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THE DIMENSION OF SOME AFFINE DELIGNE–LUSZTIG VARIETIES

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ABSTRACT. – We prove Rapoport's dimension conjecture for affine Deligne–Lusztig varieties for GL_h and superbasic b. From this case the general dimension formula for affine Deligne–Lusztig varieties for special maximal compact subgroups of split groups follows, as was shown in a recent paper by Görtz, Haines, Kottwitz, and Reuman.

RÉSUMÉ. – On démontre la conjecture de Rapoport sur la dimension des variétés de Deligne–Lusztig affines pour GL_h et b superbasique. Ce cas implique la formule générale pour la dimension des variétés de Deligne–Lusztig affines pour des sous-groupes compacts maximaux de groupes déployés, résultat démontré dans un article récent de Görtz, Haines, Kottwitz et Reuman.

1. Introduction

Let k be a finite field with $q = p^r$ elements and let $\overline{k}$ be an algebraic closure. Let $F = k((t))$ and let $L = \overline{k}((t))$. Let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by $\sigma : x \mapsto x^q$ the Frobenius of $\overline{k}$ over k and also of L over F.

Let G be a split connected reductive group over k. Let A be a split maximal torus of G and W the Weyl group of A in G. For $\mu \in X_*(A)$ let t^μ be the image of $t \in \mathbb{G}_m(F)$ under the homomorphism $\mu : \mathbb{G}_m \to A$. Let B be a Borel subgroup of G containing A. We write μ_{dom} for the dominant element in the orbit of $\mu \in X_*(A)$ under the Weyl group of A in G.

We recall the definitions of affine Deligne–Lusztig varieties from [6,1]. Let $K = G(\mathcal{O}_L)$ and let $X = G(L)/K$ be the affine Grassmannian. The Cartan decomposition shows that $G(L)$ is the disjoint union of the sets $Kt^\mu K$ where $\mu \in X_*(A)$ is a dominant coweight. For an element $b \in G(L)$ and dominant $\mu \in X_*(A)$, the affine Deligne–Lusztig variety $X_\mu(b)$ is the locally closed reduced $\overline{k}$-subscheme of X defined by

$$X_\mu(b)(\overline{k}) = \{ g \in G(L)/K \mid g^{-1} b\sigma(g) \in Kt^\mu K \}.$$

Left multiplication by $g \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(gb\sigma(g)^{-1})$. Thus the isomorphism class of the affine Deligne–Lusztig variety only depends on the σ-conjugacy class of b.

There is an algebraic group over F associated to G and b whose R-valued points (for any F-algebra R) are given by

$$J(R) = \{ g \in G(R \otimes_F L) \mid g^{-1} b\sigma(g) = b \}.$$

There is a canonical $J(F)$-action on $X_\mu(b)$.

Let ρ be the half-sum of the positive roots of G. By rk_F we denote the dimension of a maximal F-split subtorus. Let $\mathrm{def}_G(b)=\mathrm{rk}_F\,G-\mathrm{rk}_F\,J$. Let $\nu\in X_*(A)_{\mathbb{Q}}$ be the Newton point of b, compare [3]. For nonempty affine Deligne–Lusztig varieties the dimension is given by the following formula. Note that there is a simple criterion by Kottwitz and Rapoport (see [5]) to decide whether an affine Deligne–Lusztig variety is nonempty.

THEOREM 1.1. – *Assume that $X_\mu(b)$ is nonempty. Then*

$$\dim\bigl(X_\mu(b)\bigr)=\langle\rho,\mu-\nu\rangle-\frac{1}{2}\,\mathrm{def}_G(b).$$

Rapoport conjectured this in [7], Conjecture 5.10 in a different form. For the reformulation compare [4]. In [9], Reuman verifies the formula for some small groups and $b=1$. For $G=GL_n$, minuscule μ and over $\mathbb{Q}_p$ rather than over a function field, the Deligne–Lusztig varieties have an interpretation as reduced subschemes of moduli spaces of p-divisible groups. In this case, the corresponding dimension formula is shown by de Jong and Oort (see [2]) if $b\sigma$ is superbasic and in [10] for general $b\sigma$. In [1] 2.15, Görtz, Haines, Kottwitz, and Reuman prove Theorem 1.1 for all $b\in A(L)$. They also show in 5.8 that if there is a Levi subgroup M of G such that $b\in M(L)$ is basic in M and if the formula is true for M,b and μ_M in a certain subset of the set of all M-dominant coweights, then it is also true for (G,b,μ). Thus it is enough to consider superbasic elements b, that is elements for which no σ-conjugate is contained in a proper Levi subgroup of G. They show in 5.9 that it is enough to consider the case that $G=GL_h$ for some h and that b is basic with $m=v_t(\det(b))$ prime to h. In this paper we prove Theorem 1.1 for this remaining case.

The strategy of the proof is as follows: We associate to the elements of $X_\mu(b)$ discrete invariants which we call extended semi-modules. This induces a decomposition of each connected component of $X_\mu(b)$ into finitely many locally closed subschemes. Their dimensions can be written as a combinatorial expression which only depends on the extended semi-module. By estimating these expressions we obtain the desired dimension formula.

For minuscule μ, and over $\mathbb{Q}_p$, the group $J(\mathbb{Q}_p)$ acts transitively on the set of irreducible components of $X_\mu(b)$. As an application of the proof we show that for nonminuscule μ, the action of $J(F)$ on this set may have more than one orbit.

2. Notation and conventions

From now on we use the following notation: Let $G=GL_h$ and let A be the diagonal torus. Let B be the Borel subgroup of lower triangular matrices. For $\mu,\mu'\in X_*(A)_{\mathbb{Q}}$ we say that $\mu\preceq\mu'$ if $\mu'-\mu$ is a non-negative linear combination of positive coroots. As we may identify $X_*(A)_{\mathbb{Q}}$ with $\mathbb{Q}^h$, this induces a partial ordering on the latter set. An element $\mu=(\mu_1,\dots,\mu_h)\in X_*(A)\cong\mathbb{Z}^h$ is dominant if $\mu_1\leqslant\cdots\leqslant\mu_h$.

Let $N=L^h$ and let $M_0\subset N$ be the lattice generated by the standard basis $e_0,\dots,e_{h-1}$. Then $K=GL_h(\mathcal{O}_L)=\mathrm{Stab}(M_0)$ and $g\mapsto gM_0$ defines a bijection

$$X_\mu(b)(\overline{k})\cong\bigl\{M\subset N\text{ lattice }\mid\mathrm{inv}\bigl(M,b\sigma(M)\bigr)=t^\mu\bigr\}.\tag{2.1}$$

We define the volume of $M=gM_0\in X_\mu(b)$ to be $v_t(\det(g))$.

We assume b to be superbasic. The Newton point $\nu\in X_*(A)_{\mathbb{Q}}\cong\mathbb{Q}^h$ of b is then of the form $\nu=(\frac{m}{h},\dots,\frac{m}{h})\in\mathbb{Q}^h$ with $(m,h)=1$. For $i\in\mathbb{Z}$ define e_i by $e_{i+h}=te_i$. We choose b to be the representative of its σ-conjugacy class that maps e_i to e_{i+m} for all i. For superbasic b, the condition that the affine Deligne–Lusztig variety is nonempty, namely $\nu \preceq \mu$, is equivalent to $\sum \mu_i = m$. From now on we assume this.

For each central $\alpha \in X_*(A)$ there is the trivial isomorphism

$$X_\mu(b) \to X_{\mu+\alpha}\big(t^\alpha b\big).$$

We may therefore assume that all μ_i are nonnegative. For the lattices in (2.1), this implies that $b\sigma(M) \subseteq M$.

In the following we will abbreviate the right-hand side of the dimension formula for $X_\mu(b)$ by $d(b,\mu)$.

The set of connected components of X is isomorphic to $\mathbb{Z}$, an isomorphism is given by mapping $g \in GL_h(L)$ to $v_t(\det(g))$. Let $X_\mu(b)^i$ be the intersection of the affine Deligne–Lusztig variety with the i-th connected component of X. Let $\pi \in GL_h(L)$ with $\pi(e_i) = e_{i+1}$ for all $i \in \mathbb{Z}$. Then π commutes with $b\sigma$, and defines isomorphisms $X_\mu(b)^i \to X_\mu(b)^{i+1}$ for all i. Thus it is enough to determine the dimension of $X_\mu(b)^0$.

For superbasic b, an element of $J(F)$ is determined by its value at e_0. More precisely, $J(F)$ is the multiplicative subgroup of a central simple algebra over F. Hence $\operatorname{def}_G(b) = h-1$. If $v_t(\det(g)) = i$ for some $g \in J(F)$, then g induces isomorphisms between $X_\mu(b)^j$ and $X_\mu(b)^{j+i}$ for all j. On $X_\mu(b)^0$, we have an action of $\{g \in J(F) \mid v_t(\det(g)) = 0\} = J(F) \cap \operatorname{Stab}(M_0)$.

Remark 2.1. – To a vector $\psi = (\psi_i) \in \mathbb{Q}^h$ we associate the polygon in $\mathbb{R}^2$ that is the graph of the piecewise linear continuous function $f : [0,h] \to \mathbb{R}$ with $f(0) = 0$ and slope ψ_i on $[i-1,i]$. One can easily see that $d(b,\mu)$ is equal to the number of lattice points below the polygon corresponding to ν and (strictly) above the polygon corresponding to μ.

3. Extended semi-modules

In this section we describe the combinatorial invariants which are used to decompose $X_\mu(b)^0$.

DEFINITION 3.1. – (1) Let m and h be coprime positive integers. A *semi-module* for m, h is a subset $A \subset \mathbb{Z}$ that is bounded below and satisfies $m + A \subset A$ and $h + A \subset A$. Let $B = A \setminus (h + A)$. The semi-module is called *normalized* if $\sum_{a\in B} a = \frac{h(h-1)}{2}$.

(2) Let $\nu = (\frac{m}{h}, \dots, \frac{m}{h}) \in \mathbb{Q}^h$. Let $\mu' = (\mu'_1, \dots, \mu'_h) \in \mathbb{N}^h$ not necessarily dominant with $\nu \preceq \mu'$. A semi-module A for m, h is *of type* μ' if the following condition holds: Let $b_0 = \min\{b \in B\}$ and let inductively $b_i = b_{i-1} + m - \mu'_i h \in \mathbb{Z}$ for $i = 1, \dots, h$. Then $b_0 = b_h$ and $\{b_i \mid i = 0, \dots, h-1\} = B$.

Remark 3.2. – Semi-modules are also used by de Jong and Oort in [2] to define a stratification of a moduli space of p-divisible groups whose rational Dieudonné modules are simple of slope $\frac{m}{h}$. In this case μ is minuscule, and they use semi-modules for $m, h-m$ to decompose the moduli space.

LEMMA 3.3. – *If A is a semi-module, then its translate $-\frac{\sum_{a\in B} a}{h} + \frac{h-1}{2} + A$ is the unique normalized translate of A. It is called the normalization of A. There is a bijection between the set of normalized semi-modules for m, h and the set of possible types $\mu' \in \mathbb{N}^h$ with $\nu \preceq \mu'$.*

Proof. – For the first assertion one only has to notice that the fact that the h elements of B are incongruent modulo h implies that $\sum_{a\in B} a - \frac{h(h-1)}{2}$ is divisible by h. For the second assertion let A be a normalized semi-module, let $b_0 = \min\{a \in B\}$ and let inductively $b_i = b_{i-1} + m - \mu'_i h$ where μ'_i is maximal with $b_i \in A$. Then $b_h = b_0$ and $\{b_i \mid i = 0, \dots, h-1\} = B$. From

$b_0 < b_{i_0}$ for $i_0 = 1, \dots, h-1$ we obtain $\sum_{i=1}^{i_0}(m - \mu_i' h) > 0$ for all $i_0 < h$. Similarly, $b_0 = b_h$ implies $\sum_{i=1}^{h} \mu_i' = m$. This shows $\nu \preceq \mu'$. As $m + A \subset A$, the μ_i' are nonnegative. Given μ' as above, the corresponding normalized semi-module A can be constructed as follows: Let $b_0 = 0$, and inductively $b_i = b_{i-1} + m - \mu_i' h$. Then A is the normalization of $\{b_i + \alpha h \mid \alpha \in \mathbb{N}, 0 \leqslant i < h\}$. □

DEFINITION 3.4. – Let m and h be as before and let $\mu = (\mu_i) \in \mathbb{N}^h$ be dominant with $\sum \mu_i = m$. An *extended semi-module* (A, φ) for μ is a normalized semi-module A for m, h together with a function $\varphi : \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ with the following properties:

(1) $\varphi(a) = -\infty$ if and only if $a \notin A$.
(2) $\varphi(a+h) \geqslant \varphi(a) + 1$ for all a.
(3) $\varphi(a) \leqslant \max\{n \mid a + m - nh \in A\}$ for all $a \in A$. If $b \in A$ for all $b \geqslant a$, then the two sides are equal.
(4) There is a decomposition of A into a disjoint union of sequences $a_j^1, \dots, a_j^h$ with $j \in \mathbb{N}$ and the following properties:
  (a) $\varphi(a_{j+1}^l) = \varphi(a_j^l) + 1$.
  (b) If $\varphi(a_j^l + h) = \varphi(a_j^l) + 1$, then $a_{j+1}^l = a_j^l + h$. Otherwise $a_{j+1}^l > a_j^l + h$.
  (c) The h-tuple $(\varphi(a_0^l))$ is a permutation of μ.

An extended semi-module such that equality holds in (3) for all $a \in A$ is called *cyclic*.

Let A be a normalized semi-module for m, h and let μ' be its type. Let $\mu = \mu'_{\mathrm{dom}}$. Let φ be such that (1) holds and that we have equality in (3) for all $a \in A$. Then in (2) the two sides are also equal for all $a \in A$. A decomposition of A as in (4) is given by putting all elements into one sequence that are congruent modulo h. Hence (A, φ) is a cyclic extended semi-module for μ, called the *cyclic extended semi-module associated to* A.

Example 3.5. – We give an explicit example of a noncyclic extended semi-module for $m = 4$, $h = 5$, and $\mu = (0,0,0,2,2)$. Let A be the normalized semi-module of type $(0,0,1,2,1)$. Then $B = A \setminus (5 + A)$ consists of $-2, -1, 2, 5$, and 6. Let $\varphi(-1) = 0$ and $\varphi(a) = \max\{n \mid a + m - nh \in A\}$ if $a \in A \setminus \{-1\}$. See also Fig. 1 that shows elements of A marked by crosses and the corresponding values of φ. A decomposition of A is given as follows: Three sequences are given by the elements of A congruent to -2, 2, and 5 modulo 5, respectively. The forth sequence is given by all elements congruent to 4 modulo 5 and greater than -1. The last sequence consists of the remaining elements -1 and $6, 11, 16, \dots$.

LEMMA 3.6. – *If (A, φ) is an extended semi-module for μ, and if μ^0 is the type of A, then $\mu^0_{\mathrm{dom}} \preceq \mu$. If $\mu^0_{\mathrm{dom}} = \mu$, then (A, φ) is a cyclic extended semi-module.*

Proof. – Let (A, φ_0) be the cyclic extended semi-module associated to A. Let

$$\{x_1, \dots, x_n\} = \big\{a \in A \mid \varphi(a+h) > \varphi(a) + 1\big\}$$

a			-3	-2	-1	0	1	2	3	4	5	6		
	$\cdots$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$ $\cdots$
$\varphi(a)$		$\cdots$	$-\infty$	0	0	$-\infty$	$-\infty$	0	1	2	2	1	1	2 $\cdots$

Fig. 1. A noncyclic extended semi-module.

with $x_i > x_{i+1}$ for all i. For $i \in \{1, \dots, n\}$ let

$$\varphi_i(a) = \begin{cases} -\infty & \text{if } a \notin A, \\ \varphi(a) & \text{if } a \geqslant x_i, \\ \varphi_i(a+h) - 1 & \text{else.} \end{cases}$$

We show that (A, φ_i) is an extended semi-module for some μ^i with $\mu^{i-1}_{\mathrm{dom}} \preceq \mu^i_{\mathrm{dom}}$ and $\mu^{i-1}_{\mathrm{dom}} \neq \mu^i_{\mathrm{dom}}$ for all $i \geqslant 1$. As $\varphi_n = \varphi$, it then follows that $\mu^0_{\mathrm{dom}} \preceq \mu^n_{\mathrm{dom}} = \mu$ with equality if and only if $n = 0$, that is if φ is cyclic.

The decomposition of (A, φ_i) is defined as follows: For $a < x_i$, the successor of a is $a + h$. Otherwise it is the successor from the decomposition of (A, φ). From the properties of the decompositions for φ_0 and φ one deduces that the decomposition satisfies the required properties. Let $n_i \geqslant 0$ be maximal with $x_i - n_i h \in A$ and let $\alpha_i = \varphi(x_i + h) - 1 - \varphi(x_i) > 0$. Thus φ_i is obtained from φ_{i-1} by subtracting α_i from the values at $x_i, x_i - h, \dots, x_i - n_i h$. From μ^{i-1} we obtain μ^i by replacing the two entries $\varphi_{i-1}(x_i - n_i h) = \varphi_{i-1}(x_i) - n_i$ and $\varphi_{i-1}(x_i) - \alpha_i + 1$ (which is the value of φ of the successor of x_i in the sequence corresponding to φ_i) by $\varphi_{i-1}(x_i) - \alpha_i - n_i$ and $\varphi_{i-1}(x_i) + 1$. As

$$\varphi_{i-1}(x_i) - n_i, \varphi_{i-1}(x_i) - \alpha_i + 1 \in \big(\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1\big),$$

we have $\mu^{i-1}_{\mathrm{dom}} \preceq \mu^i_{\mathrm{dom}}$ and $\mu^{i-1}_{\mathrm{dom}} \neq \mu^i_{\mathrm{dom}}$. □

COROLLARY 3.7. – *If μ is minuscule, then all extended semi-modules for μ are cyclic.*

Proof. – Let (A, φ) be such an extended semi-module. Let μ' be the type of A. Then $\mu'_{\mathrm{dom}} \preceq \mu$, thus $\mu'_{\mathrm{dom}} = \mu$. Hence the assertion follows from the preceding lemma. □

LEMMA 3.8. – *There are only finitely many extended semi-modules (A, φ) for each μ.*

Proof. – Let μ' be the type of the semi-module A. As $\mu'_{\mathrm{dom}} \preceq \mu$, there are only finitely many possible types and corresponding normalized semi-modules. For fixed A, the third condition for extended semi-modules determines all but finitely many values of φ. For the remaining values we have $0 \leqslant \varphi(a) \leqslant \max\{n \mid a + m - nh \in A\}$. Thus for each A there are only finitely many possible functions φ such that (A, φ) is an extended semi-module for μ. □

4. The decomposition of the affine Deligne–Lusztig variety

Let $M \in X_\mu(b)^0$ be a lattice in N. In this section we associate to M an extended semi-module for μ. This leads to a paving of $X_\mu(b)^0$ by finitely many locally closed subschemes. For minuscule μ, this decomposition of the set of lattices is the same as the one constructed by de Jong and Oort in [2], compare also [10, Section 5.1].

Let m and h be as in Section 2. Let $v \in N$ and recall that $te_i = e_{i+h}$. Then we can write $v = \sum_{i \in \mathbb{Z}} \alpha_i e_i$ with $\alpha_i \in \overline{k}$ and $\alpha_i = 0$ for small i. Let

$$I : N \setminus \{0\} \to \mathbb{Z},$$
$$v \mapsto \min\{i \mid \alpha_i \neq 0\}.$$

For a lattice $M \in X_\mu(b)^0$ we consider the set

$$A = A(M) = \big\{I(v) \mid v \in M \setminus \{0\}\big\}.$$

Then $A(M)$ is bounded below and $h+A(M)\subset A(M)$. As $b\sigma(M)\subset M$, we have $m+A(M)\subset A(M)$, thus $A(M)$ is a semi-module for m, h. We have

$$\mathrm{vol}(M)=\left|\mathbb{N}\setminus(A\cap\mathbb{N})\right|-\left|A\setminus(\mathbb{N}\cap A)\right|=0.$$

This implies that $\sum_{a\in B}a=\sum_{i=0}^{h-1}i$, thus A is normalized.

Let further

$$\varphi=\varphi(M):\mathbb{Z}\to\mathbb{N}\cup\{-\infty\},$$
$$a\mapsto\begin{cases}\max\{n\mid\exists v\in M\text{ with }I(v)=a,t^{-n}b\sigma(v)\in M\} & \text{if }a\in A(M),\\ -\infty & \text{else.}\end{cases}$$

Note that by the definition of $A(M)$, the set on the right-hand side is nonempty. As $b\sigma(M)\subset M$, the values of φ are indeed in $\mathbb{N}\cup\{-\infty\}$.

LEMMA 4.1. – *Let $M\in X_\mu(b)^0$. Then $(A(M),\varphi(M))$ is an extended semi-module for μ.*

Proof. – We already saw that $A(M)$ is a normalized semi-module. We have to check the conditions on φ. The first condition holds by definition. Let $v\in M$ with $I(v)=a$ be realizing the maximum for $\varphi(a)$. Then $tv\in M$ with $I(tv)=a+h$ implies that $\varphi(a+h)\geqslant\varphi(a)+1$, which shows (2). Let $v\in M$ with $I(v)=a$ and $t^{-\varphi(a)}b\sigma(v)\in M$. Then $I(t^{-\varphi(a)}b\sigma(v))=a+m-\varphi(a)h\in A(M)$, whence the first part of (3). Let $b\in A$ for all $b\geqslant a$. Let $n_0=\max\{n\mid a+m-nh\in A\}$. Let $v'\in M$ with $I(v')=a+m-n_0h$ and let $v=(b\sigma)^{-1}(t^{n_0}v')\in N$. Then $I(v)=a$, thus $v=\sum_{b\geqslant a}\alpha_b e_b$ for some $\alpha_b\in\overline{k}$. As $b\in A$ for all $b\geqslant a$, we also have $e_b\in M$ for all such b. Thus $v\in M$ with $t^{-n_0}b\sigma(v)=v'\in M$. Hence $\varphi(a)=n_0$. It remains to show (4). For $a\in\mathbb{Z}$ and $\varphi_0\in\mathbb{N}$ let

$$\widetilde{V}_{a,\varphi_0}=\left\{v\in M\mid v=0\text{ or }I(v)\geqslant a,\ t^{-\varphi_0}b\sigma(v)\in M\right\}$$

and $V_{a,\varphi_0}=\widetilde{V}_{a,\varphi_0}/\widetilde{V}_{a,\varphi_0+1}$. Then V_{a_0,φ_0} is a $\overline{k}$-vector space of dimension $|\{a\geqslant a_0\mid\varphi(a)=\varphi_0\}|$. We construct the sequences by inductively sorting all elements $a\in A$ with $\varphi(a)\leqslant\varphi_0$ for some φ_0: For $\varphi_0=\min\{\varphi(a)\mid a\in A\}$ we take each element a with this value of φ as the first element of a sequence. (At the end we will see that we did not construct more than h sequences.) We now describe the induction step from φ_0 to φ_0+1: If $v_1,\dots,v_i$ is a basis of V_{a,φ_0} for some a, then the tv_j are linearly independent in V_{a+h,φ_0+1}. Thus $\dim V_{a,\varphi_0}\leqslant\dim V_{a+h,\varphi_0+1}$ for every a. Hence there are enough elements $a\in A$ with $\varphi(a)=\varphi_0+1$ to prolong all existing sequences such that conditions (a) and (b) are satisfied. We take the $a\in A$ with $\varphi(a)=\varphi_0+1$ that are not already in some sequence as first elements of new sequences. Inductively, this constructs sequences with properties (a) and (b). To show (c), let $a<b_0$. Then

$$\begin{aligned}\left|\{i\mid\mu_i=n\}\right|&=\dim_{\overline{k}}V_{a,n}-\dim_{\overline{k}}V_{a-h,n-1}\\&=\left|\left\{a_0^l\mid\varphi(a_0^l)=n\right\}\right|.\end{aligned}$$

This also shows that we constructed exactly h sequences. □

For each extended semi-module (A,φ) for μ let

$$\mathcal{S}_{A,\varphi}=\left\{M\subset N\text{ lattice}\mid A(M)=A,\ \varphi(M)=\varphi\right\}\subset X.$$

LEMMA 4.2. – *The sets $\mathcal{S}_{A,\varphi}$ are contained in $X_\mu(b)^0$. They define a decomposition of $X_\mu(b)^0$ into finitely many disjoint locally closed subschemes. Especially,* $\dim X_\mu(b)^0=\max\{\dim\mathcal{S}_{A,\varphi}\}$.

Proof. – The last property in the definition of an extended semi-module shows that (A,φ) determines μ. Thus $\mathcal{S}_{A,\varphi} \subseteq X_\mu(b)^0$. Using Lemmas 3.8 and 4.1 it only remains to show that the subschemes are locally closed. The condition that $a \in A(M)$ is equivalent to $\dim(M \cap \langle e_a, e_{a+1}, \ldots\rangle)/(M \cap \langle e_{a+1}, e_{a+2}, \ldots\rangle) = 1$. This is clearly locally closed. If a is sufficiently large, it is contained in all extended semi-modules for μ and if a is sufficiently small, it is not contained in any extended semi-module for μ. Thus fixing A is an intersection of finitely many locally closed conditions on $X_\mu(b)^0$, hence locally closed. Similarly, it is enough to show that $\varphi(a) < n$ for some $a \in A$ and $n \in \mathbb{N}$ is an open condition on $\{M \in X \mid b\sigma(M) \subset M, A(M) = A\} \subset X$. But this condition is equivalent to

$$\big(\langle e_i \mid i \geqslant a\rangle \cap M \cap t^n (b\sigma)^{-1}(M)\big)/\langle e_i \mid i \geqslant a+1\rangle = (0),$$

which is an open condition. □

Let (A,φ) be an extended semi-module for μ. Let

$$\mathcal{V}(A,\varphi) = \big\{(a,b) \in A \times A \mid b > a,\ \varphi(a) > \varphi(b) > \varphi(a-h)\big\}. \tag{4.1}$$

THEOREM 4.3. –

(1) *Let A and φ be as above. There exists a nonempty open subscheme $U(A,\varphi) \subseteq \mathbb{A}^{\mathcal{V}(A,\varphi)}$ and a morphism $U(A,\varphi) \to \mathcal{S}_{A,\varphi}$ that induces a bijection between the set of $\overline{k}$-valued points of $U(A,\varphi)$ and $\mathcal{S}_{A,\varphi}$. Especially,* $\dim(\mathcal{S}_{A,\varphi}) = |\mathcal{V}(A,\varphi)|$.

(2) *If (A,φ) is a cyclic extended semi-module, then $U(A,\varphi) = \mathbb{A}^{\mathcal{V}(A,\varphi)}$.*

Proof. – We denote the coordinates of a point x of $\mathbb{A}^{\mathcal{V}(A,\varphi)}$ by $x_{a,b}$ with $(a,b) \in \mathcal{V}(A,\varphi)$. To define a morphism $\mathbb{A}^{\mathcal{V}(A,\varphi)} \to X$, we describe the image $M(x)$ of a point $x \in \mathbb{A}^{\mathcal{V}(A,\varphi)}(R)$ where R is a $\overline{k}$-algebra. For each $a \in A$ we define an element $v(a) \in N_R = N \otimes_{\overline{k}} R$ of the form $v(a) = \sum_{b \geqslant a} \alpha_b e_b$ with $\alpha_a = 1$. The $R[\![t]\!]$-module $M(x) \subset N_R$ will then be generated by the $v(a)$. We want the $v(a)$ to satisfy the following relations: For $a \in h + A$ we want

$$v(a) = tv(a-h) + \sum_{(a,b) \in \mathcal{V}(A,\varphi)} x_{a,b} v(b). \tag{4.2}$$

Let $y = \max\{b \in B\}$. If $a = y$ we want

$$v(a) = e_a + \sum_{(a,b) \in \mathcal{V}(A,\varphi)} x_{a,b} v(b). \tag{4.3}$$

For all other elements $a \in B$, we want the following equation to hold: Let $a' \in A$ be minimal with $a' + m - \varphi(a')h = a$. Then $v' = t^{-\varphi(a')} b\sigma(v(a')) \in N_R$ with $I(v') = a$. Let

$$v(a) = v' + \sum_{(a,b) \in \mathcal{V}(A,\varphi)} x_{a,b} v(b). \tag{4.4}$$

CLAIM 1. – *For every $x \in \mathbb{A}^{\mathcal{V}(A,\varphi)}(R)$ there are uniquely determined $v(a) \in N_R$ for all $a \in A$ satisfying* (4.2) *to* (4.4)*.*

We set

$$v(a) = \sum_{j \in \mathbb{N}} \alpha_{a,j} e_{a+j}$$

with $\alpha_{a,j} \in R$ and $\alpha_{a,0} = 1$ for all a. We solve the equations by induction on j. Assume that the $\alpha_{a,j}$ are determined for $j \leqslant j_0$ and such that the equations for $v(a)$ hold up to summands of the form $\beta_j e_j$ with $j > a + j_0$. To determine the α_{a,j_0+1}, we write $a \equiv y + im \pmod h$ and proceed by induction on $i \in \{0, \ldots, h-1\}$. For $i = 0$ and $a = y$, the coefficient α_{a,j_0+1} is the uniquely determined element such that (4.3) holds up to summands of the form $\beta_j e_j$ with $j > j_0 + 1$. Note that by induction on j and as $b > a$, the coefficient of e_{y+j_0+1} on the right-hand side of the equation is determined. For $a = y + nh$ with $n > 0$, the coefficients are similarly defined by (4.2). For $i > 0$ and $a \in A$ minimal in this congruence class, the coefficient is determined by (4.4). Here, the coefficient of e_{a+j_0+1} on the right-hand side of each equation is determined by induction on i and j. For larger a in this congruence class we use again (4.2). By passing to the limit on j, we obtain the uniquely defined $v(a) \in N_R$ solving the equations.

CLAIM 2. – *Let $M(x) = \langle v(a) \mid a \in A \rangle_{R[\![t]\!]}$. Then at each specialization of x to a $\overline{k}$-valued point y we have $A = A(M(y))$ and $\varphi(M(y))(a) \geqslant \varphi(a)$ for all a.*

From the definition of M we immediately obtain $A \subseteq A(M(y))$. To show equality consider an element $v = \sum_a \alpha_a v(a) \in M(y) = M$. Write $v = \sum_{i \in \mathbb{Z}} b_i e_i$ with $b_i \in \overline{k}$. Let $i_0 = \min\{I(\alpha_a v(a))\}$. If $b_{i_0} \neq 0$, then $I(v) = i_0 \in A$. Otherwise we consider $\sum_{\{a \mid I(\alpha_a v(a)) = i_0\}} \alpha_a v(a)$. Note that $I(v(a)) \equiv i_0 \pmod h$ for all a occurring in the sum. Then (4.2) shows that this sum can be written as a sum of $v(b)$ with $b > i_0$. Thus we may replace i_0 by a larger number. As $i \in A$ for all sufficiently large i, this shows that $I(v) \in A$, so $A(M) = A$.

Let $x \in \mathbb{A}^{\mathcal{V}(A,\varphi)}(\overline{k})$ and let $M = M(x)$. We show that $t^{-\varphi(a)} b\sigma(v(a)) \in M$ for all a. This means that $\varphi(M)(a) \geqslant \varphi(a)$ for all a. Consider the elements $a' \in A$ that are minimal with $a' + m - \varphi(a')h = a$ for some $a \in B \setminus \{y\}$. For these elements, the assertion follows from (4.4). If a is minimal with $a + m - \varphi(a)h = y$, then $I(t^{-\varphi(a)} b\sigma(v(a))) = y$. As all e_i with $i \geqslant y$ are in M, this element is also contained in M. If $\varphi(a) = \varphi(a-h) + 1$ then $v(a) = tv(a-h)$ and the assertion holds for $a - h$ if and only if it holds for h. From this, we obtain the claim for all $a \in A$ with $\varphi(a) = \max\{n \mid a + m - nh \in A\}$. Especially, it follows for all sufficiently large elements of A. It remains to prove the claim for the finitely many elements $a \in A$ with $\max\{n \mid a + m - nh \in A\} > \varphi(a)$. We use decreasing induction on a: Let a be in this set, and assume that we know the assertion for all $a' > a$. From (4.2) we obtain that

$$t^{-\varphi(a)} b\sigma\big(v(a)\big) = t^{-\varphi(a)-1} b\sigma\big(tv(a)\big) = t^{-\varphi(a)-1} b\sigma\Bigg(v(a+h) - \sum_{b > a+h, \varphi(a+h) > \varphi(b) \geqslant \varphi(a)+1} x_{a+h,b} v(b)\Bigg).$$

By induction, the right-hand side is in M and Claim 2 is shown.

As all μ_i are nonnegative, we constructed a morphism from $\mathbb{A}^{\mathcal{V}(A,\varphi)}$ to the subscheme X_A of X defined by $X_A(\overline{k}) = \{M \mid A(M) = A,\ b\sigma(M) \subseteq M\}$.

CLAIM 3. – *There is a nonempty open subscheme $U(A,\varphi)$ of $\mathbb{A}^{\mathcal{V}(A,\varphi)}$ that is mapped to $\mathcal{S}_{A,\varphi}$. If (A,φ) is cyclic, then $U(A,\varphi) = \mathbb{A}^{\mathcal{V}(A,\varphi)}$.*

In general we do not have $\varphi(M)(a) = \varphi(a)$ for all a. The proof of Lemma 4.2 shows that $\varphi(M)(a) \leqslant \varphi(a)$ is an open condition on X_A, and thus on $\mathbb{A}^{\mathcal{V}(A,\varphi)}$. Let $U(A,\varphi)$ be the corresponding open subscheme, which is then mapped to $\mathcal{S}_{A,\varphi}$. We have to show that it is nonempty, thus to construct a point in $\mathbb{A}^{\mathcal{V}(A,\varphi)}$ where the corresponding function $\varphi(M)$ is equal to φ. If $\varphi(a) = \max\{n \mid a + m - nh \in A\}$, then $\varphi(M)(a) = \varphi(a)$. Especially, the two functions are equal for all a if (A,φ) is cyclic. In this case $U(A,\varphi) = \mathbb{A}^{\mathcal{V}(A,\varphi)}$. If $\varphi(a) + 1 = \varphi(a+h)$ and if $\varphi(M)(a+h) = \varphi(a+h)$, then $\varphi(M)(a+h) - 1 \geqslant \varphi(M)(a) \geqslant \varphi(a)$ implies that $\varphi(M)(a) = \varphi(a)$. Thus it is enough to find a point where $\varphi(M)(a) = \varphi(a)$ for all $a \in A$ with

$\varphi(a+h) > \varphi(a)+1$. For each such a let b_a be the successor in a decomposition of (A,φ) into sequences. Then $(a+h, b_a) \in \mathcal{V}(A,\varphi)$. Let $x_{a+h,b_a} = 1$ for these pairs and choose all other coefficients to be 0. Then for this point and a as before we have that $\varphi(M)(a) = \varphi(b_a) - 1 = \varphi(a)$. Thus $U(A,\varphi)$ is nonempty.

CLAIM 4. – *The map $U(A,\varphi) \to \mathcal{S}_{A,\varphi}$ defines a bijection on $\overline{k}$-valued points.*

More precisely, we have to show that for each $M \in \mathcal{S}_{A,\varphi}$ there is exactly one $x \in U(A,\varphi)(\overline{k})$ such that M contains a set of elements $v(a)$ for $a \in A$ with $I(v(a)) = a$ and satisfying (4.2) to (4.4) for this x. The argument is similar as the construction of $v(a)$ for given x: By induction on j we will show the following assertion: There exist $x^j = (x^j_{a,b}) \in U(A,\varphi)(\overline{k})$ and $v_j(a) \in M$ for all a with $t^{-\varphi(a)} b\sigma(v_j(a)) \in M$ and which satisfy Eqs. (4.2) to (4.4) for x^j up to summands of the form $\beta_n e_n$ with $n > a+j$. Furthermore the $x^j_{a,b}$ with $b-a \leqslant j$ and the coefficients of e_n in $v_j(a)$ for $n \leqslant a+j$ will be chosen independently of j and only depending on M.

For $j=0$ choose any $x^0 \in U(A,\varphi)(\overline{k})$ and $v_0(a) \in M$ with $I(v_0(a)) = a$, first coefficient 1 and $t^{-\varphi(a)} b\sigma(v_0(a)) \in M$. The existence of these $v_0(a)$ follows from $M \in X_\mu(b)$. Assume that the assertion is true for some j_0. For $n \leqslant j_0$ let $x^{j_0+1}_{a,a+n} = x^{j_0}_{a,a+n}$. We proceed again by induction on i to define the coefficients for $a \equiv y + im \pmod h$. Let $a = y$. Choose the coefficients $x^{j_0+1}_{y,y+n}$ with $n > j_0$ such that

$$v_{j_0+1}(y) = e_y + \sum_{(y,y+n)\in\mathcal{V}(A,\varphi)} x^{j_0+1}_{y,y+n} v_{j_0}(y+n)$$

satisfies $t^{-\varphi(y)} b\sigma(v_{j_0+1}(y)) \in M$. The definition of $\varphi = \varphi(M)$ shows that such coefficients exist and from $\varphi(y+n) < \varphi(y)$ it follows that they are unique. For the other elements $v(a)$ we proceed similarly: For those with $a-h \notin A$ we use Eq. (4.4), on the right-hand side with the values from the induction hypothesis, to define the new $v_{j_0+1}(a)$. For $a \in h+A$ we use (4.2). As we know that $t^{-\varphi(a-h)-1} b\sigma(tv_{j_0}(a-h)) \in M$, it is sufficient to consider the $b > a$ with $\varphi(a-h) < \varphi(b) < \varphi(a)$. At each step the coefficient of e_{a+j_0+1} of the right-hand side is already defined by the induction hypothesis. It only depends on the $x^{j_0}_{a,a+n}$ and the coefficients of e_{b+n} of $v_{j_0}(b)$ with $n \leqslant j_0$, hence only on M. The coefficients of x^{j_0+1} are given by requiring that $t^{-\varphi(a)} b\sigma(v_{j_0+1}(a)) \in M$. □

5. Combinatorics

In this section we estimate $|\mathcal{V}(A,\varphi)|$ to determine the dimension of the affine Deligne–Lusztig variety $X_\mu(b)$.

Remark 5.1. – For cyclic extended semi-modules we have $\varphi(a+h) = \varphi(a)+1$ for all $a \in A$. Thus

$$\mathcal{V}(A,\varphi) = \{(b_i, b) \mid b_i \in B,\ b \in A,\ b > b_i,\ \varphi(b) < \varphi(b_i)\}$$

where $B = A \setminus (h+A)$.

PROPOSITION 5.2. – *Let (A,φ) be the cyclic extended semi-module associated to the normalized semi-module A of type μ. Then $|\mathcal{V}(A,\varphi)| = d(b,\mu)$.*

Proof. – Recall that by b_0 we denote the minimal element of A or B. Let b_i be as in the definition of the type of A and let $b_h = b_0$. First we show that

$$\mathcal{V}(A,\varphi)\to\mathbb{Z},$$
$$(b_i,b)\mapsto b-b_i+b_h$$

induces a bijection between $\mathcal{V}(A,\varphi)$ and $\{a\notin A\mid a>b_h\}$. Let $b\in A$ for some $b>b_i$. Then $b-b_i+b_{i+1}\notin A$ if and only if $(b_i,b)\in\mathcal{V}(A,\varphi)$. Let $b_{i_0}=\max\{b_i\in B\}$. We have $b\in A$ for all $b\geqslant b_{i_0}$. Thus for every $b>b_h$ with $b\notin A$, there is an element $(b_i,b-b_h+b_i)\in\mathcal{V}(A,\varphi)$ for some $h>i\geqslant i_0$. Hence $\{a\notin A\mid a>b_h\}$ is in the image of the map. To show that it is injective and that its image is contained in $\{a\notin A\mid a>b_h\}$, it is enough to show that $(b_i,b)\in\mathcal{V}(A,\varphi)$ implies that $b-b_i+b_j\notin A$ for all $j\in\{i+1,\dots,h\}$. Indeed, this ensures that $(b_j,b-b_i+b_j)\notin\mathcal{V}(A,\varphi)$ for all such j and that $b-b_i+b_h\notin A$. We write $b=b_l+\alpha h$ for some l and α. Recall that $\varphi(b_i)=\mu_{i+1}$. As $(b_i,b_l+\alpha h)\in\mathcal{V}(A,\varphi)$, we have $\mu_{l+1}+\alpha<\mu_{i+1}$. Especially, $l<i$. This implies $\mu_{l+1}+\cdots+\mu_{l+\beta}+\alpha<\mu_{i+1}+\cdots+\mu_{i+\beta}$ for all $\beta\leqslant h-i$. Using the recurrence for the b_j, one sees that this implies $b-b_i+b_{i+\beta}\notin A$ for all $\beta\leqslant h-i$.

It remains to count the elements of $\{a\notin A\mid a>b_0\}$. As $h+A\subseteq A$, we have

$$\bigl|\{a\notin A\mid a>b_0\}\bigr|=\left(\sum_{i=0}^{h-1}b_i-b_0-i\right)\cdot\frac{1}{h}.$$

From the construction of A from its type we obtain

$$\begin{aligned}&=\left(\sum_{i=0}^{h-1}\sum_{j=1}^{i}(m-\mu_jh)-i\right)\cdot\frac{1}{h}\\&=\left(\sum_{i=0}^{h-1}\sum_{j=1}^{i}\frac{m}{h}-\mu_j\right)-\frac{h-1}{2}\\&=d(b,\mu).\qquad\square\end{aligned}$$

THEOREM 5.3. – *Let (A,φ) be an extended semi-module for μ. Then $|\mathcal{V}(A,\varphi)|\leqslant d(b,\mu)$.*

Proof of Theorem 5.3 for cyclic extended semi-modules. – We write $B=\{b_0,\dots,b_{h-1}\}$ as in the definition of the type μ' of A. As the extended semi-module is assumed to be cyclic, μ' is a permutation of μ. Using Remark 5.1 we see

$$\begin{aligned}\bigl|\mathcal{V}(A,\varphi)\bigr|&=\bigl|\bigl\{(b_i,a)\in B\times A\mid a>b_i,\ \varphi(a)<\varphi(b_i)\bigr\}\bigr|\\&=\sum_{\{(b_i,b_j)\in B\times B\mid b_j>b_i,\mu'_{j+1}<\mu'_{i+1}\}}\mu'_{i+1}-\mu'_{j+1}\\&\quad+\bigl|\bigl\{(b_i,b_j+\alpha h)\mid b_j<b_i<b_j+\alpha h,\ \mu'_{i+1}>\mu'_{j+1}+\alpha\bigr\}\bigr|.\end{aligned}$$

We refer to these two summands as S_1 and S_2.

Let $(\tilde b_0,\tilde\mu_1),\dots,(\tilde b_{h-1},\tilde\mu_h)$ be the set of pairs $(b_0,\mu'_1),\dots,(b_{h-1},\mu'_h)$, but ordered by the size of b_i. That is, $\tilde b_i<\tilde b_{i+1}$ for all i. Let

$$f:B\to B,$$
$$b_i\mapsto b_{i+1}=b_i+m-\mu'_{i+1}h$$

where we identify b_h with b_0. This defines a permutation of B. From the ordering of the $\tilde b_i$ we obtain $\sum_{i=0}^{i_0}f(\tilde b_i)\geqslant\sum_{i=0}^{i_0}\tilde b_i$ for all i_0. As $f(\tilde b_i)=\tilde b_i+m-\tilde\mu_{i+1}h$, this is equivalent to $\sum_{i=1}^{i_0+1}\tilde\mu_i\leqslant(i_0+1)\frac{m}{h}$ for all i_0. We thus have $\nu\preceq\tilde\mu\preceq\mu$.

Recall the interpretation of $d(b,\mu)$ from Remark 2.1. We show that S_1 is equal to the number of lattice points above μ and on or below $\tilde{\mu}$. The second summand S_2 will be less or equal to the number of lattice points above $\tilde{\mu}$ and below ν. Then the theorem follows for cyclic extended semi-modules.

We have $S_1 = \sum_{i<j} \max\{\tilde{\mu}_{i+1} - \tilde{\mu}_{j+1}, 0\}$. Consider this sum for any permutation $\tilde{\mu}$ of μ. If we interchange two entries $\tilde{\mu}_i$ and $\tilde{\mu}_{i+1}$ with $\tilde{\mu}_i > \tilde{\mu}_{i+1}$, the sum is lessened by the difference of these two values. There are also exactly $\tilde{\mu}_i - \tilde{\mu}_{i+1}$ lattice points on or below $\tilde{\mu}$ and above the polygon corresponding to the permuted vector. If $\tilde{\mu} = \mu$, both S_1 and the number of lattice points above μ and on or below $\tilde{\mu}$ are 0. Thus by induction S_1 is equal to the claimed number of lattice points.

The last step is to estimate S_2. It is enough to construct a decreasing sequence (with respect to $\preceq$) of $\psi^i \in \mathbb{Q}^h$ for $i = 0, \ldots, h-1$ with $\psi^0 = \tilde{\mu}$ and $\psi^{h-1} = \nu$ such that the number of lattice points above ψ^i and on or below ψ^{i+1} is greater or equal to the number of pairs $(\tilde{b}_{i+1}, \tilde{b}_j + \alpha h)$ contributing to S_2. Note that the ψ^i will no longer be lattice polygons. Let $f_i : B \to B$ be defined as follows: For $j > i$ let $f_i(\tilde{b}_j) = f(\tilde{b}_j)$. Let $\{f_i(\tilde{b}_j) \mid 0 \leqslant j \leqslant i\}$ be the set of $f(\tilde{b}_j)$, but sorted increasingly. Let $\psi^i = (\psi^i_j)$ be such that $f_i(\tilde{b}_j) = \tilde{b}_j + m - \psi^i_{j+1} h$, i.e.

$$\psi^i_{j+1} = \frac{\tilde{b}_j + m - f_i(\tilde{b}_j)}{h} = \frac{m}{h} - \frac{f_i(\tilde{b}_j) - \tilde{b}_j}{h}.$$

Similarly as for $\nu \preceq \tilde{\mu}$ one can show that

$$\nu \preceq \psi^{i+1} \preceq \psi^i \preceq \tilde{\mu}$$

for all i. As $f_0 = f$ and $f_{h-1} = \mathrm{id}$, we have $\psi^0 = \tilde{\mu}$ and $\psi^{h-1} = \nu$. It remains to count the lattice points between ψ^i and ψ^{i+1}. To pass from f_i to f_{i+1} we have to interchange the value $f(\tilde{b}_{i+1})$ with all larger $f_i(\tilde{b}_j)$ with $j \leqslant i$. Thus to pass from the polygon associated to ψ^i to the polygon of ψ^{i+1} we have to change the value at j by $(f_i(\tilde{b}_j) - f(\tilde{b}_{i+1}))/h$, and that for all $j \leqslant i$ with $f_i(\tilde{b}_j) > f(\tilde{b}_{i+1})$. Thus there are at least

$$\sum_{j \leqslant i, f_i(\tilde{b}_j) > f(\tilde{b}_{i+1})} \left\lfloor \frac{f_i(\tilde{b}_j) - f(\tilde{b}_{i+1})}{h} \right\rfloor = \sum_{j \leqslant i, f(\tilde{b}_j) > f(\tilde{b}_{i+1})} \left\lfloor \frac{f(\tilde{b}_j) - f(\tilde{b}_{i+1})}{h} \right\rfloor$$

lattice points above ψ^i and on or below ψ^{i+1}. For fixed i and $j < i+1$, the set of pairs $(\tilde{b}_{i+1}, \tilde{b}_j + \alpha h)$ contributing to S_2 is in bijection with $\{\alpha \geqslant 1 \mid f(\tilde{b}_j) - \alpha h > f(\tilde{b}_{i+1})\}$. The cardinality of this set is at most $\lfloor \frac{f(\tilde{b}_j) - f(\tilde{b}_{i+1})}{h} \rfloor$ which proves that S_2 is not greater than the number of lattice points between $\tilde{\mu}$ and ν. □

Example 5.4. – We give an example of a cyclic semi-module (A,φ) where the type of A is not dominant but where $|\mathcal{V}(A,\varphi)| = d(b,\mu)$. Let $m = 4$, $h = 5$, and $\mu = (0,0,1,1,2)$. Let (A,φ) be the cyclic extended semi-module associated to the normalized semi-module of type $(0,0,1,2,1)$. Note that A is the same semi-module as in Example 3.5. Then the dimension of the corresponding subscheme is

$$\left|\mathcal{V}(A,\varphi)\right| = \left|\{(-1,2),(5,6),(5,7)\}\right| = d(b,\mu).$$

Proof of Theorem 5.3. – Let (A,φ) be an extended semi-module for μ. Let φ_i and μ^i be the sequences constructed in the proof of Lemma 3.6. By induction on i we show that

$|\mathcal{V}(A,\varphi_i)| \leqslant d(b,\mu^i)$. For $i=0$, the extended semi-module (A,φ_0) is cyclic, hence the assertion is already shown.

We use the notation of the proof of Lemma 3.6. The description of the difference between μ^i and μ^{i-1} given there shows that

$$\begin{aligned} d(b,\mu^i) - d(b,\mu^{i-1}) &= \sum_{l=1}^{h}\sum_{j=1}^{l}\big(\mu^{i-1}_{\mathrm{dom},j} - \mu^{i}_{\mathrm{dom},j}\big) \\ &= \big(\big|\big\{\mu_j^{i-1} \in \big(\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i)+1\big)\big\}\big| - 1\big) \\ &\quad \times \min\{\alpha_i, n_i+1\}. \end{aligned}$$

We denote this difference by Δ. To show that $|\mathcal{V}(A,\varphi_i)| - |\mathcal{V}(A,\varphi_{i-1})| \leqslant \Delta$ we use the decomposition into sequences a_j^l of the extended semi-module (A,φ_{i-1}). Using the definition of $\mathcal{V}(A,\varphi)$ and the description of the difference between φ_i and φ_{i-1} from the proof of Lemma 3.6 one obtains

$$\big|\mathcal{V}(a,\varphi_i)\big| - \big|\mathcal{V}(a,\varphi_{i-1})\big| = S_1 + S_2 + S_3$$

where

$$\begin{aligned} S_1 &= \big|\big\{(x_i+h,b) \mid b \in A,\ b > x_i+h,\ \varphi_{i-1}(x_i)+1 > \varphi_{i-1}(b) > \varphi_{i-1}(x_i) - \alpha_i\big\}\big|, \\ S_2 &= \big|\big\{(b, x_i - \delta h) \mid b \in B \setminus \{x_i - n_i h\},\ b < x_i - \delta h,\ \delta \in \{0,\dots,n_i\} \\ &\qquad \varphi_{i-1}(x_i) - \delta - \alpha_i < \varphi_{i-1}(b) \leqslant \varphi_{i-1}(x_i) - \delta\big\}\big|, \\ S_3 &= -\big|\big\{(x_i - n_i h, b) \mid b > x_i - n_i h,\ \varphi_{i-1}(x_i) - n_i > \varphi_{i-1}(b) \geqslant \varphi_{i-1}(x_i) - n_i - \alpha_i\big\}\big|. \end{aligned}$$

Here we used that $a \leqslant x_i$ implies that $\varphi_{i-1}(a+h) = \varphi_{i-1}(a)+1$. For each sequence a_j^l of the extended semi-module (A,φ_{i-1}) we use $S_{1,l}$, $S_{2,l}$, and $S_{3,l}$ for the contributions of pairs with $b \in \{a_j^l\}$ to the three summands. Furthermore we write $S^l = S_{1,l} + S_{2,l} + S_{3,l}$. We show the following assertions: If $\varphi_{i-1}(a_0^l) \notin (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i)+1)$ or if $a_0^l = x_i - n_i h$, then $S^l = 0$. Otherwise, $S^l \leqslant \min\{\alpha_i, n_i+1\}$. Then the theorem follows from property (4c) of extended semi-modules.

To determine the S^l, we consider the following cases:

Case 1. – $\varphi_{i-1}(a_0^l) \geqslant \varphi_{i-1}(x_i)+1$. In this case it is easy to see that $S_{1,l} = S_{2,l} = S_{3,l} = 0$.

Case 2. – $a_0^l > x_i$. This implies that $S_{2,l} = 0$. If $\varphi_{i-1}(a_0^l) \leqslant \varphi_{i-1}(x_i) - n_i - \alpha_i$, then $S_{1,l} + S_{3,l} = \alpha_i - \alpha_i = 0$. Let now $\varphi_{i-1}(a_0^l) \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i)+1)$. Then

$$S_{1,l} + S_{3,l} \leqslant \big|\big\{a_j^l \mid \varphi_{i-1}(x_i)+1 > \varphi_{i-1}\big(a_j^l\big) \geqslant \max\big\{\varphi_{i-1}(x_i) - \alpha_i + 1,\ \varphi_{i-1}(x_i) - n_i\big\}\big\}\big|.$$

As $\varphi_{i-1}(a_{j+1}^l) = \varphi_{i-1}(a_j^l)+1$ for all j, the right-hand side is less or equal to $\min\{\alpha_i, n_i+1\}$.

Case 3. – $a_0^l = x_i - n_i h$. This sequence starts with $x_i - n_i h, \dots, x_i, x_i + h$. (Recall that the sequences $\{a_j^l\}$ for φ_{i-1} are of this easy form with stepwidth h as long as $a_j^l \leqslant x_i < x_{i-1}$.) Note that within one sequence $a_j^l > a_{j'}^l$ implies $\varphi_{i-1}(a_j^l) > \varphi_{i-1}(a_{j'}^l)$. Hence this special sequence does not make any contribution, as in S^l we only consider pairs where both elements are in the sequence starting with $x_i - n_i h$.

Case 4. – $a_0^l < x_i$, but not congruent to x_i modulo h. Again $a_{j+1}^l = a_j^l + h$ if $a_j^l \leqslant x_i$. We first assume that $\varphi_{i-1}(a_0^l) \leqslant \varphi_{i-1}(x_i) - n_i - \alpha_i$. Then $S_{2,l} = 0$. Assume that $b = a_j^l$ contributes to $S_{1,l}$. Then $j \geqslant n_i+1$ and $a^l_j > x_i+h$. If $a^l_0 < x_i - n_i h$, then $[x_i - n_i h, x_i + h]$ contains n_i+1 elements of the sequence. Thus in all cases $a^l_{j-n_i-1} > x_i - n_i h$. This element then leads to a contribution to $S_{3,l}$, as $\varphi_{i-1}(a^l_{j-n_i-1}) = \varphi_{i-1}(a^l_j) - n_i - 1$. In the other direction, if a^l_j contributes to $S_{3,l}$, then $a^l_{j+n_i+1}$ contributes to $S_{1,l}$. Thus $S^l = 0$. We now assume that $\varphi_{i-1}(a^l_0) \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i)+1)$. Let n be maximal with $a^l_n = a^l_0 + nh < x_i$. Then we have

$$\begin{aligned} S_{1,l} &= \bigl|\bigl\{a^l_j \mid j > n+1,\ \varphi_{i-1}(x_i) \geqslant \varphi_{i-1}\bigl(a^l_0\bigr) + j > \varphi_{i-1}(x_i) - \alpha_i\bigr\}\bigr|,\\ S_{2,l} &= \bigl|\bigl\{a^l_j \mid 0 \leqslant j \leqslant \min\{n, n_i\},\ \varphi_{i-1}(x_i) \geqslant \varphi_{i-1}\bigl(a^l_0\bigr) + j > \varphi_{i-1}(x_i) - \alpha_i\bigr\}\bigr|,\\ S_{3,l} &= -\bigl|\bigl\{a^l_j \mid j \geqslant \max\{n - n_i + 1, 0\},\ \varphi_{i-1}(x_i) - n_i > \varphi_{i-1}\bigl(a^l_0\bigr) + j\bigr\}\bigr|\\ &= -\bigl|\bigl\{a^l_j \mid j > \max\{n+1, n_i\},\ \varphi_{i-1}(x_i) \geqslant \varphi_{i-1}\bigl(a^l_0\bigr) + j\bigr\}\bigr|. \end{aligned}$$

Thus

$$S^l \leqslant S_{1,l} + S_{2,l} \leqslant \bigl\{j \mid \varphi_{i-1}(x_i) \geqslant \varphi_{i-1}\bigl(a^l_0\bigr) + j > \varphi_{i-1}(x_i) - \alpha_i\bigr\} = \alpha_i.$$

If $n+1 \geqslant n_i$, then $S_{1,l} + S_{3,l} \leqslant 0$. Thus $S^l \leqslant S_{2,l} \leqslant n_i + 1$. If $n_i > n+1$ then $S_{1,l} + S_{3,l} \leqslant n_i - n - 1$ and $S_{2,l} \leqslant n+1$. Hence in both cases $S^l \leqslant \min\{\alpha_i, n_i+1\}$. □

Example 5.5. – Example 3.5 describes a noncyclic extended semi-module (A, φ) for $\mu = (0,0,0,2,2)$ such that

$$\bigl|\mathcal{V}(A,\varphi)\bigr| = \bigl|\bigl\{(5,6),(5,7),(4,6),(4,7)\bigr\}\bigr| = d(b,\mu).$$

Proof of Theorem 1.1. – Lemma 4.2 and Theorem 4.3 imply that $\dim X_\mu(b)^0 = \max|\mathcal{V}(A,\varphi)|$. In Proposition 5.2 we give a pair with $|\mathcal{V}(A,\varphi)| = d(b,\mu)$. Theorem 5.3 shows that the maximum is at most $d(b,\mu)$. Together we obtain $\dim X_\mu(b) = d(b,\mu)$. □

6. Irreducible components

COROLLARY 6.1. – *Let $G = GL_h$, let b be superbasic and $\nu \preceq \mu$. Then the action of $J(F)$ on the set of irreducible components of $X_\mu(b)$ has only finitely many orbits.*

Proof. – It is enough to consider the intersection of the orbits with the set of irreducible components of $X_\mu(b)^0$. Theorem 4.3 implies that each $\mathcal{S}_{A,\varphi}$ is irreducible. Thus the corollary follows from Lemma 3.8. □

Example 6.2. – We give two examples to show that even for superbasic b, the irreducible components of $X_\mu(b)$ are in general not permuted transitively by $J(F)$. The description of $J(F)$ in Section 2 implies that $A(gM) = A(M)$ and $\varphi(gM) = \varphi(M)$ for each $g \in J(F)$ with $v_t(\det(g)) = 0$. First we consider the example $m = 4$, $h = 5$, and $\mu = (0,0,1,1,2)$. It is enough to find two extended semi-modules for μ leading to subschemes of dimension $d(b,\mu) = 3$. Indeed, the subschemes corresponding to different extended semi-modules are disjoint and lead to irreducible components in different $J(F)$-orbits. One such extended semi-module is the cyclic extended semi-module considered in Proposition 5.2. A second extended semi-module (A,φ) is given in Example 5.4. Here, A is of type $(0,0,1,2,1)$, hence different from the semi-module considered before.

For the second example let $m = 4$, $h = 5$, and $\mu = (0,0,0,2,2)$. Here the two extended semi-modules for μ leading to subschemes of dimension $d(b,\mu) = 4$ are the ones considered in Proposition 5.2 and Examples 3.5 and 5.5. The corresponding semi-modules are different as they are of type $(0,0,0,2,2)$ and $(0,0,1,2,1)$.

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