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Dynamics of meromorphic maps with small topological degree III: geometric currents and ergodic theory

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DYNAMICS OF MEROMORPHIC MAPS WITH SMALL TOPOLOGICAL DEGREE III: GEOMETRIC CURRENTS AND ERGODIC THEORY

BY Jeffrey DILLER, Romain DUJARDIN AND Vincent GUEDJ

ABSTRACT. – We continue our study of the dynamics of mappings with small topological degree on projective complex surfaces. Previously, under mild hypotheses, we have constructed an ergodic "equilibrium" measure for each such mapping. Here we study the dynamical properties of this measure in detail: we give optimal bounds for its Lyapunov exponents, prove that it has maximal entropy, and show that it has product structure in the natural extension. Under a natural further assumption, we show that saddle points are equidistributed towards this measure. This generalizes results that were known in the invertible case and adds to the small number of situations in which a natural invariant measure for a non-invertible dynamical system is well-understood.

RÉSUMÉ. – Nous poursuivons notre étude de la dynamique des applications rationnelles de petit degré topologique sur les surfaces complexes projectives. Dans un travail précédent nous avons construit une mesure ergodique naturelle, dite « d'équilibre », sous des hypothèses très générales. Nous étudions maintenant en détail les propriétés dynamiques de cette mesure: nous donnons des bornes optimales pour ses exposants de Lyapounov, montrons qu'elle est d'entropie maximale et qu'elle a une structure produit dans l'extension naturelle. Sous une hypothèse supplémentaire naturelle, nous montrons que cette mesure décrit la répartition des points selles. Ceci généralise des résultats qui étaient auparavant connus dans le cas inversible et vient ainsi s'ajouter au petit nombre de situations où une mesure invariante naturelle pour un système dynamique non inversible est vraiment bien comprise.

Introduction

In this article we continue our investigation, begun in [7, 8], of dynamics on complex surfaces for rational transformations with small topological degree. Our previous work culminated in the construction of a canonical mixing invariant measure for a very broad class of such mappings. We intend now to study in detail the nature of this measure. As we will show,

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the measure meets conjectural expectations concerning, among other things, Lyapunov exponents, entropy, product structure in the natural extension, and equidistribution of saddle orbits.

Before entering into the details of our results, let us recall our setting. Let X be a complex projective surface (always compact and connected), and $f : X \to X$ be a rational mapping. Our main requirement is that f has *small topological degree*:

$$\lambda_2(f) < \lambda_1(f). \tag{1}$$

Here the topological (or second dynamical) degree $\lambda_2(f)$ is the number of preimages of a generic point, whereas the first dynamical degree $\lambda_1(f) := \lim \left\|(f^n)^*|_{H^{1,1}(X)}\right\|^{1/n}$ measures the asymptotic volume growth of preimages of curves under iteration of f. We refer the reader to [7] for a more precise discussion of dynamical degrees. In particular it was observed there that the existence of maps with small topological degree imposes some restrictions on the ambient surface: either X is rational (in particular, projective), or X has Kodaira dimension zero.

Let us recall that the ergodic theory of mappings with large topological degree ($\lambda_2 > \lambda_1$) has been extensively studied, and that results analogous to our Theorems B and C are true in this context [5, 11, 24]. In dimension 1, all rational maps have large topological degree, and in this setting these results are due to [19, 28]. We note also that in the birational case $\lambda_2 = 1$, the main results of this paper are obtained in [2, 3] (for polynomial automorphisms of $\mathbf{C}^2$) in [6] (for automorphisms of projective surfaces) and in [15] (for general birational maps). Hence the focus here is on noninvertible mappings which, as the reader will see, present substantial additional difficulties.

We will work under two additional assumptions, which we now introduce.

Good birational model

We need to assume that the linear actions $(f^n)^*$ induced by f^n on cohomology are compatible with the dynamics, i.e.

(H1) $$(f^n)^* = (f^*)^n, \text{ for all } n \in \mathbf{N}.$$

This condition, often called "algebraic stability" in the literature, was first considered by Fornaess and Sibony [18]. There is some evidence that for any mapping (X, f) with small topological degree, there should exist a birationally conjugate mapping $(\tilde{X}, \tilde{f})$ that satisfies (H1). Birational conjugacy does not affect dynamical degrees, so in this case we simply replace the given system (X, f) with the "good birational model" $(\tilde{X}, \tilde{f})$.

We observed in [7] that the minimal model for X is a good birational model when X has Kodaira dimension zero. For rational X, there is a fairly explicit blowing up procedure [9] that produces a good model when $\lambda_2 = 1$. More recently, Favre and Jonsson [17] have proven that each polynomial mapping of $\mathbf{C}^2$ with small topological degree admits a good model on passing to an iterate.

Under assumption (H1), in [7], we have constructed and studied canonical invariant currents T^+ and T^-. These are defined by

$$T^+ = \lim \frac{c^+}{\lambda^n}(f^n)^*\omega \text{ and } T^- = \lim \frac{c^-}{\lambda^n}(f^n)_*\omega$$

where ω is a fixed Kähler form on X and $c^{\pm}$ are normalizing constants chosen so that in cohomology $\{T^+\}\cdot\{T^-\} = \{\omega\}\cdot\{T^-\} = \{\omega\}\cdot\{T^+\} = 1$. A fact of central importance to us is that these currents have additional geometric structures: T^+ is laminar, while T^- is woven (see §1 below for definitions).

Finite energy

Let I^+ denote the indeterminacy set of f, i.e. the collection of those points that f "blows up" to curves; and let I^- denote the analogous set of points which are images of curves under f. The invariant current T^+ (resp. T^-) typically has positive Lelong number at each point of the extended indeterminacy set $I^+_\infty = \bigcup_{n\geq 0} f^{-n}I^+$ (resp. $I^-_\infty = \bigcup_{n\geq 0} f^n I^-$). Condition (H1) is equivalent to asking that the sets I^+_∞ and I^-_∞ be disjoint.

In order to give meaning to and study the wedge product $T^+\wedge T^-$, it is desirable to have more quantitative control on how fast these sets approach one another. This is how our next hypothesis should be understood:

(H2) f has finite dynamical energy.

We refer the reader to [8] for a precise definition of finite energy and its relationship with recurrence properties of indeterminacy points. In that article we proved the following theorem.

THEOREM A ([8]). – *Let f be a meromorphic map with small topological degree on a projective surface, satisfying hypotheses (H1) and (H2). Then the wedge product $\mu := T^+\wedge T^-$ is a well-defined probability invariant measure that is f-invariant and mixing. Furthermore the wedge product is described by the geometric intersection of the laminar/woven structures of T^+ and T^-.*

The notion of "geometric intersection" will be described at length in §1.

We can now state the main results of this article. Let us emphasize that they rely on the hypotheses (H1) and (H2) only through the conclusions of Theorem A. Taking these conclusions as a starting point, one can read the proofs given here independently of [7, 8].

THEOREM B. – *Let X be a complex projective surface and $f : X \to X$ be a rational map with small topological degree. Assume that f satisfies the conclusions of Theorem A. Then the canonical invariant measure $\mu = T^+\wedge T^-$ has the following properties:*

i. *For μ-a.e. p there exists a nonzero tangent vector e^s at p, such that*

$$\limsup_{n\to\infty}\frac{1}{n}\log|df^n(e^s(p))| \leq -\frac{\log(\lambda_1/\lambda_2)}{2}. \tag{2}$$

ii. *Likewise, for μ-a.e. p there exist a tangent vector e^u at p, and a set of integers $\mathbf{N}'\subset\mathbf{N}$ of density 1 such that*

$$\liminf_{\mathbf{N}'\ni n\to\infty}\frac{1}{n}\log|df^n(e^u(p))| \geq \frac{\log\lambda_1}{2}. \tag{3}$$

iii. *μ has entropy $\log\lambda_1$; thus it has maximal entropy and $h_{\rm top}(f) = \log\lambda_1$.*
iv. *The natural extension of μ has local product structure.*

In particular it follows from *iv.* and the work of Ornstein and Weiss [31] (see also Briend [4] for useful remarks on the adaptation to the noninvertible case) that the natural extension of μ has the Bernoulli property, hence μ is mixing to all orders and has the K property. A precise definition of local product structure will be given below in §8. This is the analogue of the balanced property of the maximal measure in the large topological degree case.

Let us stress that we do not assume that $\log \operatorname{dist}(\cdot, I^+ \cup C_f)$ is μ integrable (C_f denotes the critical set). This condition is usually imposed to guarantee the existence of Lyapunov exponents and applicability of the Pesin theory of non-uniformy hyperbolic dynamical systems. However, for mappings with small topological degree, it is known to fail in general (see [8, §4.4]). This contrasts with the large topological degree case, in which the maximal entropy measure integrates all quasi-psh functions.

When the Lyapunov exponents $\chi^+(\mu) \geq \chi^-(\mu)$ are well defined, then *(i)* and *(ii)* imply that

$$\chi^+(\mu) \geq \frac{1}{2} \log \lambda_1(f) > 0 > -\frac{1}{2} \log \lambda_1(f)/\lambda_2(f) \geq \chi^-(\mu),$$

hence the measure μ is hyperbolic. These bounds are optimal and were conjectured in [23].

In order to go further and relate μ to the distribution of saddle periodic points, we use Pesin theory and must therefore invoke the above integrability hypothesis.

Theorem C. – *Under the assumptions of Theorem B, assume further that*

(H3) $$p \mapsto \log \operatorname{dist}(p, I^+ \cup C_f) \in L^1(\mu),$$

where C_f is the critical set.

Then, for every n there exists a set $\mathcal{P}_n \subset \operatorname{Supp}(\mu)$ of saddle periodic points of period n, with $\#\mathcal{P}_n \sim \lambda_1^n$, and such that

$$\frac{1}{\lambda_1^n} \sum_{q \in \mathcal{P}_n} \delta_q \longrightarrow \mu.$$

Let Per_n be the set of all isolated periodic points of f of period n. If furthermore

- *f has no curves of periodic points,*
- *or $X = \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$,*

then $\#\mathrm{Per}_n \sim \lambda_1^n$, so that asymptotically nearly all periodic points are saddles.

This theorem was proved for birational maps by the second author in [15] (though the possibility of a curve of periodic points was overlooked there). It would be interesting to prove a similar result without using Pesin Theory (i.e. without assumption (H3)).

It would also be interesting to know when saddle points might lie outside $\operatorname{Supp}(\mu)$. One can easily create isolated saddle points by blowing up an attracting fixed point with unequal eigenvalues. We then get an infinitely near saddle point in the direction corresponding to the larger multiplier, whose unstable manifold is contained in the exceptional divisor of the blow-up. We do not know any example of a saddle point outside $\operatorname{Supp}(\mu)$ whose stable and unstable manifolds are both Zariski dense.

While the results in this paper parallel those in [15], new and more elaborate arguments are needed for non-invertible maps. In particular, we are led to work in the natural extension (e.g. for establishing *iii.* and *iv.* of Theorem B), in a situation where there is no symmetry between the preimages along μ (see the examples in §3). An interesting thing to note is that additional regularity for the potentials of the invariant currents does not help our arguments much.

Under an assumption similar to (H3), De Thélin and Vigny [38] have recently found an alternate way to compute entropy, using Yomdin's Theorem [39]. Together with the work [37], this leads to an alternate proof of the bounds on Lyapunov exponents (again, [37] requires (H3)). An advantage of their method is that it works in higher dimension. On the other hand they do not obtain local product structure nor the equidistribution of saddle orbits. In any case, it seems that computing the entropy of the mappings considered here is always a delicate issue.

It is also worth noting that there are few cases in which natural invariant measures for non-invertible differentiable dynamical systems have well understood natural extensions. Measures of maximal entropy for rational mappings with large topological degree (in one and several dimensions) afford one example. Another [25] is given by absolutely continuous invariant measures for interval maps.

An important remaining question is that of uniqueness of the maximal entropy measure. Let us comment a little bit on this point. The first difficulty is that another candidate measure of maximal entropy need not integrate $\log \operatorname{dist}(\cdot, I^+ \cup C_f)$, so that it is delicate to work with. Therefore it is reasonable to restrict the uniqueness problem to measures satisfying this assumption. However, even with this restriction, and even with the additional assumption that f is birational, the problem remains unsolved.

We now discuss applicability of our assumptions. For mappings on Kodaira zero surfaces, we have seen that we can always assume that (H1) is satisfied. It is actually not very hard to prove that (H2) and (H3) are also always true (see [7, Proposition 4.8] and [8, Proposition 4.5]). Thus Theorems B and C yield the following:

Corollary D. – *Let X be a complex projective surface of Kodaira dimension zero. Let $f : X \to X$ be a rational transformation with small topological degree. Then the conclusions of Theorems B and C hold for f.*

When X is rational our results apply notably to the case where f is the rational extension of a polynomial mapping of $\mathbf{C}^2$ with small topological degree. As noted above, Favre and Jonsson have proven that a slightly weaker variation of (H1) holds for f in a suitable compactification X of $\mathbf{C}^2$. As we show in [7, §4.1.1], this variation is sufficient for our purposes. More precisely, it is explained there that though (H1) holds only for some iterate f^k, the currents T^+ and T^- are actually invariant by f. Since (H2) depends on f only through the currents T^+ and T^-, and (H3) holds for f as soon as it holds for an iterate, we conclude from [8] [(1)] that (H2) and (H3), hence the conclusions of Theorem A, hold for f. Altogether this implies the following corollary.

Corollary E. – *Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial mapping with small topological degree. Then the conclusions of Theorems B and C apply to f.*

(1) [8] was accepted before [7], so statements about polynomial maps in that article are still given in terms of iterates.

Let us emphasize that the small topological degree assumption is needed here: the reader will find in [23] an easy example of a polynomial endomorphism $f : \mathbf{C}^2 \to \mathbf{C}^2$ with $\lambda_1(f) = \lambda_2(f) > 1$ such that $h_{top}(f) = 0$.

Regarding more general rational mappings on rational surfaces, we noted above there are grounds to suspect that (H1) holds generally, after suitable birational conjugation. Among maps satisfying (H1), there is no known example of a map that violates the energy condition (H2). We offer some evidence in [8] that (H2) might only fail in very degenerate situations [(2)]. On the other hand, we give examples [8, §4.4] of mappings satisfying (H1) and (H2) but not (H3). Nevertheless, it seems plausible that (H3) is generically satisfied (see [1, Prop. 4.5]).

The structure of this paper is as follows. §1 is devoted to some (mostly non-classical) preliminaries on geometric currents, while §2 recalls some well-known facts from Ergodic Theory. The proof of Theorem B occupies §3 to §8 (we give a more precise plan of the proof in §3). Finally §9 is devoted to the proof of Theorem C.

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1. Preliminaries on geometric currents

We begin by collecting some general facts about geometric, that is laminar and woven, currents. We often use the single word "current" as a shorthand for "positive closed (1,1) current on a complex surface."

1.1. Laminations and laminar currents

Recall that a *lamination by Riemann surfaces* is a topological space such that every point admits a neighborhood U_α homeomorphic (by ϕ_α) to a product of the form $\mathbf{D} \times \tau_\alpha$ (with coordinates (z, t)), where τ_α is some locally compact set, $\mathbf{D}$ is the unit disk, and such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are of the form $(h_1(z, t), h_2(t))$, with h_1 holomorphic in the disk direction z. By definition a *plaque* is a subset of the form $\phi_\alpha^{-1}(\mathbf{D} \times \{t\})$, and a *flow box* is a subset of the form $\phi_\alpha^{-1}(\mathbf{D} \times K)$, with K a compact set. A *leaf* is a minimal connected set L with the property that every plaque intersecting L is contained in L. An *invariant transverse measure* is given by a collection of measures on the transverse sets τ_α, compatible with the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$. The survey by Ghys [21] is a good reference for these notions. We always assume that the space is separable, so that is is covered by countably many flow boxes. In this paper we will consider "abstract" laminations as well as laminations embedded in complex surfaces. In the latter case we require of course that the complex structure along the plaques is compatible with the ambient one.

Two flow boxes embedded in a manifold are said to be *compatible* if the corresponding plaques intersect along open sets. Notice that disjoint flow boxes are compatible by definition. A *weak lamination* is a countable union of compatible flow boxes. It makes sense to

(2) Note added in proof: X. Buff has recently constructed examples of birational mappings of $\mathbb{P}^2$ satisfying (H1) but not (H2).

speak of leaves and invariant transverse measures on a weak lamination. Being primarily interested in measure-theoretic properties, we need not distinguish between ordinary and weak laminations in this paper.

Let us also recall that a $(1,1)$ current T is uniformly laminar if it is given by integration over an embedded lamination endowed with an invariant transverse measure. That is, the restriction $T|_{\phi^{-1}(\tau\times\mathbf{D})}$ to a single flow box can be expressed as

$$\int_\tau [\phi^{-1}(\{t\}\times\mathbf{D})]d\mu_\tau(t),$$

where μ_τ is the measure induced by the transverse measure on τ.

The current T is *laminar* if it is an integral over a measurable family of compatible holomorphic disks. Equivalently, for each $\varepsilon > 0$ there exists an open set $X_\varepsilon \subset X$ and a uniformly laminar current $T_\varepsilon \leq T$ in X_ε such that the mass (in X) of the difference satisfies $\mathbf{M}(T - T_\varepsilon) < \varepsilon$. It is a key fact that the laminar currents we consider in this paper have some additional geometric properties. For instance, each has a natural underlying weak lamination, and the lamination carries an invariant transverse measure. We refer the reader to [14, 15] for details about this.

Our main purpose in this section is to explore the related notion of *woven current* and generalize some results of [14] that we will need afterwards.

1.2. Marked woven currents

Given an open set Q in a Hermitian complex surface, we let $Z(Q,C)$ denote the set of (codimension 1) analytic chains with volume bounded by C. We endow $Z(Q,C)$ with the topology of currents. Since there is a dense sequence of test forms, this topology is metrizable. Most often we identify a chain and its support, which is a closed analytic subset of Q. We denote by $\Delta(Q,C) \subset Z(Q,C)$ the closure of the set of analytic disks in $Z(Q,C)$.

By definition, a *uniformly woven current* in Q is an integral of integration currents over chains in $Z(Q,C)$, for some C [10]. A *woven current* is an increasing limit of sums of uniformly woven currents.

A given laminar current can be expressed as an integral of disks in an essentially unique fashion (only reparameterizations up to a set of zero measure are possible [3, Lemma 6.5]). For woven (even uniformly) currents, this is not true. For example,

$$\omega = idz\wedge d\bar z + idw\wedge d\bar w = \frac{1}{2}id(z+w)\wedge d\overline{(z+w)} + \frac{1}{2}id(z-w)\wedge d\overline{(z-w)}.$$

That is, the standard Kähler form in $\mathbf{C}^2$ may be written as a sum of uniformly laminar currents in two very different ways.

We say that a uniformly woven current is *marked* if it is presented as an integral of disks. More specifically, a marking for a uniformly woven current T in some open set Q is a positive Borel measure $m(T)$ on $Z(Q,C)$ for some C such that

$$T = \int_{Z(Q,C)} [D]dm(T)(D).$$

Abusing terminology, we will also refer to $m(T)$ as the *transverse measure* associated to T. We call the support of $m(T)$ the *web supporting* T. The woven currents considered in this paper have the additional property of being *strongly approximable* (see §1.3 below for the definition). This allows us to show (Lemmas 1.1 and 1.2) that their markings are concentrated on $\Delta(Q, C)$.

We define *strong convergence* for marked uniformly woven currents as follows: T_n strongly converges to T if the markings $m(T_n)$ are supported in $Z(Q, C)$ for a fixed C and converge weakly to $m(T)$. We leave the reader to check that this implies the usual convergence of currents.

1.3. Markings for strongly approximable woven currents

The woven structure for the invariant current T^- has several additional properties, which play a crucial role in the paper. We say that a current T is a *strongly approximable woven current* if it is obtained as a limit of divisors $[C_n]/d_n$ whose geometric genus is $O(d_n)$ (here by definition the geometric genus of a chain is the sum of the genera of its components). See [7, §3] for a proof that T^- is of this type.

Its woven structure can be constructed as follows. We fix two generically transverse linear projections $\pi_i : X \to \mathbf{P}^1$, and subdivisions by squares of the projection bases. For each square $S \subset \mathbb{P}^1$, we discard from the approximating sequence $[C_n]/d_n$ all connected components $\pi_i^{-1}(S) \cap C_n$ over S which are not graphs of area $\leq 1/2$. The geometric assumption on C_n implies that the corresponding loss of mass is small. Taken together, the two projections divide the ambient manifold into a collection $\mathcal{Q}$ of "cubes" of size r and these partition the remaining well-behaved part of $[C_n]/d_n$ into a collection of uniformly woven currents $T_{Q,n}$ whose sum $T_{\mathcal{Q},n}$ closely approximates $[C_n]/d_n$. The disks constituting $T_{\mathcal{Q},n}$ will be referred to as "good components".

More specifically we define a *cube* to be a subset of the form $Q := \pi_1^{-1}(S_1) \cap \pi_2^{-1}(S_2)$, where the S_i are squares. Near a point where π_1 and π_2 are regular and the squares S_i are small enough, Q is actually biholomorphic to an affine cube. The woven current $T_{Q,n}$, or more precisely its marking $m(T_{Q,n})$, is defined by assigning mass $1/d_n$ to each 'good' component of $C_n \cap Q$. Then we have the mass estimate $\mathbf{M}\big(T_{\mathcal{Q},n} - [C_n]/d_n\big) = O(r^2)$, independent of n, where r is the size of the cubes.

There is a subtle point here. The number of disks constituting $C_n \cap Q$ might be much larger than d_n. Thus the masses of the measures $m(T_{Q,n})$ might not be bounded above uniformly in n. However, by Lelong's theorem [27] the disks intersecting a smaller subcube Q' have volume bounded from below, so there are no more than $\leq c(Q', Q)d_n$ of these. Restricting the marking to these disks gives rise to a new current, which we continue to denote by $T_{Q,n}$, that coincides with the old one on Q'. The mass of $m(T_{Q,n})$ is now locally uniformly bounded in $Z(Q, 1/2)$, and we may extract a convergent subsequence, letting $m(T_Q)$ denote the limiting measure and T_Q the corresponding current. Let $T_{\mathcal{Q}}$ be the sum of these currents, where Q ranges over the subdivision $\mathcal{Q}$. We then have that $\mathbf{M}\big(T_{\mathcal{Q}} - T\big) = O(r^2)$.

Finally, we consider an increasing sequence of subdivisions by cubes $\mathcal{Q}_i$ of size $r_i \to 0$, and by the previous procedure we obtain an increasing sequence of currents $T_{\mathcal{Q}_i}$, converging to T by the previous estimate. Notice that in the case of T^+ the construction is the same, except that at each step the disks constituting the $T_{\mathcal{Q},n}$ are disjoint so that the $T_{\mathcal{Q}}$ are uniformly laminar.

Since the approximating disks are restrictions of graphs over each projection, the marking of T_Q has the following additional virtues.

LEMMA 1.1. – *The multiplicity with which disks in* $\operatorname{Supp} m(T_{Q,n})$ *converge to chains belonging to* $\operatorname{Supp} m(T_Q)$ *is always equal to one. Similarly, if a chain* $D \in \operatorname{Supp}(m(T_Q))$ *is a Hausdorff limit of other chains* $D_n \in \operatorname{Supp}(m(T_Q))$*, then the multiplicity of convergence is equal to one. Thus there is no folding; i.e. in either cases the corresponding tangent spaces also converge. In particular the chains in* $\operatorname{Supp}(m(T_Q))$ *are non-singular.*

On the other hand, the Hausdorff limit of a sequence of disks might not be itself a disk but rather a chain with several components. As an example, consider the family of parabolas $w = z^2 + c$, $c \in \mathbf{C}$, each restricted to the (open) unit bidisk. For $|c| < 1$ the restriction has a single simply connected component. But when $|c| \to 1$, it becomes disconnected. The following observation will be useful to us in several places.

LEMMA 1.2. – *For generic subdivisons by squares of the projection bases of* π_1 *and* π_2*, we have that* $m(T_Q)$*-almost every chain is a disk (i.e. has only one component).*

Proof. – Recall that each chain in the support of $m(T_{Q,n})$ is obtained by intersecting a graph over some square in the base of e.g. π_1 with the preimage $\pi_2^{-1}(S)$ of a square in the base of π_2. So if a sequence of disks has a disconnected limit, then the limit must be a piece of a graph tangent to some fiber of π_2. However, a graph over π_1 that is not outright contained in a fiber of π_2 will be tangent to at most countably many fibers of π_2. Since we have uncountably many choices for the subdivision by squares associated to π_2, the result follows from standard measure theory arguments. □

1.4. Geometric intersection

Let $Q \subset \mathbf{C}^2$ be an open subset, and D, D' be two holomorphic chains in Q. We define $[D]\dot{\wedge}[D']$ as the sum of point masses, counting multiplicities, at isolated intersections of D and D'. Given more generally two marked uniformly woven currents T_1 and T_2, with associated measures m_1 and m_2, we define the *geometric intersection* as

$$T_1 \dot{\wedge} T_2 = \int [D_1]\dot{\wedge}[D_2]\,(dm_1 \otimes dm_2)\,(D_1, D_2).$$

In general it is necessary to take multiplicities into account because of the possibility of persistently non transverse intersection of chains. Notice that the definition depends not only on the currents but also on the markings.

The following basic proposition says that under reasonable assumptions the wedge product of uniformly woven currents is geometric.

PROPOSITION 1.3. – *If* T_1 *and* T_2 *are as above and if* $T_1 \in L^1_{\mathrm{loc}}(T_2)$*, then* $T_1 \wedge T_2 = T_1 \dot{\wedge} T_2$*.*

See [8, Prop. 2.6] for a proof. As a corollary, if $T_1 \in L^1_{\mathrm{loc}}(T_2)$, the geometric wedge product $T_1 \dot{\wedge} T_2$ is independent of the markings.

Since woven currents are less well-behaved than laminar ones, we do not try as in the laminar case [15] to give an intrinsic definition of the geometric intersection of woven currents. Instead we focus only on the particular situation that arises in this paper: we have strongly approximable currents T^+ (laminar) and T^- (woven) in X, whose geometric structures are obtained by extracting good components of approximating curves, as explained in the previous subsection. Let $\mathcal{Q}_i$ be the increasing sequence of subdivisions by cubes constructed as in §1.3, and let $T^{\pm}_{\mathcal{Q}_i}$ be the corresponding currents. Let us write $T^{+}_{\mathcal{Q}}\dot\wedge T^{-}_{\mathcal{Q}}$ for the sum of $T^{+}_{Q}\dot\wedge T^{-}_{Q}$ over $Q \in \mathcal{Q}$.

Under the finite energy hypothesis (H2) from the introduction, we have shown in [8] that the wedge product $T^{+} \wedge T^{-}$ is a well-defined probability measure. Though the currents $T^{\pm}_{\mathcal{Q}_i}$ depend on the choice of generic linear projections, of generic subdivisions for the projection bases [(3)] and of convergent subsequences extracted from $T^{\pm}_{\mathcal{Q}_i,n}$, the central result of [8, §2.3] is that the measures $T^{+}_{\mathcal{Q}_i}\dot\wedge T^{-}_{\mathcal{Q}_i}$ increase to $T^{+} \wedge T^{-}$ regardless. We summarize the facts that the limiting measure has the correct mass and is independent of choices by saying that *the intersection of T^{+} and T^{-} is geometric*.

A slightly delicate point is that in [8], $T^{+} \wedge T^{-}$ is not always defined in the usual $L^{1}_{\rm loc}$ fashion. In particular it is not clear whether $T^{+}_{Q} \in L^{1}_{\rm loc}(T^{-}_{Q})$ for $Q \in \mathcal{Q}_i$. Nevertheless [8, Lemma 2.10] implies that this is true after negligible modification of the markings. So throughout the paper we assume that for every i and every $Q \in \mathcal{Q}_i$, $T^{+}_{Q} \in L^{1}_{\rm loc}(T^{-}_{Q})$ so that by Proposition 1.3 we can always identify $T^{+}_{Q} \wedge T^{-}_{Q}$ and $T^{+}_{Q}\dot\wedge T^{-}_{Q}$.

We now prove the useful fact that the geometric product of uniformly woven currents is lower semicontinuous with respect to the strong topology. It can further be seen that, except for boundary effects, discontinuities can occur only when the chains have common components.

Lemma 1.4. – *Let $Q \Subset Q'$ be open sets and T_1, T_2 be marked, uniformly woven currents in Q with markings supported on $\Delta(Q', 1/2)$. Then*

$$(T_1, T_2) \longmapsto \int_Q T_1 \dot\wedge T_2$$

is lower semicontinuous with respect to the strong topology.

Proof. – Observe first that for $D, D' \in \Delta(Q', 1/2)$, the mass $\int_Q [D]\dot\wedge[D']$ on Q is lower semicontinuous in the strong topology. This is just a fancy way of saying that isolated intersections between D and D' persist under small perturbation.

Now for the general case, suppose that T^j_1, T^j_2 strongly converge to T_1, T_2, and let m^j_1, m^j_2 be the corresponding transverse measures. Then

$$\int_Q T^j_1 \dot\wedge T^j_2 = \int_{\Delta(Q',1/2)^2} \left(\int_Q [D]\dot\wedge[D'] \right) d(m^j_1 \otimes m^j_2).$$

Now by strong convergence, we have that $m^j_1 \otimes m^j_2$ converges to $m_1 \otimes m_2$ weakly; and we have just seen that the inner integral is lower semicontinuous in D, D'. Hence the lemma follows from a well-known bit of measure theory: if ν_j is a sequence of positive Radon measures with uniformly bounded masses, weakly converging to some ν, and if φ is any lower semicontinuous function, then $\liminf_{j\to\infty} \langle \nu_j, \varphi\rangle \geq \langle \nu, \varphi\rangle$. □

We will also need the following lemma.

Lemma 1.5. – *If L is a generic hyperplane section of X, the wedge product $T^{+} \wedge [f^k(L)]$ is well defined for all $k \in \mathbf{N}$ (in the L^1 sense) and gives no mass to points.*

(3) Here genericity is understood in the measure-theoretic sense.

Proof. – Since $[f^k(L)] = f^k_*[L]$ for any L disjoint from $I(f^k)$, it suffices by invariance of T^+ to prove the result for $k = 0$. Fix a projective embedding $X \hookrightarrow \mathbf{P}^\ell$. Then the hyperplane sections on X are parametrized by the dual $\mathbf{P}^{\ell *}$. If dv is the Fubini-Study volume on $\mathbf{P}^{\ell *}$, then the Crofton formula says that $\alpha := \int_{\mathbf{P}^{\ell *}} [L]\, dv(L)$ is the restriction to X of the Fubini-Study Kähler form on $\mathbf{P}^\ell$. Hence $T^+ \wedge [L]$ is well-defined for almost all L and $T^+ \wedge \alpha = \int T^+ \wedge [L]\, dv(L)$.

To show that $T^+ \wedge [L]$ does not charge points for generic L, we exploit laminarity [(4)]. Since α is smooth, we have that $T^+_{Q_i} \wedge \alpha$ increases to $T^+ \wedge \alpha$ as $i \to \infty$. Thus for almost any $[L]$, $T^+_{Q_i} \wedge [L]$ increases to $T^+ \wedge [L]$. In particular, $T^+_{Q_i} \wedge [L]$ is well-defined, and since $T^+_{Q_i}$ is a sum of uniformly laminar currents, the intersection is geometric. Since T^+ does not charge curves, $T^+_{Q_i}$ is diffuse and $T^+_{Q_i} \dot\wedge [L]$ therefore gives no mass to points. Hence neither does $T^+ \wedge [L]$. □

1.5. The tautological bundle

Let T be a marked uniformly woven current in Q, with associated measure $m(T)$ as above. The *tautological bundle* over $Z(Q, C)$ is the (closed) set

$$\check{Z}(Q, C) = \{(D, p),\ p \in \mathrm{Supp}(D)\} \subset Z(Q, C) \times Q.$$

Similarly we define the tautological bundle $\check{T}$ over T by restricting to $\mathrm{Supp}\, m(T)$. Defining the tautological bundle is a somewhat artificial way of separating the disks of T, which will nevertheless be quite useful conceptually.

In particular, when passing to the tautological bundle, the web supporting T becomes a weak lamination with transverse measure $m(T)$. When (as in our situation) there is no folding, we get a lamination.

Let $\check{\sigma}_T$ be the product of the area measure along the disks D with $m(T)$. Let $\check{\pi} : \check{T} \to Q$ be the natural projection. Then $\check{\pi}_* \check{\sigma}_T$ is the usual trace measure σ_T of T. For σ_T a.e. $p \in Q$, $\check{\sigma}_T$ induces a conditional measure $\check{\sigma}_T(\cdot|p)$ on the fiber $\check{\pi}^{-1}(p)$, which records as a measure the set of disks passing through p.

1.6. Analytic continuation property

The dynamical results in [15] depend on fine properties of strongly approximable uniformly laminar currents proved in [14]. The situation is similar here. Therefore we now state and sketch the proof of an "analytic continuation" property for curves subordinate to strongly approximable woven currents. This is a fairly straightforward extension of [14] (see in particular Remark 3.13 there). Hence our brief account closely follows the more detailed presentation in §3 of that paper.

Recall our notation from §1.3: we have a normalized sequence $[C_n]/d_n$ of curves with slowly growing genera converging to our current T, a generic linear projection $\pi : X \to \mathbf{P}^1$, and approximations $T_{\mathcal{Q}_i,n}, T_{\mathcal{Q}_i}$ of $[C_n]/d_n$ and T corresponding to a sequence of subdivisions $\mathcal{Q}_i$ of $\mathbf{P}^1$ into squares Q. Note that in order to better cohere with notation in [14], we are departing from our overall convention by letting Q denote a *square* in $\mathbf{P}^1$ rather than a cube in X. Likewise, the notation T_Q (resp. $T_{\mathcal{Q}}$, etc.) refers to a current in $\pi^{-1}(Q)$ that is a limit

(4) This fact does not seem to follow trivially from the fact that T^+ does not charge curves.

of families of good components of C_n over Q. We may assume that each fiber of π has unit area and intersects C_n in d_n points counting multiplicity.

Let $\mathcal{W}$ be a *web*, i.e. an arbitrary union of smooth curves in some open set [(5)]. For any square $Q \subset \mathbf{P}^1$, we define the restriction $T_Q|_{\mathcal{W}}$ by restricting the marking measure for T_Q to the set of disks contained in $\mathcal{W}$. We then take $T_{\mathcal{Q}_i}|_{\mathcal{W}} = \sum_{Q\in\mathcal{Q}_i} T_Q|_{\mathcal{W}}$ and define $T|_{\mathcal{W}}$ to be the increasing limit of $T_{\mathcal{Q}_i}|_{\mathcal{W}}$. Thus $T|_{\mathcal{W}}$ depends on the choice of marking. One can check, however, that it does not depend on the sequence of subdivisions $\mathcal{Q}_i$.

Here is the analytic continuation statement we will need.

THEOREM 1.6. – *$T|_{\mathcal{W}}$ is a uniformly woven current.*

Proof (sketch). – We assume without loss of generality that the leaves Γ_α of $\mathcal{W}$ are graphs over a disk U of area less than $1/2$. Note that this allows us to view a restriction $T_Q|_{\mathcal{W}}$ as the restriction $T_{Q,\mathcal{W}}|_Q$ of a marked uniformly woven current $T_{Q,\mathcal{W}}$ defined over all of U. Moreover, rather than work with a single sequence of subdivisions, we choose three such sequences $(\mathcal{Q}_i^j)_{i\in\mathbf{N}}$, $j = 1, 2, 3$ so that the set of all squares forms a neighborhood basis for $\mathbf{P}^1$. So given any $x \in \mathbf{P}^1$, we can choose squares Q decreasing to x and define the marked uniformly woven current $T_{x,\mathcal{W}}$ to be the increasing limit of the currents $T_{Q,\mathcal{W}}$. Given another uniformly woven current $S = \int \Gamma_\alpha\, ds(\alpha)$ supported on $\mathcal{W}$, we will say that T *strongly dominates* S over Q (resp. over x) if the marking for $T_{Q,\mathcal{W}}$ (resp. for $T_{x,\mathcal{W}}$) dominates ds.

Let us also recall the notion of "defect". If $Q \subset \mathbf{P}^1$ is a square, then the defect $\mathrm{dft}(Q, n)$ is the fraction of those components of C_n over Q that are bad. By strong convergence, $\mathrm{dft}(Q, n)$ converges to a limit $\mathrm{dft}(Q)$ as $n \to \infty$. Slow growth of genera implies that the total number of bad components over all squares in a subdivision is not larger than Cd_n. Hence the total defect $\sum_{Q\in\mathcal{Q}_i} \mathrm{dft}(Q)$ of $T_{\mathcal{Q}_i}$ is bounded uniformly in i. Then we can define the defect $\mathrm{dft}(x) = \lim_{Q\searrow x} \mathrm{dft}(Q)$ of T over a point $x \in \mathbf{P}^1$. The total defect bound becomes $\sum_{x\in\mathbf{P}^1} \mathrm{dft}(x) \leq 3C$. In particular, $\mathrm{dft}(x)$ is positive for only countably many $x \in X$.

The key fact underlying Theorem 1.6 is (compare [14, Prop. 3.10] and also [16])

LEMMA 1.7. – *Suppose that $x_0, x_1 \in \mathbf{P}^1$ are points satisfying $\mathrm{dft}(x_0) = \mathrm{dft}(x_1) = 0$. Then $T_{x_0,\mathcal{W}} = T_{x_1,\mathcal{W}}$ as marked uniformly woven currents.*

One proves this lemma by choosing a path γ from x_0 to x_1 along which T has zero defect at every point. Given $\varepsilon > 0$, one can then choose a finite cover of γ by squares $Q_0 \ni x_0, Q_1, \ldots, Q_N \ni x_1$ such that $\sum \mathrm{dft}(Q_j) < \varepsilon$, and such that the mass of $T_{Q_0,\mathcal{W}}$ is within ε of that of $T_{x_0,\mathcal{W}}$. Beginning with $T_{Q_0,\mathcal{W}}$ and Q_0, one then proceeds from Q_0 to Q_1 and so on, keeping only that part of $T_{Q_0,\mathcal{W}}$ supported on leaves that remain good over the new square. At each step one loses mass proportional to the defect over the squares involved. In the end, one arrives at Q_N with a marked uniformly woven current S strongly dominated by each of the $T_{Q_j,\mathcal{W}}$ but with the mass bound $\mathbf{M}(T_{Q_0,\mathcal{W}} - S) < C\varepsilon$. Letting ε decrease to zero, one then infers that $T_{x_1,\mathcal{W}}$ strongly dominates $T_{x_0,\mathcal{W}}$. By symmetry, we have equality.

From the lemma, we have a marked uniformly woven current $T_{\mathcal{W}} = T_{x,\mathcal{W}}$ that is independent of which zero defect point we use to define it. Since each square contains points with

(5) We will only need the statement for webs of smooth curves, but the case of singular $\mathcal{W}$ could be of interest.

zero defect, we have that $T_{\mathcal{W}}$ strongly dominates $T_{Q,\mathcal{W}}$ for every Q. In particular $T|_{\mathcal{W}} \leq T_{\mathcal{W}}$. On the other hand, the argument used to prove the lemma gives a slightly different statement: *if* Q *is a rectangle with* $\mathrm{dft}(Q) < \delta$, *then there is a marked uniformly woven current* S *strongly dominated by* $T_{Q,\mathcal{W}}$ *such that* $\mathbf{M}(T_{\mathcal{W}} - S) < C\delta$. To prove this, one applies the argument from the previous paragraph to the trivial path from some zero defect point $x_0 = x_1 \in Q$ "covered" by two squares Q_0, Q_1 such that $Q_1 = Q$ and $\mathrm{dft}(Q_0)$ is arbitrarily small.

Now we can choose a finite number of points $x_1, \ldots, x_N \in \mathbf{P}^1$ such that $\sum_{x \neq x_j} \mathrm{dft}(x) < \varepsilon$. Since neither T nor $T_{\mathcal{W}}$ charge fibers of π, the modified version of the lemma tells that $\sum_{Q \in \mathcal{Q}_i^j, x_j \notin Q} T_Q|_{\mathcal{W}}$, which is a woven current dominated by both $T|_{\mathcal{W}}$ and $T_{\mathcal{W}}$, is within mass $C\varepsilon$ of $T_{\mathcal{W}}$ when i is large. Shrinking ε, we infer $T|_{\mathcal{W}} = T_{\mathcal{W}}$ as desired. □

2. Preliminaries on ergodic theory

We now collect some standard facts from measurable dynamics that will be useful to us.

2.1. The natural extension

A good reference for this paragraph is the book of F. Przytycki and M. Urbański [33, Chapter 1].

The natural extension of a (non-invertible) measurable dynamical system (X, μ, f) is the (unique up to isomorphism) invertible system $(\hat{X}, \hat{\mu}, \hat{f})$ semiconjugate to (X, μ, f) by a projection $\hat{\pi} : \hat{X} \to X$ with the universal property that any other semiconjugacy $\varpi : Y \to X$ of an invertible system (Y, ν, g) onto (X, μ, f) factors through $\hat{\pi}$.

The natural extension $\hat{X}$ may be presented concretely as the space of *histories*, i.e. sequences $(x_n)_{n \leq 0}$ such that $f(x_n) = x_{n+1}$. Here $\hat{\pi} = \pi_0$ is the projection $((x_n)_{n \leq 0}) = x_0$ onto the 0^{th} factor; $\hat{f}$ is the shift map $(x_j) \mapsto (x_{j+1})$; and $\hat{\mu}$ is the unique $\hat{f}$-invariant measure such that $\hat{\mu}(\pi_0^{-1}(A)) = \mu(A)$. From this point of view, the factorization of a semiconjugacy $\eta : Y \to X$ by some other invertible system is easy to describe. For each $y \in Y$, we have $\varpi(y) = \hat{\pi} \circ \eta(y)$, where $\eta(y)$ is the sequence $(\varpi(g^n(y)))_{n \leq 0}$.

The natural extension preserves entropy: $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$. Also $(\hat{f}, \hat{\mu})$ is ergodic iff (f, μ) is.

Another characterization of $\hat{\mu}$ is the following. It corresponds to the presentation of $(\hat{X}, \hat{\mu}, \hat{f})$ as the inverse limit of the system of measure preserving maps $(\cdots X \xrightarrow{f} X \cdots)$. Consider the standard model of the natural extension, and denote by π_{-n} the projection on the $(-n)^{th}$ factor.

LEMMA 2.1. – *If* ν *is any probability measure on* $\hat{X}$ *such that for every* $n \geq 0$, $(\pi_{-n})_* \nu = \mu$, *then* $\nu = \hat{\mu}$.

Proof. – Let $C_{A_{-n}, \ldots, A_0} = \left\{ \hat{x} \in \hat{X}, \ \forall i \leq n, \ x_{-i} \in A_{-i} \right\}$ be a cylinder of depth n. One verifies easily that under the assumption of the lemma, $\nu(C_{A_{-n}, \ldots, A_0}) \leq \hat{\mu}(C_{A_{-n}, \ldots, A_0})$. From this we infer that $\nu \leq \hat{\mu}$, hence $\nu = \hat{\mu}$ by equality of the masses. □

2.2. Measurable partitions and conditional measures

We will use the formalism of measurable partitions and conditional measures, so we recall a few facts (see [3] for a short presentation, and [33, 35] for a more systematic treatment). Recall first that a *Lebesgue space* is a probability space isomorphic to the unit interval with Lebesgue measure, plus countably many atoms. All the spaces we will consider in the paper are Lebesgue. A *measurable partition* of a Lebesgue space is the partition into the fibers of some measurable function. If ξ is a measurable partition, a probability measure ν may be *disintegrated* with respect to ξ, giving rise to a probability measure $\nu_{\xi(x)}$ on almost every atom of ξ (the conditional measure). The function $x \mapsto \int \phi(y) d\nu_{\xi(x)}(y)$ is measurable and we have the following disintegration formula: for every continuous function ϕ,

$$\int \left(\int \phi(y) d\nu_{\xi(x)}(y) \right) d\nu(x) = \int \phi d\nu.$$

Conversely, the validity of this formula for all ϕ characterizes the conditional measures.

Given partitions ξ_i, we denote by $\bigvee \xi_i$ the *joint partition*, i.e. $\left(\bigvee \xi_i\right)(x) = \bigcap (\xi_i(x))$.

If π is a measurable map (possibly between different spaces) and ξ a measurable partition, we define the (measurable) partition $\pi^{-1}\xi$ by $(\pi^{-1}\xi)(p) = \pi^{-1}(\xi(p))$. We have the following easy lemma, whose proof is left to the reader.

LEMMA 2.2. – *Let $(\tilde{Y}, \tilde{\nu})$, (Y, ν) be two probability spaces with a measure preserving map $\pi : \tilde{Y} \to Y$. Assume that ξ is a measurable partition of Y, and denote by $\tilde{\xi}$ the measurable partition $\pi^{-1}(\xi)$.*

Then for $\tilde{\nu}$-a.e. $p \in \tilde{Y}$, $\pi_*(\tilde{\nu}_{\tilde{\xi}(p)}) = \nu_{\xi(\pi(p))}$.

3. Outline of proof of Theorem B

Before embarking to the proof, we give an overview of the main arguments leading to Theorem B. The proof of *i.* (§4) is based on the study of the contraction properties along disks subordinate to the laminar current T^+. This is delicate but fairly similar to [15], and it is achieved in §4.

The proof of *ii.* is in the same spirit but with many more differences. The fact that the current T^- is only woven leads to substantial difficulties, the first of which is that there is no natural web associated to such a current. This has been overcome by De Thélin in [36] who has given a short argument leading to *ii.*

Nevertheless to compute the entropy and establish local product structure we need a finer analysis of μ and its natural extension. We therefore take a longer path. For the bounds on both positive and negative exponents, we use an argument "à la Lyubich" that is suitable only for showing contraction. So for the positive exponent, we need to iterate backward. To make this possible, we first carefully select (§5) a set of *distinguished histories* which has full measure in the natural extension, and exhibits exponential contraction along disks. Then (§6) we use the tautological extension of T^- to construct a dynamical system $(\check{f}, \check{\mu})$ that refines (f, μ) by making disks subordinate to T^- disjoint. In particular, we may apply the theory of measurable partitions and conditional measures to $\check{f}$. However, we cannot compute the entropy of $\check{f}$ using the Rokhlin formula as in [15], because constructing invariant partitions requires invertible maps. For entropy we consider finally the natural extension of $\check{f}$. Happily, the natural extension of $\check{f}$ turns out to be isomorphic to that of f (§7; see also the figure on p. 263 for a synthetic picture of the relationship between these spaces). Hence $h(f,\mu) = h(\check{f},\check{\mu})$. Once we know that $h(f,\mu) = \log\lambda_1$, the fact that μ is a measure of maximal entropy follows from Gromov's inequality [22].

The product structure of the natural extension of μ follows from the above analysis and the analytic continuation property of the disks subordinate to strongly approximable woven/laminar currents (§8).

To understand some of the subtleties to come, we encourage the reader to consider the following simple examples.

Example 3.1. – Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be a monomial map with small topological degree, that is of the form $(z,w) \mapsto (z^a w^b, z^c w^d)$ where the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbf{Z})$ satisfies $|\det(A)| < \rho(A)$, e.g. $A = \left(\begin{smallmatrix} n & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with $n \geq 3$. In this case the dynamics of f is well known and the unique measure of maximal entropy is the Lebesgue measure on the unit torus $\{|z| = |w| = 1\}$, which is a *totally invariant* subset. Notice also that for this mapping, unstable manifolds do not depend on histories.

Example 3.2. – Let f_0 be a complex Hénon map of degree $d \geq 3$, of the form $f_0(z,w) = (aw + p(z), az)$. Consider now an integer $1 < e < d$ and for $b \ll a$ consider the map $f_b(z,w) = (aw + p(z), az + bz^e)$. It is easy to prove that f_b is algebraically stable in $\mathbf{C}^2$, with $\lambda_1 = d$ and $\lambda_2 = e$. Now there exists a large bidisk B where f_b is a Hénon-like map in the sense of [12], in particular injective. So it has a unique measure of maximal entropy $\log d$ in B, and it is easy to prove that this measure is actually $\mu = T^+(f_b) \wedge T^-(f_b)$, where $T^\pm(f_b)$ are the global currents constructed above.

So in this case most preimages of points in $\mathrm{Supp}(\mu)$ actually escape $\mathrm{Supp}(\mu)$ and therefore do not contribute to the dynamics in the natural extension. We see with this example that it is insufficient to prove a statement like: "for a disk $\Delta \subset f^n(L)$, we have at least $(1-\varepsilon)\lambda_2^n$ contracting inverse branches", since the remaining histories could have full measure in the natural extension. Our process of selecting distinguished inverse branches in Proposition 5.1 will give a way of identifying the "good" preimages.

In view of these examples, one may wonder whether there exists a rational map with small topological degree such that the number of preimages of points on the support of μ is a constant strictly between 1 and $\lambda_2(f)$.

Likewise, is there an example where the number of preimages of points on the support of μ is essentially non-constant? By "essentially" we mean that the degree does not only vary on a set of zero measure. Notice that this is not incompatible with ergodicity or mixing; indeed it is easy to construct unilateral subshifts of finite type with this property: consider for instance the subshift on two symbols $0 \mapsto 1$, $1 \mapsto 0$ or 1. This situation can also occur for basic sets of Axiom A endomorphisms.

4. The negative exponent

In this section we study the contraction properties along T^- to estimate the negative exponent.[6] The main lemma is the following. Recall that $T^+ \wedge \tau$ denotes the transverse measure induced by T^+ on τ.

LEMMA 4.1 (à la Lyubich). – *Let $\mathscr{L} = \{D_t,\ t \in \tau\}$ be a flow box subordinate to T^+. Then for every $\varepsilon > 0$, there exist a positive constant $C(\varepsilon)$ and a transversal $\tau(\varepsilon) \subset \tau$ such that $\mathbf{M}(T^+ \wedge \tau(\varepsilon)) \geq (1-\varepsilon)\mathbf{M}(T^+ \wedge \tau)$ and*

$$\forall n \geq 1,\ \forall t \in \tau(\varepsilon),\ \mathrm{Area}(f^n(D_t)) \leq \frac{C(\varepsilon) n^2 \lambda_2^n}{\lambda_1^n}.$$

Proof. – The method is similar to [15], though it requires substantial adaptation. We freely use the structure properties of strongly approximable laminar currents; the reader is referred to [14, 15] for more details.

It is enough to prove that for every α and every fixed n, the transverse measure of the set of disks D_t such that $\mathrm{Area}(f^n(D_t)) > \alpha$ is smaller than $\frac{\lambda_2^n}{\alpha \lambda_1^n}$. Indeed if this is the case, then for every $n \geq 1$ and every $c > 0$,

$$(T^+ \wedge \tau)\left(\left\{t,\ \mathrm{Area}(f^n(D_t)) > \frac{cn^2\lambda_2^n}{\lambda_1^n}\right\}\right) < \frac{1}{cn^2},$$

and it will suffice to sum over all integers n and adjust the constant c to get the conclusion of the lemma.

We first need to analyze the action of f on the transverse measure. This is a local problem. Recall that T^+ gives no mass to the critical set [7], and consider an open set U such that $f : U \to f(U)$ is a biholomorphism. If τ is a transversal to some flow box $\mathscr{L}$ contained in U, then $f(\tau)$ is a transversal to $f(\mathscr{L})$ and from the invariance relation $f^*T^+ = \lambda_1 T^+$ we infer that $\mathbf{M}(T^+ \wedge f(\tau)) = \lambda_1 \mathbf{M}(T^+ \wedge \tau)$.

Consider our original flow box $\mathscr{L}$ and fix n and α. Let $\tau_\alpha \subset \tau$ be the set of disks such that $\mathrm{Area}\, f^n(D_t) > \alpha$, and $\mathscr{L}_\alpha$ be the corresponding flow box. Sliding τ along the lamination and discarding a set of transverse measure zero if necessary, we can arrange that

- τ is contained in a holomorphic disk transverse to $\mathscr{L}$;
- $\tau \cap I(f^n) = \varnothing$ and $\tau \cap C(f^n)$ is a finite set;
- for every $p \in \tau_\alpha$, p is the unique preimage of $f^n(p)$ in τ_α.

We want to estimate the transverse measure of τ_α. We exhaust τ_α by compact subsets τ'_α, such that $\tau'_\alpha \cap C(f^n) = \varnothing$, and for simplicity rename τ'_α into τ_α. Observe that $f^n(\mathscr{L}_\alpha)$ is contained in T^+ so its total mass relative to T^+, $\mathbf{M}(T^+ \wedge f^n(\mathscr{L}_\alpha))$ is bounded by 1. On the other hand we will prove that

$$\mathbf{M}(T^+ \wedge f^n(\mathscr{L}_\alpha)) \geq \frac{\alpha \lambda_1^n}{\lambda_2^n} \mathbf{M}(T^+ \wedge \tau_\alpha),$$

hence giving the desired result. In contrast to the birational case ($\lambda_2 = 1$), we have to take into account the fact that $f^n(\mathscr{L}_\alpha)$ will overlap itself, in a way controlled by the topological degree λ_2^n.

[6] Lyapunov exponents are well-defined only when (H3) is satisfied: of course here we mean item *i.* of Theorem B.

To give the idea for the rest of the proof, we first consider a model situation: imagine a single disk D which admits a partition into λ_1^n pieces D_i such that $f^n(D_i)$ has area greater than α. Then $f^n(D) = \bigcup f^n(D_i)$ has area greater than $\alpha\lambda_1^n/\lambda_2^n$ because

$$\forall p \in f^n(D),\ \#\{i,\ p \in f^n(D_i)\} \leq \lambda_2^n.$$

The following computation is a "foliated" version of this counting argument.

If $p \in \mathscr{L}\backslash C(f^n)$, f^n is a biholomorphism in some neighborhood N of p. If D_p denotes the disk of $\mathscr{L}$ through p, which is subordinate to T^+, then $f^n(D_p \cap N)$ is a disk subordinate to T^+. Recall from [14] that disks subordinate to T^+ do not intersect (i.e. they overlap when they intersect), so there is an unambiguous notion of leaf subordinate to T^+ (union of overlapping subordinate disks), and a disk subordinate to T^+ is contained in a unique leaf. Notice that for every $t \in \tau_\alpha$, $D_t \cap C(f^n)$ is an at most countable number of points. Since $f^n(D_t)$ has area greater than α, we can remove a small neighborhood N of $C(f^n)$ such that for every t, $\mathrm{Area}(f^n(D_t \setminus N)) > \alpha$. We further assume that $N \cap \tau_\alpha = \varnothing$. Now if $p \in f^n(\mathscr{L} \setminus N)$, locally there is a unique disk through p, subordinate to T^+ (namely $f^n(D_q \cap N(q))$ where $q \in \mathscr{L} \setminus N$, is any preimage of p and $N(q)$ is a small neighborhood of q), and we conclude that $f^n(\mathscr{L} \setminus N)$ is a piece of lamination subordinate to T^+. If $p \in f^n(\mathscr{L} \setminus N)$, we denote by L_p the leaf subordinate to T^+ through p.

Consider the closed set $f^n(\tau_\alpha)$. This is locally a transversal to the lamination $f^n(\mathscr{L} \setminus N)$, but globally it can intersect a leaf many times. Its total transverse mass is $\lambda_1^n \mathbf{M}(T^- \wedge \tau_\alpha)$. Recall that every $p \in f^n(\tau_\alpha)$ admits a unique preimage $q \in \tau_\alpha$, and $f^n(D_q \setminus N)$ is a piece of a holomorphic curve through p of area greater than α, that we will denote by Δ_p. By construction, a point in $f^n(\mathscr{L} \setminus N)$ belongs to at most λ_2^n such Δ's.

Let $dA_L(x)$ denote area measure along the leaf L. We have

$$\begin{aligned}
(4) \qquad \alpha\lambda_1^n \mathbf{M}(T^+ \wedge \tau_\alpha) &= \alpha \mathbf{M}(T^+ \wedge f^n(\tau_\alpha)) \\
&\leq \int_{f^n(\tau_\alpha)} \mathrm{Area}(\Delta_p) d(T^+ \wedge f^n(\tau_\alpha))(p) \\
&= \int_{f^n(\tau_\alpha)} \left(\int_{L_p} \mathbf{1}_{\Delta_p}(x) dA_{L_p}(x) \right) d(T^+ \wedge f^n(\tau_\alpha))(p).
\end{aligned}$$

Now, take a partition of $f^n(\mathscr{L} \setminus N)$ into finitely many flow boxes, and in each flow box, project $f^n(\tau_\alpha)$ on a reference transversal τ_{ref}. For simplicity, we will assume that there is only one such flow box. The general case follows easily. We can decompose $f^n(\tau_\alpha)$, up to a set of zero transverse measure, into at most countably many subsets τ_i intersecting each leaf in a single point in the flow box, so that we get an injective map $h_i : \tau_i \to \tau_{\mathrm{ref}}$. Since the transverse measure is by definition invariant under holonomy, $(h_i)_*(T^+ \wedge \tau_i) = (T^+ \wedge \tau_{\mathrm{ref}})|_{h_i(\tau_i)}$. We can thus resume computation (4) as follows:

$$
\begin{aligned}
\int_{f^n(\tau_\alpha)}\int_{L_p}\mathbf{1}_{\Delta_p}(x)dA_{L_p}(x)d(T^+\wedge f^n(\tau_\alpha))(p) &= \sum_i\int_{\tau_i}\int_{L_p}\mathbf{1}_{\Delta_p}(x)dA_{L_p}(x)d(T^+\wedge\tau_i)(p)\\
&= \int_{\tau_{\mathrm{ref}}}\int_{L_q}\sum_i \mathbf{1}_{\Delta_{h_i^{-1}(q)}}(x)dA_{L_q}(x)d(T^+\wedge\tau_{\mathrm{ref}})(q)\\
&\leq \int_{\tau_{\mathrm{ref}}}\int_{L_q}\lambda_2^n\mathbf{1}_{\bigcup_i\Delta_{h_i^{-1}(q)}}(x)dA_{L_q}(x)d(T^+\wedge\tau_{\mathrm{ref}})(q)\\
&= \lambda_2^n\int_{\tau_{\mathrm{ref}}}\mathrm{Area}\left(\bigcup_i\Delta_{h_i^{-1}(q)}\right)d(T^+\wedge\tau_{\mathrm{ref}})(q)\\
&= \lambda_2^n\mathbf{M}_{T^+}(f^n(\mathcal{L}\setminus N))\leq\lambda_2^n,
\end{aligned}
$$

where the inequality on the third line comes from the fact that a given point belongs to at most λ_2^n disks Δ. We conclude that $\mathbf{M}(T^+\wedge\tau_\alpha)\leq\lambda_2^n/\alpha\lambda_1^n$ which was the desired estimate. □

Proof of item i. in Theorem B. – The conclusion now follows directly from [15, p. 236]. Here we give a simpler argument that avoids using the Birkhoff Ergodic Theorem. This shows that taking sets of density 1 in [15] was superflous.

Since μ is the geometric intersection of T^+ and T^-, we can fix a finite disjoint union $A=\mathcal{L}_1\cup\cdots\cup\mathcal{L}_N$ of flow boxes $\mathcal{L}_i$ for T^+ such that $\mu(\mathcal{L}_1\cup\cdots\cup\mathcal{L}_N)\geq 1-\varepsilon$. For each $p\in\mathcal{L}_i$ let D_p be the disk of $\mathcal{L}_i$ through p, and let $e^s(p)$ be the unit tangent vector to D_p at p. By the previous lemma, we may discard from A a set of plaques of small transverse measure (hence of small μ measure) to arrange that $\mathrm{Area}(f^n(D_p))\leq\frac{c(\varepsilon)n^2\lambda_2^n}{\lambda_1^n}$ for every $p\in A$ and *every* $n\geq 1$. By slightly reducing the disks (hence losing one more ε of mass) and applying the Briend-Duval area-diameter estimate [5], we further arrange that the diameter of $f^n(D_p)$ is controlled by $C(\varepsilon)n\lambda_2^{n/2}/\lambda_1^{n/2}$. Here, the constant depends on the geometry of the disks D_p, hence ultimately on ε since the disks in $\mathcal{L}_1\cup\cdots\cup\mathcal{L}_N$ have bounded geometry. So for n large enough, $f^n(D_p)$ is contained in a single coordinate chart of X, and we infer that the derivative $df^n\,(e^s(p))$ has norm controlled by $C(\varepsilon)n\lambda_2^{n/2}/\lambda_1^{n/2}$. Thus

$$
\limsup_{n\to\infty}\frac{1}{n}\log|df^n\,(e^s(p))|\leq-\frac{\log(\lambda_1/\lambda_2)}{2}\tag{5}
$$

for all $p\in A$. Letting $\varepsilon\to 0$, we obtain that the same inequality holds for μ-almost every p. □

5. Distinguished inverse branches

In this section we study the dynamics along the current T^-. This will lead to the estimate on the positive exponent and lay the groundwork for computing entropy and proving local product structure. Our goal is to prove the following assertion about backward iteration on which nearly all subsequent results depend.

PROPOSITION 5.1. – *For any $\varepsilon>0$, there exist a family $\mathcal{Q}$ of disjoint cubes, a current $T^-_Q\leq T^-$ (resp. $T^+_Q\leq T^+$) marked and uniformly woven (resp. uniformly laminar) in each $Q\in\mathcal{Q}$, and a set of distinguished histories such that*

i. $\mathbf{M}(T_Q^+ \dot\wedge T_Q^-) \geq 1-\varepsilon$;
ii. *the set of distinguished histories* (x_j) *with* $x_0 \in \operatorname{Supp} T_Q^+ \dot\wedge T_Q^-$ *has measure* $\geq 1-\varepsilon$ *in the natural extension* $\hat\mu$ *of* μ;
iii. *if* f_{-n} *is a distinguished inverse branch of* f^n *along a disk* D *in the web supporting* T_Q^-, *then the derivative of* f_{-n} *is controlled by* $C(\varepsilon)n/\lambda_1^{n/2}$;
iv. *for every disk* D *in the web supporting* T_Q^-, *there exists at least one compatible sequence of distinguished inverse branches along* D.

The proof of the bound in *iii.* originates in [15] and [36], but to allow for further applications, we undertake a more elaborate analysis of μ. The exact meaning of the word "distinguished" appearing throughout the statement will be made clear during the proof.

Proof of Proposition 5.1. – For a generic hyperplane section L, $S_k := \lambda_1^{-k}[f^k(L)]$ converges to T^-. As described earlier, we choose a generic subdivision $\widetilde{Q}$ by cubes and extract from $S_k^- := \lambda_1^{-k}[f^k(L)]$ a (uniquely marked) uniformly woven current $S^-_{k,\widetilde{Q}}$ which is the restriction of S_k to disks (graphs over one projection in Q) of area not greater than $1/2$.

Let ε be a small positive number. Fix the subdivision $\widetilde{Q}$ as above, together with a corresponding family Q of slightly smaller concentric cubes, homothetic to those of $\widetilde{Q}$ by a factor $(1-\delta)$. Then for an appropriate choice of $\widetilde{Q}$,

$$\mu(\widetilde{Q}\setminus Q) \leq 2(1-(1-\delta)^4). \tag{6}$$

See [13, Lemma 4.5] for a proof.

Now remove from $S^-_{k,\widetilde{Q}}$ all components not intersecting Q, and denote by $S^-_{k,Q}$ the remaining current. Observe that $S^-_{k,\widetilde{Q}}|_Q = S^-_{k,Q}|_Q$. As explained in §1.3, the currents $S^-_{k,Q}$ are marked by measures on $Z(\widetilde{Q},1/2)$ whose masses are bounded uniformly in k.

For reasons that will become clear in Theorem 6.1 below (roughly speaking, to get some invariance for the web supporting T^-), we do some further averaging, by setting $T_k^- = \frac{2}{k}\sum_{i=\lfloor k/2\rfloor}^k S_i^-$, where $\lfloor\cdot\rfloor$ denotes the integer part function. Observe that $T_k^- \to T^-$ as $k\to\infty$, and that restricting T_k^- to the good components (as defined in §1.3) of T_k^-, we get the uniformly woven current $T^-_{k,Q} = \frac{2}{k}\sum_{i=\lfloor k/2\rfloor}^k S^-_{i,Q}$.

Since the transverse measures of $T^-_{k,Q}$ have uniformly bounded mass, we may extract a strongly converging subsequence of $T^-_{k,Q}$. The limiting current T_Q^- is automatically uniformly woven (in each cube). We similarly construct currents T_Q^+ associated to backward images of a generic line, though in this case, the extra averaging step, while harmless, is not necessary.

Recall that the $T_Q^\pm$ are actually defined in the slightly bigger subdivision $\widetilde{Q}$. If the size of the cubes is small enough, then

$$\mathbf{M}\left(T^+\wedge T^- - (T_Q^+\dot\wedge T_Q^-)\right) \tag{7}$$

will be small. So in the following we fix $\widetilde{Q}\supset Q$ so that the sum of the error terms in (6) and (7), together with the loss of mass coming from a neighborhood of the set of points where the two projections are not transverse fibrations is less than $\varepsilon/10$. For technical reasons, we further require that $\mu(\partial Q)=0$.

By Lemma 1.5, we can define the measures $\mu_k = T^+ \wedge T^-_k$ and $T^+ \wedge T^-_{k,\mathcal{Q}}$ for every $k \geq 0$. The mass of μ_k tends to 1 for cohomological reasons. We claim that in fact $\mu_k \to \mu$ as $k \to \infty$. Indeed, the wedge product of uniformly woven currents is geometric, so from Lemma 1.4 we have for large k (depending only on $\mathcal{Q}$, hence on ε) that

$$\mathbf{M}\left((T^+ \wedge T^-_{k,\mathcal{Q}})|_{\mathcal{Q}}\right) \geq \mathbf{M}\left((T^+_{\mathcal{Q}} \wedge T^-_{k,\mathcal{Q}})|_{\mathcal{Q}}\right) = \mathbf{M}\left((T^+_{\mathcal{Q}} \dot{\wedge} T^-_{k,\mathcal{Q}})|_{\mathcal{Q}}\right) \geq 1 - \frac{2\varepsilon}{10}. \tag{8}$$

More generally, if k_j is any sequence such that $T^-_{k_j,\mathcal{Q}}$ strongly converges to some $T_{\mathcal{Q}}$, then $\lim T^+_{\mathcal{Q}} \dot{\wedge} T^-_{k_j,\mathcal{Q}} \geq T^+_{\mathcal{Q}} \dot{\wedge} T^-_{\mathcal{Q}}$. So any cluster value of μ_k is larger than $T^+_{\mathcal{Q}} \dot{\wedge} T^-_{\mathcal{Q}}$, whence $\mu_k \to \mu$.

Denote by $\mu_{k,\mathcal{Q}}$ the measure $T^+_{\mathcal{Q}} \wedge T^-_{k,\mathcal{Q}} = T^+_{\mathcal{Q}} \dot{\wedge} T^-_{k,\mathcal{Q}}$.

Backward contraction for $\mu_{k,\mathcal{Q}}$

Fix an integer $n \geq 1$. Then for generic L and any $k \geq n$, we have that $f^n : f^{k-n}(L) \to f^k(L)$ is 1-1 outside some finite set. So for every disk $D \hookrightarrow f^k(L)$, f^n admits a unique ("distinguished") inverse branch $f_{-n} : D \to f^{k-n}(L)$. Since the area of $f^k(L)$ is no greater than $C\lambda_1^k$ for some $C > 0$, we have

$$\#\left\{\text{plaques } D \text{ of } S^-_{k,\mathcal{Q}}, \text{ s.t. Area}(f_{-n}(D)) \geq \frac{An^2}{\lambda_1^n}\right\} \leq \frac{C\lambda_1^k}{An^2},$$

and thus

$$\#\left\{\text{plaques } D \text{ of } S^-_{k,\mathcal{Q}}, \text{ s.t. } \exists n \leq k, \text{Area}(f_{-n}(D)) \geq \frac{An^2}{\lambda_1^n}\right\} \leq \sum_{n=1}^{k} \frac{C\lambda_1^k}{An^2} \leq \frac{C'\lambda_1^k}{A}.$$

Discarding the plaques in the latter set from $S^-_{k,\mathcal{Q}}$, we get a new uniformly woven current, that we denote by $S^-_{k,\mathcal{Q}}(A)$. In terms of the transverse measure, we have removed at most $C'\lambda_1^k/A$ Dirac masses [(7)]. Hence

$$\mathbf{M}\left(m(S^-_{k,\mathcal{Q}}) - m(S^-_{k,\mathcal{Q}}(A))\right) \leq \frac{C}{A}. \tag{9}$$

We then form the current $T^-_{k,\mathcal{Q}}(A) = \frac{2}{k}\sum_{i=\lfloor k/2\rfloor}^{k} S^-_{i,\mathcal{Q}}(A)$ (which also satisfies (9)). Extracting a strongly convergent subsequence, we get a current $T^-_{\mathcal{Q}}(A) \leq T^-_{\mathcal{Q}}$. This family of currents increases to $T^-_{\mathcal{Q}}$ as A increases to infinity (technically we need to take a sequence of A's and a diagonal extraction), hence by geometric intersection $T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}}(A)$ increases to $T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}}$. We fix A so that $M(T^+_{\mathcal{Q}} \wedge T^-_{\mathcal{Q}}(A)) \geq 1 - \varepsilon/2$. Relabeling, we now use $T^-_{\mathcal{Q}}$ to denote $T^-_{\mathcal{Q}}(A)$ and $T^-_{k,\mathcal{Q}}$ to denote $T^-_{k,\mathcal{Q}}(A)$. We will show that the conclusions of Proposition 5.1 hold for this current $T^-_{\mathcal{Q}}$ and the current $T^+_{\mathcal{Q}}$ constructed above.

If n is fixed, then as soon as $k/2 \geq n$, the "distinguished" preimage of each plaque of $T^-_{k,\mathcal{Q}}$ under f^n has small area by construction. By lower semicontinuity (Lemma 1.4), we get that for large k, $\mathbf{M}(\mu_{k,\mathcal{Q}}) \geq 1 - \varepsilon/2$.

From the area-diameter estimate of [5] and the Cauchy estimates, we see that the modulus of the derivative of f_{-n} along the plaques of $T^-_{k,\mathcal{Q}}$ is small in $\mathcal{Q}$. To be more specific, if $Q \in \mathcal{Q}$ and D is a plaque of $T^-_{k,\mathcal{Q}}$ in $\widetilde{\mathcal{Q}}$, the modulus of the derivative of f_{-n} along $D \cap Q$ will be

(7) It is important here that the transverse measure is a sum of Dirac masses on disks, not on arbitrary chains.

controlled by $C(\varepsilon)n/\lambda_1^{n/2}$. Indeed, recall that D has area $\leq 1/2$ so by a simple compactness argument, the moduli of annuli surrounding the connected components of $D \cap Q$ in D are bounded from below by a constant depending only on the geometry of Q and $\widetilde{Q}$, hence ultimately on ε. The estimate on the derivative then follows from the original estimate of Briend and Duval.

Distinguished inverse branches

For each $Q \in \mathcal{Q}$ and each plaque D of T_Q^- in Q, D is the Hausdorff limit of a sequence of disks D_k with the property that for every $1 \leq n \leq k/2$, f^n admits a *natural* inverse branch f_{-n} over D_k, with a uniform control on the derivative along D_k, of the form

$$|d(f_{-n}|_{D_k})| \leq C(\varepsilon)n/\lambda_1^{n/2}. \tag{10}$$

As already explained (see the examples in §3) it will be important to isolate a set of "meaningful" histories in the natural extension. Here is the crucial definition.

Definition 5.2. – *Let D be a disk subordinate to T_Q^-. We call an inverse branch f_{-n} of f^n along D* distinguished *if there exists a sequence of disks D_k subordinate to $T_{k,Q}^-$ converging to D such that the natural branches $f_{-n}|_{D_k}$ converge normally to $f_{-n}|_D$.*

We say that a history $(x_{-j})_{0 \leq j \leq n}$ of length n is $\mathcal{Q}$-distinguished *if $x_0 \in \operatorname{Supp}(T_Q^+ \dot{\wedge} T_Q^-)$ and $x_{-n} = f_{-n}(x_0)$ for some distinguished inverse branch f_{-n}. An infinite history $(x_{-i})_{i \geq 0}$ is $\mathcal{Q}$-distinguished if all its subhistories of length n are $\mathcal{Q}$-distinguished.*

When there is no danger of confusion we will omit the '$\mathcal{Q}$' in $\mathcal{Q}$-distinguished; more generally, "distinguished" will stand for "$\mathcal{Q}$-distinguished for some $\mathcal{Q}$". Taking normal limits of the $f_{-n}|_{D_k}$, we see that f^n admits distinguished inverse branches on every disk subordinate to T_Q^-. By diagonal extraction, we further find that every $x_0 \in \operatorname{Supp}(T_Q^+ \dot{\wedge} T_Q^-)$ admits a distinguished (full) history. Moreover, attached to every $\mathcal{Q}$-distinguished history of x_0, there is a disk $D \ni x_0$ subordinate to T_Q^- and a sequence of distinguished inverse branches $f_{-n}|_D$, with $f_{-n}(x_0) = x_{-n}$, compatible in the sense that $f \circ f_{-n-1} = f_{-n}$. Since the estimate in (10) is uniform in k we have the estimate $|d(f_{-n}|_D)| \leq C(\varepsilon)n/\lambda_1^{n/2}$. This proves items *iii.* and *iv.* of Proposition 5.1.

It remains to show that distinguished histories are overwhelming in the natural extension of μ. Let $\hat{X}_{\mathcal{Q}}^{\mathrm{dist}} \subset \hat{X}$ be the set of distinguished histories (x_j) of points $x_0 \in \operatorname{Supp}(\mu_{\mathcal{Q}})$; likewise, let $\hat{X}^{\mathrm{dist}}$ be the increasing union of $\hat{X}_{\mathcal{Q}}^{\mathrm{dist}}$ as the diameter of the cubes in $\mathcal{Q}$ goes to zero. We prove that $\hat{\mu}(\hat{X}^{\mathrm{dist}}) = 1$ by proving that $\hat{\mu}(\hat{X}_{\mathcal{Q}}^{\mathrm{dist}}) \geq 1 - \varepsilon$.

Let $\hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}} \subset \hat{X}$ consist of histories of points $x_0 \in \operatorname{Supp}\mu$ that are distinguished up to time $-N$. This is a decreasing sequence of subsets of $\hat{X}$, and $\bigcap_{N \geq 1} \hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}} = \hat{X}_{\mathcal{Q}}^{\mathrm{dist}}$. It is enough to prove that $\hat{\mu}(\hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}}) \geq 1 - \varepsilon$ for all $N \geq 1$. Now, $\hat{\mu}(\hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}}) = \hat{\mu}\big(\hat{f}^N(\hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}})\big)$ and by definition $\hat{f}^N(\hat{X}_{-N,\mathcal{Q}}^{\mathrm{dist}})$ is the set of sequences (x_n) such that x_0 is a cluster value of a sequence $f_{-N}(x_N^{(k)})$, with $\operatorname{Supp}(S_{k,\mathcal{Q}}^-) \ni x_N^{(k)} \to x_N$.

Recall that the measure $\mu_{k,\mathcal{Q}}$ has mass larger than $1 - \varepsilon$, and satisfies

$$\mu_{k,\mathcal{Q}} := T_{\mathcal{Q}}^+ \wedge T_{k,\mathcal{Q}}^- \leq T^+ \wedge T_{k,\mathcal{Q}}^- = \frac{2}{k} \sum_{i=\lfloor k/2 \rfloor}^{k} T^+ \wedge S_{i,\mathcal{Q}}^-. \tag{11}$$

We come now to the crucial point. Since there is a natural f_{-N} on $f^k(L)$ for $k > 2N$, we may consider the measure $(f_{-N})_*\mu_{k,Q}$, which has mass larger than $1-\varepsilon$. From (11) we infer that $(f_{-N})_*\mu_{k,Q} \leq T^+ \wedge T^-_{k-N} + \sigma_k$, where σ_k is a signed measure of total mass $O(\frac{N}{k})$. Consider any cluster value of this sequence of measures as $k \to \infty$ and denote it by $(f_{-N})_*\mu_Q$ (this notation is convenient but somewhat improper). Then $(f_{-N})_*\mu_Q \leq \mu$ because $\mu_k \to \mu$ and $\mathbf{M}((f_{-N})_*\mu_Q) \geq 1-\varepsilon$. Now if $(x_n) \in \hat{X}$ is *any* sequence such that $x_0 \in \mathrm{Supp}((f_{-N})_*\mu_Q)$, then by definition $(x_n) \in \left(\hat{f}^N(\hat{X}^{\mathrm{dist}}_{-N,Q})\right)$ (note that the tail $(x_j)_{j<-N}$ of a history in $\hat{X}^{\mathrm{dist}}_{-N,Q}$ is arbitrary). Hence

$$\hat{\mu}\left(\hat{f}^N(\hat{X}^{\mathrm{dist}}_{-N,Q})\right) \geq \hat{\mu}\left(\hat{\pi}_0^{-1}\left(\mathrm{Supp}((f_{-N})_*\mu_Q)\right)\right) = \mu\left(\mathrm{Supp}((f_{-N})_*\mu_Q)\right) \geq 1-\varepsilon.$$

This finishes the proof of the proposition. □

REMARK 5.3. – i. From the first part of the proof we could directly deduce the existence of the positive exponent (expansion in forward time), in the spirit of [36]. However, to obtain more complete results, we first construct the tautological extension.

ii. The laminarity of T^+ is only used in the proof to get lower semicontinuity properties of the wedge products. So if, for instance, T^+ has continuous potential, we can drop the laminarity assumption and get the same conclusion. Notice additionally in this case that the wedge product $T^+ \wedge T^-$ is "semi-geometric" in the sense that it is approximated from below by $T^+ \wedge T^-_Q$ (see the proof of [13, Remarque 4.6] or [15, Remark 5.3]). This is the setting of [36], and might be useful for further applications.

6. The tautological extension

So far we have constructed a family of marked uniformly woven currents $T^-_Q = \sum T^-_Q$, increasing to T^-, with the property that for any disk appearing in the markings, f^n admits exponentially contracting inverse branches. We say that such disks are *subordinate* to T^-.

If D is a disk subordinate to T^-, we define the measure $T^+ \dot{\wedge} D$ as the increasing limit of the measures $T^+_{Q_i} \dot{\wedge} D$ for our choice of increasing subdivisions Q_i. Since $T^+ \wedge T^-$ is a geometric intersection, if D is a generic (relative to the marking) disk subordinate to T^-, then $T^- \wedge [D]$ is well defined and $T^+ \wedge [D] = T^+ \dot{\wedge} D$.

We now construct the tautological extension $(\check{X}, \check{\mu}, \check{f})$ of (X, μ, f). This is roughly speaking the "smallest" space [(8)] where the unstable leaves become separated. It will be realized as a disjoint union of flow boxes, foliated [(9)] by the lifts of the disks subordinate to T^- (which play the role of unstable manifolds). The marking data give us a lift of T^- to a "laminar current" on $\check{X}$. Hence we get a lift $\check{\mu}$ of μ, whose conditionals on unstable manifolds are well understood. This will be our main technical step towards the understanding the conditionals of $\hat{\mu}$ on unstable manifolds in the *natural* extension. As suggested by the referee, it might be possible to analyze the conditionals of $\hat{\mu}$ directly from μ but we do not know how to do it.

(8) This space does not depend canonically on (X, μ, f). On the other hand it depends canonically on μ viewed as a geometric intersection of marked woven currents.

(9) We make no attempt to connect the flow boxes, as this would certainly lead to unwelcome topological complications.

THEOREM 6.1. – *There exists a locally precompact and separable space $\check{X}$, which is a countable union of compatible flow boxes, together with a Borel probability measure $\check{\mu}$ and a measure preserving map $\check{f}$, with the following properties.*

i. *There exists a projection $\check{\pi} : (\check{X}, \check{\mu}) \to (X, \mu)$ semiconjugating f and $\check{f}$, i.e. $\check{\pi} \circ \check{f} = f \circ \check{\pi}$.*
ii. *$\check{X}$ admits a measurable partition $\check{D}$ whose atoms $\check{D}(\check{x})$, $\check{x} \in \check{X}$, are mapped homeomorphically by $\check{\pi}$ onto disks subordinate to T^-.*
iii. *The conditional measure of $\check{\mu}$ on almost any atom $\check{D}(\check{x})$ is induced by the current T^+ as follows: it is equal up to normalization to $\left((\check{\pi}|_{D(\check{x})})^{-1}\right)_* (T^+ \dot{\wedge} D)$ where $D = \check{\pi}(\check{D}(\check{x}))$.*

Proof. – As before, let Q be one among a fixed increasing sequence $(\mathcal{Q}_i)$ of subdivisions with $\mu(\partial \mathcal{Q}_i) = 0$. The current T_Q is marked by a measure on the disjoint union $\coprod_{\tilde{Q} \in \tilde{\mathcal{Q}}} Z(\tilde{Q}, 1/2)$. To simplify notation, we will omit the $\tilde{}$ and the $1/2$ in the sequel. Recall from §1 that points in the tautological extension $\check{Z}(Q)$ of $Z(Q)$ are pairs (x, D) with $x \in D \in Z(Q)$, and that the projection $\check{\pi} : \check{Z}(Q) \to Z(Q)$ is given by $\check{\pi}(x, D) = x$.

The principle of the proof is quite simple. Each $D \in \operatorname{Supp} m(T_Q^-)$ admits a natural lift $\check{D}$ to $\check{Z}(Q)$, and T^+ induces a measure on $\check{D}$ according to the formula

$$T^+ \dot{\wedge} \check{D} := \left((\check{\pi}|_{\check{D}})^{-1}\right)_* (T^+ \dot{\wedge} [D]).$$

Averaging with respect to the markings then gives a measure

$$\check{\mu}_Q := \int (T^+ \dot{\wedge} \check{D}) \, d(m(T_Q^-))(D)$$

that projects on μ_Q. We call $\check{\mu}_Q$ the *tautological extension* of $T^+ \dot{\wedge} T_Q^-$. What remains is to make sense of the "increasing limit" $\check{\mu}$ of $\sum_{Q \in \mathcal{Q}_i} \mu_Q$ as $i \to \infty$. In particular, we need to construct the space $\check{X}$ that carries $\check{\mu}$, and then show that $\check{\mu}$ is invariant under some naturally associated map $\check{f}$.

To construct $\check{X}$ we recall from Lemma 1.2 that for generic subdivisions, almost all chains appearing in the markings $m(T_Q^-)$ are disks transverse to the boundary (this is an open subset of chains). Let $O_1 \subset \operatorname{Supp}(m(T_{\mathcal{Q}_1}^-))$ be the relatively open, full measure subset of boundary-transverse disks. Set $E_1 = O_1$ and let $\check{E}_1$ be the tautological bundle over E_1.

Now assume that for $1 \leq j \leq i-1$ the sets E_j and $\check{E}_j$ have been constructed. Consider the marked current $T_{\mathcal{Q}_i}^-$, and as before denote by O_i the open subset of $\operatorname{Supp}(m(T_{\mathcal{Q}_i}^-))$ made of boundary-transverse disks. We add to $\bigcup_{j \leq i-1} E_j$ the smaller set $E_i \subset O_i$ consisting only of those disks which have not previously appeared: i.e.

$$E_i = O_i \setminus \left\{ D : D \subset Z \in \operatorname{Supp}(m(T_{\mathcal{Q}_j}^-)) \right\}.$$

As is easily verified, E_i is open in O_i and therefore also in the locally compact metric space $\coprod_{Q \in \mathcal{Q}_i} Z(Q)$.

Now take $\check{X}$ to be the disjoint union $\check{X} = \coprod_i \check{E}_i$. By Lemma 1.1, there is no folding in $m(T_{\mathcal{Q}_j}^-)$ so each $\check{E}_i$ is laminated by the disks of E_i. We can endow $\check{X}$ with a natural topology by putting the natural topology on each $\check{E}_j$, and declaring that each $\check{E}_j$ is open and closed. This topology is even induced by a metric where each of the $\check{E}_i$ is bounded, and at definite (positive) distance from the others. This makes $\check{X}$ a locally precompact and separable space. We could also take its completion to get a locally compact and separable space but we will not need it. Each E_i is naturally partitioned by the disks of E_i, giving rise to the measurable partition $\check{D}$ of *ii.*, and can be covered by a countable family of compatible flow boxes.

To define the measure $\check{\mu}$, we note that if $D \subset D'$ with $D \in Z(Q)$ (resp. $D' \in Z(Q')$), then there is a natural inclusion $\check{D} \hookrightarrow \check{D}'$. Hence we can view the tautological extension of $T^+ \dot{\wedge} T^-_{Q_i}$ as a measure $\check{\mu}_{Q_i}$ supported on $E_1 \cup \cdots \cup E_i$ (rather than $Z(Q_i)$). This defines an increasing sequence of measures[10] in $\check{X}$. The limit $\check{\mu}$ is a Borel probability measure. The conditional of the measure $\check{\mu}_{Q_i}$ on a disk D of $\check{E}_j$, is by definition induced by T^+ (and independent of $i \geq j$), so statement *iii.* follows.

Finally, we seek to construct the measurable map $\check{f}$ projecting onto f and leaving $\check{\mu}$ invariant. We want to define $\check{f}$ as follows: $\check{f}(x, D) = (x', D')$ if $f(x) = x'$ and $\mathrm{germ}_{x'}(f(D)) = \mathrm{germ}_{x'}(D')$. The details of this are a little involved, however.

To begin with, let us recall some notation from Proposition 5.1. In the course of proving this result, we introduced currents $T^-_k = \frac{2}{k} \sum_{j=k/2}^{k} \frac{1}{\lambda_1^j} S_j$ together with the restriction $T^-_{k,Q}$ of T^-_k to disks which were 'good' relative to Q. We also had measures $\mu_k = T^+ \wedge T^-_k$ and $\mu_{k,Q} = T^+ \wedge T^-_{k,Q}$ (note the slight change in the definition of the latter).

It is immediate from the definitions that the difference $\sigma_k := f_*\mu_k - \mu_k$ has mass no greater than $4/k$. Another useful observation is that, for purposes of comparing the various measures we have defined, we can be a little flexible concerning their domains. Since μ_k gives no mass to points (Lemma 1.5) and $\mu_{k,Q}$ is concentrated on countably many disks of $Z(Q)$ with discrete intersections, we can lift $\mu_{k,Q}$ canonically to a measure $\check{\mu}_{k,Q}$ on $\check{Z}(Q)$. So we may regard $\mu_{k,Q}$ as a measure on the cubes of Q or alternatively as a measure on the the tautological bundle $\check{Z}(Q)$. In a similar vein, we may regard $\check{\mu}|_{\check{E}_1 \cup \cdots \cup \check{E}_i}$ as a measure on $\check{Z}(Q_i)$ rather than $\check{X}$.

LEMMA 6.2. – *Let $\check{\nu}_{Q_i}$ be any cluster value of the sequence of measures $\check{\mu}_{k,Q_i}$. Then $\check{\nu}_{Q_i} - \check{\mu}_{Q_i}$ is a signed measure with*

$$\mathbf{M}\left(\check{\nu}_{Q_i} - \check{\mu}_{Q_i}\right) = \varepsilon(Q_i)$$

where $\varepsilon(Q_i)$ depends only on i and tends to zero as $i \to \infty$.

Proof of Lemma 6.2. – Since $T^+ \geq T^+_{Q_i}$, we have

$$\check{\mu}_{k,Q_i} \geq \int (T^+_{Q_i} \dot{\wedge} \check{D})\, d(m(T^-_{k,Q_i}))(D).$$

Taking cluster values on both sides, and using lower semicontinuity (Lemma 1.4) we infer that

$$\check{\nu}_{Q_i} \geq \int (T^+_{Q_i} \dot{\wedge} \check{D})\, d(m(T^-_{Q_i}))(D).$$

On the other hand,

$$\check{\mu}_{Q_i} = \int (T^+ \dot{\wedge} \check{D})\, d(m(T^-_{Q_i}))(D) \geq \int (T^+_{Q_i} \dot{\wedge} \check{D})\, d(m(T^-_{Q_i}))(D).$$

Hence $\check{\mu}_{Q_i}, \check{\nu}_{Q_i}$ are both measures with at most unit mass, and both are bounded below by a measure which, by geometric intersection, has mass at least $1 - \varepsilon(Q_i)$. □

[10] Note that $\check{\mu}_{Q_i}$ might exceed $\check{\mu}_{Q_{i-i}}$ not only on E_i but also on those earlier sets E_j that contain disks in O_i.

Continuing with the proof of the theorem, we let $A_{k,Q_i} = \operatorname{Supp} T^-_{k,Q_i} \cap f^{-1}(\operatorname{Supp} T^-_{k,Q_i})$. That is, A_{k,Q_i} consists of points that 'go from large disks to large disks.' Since $f_*\mu_k = \mu_k + \sigma_k$ and f is essentially 1-1 on $\operatorname{Supp} T^-_k$, we have that A_{k,Q_i} has almost full mass:

$$\mu_k(A_{k,Q_i}) \geq \mu_k(\operatorname{Supp} T^-_{k,Q_i}) - \mu_k(\operatorname{Supp} T^-_k - \operatorname{Supp} T^-_{k,Q_i}) - \frac{4}{k}.$$

By geometric intersection, the second term on the right hand side is of the form $\varepsilon(k, Q_i)$ with $\lim_{k\to\infty} \varepsilon(k, Q_i) = \varepsilon(Q_i)$ and $\varepsilon(Q_i) \to 0$ as $i \to \infty$. So since $\mu_{k,Q_i} = \mu_k|_{\operatorname{Supp} T_{k,Q_i}}$, we infer that

$$\mathbf{M}\left(\mu_{k,Q_i} - \mu_{k,Q_i}|_{A_{k,Q_i}}\right) \leq \mathbf{M}\left(\mu_k - \mu_{k,Q_i}|_{A_{k,Q_i}}\right) \leq \varepsilon(k, Q_i) + \frac{4}{k}. \tag{12}$$

In the same way we obtain that

$$\mathbf{M}\left(\mu_{k,Q_i} - f_*\mu_{k,Q_i}|_{A_{k,Q_i}}\right) \leq \varepsilon(k, Q_i) + O(\frac{1}{k}). \tag{13}$$

Lifting to the tautological bundle, we get a set $\check{A}_{k,Q_i}$. Consider a cluster value ν_A of the sequence of measures $\check{\mu}_{k,Q_i}|_{\check{A}_{k,Q_i}}$. From Lemma 6.2 and (12), and since there is no loss of mass in the boundary, we get that

$$\mathbf{M}\left(\nu_A - \check{\mu}_{Q_i}\right) \leq \mathbf{M}\left(\nu_A - \check{\nu}_{Q_i}\right) + \mathbf{M}\left(\check{\nu}_{Q_i} - \check{\mu}_{Q_i}\right) = \varepsilon(Q_i),$$

where $\check{\nu}_{Q_i}$ is a cluster value of the sequence of measures $\check{\mu}_{k,Q_i}$.

Now set $\check{A}_{Q_i} := \limsup_{k\to\infty} A_{k,Q_i}$. Since this contains $\operatorname{Supp}(\nu_A)$; we infer that $\check{\mu}_{Q_i}(\check{A}_{Q_i}) \geq 1-\varepsilon(Q_i)$. We claim moreover that there is a well-defined map $\check{f}_i : \check{A}_{Q_i} \to Z(Q_i)$ given by $f_i(x, D) = (f(x), D')$ where $D' \in Z(Q_i)$ coincides with $f(D)$ near $f(x)$. To see that this works, observe that by definition of $\check{A}_{Q_i}$, there is a sequence of pointed disks $(x_k, D_k) \to (x, D)$ such that D_k is subordinate to T^+_{k,Q_i} and $f(D_k)$ coincides near $f(x_k)$ with some other disk D'_k subordinate to T^+_{k,Q_i}. Taking D' to be a cluster value of the $(D'_k)_{k\in\mathbf{N}}$, we see that $\check{f}_i$ is indeed well-defined. Lemma 6.2 and (13) tell us additionally that

$$\mathbf{M}\left((\check{f}_i)_*(\check{\mu}_{Q_i}|_{\check{A}_{Q_i}}) - \check{\mu}_{Q_i}\right) = \varepsilon(Q_i). \tag{14}$$

Rephrasing the preceding construction in terms of $\check{X}$, we have constructed a set $\check{A}_{Q_i} \subset \check{E}_1 \cup \cdots \cup \check{E}_i$, with $\check{\mu}$-mass $\geq 1-\varepsilon(Q_i)$, together with a map $\check{f}_i : A_{Q_i} \to \check{E}_1 \cup \cdots \cup \check{E}_i$, that coincides with the action of f on the space of germs. If, when refining the subdivision, we are careful to extract our subsequences from those chosen for earlier subdivisions, then we will obtain an increasing sequence of subsets $(\check{A}_{Q_i})$, with the compatibility (say Q_j is finer than Q_i) $\check{f}_j|_{\check{A}_{Q_i}} = \check{f}_i$. So the maps $\check{f}_i$ piece together to form a single map $\check{f}$ defined on a full measure subset of $\check{X}$. Furthermore, since

$$\mathbf{M}\left((\check{f}_i)_*\check{\mu}|_{\check{A}_{Q_i}} - \check{\mu}|_{\check{E}_1\cup\cdots\cup\check{E}_i}\right) = \varepsilon(Q_i)$$

we infer that $\check{\mu}$ is $\check{f}$-invariant. □

The following proposition clarifies the relationship between iteration on $\check{X}$ and the construction of the previous section. We say that a disk has size $\geq r$ if it belongs to $Z(Q)$ for some cube Q of size $\geq r$.

Proposition 6.3. – *For every fixed positive integer ℓ, there exists a set $\check{A}_\varepsilon(\ell) \subset \check{X}$ of $\check{\mu}$-measure $\geq 1-\varepsilon$ such that if $\check{x} = (x, D) \in \check{A}_\varepsilon(\ell)$ then $\check{f}^\ell(\check{x})$ has the following properties:*

i. *$f^\ell(D)$ coincides near $f^\ell(x)$ with a disk of size $\geq r(\varepsilon) > 0$;*
ii. *$f^\ell : D \to f^\ell(D)$ is univalent, with derivative larger than $C(\varepsilon)\lambda_1^{\ell/2}\ell^{-1}$;*
iii. *the orbit segment $x, f(x), \dots, f^\ell(x)$ is distinguished.*

Proof. – Replace the set A_{k,Q_i} used in the previous proof with the analogous set $A_{k,Q_i}(\ell)$ of points $x \in \operatorname{Supp} T^-_{k,Q_i}$ such that $f^\ell(x) \in \operatorname{Supp} T^-_{k,Q_i}$. Define $A_{Q_i}(\ell) = \limsup(A_{k,Q_i}(\ell))$ and consider as before the tautological bundles over these sets. As we now explain, it suffices to take $\check{A}_\varepsilon(\ell) = \check{A}_{Q_i}(\ell)$ for large enough i.

We first need to check that $\check{f}^\ell(\check{x})$ is well defined for almost every point in $\check{A}_{Q_i}(\ell)$. The point is that there is a piece of disk in $Z(Q_i)$ sent by f^ℓ to a piece of disk of $Z(Q_i)$, however along the branch $x, f(x), \dots, f^\ell(x)$ the disk can become small. Nevertheless if $x \notin \operatorname{Crit}(f^\ell)$, f^q is locally invertible at x for $1 \leq q \leq \ell$, so all[11] the germs $f^q(D)$ are traced on disks of *some*, possibly much smaller, size $Q_{i'}$ depending on ℓ. The same holds for disks subordinate to T^-_{k,Q_i} that approximate D. We conclude that ℓ successive iterates of $\check{f}$ are defined at $\check{x}$.

Now, the conclusions of the lemma follow easily from the analysis of distinguished inverse branches in the proof of Proposition 5.1. The only point that needs explanation is *ii.* If $(x, D) \in \check{A}_{Q_i}(\ell)$, then by construction, (x, D) is the limit of a sequence of pointed disks (x_k, D_k) with $\left\| df^\ell_{x_k}(e(\check{x}_k)) \right\| \geq C(Q_i)\lambda_1^{\ell/2}\ell^{-1}$, where $e(\check{x}_k)$ is the unit tangent vector to D_k at x_k. Recall from Lemma 1.1 that since the current T^- is strongly approximable, there is no multiplicity in the convergence of the disks subordinate to $T^-_{Q,k}$. So if $\check{x}_k \to \check{x}$ in this construction, then $e(\check{x}_k) \to e(\check{x})$. It follows for $x \notin I(f^\ell)$ that $df^\ell_{x_k}(e(\check{x}_k)) \to df^\ell_x(e(\check{x}))$, giving the desired estimate. We note that since the estimate extends across finite sets, it holds even at points in $I(f^\ell)$. □

Now we can estimate the positive exponent. For convenience here we take for granted that $\check{\mu}$ is ergodic, a fact we will prove in Corollary 7.4 below.

Corollary 6.4. – *For μ-a.e. x there exist a tangent vector e^u at x and a set of integers $\mathbf{N}' \subset \mathbf{N}$ of density 1 such that*

$$\liminf_{\mathbf{N}' \ni n \to \infty} \frac{1}{n} \log |df^n(e^u(x))| \geq \frac{\log \lambda_1}{2}. \tag{15}$$

Given $\check{x} = (x, D) \in \check{X}$, we let $e(\check{x})$ denote the tangent vector to D at x. The proof makes evident that one can take $e_u = e(\check{x})$ for $\check{\mu}$ a.e. $\check{x}$.

Proof. – We have the following lemma from elementary measure theory (see below for the proof).

Lemma 6.5. – *Let (Y, m) be a probability space, and $(A_n)_{n\geq 1}$ a collection of sets of measure $\geq 1 - \varepsilon$. Then for every $\delta > \varepsilon$,*

$$m\left(\{y \in Y, y \in A_n \textit{ for a set of integers } n \textit{ of density } \geq 1 - \delta\}\right) \geq 1 - \frac{\varepsilon}{\delta}.$$

[11] Technically, we need to avoid the measure zero set of x sent by f^q onto boundaries of cubes.

With notation as in Proposition 6.3, let

$$\check{A}_\varepsilon = \left\{\check{x},\ \check{x} \in \check{A}_\varepsilon(\ell) \text{ for a set of integers } \ell \text{ of density } \geq 1-\sqrt{\varepsilon}\right\}.$$

By the previous lemma, $\check{\mu}(\check{A}_\varepsilon) \geq 1-\sqrt{\varepsilon}$. If $\check{x} \in \check{A}_\varepsilon$, then (15) holds for a set of integers $\mathbf{N}_\varepsilon$ of density $\geq 1-\sqrt{\varepsilon}$. To conclude, observe that $\check{A}_\varepsilon$ is an invariant set, so that by ergodicity it has full measure. □

Proof of Lemma 6.5. – Consider the function $\varphi_N = \sum_{n=1}^{N} \mathbf{1}_{A_n}$ (with possibly $N=\infty$). We have that $0 \leq \varphi_N \leq N$ and $\int \varphi_N dm \geq (1-\varepsilon)N$. We leave the reader prove that for every $\delta > \varepsilon$,

$$m(B_N) \geq 1 - \frac{\varepsilon}{\delta} \text{ where } B_N = \{y,\ \varphi_N(y) \geq (1-\delta)N\}.$$

In particular taking δ close to 1, at this point we conclude that for every fixed C and N large enough, $m(\{\varphi_N \geq C\}) \geq 1-\varepsilon$. In particular for every C, $m(\{\varphi_\infty \geq C\}) \geq 1-\varepsilon$, so the set of points belonging to infinitely many A_n has measure $\geq 1-\varepsilon$.

Now the (B_N) themselves form an infinite collection of sets of measure $\geq 1-\varepsilon/\delta$ so applying the same reasoning proves that the set of y belonging to infinitely many B_N's has measure $\geq 1-\varepsilon/\delta$ which is the desired statement. □

7. The natural extension and entropy

In this section we analyze the natural extension of (X,μ,f). There are two main steps: prove that the natural extension of (X,μ,f) is the same as that of $(\check{X},\check{\mu},\check{f})$, and analyze the conditional measures of $\hat{\mu}$ relative to the unstable partition in the natural extension.

We first show that different disks subordinate to T^- correspond to different histories, as one would expect for unstable manifolds.

PROPOSITION 7.1. – *Let $\hat{x} \in \hat{X}$ be a $\hat{\mu}$-generic point. Then there exists a* unique *disk D subordinate to some T_Q^-, together with a sequence of inverse branches $f_{\hat{x},-n}$ defined on D, such that $f_{\hat{x},-n}(x_0) = x_{-n}$ and $f_{\hat{x},-n}$ contracts exponentially on D.*

Proof. – Proposition 5.1 tells us that distinguished histories have full measure in the natural extension. So we may assume that $\hat{x}$ is Q-distinguished for some Q. We therefore have a disk $D \ni x_0$ that is subordinate to T_Q^- and equipped with a compatible sequence (f_{-n}) of inverse branches such that $f_{-n}(x_0) = x_{-n}$. Proposition 5.1 further guarantees that f_{-n} is exponentially contracting on D. It remains to show D is unique.

Consider the measure $\mu_Q = T_Q^+ \dot{\wedge} T_Q^-$. By construction, for each $x \in \mathrm{Supp}(\mu_Q)$ there exists a radius $r = r(x)$ such that the disks subordinate to T_Q^+ are submanifolds in $B(x,r)$ and get contracted at uniform exponential speed $O(n\lambda_2^n/\lambda_1^n)$ under forward iteration. Every point in $\mathrm{Supp}(\mu_Q)$ has such a local "stable manifold", which is then unique because T^+ is strongly approximable and laminar [14, Theorem 1.1]. Let $\mathscr{L}^s \cap B(x,r)$ be the stable lamination near $x \in \mathrm{Supp}\,\mu_Q$, that is, the union of stable manifolds of points in $\mathrm{Supp}(\mu_Q) \cap B(x,r)$. Since the image of a disk subordinate to T^+ under f (resp. under a branch of f^{-1} defined in an open set) is a disk subordinate to T^+, the stable lamination is invariant under the dynamics: $f(\mathscr{L}^s \cap B(x,r)) \subset \mathscr{L}^s$.

Now suppose for some generic history $\hat{x}$, that D_1, D_2 are two disks through x_0 satisfying the conclusions of the proposition. Then for n large enough, we have $f_{-n}(D_j) \subset B(x_{-n}, r_0/2)$ where $r_0 = r(x_0)$. And since $x_0 \in \operatorname{Supp} \mu_Q \subset \operatorname{Supp} \mu$, Poincaré recurrence for $\hat{\mu}$ gives an infinite set $S \subset \mathbf{N}$ of n such that $x_{-n} \subset B(x_0, r_0/2)$. So if L is any leaf in $\mathscr{L}^s \cap B(x_0, r)$ that meets both D_1 and D_2, then for every $n \in S$, $f_{-n}(L) \cap B(x_0, r_0)$ is contained in a leaf L' intersecting the preimages of either disk $f_{-n}(D_j)$. Contraction of stable leaves then gives.

$$\operatorname{distance}(L \cap D_1, L \cap D_2) \leq Cn \left(\frac{\lambda_2}{\lambda_1}\right)^n \operatorname{diameter}(L') = O\left(\frac{n\lambda_2^n}{\lambda_1^n}\right).$$

As this is true for all $n \in S$, we conclude that $L \cap D_1 = L \cap D_2$. Similarly, neither D_j can be entirely contained in the stable leaf through x_0, because pulling back and pushing forward would, in the same fashion, show that the diameter of D_j vanishes.

Now we can conclude. Leaves in $\mathscr{L}^s \cap B(x_0, r_0)$ accumulate on the one through x_0. Hence there are infinitely many such leaves intersecting both D_1 and D_2. Since the intersection points all lie in $D_1 \cap D_2$, we see that x_0 is an accumulation point of $D_1 \cap D_2$. It follows that $D_1 = D_2$; i.e. the disk D in the proposition is unique. □

Remark 7.2. – As is clear from the proof, the uniqueness assertion of the proposition holds for a given distinguished history $\hat{x}$ as long as there are infinitely many n for which $x_{-n} \in \operatorname{Supp}(\mu_Q)$, and we only need $f_{\hat{x},-n}$ to be contracting for these n.

The proof also makes clear that the disk D is unique *regardless of whether it is subordinate to T_Q^-*. This shows that the web supporting T_Q^- is essentially independent, along histories in $\hat{X}$, of the manner in which it was constructed. It also shows that for almost any $\hat{x} \in \hat{X}$, the disk D is the only reasonable candidate for a 'local unstable manifold' associated to $\hat{x}$. We will therefore refer to such disks as local unstable manifolds and to the resulting partitions of $\hat{\mu}$ and $\check{\mu}$ as the 'unstable partition' of each. We should emphasize, however, that we do not know whether D is the full local unstable set of $\hat{x}$—i.e. whether something like the local unstable manifold theorem holds in the present context.

Proposition 7.3. – *The natural extension $(\widehat{\check{X}}, \widehat{\check{\mu}}, \widehat{\check{f}})$ of $(\check{X}, \check{\mu}, \check{f})$ is measurably isomorphic to that of (X, μ, f).*

Proof. – The natural extension of $(\check{X}, \check{\mu}, \check{f})$ is the set of sequences $(\check{x}_n) = (x_n, D_n)$, indexed by $\mathbf{Z}$, with $\check{f}(\check{x}_n) = \check{x}_{n+1}$. Observe that if x_0 does not belong to $\bigcup_{n\geq 0} f^n(C(f^n))$, then for positive n, there is a unique germ D_{-n} at x_{-n} such that $f^n(D_{-n}) = D_0$. In particular the whole sequence (D_n) is determined by D_0 and (x_n).

Furthermore, since $(\widehat{\check{X}}, \widehat{\check{\mu}}, \widehat{\check{f}})$ is an invertible dynamical system projecting onto (X, μ, f), the universal property of $(\hat{X}, \hat{\mu}, \hat{f})$ gives us an intermediate semiconjugacy $\eta : (\widehat{\check{X}}, \widehat{\check{\mu}}, \widehat{\check{f}}) \to (\hat{X}, \hat{\mu}, \hat{f})$. In explicit terms, $\eta(x_n, D_n) = (x_n)$.

Since the set of distinguished histories $\hat{X}^{\mathrm{dist}}$ has full measure in $\hat{X}$, we get that for $\widehat{\check{\mu}}$-a.e. (x_n, D_n), $(x_n) \in \hat{X}^{\mathrm{dist}}$. By Proposition 7.1 above, associated to the sequence $\hat{x} = (x_n) \in \hat{X}^{\mathrm{dist}}$, there is a unique germ of disk $D(\hat{x})$ through x_0 which is contracted exponentially in the past along the branch x_{-n}. The proof will be finished if we show that for $\widehat{\mu}$-a.e. (x_n, D_n), $D_0 = D(\hat{x})$. Indeed, since (D_n) depends only on D_0, we will have found an inverse for η.

Consider the sets $\check{A}_\varepsilon(\ell)$ as defined in Proposition 6.3, and define $\widehat{\check{A}}_\varepsilon(\ell)$ to be the set of sequences $(\check{x}_n) \in \widehat{\check{X}}$ with $\check{x}_{-\ell} \in \check{A}_\varepsilon(\ell)$. We have that $\widehat{\mu}(\widehat{\check{A}}_\varepsilon(\ell)) \geq 1-\varepsilon$. Hence by Lemma 6.5 there is a set $\widehat{\check{A}}_\varepsilon \subset \widehat{\check{X}}$ of measure $\geq 1-\sqrt{\varepsilon}$ of points belonging to $\widehat{\check{A}}_\varepsilon(\ell)$ for infinitely many ℓ. By definition, if $(x_n, D_n) \in \widehat{\check{A}}_\varepsilon$, then for infinitely many integers ℓ, the (germ of) disk D_0 is contracted by the branch of $f^{-\ell}$ sending D_0 to $D_{-\ell}$. By Proposition 7.1 and Remark 7.2 we conclude that $D_0 = D(\hat{x})$. □

Corollary 7.4. – *The measure $\check{\mu}$ is ergodic under $\check{f}$, and we have equality between entropies $h(f,\mu) = h(\check{f},\check{\mu})$.*

Proof. – Since μ is ergodic, so is $\hat{\mu}$, and therefore $\check{\mu}$. Also we have that $h(f,\mu) \leq h(\check{f},\check{\mu}) \leq h(\hat{f},\hat{\mu}) = h(f,\mu)$. □

From now on, depending on the context, we can think of $(\hat{X}, \hat{\mu}, \hat{f})$ as the natural extension of either (X, μ, f) or $(\check{X}, \check{\mu}, \check{f})$. Figure 1 illustrates the relationship between the natural and tautological extensions. A disk subordinate to T^- and its preimages under $\check{\pi}$ and π_0 has been underlined. The notation $\check{\mathscr{L}}^s$, $\check{\mathscr{L}}^u$ is introduced in the proof of Theorem 8.1. The existence of the dashed arrow is ensured by the previous proposition, i.e. by the fact that η is a measurable isomorphism. With notation as above, it is defined by $\hat{x} \mapsto (x_0, D(\hat{x}))$.

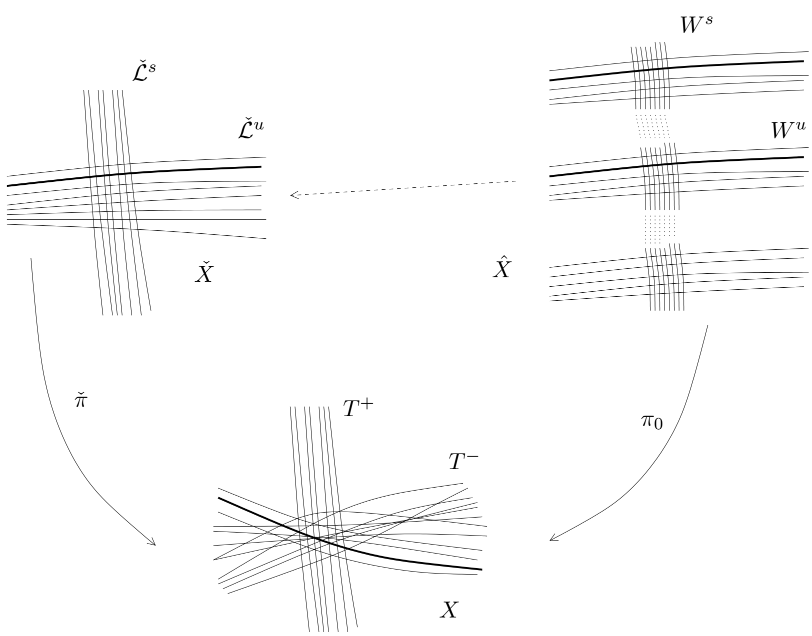

Figure 1. Schematic picture of the tautological and natural extensions

Aside: unstable manifolds and the natural extension

As it is well known, defining unstable manifolds for a non invertible system requires working in the natural extension. The unstable set of $\hat{x} \in \hat{X}$ is

$$W^u(\hat{x}) = \left\{ \hat{y} \in \hat{X}, \lim_{n\to\infty} \operatorname{dist}(\hat{f}^{-n}(x), \hat{f}^{-n}(y)) = 0 \right\}.$$

Under some hyperbolicity assumptions on $\hat{x}$, $W^u_{\mathrm{loc}}(\hat{x})$ projects isomorphically onto a submanifold embedded in a neighborhood of $x_0 \in X$. Different histories generally give rise to different local unstable manifolds in X. In our situation, the disks subordinate to T^- play the role of unstable disks (Proposition 7.1 and the remark thereafter).

Definition 7.5. – *We say that the unstable manifolds of f fully depend on histories if the assignment $\hat{x} \mapsto (x_0, D(\hat{x}))$ is 1-1 on a set of full measure; in other words, if the intermediate projection $\hat{X} \to \check{X}$ is a measurable isomorphism.*

We say that the unstable manifolds do not depend on histories if $\check{\pi}$ is a measurable isomorphism but $\pi_0 : \hat{X} \to X$ is not (i.e. f is not essentially invertible).

Przytycki proved in [32] that for C^1-generic Anosov endomorphisms of tori, the unstable manifolds depend on histories. Mihailescu and Urbanski [29] have proved the dependence on histories for certain generic perturbations of saddle sets for holomorphic endomorphisms of $\mathbf{P}^2$. It is natural to expect that for generic mappings with small topological degree (and not essentially invertible, see Example 3.2), the unstable manifolds (fully) depend on histories. However we do not know how to verify this, even on a single example!

We return to establishing the ergodic properties of μ. In the next theorem we analyze the conditionals induced by $\hat{\mu}$ on the partition by local unstable manifolds in $\hat{X}$.

Theorem 7.6. – *The conditional measures of $\hat{\mu}$ along the unstable partition in $\hat{X}$ are induced by the current T^+.*

The unstable conditionals are well understood in $\check{X}$, so we will take advantage of the fact that $\hat{X}$ is the natural extension of $\check{X}$: if $\hat{p} \in \hat{X}$ is a generic point, it has a local unstable manifold $W^u_{\mathrm{loc}}(\hat{p})$, and π_0 is a homeomorphism from $W^u_{\mathrm{loc}}(\hat{p})$ to a disk D through $\pi_0(p)$. The statement of the theorem is that $\hat{\mu}_{W^u_{\mathrm{loc}}(\hat{p})} = c(\pi_0^{-1})_*(T^+ \dot{\wedge} D)$ (c is a normalization constant). Notice here that it is important to consider the natural extension as a topological, and not only measurable, object. To address topological issues in the natural extension, we use the ordinary model of the shift acting on the space of histories, which is naturally a compact metric space. Note also that if the unstable manifolds fully depend on histories, then the result is obvious since $\hat{X} \simeq \check{X}$.

Proof. – We view $\hat{X}$ as the natural extension of $\check{X}$. So for notational ease we let π_0 denote the natural mapping $\hat{X} \to \check{X}$. Consider also the sequence of projections $(\pi_n)_{n\in\mathbf{Z}}$ with $\pi_{n+1} = \check{f} \circ \pi_n$.

By construction, $\check{X}$ admits a measurable partition into local unstable disks. Consider a flow box P of positive measure in $\check{X}$, that is, a sub-lamination of one of the sets $\check{E}_k$ of Theorem 6.1, made up of disks of size Q. We denote by p the generic point of P, and by ξ the natural partition of P by disks, so that $\xi(p)$ is the disk through p.

From the discussion following Definition 5.2, we know that every Q-distinguished history of $p \in P$ comes along with a sequence of inverse branches of $\xi(p)$. Reducing the size of P if necessary, the set of Q-distinguished inverse branches of the disks of P forms a set of positive measure (a flow box) in $\hat{X}$, that we will denote by $\hat{P}$ and which is naturally laminated by unstable manifolds. We denote by $\hat{\xi}^u$ the partition of $\hat{P}$ into unstable disks. By transitivity of the conditional expectation, the conditionals induced by $\hat{\mu}$ on the atoms $\hat{\xi}$ are also induced by $\hat{\mu}_{\hat{P}} = \frac{1}{\hat{\mu}(\hat{P})}\hat{\mu}|_{\hat{P}}$. Since we can exhaust $\hat{X}$ up to a set of arbitrarily small measure with flow boxes, it will be enough to understand the conditionals on $\hat{P}$.

The partition ξ induces a (coarse) partition $\hat{\xi}_0 = \pi_0^{-1}(\xi)$ on $\hat{P}$, defined by $\hat{\xi}_0(\hat{p}) = \pi_0^{-1}\xi(\pi_0(\hat{p}))$. Consider an atom D of the partition ξ on $\check{X}$ and look at the part of $\check{f}^{-n}(D)$ corresponding to the branches belonging to $\hat{P}$. This is a union of univalent inverse branches of $\check{f}^n$, so it inherits a natural finite partition into inverse branches. We can reformulate this as follows: given an atom $C \in \hat{\xi}_0$, $\pi_{-n}(C)$ admits a partition into finitely many pieces corresponding to the inverse branches of $\check{f}^n$ along $\pi_0(C)$. This induces a refinement of $\hat{\xi}_0$ that we denote by $\hat{\xi}_{-n}$ ("separating inverse branches of order n", see Figure 2). It is clear that $\hat{\xi}_{-n}$ is an increasing sequence of partitions such that $\bigvee_{n=0}^{\infty} \hat{\xi}_{-n} = \hat{\xi}^u$, the partition of $\hat{P}$ into unstable leaves, up to a set of zero measure.

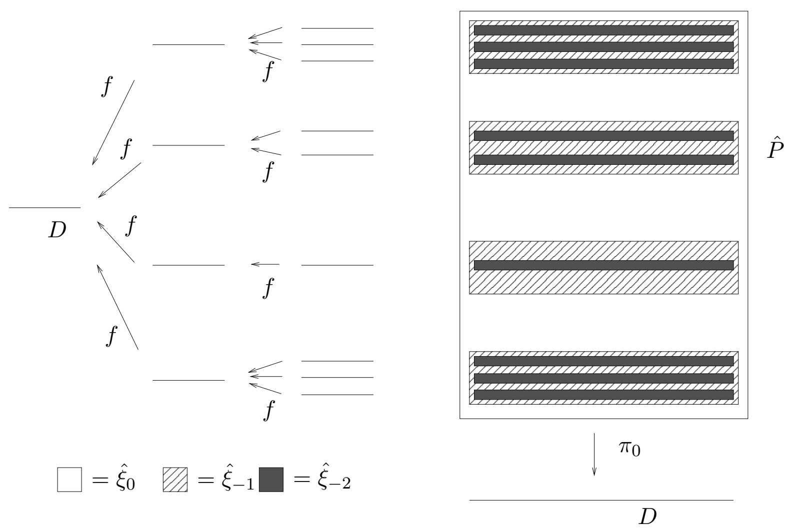

FIGURE 2. Construction of $\hat{\xi}^u$. Inverse branches of D in $\hat{P}$ are depicted on the left, and the inductive construction of the partition on the right

By Theorem 6.1, the conditionals of $\check{\mu}$ on the atoms of ξ are induced by T^+, i.e. for a.e. p, $\check{\mu}_{\xi(p)} = c(p)(T^+ \dot{\wedge} [\xi(p)])$. Consider the conditionals induced by $\hat{\mu}$ on $\hat{\xi}_0$. Applying the projection Lemma 2.2 with $(\tilde{Y}, \nu) = (\hat{X}, \hat{\mu})$ and the projection π_0, we get that for almost every atom C of $\hat{\xi}_0$, $(\pi_0)_*(\hat{\mu}_C) = \check{\mu}_{\pi_0(C)}$.

Consider now the disintegration of $\hat{\mu}$ relative to the refined partition $\hat{\xi}_{-n}$. We will prove that for every n, if C is a generic atom of $\hat{\xi}_{-n}$, $(\pi_0)_*\hat{\mu}_C = \check{\mu}_{\pi_0(C)}$. Let us first see why this implies the statement of the theorem. Recall that $\hat{\xi}_{-n}$ increases to $\hat{\xi}^u$. Hence for

$\hat{\mu}$ a.e. $\hat{p}$ and every measurable function ψ, $\hat{\mu}_{\hat{\xi}_{-n}(\hat{p})}(\psi) \to \hat{\mu}_{\hat{\xi}^u(\hat{p})}(\psi)$ (this is the Martingale Convergence Theorem, see [33]). Now if ψ is of the form $\varphi \circ \pi_0$, the statement that $(\pi_0)_*\hat{\mu}_{\hat{\xi}_{-n}(\hat{p})} = \check{\mu}_{\xi(\pi_0(\hat{p}))}$, implies that for every n, $\hat{\mu}_{\hat{\xi}_{-n}(\hat{p})}(\psi) = \check{\mu}_{\xi(\pi_0(\hat{p}))}(\varphi)$ is independent of n. So we get that $\hat{\mu}_{\hat{\xi}(\hat{p})}(\psi) = \check{\mu}_{\xi(\pi_0(\hat{p}))}(\varphi)$. In other words, $(\pi_0)_*\hat{\mu}_{\hat{\xi}^u(\hat{p})} = \check{\mu}_{\xi(\pi_0(\hat{p}))}$. But now π_0 is a measurable isomorphism $\pi_0 : \hat{\xi}^u(\hat{p}) \to \xi(\pi_0(\hat{p}))$ and $\check{\mu}_{\xi(\pi_0(\hat{p}))}$ is induced by T^+, so the proof is finished.

It remains to prove our claim that if C is a generic atom of $\hat{\xi}_{-n}$, $(\pi_0)_*\hat{\mu}_C = \check{\mu}_{\pi_0(C)}$. For this, denote by D the disk $D = \pi_0(C)$ and notice that $C = (\pi_n)^{-1}(D_{-n})$, where D_{-n} is the image of D by some inverse branch of f^n. By Lemma 2.2 applied to π_{-n}, we get that $(\pi_{-n})_*\hat{\mu}_C = \check{\mu}_{D_{-n}}$. Since $\pi_0 = f^n \circ \pi_{-n}$, we thus obtain that $(\pi_0)_*\hat{\mu}_C = (f^n)_*\check{\mu}_{D_{-n}}$. Now we know that the conditional $\check{\mu}_{D_{-n}}$ is induced by T^+, and by the invariance property of T^+ we have that $T^+ \wedge [D] = \lambda_1^n (f^n)_*(T^+ \wedge [D_{-n}])$. After normalization, we conclude that $(f^n)_*\check{\mu}_{D_{-n}} = \check{\mu}_D$ and the result is proved. □

We can now compute the entropy, using the Rokhlin formula and the invariant "Pesin partition" of [26], as in [15]. See [3] for a nice presentation of the material needed here. The partition ξ is said to be f^{-1}-*invariant* if $f^{-1}\xi$ is a refinement of ξ, i.e. for every x, $f^{-1}(\xi(f(x))) \subset \xi(x)$. It is generating if $\bigvee_{n\geq 0} f^{-n}\xi$ is the partition into points; such partitions allow the computation of entropy.

Proposition 7.7 ([26]). – *There exists a measurable $\hat{f}^{-1}$-invariant and generating partition of $\hat{X}$, whose atoms are open subsets of local unstable manifolds a.s.*

Proof. – The proposition is stated in the context of Pesin's theory applied to diffeomorphisms of manifolds in [26], but it is well adapted to our situation. The exact requirements are listed in [26, Prop. 3.3]. What is needed is a family of local manifolds $V_{\rm loc}(x)$, and a set Λ_ℓ of measure $\geq 1-\varepsilon(\ell)$ such that

– for $x \in \Lambda_\ell$, the manifolds $V_{\rm loc}(x)$ have uniformly bounded geometry, and move continuously;
– $\hat{f}^{-n}$ is uniformly exponentially contracting on $V_{\rm loc}(x)$, $x \in \Lambda_\ell$.

In our situation, we know that $\check{X}$ can be written as a countable union of flow boxes as follows: first write $\check{X}$ as a countable union of flow boxes P. Then write $\pi_0^{-1}(X)$ as a countable union of flow boxes, up to a set of zero measure as follows: consider the increasing sequence of subdivisions Q_i, subdivide P into smaller flow boxes of size Q_i, and consider the Q_i-distinguished histories of the smaller flow boxes.

Now if P is a flow box of size Q and $\hat{P}$ is the set of its Q-distinguished histories, then $\hat{P}$ is naturally a flow box of $\hat{X}$, where the plaques are unstable manifolds and the transversal is the set of Q-distinguished histories of a transversal of P. Moreover, the dynamics of $\hat{f}^{-n}$ is uniformly exponentially contracting along the leaves. We leave the reader check that [26, Prop. 3.1] can now be adapted easily –see also [34] for an adaptation of [26] to a non-invertible situation, using Pesin's theory. □

Corollary 7.8. – $h(f,\mu) = \log \lambda_1$.

Proof. – The proof now follows from a classical argument, which we include for the reader's convenience. We compute the entropy in the natural extension. Let $\hat{\xi}^u$ be the invariant partition constructed above. Since $\hat{\xi}^u$ is a generator and $h_{\hat{\mu}}(\hat{f}) \leq \log \lambda_1$ is finite, $h_{\hat{\mu}}(\hat{f})$ equals the conditional entropy $h_{\hat{\mu}}(\hat{f}, \hat{\xi}^u)$. Now the conditional entropy may computed by the Rokhlin formula:

$$h_{\hat{\mu}}(\hat{f}, \hat{\xi}^u) = -\int \log \mu_{\hat{\xi}^u(x)}(\hat{f}^{-1}\hat{\xi}^u(x)))d\hat{\mu}(x) = \int \log J^u_{\hat{\mu}}(x) d\hat{\mu}(x),$$

where $J^u_{\hat{\mu}}(x) := \left(\hat{\mu}_{\hat{\xi}^u(x)}\big(\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(x)))\big)\right)^{-1}$ is the unstable Jacobian.

Since the $\hat{\xi}^u(x)$ are open subsets of unstable manifolds in $\hat{X}$, by Theorem 7.6 the conditionals $\hat{\mu}_{\hat{\xi}^u(x)}$ are induced by T^+, that is, with the usual abuse of notation,

$$\hat{\mu}_{\hat{\xi}^u(x)} = \frac{T^+ \dot{\wedge} [\hat{\xi}^u(x)]}{\mathbf{M}\big(T^+ \dot{\wedge} [\hat{\xi}^u(x)]\big)}.$$

From the invariance relation $f^*T^+ = \lambda_1 T^+$ we deduce that

$$T^+ \dot{\wedge} [\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(x)))] = \big(T^+ \dot{\wedge} [\hat{\xi}^u(x)]\big)\big|_{\hat{f}^{-1}(\hat{\xi}^u(\hat{f}(x)))} = \frac{1}{\lambda_1}\hat{f}^*(T^+ \dot{\wedge} [\hat{\xi}^u(\hat{f}(x))]),$$

hence the unstable Jacobian $J^u_{\hat{\mu}}$ satisfies the multiplicative cohomological equation

$$J^u_{\hat{\mu}}(x) = \lambda_1 \frac{\rho(x)}{\rho(\hat{f}(x))} \text{ a.e. , where } \rho(x) = \mathbf{M}\big(T^+ \dot{\wedge} [\hat{\xi}^u(x)]\big).$$

To prove that the integral of $\log J^u_{\hat{\mu}}$ equals $\log \lambda_1$, we want to use Birkhoff's Ergodic Theorem. The (additive) coboundary $\log \rho(x) - \log \rho(\hat{f}(x))$ need not be integrable, nevertheless by the invariance of the partition it is bounded from below (see [3, Proposition 3.2] for more details). So Birkhoff's Theorem applies and we conclude that $h_{\hat{\mu}}(\hat{f}) = \log \lambda_1$. □

8. Product structure

We say that a measure has local product structure with respect to stable and unstable manifolds if there is a covering by product subsets of positive measure, in which the vertical (resp. horizontal) fibers are exponentially contracted (resp. expanded) by the dynamics, and the measure is isomorphic to a product measure in each of these subsets. This property is known to have strong ergodic consequences, like the K and Bernoulli property (see [31] for more details).

THEOREM 8.1. – *The measure $\hat{\mu}$ has local product structure with respect to local stable and unstable manifolds in $\hat{X}$.*

Proof. – The proof is in two steps. We first show that $\check{\mu}$ has local product structure. This is actually a statement about the geometry of the currents $T^{\pm}$. Next we pass to the natural extension. What is delicate in this step is to analyze the stable conditionals. Observe also that when unstable manifolds fully depend on histories, the second step is automatic.

Step 1

We begin by identifying product subsets in $\check{X}$. We let $\check{\mathscr{L}}^u$ denote a "flow box for T^-", that is, a sublamination of a flow box in $\check{X}$. We may write $\check{\mathscr{L}}^u = \bigcup_{D^u\in\tau^u} D^u$, where D^u are local unstable disks and τ^u is an (abstract) transversal. Now suppose that D^s_0 is a disk subordinate to T^+ in X and that D^s_0 meets some disk in $\check{\mathscr{L}}^u$ transversely in a single point. By definition of subordinate disks, there is a flow box $\mathscr{L}^s = \bigcup_{D^s\in\tau^s} D^s$ of positive mass for T^+ made of disks D^s close to D^s_0. Restricting $\check{\mathscr{L}}^u$ if necessary, we may assume that each disk in $\mathscr{L}^s$ intersects each disk in $\check{\mathscr{L}}^u$ transversely in a single point.

We lift leaves $D^s\in\tau^s$ to $\check{\mathscr{L}}^u$ by setting[12] $\check{D}^s := \check{\pi}^{-1}(D^s)|_{\check{\mathscr{L}}^u}$ and $\check{\mathscr{L}}^s = \bigcup_{D^s\in\check{\tau}^s}\check{D}^s$. It is clear that $\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s$ is a product set, homeomorphic to $\tau^s\times\tau^u$. For later convenience we identify the abstract transversal τ^s with a subset of some leaf $D^u\in\tau^u$ and likewise τ^u with a subset of some $\check{D}^s$. Since T^+ and T^- intersect geometrically, there is a countable family of disjoint product sets $\tau^s\times\tau^u$ as above, whose union has full $\check{\mu}$ measure.

By the analytic continuation theorem for T^+ [14, Theorem 1.1] we know that $T^+|_{\mathscr{L}^s}$ is uniformly laminar. Similarly, Theorem 1.6 implies that $T^-|_{\check{\pi}(\check{\mathscr{L}}^u)}$ is uniformly woven and can hence be lifted as a uniformly laminar current on the "abstract" lamination $\check{\mathscr{L}}^u$. Abusing notation, we denote this lift by $T^-|_{\check{\mathscr{L}}^u}$. Taking tautological extensions gives rise to a natural product measure on $\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s$, which, abusing notation again, we denote by $T^+|_{\check{\mathscr{L}}^s}\dot{\wedge}T^-|_{\check{\mathscr{L}}^u}$. To conclude that $\check{\mu}$ has local product structure, it remains to prove that this product measure coincides with $\check{\mu}|_{\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s}$.

Recall from Theorem 6.1 that we have constructed $\check{\mu}$ as the increasing limit of the tautological extensions of $T^+\dot{\wedge}T^-_{Q_i}$. The measure $T^+\dot{\wedge}T^-_{Q_i}$ is in turn defined as an increasing limit of $T^+_{Q'}\dot{\wedge}T^-_{Q_i}$. Therefore, the restriction $\check{\mu}|_{\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s}$ is an increasing limit of restrictions of the tautological extensions $(T^+_{Q'}\dot{\wedge}T^-_{Q_i})|_{\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s}$. By definition of the restriction of $T^{+/-}$ to $\mathscr{L}^{s/u}$, all these measures are dominated by $T^+|_{\check{\mathscr{L}}^s}\dot{\wedge}T^-|_{\check{\mathscr{L}}^u}$. So we infer that $\check{\mu}|_{\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s}\leq T^+|_{\check{\mathscr{L}}^s}\dot{\wedge}T^-|_{\check{\mathscr{L}}^u}$. On the other hand $T^+|_{\check{\mathscr{L}}^s}\dot{\wedge}T^-|_{\check{\mathscr{L}}^u}$ is a measure supported on $\check{\mathscr{L}}^u\cap\check{\mathscr{L}}^s$ and dominated by $\check{\mu}$ so the converse inequality is obvious.

Step 1, reinforced

To pass to the natural extension we actually need a stronger version of the product structure of $\check{\mu}$, where stable and unstable pieces are allowed to intersect many times. Consider a flow box $\check{\mathscr{L}}^u$ as above, endowed with the abstract uniformly laminar current $T^-|_{\check{\mathscr{L}}^u}$, or equivalently, with an invariant transverse measure μ^-. Define a *measurable transversal* as a measurable set in $\check{\mathscr{L}}^u$ intersecting each unstable leaf in $\check{\mathscr{L}}^u$ along a discrete set. The transverse measure induces a positive measure μ^-_τ on each measurable transversal τ, invariant under the equivalence relation defined by the unstable leaves (see [30, p. 102]). More precisely, if τ_1 and τ_2 are two measurable transversals and $\phi:\tau_1\to\tau_2$ is a measurable isomorphism preserving the leaves, then $\phi_*\mu^-_{\tau_1}=\mu^-_{\tau_2}$.

[12] This might look quite cumbersome at first sight, but be careful that by definition the natural lifts of disks to the tautological extension never intersect! Here we play with the distinction between total and proper transform: we intersect the proper transform under $\check{\pi}$ of D^u with the total transform of D^s. We put a check on $\check{D}^s$ and not on D^u to emphasize the fact that these two objects do not have the same status.

Using this formalism and the geometric intersection of the currents, we have a finer understanding of the stable conditionals. Consider a set of positive measure, endowed with a partition ξ^s, such that each piece ξ^s is contained in a countable union of disks subordinate to T^+. We can thus consider the trace $\check{\xi}^s$ of this partition on $\check{\mathscr{L}}^u$, as done before. We further assume that each atom of $\check{\xi}^s$ intersects the leaves of $\check{\mathscr{L}}^u$ along a discrete set. So it is a measurable transversal to $\check{\mathscr{L}}^u$. The conclusion is that the conditionals of $\check{\mu}$ (or equivalently, $\check{\mu}_{\check{\mathscr{L}}^u}$) on the pieces $\check{\xi}^s$ are induced by the transverse measure μ^-, i.e. $\check{\mu}_{\check{\xi}^s} = c\mu^-_{\check{\xi}^s}$, with c a normalization constant.

Step 2

As above we view $\hat{X}$ as the natural extension of $\check{X}$, and denote by $(\pi_n)_{n\in\mathbf{Z}}$ the natural projections $\hat{X} \to \check{X}$. Consider a product set $P \simeq \tau^s \times \tau^u$ as defined above. For fine enough Q, the set of Q-distinguished histories of points in P has positive $\hat{\mu}$ measure. Reducing P if necessary, we can assume that P is contained in a flow box of size Q, and denote by $\hat{P}$ the set of Q-distinguished histories with $x_0 \in P$. We will first prove that $\hat{P}$ is a product set (see also Figure 1) and next that $\hat{\mu}|_{\hat{P}}$ has product structure.

Denote by $\hat{\tau}^u$ the set of Q-distinguished histories of points in τ^u. Recall that every Q-distinguished history of $p \in P$ comes along with a sequence of exponentially contracting inverse branches defined on the disk of size Q on which p sits. So for every $\hat{p} \in \hat{\tau}^u$ there exists a lift $\hat{D}^u(\hat{p})$ of $D^u(\pi_0(\hat{p}))$ such that $\pi_0 : \hat{D}^u(\hat{p}) \to D^u(\pi_0(\hat{p}))$ is an isomorphism. Clearly, $\hat{P} \subset \bigcup_{\hat{p}\in\hat{\tau}^u} \hat{D}^u(\hat{p})$.

On the other hand, for every $\hat{p} \in \hat{P}$, the local stable manifold of $\hat{p}$ is the full preimage under π_0 of the local stable manifold of $\pi_0(p)$. More precisely, let $\hat{D}^s(\hat{p}) = \pi_0^{-1}(\check{D}^s(\pi_0(\hat{p}))) \cap \hat{P}$. The dynamics are exponentially contracting along the pieces $\hat{D}^s$. The set of pieces is parameterized by τ^s, or equivalently by the lift of τ^s to some unstable disk $\hat{D}^u$ (recall that τ^s is identified with a subset of some D^u). Since π_0 is injective on $\hat{D}^u(\hat{p})$ and $\pi_0(\hat{D}^u(\hat{p}) \cap \hat{D}^s(\hat{p})) \subset \{\pi_0(\hat{p})\}$, we infer that $\hat{D}^u(\hat{p}) \cap \hat{D}^s(\hat{p}) = \{\hat{p}\}$. We conclude that $\hat{P} \simeq \tau^s \times \hat{\tau}^u$ is a product set.

We now show that $\hat{\mu}$ is a product measure in $\hat{P}$. Since $\hat{P}$ is a product, we have two partitions $\hat{D}^s$ and $\hat{D}^u$, with a natural holonomy map between stable (resp. unstable) pieces. We have to prove that the conditionals induced by $\hat{\mu}$ on the stable (resp. unstable) pieces are invariant under holonomy.

This is easier for unstable conditionals since we know them explicitly. Indeed, let $\hat{D}^u_i$, $i = 1, 2$ be unstable pieces, and $\hat{h} : \hat{D}^u_1 \to \hat{D}^u_2$ be the holonomy map. Define the corresponding objects D^u_i and h in P by projecting under π_0. Recall that π_0 is an isomorphism $\hat{D}^u_i \to D^u_i$. We have proved in Theorem 7.6 that $(\pi_0)_*\hat{\mu}_{\hat{D}^u_i} = \check{\mu}_{D^u_i}$. In addition, by the product structure of $\check{\mu}$ in P, we know that $h_*\check{\mu}_{D^u_1} = \check{\mu}_{D^u_2}$. Since $h \circ \pi_0 = \pi_0 \circ \hat{h}$, we conclude that $\hat{h}_*\hat{\mu}_{\hat{D}^u_1} = \hat{\mu}_{\hat{D}^u_2}$.

Due to possible asymmetry between preimages, we cannot give an explicit description of the conditionals of $\hat{\mu}$ on the stable partition $\hat{D}^s$. We nevertheless have enough information to prove holonomy invariance. That is, if $\hat{h} : D^s_1 \to D^s_2$ is the holonomy map between two stable pieces, we will show that $\hat{h}_*\hat{\mu}_{\hat{D}^s_1} = \hat{\mu}_{\hat{D}^s_2}$. Consider the measurable set $P_{-n} := \pi_{-n}(\hat{P})$.

We will use the following principle "the conditionals $\hat{\mu}_{\hat{D}^s}$ are completely determined by the conditionals induced by $\check{\mu}$ on $\check{f}^{-n}\check{D}^s \cap P_{-n}$, and the latter are invariant under holonomy".

To make the argument more accessible, we first make the following simplifying hypotheses:

(i) $\hat{P} = \pi_{-k}^{-1}(P_{-k})$ for some k;

(ii) for every unstable piece $\hat{D}^u$ of the unstable partition of $\hat{P}$, $\pi_{-n}(\hat{D}^u)$ is contained in a single flow box of $\check{X}$.

Restricting our attention to $n \geq k$, we observe that assumption (i) implies $\hat{P} = \pi_{-n}^{-1}(P_{-n})$. From this and the defining properties of $\hat{\mu}$ we get that $(\pi_{-n})_*\hat{\mu}_{\hat{P}} = \check{\mu}_{P_{-n}}$. Let $\check{D}^{s,-n}$ denote the pullback partition $\check{f}^{-n}\check{D}^s$ of P_{-n}. Since $\pi_0 = f^n \circ \pi_{-n}$ we infer that for every $p \in P_{-n}$, we have $\pi_0^{-1}\check{D}^s(f^n(p)) \cap \hat{P} = \pi_{-n}^{-1}(\check{f}^{-n}\check{D}^s(p) \cap P_{-n})$. Or rather, in our notation, $\hat{D}^s = \pi_{-n}^{-1}(\check{D}^{s,-n})$. Passing to the conditionals, by Lemma 2.2 we infer that for a.e. $\hat{p}$ (with $p_{-n} = \pi_{-n}(\hat{p})$)

$$(\pi_{-n})_*\hat{\mu}_{\hat{D}^s(\hat{p})} = \check{\mu}_{\check{D}^{s,-n}(p_{-n})}. \tag{16}$$

We define the holonomy map (depending on n) between pieces of $\check{D}^{s,-n}$ naturally as follows: let $\hat{D}_1^s$ and $\hat{D}_2^s$ be two stable pieces and $\hat{h}$ be the holonomy map between them. For $p_{-n} = \pi_{-n}(\hat{p}) \in \check{D}_1^{s,-n}$, let $h(p_{-n}) = \pi_{-n}(\hat{h}(\hat{p})) \in \check{D}_2^{s,-n}$ (which is independent of the choice of $\hat{p}$ mapping to p_{-n}). By (ii), the points p_{-n} and $h(p_{-n})$ belong to the same flow box of $\check{X}$, and of course correspond under holonomy in this flow box.

Let us prove that the conditionals $\check{\mu}_{\check{D}^{s,-n}}$ are invariant under h. It is enough to restrict $\check{D}^{s,-n}$ to some flow box $\check{\mathcal{L}}^u$. To simplify notation, we continue to denote the restriction by $\check{D}^{s,-n}$. As the trace in $\mathcal{L}^u$ of a holomorphic disk, $\check{D}^{s,-n}$ intersects unstable disks along discrete sets (the proof of Proposition 7.1 shows that no open subset of $D^{s,-n}$ can be *contained* in an unstable leaf). Hence we are in position to apply the reinforced version of Step 1: the conditionals $\check{\mu}_{\check{D}^{s,-n}}$ are induced by the transverse measure associated to $T^-|_{\check{\mathcal{L}}^u}$, and by assumption (ii), intersection points do not escape $\check{\mathcal{L}}^u$ by flowing under h; hence h defines a measurable isomorphism $\check{D}_1^{s,-n} \to \check{D}_2^{s,-n}$, respecting the leaves. It follows that the conditionals $\check{\mu}_{\check{D}^{s,-n}}$ are invariant under h.

From (16) and this discussion, we deduce for every n and almost every pair of atoms $\hat{D}_1^s$, $\hat{D}_2^s$, that $(\pi_{-n})_*\hat{h}_*\hat{\mu}_{\hat{D}_1^s} = (\pi_{-n})_*\hat{\mu}_{\hat{D}_2^s}$, where $\hat{h}$ is the holonomy map $\hat{D}_1^s \to \hat{D}_2^s$. It follows that $\hat{h}_*\hat{\mu}_{\hat{D}_1^s} = \hat{\mu}_{\hat{D}_2^s}$. Indeed we have two measures $\hat{h}_*\hat{\mu}_{\hat{D}_1^s}$ and $\hat{\mu}_{\hat{D}_2^s}$ on $\hat{D}_2^s$, agreeing on the σ-algebra $\mathcal{F}_n$ generated by sets of the form $\pi_{-n}^{-1}(A)$. For every $A \subset \hat{D}_2^s$, we have $A = \bigcap_{n \geq k} \pi_{-n}^{-1}(\pi_{-n}(A))$. Hence the smallest σ-algebra containing all $\mathcal{F}_n$, $n \geq k$, is the Borel σ-algebra. The assertion now follows from standard measure theory.

What remains now is to remove the simplifying assumptions. We will show that the simplifying assumptions are true "up to subsets of small measure". The details are a bit intricate; we start with a simple observation.

Lemma 8.2. – *Let (A, ν) be a probability space with a measurable partition ξ. Assume that B_n is a sequence of sets with $\nu(B_n) \to 1$. Then $\nu_{\xi(p)}(B_n \cap \xi(p)) = 1 - \varepsilon(p, n)$, with $\varepsilon(\cdot, n) \to 0$ in probability as $n \to \infty$. In particular there is a subsequence such that $\varepsilon(\cdot, n) \to 0$ a.e.*

Consider the product set $\hat{P}$ as above, endowed with the partition $\hat{D}^s$. Since we do not necessarily have that $(\pi_{-n})_*\mu_{\hat{P}} = \mu_{P_n}$, we consider the sequence of sets $\pi_{-n}^{-1}(P_n) = \pi_{-n}^{-1}(\pi_{-n}(P))$ decreasing to $\hat{P}$. We let E_n^s denote the partition $\pi_0^{-1}(\check{D}^s)$ of $\pi_{-n}^{-1}(P_n)$. For every $p \in \hat{P}$, the sequence $E_n^s(\hat{p})$ decreases to $\hat{D}^s(p)$, so $\hat{\mu}_{\pi_{-n}^{-1}(P_n)}|_{\hat{P}}$ is proportional to $\mu_{\hat{P}}$. In terms of the conditionals induced on $\pi_0^{-1}(\check{D}^s)$, this implies that

$$\mu_{E_n^s(\hat{p})}|_{\hat{D}^s(p)} = u(\hat{p}, n)\mu_{\hat{D}^s(\hat{p})} \tag{17}$$

where $u(\cdot, n)$ is constant on $\hat{D}^s(p)$ and increases to 1 a.e. With notation as before, observe that the analogue of (16) is now

$$(\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p})} = \check{\mu}_{\check{D}^{s,-n}(p_{-n})}. \tag{18}$$

We face two problems regarding holonomy. First, $\hat{h}$ is not defined everywhere on $\pi_{-n}^{-1}P_n$. Second, the holonomy h is not defined everywhere in P_{-n} because points can escape flow boxes. Let $\hat{R}$ be the set (depending on n) of points $\hat{p} \in \hat{P}$ such that $\pi_{-n}(\hat{D}^u(\hat{p}))$ is contained in a single flow box of $\check{X}$. By construction, this is a product set. Furthermore, the diameters of the disks $\pi_{-n}(\hat{D}^u(\hat{p}))$ are bounded above by $Cn\lambda_1^{-n/2}$, with C uniform in $\hat{P}$. Thus as soon as $\pi_{-n}(\hat{p})$ has distance at least $Cn\lambda_1^{-n/2}$ from the boundary of a flow box (in the leafwise direction), we have $D^u(\hat{p}) \subset \hat{R}$. Since $\check{\mu}$ concentrates no mass on the boundary, we conclude that the relative measure of $\hat{R}$ in $\hat{P}$ tends to 1 as $n \to \infty$. If $R_{-n} = \pi_{-n}\hat{R}$, then the holonomy map h along the leaves of $\check{X}$ is well defined on R_{-n} and preserves the conditionals induced by μ on the induced partition $D^{s,-n} \cap R_{-n}$. The relationships among the sets we have introduced are summed up as follows:

$$\pi_{-n}^{-1}P_n \supset \hat{P} \supset \hat{R} \subset \pi_{-n}^{-1}R_n \subset \pi_{-n}^{-1}P_n.$$

Moreover, $\hat{h}$ is well-defined on $\hat{P}$, h is well-defined on R_n, and $\pi_{-n} \circ \hat{h} = h \circ \pi_{-n}$ on $\hat{P} \cap \pi_{-n}^{-1}R_n$. Using (18) we deduce for a.e. $\hat{p}_1, \hat{p}_2$ that

$$h_*\left[\left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_1)}\right)|_{R_{-n}}\right] = \left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_2)}\right)|_{R_{-n}}. \tag{19}$$

Restricting measures does not commute with π_{-n} but gives at least an inequality. Combining it with (17), we find

$$\left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_2)}\right)|_{R_{-n}} \geq (\pi_{-n})_*\left(\hat{\mu}_{E_n^s(\hat{p}_2)}|_{\hat{P}\cap\pi_{-n}^{-1}R_n}\right) = u(\hat{p}_2, n)(\pi_{-n})_*\left(\hat{\mu}_{\hat{D}^s(\hat{p}_2)}|_{\pi_{-n}^{-1}R_n}\right).$$

The left side of the inequality is a measure of at most unit mass, whereas by Lemma 8.2 the mass of the right side is of the form $1 - \varepsilon(\hat{p}_2, n)$ (here $\varepsilon(\cdot, n)$ denotes a sequence of functions converging in probability to zero, possibly changing from line to line). So the right and left sides differ by a measure of mass of at most $\varepsilon(\hat{p}_2, n)$. By Lemma 8.2 again, $\hat{\mu}_{\hat{D}^s(\hat{p}_2)}|_{\pi_{-n}^{-1}R_n}$ is close to $\hat{\mu}_{\hat{D}^s(\hat{p}_2)}$ in mass. Therefore, finally we see that

$$\mathbf{M}\left(\left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_2)}\right)|_{R_{-n}} - (\pi_{-n})_*\hat{\mu}_{\hat{D}^s(\hat{p}_2)}\right) = \varepsilon(\hat{p}_2, n). \tag{20}$$

Since $\pi_{-n} \circ \hat{h} = h \circ \pi_{-n}$ on $\hat{P} \cap \pi_{-n}^{-1}R_n$ the left side of (19) similarly satisfies

$$h_*\left[\left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_1)}\right)|_{R_{-n}}\right] \geq u(\hat{p}_1, n)(\pi_{-n})_*\hat{h}_*\left(\hat{\mu}_{\hat{D}^s(\hat{p}_1)}|_{\pi_{-n}^{-1}R_n}\right).$$

Applying the same reasoning on masses yields

$$\mathbf{M}\left(h_*\left[\left((\pi_{-n})_*\hat{\mu}_{E_n^s(\hat{p}_1)}\right)|_{R_{-n}}\right] - (\pi_{-n})_*\hat{h}_*\hat{\mu}_{\hat{D}^s(\hat{p}_1)}\right) = \varepsilon(\hat{p}_1, n). \tag{21}$$

From (20) and (21) we conclude that

$$\mathbf{M}\left((\pi_{-n})_*\hat{h}_*\hat{\mu}_{\hat{D}^s(\hat{p}_1)} - (\pi_{-n})_*\hat{\mu}_{\hat{D}^s(\hat{p}_2)}\right) = \varepsilon(\hat{p}_1, n) + \varepsilon(\hat{p}_2, n).$$

Consider as before the σ-algebra $\mathscr{F}_n$ generated by π_n. Given A in $\mathscr{F}_n$, we get that $\hat{h}_*\hat{\mu}_{\hat{D}^s(\hat{p}_2)}(A)$ and $(\pi_{-n})_*\hat{\mu}_{\hat{D}^s(\hat{p}_2)}(A)$ differ by at most $\varepsilon(\hat{p}_1, n)+\varepsilon(\hat{p}_2, n)$. But since $A \in \mathscr{F}_m$ for all $m \geq n$, we may pass to a subsequence and arrange for a.e. $\hat{p}_1$, $\hat{p}_2$ that the difference tends to zero. Hence $\hat{h}_*\hat{\mu}_{\hat{D}^s(\hat{p}_1)} = \hat{\mu}_{\hat{D}^s(\hat{p}_2)}$, which is what we wanted to prove. □

9. Saddle points

In this section we prove Theorem C. The proof is more classical and follows [2] closely.

Step 1: Pesin theory

As explained in the introduction, the proof relies on Pesin's theory of non-uniformly hyperbolic dynamical systems. The applicability of this theory in our context only requires the assumption that $p \mapsto \log d(p, I^+ \cup C_f) \in L^1(\mu)$, as is neatly shown in [37].

By using the Osedelets theorem or the foregoing study, we have a set of full $\hat{\mu}$ measure provided with an invariant splitting of the tangent space $T_pX = E^u(\hat{p}) \oplus E^s(p)$ into an (exponentially) expanding direction and a contracting direction which depend measurably on $\hat{p}$. As indicated by the notation, $E^s(p)$ depends only on $p = \pi_0(\hat{p})$. By Pesin Theory, there exists an invariant set $\hat{\mathscr{R}} \subset \hat{X}$ of full $\hat{\mu}$ measure such that for each $\hat{p} \in \hat{\mathscr{R}}$, there exists a *Lyapunov chart* $L(\hat{p})$, which is a topological bidisk in X (in the terminology of [2]) centered at $p = \pi_0(\hat{p})$. The bidisk can be chosen to be the image, under the Riemannian exponential map, of an affine bidisk with axes $E^u(\hat{p})$, $E^s(p)$ and measurably varying size $r(\hat{p})$. We can further assume that $L(\hat{p})$ does not intersect $I^+ \cup C_f$ or $f(I^+ \cup C_f)$ and that $f|_{L(\hat{p})}$ is injective.

The Lyapunov charts have the fundamental property that $f : L(\hat{p}) \to L(\hat{f}\hat{p})$ defines a Hénon-like map of degree 1 (we use the terminology of [12]). That is, the cut-off image of a graph over the horizontal (i.e. unstable) direction, is a graph. This property is referred to in [2] as the "u-overflowing property" of Lyapunov charts. The branches of f^{-1} have the overflowing property in the vertical (i.e. stable) direction.

We can also consider the sets

$$L_n^s(\hat{p}) := \left\{y \in L(\hat{p}),\ \forall 1 \leq j \leq n,\ f^j(y) \in L(\hat{f}^j\hat{p})\right\} \text{ and } L_n^u(\hat{p}) := f^n L_n^s(\hat{f}^{-n}\hat{p}),$$

which converge exponentially fast to the local stable manifold $W^s_{\mathrm{loc}}(p)$ and unstable manifold $W^u_{\mathrm{loc}}(\hat{p})$, respectively. Note that depending on the context, we will sometimes regard local stable and unstable manifolds as subsets of X and sometimes as subsets of $\hat{X}$.

We have shown in the proof of Theorem B*i.* that disks subordinate to T^+ are μ-a.e. exponentially contracted by f, while disks subordinate to T^- are $\hat{\mu}$-a.e. exponentially contracted by distinguished preimages of f. On the other hand, the Pesin stable and unstable manifolds are unique. Therefore, Pesin stable and unstable manifolds coincide a.e. with disks subordinate to T^+ and T^-.

If $\hat{p}$ and $\hat{q}$ are sufficiently close in $\hat{X}$, then $W^s_{\mathrm{loc}}(p) \cap W^u_{\mathrm{loc}}(\hat{p})$ is a single point classically denoted by $[\hat{p}, \hat{q}]$. A subset is said to have product structure if it is closed under the operation

$[\cdot,\cdot]$. A *Pesin box* is a compact, positive measure subset of $\hat{X}$ with product structure and a positive lower bound on the size of the associated Lyapunov charts.

Step 2: constructing saddle points

The basic step in the argument is the following: if g is a Hénon-like map of degree 1 in some topological bidisk B, then $\bigcap_{k\in\mathbf{Z}} g^k(B)$ is a single saddle fixed point q of g. Similarly $\bigcap_{k\geq 0} g^k(B) = W^u_B(p)$ and $\bigcap_{k\leq 0} g^k(B) = W^s_B(p)$. Here we employ truncated iteration, in which points are omitted once they leave B.

Fix $\varepsilon > 0$. There exists a compact set $\hat{\mathcal{R}}_\varepsilon$ with $\hat{\mu}\big(\hat{\mathcal{R}}_\varepsilon\big) > 1-\varepsilon$, and where all constants appearing above, as well as the stable and unstable directions and manifolds, vary continuously. As argued in [2, Lemma 1] (stated in the context of polynomial automorphisms, but the proof extends without change to our situation), given $\eta > 0$, there exist finitely many Pesin boxes, each of diameter smaller than η, covering $\hat{\mathcal{R}}_\varepsilon$. Let $\hat{P}$ be one of these Pesin boxes. Since the stable and unstable directions of points belonging to $\hat{P}$ are almost parallel, if η is sufficiently small, there exists a "common Lyapunov chart" B, which is a topological bidisk such that

$$\pi_0(\hat{P}) \subset B \subset \bigcap_{\hat{p}\in\hat{P}} L(\hat{p}).$$

Now if n is large enough and $\hat{p} \in \hat{P}\cap \hat{f}^{-n}(\hat{P})$ (of course there are infinitely many such n), f^n induces a Hénon-like map of degree 1 in B that sends $p = \pi_0(\hat{p})$ to $f^n(p)$. We infer that f^n has a saddle fixed point q exponentially (in n) close to $\pi_0([\hat{p}, \hat{f}^n(\hat{p})]) \subset \pi_0(\hat{P})$.

Now let $\hat{q}$ be the unique periodic point in $\hat{X}$ projecting to q. We claim that $\hat{q}$ is close to $\hat{P}$. By construction, for $k \in \mathbf{N}$ and $0 \leq i \leq n-1$, $f^{i+kn}(q) \in L(\hat{f}^i\hat{p})$. Since $\hat{q}$ is periodic we infer that for $k \in \mathbf{Z}$, $\pi_0\hat{f}^{i+kn}(\hat{q}) \in L(\hat{f}^i\hat{p})$. In particular if i_0 is a fixed integer we can arrange so that for $|i| \leq i_0$, $\pi_0\hat{f}^{-i}(\hat{q})$ is close to $\pi_0\hat{f}^{-i}(\hat{P})$. Thus $\hat{q}$ is close to $\hat{P}$ relative to the product metric on $\hat{X}$, as claimed.

At this stage we know that saddle periodic points accumulate everywhere on $\mathrm{Supp}(\mu)$ and $\mathrm{Supp}(\hat{\mu})$.

Step 3: equidistribution

Given $\hat{p} \in \hat{P}\cap\hat{f}^{-n}(\hat{P})$ as in Step 2, let $B^s_n(p)$ be the connected component of $B\cap f^{-n}B$ containing p. Let us first show that $B^s_n(p) \subset L^s_n(\hat{p})$. For this, suppose that there exist $p' \in B^s_n(p)$ and a smallest integer $i \geq 0$ such that $f^i(p') \notin L(\hat{f}^i(\hat{p}))$. Let γ be a path joining p and p'; $f^i(\gamma)$ is not contained in $L(\hat{f}^i(\hat{p}))$, so it must intersect the vertical boundary of that polydisk. Hence by the Hénon-like property, for all subsequent iterates $j \geq i$, $f^j(\gamma)$ intersects the vertical boundary of $L(\hat{f}^j(\hat{p}))$, which contradicts the fact that $f^n(B^s_n(p)) \subset B$.

Since $B^s_n(p) \subset L^s_n(\hat{p})$, the behavior of f^n on $B^s_n(p)$ is that of n successive iterates of a Hénon-like map of degree 1. In particular we infer that $B^s_n(p)$ is a vertical sub-bidisk of B, while $f^n(B^s_n(p))$ is horizontal. Also, exactly as in [2, Lemma 3] we get that

$$W^s_{loc}(p)\cap B \subset B^s_n(p) \text{ and } W^u_{loc}(\hat{p})\cap B^s_n(p) \subset \left(f^n|_{L^s_n(\hat{p})}\right)^{-1}\left(W^u_{loc}(\hat{f}^n\hat{p})\cap B\right). \tag{22}$$

The sets $\pi_0^{-1}(B^s_n)$ induce a partition of $\hat{P}\cap\hat{f}^{-n}(\hat{P})$. If T is an atom of this partition, it is clear that the construction of Step 2 associates to all points $\hat{p} \in T$ the same saddle point $q = q(T)$, and that the mapping $T \mapsto q(T)$ is injective. From (22) and the fact that the local stable manifold of $\hat{p}$ in $\hat{X}$ is the full preimage under π_0 of $W^s_{\mathrm{loc}}(p) \subset X$, we get that T has product structure. That is, if $\hat{p}_1, \hat{p}_2 \in T$, then $[\hat{p}_1, \hat{p}_2] \in T$.

Now since $\hat{\mu}$ has product structure and its unstable conditionals are induced by the current T^+ (Theorems 7.6 and 8.1), we can reproduce [2, Lemma 5] and conclude that for any atom T of the partition, $\hat{\mu}(T) \leq \lambda_1^{-n}\hat{\mu}(\hat{P})$.

Let SFix_n be the set of saddle periodic points of period (dividing) n in $\hat{X}$, and $\hat{\nu}_n = \lambda_1^{-n}\sum_{q\in \mathrm{SFix}_n}\delta_q$. If $\hat{P}_\delta$ denotes the δ-neighborhood of $\hat{P}$, we obtain that for large n,

$$\lambda_1^{-n}\hat{\mu}(\hat{P})\#\left\{\mathrm{SFix}_n \cap \hat{P}_\delta\right\} = \hat{\mu}(\hat{P})\hat{\nu}_n(\hat{P}_\delta) \geq \sum \hat{\mu}(T) = \hat{\mu}(\hat{P}\cap \hat{f}^{-n}\hat{P}) \longrightarrow \hat{\mu}(\hat{P})^2, \tag{23}$$

whence $\liminf \hat{\nu}_n(\hat{P}_\delta) \geq \hat{\mu}(\hat{P})$. If $\hat{\nu}$ is any cluster point of the sequence $\hat{\nu}_n$, we conclude that for any $\delta > 0$, $\hat{\nu}(\hat{P}_\delta) \geq \hat{\mu}(\hat{P})$, thus $\hat{\nu}(\hat{P}) \geq \hat{\mu}(\hat{P})$. Since any open subset in $\hat{X}$ can be covered up to a set of small $\hat{\mu}$ measure by disjoint Pesin boxes, we conclude that $\hat{\nu} \geq \hat{\mu}$

Now assume (see Step 5 below) that we have an estimate $\#\mathrm{SFix}_n \leq \lambda_1^n + o(\lambda_1^n)$. Then $\limsup \nu_n(X) \leq 1 \leq \mu(X)$. From this and the previous paragraph we conclude that $\mu = \lim \nu_n$.

In the (unexpected?) event that the estimate $\#\mathrm{SFix}_n \leq \lambda_1^n + o(\lambda_1^n)$ fails, then we can certainly replace SFix_n with a subset $\mathscr{P}_n \subset \mathrm{SFix}_n$ of size $\#\mathscr{P}_n \sim \lambda_1^n$ for which (23) remains valid relative to a countable collection of Pesin boxes sufficient to provide disjoint "near covers" of any open set. Hence the measures ν_n defined using $\mathscr{P}_n$ instead of SFix_n will again converge to μ.

Step 4: saddle points in $\mathrm{Supp}(\mu)$

We now prove that the saddle points constructed in Step 2 lie inside $\mathrm{Supp}(\mu)$. Here the argument is identical to [15], we sketch it for completeness (see also [3]).

Given a Pesin box $\hat{P}$ as above, let $\mathscr{L}^s \subset X$ be the local stable lamination of $\pi_0(\hat{P})$, that is, the union of local stable manifolds. Likewise we can define the local unstable web $\mathscr{L}^u = \bigcup_{\hat{p}\in\hat{P}} W^u_{\mathrm{loc}}(\hat{p})$. Let $S^+ = T^+|_{\mathscr{L}^s}$ and $S^- = T^-|_{\mathscr{L}^u}$. These currents are uniformly geometric and dominated by T^+ and T^-, respectively. Furthermore, from our understanding of μ we know that $S^+ > 0$ and $S^- > 0$. With notation as in Step 2, let now g be the Hénon-like map in B induced by f^n. Therefore, $\lambda_1^{-nk}(g^k)^*S^+$ is a non-trivial, uniformly laminar current dominated by T^+. As $n \to \infty$ its support converges in the Hausdorff sense to the local stable manifold of q, where $q = \bigcap_{n\in\mathbf{Z}} g^k(B)$ is the saddle point that we have just constructed. Similarly, we have corresponding currents $0 < \frac{1}{\lambda_1^{kn}}(g^k)_*S^- \leq T^-$ with supports converging to the local unstable manifold of (the periodic history of) $\hat{q}$. Hence we obtain a sequence of measures

$$0 < \mu_k = \frac{1}{\lambda_1^{kn}}(g^k)^*S^+ \wedge \frac{1}{\lambda_1^{kn}}(g^k)_*S^- \leq \mu$$

with supports converging to q. We conclude that $q \in \mathrm{Supp}(\mu)$.

Step 5: counting periodic points

The second statement in Theorem C requires a bound of the form $\#Per_n \leq \lambda_1^n + o(\lambda_1^n)$ on the number of isolated periodic points. When f has no curves of periodic points, this is classical. The Lefschetz Fixed Point Formula [20, p.314] asserts that

$$\{\Delta\} \cdot \left\{\mathrm{Graph}(f^k)\right\} = \sum_{0 \leq p,q \leq 2} (-1)^{p+q} \mathrm{Trace}\left((f^k)^*|_{H^{p,q}(X,\mathbf{R})}\right),$$

where Δ is the diagonal in $X \times X$. If f^k has no curve of fixed points, the intersection product is the sum of the multiplicities of periodic points plus a (nonnegative) term coming from the indeterminacy set (see e.g. [9, §8]). All components of $\Delta \cap \mathrm{Graph}(f^k)$ have dimension 0, hence give positive contribution to the intersection product. In particular the left hand side of this inequality dominates the number of fixed points of f^k.

On the other hand, the dominating term on the right hand side is given by the trace of the action on $H^{1,1}$, which is $\lambda_1^n + o(\lambda_1^n)$. Indeed, observe first that by the small topological degree assumption, the action on H^2 predominates. Now, when X is rational, $\dim H^{0,2} = \dim H^{2,0} = 0$ so we are done. When X is irrational it can be checked directly that $\left\|(f^k)^*|_{H^{0,2}}\right\| \lesssim \lambda_2^{n/2}$ (a general argument for this is given in [10, Proposition 5.8]), and we are also done in this case.

When f admits curves of periodic points and $X = \mathbf{P}^2$ or $X = \mathbf{P}^1 \times \mathbf{P}^1$, the result follows from a slightly more sophisticated argument from Intersection Theory [20, §12.2]. We thank Charles Favre for indicating this to us. Indeed in this case, the intersection in the Lefschetz fixed point formula takes place in $X \times X = \mathbf{P}^2 \times \mathbf{P}^2$ or $(\mathbf{P}^1 \times \mathbf{P}^1)^2$. This manifold has the property that its tangent bundle is generated by its sections (see [20, Example 12.2.1]), in which case the contribution of every irreducible component of $\Delta \cap \mathrm{Graph}(f^k)$ to the intersection product $\{\Delta\} \cdot \left\{\mathrm{Graph}(f^k)\right\}$ is nonnegative [20, Corollary 12.2] (see also [20, Example 16.2.2]). As before we conclude that the number of isolated fixed points of f^n is controlled by $\lambda_1^n + o(\lambda_1^n)$.

Observe that if X is any rational surface, by using a birational conjugacy between f and a (possibly non algebraically stable) map on $\mathbf{P}^2$, this argument shows that the number of isolated fixed points of f^n is controlled by $C\lambda_1^n + o(\lambda_1^n)$ for some C.

REMARK 9.1. – Under the same assumptions as in Theorem C we can also adapt [2, Theorem 2] and obtain that the Lyapunov exponents of μ can be evaluated by averaging on saddle orbits, which is an important fact in bifurcation theory.

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