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TAYLOR EXPANSION OF A POISSON MEASURE

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Abstract. Denote by $\mathcal{Y}(\varrho)$ the Poisson measure associated to a positive Radon measure ϱ on a locally compact space countable at infinity. If ϱ is bounded, $\mathcal{Y}(\varrho)$ can be expressed as a power series in ϱ. If ϱ becomes non-bounded this expansion keeps its sense at least for some $\mathcal{Y}(\varrho)$-integrable functions (Theorem). These functions can be explicitly characterized (Additional Remark).

A Poisson measure is a generalization of the Poisson process on the real line to arbitrary locally compact spaces countable at infinity. A Poisson process on a finite interval $I \subset \mathbb{R}$ is given by its jumping points $\tau_1, \dots \tau_N$ in I, where N is a random number. The probability that $N = n$ is equal to $c^n T^n e^{-cT}/n!$, where T is the length of the interval and c is the parameter describing the Poisson process, i.e. the mean frequency of jumping points. Given that the number N of jumping points is equal to n, the n jumping points are distributed independently and uniformly on the interval I. Be $\mathcal{f}(I)$ the topological sum

$$\mathcal{f}(I) = I^0 \,\dot\cup\, I^1 \,\dot\cup\, I^2 \,\dot\cup\, I^3 \,\dot\cup$$

where $I^0 = \{e\}$, $I^1 = I$, $I^2 = I \times I$, ..., and e is an arbitrary additional point. Be $f \geq 0$ a function on $\mathcal{f}(I)$, whose components $f_n : I^n \to \mathbb{R}_+$ are Lebesgue-measurable, then $E f(\tau_1, \dots, \tau_N)$ can be calculated and is equal to

$$E\, f(\tau_1, \dots, \tau_N) = \sum_{n=0}^{\infty} \mathrm{Prob}\{N = n\} \frac{1}{T^n} \int_{I^n} \cdots \int f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

or

$$E\, f(\tau_1, \dots, \tau_N) = e^{-cT} \Big(f(e) + \sum_{n=1}^{\infty} \frac{c^n}{n!} \int_{I^n} \cdots \int f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n \Big)$$

This formula can easily be extended to any compact space $\mathfrak{X}$ and to any positive measure ϱ on $\mathfrak{X}$. Be $f \geqslant 0$ a function on $\mathfrak{f}(\mathfrak{X})$, with the property that $f_n : \mathfrak{X}^n \to \mathbb{R}_+$ is $\varrho^{\otimes n}$ -measurable, then the application of the <u>Poisson measure</u> $p(\varrho)$ on f is defined by

$$(1) \quad \langle p(\varrho), f \rangle = e^{-\rho(\mathfrak{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\otimes n}, f_n \rangle$$

where $\varrho^{\otimes 0} = \delta_e$, the Dirac measure in e the unique point of $\mathfrak{X}_o$.

Now $\mathfrak{f}(\mathfrak{X})$ can be interpreted as the free monoid generated by $\mathfrak{X}$ with neutral element e, the product being defined by juxtaposition. $\mathfrak{f}(\mathfrak{X})$ is locally compact containing $\mathfrak{X}$ as a compact open subset. The measure ϱ on $\mathfrak{X}$ can be interpreted as a measure on $\mathfrak{f}(\mathfrak{X})$. The product in $\mathfrak{f}(\mathfrak{X})$ induces a convolution for measures. The n-th convolution power $\varrho^{\star n}$ of ϱ is exactly $\varrho^{\otimes n}$ carried by $\mathfrak{X}^n \subset \mathfrak{f}(\mathfrak{X})$. So the probability measure $p(\varrho)$ can be written

$$\langle p(\varrho), f \rangle = e^{-\varrho(\mathfrak{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varrho^{\star n}, f \rangle$$

or

$$p(\varrho) = e^{-\varrho(\mathfrak{X})} \exp_{\star} \rho$$

$$(2) \quad p(\varrho) = \exp_{\star} a(\varrho)$$

with

$$(2') \quad a(\rho) = \varrho - \varrho(\mathfrak{X})\delta_e = \int \varrho(dx)(\delta_x - \delta_e).$$

as δ_e is the unit element in the convolution algebra.

As $\varrho^{\otimes n}(dx_1, \dots, dx_n) = \varrho(dx_1) \dots \varrho(dx_n)$ is symmetric in $x_1, \dots, x_n$ only the symmetric part of f_n gives a contribution to the integral. So we can switch as well to $\mathfrak{f}_c(\mathfrak{X})$, the free commutative monoid generated by $\mathfrak{X}$. $p(\rho)$ can be defined by the same formula as a measure on $\mathfrak{f}_c(\mathfrak{X})$, formulae (2) and (3) hold as well. We denote by $\mathfrak{X}_c^k$ the compact open subspace of $\mathfrak{f}_c(\mathfrak{X})$ formed by the monomials of degree k.

Let $\mathcal{M}(\mathfrak{X})$ be the space of all positive measures on $\mathfrak{X}$ with the vague topology and let $\mathcal{M}_c(\mathfrak{X})$ be the subspace of positive counting measures, i.e. the space of all $\mu \in \mathcal{M}(\mathfrak{X})$ of the form

$$\mu = \sum_{j=1}^{n} \delta_{x_j}$$

$x_j \in \mathfrak{X}, j = 1, \dots, n$ and variable n. Of course $\mathcal{M}_c(\mathfrak{X})$ is a submonoid of the additive monoid $\mathcal{M}(\mathfrak{X})$. It can be proved that the application

$$(x_1, \dots, x_n) \in \mathfrak{f}_c(\mathfrak{X}) \longmapsto \delta_{x_1} + \dots + \delta_{x_n} \in \mathcal{M}_c(\mathfrak{X})$$

is a topological isomorphism. So $\rho(\rho)$ can be interpreted, as well, as a measure on $\mathcal{M}_c(\mathfrak{X})$ denoted by $\varphi(\rho)$ and $\varphi(\varrho)$ is given by

$$(4) \quad \langle \varphi(\varrho), f \rangle = e^{-\varrho(\mathfrak{X})} \Big(f(\sigma) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \varrho(dx_1) \cdots \varrho(dx_n) f(\delta_{x_1} + \dots + \delta_{x_n}) \Big)$$

$$(5) \quad \varphi(\varrho) = \exp_{*} \alpha(\varrho)$$

$$(5') \quad \alpha(\varrho) = \int \varrho(dx) \left(\vartheta_{\delta_x} - \vartheta_0 \right)$$

There 0 is the zero-measure, ϑ_{δ_x} signifies the Dirac measure on $\mathcal{M}(\mathfrak{X})$ in the point $\delta_x \in \mathcal{M}(\mathfrak{X})$ and ϑ_0 the Dirac measure on $\mathcal{M}(\mathfrak{X})$ in the point 0.

As $\mathcal{M}_c(\mathfrak{X})$ is a part of the dual of $C(\mathfrak{X})$ the space of all continuous real-valued function on $\mathfrak{X}$, a Fourier transform for measures on $\mathcal{M}_c(\mathfrak{X})$ can be defined. Be $\varphi \in C(\mathfrak{X})$, then the Fourier transform of $\varphi(\rho)$ in the point φ is given by the $\varphi(\rho)$-integral of the function $\mu \in \mathcal{M}_c(\mathfrak{X}) \longmapsto e^{i\langle \mu, \varphi \rangle}$

So

(6) $$\mathcal{Y}(\rho)^{\wedge}(\varphi) = \int \mathcal{Y}(\rho)(d\mu)\, e^{i\langle\mu,\varphi\rangle}$$
$$= \exp\, \mathcal{U}(\rho)^{\wedge}(\varphi)$$

(6') $$\mathcal{U}(\rho)^{\wedge}(\varphi) = \int \rho(dx)\,(e^{i\varphi(x)} - 1).$$

If $\mathfrak{X}$ becomes non-compact and ρ a non-bounded measure on $\mathfrak{X}$, then formulae (1) - (5) fail, but formula (6) keeps its sense. Consider the space $\mathcal{M}_c(\mathfrak{X})$ of all positive counting measures on $\mathfrak{X}$, i.e. the space of all measures of the form

$$\sum_{\iota\in I} \delta_{x_\iota}$$

where $(x_\iota)_{\iota\in I}$ is locally finite: only finitely many of the x_ι are contained in a compact subset of $\mathfrak{X}$. We assume the vague topology on $\mathcal{M}_c(\mathfrak{X})$. Then $\mathcal{M}_c(\mathfrak{X})$ can be considered as a part of the dual space of $C_0(\mathfrak{X})$, the space of all continuous real-valued functions on $\mathfrak{X}$ with compact support. If $\mathfrak{X}$ is countable at infinity and ρ a positive measure on $\mathfrak{X}$, then there exists a unique Radon measure $\mathcal{Y}(\rho)$ on $\mathcal{M}_c(\mathfrak{X})$ with the Fourier transform (cf. [2], [3])

(7) $$\mathcal{Y}(\rho)^{\wedge}(\varphi) = \exp \int \rho(dx)\,(e^{i\varphi(x)} - 1)$$

Further investigation shows that formula (2) may keep its sense as well. This can be seen by writing (2) in a more explicit way

$$\begin{aligned}\langle \mathcal{Y}(\rho), f\rangle &= f(e) + \int \rho(dx)\,(f(x) - f(e)) \\ &+ \frac{1}{2!}\iint \rho(dx_1)\rho(dx_2)\,(f(x_1,x_2) - f(x_1) - f(x_2) + f(e)) \\ &+ \frac{1}{3!}\iiint \rho(dx_1)\rho(dx_2)\rho(dx_3)\,\big(f(x_1,x_2,x_3) - f(x_1,x_2) \\ &\quad - f(x_1,x_3) - f(x_2,x_3) + f(x_1) + f(x_2) + f(x_3) - f(e)\big) \\ &+ \cdots\end{aligned}$$

In fact, the following theorem holds.

Theorem: Assume $\mathfrak{X}$ to be a locally compact space countable at infinity and ϱ a positive Radon measure on $\mathfrak{X}$. Let f be a function on $\mathcal{M}_c(\mathfrak{X})$ with the property: The functions

$$
\begin{aligned}
(8)\qquad g_0(e) &= f(0) \\
g_1(x) &= f(\delta_x) - f(0) \\
g_2(x_1, x_2) &= f(\delta_{x_1} + \delta_{x_2}) - f(\delta_{x_1}) - f(\delta_{x_2}) + f(0) \\
&\vdots \\
g_n(x_1, \dots, x_n) &= \sum_{I \subset \{1,2,\dots,n\}} (-1)^{n-|I|} f\Big(\sum_{i \in I} \delta_{x_i}\Big) \\
&\vdots
\end{aligned}
$$

are $\varrho^{\otimes n}$ -measurable on $\mathfrak{X}^n$ and

$$(9)\qquad \sum_{n=0}^{\infty} \frac{1}{n!} \langle \rho^{\otimes n}, |g_n| \rangle < \infty$$

Denote by μ_K the restriction of $\mu \in \mathcal{M}_c(K)$ to a compact subspace $K \subset \mathfrak{X}$ and suppose that $f(\mu_K) \to f(\mu)$ in $\mathcal{Y}(\rho)$ -measure for $K \uparrow \mathfrak{X}$ (that is the case if e.g. f is vaguely continuous). Then f is $\mathcal{Y}(\varrho)$ -integrable and

$$(10)\qquad \langle \mathcal{Y}(\rho), f \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \rho^{\otimes n}, g_n \rangle$$

In order to understand the theorem let us investigate the connection between f and the function $g_n, n = 0, 1, 2, \dots$

One finds

$$
\begin{aligned}
f(0) &= g_0(e) \\
f(\delta_x) &= g_0(e) + g_1(x) \\
f(\delta_{x_1} + \delta_{x_2}) &= g_0(e) + g_1(x_1) + g_1(x_2) + g_2(x_1, x_2) \\
&\vdots \\
f(\delta_{x_1} + \dots + \delta_{x_n}) &= \sum_{k=0}^{n} \sum_{i_1 < i_2 < \dots < i_k} g_k(x_{i_1}, \dots, x_{i_k}).
\end{aligned}
$$

Taking into account that the functions $g_k(x_1, \dots, x_k)$ are symmetric in their arguments $x_1, \dots, x_k$ observe

$$\sum g_1(x_i) = \langle \mu, g \rangle$$

$$\sum_{i<j} g_2(x_i, x_j) = \frac{1}{2} \iint \mu(d\xi_1)\mu(d\xi_2)\, g_2(\xi_1, \xi_2) - \frac{1}{2} \int \mu(d\xi)\, g(\xi, \xi)$$

$$\sum_{i<j<k} g_3(x_i, x_j, x_k) = \frac{1}{6} \iiint \mu(d\xi_1)\mu(d\xi_2)\mu(d\xi_3)\, g_3(\xi_1, \xi_2, \xi_3) - \frac{1}{2} \iint \mu(d\xi_1)\mu(d\xi_2)\, g_3(\xi_1, \xi_1, \xi_2) + \frac{1}{3} \int \mu(d\xi)\, g_3(\xi, \xi, \xi),$$

for $\mu = \delta_{x_1} + \dots + \delta_{x_n}$.

This leads to the assumption that any such sum can be expressed by μ. We begin with a well-known lemma from elementary algebra.

Lemma 1 (Newton). Let $\mathcal{R}[x_1, \dots, x_n]$ be the ring of polynomials in n commutative indeterminates over the rational numbers. Then the symmetric functions

$$\sigma_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} \cdots x_{i_k}$$

can be expressed as polynomials with rational coefficients of the power sums

$$\Delta_k = \sum_{j=1}^{n} x_j^k$$

These polynomials are independent of the number n of indeterminates and are given by the formal power series

$$1 + \sigma_1 \xi + \sigma_2 \xi^2 + \sigma_3 \xi^3 + \dots = \exp\left[\Delta_1 \xi - \Delta_2 \xi^2/2 + \Delta_3 \xi^3/3 - + \right].$$

Proof. We give the proof as it is very short and not very known. One has $(1 + x_1\xi)(1 + x_2\xi) \cdots (1 + x_n\xi) = 1 + \sigma_1\xi + \sigma_2\xi^2 + \dots + \sigma_n\xi^n$

and $1 + x_i\xi = \exp \log(1 + x_i\xi)$.

So

$$1+\sigma_1\xi+\sigma_2\xi^2+\cdots$$
$$=\exp\sum_{i=1}^{n}\log(1+x_i\xi)$$
$$=\exp\sum_{i=1}^{n}\sum_{k=1}^{\infty}(-1)^k\xi^k x_i^k/k$$
$$=\exp\sum_{k=1}^{\infty}(-1)^k\xi^k\Delta_k/k.$$

We recall the definition of $\mathfrak{f}_c(\mathfrak{X})=\sum_{k=0}^{\infty}\mathfrak{X}_c^k$ the free commutative monoid generated by $\mathfrak{X}$. If $\mathfrak{X}$ is locally compact, $\mathfrak{f}_c(\mathfrak{X})$ is locally compact, too. Any measure μ on $\mathfrak{X}$ can be considered as a measure on $\mathfrak{f}_c(\mathfrak{X})$. The convolution powers $\mu^{\star n}=\mu^n$ of μ are measures on $\mathfrak{X}_c^n$.

Denote the restriction to $\mathfrak{X}_c^n$ of a function g on $\mathfrak{f}_c(\mathfrak{X})$ by g_n, then

$$\langle\mu^n,g\rangle=\int\cdots\int\mu(dx_1)\cdots\mu(dx_n)\,g_n(x_1,\dots,x_n)$$

Another measure on $\mathfrak{f}_c(\mathfrak{X})$ carried by $\mathfrak{X}_c^n$ and related to μ is

$$\Delta_n(\mu):\ \langle\Delta_n(\mu),g\rangle=\int\mu(dx)\,g_n(x,\dots,x).$$

We define now a third measure $\mu^{(n)}$ on $\mathfrak{f}_c(\mathfrak{X})$ carried by $\mathfrak{X}_c^n$ by the formal power series

$$1+\mu^{(1)}\xi+\mu^{(2)}\xi^2+\cdots=\exp_\star\left(\Delta_1(\mu)\xi-\Delta_2(\mu)\xi^2/2+\Delta_3(\mu)\xi^3/3\mp\right)$$

<u>Lemma 2.</u> If $\mu=\delta_{x_1}+\cdots+\delta_{x_n}$ and if g is a function on $\mathfrak{X}_c^k$ then

$$\langle\mu^{(k)},g\rangle=\sum_{1\le i_1<i_2<\cdots<i_k\le n}g(x_{i_1},\dots,x_{i_k}).$$

<u>Proof.</u> Let $x_1,\dots,x_n\in\mathfrak{X}$. The application $x_i\longmapsto\delta_{x_i}$ can be extended to a homomorphism from $\mathbb{R}[x_1,\dots,x_n]$ into the convolution algebra of measures on $\mathfrak{f}_c(\mathfrak{X})$. The image of $x_1+\cdots+x_n$

is μ and the image of $\Delta_k = \sum x_i^{k}$ is $\sum (\delta_{x_i})^k = \Delta_k(\mu)$

as
$$\left\langle \sum (\delta_{x_i})^k, g \right\rangle = \sum_{i=1}^{m} g(x_i, .., x_i) = \langle \Delta_k(\mu), g \rangle$$

By lemma 1 the image of $\sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$ is $\mu^{(k)}$. This proves lemma 2.

Lemma 3. If μ is a counting measure, then $\mu^{(k)}$ is a positive measure on $\mathfrak{X}$.

Proof. If $g \geq 0$ of compact support, then $\langle \mu^{(k)}, g \rangle = \langle \mu_K^{(k)}, g \rangle$, if K is compact and contains the support of g. As μ_K is a finite counting measure, lemma 2 applies.

An immediate consequence of lemma 2 is

Lemma 4. On the assumptions of the theorem if μ is a finite counting measure

$$f(\mu) = g_0(e) + \langle \mu^{(1)}, g \rangle + \langle \mu^{(2)}, g \rangle + \cdots$$

If ρ is a bounded measure on $\mathfrak{X}$, then $\varphi(\rho)$ can be defined as in (5) and (5'). If $K \subset \mathfrak{X}$ is compact and μ a positive measure on $\mathfrak{X}$, its restriction to K will be denoted by μ_K. The measure can be considered as a bounded measure on $\mathfrak{X}$.

Lemma 5. For any compact $K \subset \mathfrak{X}$ the mapping $\mu \mapsto \mu_K$ is $\varphi(\rho)$ -measurable and the image of $\varphi(\rho)$ is equal to $\varphi(\rho_K)$.

Proof. We show at first that the mapping is measurable. Let $\mathcal{U}$ be an open neighborhood of K and let ψ be a continuous function $\mathfrak{X} \to [0,1]$ with compact support in $\mathcal{U}$ such that $\psi = 1$ on K. Then $\mu \mapsto \mu\psi$ is continuous and $\mu\psi = \mu_K$ if $\mu(\mathcal{U} - K) = 0$. But
$\varphi(\rho)\{\mu : \mu(\mathcal{U} - K) = 0\} = \exp(-\rho(\mathcal{U} - K))$.
So $\mu \mapsto \mu_K$ is continuous on the closed subset of all μ with $\mu(\mathcal{U} - K) = 0$, whose $\varphi(\rho)$ -measure approximates 1 if $\rho(\mathcal{U} - K)$ goes to zero.

The Fourier transform of the image is

$$\int \mathcal{g}(\rho)(d\mu)\, e^{i\langle \mu_K, \varphi\rangle} = \int \mathcal{g}(\rho)(d\mu)\, e^{i\langle \mu, \varphi_K\rangle}$$
$$= \exp\langle \rho, e^{i\varphi_K} - 1\rangle = \exp\langle \rho_K, e^{i\varphi} - 1\rangle$$
$$= \mathcal{g}(\rho_K)^{\wedge}(\varphi).$$

This proves the lemma.

Lemma 6. If g is a ρ^k -integrable function on $\mathfrak{X}_c^k$, then for $\mathcal{g}(\rho)$ -almost every μ the function g is $\mu^{(k)}$ -integrable. The function $\mu \mapsto \langle \mu^{(k)}, g\rangle$ is $\mathcal{g}(\rho)$-integrable and

$$\int \mathcal{g}(\rho)(d\mu)\, \langle \mu^{(k)}, g\rangle = \frac{1}{k!}\langle \rho^k, g\rangle.$$

Proof. Assume a continuous function $\varphi \geqslant 0$ on $\mathfrak{X}_c^k$ whose support is contained in K_c^k where $K \subset \mathfrak{X}$ compact. Then $\mu \in \mathcal{M}_c(\mathfrak{X}) \mapsto \langle \mu^{(k)}, \varphi\rangle$ ist continuous and $\geqslant 0$,

$$\int \mathcal{g}(\rho)(d\mu)\, \langle \mu^{(k)}, \varphi\rangle = \int \mathcal{g}(\rho)(d\mu)\, \langle \mu_K^{(k)}, \varphi\rangle$$
$$= e^{-\rho(K)} \sum_{n \geqslant k} \frac{1}{n!} \int \cdots \int \rho(dx_1)\cdots\rho(dx_n) \sum_{i_1 < i_2 < \cdots i_k} \varphi(x_{i_1}, \ldots, x_{i_k})$$
$$= \frac{1}{k!}\langle \rho^{(k)}, \varphi\rangle.$$

This formula extends to any continuous φ of compact support.

If $\varphi \geqslant 0$ is lower semi-continuous, there exists a net $\varphi_\iota \in C_0(\mathfrak{X}),\ \varphi_\iota \uparrow \varphi$.

So

$$0 \le \langle \mu^{(k)}, \varphi_\iota\rangle \uparrow \langle \mu^{(k)}, \varphi\rangle$$
$$\int \mathcal{g}(\rho)(d\mu)\, \langle \mu^{(k)}, \varphi_\iota\rangle \uparrow \int \mathcal{g}(\rho)(d\mu)\, \langle \mu^{(k)}, \varphi\rangle$$
$$\langle \rho^k, \varphi_\iota\rangle \uparrow \langle \rho^k, \varphi\rangle.$$

So $\mu \mapsto \langle \mu^{(k)}, \varphi \rangle$ is lower semi-continuous, its $\mathcal{G}(\rho)$ -integral is $1/k! \ \langle \rho^{k}, \varphi \rangle$ and φ is $\mu^{(k)}$ -integrable $\mathcal{G}(\rho)$ -a.e. if $\langle \rho^{k}, \varphi \rangle < \infty$.

Assume now that $\varphi \geq 0$ is a ϱ -null function. Then there exists a sequence of lower semi-continuous functions $\varphi_n \downarrow \tilde{\varphi} \geq \varphi$ such that $\langle \rho^{k}, \varphi_n \rangle \downarrow 0$. For $\mathcal{G}(\rho)$ -almost every μ the functions $\varphi_1, \varphi_2, \dots$ are $\mu^{(k)}$ -integrable and $\langle \mu^{(k)}, \varphi_n \rangle \downarrow \langle \mu^{(k)}, \tilde{\varphi} \rangle$ Therefore $\int \mathcal{G}(\rho)(d\mu) \langle \mu^{(k)}, \varphi_n \rangle \downarrow \int \mathcal{G}(\rho)(d\mu) \langle \mu^{(k)}, \tilde{\varphi} \rangle$ and $\tilde{\varphi}$ and φ are $\mu^{(k)}$ -null functions for a.e. μ .

Assume finally $\varphi \in L^1(\rho^{k})$. Then there exists a sequence $\varphi_n \in C_0(\mathcal{X}), \varphi_n \to \varphi$ ϱ^{k} - a.e. and $|\varphi_n| \leq \Psi$ where Ψ is lower semi-continuous and integrable. Then φ_n converges to φ $\mu^{(k)}$-a.e. for almost all μ . As $|\varphi_n| \leq \Psi$ and Ψ is $\mu^{(k)}$- integrable a.e., the function φ is $\mu^{(k)}$ -integrable a.e. and $\langle \mu^{(k)}, \varphi_n \rangle \to \langle \mu^{(k)}, \varphi \rangle$ a.e. The theorem of Lebesgue yields the end of the proof.

<u>Proposition.</u> Assume a sequence g_k, $k = 0, 1, 2, \ldots$ of ρ^{k} -integrable functions on $\mathcal{X}^{k}_{c}$ such that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \langle \rho^{k}, |g_k| \rangle < \infty .$$

Then the function

$$f(\mu) = \sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

is $\mathcal{G}(\rho)$ -almost everywhere defined and

$$\text{(11)} \quad \int \mathcal{G}(\rho)(d\mu) f(\mu) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \rho^{k}, g_k \rangle .$$

<u>Proof.</u> Immediate.

Proof of the theorem. By the assumption of the theorem and the proposition

$$\tilde{f}(\mu) = \sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

is $\varphi(\rho)$ -integrable and its integral is given by (11). By lemma 4 one has for any compact $K \subset \mathfrak{X}$

$$f(\mu_K) = \tilde{f}(\mu_K)$$

If $K \uparrow \mathfrak{X}$ which can be done by a sequence as $\mathfrak{X}$ is countable at infinity, $f(\mu_K) \to f(\mu)$ i. m. by assumption and $\tilde{f}(\mu_K) \to \tilde{f}(\mu)$ $\varphi(\rho)$ -almost everywhere. Hence $f(\mu) = \tilde{f}(\mu)$ $\varphi(\rho)$ -a.e. This proves the theorem.

Additional remark to the theorem. The function $f(\mu)$ is $\varphi(\rho)$-a.e. equal to the function

$$\sum_{k=0}^{\infty} \langle \mu^{(k)}, g_k \rangle$$

and $f(\mu_K)$ converges to $f(\mu)$ $\varphi(\rho)$ -almost everywhere.

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