i| \leqslant c_1 \text{ for } i \leqslant \ell\} \\
&\quad \geqslant P\{S_\ell = n - n'\} - P\{|S_i| > c_1 \text{ for some } i \leqslant m^+(n - n')\} \\
&\quad \geqslant c_3.
\end{aligned}$$

Finally we take

$$g^+(n) = \sum_{i=0}^{i_0} t_i + \ell(n - n').$$

Clearly (4.2) is satisfied, by virtue of the preceding estimate for $\sum t_i$. In addition, by virtue of (4.6) and the fact that all k_i are strictly positive,

$$\begin{aligned}
&P\{S_{g^+(n)} = n, -c_1 \leqslant S_i \leqslant S_{g^+(n)} + c_1 \text{ for } i \leqslant g^+(n)\} \\
&\quad \geqslant \prod_{i=1}^{i_0} [P\{X = k_i\}]^{t_i} P\{S_{\ell(n-n')} = n - n', |S_i| \leqslant c_1 \text{ for } i \leqslant \ell(n - n')\} \\
&\quad \geqslant c_3 \exp\left[-\sum_{i=1}^{i_0} t_i\left(\chi^+ k_i + \mathrm{o}(k_i)\right)\right] \\
&\quad = \exp\left[-\chi^+ n + \mathrm{o}(n)\right].
\end{aligned}$$

Thus also (4.3) holds.

Finally, (4.4) follows from (4.3) and

$$\begin{aligned} &P\{R_{n+g^+(u)} \geqslant nx,\, S_{n+g^+(u)} = 0\} \\ &\quad \geqslant P\{R_n \geqslant nx,\, S_n = -u\} P\left\{\sum_{i=n+1}^{n+g^+(u)} X_i = u\right\}. \qquad \square \end{aligned} \tag{4.7}$$

Remark 5. – Define

$$\chi^- := \liminf_{n\to\infty} \frac{-1}{n} \log P\{X = -n\}. \tag{4.8}$$

Of course, when $\chi^- < \infty$, then the analogous results to Lemma 4 with positive and negative interchanged hold. We then have to replace $g^+(n)$ by some $g^-(n) = \mathrm{o}(n)$.

Note that $\chi^\pm = \infty$ is possible, for instance when X is bounded on one or both sides. In this case (4.4) should be replaced by

$$\begin{aligned} P\{R_n \geqslant nx,\ S_n = -u\} &\leqslant [c_2]^{-u} P\{R_{n+um^+} \geqslant nx,\, S_{n+um^+} = 0\} \\ &= [c_2]^{-u} P\left\{C_{n+um^+}\left(\frac{xn}{n+um^+}\right)\right\} \end{aligned} \tag{4.9}$$

for $u \geqslant 0$. This inequality again follows from (4.7) with $g^+(u)$ replaced by um^+, if one takes into account that $P\{S_{um^+} = u\} \geqslant [P\{S_{m^+} = 1\}]^u$ (see (4.5) for m^+ and c_2).

With the help of Lemma 4 we can now prove an analogue of Lemma 1 in [4], for the case (1.12).

LEMMA 5. – *Assume that* (1.12) *and* (2.23) *hold. Then, there exists a constant M and a function $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ with the following properties*:

$$r(p) \to \infty \quad \text{but } r(n) = \mathrm{o}(n) \quad \text{as } n \to \infty, \tag{4.10}$$

$$r(n+1) - r(n) \leqslant 1, \quad n \geqslant 0, \tag{4.11}$$

and for all integers $n, m \geqslant M$ and $y, z \in [0, \infty)$ it holds that

$$P\bigl\{R_{n+m+r(n+m)} \geqslant y + z - r(n+m)\bigr\} \geqslant \mathrm{e}^{-r(n+m)} P\{R_n \geqslant y\} P\{R_m \geqslant z\}. \tag{4.12}$$

Proof. – This proof is essentially the same as that of Lemma 1 in [4]. Suppose that we can find a set $\Xi = \Xi(n, m) \subset \mathbb{Z}$ and functions $g, r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ such that

$$g(w) \leqslant r(n+m), \quad w \in \Xi, \tag{4.13}$$

$$\text{for each } w \in \Xi, \quad P\{S_{g(w)} = w\} \geqslant 2\mathrm{e}^{-r(n+m)} \tag{4.14}$$

and

$$\frac{nm}{r(n+m)} \leqslant \frac{1}{2}|\Xi|. \tag{4.15}$$

Then a few simple modifications in the proof of inequality (2.11) in [4] show that for such n, m

$$P\{R_{n+m+r(n+m)} \geqslant y + z - r(n+m)\} \geqslant \frac{2\mathrm{e}^{-r(n+m)}}{|\Xi|} \frac{1}{2} |\Xi| P\{R_n \geqslant y\} P\{R_m \geqslant z\}.$$

We therefore merely have to find the function r and a replacement for the set Ξ_q which we used in [4], in such a way that (4.10), (4.11) and (4.13)–(4.15) are satisfied for $n, m \geqslant M$

To find the required r and Ξ we appeal to Lemma 4. If (1.12) holds, then $\chi^+ = 0$ or $\chi^- = 0$. For the sake of argument we assume that $\chi^+ = 0$. By Lemma 4 there then exists a funtion g^+ such that

$$g^+(p) = \mathrm{o}(p) \quad \text{and} \quad \lim_{p\to\infty} \frac{1}{p} \log P\{S_{g^+(p)} = p\} = 0.$$

But then we can find a p_0 and a nondecreasing function $t : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ which increases so slowly that

$$t(p) \to \infty \quad \text{but } t(p) = \mathrm{o}(p) \quad \text{as } p \to \infty, \tag{4.16}$$

$$g^+(w) \leqslant \frac{p}{t(p)} \quad \text{for all } p_0 \leqslant w \leqslant 4p\big(t(p)+1\big) \tag{4.17}$$

and

$$P\{S_{g^+(w)} = w\} \geqslant 2\mathrm{e}^{-p/t(p)} \quad \text{for } p_0 \leqslant w \leqslant 4p\big(t(p)+1\big). \tag{4.18}$$

Finally we take g^+ for g and

$$r(p) = \begin{cases} \lceil \frac{p_0}{t(p_0)} \rceil & \text{if } p \leqslant p_0, \\ \lceil \frac{p}{t(p)} \rceil & \text{if } p > p_0, \end{cases}$$

and

$$\Xi = \left[p_0, \frac{4(n+m)^2}{r(n+m)}\right] \subset \big[p_0, 4(n+m)t(n+m)\big].$$

It is easy to see that this choice satisfies all requirements for some suitable M and $n, m \geqslant M$. We merely comment on the requirement (4.11). This follows from the fact that t is nondecreasing. Indeed this monotonicity implies $(p+1)/t(p+1) - p/t(p) \leqslant 1$, and hence also $\lceil (p+1)/t(p+1) \rceil - \lceil p/t(p) \rceil \leqslant 1$. □

We can now prove (1.2) in the case of (1.12).

LEMMA 6. – *Assume that* (1.12) *holds and that* $0 \leqslant x \leqslant 1$. *Then*

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} = \psi^*(x). \tag{4.19}$$

Proof. – If $p^- = P\{X < 0\} = 0$ (and hence $p^+ = P\{X > 0\} > 0$, because $P\{X = 0\} < 1$), then $P\{A_n^-(x)\} = 0$ for $n \geqslant 1$ and $\psi^-(x) = \infty$. Thus the right hand side of (4.19) equals $\psi^+(x)$. Moreover, $P\{R_n \geqslant nx\} = P\{\widetilde{A}_n^+(x)\}$, since S_k is nondecreasing with probability 1. Thus (4.19) is included in (2.6) in this case. A similar argument applies if $p^+ = 0$, so that we may assume that (2.23) holds for the remainder of this proof.

In view of (3.2) it suffices for (4.19) to prove

$$\limsup_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} \leqslant \psi^*(x). \tag{4.20}$$

First we consider the special case of this when $x = 1$. In this case, the only convex combinations $\alpha y + (1-\alpha)z$ with $0 \leqslant \alpha, y, z \leqslant 1$ which equal x are combinations with $y = z = 1$ or $\alpha \in \{0, 1\}$. Thus, for $x = 1$, the right hand side of (4.20) equals $\min\{\psi^+(1), \psi^-(1)\}$. The inequality (4.20) for $x = 1$ therefore follows from (2.3) and the fact that

$$P\{R_n \geqslant nx\} \geqslant \max\bigl\{P\{A_n^+(1)\}, P\{A_n^-(1)\}\bigr\} \geqslant p^n \tag{4.21}$$

for each $x \in [0, 1]$ (see (2.11) and recall that $p = \min\{p^+, p^-\}$).

For the remainder of this proof we fix $0 \leqslant x < 1$ and $\varepsilon > 0$. Since we already know from Lemma 1 that $\psi^\pm$ are bounded on [0, 1], we can also fix $\alpha, y, z \in [0, 1]$ such that

$$x = \alpha y + (1-\alpha)z, \tag{4.22}$$

and

$$\psi^*(x) \geqslant \alpha\psi^+(y) + (1-\alpha)\psi^-(z) - \varepsilon. \tag{4.23}$$

Necessarily $\alpha > 0, y \leqslant x$, or $\alpha < 1, z \leqslant x$. For the sake of argument we assume that $\alpha > 0$ and $y \leqslant x < 1$. Since ψ^+ is continuous on [0, 1) (by virtue of Lemma 1), we can further choose $y' > y$ such that even

$$\psi^*(x) \geqslant \alpha\psi^+(y') + (1-\alpha)\psi^-(z) - 2\varepsilon. \tag{4.24}$$

We now apply (4.12) with n and m replaced by $\lceil \alpha n\rceil$ and $\lceil (1-\alpha)n\rceil$, respectively, and with y and z replaced by $\lceil \alpha n\rceil y'$ and $\lceil (1-\alpha)n\rceil z$, respectively. This gives for large n

$$\begin{aligned}
&P\bigl\{R_{n+2+r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)} \geqslant \lceil \alpha n\rceil y' + \lceil (1-\alpha)n\rceil z - r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)\bigr\}\\
&\quad\geqslant P\bigl\{R_{\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil + r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)}\\
&\qquad\qquad \geqslant \lceil \alpha n\rceil y' + \lceil (1-\alpha)n\rceil z - r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)\bigr\}\\
&\quad\geqslant \mathrm{e}^{-r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)} P\bigl\{R_{\lceil \alpha n\rceil} \geqslant \lceil \alpha n\rceil y'\bigr\} P\bigl\{R_{\lceil (1-\alpha)n\rceil} \geqslant \lceil (1-\alpha)n\rceil z\bigr\}\\
&\quad\geqslant \mathrm{e}^{-r(\lceil \alpha n\rceil + \lceil (1-\alpha)n\rceil)} P\bigl\{A^+_{\lceil \alpha n\rceil}(y')\bigr\} P\bigl\{A^-_{\lceil (1-\alpha)n\rceil}(z)\bigr\}\\
&\quad\geqslant \exp\bigl[\mathrm{o}(n) - \alpha n\psi^+(y') - (1-\alpha n)\psi^-(z)\bigr] \quad \text{(by (2.3) and (4.10))}\\
&\quad\geqslant \exp\bigl[\mathrm{o}(n) - n\psi^*(x) - 2n\varepsilon\bigr].
\end{aligned} \tag{4.25}$$

Now, for a given large ℓ, find n such that

$$\begin{aligned} n+2+r(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil) &\leqslant \ell \\ &\leqslant n+1+2+r(\lceil \alpha(n+1)\rceil+\lceil (1-\alpha)(n+1)\rceil). \end{aligned} \tag{4.26}$$

By virtue of (4.11) we then have

$$n+2+r\bigl(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil\bigr) \leqslant \ell \leqslant n+3+r\bigl(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil\bigr)+2.$$

Hence, for large ℓ,

$$\ell x=\ell\bigl(\alpha y+(1-\alpha)z\bigr) \leqslant \lceil \alpha n\rceil y'+\lceil (1-\alpha)n\rceil z-r\bigl(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil\bigr).$$

Consequently, by the monotonicity properties of $P\{R_\ell \geqslant z\}$,

$$\begin{aligned} P\{R_\ell \geqslant \ell x\} \geqslant P\bigl\{&R_{n+2+r(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil)} \\ &\geqslant \lceil \alpha n\rceil y'+\lceil (1-\alpha)n\rceil z-r\bigl(\lceil \alpha n\rceil+\lceil (1-\alpha)n\rceil\bigr)\bigr\}. \end{aligned}$$

Together with (4.25) this implies

$$\limsup_{\ell\to\infty} \frac{-1}{\ell}\log P\{R_\ell \geqslant \ell x\} \leqslant \limsup_{\ell\to\infty} \frac{1}{\ell}\bigl[n(\psi^*(x)+2\varepsilon)+\mathrm{o}(n)\bigr],$$

where n is determined as a function of ℓ by (4.26). In particular, the latter relation, together with (4.10), shows that $\lim \ell/n=1$. Finally, this implies (4.20) and (4.19), since $\varepsilon>0$ was arbitrary. □

5. Another upper bound for $P\{R_n \geqslant nx\}$

In this section we derive an alternative to the upper bound for $P\{R_n \geqslant nx\}$ of Lemma 3. This bound will be used only in the case when the distribution of X has an exponentially bounded tail, that is when (1.13) holds. For $x \in [0,1)$ we define

$$\begin{aligned} \rho^+(x) &= \inf_{\substack{0\leqslant \alpha\leqslant 1,\, y<1,\, z<x_0 \\ \alpha y+(1-\alpha)z=x}} \bigl[\alpha\psi^+(y)+(1-\alpha)\sigma(z)\bigr], \\ \rho^-(x) &= \inf_{\substack{0\leqslant \alpha\leqslant 1,\, y<1,\, z<x_0 \\ \alpha y+(1-\alpha)z=x}} \bigl[\alpha\psi^-(y)+(1-\alpha)\sigma(z)\bigr] \end{aligned} \tag{5.1}$$

and

$$\psi^\sharp(x)=\min\bigl\{\rho^+(x),\rho^-(x)\bigr\}. \tag{5.2}$$

LEMMA 7. – *Assume that* (2.23) *holds and that* $\chi^+>0$ *and* $\chi^->0$. *Finally, let* $x<1$. *Then*

$$\liminf_{n\to\infty} \frac{-1}{n}\log P\{R_n \geqslant nx\} \geqslant \psi^\sharp(x). \tag{5.3}$$

Proof. – Analogously to the proof of Lemma 3 we introduce times at which the sample path of $\{S_n\}$ achieves its maximum and minimum. More precisely, we let

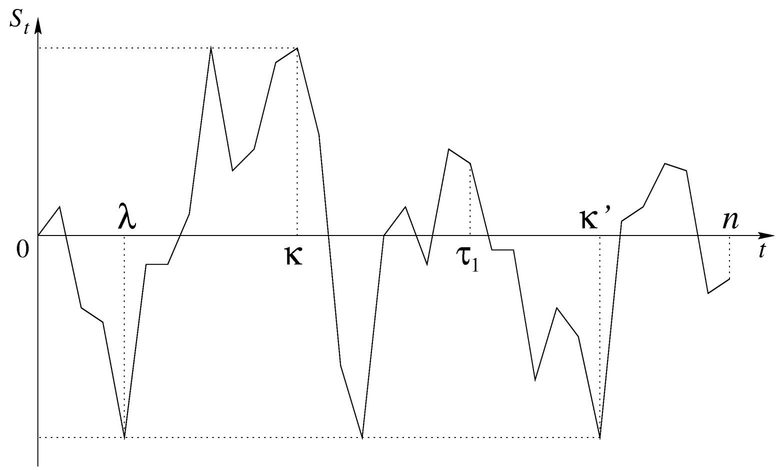

Fig. 2. Illustration of the location of κ, κ', λ and τ_1.

$$\kappa = \max\bigl\{i \leqslant n\colon\ S_i = \max\{0, S_1, S_2, \ldots, S_n\}\bigr\},$$
$$\lambda = \min\bigl\{i \geqslant 0\colon\ S_i = \min\{0, S_1, S_2, \ldots, S_n\}\bigr\}.$$

For the sake of argument we consider the case $\lambda \leqslant \kappa$; the case $\lambda > \kappa$ is similar. If $\lambda \leqslant \kappa$ we also introduce

$$\kappa' = \max\bigl\{i \geqslant \kappa\colon\ S_i = \min\{S_\kappa, S_{\kappa+1}, \ldots, S_n\}\bigr\}.$$

For the time being we make the extra assumption

$$0 < \chi^+ < \infty \quad \text{and} \quad 0 < \chi^- < \infty. \tag{5.4}$$

We remind the reader of (4.21). This implies that $P\{R_n \geqslant nx\} \geqslant \exp(-c_4 n)$ for some constant $c_4 < \infty$. Since we assumed $\chi^\pm > 0$, we can find a constant $c_5 < \infty$ such that

$$P\{R_n \geqslant nx\} \leqslant 2P\bigl\{R_n \geqslant nx, |X_i| \leqslant c_5 n, 1 \leqslant i \leqslant n\bigr\}. \tag{5.5}$$

We shall estimate the right hand side in two pieces:

$$\text{(i)} \quad P\bigl\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n, 1 \leqslant i \leqslant n,\ S_n \leqslant 0\bigr\},$$
$$\text{(ii)} \quad P\bigl\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n, 1 \leqslant i \leqslant n,\ S_n > 0\bigr\}.$$

For the time being we shall work on the first piece. For this estimate we first define

$$\tau_1 = \max\{i\colon\ \kappa \leqslant i \leqslant \kappa', S_i \geqslant 0\}$$

(see Fig. 2). We note first that $S_\kappa = \max\{0, S_1, \ldots, S_n\} \geqslant 0$, while $S_{\kappa'} \leqslant S_n \leqslant 0$ in case (i). Thus, the set on the right in this definition is non-empty, so that τ_1 is well defined. If $\tau_1 = \kappa'$, then $0 \leqslant S_{\tau_1} = S_{\kappa'} \leqslant S_n \leqslant 0$, and hence $S_n = 0$. But $P\{R_n \geqslant nx, S_n = 0\} = P\{C_n(x)\}$. If $x > x_0$ this vanishes for large n, and if $x \leqslant x_0$, then $\liminf(-1/n)\log P\{C_n(x)\} \geqslant \sigma(x-) \geqslant \psi^\sharp(x)$ (compare the lines before (5.24) below).

We may therefore ignore the case $S_n = 0$ and restrict ourselves to $\tau < \kappa'$, and $S_{\tau_1} \leqslant 0$. We next show that we may assume $S_{\tau_1} = 0$ at a "small cost in probability". Specifically we shall prove (5.17) below, which corresponds to taking $S_{\tau_1} = 0$ and even $S_{\tau_2} = 0$, where τ_2 is defined in (5.11). If $|X_{\tau_1+1}| < c_5 n$ and $S_{\tau_1} \geqslant 0 \geqslant S_{\tau_1+1}$, then we must have $0 \leqslant S_{\tau_1} \leqslant c_5 n$ and $0 \geqslant S_{\tau_1+1} \geqslant -c_5 n$. There must therefore exist a constant c_6 (independent of n) and integers p_i, s, t, r_i such that

$$\begin{aligned} &0 \leqslant p_1, p_2 \leqslant c_5 n, \quad 0 \leqslant s \leqslant t \leqslant n, \\ &0 \leqslant r_1 \leqslant s, \quad 0 \leqslant r_2 \leqslant n - s, \quad r_1 + r_2 \geqslant nx, \end{aligned} \tag{5.6}$$

and such that

$$\begin{aligned} &P\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n,\ 1 \leqslant i \leqslant n, \\ &\quad \tau_1 = s,\ S_{\tau_1} = p_1,\ S_{\tau_1+1} = -p_2,\ \kappa' = t, \\ &\quad R[0, s-1] = r_1,\ R[s, n-1] = r_2,\ S_n \leqslant 0\} \\ &\quad \geqslant \frac{c_6}{(n+1)^6} P\{R_n \geqslant nx,\ S_n < 0,\ |X_i| \leqslant c_5 n,\ 1 \leqslant i \leqslant n\}. \end{aligned} \tag{5.7}$$

(Note that we can take $c_6 = [c_5]^{-2}$.) If the event in the left hand side here occurs, then $X_1, \ldots, X_n$ are such that

$$\begin{aligned} &R_s = r_1, \qquad S_s = p_1, \qquad X_{s+1} = -p_1 - p_2, \\ &\sum_{i=s+1}^{j} X_i \leqslant -p_1, \quad s+1 \leqslant j \leqslant t, \qquad \sum_{i=s+1}^{t} X_i \leqslant \sum_{i=s+1}^{n} X_i \leqslant -p_1, \\ &R[s, n-1] = r_2, \qquad |X_h| \leqslant c_5 n, \quad 1 \leqslant h \leqslant n. \end{aligned} \tag{5.8}$$

Thus, by virtue of (4.4) (with positive and negative interchanged) and the definition of χ^-, the left hand side of (5.7) is at most

$$\begin{aligned} &P\{R_s = r_1, S_s = p_1\} P\{X_{s+1} = -p_1 - p_2\} \\ &\times P\left\{ \sum_{i=s+2}^{j} X_i \leqslant p_2,\ s+2 \leqslant j \leqslant t,\ \sum_{i=s+2}^{t} X_i \leqslant \sum_{i=s+2}^{n} X_i \leqslant p_2, \right. \\ &\qquad \left. R[s+1, n-1] \geqslant r_2 - 1,\ |X_h| \leqslant c_5 n,\ s+2 \leqslant h \leqslant n \right\} \\ &\leqslant \mathrm{e}^{\chi^- p_1 + \mathrm{o}(p_1)} P\{R_{s+g^-(p_1)} \geqslant r_1,\ S_{s+g^-(p_1)} = 0\} \mathrm{e}^{-\chi^-[p_1 + p_2 + \mathrm{o}(p_1+p_2)]} \\ &\quad \times P\left\{ \sum_{i=s+2}^{j} X_i \leqslant p_2,\ s+2 \leqslant j \leqslant t,\ \sum_{i=s+2}^{t} X_i \leqslant \sum_{i=s+2}^{n} X_i \leqslant p_2, \right. \\ &\qquad \left. R[s+1, n-1] \geqslant r_2 - 1,\ |X_h| \leqslant c_5 n,\ s+2 \leqslant h \leqslant n \right\}. \end{aligned} \tag{5.9}$$

Finally, by (4.3) (again with positive and negative interchanged), the left hand side of (5.7) is bounded by

$$
\begin{aligned}
&\mathrm{e}^{o(n)} P\{R_{s+g^-(p_1)} \geqslant r_1, S_{s+g^-(p_1)} = 0\} \\
&\times P\{S_{g^-(p_2)} = -p_2, c_1 \geqslant S_i \geqslant -p_2 - c_1, 0 \leqslant i \leqslant g^-(p_2)\} \\
&\times P\Bigg\{\sum_{i=s+2}^{j} X_i \leqslant p_2 \text{ for } s+2 \leqslant j \leqslant t, \sum_{i=s+2}^{t} X_i \leqslant \sum_{i=s+2}^{n} X_i \leqslant p_2, \\
&\qquad R[s+1, n-1] \geqslant r_2 - 1,\ |X_h| \leqslant c_5 n,\ s+2 \leqslant h \leqslant n\Bigg\} \\
&\quad \leqslant \mathrm{e}^{o(n)} P\{R_{s+g^-(p_1)} \geqslant r_1, S_{s+g^-(p_1)} = 0\} \\
&\qquad \times P\{S_{g^-(p_2)} = -p_2,\ S_h \leqslant c_1,\ 0 \leqslant h \leqslant g^-(p_2), \\
&\qquad\qquad S_j \leqslant 0, g^-(p_2) \leqslant j \leqslant g^-(p_2) + t - s - 1, \\
&\qquad\qquad S_{g^-(p_2)+t-s-1} \leqslant S_{g^-(p_2)+n-s-1} \leqslant 0, \\
&\qquad\qquad R_{g^-(p_2)+n-s-1} \geqslant r_2 - 1,\ |X_i| \leqslant c_5 n \vee (p_2 + 2c_1) \leqslant 2c_5 n, \\
&\qquad\qquad 1 \leqslant i \leqslant g^-(p_2) + n - s - 1\}.
\end{aligned} \tag{5.10}
$$

Here and in the rest of this proof $o(n)$ is such that $n^{-1}o(n) \to 0$ as $n \to \infty$, uniformly in the p_i, r_i, s, t and u (some of these will only be chosen below). However, the precise value of the $o(n)$ expressions may vary from one appearance to another. To handle the last probability in the right hand side here, we introduce

$$
\tau_2 = \min\{i\colon\ S_i \leqslant S_{g^-(p_2)+n-s-1}\}. \tag{5.11}
$$

Note that $\tau_2 \leqslant g^-(p_2) + t - s - 1$ if the event in the last probability occurs. Arguing as above one now decomposes the sample path $0, S_1, \ldots, S_{g^-(p_2)+n-s-1}$ into the piece $0, S_1, \ldots, S_{\tau_2-1}$, the jump X_{τ_2}, and the piece $S_{\tau_2}, \ldots, S_{g^-(p_2)+n-s-1}$. One can now find integers u, p_3, p_4, r_3, r_4 such that

$$
\begin{aligned}
&0 \leqslant p_3, p_4 \leqslant 2c_5 n, \quad u \leqslant g^-(p_2) + t - s - 1 \leqslant n + g^-(p_2) \\
&r_3 \leqslant u, \quad r_4 \leqslant n - s - 1 - \tau_2 + g^-(p_2) \leqslant n - s - u + g^-(p_2), \\
&r_3 + r_4 \geqslant r_2 - 2,
\end{aligned} \tag{5.12}
$$

and such that the last probability in (5.10) is at most

$$
\begin{aligned}
&\mathrm{e}^{o(n)} P\{c_1 \geqslant S_i \geqslant S_u - p_3,\ 0 \leqslant i \leqslant u, R_u = r_3\} \\
&\times \mathrm{e}^{-\chi^-(p_3+p_4)} P\{R_{g^-(p_2)+n-s-u-2} \geqslant r_4,\ S_{g^-(p_2)+n-s-u-2} = p_4\}.
\end{aligned} \tag{5.13}
$$

Here we fixed some random variables as follows: $\tau_2 - 1 = u$, $S_{\tau_2-1} - S_{g^-(p_2)+n-s-1} = p_3, X_{\tau_2} = -p_3 - p_4$, and consequently $S_{g^-(p_2)+n-s-1} - S_{\tau_2} = p_4, R[0, \tau_2 - 2] = r_3, R[\tau_2, g^-(p_2) + n - s - 2] = r_4$. By using (4.4) once more we see that (5.13) is at most

$$
\begin{aligned}
&\mathrm{e}^{o(n)} P\{c_1 \geqslant S_i \geqslant S_u - p_3,\ 0 \leqslant i \leqslant u, R_u = r_3\} \\
&\times \mathrm{e}^{-\chi^- p_3} P\{R_{g^-(p_2)+g^-(p_4)+n-s-u-2} \geqslant r_4, S_{g^-(p_2)+g^-(p_4)+n-s-u-2} = 0\}.
\end{aligned} \tag{5.14}
$$

Further, by (4.3) we can write

$$\begin{aligned} \mathrm{e}^{-\chi^- p_3} \leqslant \mathrm{e}^{\mathrm{o}(n)} P\big\{ & S_{u+g^-(p_3)} - S_u = -p_3, \\ & c_1 \geqslant S_i - S_u \geqslant -p_3 - c_1,\ u \leqslant i \leqslant u + g^-(p_3)\big\}. \end{aligned} \tag{5.15}$$

By putting together a sample path $X_1, \dots, X_u$ for which the first event in (5.14) occurs and a sample path $X_{u+1}, \dots, X_{u+g^-(p_3)}$ for which the event in the right hand side of (5.15) occurs, we see that (5.14) is bounded by

$$\begin{aligned} & \mathrm{e}^{\mathrm{o}(n)} P\big\{2c_1 \geqslant S_i \geqslant S_{u+g^-(p_3)} - c_1, 0 \leqslant i \leqslant u + g^-(p_3), R_{u+g^-(p_3)} \geqslant r_3\big\} \\ & \quad \times P\big\{R_{g^-(p_2)+g^-(p_4)+n-s-u-2} \geqslant r_4, S_{g^-(p_2)+g^-(p_4)+n-s-u-2} = 0\big\}. \end{aligned} \tag{5.16}$$

Combining (5.5)–(5.16) we finally obtain that

$$\begin{aligned} & P\big\{R_n \geqslant nx,\ S_n < 0,\ |X_i| \leqslant c_5 n,\ 1 \leqslant i \leqslant n\big\} \\ & \quad \leqslant \mathrm{e}^{\mathrm{o}(n)} P\big\{R_{s+g^-(p_1)} \geqslant r_1, S_{s+g^-(p_1)} = 0\big\} \\ & \qquad \times P\big\{2c_1 \geqslant S_i \geqslant S_{u+g^-(p_3)} - c_1,\ 0 \leqslant i \leqslant u + g^-(p_3),\ R_{u+g^-(p_3)} \geqslant r_3\big\} \\ & \qquad \times P\big\{R_{g^-(p_2)+g^-(p_4)+n-s-u-2} \geqslant r_4, S_{g^-(p_2)+g^-(p_4)+n-s-u-2} = 0\big\}. \end{aligned} \tag{5.17}$$

Now let $a \geqslant 1$ be such that $P\{X = -a\} > 0$. As in (2.12), if

$$\big\{2c_1 \geqslant S_i \geqslant S_{u+g^-(p_3)} - c_1, 0 \leqslant i \leqslant u + g^-(p_3), R_{u+g^-(p_3)} \geqslant r_3\big\}$$

occurs, then we can insert $2c_1$ steps of value $-a$ in front of $X_1, \dots, X_{u+g^-(p_3)}$ and c_1 such steps at the end. If the original path took steps $X_1, \dots, X_{u+g^-(p_3)}$, then the new path takes steps $-a, -a, \dots, -a$ ($2c_1$ times), $X_1, \dots, X_{u+g^-(p_3)}, -a, -a, \dots, -a$ (c_1 times). Clearly the partial sums of this extended path of length $u + g^-(p_3) + 3c_1$ lie between 0 and the last sum, and therefore this extended path lies in $\widetilde{A}^-_{u+g^-(p_3)+3c_1}(r_3/(u + g^-(p_3) + 3c_1))$. In other words

$$\begin{aligned} & P\big\{2c_1 \geqslant S_i \geqslant S_{u+g^-(p_3)} - c_1,\ 0 \leqslant i \leqslant u + g^-(p_3),\ R_{u+g^-(p_3)} \geqslant r_3\big\} \\ & \quad \leqslant [P\{X = -a\}]^{-3c_1} P\left\{\widetilde{A}^-_{u+g^-(p_3)+3c_1}\left(\frac{r_3}{u + g^-(p_3) + 3c_1}\right)\right\} \\ & \quad \leqslant \exp\left[\mathrm{o}(n) - \big(u + g^-(p_3) + 3c_1\big)\psi^-\left(\frac{r_3}{u + g^-(p_3) + 3c_1}\right)\right] \end{aligned} \tag{5.18}$$

(by (2.13) and (2.4)). We define α, y and z by

$$\alpha = \frac{u}{n} \wedge 1, \qquad y = \frac{r_3}{u} \quad \text{if } u \neq 0, \quad y = 0 \quad \text{if } u = 0,$$
$$z = \frac{r_1 + r_4}{n - u} \wedge 1 \quad \text{if } n - u \neq 0, \qquad z = 0 \quad \text{if } n - u = 0.$$

Note that these quantities depend on n, u, s, t, p_i and r_i, but we do not indicate this in our notation. Then the right hand side of (5.18) is bounded above by

$$\exp\left[\mathrm{o}(n) - \alpha n \psi^-\left(y + \mathrm{o}(1)\right)\right],$$

where the $\mathrm{o}(1)$ here (in the argument of ψ^-) and later $\mathrm{o}(1)$ terms in this proof tend to 0 as $n \to \infty$, uniformly in u, s, t, and in the p_i, r_i. Indeed, we know that $\psi^-(\cdot)$ is increasing and bounded on $0 \leqslant t \leqslant 1$ by $-\log p^-$ (compare (2.11)). Moreover, $g^-(p) = \mathrm{o}(p) = \mathrm{o}(n)$ for $p \leqslant n$, and $r_3 \leqslant u \leqslant n + g^-(p_2)$, so that $u = \alpha n + \mathrm{o}(n)$ (see (5.12)).

On the other hand, the product of the first and third probability in the right hand side of (5.17) is, by virtue of (2.26), at most

$$\begin{aligned}\left(n + \mathrm{o}(n)\right) P\big\{&R_{s+g^-(p_1)+g^-(p_2)+g^-(p_4)+n-s-u-2} \geqslant r_1 + r_4 - 1,\\ &S_{s+g^-(p_1)+g^-(p_2)+g^-(p_4)+n-s-u-2} = 0\big\}.\end{aligned} \tag{5.19}$$

To estimate this define

$$G^-(n) = \left\{n\left[1 + \max_{p \leqslant 2c_5 n} g^-(p)\right]\right\}^{1/2}.$$

Note that as $n \to \infty$,

$$G^-(n) \to \infty, \quad \frac{G^-(n)}{n} \to 0 \quad \text{and} \quad \frac{g^-(p_1) + g^-(p_2) + g^-(p_4)}{G^-(n)} \to 0.$$

Now let $0 \leqslant \eta < x_0$ and let M_η be as in the lines following (2.34). If $n - u - 2 = (1-\alpha)n + \mathrm{o}(n) \geqslant G^-(n)$, then the expression in (5.19) is at most

$$\left(n + \mathrm{o}(n)\right) P\left\{C_{(1-\alpha)n+\mathrm{o}(n)}\left(z + \mathrm{o}(1)\right)\right\} \leqslant \exp\left[\mathrm{o}(n) - (1-\alpha)n\sigma_\eta\left(z + \mathrm{o}(1)\right)\right],$$

by virtue of (2.35) and the fact that $r_1 + r_4 \leqslant n - u + g^-(p_2)$ (see (5.6) and (5.12)). The final estimate here for (5.19) remains valid even if $n \to \infty$ through a subsequence for which $n - u - 2 \leqslant G^-(n)$, for then we can simply estimate the probability in (5.19) by

$$1 \leqslant \exp\left[\mathrm{o}(n) - (1-\alpha)n\sigma_\eta(z)\right],$$

provided we take $\mathrm{o}(n) \geqslant 2G^-(n)\sigma_\eta(1)$. We conclude that for large n

$$\begin{aligned}&-\log P\left\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n,\ 1 \leqslant i \leqslant n,\ S_n < 0\right\}\\ &\quad \geqslant \mathrm{o}(n) + \alpha n \psi^-\left(y + \mathrm{o}(1)\right) + (1-\alpha)n\sigma_\eta\left(z + \mathrm{o}(1)\right).\end{aligned}$$

Before we take the limit over n we replace ψ^- by its left continuous modification

$$\psi_c^-(y) := \lim_{\varepsilon \downarrow 0} \psi^-(y - \varepsilon).$$

Since we already knew that ψ^- is continuous on $[0,1)$ (and in fact on $(-\infty,1)$ as one easily checks), ψ_c^- is continuous on $[0,1]$ and

$$\psi_c^-(y)=\psi^-(y) \quad \text{for } y<1 \quad \text{and} \quad \psi_c^-(1)\leqslant\psi^-(1).$$

Similarly we introduce

$$\sigma_c(z)=\begin{cases}\lim_{\varepsilon\downarrow 0}\sigma(z-\varepsilon) & \text{if } z\leqslant x_0,\\ \infty & \text{if } z>x_0.\end{cases}$$

For the same reasons as in the preceding lines, $\sigma_c(z)=\sigma(z)$ for $z<x_0$.

It follows from our estimates for (5.18) and (5.19) that for each fixed $0<\eta<x_0$

$$\begin{aligned}&\liminf_{n\to\infty}\frac{-1}{n}\log P\bigl\{R_n\geqslant nx,\ |X_i|\leqslant c_5n,\ 1\leqslant i\leqslant n,\ S_n\leqslant 0\bigr\}\\ &\quad\geqslant \inf\bigl[\alpha\psi_c^-(y)+(1-\alpha)\sigma_\eta(z)\bigr],\end{aligned}\tag{5.20}$$

where the inf is over

$$0\leqslant\alpha,y,z\leqslant 1,\quad \alpha y+(1-\alpha)z\geqslant x-\eta.$$

The last inequality here comes from $n\alpha=u+\mathrm{o}(n)$ and $uy=r_3$, whence $\alpha ny\geqslant r_3-\eta n/3$ for large n (see (5.12)), and similarly $(1-\alpha)nz\geqslant r_1+r_4-\eta n/3$ (note that $r_1+r_4\leqslant n-u+\mathrm{o}(n)$ by (5.6) and (5.12)), and finally, $r_1+r_3+r_4\geqslant r_1+r_2-2\geqslant nx-2$ (see (5.12) and (5.6)).

The inequality (5.20) is close to what we want in (5.3). In order to complete the estimate of piece i) we shall now show that for $x<1$

$$\liminf_{\eta\downarrow 0}\inf_{\substack{0\leqslant\alpha,y,z\leqslant 1\\ \alpha y+(1-\alpha)z\geqslant x-\eta}}\bigl[\alpha\psi_c^-(y)+(1-\alpha)\sigma_\eta(z)\bigr]\geqslant\rho^-(x).\tag{5.21}$$

(The definition of $\rho^-(x)$ is given in (5.1).) To see this, let $\alpha_\eta, y_\eta, z_\eta$ be such that for $\eta\downarrow 0$ along a suitable seqence we have

$$\begin{aligned}&\bigl[\alpha_\eta\psi_c^-(y_\eta)+(1-\alpha_\eta)\sigma_\eta(z_\eta)\bigr]\\ &\quad\to\liminf_{\eta\downarrow 0}\inf_{\substack{0\leqslant\alpha,y,z\leqslant 1\\ \alpha y+(1-\alpha)z\geqslant x-\eta}}\bigl[\alpha\psi_c^-(y)+(1-\alpha)\sigma_\eta(z)\bigr],\end{aligned}\tag{5.22}$$

and such that $\alpha_\eta y_\eta+(1-\alpha_\eta)z_\eta\geqslant x-\eta$. By going over to a subsequence we may assume that $(\alpha_\eta,y_\eta,z_\eta)\to(\alpha_0,y_0,z_0)$ for some α_0,y_0,z_0 with $\alpha_0y_0+(1-\alpha_0)z_0\geqslant x$. Since ψ_c^- is continuous on $[0,1]$ we have $\alpha_\eta\psi_c^-(y_\eta)\to\alpha_0\psi_c^-(y_0)$. Similarly, the monotonicity of σ and the definition of σ_η show that

$$\liminf_{\eta\downarrow 0}(1-\alpha_\eta)\sigma_\eta(z_\eta)\geqslant(1-\alpha_0)\sigma_c(z_0).$$

Thus, the liminf in the left hand side of (5.21) is at least

$$\alpha_0\psi_c^-(y_0)+(1-\alpha_0)\sigma_c(z_0),\tag{5.23}$$

for some $0 \leqslant \alpha_0, y_0, z_0 \leqslant 1$ with $\alpha_0 y_0 + (1-\alpha_0) z_0 \geqslant x$. We can and shall even assume that $\alpha_0 y_0 + (1-\alpha_0) z_0 = x$ because ψ_c^- and σ_c are nondecreasing (compare (3.39)). We complete the proof by showing that the expression (5.23) is at least as large as the right hand side of (5.21). A number of cases have to be distinguished.

If $\alpha_0 < 1$ and $z_0 > x_0$, then (5.23) equals ∞ and (5.21) certainly holds.

If $\alpha_0 = 1$, then we must have $y_0 = \alpha_0 y_0 + (1-\alpha_0) z_0 = x$. Then (5.23) is at least

$$\psi_c^-(y_0) = \psi_c^-(x) = \lim_{k\to\infty} \left\{ \frac{k}{k+1} \psi^- \left(\frac{x(k+1)}{k} \right) + \frac{1}{k+1} \sigma(0) \right\},$$

by the continuity of ψ^- on $[0,1)$. Note also that we proved $x_0 > 0$ in the lines following (2.23). This is at least as large as the right hand side of (5.21), so that (5.21) again holds.

If $z_0 < x_0$ and $y_0 < 1$, then the continuity of ψ^- and σ on $[0,1)$ and $[0,x_0)$, respectively, shows that (5.23) equals

$$\alpha_0 \psi^-(y_0) + (1-\alpha_0)\sigma(z_0),$$

which again is at least as large as the right hand side of (5.21).

If $z_0 < x_0,\ y_0 = 1,\ \alpha_0 < 1$, then we use that (5.23) is at least

$$\lim_{k\to\infty} \left\{ \alpha_0 \psi^- \left(1 - \frac{1-\alpha_0}{k} \right) + (1-\alpha_0)\sigma\left(z_0 + \frac{\alpha_0}{k} \right) \right\}.$$

Thus (5.21) again holds.

We still need to verify (5.21) when $\alpha_0 < 1, z_0 = x_0$. If $\alpha_0 < 1, z_0 = x_0 < y_0$, then (5.23) equals

$$\lim_{k\to\infty} \left\{ \left(\alpha_0 + \frac{1}{k} \right) \psi^- \left(y_0 - \frac{1}{k}(y_0 - x_0) \right) + \left(1 - \alpha_0 - \frac{1}{k} \right) \sigma \left(x_0 - \frac{1}{k}(y_0 - x_0) \right) \right\},$$

which again implies (5.21).

If $0 < \alpha_0 < 1, z_0 = x_0 \geqslant y_0$, then $1 > x = \alpha_0 y_0 + (1-\alpha_0) z_0 \geqslant y_0$ and therefore (5.23) equals

$$\lim_{k\to\infty} \left\{ \alpha_0 \psi^- \left(y_0 + \frac{1-\alpha_0}{k} \right) + (1-\alpha_0)\sigma\left(x_0 - \frac{\alpha_0}{k} \right) \right\}.$$

Finally, if $\alpha_0 = 0, z_0 = x_0$, then $z_0 = x_0 = x$ and $0 < x_0 = x < 1$. In this case (5.23) equals

$$\sigma_c(x_0) = \lim_{k\to\infty} \left\{ \frac{1}{k} \psi^- \left(x_0 + \frac{1}{k} \right) + \frac{k-1}{k} \sigma \left(x_0 - \frac{1}{k(k-1)} \right) \right\}.$$

This proves (5.21) in all cases and hence

$$\liminf_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n,\ 1 \leqslant i \leqslant n,\ S_n \leqslant 0\} \geqslant \rho^-(x), \tag{5.24}$$

under (5.4).

Essentially the same argument takes care of piece ii), that is, of $P\{R_n \geqslant nx, |X_i| \leqslant c_5 n, 1 \leqslant i \leqslant n, S_n > 0\}$. This will lead to

$$\liminf_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx,\ |X_i| \leqslant c_5 n, 1 \leqslant i \leqslant n,\ S_n > 0\} \geqslant \rho^+(x) \tag{5.25}$$

(see (5.1) for the definition of $\rho^+(x)$). We only need a small change in the definitions of τ_1 and τ_2. This time they have to be chosen in $[\lambda, \kappa]$ instead of in $[\kappa, \kappa']$. The definitions for this case should be

$$\tau_1 = \max\{i\colon \lambda \leqslant i \leqslant \kappa, S_i \leqslant 0\},$$

and on the event $\{\tau_1 = s, S_{s+1} = p_2\}$,

$$\tau_2 = \min\{i\colon\ S_i \geqslant S_{g^+(p_2)+n-s-1}\}.$$

We leave the details to the reader. (5.24), (5.25) and (5.5) together imply (5.3).

So far we have worked under the extra assumption (5.4). To illustrate the small changes needed when this fails, let us assume that $\chi^- = \infty$. We can no longer use (4.3) and (4.4) (or rather their analogues with positive and negative interchanged). However, if $\chi^- = \infty$, then $P\{X_i \leqslant -n\}$ goes to 0 faster than exponentially in n. Thus there exists a function $k(n)$ which is $o(n)$ and such that

$$P\{R_n \geqslant nx\} \leqslant 2P\{R_n \geqslant nx,\ c_5 n \geqslant X_i \geqslant -k(n), 1 \leqslant i \leqslant n\}.$$

In other words, we may replace the restriction $|X_i| \leqslant c_5 n$ in (5.5)–(5.10) by $c_5 n \geqslant X_i \geqslant -k(n)$. Now, if we want to estimate the analogue of the first piece, that is, $P\{R_n \geqslant nx,\ c_5 n \geqslant X_i \geqslant -k(n), 1 \leqslant i \leqslant n,\ S_n \leqslant 0\}$, we define τ_i as before, and note that if $X_i \geqslant -k(n)$ for $1 \leqslant i \leqslant n$, then necessarily $X_{\tau_i} \geqslant -k(n),\ i = 1, 2$, so that we may take $0 \leqslant p_1, p_2 \leqslant k(n)$ in (5.6) and $0 \leqslant p_3, p_4 \leqslant 2k(n)$ in (5.13)–(5.18). Instead of (4.3) and (4.4) we can now use (4.9) and the trivial estimates $P\{X_{s+1} = -p_1 - p_2\} \leqslant 1$, $P\{X_u = -p_3 - p_4\} \leqslant 1$. We can then take $g^-(p) = m^- p$ and replace (5.7)–(5.10) with the estimate

$$\begin{aligned}
&P\{R_n \geqslant nx,\ c_5 n \geqslant X_i \geqslant -k(n), 1 \leqslant i \leqslant n, \tau_1 = s,\ S_{\tau_1} = p_1,\\
&\quad S_{\tau_1+1} = -p_2,\ \kappa' = t, R[0, s-1] = r_1,\ R[s, n-1] = r_2,\ S_n \leqslant 0\}\\
&\quad \leqslant [c_2]^{-p_1-p_2} P\{R_{s+m^- p_1} \geqslant r_1,\ S_{s+m^- p_1} = 0\}\\
&\qquad \times P\{S_{m^- p_2} = -p_2,\ S_h \leqslant c_1,\ 0 \leqslant h \leqslant m^- p_2,\\
&\qquad\qquad S_j \leqslant 0,\ m^- p_2 \leqslant j \leqslant m^- p_2 + t - s - 1,\\
&\qquad\qquad S_{m^- p_2+t-s-1} \leqslant S_{m^- p_2+n-s-1} \leqslant 0, R_{m^- p_2+n-s-1} \geqslant r_2 - 1,\\
&\qquad\qquad c_5 n \geqslant X_i \geqslant -k(n), 1 \leqslant i \leqslant m^- p_2 + n - s - 1\}.
\end{aligned}$$

The later estimates can be changed similarly and we end up with (5.24) as before. Note that finiteness of χ^+ plays no role in estimating $P\{R_n \geqslant nx,\ c_5 n \geqslant X_i \geqslant -k(n), 1 \leqslant i \leqslant n,\ S_n \leqslant 0\}$, and similarly, the finiteness of χ^- is not needed to estimate $P\{R_n \geqslant nx,\ k(n) \geqslant X_i \geqslant -c_5 n, 1 \leqslant i \leqslant n,\ S_n > 0\}$. Therefore (5.3) holds (under (2.23)) as soon as $\chi^+ \chi^- > 0$. $\square$

6. Lower bound for $P\{R_n \geqslant nx\}$ in case (1.13)

We shall show in the next lemma, that under (1.13) the right hand side of (5.3) is also an upper bound for $\limsup_{n\to\infty}(-1/n)\log P\{R_n \geqslant nx\}$.

LEMMA 8. – *Assume that* $0 \leqslant x < 1$ *and that* (1.13) *and* (2.23) *hold. Then*

$$\limsup_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} \leqslant \psi^{\sharp}(x). \tag{6.1}$$

Proof. – The right hand side of (6.1) is finite because $\psi^{\sharp}(x)$ is bounded by $\min\{\psi^{+}(x), \psi^{-}(x)\}$ (see (5.2) for the definition of $\psi^{\sharp}$). Hence, for all $\eta > 0$ there exist $\alpha_\eta \in [0,1]$, $y_\eta < 1$, $z_\eta < x_0$ such that

$$\alpha_\eta y_\eta + (1-\alpha_\eta) z_\eta = x$$

and

$$\alpha_\eta \psi^{+}(y_\eta) + (1-\alpha_\eta)\sigma(z_\eta) \leqslant \eta + \psi^{\sharp}(x), \tag{6.2}$$

or the last inequality holds with ψ^{+} replaced by ψ^{-}. For the sake of argument assume that (6.2) holds for a certain η. It will suffice to show that for each such η

$$\limsup_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} \leqslant \alpha_\eta \psi^{+}(y_\eta) + (1-\alpha_\eta)\sigma(z_\eta). \tag{6.3}$$

If $\alpha_\eta = 1$, then $y_\eta = x$ and (6.3) certainly holds, because the left hand side here is at most $\psi^{+}(x)$ by (2.3). For the remainder of this proof we therefore may restrict ourselves to $\alpha_\eta < 1$.

Now fix a $z \in (z_\eta, x_0)$ and let $0 \leqslant r \leqslant \vartheta =$ the period of $\{S_n\}$. Then combine a cylinder path $S_1, S_2, \ldots, S_{\lceil \alpha_\eta n\rceil + r}$ for which

$$R\big[0, \lceil \alpha_\eta n\rceil + r - 1\big] \geqslant (\lceil \alpha_\eta n\rceil + r) y_\eta$$
$$\text{and} \quad 0 \leqslant S_i \leqslant S_{\lceil \alpha_\eta n\rceil + r}, \quad 1 \leqslant i \leqslant \lceil \alpha_\eta n\rceil + r,$$

with a circuit $S_{\lceil \alpha_\eta n\rceil + r}, \ldots, S_n$ for which

$$R\big[\lceil \alpha_\eta n\rceil + r, n-1\big] \geqslant \big(n - \lceil \alpha_\eta n\rceil - r\big) z$$
$$\text{and} \quad \sum_{i=\lceil \alpha_\eta n\rceil + r+1}^{n} X_i = 0, \quad \sum_{i=\lceil \alpha_\eta n\rceil + r+1}^{p} X_i \geqslant 0, \quad \lceil \alpha_\eta n\rceil + r + 1 \leqslant p \leqslant n.$$

In order for such a circuit to exist we must take r such that $n - \lceil \alpha_\eta n\rceil - r$ is divisible by ϑ. It is easy to see (compare the argument for (2.26)) that the combined path $S_1, \ldots, S_n$ will have

$$R_n \geqslant R\big[0, \lceil \alpha_\eta n\rceil + r - 1\big] + R\big[\lceil \alpha_\eta n\rceil + r, n-1\big] - 1$$
$$\geqslant (\lceil \alpha_\eta n\rceil + r) y_\eta + (n - \lceil \alpha_\eta n\rceil - r) z - 1.$$

Since $\alpha_\eta < 1$ and $z > z_\eta$, the right hand side here is, for large n, greater than $n[\alpha_\eta y_\eta + (1-\alpha_\eta) z_\eta] = nx$. It follows that

$$\begin{aligned}
&P\{R_n \geqslant nx\}\\
&\quad\geqslant P\big\{R_{\lceil\alpha_\eta n\rceil+r} \geqslant (\lceil\alpha_\eta n\rceil + r)y_\eta,\ 0 \leqslant S_i \leqslant S_{\lceil\alpha_\eta n\rceil+r}, 1 \leqslant i \leqslant \lceil\alpha_\eta n\rceil + r\big\}\\
&\qquad\times P\big\{R_{n-\lceil\alpha_\eta n\rceil-r} \geqslant (n - \lceil\alpha_\eta n\rceil - r)z,\\
&\qquad\qquad S_{n-\lceil\alpha_\eta n\rceil-r} = 0,\ S_p \geqslant 0,\ 0 \leqslant p \leqslant n - \lceil\alpha_\eta n\rceil - r\big\}\\
&\quad\geqslant P\big\{A^+_{\lceil\alpha_\eta n\rceil+r}(y_\eta)\big\}[n - \lceil\alpha_\eta n\rceil - r]^{-1} P\big\{C_{n-\lceil\alpha_\eta n\rceil-r}(z)\big\} \quad \text{(by (2.25))}.
\end{aligned}$$

It therefore follows from (2.3) and (2.20) that

$$\limsup_{n\to\infty} \frac{-1}{n} \log P\{R_n \geqslant nx\} \leqslant \alpha_\eta\psi^+(y_\eta) + (1-\alpha_\eta)\sigma(z).$$

(6.3) now follows by letting $z \downarrow z_\eta$, because $\sigma(\cdot)$ is continuous at $z_\eta < x_0$. □

7. Proof of Theorem 1

The limit $\lim_{n\to\infty}(-1)/n \log P\{R_n \geqslant n\}$ exists because

$$P\{R_{n+m} \geqslant n + m\} \leqslant P\{R_n \geqslant n\}P\{R_m \geqslant m\}.$$

$\lim_{n\to\infty}(-1)/n \log P\{R_n \geqslant nx\}$ also exists if $P\{X < 0\} = 0$ (or $P\{X > 0\} = 0$). Indeed in this case $P\{R_n \geqslant nx\} = P\{\widetilde{A}^+_n(x)\}$ (or $P\{\widetilde{A}^-_n(x)\}$, respectively), so that (1.2) is a consequence of (2.6) and (2.8). Thus we may assume that (2.23) applies. If (1.12) holds, then Lemma 6 proves (1.2) with ψ equal to ψ^* defined in (3.1). We therefore only have left the case $0 \leqslant x < 1$ under the assumptions (1.13) and (2.23). In this situation (1.2) holds by virtue of Lemmas 7 and 8, this time with ψ equal to $\psi^\sharp$ defined in (5.2).

It remains to prove the properties (1.3)–(1.7). We first prove (1.6) as a separate lemma.

LEMMA 9. – *Under the assumptions of Theorem* 1*,* $x \mapsto \psi(x)$ *is continuous on* $[0, 1]$*. This function is also convex if* (1.12) *holds.*

Proof. – Let us first assume that (2.23) holds. Then, as we saw in Lemma 1, $\psi^\pm(x) < \infty$ for $0 \leqslant x \leqslant 1$, and $\psi^\pm$ are nondecreasing, bounded and convex on $[0, 1]$ and continuous on $[0, 1)$. Clearly ψ^* (which is defined in (3.1)) is then also nondecreasing and bounded on $[0, 1]$. To show that ψ^* is convex, let $x_1, x_2, \gamma \in [0, 1]$ and let

$$(\alpha_1, y_1, z_1) \in L(x_1), \quad (\alpha_2, y_2, z_2) \in L(x_2), \tag{7.1}$$

where $L(x) = \{(\alpha, y, z) \in [0, 1]^3\colon \alpha y + (1-\alpha)z = x\}$. Then, by the definition of ψ^* and the convexity of $\psi^\pm$,

$$\begin{aligned}
&\psi^*(\gamma x_1 + (1-\gamma)x_2)\\
&\quad= \psi^*\big(\gamma\alpha_1 y_1 + (1-\gamma)\alpha_2 y_2 + \gamma(1-\alpha_1)z_1 + (1-\gamma)(1-\alpha_2)z_2\big)\\
&\quad\leqslant \big[\gamma\alpha_1 + (1-\gamma)\alpha_2\big]\psi^+\left(\frac{\gamma\alpha_1 y_1 + (1-\gamma)\alpha_2 y_2}{\gamma\alpha_1 + (1-\gamma)\alpha_2}\right)\\
&\qquad+ \big[\gamma(1-\alpha_1) + (1-\gamma)(1-\alpha_2)\big]\psi^-\left(\frac{\gamma(1-\alpha_1)z_1 + (1-\gamma)(1-\alpha_2)z_2}{\gamma(1-\alpha_1) + (1-\gamma)(1-\alpha_2)}\right)
\end{aligned}$$

$$\leqslant \gamma\alpha_1\psi^+(y_1) + (1-\gamma)\alpha_2\psi^+(y_2) + \gamma(1-\alpha_1)\psi^-(z_1) + (1-\gamma)(1-\alpha_2)\psi^-(z_2).$$

Now taking the inf over $\alpha_1, \alpha_2, y_1, y_2, z_1, z_2$ satisfying (7.1) gives

$$\psi^*(\gamma x_1 + (1-\gamma)x_2) \leqslant \gamma\psi^*(x_1) + (1-\gamma)\psi^*(x_2).$$

Thus if (2.23) holds, then ψ^* is convex on $[0, 1]$. If $P\{X < 0\} = 0$ (but $P\{X > 0\} > 0$), then $\psi^-(x) = \infty$ and $\psi^*(x) = \psi^+(x)$ for all $x \in [0, 1]$. By Lemma 1, ψ^* is still convex and bounded on $[0, 1]$ in this case. A similar argument applies when $P\{X > 0\} = 0$.

Essentially the same argument proves that ρ^+ and ρ^- defined in (5.1) are convex and bounded on $[0, 1)$ if (2.23) holds. (Note that $\rho^\pm \leqslant \psi^\pm$.)

Since ψ^* and $\rho^\pm$ are also nondecreasing we conclude from their convexity that all these functions are continuous on $[0, 1)$ when (2.23) holds. We already proved the same property for ψ^+ when $P\{X > 0\} > 0$ and for ψ^- when $P\{X < 0\} > 0$. But ψ equals ψ^* or $\psi^\pm$ or $\psi^\sharp = \min\{\rho^+, \rho^-\}$ (see (5.2)), depending on which of $P\{X > 0\}$ and $P\{X < 0\}$ is strictly positive, and on which of (1.12) or (1.13) holds. One easily checks from the above that in all cases $x \mapsto \psi(x)$ is bounded on $[0, 1]$ and continuous on $[0, 1)$. We also have $\psi(x) = \psi^*(x)$ convex on $[0, 1]$ if (1.12) holds. Finally, the proof that $\psi(x)$ is also continuous at $x = 1$ is the same as for (1.14) in [4]. □

Proof of the properties (1.3)–(1.5) and (1.7). – All but (1.7) are proven in exactly the same way as the corresponding results in Theorem 1 of [4]. If (1.12) holds, then we even know from the preceding lemma that ψ is convex on $[0, 1]$ and (1.7) is then immediate from (1.4) as in [4]. Thus here we only have to prove (1.7) under (1.13). We may also assume that there exist integers $a, b \geqslant 1$ such that $P\{X = -a\}P\{X = b\} > 0$ (i.e., (2.23) holds), for if this fails, then ψ equals ψ^+ or ψ^- and then ψ is convex again. Thus we are in the situation where $\psi = \psi^\sharp$ (see Lemmas 7 and 8). Unfortunately we only have a somewhat circuitous proof of (1.7) in this case.

To show that ψ is indeed strictly increasing on $[\pi, 1]$ we are first going to compute $\psi^\pm(0)$. We distinguish the following two subcases: i) $EX = 0$ and ii) $EX \neq 0$ (recall that we are now assuming (1.13), so that $P\{|X| \geqslant t\} \to 0$ exponentially fast in t, so that X has all moments). In case i) the random walk is recurrent, that is $\pi = 0$. Moreover, by the strong law of large numbers, $S_k/k \to 0$ a.s. Therefore, there exists a function $q(n)$ such that $q(n)/n \to 0$ and such that

$$P\{\max_{i \leqslant n} |S_i| \leqslant q(n)\} \geqslant \frac{1}{2}. \tag{7.2}$$

If $|S_i| \leqslant q(n)$ for $0 \leqslant i \leqslant n$, then we can insert $q(n)$ steps b before X_1 and also $2q(n)$ steps b after X_n. The resulting path of length $n + 3q(n)$ will have all its partial sums between 0 and $S_{n+3q(n)}$ and will therefore lie in $\widetilde{A}^+_{n+3q(n)}(0)$. Thus

$$P\{\widetilde{A}^+_{n+3q(n)}(0)\} \geqslant \frac{1}{2}[P\{X = b\}]^{3q(n)}, \tag{7.3}$$

and consequently $\psi^+(0) = 0$. The same argument can be used to show $\psi^-(0) = 0$ and $\sigma(0) = 0$. Hence $\rho^\pm(0) = 0$ and also $\psi^\sharp(0) = 0$. On the other hand, by (1.4),

$\psi^\sharp(x) = \psi(x) > 0$ for $x > 0$. As before with ψ, this shows that the convex functions $\rho^\pm$ are both strictly increasing on $[\pi, 1) = [0, 1)$. Since the minimum of two strictly increasing functions is strictly increasing, this implies (1.7) in subcase i).

We turn to subcase ii). For the sake of argument we assume that $EX > 0$. We claim that then

$$\psi^+(x) = 0 \quad \text{for } 0 \leqslant x < \pi. \tag{7.4}$$

Indeed, it is immediate from the strong law of large numbers that $\min_{k \geqslant 0} S_k$ is almost surely finite if $EX > 0$. By reversing the random walk this shows that also $\{\max_{k \leqslant n}(S_k - S_n) : n \geqslant 1\}$ is a tight family. Thus, there exists a constant c_7 such that

$$P\{-c_7 \leqslant S_k \leqslant S_n + c_7,\ 1 \leqslant k \leqslant n\} \geqslant \frac{1}{2}$$

for all $n \geqslant 1$. Since $R_n/n \to \pi$ a.s. (see [7] and [8, pp. 38–40]) this shows that for $x < \pi$ and n sufficiently large

$$P\bigl\{R_n \geqslant nx, -c_7 \leqslant S_k \leqslant S_n + c_7,\ 1 \leqslant k \leqslant n\bigr\} \geqslant \frac{1}{4}.$$

(7.4) follows from this as in the argument for (7.3) or (5.18).

On the other hand,

$$\begin{aligned} \psi^-(0) &= \lim_{n\to\infty} \frac{-1}{n} \log P\{0 \geqslant S_k \geqslant S_n,\ 1 \leqslant k \leqslant n\} \\ &\geqslant \lim_{n\to\infty} \frac{-1}{n} \log P\{S_n \leqslant 0\} > 0. \end{aligned} \tag{7.5}$$

All but the last inequality here are obvious. The last inequality follows from standard large deviation estimates (see Cramér's theorem in [2, Section 2.2.1], or (7.9) below). Unfortunately we need to go into more detail about the Cramér transform in order to show that in fact the first inequality in (7.5) is an equality in the present situation, that is,

$$\psi^-(0) = \lim_{n\to\infty} \frac{-1}{n} \log P\{S_n \leqslant 0\}. \tag{7.6}$$

To prove this we set

$$\Phi(\lambda) = E\bigl\{e^{\lambda X}\bigr\} = \int_{\mathbb{R}} e^{\lambda x}\, dF(x),$$

where F is the distribution function of X, and if $\Phi(\lambda) < \infty$, then we also define the distribution

$$dF^\lambda(x) = \frac{e^{\lambda x}\, dF(x)}{\Phi(\lambda)}.$$

In view of (1.13) $\Phi(\lambda) < \infty$ on some interval J which contains 0 in its interior. On J the expectation of F^λ, that is,

$$\frac{1}{\Phi(\lambda)} \int_{\mathbb{R}} x \mathrm{e}^{\lambda x} \, \mathrm{d}F(x),$$

is continuous and strictly increasing in λ (provided we allow the values $+\infty$ and $-\infty$ at the right and left hand endpoint, respectively, of J). Thus there either is a unique value of $\lambda_0 \in J$ such that this expectation at λ_0 is 0, or J has a finite left endpoint where the expectation of F^λ is still finite but strictly positive. In the latter case we take λ_0 equal to the left endpoint of J. Note that $EX > 0$ implies that $-\infty < \lambda_0 < 0$. In any case we define F^* as F^{λ_0} and let $X^*, X_i^*, i \geqslant 1$, be i.i.d. random variables with distribution F^*. We also take $S_0^* = 0$ and $S_n^* = \sum_{i=1}^n X_i^*$. It is easy to see that

$$\begin{aligned} &P\{S_n = r,\ |S_k| \leqslant q,\ 0 \leqslant k \leqslant n\} \\ &\quad = \mathrm{e}^{-\lambda_0 r}[\Phi(\lambda_0)]^n P\{S_n^* = r,\ |S_k^*| \leqslant q,\ 0 \leqslant k \leqslant n\}, \end{aligned} \tag{7.7}$$

and hence

$$P\{|S_k| \leqslant q, 0 \leqslant k \leqslant n\} \geqslant \mathrm{e}^{-|\lambda_0| q}[\Phi(\lambda_0)]^n P\{|S_k^*| \leqslant q,\ 0 \leqslant k \leqslant n\}. \tag{7.8}$$

In particular, by taking $q = \infty$ in (7.7),

$$P\{S_n \leqslant 0\} \leqslant \sum_{r \leqslant 0} \mathrm{e}^{-\lambda_0 r}[\Phi(\lambda_0)]^n.$$

Thus

$$\lim_{n\to\infty} \frac{-1}{n} \log P\{S_n \leqslant 0\} \geqslant -\log \Phi(\lambda_0). \tag{7.9}$$

In the other direction we claim that there exists a function $q(n)$ with $q(n)/n \to 0$ and for which

$$\lim_{n\to\infty} \frac{1}{n} \log P\{|S_k^*| \leqslant q(n),\ 0 \leqslant k \leqslant n\} = 0. \tag{7.10}$$

Once this is proved, with the help of (7.8), the same argument as used for (7.3) will yield

$$\begin{aligned} \psi^-(0) &= \lim_{n\to\infty} \frac{-1}{n + 3q(n)} \log P\{\widetilde{A}^-_{n+3q(n)}(0)\} \\ &\leqslant -\log \Phi(\lambda_0) - \lim_{n\to\infty} \frac{1}{n} \log P\{|S_k^*| \leqslant q(n),\ 0 \leqslant k \leqslant n\} + \lim_{n\to\infty} \frac{|\lambda_0| q(n)}{n} \\ &= -\log \Phi(\lambda_0), \end{aligned}$$

which, together with (7.5) and (7.9), proves the desired (7.6). Now, if $EX^* = 0$, then (7.10) follows from the strong law of large numbers as in (7.2). The remaining possibility is that $0 < EX^* < \infty$ and λ_0 is the left endpoint of J. That means that $\Phi(\lambda) = \infty$ for all $\lambda < \lambda_0$, whence

$$\liminf_{n\to\infty} \frac{-1}{n} \log P\{X^* = -n\} = 0.$$

Thus, by Lemma 4 with positive and negative interchanged, applied to X^*, it follows that there exists a function $g^-(p)$ and a constant c_1 such that $g^-(p)/p \to 0$ and as $p \to \infty$,

$$P\{S^*_{g^-(p)} = -p,\ c_1 \geqslant S^*_i \geqslant -p - c_1,\ 0 \leqslant i \leqslant g^-(p)\} \geqslant \mathrm{e}^{\mathrm{o}(p)}. \tag{7.11}$$

Finally, we can build a path which satisfies $\{|S^*_k| \leqslant q(n), 0 \leqslant k \leqslant n\}$, by putting together roughly $n/(g^-(p) + p/EX^*)$ pieces which go down $-p$ units in $g^-(p)$ steps to end in $[-p, -p/2]$, and then go back up to $[0, p/2]$ in at most $4p/EX^*$ steps along which the path has standard weak law of large numbers behavior. The details are as follows. For $k \geqslant 0$ we define τ_k by $\tau_0 = 0$ and

$$\tau_{k+1} = \inf\left\{i \geqslant \tau_k + g^-(p)\colon\ S_i - S_{\tau_k + g^-(p)} \in \left[0, \frac{p}{2}\right]\right\}, \quad k \geqslant 0.$$

Let μ be the smallest integer satsfisying $\tau_\mu \geqslant n$. Then we have that

$$\mu \leqslant \ell(n) := \left\lceil \frac{n}{g^-(p) + p/(4EX^*)} \right\rceil$$

if $p/(4EX^*) \leqslant \tau_{k+1} - \tau_k - g^-(p) \leqslant 2p/EX^*$ occurs for all $0 \leqslant k \leqslant \ell(n)$. Therefore we obtain for $2p + c_1 \leqslant q(n)$ that

$$\begin{aligned}
&P\{|S^*_k| \leqslant q(n),\ 0 \leqslant k \leqslant n\}\\
&\quad\geqslant P\{S^*_{\tau_k+g^-(p)} - S^*_{\tau_k} = -p,\ 0 \leqslant k \leqslant \mu - 1,\\
&\qquad\quad \mu \leqslant \ell(n),\ |S^*_h| \leqslant 2p + c_1,\ 0 \leqslant h \leqslant n\}\\
&\quad\geqslant P\bigg\{S^*_{\tau_k+g^-(p)} - S^*_{\tau_k} = -p,\ 0 \leqslant k \leqslant \ell(n) - 1,\\
&\qquad\quad \frac{p}{4EX^*} \leqslant \tau_{j+1} - \tau_j - g^-(p) \leqslant \frac{2p}{EX^*},\ 0 \leqslant j \leqslant \ell(n),\\
&\qquad\quad |S^*_h| \leqslant 2p + c_1,\ 0 \leqslant h \leqslant n\bigg\}.
\end{aligned}$$

Note that if $S^*_{\tau_k+g^-(p)} \in [-p, -p/2]$ and $|S^*_{\tau_k+g^-(p)+i} - S^*_{\tau_k+g^-(p)} - iEX^*| \leqslant p/8$ for $0 \leqslant i \leqslant \tau_{k+1} - \tau_k - g^-(p)$, then $p/(4EX^*) \leqslant \tau_{k+1} - \tau_k - g^-(p) \leqslant 2p/EX^*$. Therefore, the right hand side of the previous inequality is at least

$$\begin{aligned}
&P\{S^*_{\tau_k+g^-(p)} - S^*_{\tau_k} = -p,\\
&\quad |S^*_{\tau_k+g^-(p)+i} - S^*_{\tau_k+g^-(p)} - iEX^*| \leqslant \frac{p}{8},\ 0 \leqslant i \leqslant \tau_{k+1} - \tau_k - g^-(p),\\
&\quad -p - c_1 \leqslant S^*_h - S^*_{\tau_k} \leqslant c_1,\ \tau_k \leqslant h \leqslant \tau_k + g^-(p) \text{ for all } 0 \leqslant k \leqslant \ell(n)\}.
\end{aligned}$$

Consequently we have that

$$\begin{aligned}
P\{|S^*_k| \leqslant q(n), 0 \leqslant k \leqslant n\} \geqslant \bigg[&P\{S^*_{g^-(p)} = -p,\ c_1 \geqslant S^*_i \geqslant -p - c_1,\ 0 \leqslant i \leqslant g^-(p)\}\\
&\times P\left\{|S^*_i - iEX^*| \leqslant \frac{p}{8}, 0 \leqslant i \leqslant \frac{2p}{EX^*}\right\}\bigg]^{\ell(n)+1}.
\end{aligned}$$

The second factor in square brackets here tends to 1 as $p \to \infty$, by the strong law of large numbers (or even just the weak law of large numbers). Thus we can take any $p(n), q(n)$ such that $p(n), q(n) \to \infty$, $q(n)/n \to 0$ and $p(n) \leqslant q(n)/4$ to obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log P\{|S_k^*| \leqslant q(n),\ 0 \leqslant k \leqslant n\} \\
&\quad\geqslant \liminf_{p\to\infty} \frac{1}{g^-(p) + p/(4EX^*)} \\
&\qquad \times \log P\{S^*_{g^-(p)} = -p,\ c_1 \geqslant S_i^* \geqslant -p - c_1,\ 0 \leqslant i \leqslant g^-(p)\} \\
&\quad= 0
\end{aligned}$$

(by (7.11)). This proves (7.10), and hence (7.6), when $EX > 0$.

Recall that ϑ is the period of the random walk $\{S_k\}$. By the definition of $\{S_k^*\}$, this is also the period of $\{S_k^*\}$. A small variation on the last argument shows that even

$$\lim_{n\to\infty} \frac{1}{n\vartheta} \log P\{S^*_{n\vartheta} = 0\} = 0,$$

and hence, by (7.7),

$$\sigma(0) = \lim_{n\to\infty} \frac{-1}{n\vartheta} \log P\{S_{n\vartheta} = 0\} = -\log \Phi(\lambda_0) = \psi^-(0). \tag{7.12}$$

(This is more or less contained in Cramér's theorem and probably well known.)

Our next task is to show that

$$\psi^-(x) \geqslant \psi^-(0) - \lambda_0 x > \psi^-(0) \quad \text{for } 0 < x \leqslant 1. \tag{7.13}$$

Fortunately, this is almost immediate from (7.7). Indeed, if $0 > S_k \geqslant S_n$ for $1 \leqslant k \leqslant n$, then only the integers in $[S_n, -1]$ are possible for S_k, $1 \leqslant k \leqslant n-1$. Therefore, if also $R_n \geqslant nx$, then it must be the case that $-S_n + 1 \geqslant nx$. Therefore, by (7.7) with $q = \infty$,

$$P\{A_n^-(x)\} \leqslant P\{S_n \leqslant -nx + 1\} \leqslant [\Phi(\lambda_0)]^n \sum_{r \leqslant -nx+1} \mathrm{e}^{-\lambda_0 r},$$

and (7.13) follows (recall that $\lambda_0 < 0$).

It would simplify our proof if we could also show $\sigma(x) > \sigma(0)$ for $0 < x < x_0$, but we do not know how to do this in general. Let us summarize what we already know. In the case when ψ is equal to $\psi^\sharp$,

$$\psi^\sharp(x) = 0 \quad \text{for } 0 \leqslant x \leqslant \pi \quad \text{and} \quad \psi^\sharp(x) > 0 \quad \text{for } \pi < x \leqslant 1 \tag{7.14}$$

(see (1.3) and (1.4)). But from the definition of $\psi^\sharp$ given in (5.2) we see that $\psi^\sharp(x) > 0$ implies $\rho^+(x) > 0$ and $\rho^-(x) > 0$. In particular, this holds for $x > \pi$. Together with the convexity of ρ^+ and $\rho^+(0) = \min\{\psi^+(0), \sigma(0)\} = 0$ this implies

$$\rho^+ \text{ is strictly increasing on } [\pi, 1). \tag{7.15}$$

Similarly, the convexity of ψ^-, together with (7.13), shows that

$$\psi^- \text{ is strictly increasing on } [0,1]. \tag{7.16}$$

Further, from the definition of ρ^- and the fact that ψ^- and σ are nondecreasing, we see that ρ^- is nondecreasing and $\rho^-(x) \geqslant \rho^-(0) = \min\{\psi^-(0),\ \sigma(0)\} = \sigma(0)$. Moreover, the fact that ψ^- is strictly increasing shows that $\rho^-(x) > \rho^-(0) = \sigma(0)$ for $x > x_0$. The convexity of ρ^- then shows that ρ^- is strictly increasing on $[x_0, 1)$. In fact, we can say more. Let

$$x_1 := \sup\{z < x_0\colon\ \sigma(z) = \sigma(0)\}.$$

We claim that even

$$\rho^- \text{ is strictly increasing on } [x_1, 1). \tag{7.17}$$

In view of the preceding statement we only have to prove this if $x_1 < x_0$. By continuity of σ on $[0, x_0)$ we have $\sigma(x_1) = \sigma(0)$ in that case. Then, if $x = \alpha y + (1-\alpha)z > x_1$, it must be the case that $\alpha > 0$, $y \geqslant x > x_1$, $\psi^-(y) > \psi^-(0) = \sigma(0)$ or $\alpha < 1, z > x_1, \sigma(z) > \sigma(x_1)$. In either case $\rho^-(x) > \rho^-(0) = \sigma(0)$ for $x > x_1$, so that (7.17) indeed follows. Again, since the minimum of two strictly increasing functions is strictly increasing, it follows that ψ is strictly increasing on $[x_1 \vee \pi, 1)$. Of course, this also makes ψ strictly increasing on $[x_1 \vee \pi, 1]$, since ψ is nondecreasing. If $\pi \geqslant x_1$, this proves (1.7), so we may assume $\pi < x_1$ from now on.

Finally, let

$$x_2 = \inf\bigl\{x < x_1\colon\ \rho^+(x) = \sigma(x)\bigr\} \tag{7.18}$$

if the set on the right here is nonempty. We know from (7.4), (7.5) and (7.12) that $\sigma(0) > 0 = \rho^+(0)$. Thus if the set on the right in (7.18) is empty, then $\rho^+(x) < \sigma(x) = \sigma(0) \leqslant \rho^-(x)$ for $x < x_1$, and hence

$$\psi^\sharp(x) = \min\bigl\{\rho^+(x), \rho^-(x)\bigr\} = \rho^+(x) \quad \text{for } x < x_1. \tag{7.19}$$

But then $\psi^\sharp$ is strictly increasing on $[\pi, x_1]$, by (7.15), as well as on $[x_1, 1]$, by the preceding lines. In this case (1.7) holds. We therefore only have to consider the case when $x_2 < x_1$.

We conclude our proof of (1.7) by showing that $x_2 < x_1$ is impossible. To see this, recall that $\sigma(0) > 0$ by (7.12) and (7.5). As in (7.19) we have

$$\psi^\sharp(x) = \rho^+(x) \quad \text{for } x < x_2.$$

Moreover, by continuity of $\psi^\sharp$ and the convex functions ρ^+ on $[0,1)$ and of σ on $[0, x_0)$ we must have $\psi^\sharp(x_2) = \rho^+(x_2) = \sigma(x_2) > 0$ if $x_2 < x_1 \leqslant x_0$. In this case we would have also $x_2 > \pi$ by (7.14), and $\rho^+(x) > \rho^+(x_2) = \sigma(0)$ for any $x > x_2$, by (7.15). But the definition of ρ^+ implies that $\rho^+(x) \leqslant \sigma(x)$ for $x < x_0$, and hence $\sigma(x) \geqslant \rho^+(x) > \rho^+(x_2) = \sigma(x_2) = \sigma(0)$ for $x_2 < x < x_1$. This contradicts the definition of x_1.

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