

# RIESZ TRANSFORM ON MANIFOLDS AND HEAT KERNEL REGULARITY

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ABSTRACT. – One considers the class of complete non-compact Riemannian manifolds whose heat kernel satisfies Gaussian estimates from above and below. One shows that the Riesz transform is  $L^p$  bounded on such a manifold, for  $p$  ranging in an open interval above 2, if and only if the gradient of the heat kernel satisfies a certain  $L^p$  estimate in the same interval of  $p$ 's.

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RÉSUMÉ. – On considère la classe des variétés riemanniennes complètes non compactes dont le noyau de la chaleur satisfait une estimation supérieure et inférieure gaussienne. On montre que la transformée de Riesz y est bornée sur  $L^p$ , pour un intervalle ouvert de  $p$  au-dessus de 2, si et seulement si le gradient du noyau de la chaleur satisfait une certaine estimation  $L^p$  pour le même intervalle d'exposants  $p$ .

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### 1. Introduction

The aim of this article is to give a necessary and sufficient condition for the two natural definitions of homogeneous first order  $L^p$  Sobolev spaces to coincide on a large class of Riemannian manifolds, for  $p$  in an interval  $(q_0, p_0)$ , where  $2 < p_0 \leq \infty$  and  $q_0$  is the conjugate exponent to  $p_0$ . On closed manifolds, these definitions are well-known to coincide for all  $1 < p < \infty$ . For non-compact manifolds, and again  $p_0 = \infty$ , a sufficient condition has been asked for by Robert Strichartz in 1983 [92] and many partial answers have been given since. We shall review them in Section 1.3 below. The condition we propose is in terms of regularity of the heat kernel, more precisely in terms of integral estimates of its gradient. We are able to treat manifolds with the doubling property together with natural heat kernel bounds, as well as the ones with locally bounded geometry where the bottom of the spectrum of the Laplacian is positive.

#### 1.1. Background

Let  $M$  be a complete non-compact connected Riemannian manifold,  $\mu$  the Riemannian measure,  $\nabla$  the Riemannian gradient. Denote by  $|\cdot|$  the length in the tangent space, and by  $\|\cdot\|_p$  the norm in  $L^p(M, \mu)$ ,  $1 \leq p \leq \infty$ . One defines  $\Delta$ , the Laplace–Beltrami operator, as a self-adjoint positive operator on  $L^2(M, \mu)$  by the formal integration by parts

$$(\Delta f, f) = \|\nabla f\|_2^2$$

for all  $f \in C_0^\infty(M)$ , and its positive self-adjoint square root  $\Delta^{1/2}$  by

$$(\Delta f, f) = \|\Delta^{1/2} f\|_2^2.$$

As a consequence,

$$(1.1) \quad \|\nabla f\|_2^2 = \|\Delta^{1/2} f\|_2^2.$$

To identify the spaces defined by (completion with respect to) the seminorms  $\|\nabla f\|_p$  and  $\|\Delta^{1/2} f\|_p$  on  $C_0^\infty(M)$  for some  $p \in (1, \infty)$ , it is enough to prove that there exist  $0 < c_p \leq C_p < \infty$  such that for all  $f \in C_0^\infty(M)$

$$(1.2) \quad c_p \|\Delta^{1/2} f\|_p \leq \|\nabla f\|_p \leq C_p \|\Delta^{1/2} f\|_p.$$

Note that the right-hand inequality may be reformulated by saying that the Riesz transform  $\nabla \Delta^{-1/2}$  is bounded from  $L^p(M, \mu)$  to the space of  $L^p$  vector fields,<sup>1</sup> in other words

$$(R_p) \quad \|\nabla \Delta^{-1/2} f\|_p \leq C_p \|f\|_p$$

for some constant  $C_p$  and all  $f \in C_0^\infty(M)$ . Note that  $(R_2)$  is trivial from (1.1). It is well-known (see [6, Section 4], or [20, Section 2.1]) that the right-hand inequality in (1.2) implies the reverse

<sup>1</sup> In the case where  $M$  has finite measure, instead of  $L^p(M)$ , one has to consider the space  $L^p_0(M)$  of functions in  $L^p(M)$  with mean zero; this modification will be implicit in what follows.

inequality

$$\|\Delta^{1/2} f\|_q \leq C_p \|\nabla f\|_q,$$

for all  $f \in C_0^\infty(M)$ , where  $q$  is the conjugate exponent of  $p$ . Hence, (1.2) for all  $p$  with  $1 < p < \infty$  follows from  $(R_p)$  for all  $p$  with  $1 < p < \infty$ . More generally, if  $(R_p)$  holds for  $1 < p < p_0$  (with  $2 < p_0 \leq \infty$ ), one obtains the equivalence (1.2) and the identification of first order Sobolev spaces for  $q_0 < p < p_0$ ,  $q_0$  being the conjugate exponent to  $p_0$ .

Under local assumptions on the manifold, one can hope for the inhomogeneous analog of the equivalence (1.2), namely

$$(1.3) \quad c_p (\|\Delta^{1/2} f\|_p + \|f\|_p) \leq \|\nabla f\|_p + \|f\|_p \leq C_p (\|\Delta^{1/2} f\|_p + \|f\|_p),$$

for all  $f \in C_0^\infty(M)$ . It suffices then to study the boundedness on  $L^p$  of the local Riesz transform  $\nabla(\Delta + a)^{-1/2}$  for some  $a > 0$  large enough. If, in addition, the bottom of the spectrum of the Laplace–Beltrami is positive, that is, if

$$(1.4) \quad \|\nabla f\|_2 \geq \lambda \|f\|_2$$

for some positive real number  $\lambda$  and all  $f \in C_0^\infty(M)$ , one can recover (1.2) from (1.3) (see [18, p. 1154]).

### 1.2. Main results

Let us first recall the result of [18] which deals with  $(R_p)$  for  $1 < p < 2$ . Denote by  $B(x, r)$  the open ball of radius  $r > 0$  and center  $x \in M$ , and by  $V(x, r)$  its measure  $\mu(B(x, r))$ . One says that  $M$  satisfies the doubling property if for all  $x \in M$  and  $r > 0$

$$(D) \quad V(x, 2r) \leq C V(x, r).$$

Denote by  $p_t(x, y)$ ,  $t > 0$ ,  $x, y \in M$ , the heat kernel of  $M$ , that is the kernel of the heat semigroup  $e^{-t\Delta}$ . One says that  $M$  satisfies the on-diagonal heat kernel upper estimate if

$$(DUE) \quad p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})},$$

for all  $x \in M$ ,  $t > 0$  and some constant  $C > 0$ .

**THEOREM 1.1.** – *Let  $M$  be a complete non-compact Riemannian manifold. Assume that (D) and (DUE) hold. Then the Riesz transform is bounded on  $L^p$  for  $1 < p < 2$ .*

It is also shown in [18] that the Riesz transform is unbounded on  $L^p$  for every  $p > 2$  on the manifold consisting of two copies of the Euclidean plane glued smoothly along their unit circles, although it satisfies (D) and (DUE). A stronger assumption is therefore required in order to extend Theorem 1.1 to the range  $p > 2$ .

It is well-known [46, Theorem 1.1] that, under (D), (DUE) self-improves into the off-diagonal upper estimate:

$$(UE) \quad p_t(x, y) \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right),$$

for all  $x, y \in M, t > 0$  and some constants  $C, c > 0$ . A natural way to strengthen the assumption is to impose a lower bound of the same size, that is the full Li–Yau type estimates

$$(LY) \quad \frac{c}{V(y, \sqrt{t})} \exp\left(-C \frac{d^2(x, y)}{t}\right) \leq p_t(x, y) \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right),$$

for all  $x, y \in M, t > 0$  and some constants  $C, c > 0$ . It is known since [63] that such estimates hold on manifolds with non-negative Ricci curvature. Later, it has been proved in [84] that (LY) is equivalent to the conjunction of (D) and the Poincaré inequalities (P) which we introduce next.

We say that  $M$  satisfies the (scaled) Poincaré inequalities if there exists  $C > 0$  such that, for every ball  $B = B(x, r)$  and every  $f$  with  $f, \nabla f$  locally square integrable,

$$(P) \quad \int_B |f - f_B|^2 d\mu \leq Cr^2 \int_B |\nabla f|^2 d\mu,$$

where  $f_E$  denotes the mean of  $f$  on  $E$ .

However, it follows from the results in [58] and [22] that even (D) and (P) do not suffice for the Riesz transform to be bounded on  $L^p$  for all  $p > 2$ .

In fact, there is also an easy necessary condition for  $(R_p)$  to hold. Indeed,  $(R_p)$  implies

$$\|\|\nabla e^{-t\Delta} f\|\|_p \leq C_p \|\Delta^{1/2} e^{-t\Delta} f\|_p \leq \frac{C'_p}{\sqrt{t}} \|f\|_p,$$

for all  $t > 0, f \in L^p(M, \mu)$ , since, according to [89], the heat semigroup is analytic on  $L^p(M, \mu)$ . And this estimate may not hold for all  $p > 2$ , even in presence of (D) and (P).

Our main result is that, under (D) and (P), this condition is sufficient for  $(R_q), 2 < q < p$ . Denote by  $\|T\|_{p \rightarrow p}$  the norm of a bounded sublinear operator  $T$  from  $L^p(M, \mu)$  to itself.

**THEOREM 1.2.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (D) and (P) (or, equivalently, (LY)). If for some  $p_0 \in (2, \infty]$  there exists  $C > 0$  such that, for all  $t > 0$ ,*

$$(G_{p_0}) \quad \|\|\nabla e^{-t\Delta}\|\|_{p_0 \rightarrow p_0} \leq \frac{C}{\sqrt{t}},$$

*then the Riesz transform is bounded on  $L^p$  for  $2 < p < p_0$ .*

We therefore obtain the announced necessary and sufficient condition as follows.

**THEOREM 1.3.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (D) and (P) (or, equivalently, (LY)). Let  $p_0 \in (2, \infty]$ . The following assertions are equivalent:*

- (1) *For all  $p \in (2, p_0)$ , there exists  $C_p$  such that*

$$\|\|\nabla e^{-t\Delta}\|\|_{p \rightarrow p} \leq \frac{C}{\sqrt{t}},$$

*for all  $t > 0$  (in other words,  $(G_p)$  holds for all  $p \in (2, p_0)$ ).*

- (2) *The Riesz transform  $\nabla \Delta^{-1/2}$  is bounded on  $L^p$  for  $p \in (2, p_0)$ .*

Notice that we do not draw a conclusion for  $p = p_0$ . It has been shown in [58] that there exist (singular) manifolds, namely conical manifolds with closed basis, such that  $(R_p)$  hold if

$1 < p < p_0$ , but not for  $p \geq p_0$ , for some  $p_0 \in (2, \infty)$ , and it has been observed in [22] that these manifolds do satisfy (D) and (P); in that case,  $(G_{p_0})$  does not hold either. Strictly speaking, these manifolds are not complete, since they have a point singularity. But one may observe that our proofs do not use completeness in itself, but rather stochastic completeness, that is the property

$$(1.5) \quad \int_M p_t(x, y) d\mu(y) = 1, \quad \forall x \in M, t > 0,$$

which does hold for complete manifolds satisfying (D), or more generally condition (E) below (see [43]), but also for conical manifolds with closed basis.

It follows from Theorems 1.3 and 1.1 that the assumptions of Theorem 1.3 are sufficient for  $(R_p)$  to hold for  $p \in (1, p_0)$ , and for the equivalence (1.2) to hold for  $p \in (q_0, p_0)$  where  $q_0$  is the conjugate exponent to  $p_0$ . In the case  $p_0 = \infty$ , one can formulate a sufficient condition for  $(R_p)$  and (1.2) in the full range  $1 < p < \infty$  in terms of pointwise bounds of the heat kernel and its gradient.

**THEOREM 1.4.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (D) and (DUE). If there exists  $C$  such that, for all  $x, y \in M, t > 0$ ,*

$$(G) \quad |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}[V(y, \sqrt{t})]},$$

*then the Riesz transform is bounded on  $L^p$  and the equivalence (1.2) holds for  $1 < p < \infty$ .*

We have seen that under (D), (DUE) implies (UE), which, together with (G), implies the full estimate (LY) (see for instance [63]). We shall see in Section 3.3 that (G) implies  $(G_p)$  for all  $p \in (2, \infty)$ . This is why Theorem 1.4 is a corollary of Theorems 1.3 and 1.1.

Our results admit local versions. We say that  $M$  satisfies the exponential growth property (E) if for all  $r_0 > 0$ , for all  $x \in M, \theta > 1, r \leq r_0$ ,

$$(E) \quad V(x, \theta r) \leq m(\theta)V(x, r),$$

where  $m(\theta) = Ce^{c\theta}$  for some  $C \geq 0$  and  $c > 0$  depending on  $r_0$ . Note that this implies the local doubling property  $(D_{loc})$ : for all  $r_0 > 0$  there exists  $C_{r_0}$  such that for all  $x \in M, r \in (0, r_0)$ ,

$$(D_{loc}) \quad V(x, 2r) \leq C_{r_0} V(x, r).$$

We write  $(DUE_{loc})$  for the property (DUE) restricted to small times (say,  $t \leq 1$ ).

We say that  $M$  satisfies the local Poincaré property  $(P_{loc})$  if for all  $r_0 > 0$  there exists  $C_{r_0}$  such that for every ball  $B$  with radius  $r \leq r_0$  and every function  $f$  with  $f, \nabla f$  square integrable on  $B$ ,

$$(P_{loc}) \quad \int_B |f - f_B|^2 d\mu \leq C_{r_0} r^2 \int_B |\nabla f|^2 d\mu.$$

**THEOREM 1.5.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (E) and  $(P_{loc})$ . If for some  $p_0 \in (2, \infty]$  and  $\alpha \geq 0$ , and for all  $t > 0$ ,*

$$(G_{p_0}^{loc}) \quad \|\nabla e^{-t\Delta}\|_{p_0 \rightarrow p_0} \leq \frac{Ce^{\alpha t}}{\sqrt{t}},$$

then the local Riesz transform  $\nabla(\Delta + a)^{-1/2}$  is bounded on  $L^p$  for  $2 < p < p_0$  and  $a > \alpha$ .

As a consequence, we can state the

**THEOREM 1.6.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (E) and  $(P_{loc})$ . Let  $p_0 \in (2, \infty]$ . Then the following assertions are equivalent:*

- (1) *For all  $p \in (2, p_0)$ , all  $t > 0$  and some  $\alpha \geq 0$*

$$\|\nabla e^{-t\Delta}\|_{p \rightarrow p} \leq \frac{C_p e^{\alpha t}}{\sqrt{t}}.$$

- (2) *The local Riesz transform  $\nabla(\Delta + a)^{-1/2}$  is bounded on  $L^p$  for  $2 < p < p_0$  and some  $a > 0$ .*

Taking into account the local result in [18], and denoting by  $(G_{loc})$  condition  $(G)$  restricted to small times, the main corollary for the full range  $1 < p < \infty$  is

**THEOREM 1.7.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (E),  $(DUE_{loc})$  and  $(G_{loc})$ . Then, for  $a > 0$  large enough, the local Riesz transform  $\nabla(\Delta + a)^{-1/2}$  is bounded on  $L^p$  and the equivalence (1.3) holds for  $1 < p < \infty$ .*

Finally, thanks to the argument in [18, p. 1154], one obtains

**THEOREM 1.8.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (E),  $(P_{loc})$  and (1.4). Assume that  $(G_{p_0}^{loc})$  holds for some  $p_0 \in (2, \infty]$ . Then  $(R_p)$  holds for all  $p \in (1, p_0)$ , and (1.2) holds for all  $p \in (q_0, p_0)$ , where  $q_0$  is the conjugate exponent to  $p_0$ .*

and, in particular,

**THEOREM 1.9.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (E),  $(DUE_{loc})$ ,  $(G_{loc})$  and (1.4). Then (1.2) holds for all  $p$ ,  $1 < p < \infty$ .*

The core of this paper is concerned with the proof of Theorems 1.2 and 1.4 and of their local versions Theorems 1.5 and 1.7. Before going into details, we comment on anterior results, on the nature of our assumptions, and on our method.

### 1.3. Anterior results

The state of the art consists so far of a list of (quite interesting and typical) examples with *ad hoc* proofs rather than a general theory. These examples essentially fall into three categories:

- I. Global statements for manifolds with at most polynomial growth
  - 1. manifolds with non-negative Ricci curvature [7,8];
  - 2. Lie groups with polynomial volume growth endowed with a sublaplacian [1];
  - 3. co-compact covering manifolds with polynomial growth deck transformation group [32];
  - 4. conical manifolds with compact basis without boundary [58].
- II. A local statement
  - 5. manifolds with Ricci curvature bounded below [7,8].
- III. Global statements for manifolds where the bottom of the spectrum is positive
  - 6. Cartan–Hadamard manifolds where the Laplace operator is strictly positive, plus bounds on the curvature tensor and its two first derivatives [65];
  - 7. unimodular, non-amenable Lie groups [68].

Note that, for results concerning Lie groups in the above list, one can consider not only the case where they are endowed with a translation-invariant Riemannian metric, but also the case where they are endowed with a sublaplacian, that is a sum of squares of invariant vector fields satisfying the Hörmander condition. For more on this, see for instance [1]. Although we did not introduce this framework, for the sake of brevity, our proofs do work without modification in this setting also, as well as, more generally, on a manifold endowed with a subelliptic sum of squares of vector fields.

In cases I and III, the conclusion is the boundedness of the Riesz transform, hence the seminorms equivalence (1.2), for all  $p \in (1, \infty)$ .

In case II, the conclusion involves local Riesz transforms, or the equivalence (1.3) of inhomogeneous Sobolev norms. Note that an important feature of Bakry's result in this case (say, [7, Theorem 4.1]) is the weakness of the assumption: neither positivity of the injectivity radius nor bounds on the derivatives of the curvature tensor are assumed. By contrast, for bounded geometry manifolds, (1.3) follows easily from the Euclidean result by patching.

We feel that there is a logical order between I, II, III: the results in II are nothing but local versions of I, and III follows easily from II if one uses the additional assumption on the spectrum of the Laplace operator. One may observe that the above results were in fact obtained in a quite different chronological order.

The results in I are covered by Theorem 1.4, the one in II by Theorem 1.7, and the ones in III by Theorem 1.9. Let us explain now in each of the above situations where the required assumptions come from.

In case 1, the doubling property follows from Bishop–Gromov volume comparison [12, Theorem 3.10], and the heat kernel bounds including  $(G)$  from [63]. Note however that an important additional outcome of Bakry's method in [7,8] (see also [6] for a more abstract setting) is the independence of constants with respect to the dimension. See the comments in Section 6.

In case 2, the doubling property is obvious, the heat kernel upper bound follows from [98] (but one has nowadays much simpler proofs, see for instance [21] for an exposition), and the gradient bound from [83]. The proof in [1] is much more complicated than ours; it requires some structure theory of Lie groups as well as a substantial amount of homogenization theory.

In case 3,  $(D)$  is again obvious since such a manifold has polynomial volume growth,  $(DUE)$  is well known (it can be extracted from the work of Varopoulos, see for instance [99], but nowadays one can write down a simpler proof by using [16] or [24]; see for instance [47, Theorem 7.12]) and  $(G)$  is proved in [32]. (*Added after acceptance:* another simpler proof of  $(G)$  has been proposed recently in [34].) The boundedness of the Riesz transform is directly deduced in [32] by using further specific properties of this situation.

In case 4, the boundedness of the Riesz transform is obtained for a range  $(1, p_0)$  of values of  $p$ , and is shown to be false outside this range. It follows from [59] that  $(G_p)$  holds for  $1 < p < p_0$ , hence yielding with our result a simple proof of the main results in [58]. By direct estimates from below on  $\nabla e^{-t\Delta}$  as in [59], one can also recover the negative results for  $p \geq p_0$  (see [22]).

Cases 6, 7 are covered by Theorem 1.9. In case 6, we get rid of specific regularity assumptions on the curvature tensor. As far as case 7 is concerned, for more recent results related to Lie groups with exponential growth, see [69,70,42,52]. Note that the groups considered there are either non-unimodular or non-amenable, which allows reduction to a local problem by use of the positivity of the bottom of the spectrum.

Let us finally mention a few results which are *not* covered by our methods: in [58], conical manifolds with compact basis with boundary are considered; in that case, the conical manifold is not complete. The case where the basis is non-compact has been considered in [58], and studied further in [61]; here the volume of balls with finite radius may even be infinite. In [60], the  $L^p$  boundedness of the Riesz transform for all  $p \in (1, \infty)$  is obtained for a specific class

of manifolds with exponential volume growth, namely cuspidal manifolds with compact basis without boundary. In [64], Theorem 2.4, the boundedness of Riesz transform for  $p > 2$  is proved for a class of Riemannian manifolds with a certain amount of negative curvature; here doubling is not assumed, and the main tool is Littlewood–Paley theory, as in [20].

**1.4. About our assumptions**

We discuss here the meaning and relevance of our assumptions.

Let us begin with the basic assumptions on the heat kernel. The two assumptions  $(D)$  and  $(DUE)$  in Theorem 1.1 are equivalent, according to [44], to the so-called relative Faber–Krahn inequality

$$(FK) \quad \lambda_1(\Omega) \geq \frac{c}{r^2} \left( \frac{V(x, r)}{\mu(\Omega)} \right)^{2/\nu},$$

for some  $c, \nu > 0$ , all  $x \in M$ ,  $r > 0$ ,  $\Omega$  smooth subset of  $B(x, r)$ . Here  $\lambda_1(\Omega)$  is the first eigenvalue of the Laplace operator on  $\Omega$  with Dirichlet boundary conditions:

$$\lambda_1(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}, u \in C_0^{\infty}(\Omega) \right\}.$$

In the sequel, we shall sometimes denote by  $(FK)$  the conjunction of  $(D)$  and  $(DUE)$ . Also, we have recalled that the conjunction of  $(D)$  and  $(P)$  is equivalent to  $(LY)$ . It is worthwhile to note that, contrary to the non-negativity of the Ricci curvature,  $(D)$  and  $(P)$  are invariant under quasi-isometry, which is not obvious to check directly on  $(LY)$ .

We may question the relevance of this group of assumptions to Riesz transform bounds; as a matter of fact,  $(G_p)$  is needed but neither  $(FK)$  for  $p < 2$  in [18] nor  $(D)$  and  $(P)$  for  $p > 2$  are known to be necessary. However, it seems out of reach as of today to prove such bounds without some minimal information on the heat kernel. These assumptions are reasonable for the moment but we think they can be weakened. One direction is to replace the doubling condition by exponential volume growth (without positivity of the bottom of the spectrum). This would mean extending the Calderón–Zygmund theory to the exponential growth realm. A very promising tentative in this direction is in [52], although the method has been so far only applied to a (typical) class of Lie groups having a positive bottom of the spectrum. Another direction is to pursue [20] by using Littlewood–Paley–Stein functionals and prove the conjecture stated there. Again, see the comments in Section 6.

Concerning  $(P)$ , a minor improvement of our assumptions is that, under  $(FK)$  and  $(G_{p_0})$ , it may certainly be relaxed to  $L^r$  Poincaré inequalities for  $r$  large enough so as to guarantee some control on the oscillation of the heat kernel. We have not tried to go into this direction here. On the other hand, if the manifold has polynomial volume growth, then as soon as such weak Poincaré inequalities hold,  $(P)$  is necessary for the Riesz transform to be bounded on  $L^p$  for  $p$  larger than the volume growth exponent (see [17, Section 5]).

We continue with estimates on the gradient of the heat kernel.

First, we can reformulate the necessary and sufficient condition in Theorem 1.3 in terms of integral bounds on the gradient of the heat kernel, thanks to the following proposition proved in Section 3.3.

**PROPOSITION 1.10.** – *Assume that  $M$  is a complete non-compact Riemannian manifold satisfying  $(FK)$ . Let  $2 < p_0 \leq \infty$ . The following assertions are equivalent:*

- (1)  $(G_p)$  holds for all  $2 < p < p_0$ .



(2) For all  $2 < p < p_0$ , for all  $y \in M$  and  $t > 0$

$$(1.6) \quad \|\|\nabla_x p_t(\cdot, y)\|\|_p \leq \frac{C_p}{\sqrt{t}[V(y, \sqrt{t})]^{1-\frac{1}{p}}}.$$

As one can see, the case  $p = \infty$  is excluded from this statement. When  $p = \infty$ , (1.6) is precisely (G) in Theorem 1.4 on which we concentrate now. Consider two other conditions on  $\nabla_x p_t(x, y)$ :

$$(1.7) \quad \sup_{t>0, x \in M} \sqrt{t} \int_M |\nabla_x p_t(x, y)| d\mu(y) < \infty.$$

$$(1.8) \quad |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right),$$

for all  $t > 0, x, y \in M$ .

It is easy to show that, under (FK), (1.8)  $\Rightarrow$  (1.7)  $\Rightarrow$  (G). Indeed (1.8)  $\Rightarrow$  (1.7) is immediate by integration using (D). Let us note that (1.7) is equivalent to

$$(G_\infty) \quad \|\|\nabla e^{-t\Delta}\|\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t}}.$$

Note in passing that, by interpolation with (G<sub>2</sub>), (G<sub>∞</sub>) implies (G<sub>p</sub>) for all  $p \in (2, \infty)$ . Then, (1.7)  $\Rightarrow$  (G) follows by writing

$$\nabla_x p_t(x, y) = \int_M \nabla_x p_{t/2}(x, z) p_{t/2}(z, y) d\mu(z)$$

and by direct estimates using (UE). One can see that in fact, under (D), (1.7) is equivalent to (1.8), but this is another story (see [25]).

Next, it is interesting to observe that the size estimates (LY) do include already some regularity estimates for the heat kernel, and that (G) is nothing but a slightly stronger form of this regularity. More precisely, the estimates (LY) are equivalent to a so-called uniform parabolic Harnack principle (see [85]) and, by the same token, they imply

$$(1.9) \quad |p_t(x, y) - p_t(z, y)| \leq \left(\frac{d(x, z)}{\sqrt{t}}\right)^\alpha \frac{C}{V(y, \sqrt{t})},$$

for some  $C, c > 0, \alpha \in (0, 1)$ , and all  $x, y, z \in M, t > 0$ . The additional assumption (G) is nothing but the limit case  $\alpha = 1$  of (1.9). One can therefore summarize the situation by saying that the Hölder regularity of the heat kernel, yielded by the uniform parabolic Harnack principle, is not enough in general for the Riesz transform to be bounded on all  $L^p$  spaces, whereas Lipschitz regularity does suffice.

Unfortunately, (G) does not have such a nice geometric characterization as (D) and (P). In fact, it is unlikely one can find a geometric description of (G) that is invariant under quasi-isometry. It may however be the case that (G), and even more (G<sub>p</sub>), are stable under some kind of perturbation of the manifold, and this certainly deserves investigation.

Let us make a digression. We already observed that  $(G)$  and  $(FK)$  imply the full  $(LY)$  estimates, hence, under  $(FK)$ , (1.8) is equivalent to

$$(1.10) \quad |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}} p_{C't}(x, y)$$

for some constants  $C, C' > 0$ . Note that, in the case where  $C' = 1$ , this can be reformulated as

$$(1.11) \quad |\nabla_x \log p_t(x, y)| \leq \frac{C}{\sqrt{t}},$$

which is one of the fundamental bounds for manifolds with non-negative Ricci curvature (see [63]).

Known methods to prove pointwise gradient estimates include the Li–Yau method ([63], see also [79] for generalizations), as well as coupling [26], and other probabilistic methods (see for instance [76]) including the derivation of Bismut type formulae which enable one to estimate the logarithmic derivative of the heat kernel as in (1.11) (see for instance [39,95,96] and references therein). Unless one assumes non-negativity of the curvature, all these methods are limited so far to small time, more precisely they yield the crucial factor  $\frac{1}{\sqrt{t}}$  only for small time. One may wonder which large scale geometric features, more stable and less specific than non-negativity of the Ricci curvature, would be sufficient to ensure a large time version of such estimates. A nice statement is that if  $M$  satisfies  $(FK)$  and if for all  $t > 0, x, y \in M$ ,

$$|\nabla_x p_t(x, y)| \leq C |\nabla_y p_t(x, y)|$$

then (1.8) (therefore  $(G)$ ) holds ([45, Theorem 1.3], see also the first remark after Lemma 3.3 in Section 3.2). Another interesting approach is in [32], where  $(G)$  is deduced from a discrete regularity estimate, but here a group invariance is used in a crucial way; see also related results in [54]. The question of finding weaker sufficient conditions for the integrated estimates  $(G_p)$  is so far completely open.

We note that our work is not the first example of the phenomenon that higher integrability of gradients of solutions is related to the  $L^p$  boundedness of singular integrals. Indeed, the property that the gradient of the heat kernel satisfies an  $L^p$  bound can be thought of as analogous to the estimate of Norman Meyers [75] concerning the higher integrability of gradients of solutions, which in turn is connected to Caccioppoli inequalities and reverse Hölder inequalities. It has been pointed out by T. Iwaniec in [56] that  $L^p$  reverse Hölder/Caccioppoli inequalities for solutions to a divergence form elliptic equation  $Lu = f$  are equivalent (at least up to endpoints) to the  $L^p$  boundedness of the Hodge projector  $\nabla L^{-1} \operatorname{div}$ .<sup>2</sup> Our result, which says that the  $L^p$  boundedness of the gradient of the heat semigroup is equivalent (again up to endpoints), to the  $L^p$  boundedness of the Riesz transform, is thus in the same spirit.

Let us finally connect (1.8) with properties of the heat kernel on 1-forms. In [19,20], the boundedness of the Riesz transform on  $L^p$  is proved for  $2 < p < \infty$  under  $(FK)$  and the assumption that the heat kernel on 1-forms is dominated by the heat kernel on functions: for

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<sup>2</sup> *Added after acceptance:* in fact, a recent work by Z. Shen [87] shows that reverse Hölder inequalities are equivalent to  $L^p$  boundedness of the Hodge projector and also to the  $L^p$  boundedness of the Riesz transform  $\nabla L^{-1/2}$  when  $p > 2$  and  $L$  is a real symmetric uniformly elliptic operator  $-\operatorname{div}(A\nabla)$  on Lipschitz domains of  $\mathbb{R}^n$ . For this, he states a general theorem akin to our Theorem 2.1 and attributes the method of proof to ideas of Caffarelli–Peral [10]. In a subsequent paper [5], the two first-named authors of the present article will extend these ideas to the manifold setting and prove actually that there always is some  $p_0 > 2$  for which our Theorem 1.3 applies.

all  $t > 0$ ,  $\omega \in C^\infty T^*M$ ,

$$|e^{-t\bar{\Delta}}\omega| \leq C e^{-ct\Delta}|\omega|$$

(the case  $C = c = 1$  of this estimate corresponds to non-negative Ricci curvature). It would certainly be interesting to investigate the class of manifolds where this domination condition holds; unfortunately, this is in general too strong a requirement, since for instance it does not hold for nilpotent Lie groups, as is shown in [80,81], whereas on such groups the Riesz transforms are known to be bounded on  $L^p$  for all  $p \in (1, \infty)$  [1].

It is conjectured in [20] that the same result is true under a weak commutation between the gradient and the semigroup (since commutation is too much to ask, of course) that is the restriction of the above domination condition to *exact* forms: for all  $t > 0$ ,  $f \in C_0^\infty(M)$ ,

$$|\nabla e^{-t\Delta}f| \leq C e^{-ct\Delta}|\nabla f|,$$

and even its weaker, but more natural,  $L^2$  version:

$$(1.12) \quad |\nabla e^{-t\Delta}f|^2 \leq C e^{-ct\Delta}(|\nabla f|^2).$$

(*Added after acceptance:* we mention a paper by Driver and Melcher [31] where the  $L^p$  versions of such inequalities (with  $c = 1$ ) are proved on the Heisenberg group  $\mathbb{H}^1$  for all  $p > 1$  by probabilistic methods.) This is what we prove here, in the class of manifolds with  $(FK)$ , as a consequence of Theorem 1.4, since then (1.12) is equivalent to (1.8) as we show in Lemma 3.3, Section 3.2, when we give a simpler argument for proving Theorem 1.4 with (1.8) instead of  $(G)$ .

## 1.5. About our method

Let us emphasize several features of our method.

First, we develop an appropriate machinery to treat operators beyond the classical Calderón–Zygmund operators. Indeed, our operators no longer have Hölder continuous kernels, as this is often too demanding in applications: the kernel of the Riesz transform is formally given by

$$\int_0^\infty \nabla_x p_t(x, y) \frac{dt}{\sqrt{t}},$$

and condition  $(G)$  is just an upper bound which does not require Hölder regularity on  $\nabla_x p_t(x, y)$  in a spatial variable. Geometrically, this makes a big difference since pointwise upper bounds on the oscillation in  $x$  of  $\nabla_x p_t(x, y)$  seem fairly unrealistic for large time (see [62]). The loss of Hölder continuity is compensated by a built-in regularity property from the semigroup  $e^{-t\Delta}$ . Such an idea, which originates from [51], has been formalised in [35] for boundedness results in the range  $1 < p < 2$  and is actually used in [18] to derive Theorem 1.1. However, this method does not apply to our situation as  $p > 2$ ; a duality argument would not help us either as we would have to make assumptions on the semigroup acting on 1-forms as explained in Section 1.4; this would bring us back to the state of the art in [20] (see [88] for this approach to the results in [20]). But recently, it was shown in [72] that this regularity property can be used for  $L^p$  results in the range  $p > 2$  by employing good- $\lambda$  inequalities as in [41] for an *ad hoc* sharp maximal function; this may be seen as the basis to an  $L^\infty$  to  $BMO$  version of the  $L^1$  to weak  $L^1$  theory in [35]. Note that, in this connection, the usual  $BMO$  theory requires too strong assumptions and can only work in very special situations (see [13]).

Second, our method works for the usual full range  $2 < p < \infty$  of values of  $p$  and also for a limited range  $2 < p < p_0$ . This is important as in applications (to Riesz transforms on manifolds or to other situations, see [2]) the operators may no longer have kernels with pointwise bounds! This is akin to results recently obtained in [9] for  $p < 2$  and non-integral operators, which generalize [35]; in this circle of ideas, see also [53]. Here, we state a general theorem valid in arbitrary range of  $p$ 's above 2, and its local version (Theorems 2.1 and 2.4, Section 2).

Third, we use very little of the differential structure on manifolds, and in particular we do not use the heat kernel on 1-forms as in [7], [8], or [20]. As a matter of fact, our method is quite general, and enables one to prove the  $L^p$  boundedness of a Riesz transform of the form  $\nabla L^{-1/2}$  as soon as the following ingredients are available:

1. doubling measure,
2. scaled Poincaré inequalities,
3.  $e^{-tL}1 = 1$ ,
4. ellipticity, a divergence form structure, and “integration by parts” (in other words, the ingredients necessary to prove Caccioppoli type inequalities),
5.  $L^p$  boundedness of  $\sqrt{t}\nabla e^{-tL}$ ,
6.  $L^2$  bound for the Riesz transform.

In particular, the method applies equally well to accretive (i.e. elliptic) divergence form operators on  $\mathbb{R}^n$ , in which case the  $L^2$  bound is equivalent to the solution of the square root problem of Kato [3]. One of the present authors (Auscher) will present the details of this case (as well as related results) in a forthcoming article [2].

The method is also subject to further extensions to other settings such as general Markov diffusion semigroups on metric measure spaces, or discrete Laplacians on graphs. See Section 6.

## 2. Singular integrals and a variant of the sharp maximal function

In this section,  $(M, d, \mu)$  is a measured metric space. We denote as above by  $B(x, r)$  the open ball of radius  $r > 0$  and center  $x \in M$ , which we assume to be always of finite  $\mu$ -measure. We state and prove a criterion for  $L^p$  boundedness, with  $p > 2$ , for operators such as singular integrals or quadratic expressions. We also give a local analog of this criterion.

### 2.1. The global criterion

We say that  $M$  satisfies the doubling property (that is  $(M, d, \mu)$  is of homogeneous type in the terminology of [15]) if there exists a constant  $C$  such that, for all  $x \in M, r > 0$ ,

$$(D) \quad \mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Consider a sublinear operator acting on  $L^2(M, \mu)$ . We are going to prove a general statement that allows one to obtain a bound for its operator norm on  $L^p(M, \mu)$  for a fixed  $p > 2$ . Such techniques originate, in a Euclidean setting, in [41] (see also [94]) by use of the sharp maximal function and good- $\lambda$  inequalities. It is proved in [72] that, in the definition of the sharp function, the average over balls can be replaced by more general averaging operators depending on the context, and that the ideas of [41] can be adapted. Our method is based on that of [72].

Denote by  $\mathcal{M}$  the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where  $B$  ranges over all open balls containing  $x$ .

**THEOREM 2.1.** – *Let  $(M, d, \mu)$  satisfy (D) and let  $T$  be a sublinear operator which is bounded on  $L^2(M, \mu)$ . Let  $p_0 \in (2, \infty]$ . Let  $A_r$ ,  $r > 0$ , be a family of linear operators acting on  $L^2(M, \mu)$ . Assume*

$$(2.1) \quad \left( \frac{1}{\mu(B)} \int_B |T(I - A_{r(B)})f|^2 d\mu \right)^{1/2} \leq C(\mathcal{M}(|f|^2))^{1/2}(x),$$

and

$$(2.2) \quad \left( \frac{1}{\mu(B)} \int_B |TA_{r(B)}f|^{p_0} d\mu \right)^{1/p_0} \leq C(\mathcal{M}(|Tf|^2))^{1/2}(x),$$

for all  $f \in L^2(M, \mu)$ , all  $x \in M$  and all balls  $B \ni x$ ,  $r(B)$  being the radius of  $B$ . If  $2 < p < p_0$  and  $Tf \in L^p(M, \mu)$  when  $f \in L^p(M, \mu)$ , then  $T$  is of strong type  $(p, p)$  and its operator norm is bounded by a constant depending only on its  $(2, 2)$  norm, on the constant in (D), on  $p$  and  $p_0$ , and on the constants in (2.1) and (2.2).

*Remarks.* –

- If  $p_0 = \infty$ , the left-hand side of (2.2) should be understood as the essential supremum  $\sup_{y \in B} |TA_{r(B)}f(y)|$ .
- The operators  $A_r$  play the role of approximate identities (as  $r \rightarrow 0$ ). Notice that the regularized version  $TA_r$  of  $T$  is controlled by the maximal function of  $|Tf|^2$  which may be surprising at first sight since  $T$  is the object under study. The improvement from 2 to  $p_0$  in the exponents expresses a regularizing effect of  $A_r$ .
- Define, for  $f \in L^2(M, \mu)$ ,

$$\mathcal{M}_{T,A}^\# f(x) = \sup_{B \ni x} \left( \frac{1}{\mu(B)} \int_B |T(I - A_{r(B)})f|^2 d\mu \right)^{1/2},$$

where the supremum is taken over all balls  $B$  in  $M$  containing  $x$ , and  $r(B)$  is the radius of  $B$ . This is (a variant of) the substitute to the sharp function alluded to above. Assumption (2.1) means that  $\mathcal{M}_{T,A}^\# f$  is controlled pointwise by  $(\mathcal{M}(|f|^2))^{1/2}$ . In fact, rather than the exact form of the control, what matters is that  $\mathcal{M}_{T,A}^\#$  is of strong type  $(p, p)$  for the desired values of  $p$ .

- Note that we assumed that  $T$  was already acting on  $L^p(M, \mu)$  and then we obtained boundedness and a bound of its norm. In practice, this theorem is applied to suitable approximations of  $T$ , the uniformity of the bound allowing a limiting argument to deduce  $L^p$  boundedness of  $T$  itself.
- A careful reader will notice that in the proof below, the  $L^2$  bound for  $T$  is explicitly used only if  $M$  has finite volume; but in practice, the verification of the assumptions (2.1) and (2.2) requires the  $L^2$  boundedness of  $T$  (and  $A_r$ ) anyway.

Let us now prove two lemmas inspired from [72] but with modifications to allow a treatment at a given exponent  $p$  and for a right regularization (see the remark after the proof). The first one is a so-called good- $\lambda$  inequality. For simplicity, we normalize in the following proof the constants in assumptions (2.1) and (2.2) to one.

LEMMA 2.2. – *Let  $(M, d, \mu, A_r, p_0)$  and  $T$  be as above. Assume that (2.2) holds. There exist  $K_0 > 1$  and  $C > 0$  only depending on  $p_0$  and the constant in (D), such that, for every  $\lambda > 0$ , every  $K > K_0$  and  $\gamma > 0$ , for every ball  $B_0$  in  $M$  and every function  $f \in L^2(M, \mu)$  such that there exists  $x_0 \in B_0$  with  $\mathcal{M}(|Tf|^2)(x_0) \leq \lambda^2$ , then*

$$(2.3) \quad \mu(\{x \in B_0; \mathcal{M}(|Tf|^2)(x) > K^2\lambda^2, \mathcal{M}_{T,A}^\# f(x) \leq \gamma\lambda\}) \leq C(\gamma^2 + K^{-p_0})\mu(B_0).$$

*Proof.* – Let us assume first that  $p_0 < \infty$ . Set

$$E = \{x \in B_0; \mathcal{M}(|Tf|^2)(x) > K^2\lambda^2, \mathcal{M}_{T,A}^\# f(x) \leq \gamma\lambda\}.$$

From (2.2) and the hypothesis that  $\mathcal{M}(|Tf|^2)(x_0) \leq \lambda^2$ , one has

$$\int_{3B_0} |TA_{r_0}f|^{p_0} d\mu \leq \lambda^{p_0} \mu(3B_0),$$

where  $r_0 = r(3B_0)$ . Denote

$$\Omega = \{x \in M; \mathcal{M}(|TA_{r_0}f|^2 \chi_{3B_0})(x) > J^2\lambda^2\},$$

where  $J$  is a positive constant to be chosen. By the weak type  $(p_0/2, p_0/2)$  of the maximal operator, we have

$$\mu(\Omega) \leq \frac{C}{J^{p_0} \lambda^{p_0}} \int_{3B_0} |TA_{r_0}f|^{p_0} d\mu \leq CJ^{-p_0} \mu(3B_0).$$

Now, we want to estimate  $\mu(E \setminus \Omega)$ . We remark that by definition, if  $x \in E \setminus \Omega$ , then

$$(2.4) \quad \mathcal{M}(|TA_{r_0}f|^2 \chi_{3B_0})(x) \leq J^2\lambda^2.$$

We first prove that there exists  $c_0$  only depending on (D) such that, if  $c_0K^2 > 1$ , then for every  $x \in E$ ,

$$(2.5) \quad \mathcal{M}(|Tf|^2 \chi_{3B_0})(x) > K^2\lambda^2.$$

Indeed, let  $x \in E$ . Since  $\mathcal{M}(|Tf|^2)(x) > K^2\lambda^2$ , there is a ball  $B$  containing  $x$  such that

$$(2.6) \quad \int_B |Tf|^2 d\mu > K^2\lambda^2 \mu(B).$$

If  $r = 2r(B)$ , one has  $B \subset B(x, r) \subset 3B$ , hence for  $c_0$  only depending on the doubling condition (D) one has  $\mu(B) \geq c_0\mu(B(x, r))$ . Therefore,

$$(2.7) \quad \int_{B(x,r)} |Tf|^2 d\mu > c_0K^2\lambda^2 \mu(B(x, r)).$$

Since  $\mathcal{M}(|Tf|^2)(x_0) \leq \lambda^2$  and  $c_0K^2 > 1$ , one can infer that  $x_0$  does not belong to  $B(x, r)$ . Therefore  $r < 2r(B_0)$  and one concludes that  $B \subset 3B_0$ . Together with (2.6), this yields (2.5).

Next, choose  $J$  such that  $2(J^2 + 1) = K^2$ . Then we have, for  $x \in E \setminus \Omega$ ,

$$\begin{aligned} 2(J^2 + 1)\lambda^2 &< \mathcal{M}(|Tf|^2 \chi_{3B_0})(x) \\ &\leq 2\mathcal{M}(|T(f - A_{r_0}f)|^2 \chi_{3B_0})(x) + 2\mathcal{M}(|TA_{r_0}f|^2 \chi_{3B_0})(x) \\ &\leq 2\mathcal{M}(|T(I - A_{r_0})f|^2 \chi_{3B_0})(x) + 2J^2\lambda^2, \end{aligned}$$

and so

$$\mathcal{M}(|T(I - A_{r_0})f|^2 \chi_{3B_0})(x) > \lambda^2.$$

Therefore

$$E \setminus \Omega \subset \{x \in M; \mathcal{M}(|T(I - A_{r_0})f|^2 \chi_{3B_0})(x) > \lambda^2\}.$$

The weak type  $(1, 1)$  inequality for the Hardy–Littlewood maximal function yields

$$\begin{aligned} \mu(E \setminus \Omega) &\leq \mu(\{x \in M; \mathcal{M}(|T(I - A_{r_0})f|^2 \chi_{3B_0})(x) > \lambda^2\}) \\ &\leq \frac{C}{\lambda^2} \int_M |T(I - A_{r_0})f|^2 \chi_{3B_0} d\mu = \frac{C}{\lambda^2} \int_{3B_0} |T(I - A_{r_0})f|^2 d\mu \\ &\leq \frac{C}{\lambda^2} \mu(3B_0) (\mathcal{M}_{T,A}^\# f(x))^2 \leq C\gamma^2 \mu(3B_0). \end{aligned}$$

In the last two inequalities, we have used that  $3B_0$  contains  $x$ , and that  $x \in E$ . Note that  $C$  is the weak type  $(1, 1)$  bound of the maximal operator and, therefore, only depends on  $(D)$ .

Altogether, we have obtained that

$$\mu(E) \leq C(J^{-p_0} + \gamma^2)\mu(3B_0)$$

provided  $K > 1$ ,  $c_0K^2 > 1$  and  $K^2 = 2(J^2 + 1)$ . This proves the lemma when  $p_0 < \infty$ .

If  $p_0 = \infty$ , one deduces from (2.2) that

$$|TA_{r_0}f(x)| \leq \mathcal{M}(|Tf|^2)^{1/2}(x_0) \leq \lambda$$

for  $\mu$ -a.e.  $x \in 3B_0$ . Hence

$$\mathcal{M}(|TA_{r_0}f|^2 \chi_{3B_0})(x) \leq \lambda^2$$

for all  $x \in M$ , and the set  $\Omega$  is empty if  $J \geq 1$ . The rest of the proof proceeds as before.  $\square$

As in [91, Lemma 2, p. 152], the good- $\lambda$  inequality yields comparisons of  $L^p$  norms as used in [72, Theorem 4.2].

LEMMA 2.3. – *Let  $(M, d, \mu, A_r)$  and  $T$  be as above. Assume that (2.2) holds. Then, for  $0 < p < p_0$ , there exists  $C_p$  such that*

$$(2.8) \quad \|(\mathcal{M}(|Tf|^2))^{1/2}\|_p \leq C_p (\|\mathcal{M}_{T,A}^\# f\|_p + \|f\|_p),$$

for every  $f \in L^2(M, \mu)$  for which the left-hand side is finite (if  $\mu(M) = \infty$ , the term  $C_p\|f\|_p$  can be dispensed with in the right-hand side of (2.8)).

*Proof.* – Let  $f \in L^2(M, \mu)$  be such that  $\|(\mathcal{M}(|Tf|^2))^{1/2}\|_p < \infty$ . For  $\lambda > 0$ , set

$$E_\lambda = \{x \in M; \mathcal{M}(|Tf|^2)(x) > \lambda^2\}.$$

Set  $\lambda_0 = 0$  if  $\mu(M) = \infty$ ,  $\lambda_0 = \frac{1}{\mu(M)} \int_M \mathcal{M}(|Tf|^2) d\mu$  if  $\mu(M) < \infty$ . In the latter case, by the Kolmogorov inequality (see [74, p. 250]), (D), and the weak type (1, 1) of  $\mathcal{M}$ ,

$$\int_M \mathcal{M}(|Tf|^2)^{1/2} d\mu \leq C\mu(M)^{1/2} \| |Tf|^2 \|_1^{1/2} = C\mu(M)^{1/2} \|Tf\|_2.$$

Using then the  $L^2$  boundedness of  $T$ , we obtain

$$\lambda_0 \leq \frac{C}{\mu(M)^{1/2}} \|f\|_2 \leq \frac{C'}{\mu(M)^{1/p}} \|f\|_p.$$

Now fix  $K > 0$  to be chosen later, and write

$$\|(\mathcal{M}(|Tf|^2))^{1/2}\|_p^p = I_1 + I_2,$$

with

$$I_1 = \int_{\mathcal{M}(|Tf|^2) \leq K^2 \lambda_0^2} (\mathcal{M}(|Tf|^2))^{p/2} d\mu,$$

$$I_2 = \int_{\mathcal{M}(|Tf|^2) > K^2 \lambda_0^2} (\mathcal{M}(|Tf|^2))^{p/2} d\mu.$$

Clearly,  $I_1$  is bounded above by

$$K^p \lambda_0^p \mu(M) \leq K^p C^p \|f\|_p^p$$

with  $C$  depending only on the constant in (D) and the  $L^2$  norm of  $T$ . One can treat  $I_2$  as follows. The Whitney decomposition [15, Chapter III, Theorem 1.3] for  $E_\lambda$  yields, for  $\lambda > \lambda_0$ , a family of boundedly overlapping balls  $B_i$  such that  $E_\lambda = \bigcup_i B_i$ . There exists  $c > 1$  such that, for all  $i$ ,  $cB_i$  contains at least one point  $x_i$  outside  $E_\lambda$ , that is

$$\mathcal{M}(|Tf|^2)(x_i) \leq \lambda^2.$$

Therefore, according to Lemma 2.2 for the balls  $cB_i$ , for every  $\gamma > 0$  and  $K > K_0$

$$(2.9) \quad \mu(U_{\lambda,i}) \leq C(\gamma^2 + K^{-p_0})\mu(B_i),$$

where  $U_{\lambda,i} = \{x \in cB_i; \mathcal{M}(|Tf|^2)(x) > K^2 \lambda^2, \mathcal{M}_{T,A}^\# f(x) \leq \gamma \lambda\}$ .

Let

$$U_\lambda = \{x \in M; \mathcal{M}(|Tf|^2)(x) > K^2 \lambda^2, \mathcal{M}_{T,A}^\# f(x) \leq \gamma \lambda\}.$$

Then, since  $K > 1$ ,

$$U_\lambda \subset E_\lambda \subset \bigcup_i (cB_i),$$

thus, for all  $\lambda > \lambda_0$ ,

$$\mu(U_\lambda) = \sum_i \mu(U_{\lambda,i}) \leq C(\gamma^2 + K^{-p_0}) \sum_i \mu(B_i) \leq C'(\gamma^2 + K^{-p_0})\mu(E_\lambda).$$



Now

$$\begin{aligned}
 I_2 &= K^p \int_{\lambda_0}^{\infty} p\lambda^{p-1} \mu\{\mathcal{M}(|Tf|^2) > K^2\lambda^2\} d\lambda \\
 &\leq K^p \int_{\lambda_0}^{\infty} p\lambda^{p-1} (\mu(U_\lambda) + \mu(\{\mathcal{M}_{T,A}^\# f > \gamma\lambda\})) d\lambda \\
 &\leq K^p \int_{\lambda_0}^{\infty} p\lambda^{p-1} (C'(\gamma^2 + K^{-p_0})\mu(E_\lambda) + \mu(\{\mathcal{M}_{T,A}^\# f > \gamma\lambda\})) d\lambda \\
 &= C'(K^{p-p_0} + K^p\gamma^2) \|(\mathcal{M}(|Tf|^2))^{1/2}\|_p^p + K^p\gamma^{-p} \|\mathcal{M}_{T,A}^\# f\|_p^p.
 \end{aligned}$$

Since  $p < p_0$ , one obtains the lemma by choosing first  $K$  large enough and then  $\gamma$  small enough.  $\square$

Now we are ready to prove Theorem 2.1. Let  $f \in L^2(M, \mu) \cap L^p(M, \mu)$ . Then  $Tf \in L^p(M, \mu)$  and by Lemma 2.3

$$\|Tf\|_p \leq \|(\mathcal{M}(|Tf|^2))^{1/2}\|_p \leq C(\|\mathcal{M}_{T,A}^\# f\|_p + \|f\|_p).$$

Using (2.1) and the strong type  $(p/2, p/2)$  of the maximal function yields

$$\|Tf\|_p \leq C\|f\|_p$$

and the conclusion follows by density.

*Remark.* – We implemented an algorithm with a right regularization of the operator  $T$  by looking at  $TA_r$  and  $T(I - A_r)$ . It is also possible to obtain a result with a left regularization by making assumptions on  $A_r$  and  $(I - A_r)T$ .

If one can use duality (that is, if  $T$  is linear), one can try to prove  $L^p$  boundedness of  $T^*$  for some  $p < 2$ . Then one can invoke a result in [35] if one wants a result for the full range  $1 < p < 2$  or its generalization to a limited range  $p_0 < p < 2$  in [9].

Another way (which covers the sublinear case as well), would be to mimic what we just did using instead the sharp function introduced in [72]. Define for  $f \in L^2(M, \mu)$ ,

$$\mathcal{M}_A^\# f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f - A_{r(B)}f| d\mu,$$

where the supremum is taken over all balls  $B$  in  $M$  containing  $x$ , and  $r(B)$  is the radius of  $B$ . Regularizing  $T$  from the left means considering  $\mathcal{M}_A^\#(Tf)$ .

However, if we were to apply left regularization to the boundedness of Riesz transforms on manifolds, since  $T$  takes functions to vector fields (or to 1-forms),  $A_r$  would have to be vector-valued, say, the heat semigroup on 1-forms instead of the heat semigroup on functions as in Section 3 below. Since only the action of  $A_r$  on the image of  $T$  comes into play, one sees that the needed assumptions on the heat semigroup on 1-forms would only concern its action on *exact* forms; this fits with our purpose, see the discussion at the end of Section 1.4. The advantage of the right regularization is that it is susceptible to be applied to more general situations, where the notion of differential forms is not available.

For more on the applications of the above sharp function, the associated space, and its potential uses in harmonic analysis, see [37].

**2.2. The local criterion**

The theorem above admits variations towards localization.

Denote by  $\mathcal{M}_E$  the Hardy–Littlewood maximal operator relative to a measurable subset  $E$  of  $M$ , that is, for  $x \in E$  and  $f$  a locally integrable function on  $M$ ,

$$\mathcal{M}_E f(x) = \sup_{B \ni x} \frac{1}{\mu(B \cap E)} \int_{B \cap E} |f| d\mu,$$

where  $B$  ranges over all open balls of  $M$  containing  $x$  and centered in  $E$ . If in particular,  $E$  is a ball with radius  $r$ , it is enough to consider balls  $B$  with radii not exceeding  $2r$ .

We say that a subset  $E$  of  $M$  has the relative doubling property if there exists a constant  $C_E$  such that for all  $x \in E$  and  $r > 0$  we have

$$\mu(B(x, 2r) \cap E) \leq C_E \mu(B(x, r) \cap E).$$

In other words,  $E$  endowed with the induced distance and measure has the doubling property. The constant  $C_E$  is called the relative doubling constant of  $E$ . On such a set,  $\mathcal{M}_E$  is weak type  $(1, 1)$  and bounded on  $L^p(E, \mu)$ ,  $1 < p \leq \infty$ .

**THEOREM 2.4.** – *Let  $(M, d, \mu)$  be a measured metric space. Let  $p_0 \in (2, \infty]$ . Suppose that  $T$  is a bounded sublinear operator which is bounded on  $L^2(M, \mu)$ , and let  $A_r, r > 0$ , be a family of linear operators acting on  $L^2(M, \mu)$ . Let  $E_1$  and  $E_2$  be two subsets of  $M$  such that  $E_2$  has the relative doubling property,  $\mu(E_2) < \infty$  and  $E_1 \subset E_2$ . Assume that  $f \mapsto \mathcal{M}_{E_2, T, A}^\# f$  is bounded from  $L^p(E_1, \mu)$  into  $L^p(E_2, \mu)$  for all  $p \in (2, p_0)$ , where*

$$(2.10) \quad \begin{aligned} & (\mathcal{M}_{E_2, T, A}^\# f)^2(x) \\ &= \sup_{\substack{B \text{ ball in } M, \\ B \ni x}} \frac{1}{\mu(B \cap E_2)} \int_{B \cap E_2} |T(I - A_{r(B)})f|^2 d\mu, \quad x \in E_2, \end{aligned}$$

and, for some sublinear operator  $S$  bounded from  $L^p(E_1, \mu)$  into  $L^p(E_2, \mu)$  for all  $p \in (2, p_0)$ ,

$$(2.11) \quad \left( \frac{1}{\mu(B \cap E_2)} \int_{B \cap E_2} |T A_{r(B)} f|^{p_0} d\mu \right)^{1/p_0} \leq C (\mathcal{M}_{E_2}(|Tf|^2) + (Sf)^2)^{1/2}(x),$$

for all  $f \in L^2(M, \mu)$  supported in  $E_1$ , all balls  $B$  in  $M$  and all  $x \in B \cap E_2$ , where  $r(B)$  is the radius of  $B$ . If  $2 < p < p_0$  and  $Tf \in L^p(E_2, \mu)$  whenever  $f \in L^p(E_1, \mu)$ , then  $T$  is bounded from  $L^p(E_1, \mu)$  into  $L^p(E_2, \mu)$  and its operator norm is bounded by a constant depending only on the operator norm of  $T$  on  $L^2(M, \mu)$ ,  $C_{E_2}$ ,  $p$ ,  $p_0$ , the operator norms of  $\mathcal{M}_{E_2, T, A}^\#$  and  $S$  on  $L^p$ , and the constant in (2.11).

Again, if  $p_0 = \infty$  the left-hand side of (2.11) is understood as the essential supremum on  $B$ .

The proof of this result is almost identical to that of Theorem 4 once we make some adjustments. The first one is to forget about  $M$  and to work directly in the relative space  $E_2$  by replacing systematically  $T$  and  $T A_r$  by truncations  $\chi_{E_2} T \chi_{E_1}$  and  $\chi_{E_2} T A_r \chi_{E_1}$ . Thus the maximal operator relative to  $E_2$  becomes the maximal operator on  $E_2$ .

The second one is that Lemma 2.2 becomes

LEMMA 2.5. – Let  $(M, d, \mu, A_r, p_0, E_2, E_1)$  and  $T$  be as above. Assume that (2.11) holds. There exist  $K_0 > 1$  and  $C > 0$  only depending on  $C_{E_2}$  and  $p_0$ , such that, for every  $\lambda > 0$ , every  $K > K_0$  and  $\gamma > 0$ , for every ball  $B_0$  in  $M$  and every function  $f \in L^2(E_1, \mu)$  such that there exists  $x_0 \in B_0 \cap E_2$  with  $\mathcal{M}_{E_2}(|Tf|^2)(x_0) + (Sf)^2(x_0) \leq \lambda^2$ , then

$$\begin{aligned} &\mu(\{x \in B_0 \cap E_2; \mathcal{M}_{E_2}(|Tf|^2)(x) > K^2\lambda^2, \mathcal{M}_{E_2, T, A}^\# f(x) \leq \gamma\lambda\}) \\ &\leq C(\gamma^2 + K^{-p_0})\mu(B_0 \cap E_2). \end{aligned}$$

The proof is the same since  $Sf \geq 0$  implies  $\mathcal{M}_{E_2}(|Tf|^2)(x_0) \leq \lambda^2$ .

The third one is that the term  $Sf$  brings a modification in Lemma 2.3 which becomes

LEMMA 2.6. – Let  $(M, d, \mu, A_r, E_2, E_1)$  and  $T$  be as above. Assume that (2.11) holds. Then, for  $0 < p < p_0$ ,

$$\|(\mathcal{M}_{E_2}(|Tf|^2))^{1/2}\|_{L^p(E_2)} \leq C(\|\mathcal{M}_{E_2, T, A}^\# f\|_{L^p(E_2)} + \|Sf\|_{L^p(E_2)} + \|f\|_{L^p(E_2)}),$$

for every  $f \in L^2(E_1, \mu)$  for which the left-hand side is finite, where  $C$  depends only on  $p, p_0$  and the doubling constant of  $E_2$  (but not on  $\mu(E_2)$ ).

Since  $\mu(E_2) < \infty$ , the Whitney decomposition in the proof of Lemma 2.3 can be performed for  $\lambda > \lambda_0$  with  $\lambda_0 = \frac{1}{\mu(E_2)} \int_{E_2} \mathcal{M}_{E_2}(|Tf|^2)^{1/2} d\mu$ . Again, by Kolmogorov inequality, the doubling property of  $E_2$ , and the weak type  $(1, 1)$  of  $\mathcal{M}_{E_2}$  (with constant independent of the size of  $\mu(E_2)$ ),

$$\int_{E_2} \mathcal{M}_{E_2}(|Tf|^2)^{1/2} d\mu \leq C\mu(E_2)^{1/2} \| |Tf|^2 \|_{L^1(E_2)}^{1/2} = C\mu(E_2)^{1/2} \|Tf\|_{L^2(E_2)}.$$

Using then the  $L^2$  boundedness of  $T$  and the support condition of  $f$  (this is where we use  $E_1 \subset E_2$ ), we obtain

$$\lambda_0 \leq \frac{C}{\mu(E_2)^{1/2}} \|f\|_{L^2(E_2)} \leq \frac{C}{\mu(E_2)^{1/p}} \|f\|_{L^p(E_2)}.$$

Now write with the notation of the proof in Lemma 2.3

$$\begin{aligned} \|(\mathcal{M}_{E_2}(|Tf|^2))^{1/2}\|_{L^p(E_2)}^p &= \int_{\mathcal{M}_{E_2}(|Tf|^2) \leq K^2\lambda_0^2} (\mathcal{M}_{E_2}(|Tf|^2))^{p/2} d\mu \\ &\quad + \int_{\mathcal{M}_{E_2}(|Tf|^2) > K^2\lambda_0^2} (\mathcal{M}_{E_2}(|Tf|^2))^{p/2} d\mu. \end{aligned}$$

The last integral can be treated as before, using Lemma 2.5. The first integral is bounded above by

$$K^p \lambda_0^p \mu(E_2) \leq K^p C^p \|f\|_{L^p(E_2)}^p$$

with  $C$  depending only on the  $L^2$  norm of  $T$  and the doubling constant of  $E_2$ . Further details are left to the reader.

### 3. Application to the Riesz transform

In this section,  $M$  is a complete non-compact Riemannian manifold,  $\Delta$  denotes the Laplace–Beltrami operator,  $e^{-t\Delta}$ ,  $t > 0$ , the heat semigroup and  $p_t(x, y)$ ,  $t > 0$ ,  $x, y \in M$ , the heat kernel. The measure  $\mu$  is the induced Riemannian volume. The measure of the ball  $B(x, r)$ ,  $x \in M$ ,  $r > 0$  is also written  $V(x, r)$ .

We prove the statements corresponding to the (global) Riesz transform  $\nabla\Delta^{-1/2}$ . We set  $Tf = |\nabla\Delta^{-1/2}f|$  (remember that, in the finite volume case, we restrict ourselves to functions with mean zero; in other words, in order to apply verbatim Theorem 2.1, we define  $\nabla\Delta^{-1/2}$  by zero on constants). The boundedness of  $T$  on  $L^2$  has been already observed.

#### 3.1. Proof of the main result

We now show Theorem 1.2, namely the fact that, under  $(D)$  and  $(P)$ ,  $(G_{p_0})$  implies the  $L^p$  boundedness of the Riesz transform for  $2 < p < p_0$ , as a consequence of Theorem 2.1.

Recall that Theorem 2.1 applies if  $T$  is assumed to act on  $L^p(M)$ . However, we have

$$\nabla\Delta^{-1/2} = c \int_0^\infty \nabla e^{-t\Delta} \frac{dt}{\sqrt{t}}.$$

If we set  $T_\varepsilon f = |c \int_\varepsilon^{1/\varepsilon} \nabla e^{-t\Delta} f \frac{dt}{\sqrt{t}}|$  for  $0 < \varepsilon < 1$ , then for  $f \in L^2(M)$  we have  $\|T_\varepsilon f\|_2 \leq \|f\|_2$  (this follows from (1.1) and spectral theory) and  $T_\varepsilon f \rightarrow Tf$  in  $L^2(M)$  as  $\varepsilon \rightarrow 0$ , while  $\|T_\varepsilon f\|_p \leq C_\varepsilon \|f\|_p$  for  $f \in L^p(M)$ . As the application of Theorem 2.1 to  $T_\varepsilon$  gives us a uniform bound with respect to  $\varepsilon$ , a limiting argument yields the  $L^p$  boundedness of  $T$  on  $L^2(M) \cap L^p(M)$ , hence on  $L^p(M)$ . Henceforth, we ignore this approximation step and our goal is now to establish (2.1) and (2.2) for  $T$ .

The first ingredient is Gaffney off-diagonal estimates valid in a general Riemannian manifold: There exist two constants  $C \geq 0$  and  $\alpha > 0$  such that, for every  $t > 0$ , every closed subsets  $E$  and  $F$  of  $M$ , and every function  $f$  supported in  $E$ , one has

$$(3.1) \quad \begin{aligned} & \|e^{-t\Delta} f\|_{L^2(F)} + \|t\Delta e^{-t\Delta} f\|_{L^2(F)} + \|\sqrt{t}|\nabla e^{-t\Delta} f|\|_{L^2(F)} \\ & \leq C e^{-\alpha d(E,F)^2/t} \|f\|_{L^2(E)}. \end{aligned}$$

Here,  $d(E, F)$  is the distance between the sets  $E$  and  $F$ . The inequality for the first term of the left-hand side is classical (see, e.g., [27]). The estimate for the second one follows from essentially the same proof (see [28, Lemma 7]). We give a proof for the third one as we could not find it in the literature in this situation (it is proved in [4] for elliptic operators on  $\mathbb{R}^n$ ).

We assume  $d(E, F) > \sqrt{t}$  as otherwise there is nothing to prove. Let  $\tilde{F}$  be the set of those  $x \in M$  for which  $d(x, F) \leq d(E, F)/3$ . Let  $\varphi$  be a smooth function on  $M$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi$  is supported in  $\tilde{F}$ ,  $\varphi \equiv 1$  on  $F$ , and  $|\nabla\varphi| \leq 6/d(E, F)$ . Set  $A = \|\sqrt{t}|\varphi|\nabla e^{-t\Delta} f\|_2 \geq \|\sqrt{t}|\nabla e^{-t\Delta} f|\|_{L^2(F)}$ . Integrating by parts,

$$\begin{aligned} A^2 &= t\langle \varphi^2 \nabla e^{-t\Delta} f, \nabla e^{-t\Delta} f \rangle \\ &= -2t\langle (e^{-t\Delta} f) \nabla\varphi, \varphi \nabla e^{-t\Delta} f \rangle + t\langle \varphi^2 e^{-t\Delta} f, \Delta e^{-t\Delta} f \rangle \\ &\leq 2\sqrt{t} \|\nabla\varphi|e^{-t\Delta} f\|_2 A + \|\varphi e^{-t\Delta} f\|_2 \|t\varphi \Delta e^{-t\Delta} f\|_2. \end{aligned}$$

Using the properties of  $\varphi$  and the bounds for the first two terms in (3.1) we obtain the desired conclusion.

We now introduce the regularizing operator  $A_r, r > 0$ , by setting

$$I - A_r = (I - e^{-r^2\Delta})^n$$

for some integer  $n$  to be chosen. Observe that  $A_r$  is bounded on  $L^2(M)$  with norm 1. We prove (2.1) in the following lemma.

LEMMA 3.1. – Assume that (D) holds. Then, for some  $n$  large enough depending only on (D), for every ball  $B$  with radius  $r > 0$  and all  $x \in B$ ,

$$(3.2) \quad \|\|\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n f\|\|_{L^2(B)} \leq C\mu(B)^{1/2}(\mathcal{M}(|f|^2)(x))^{1/2}.$$

*Proof.* – Let  $f \in L^2(M)$ . Take a ball  $B$  with radius  $r = r(B)$  and  $x$  a point in  $B$ . Denote by  $C_i$  the ring  $2^{i+1}B \setminus 2^iB$  if  $i \geq 2$  and let  $C_1 = 4B$ . Decompose  $f$  as  $f_1 + f_2 + f_3 + \dots$  with  $f_i = f\chi_{C_i}$ . By the Minkowski inequality we have that

$$\|\|\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n f\|\|_{L^2(B)} \leq \sum_{i \geq 1} \|\|\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n f_i\|\|_{L^2(B)}.$$

For  $i = 1$  we use the  $L^2$  boundedness of  $\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n$ :

$$\|\|\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n f_1\|\|_{L^2(B)} \leq \|f\|_{L^2(4B)} \leq \mu(4B)^{1/2}(\mathcal{M}(|f|^2)(x))^{1/2}.$$

For  $i \geq 2$  we use the integral representation of  $\Delta^{-1/2}$ :

$$\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n = c \int_0^\infty \nabla e^{-t\Delta}(I - e^{-r^2\Delta})^n \frac{dt}{\sqrt{t}} = c \int_0^\infty g_r(t)\nabla e^{-t\Delta} dt,$$

where using the usual notation for the binomial coefficient,

$$g_r(t) = \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{\chi_{\{t > kr^2\}}}{\sqrt{t - kr^2}}.$$

By the Minkowski integral inequality and Gaffney estimates (3.1), using the support of  $f_i$ , we have that

$$\|\|\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^n f_i\|\|_{L^2(B)} \leq C \left( \int_0^\infty |g_r(t)| e^{-\frac{c'4^i r^2}{t}} \frac{dt}{\sqrt{t}} \right) \|f\|_{L^2(C_i)}.$$

The latter integral can be estimated as follows. Elementary analysis yields the following estimates for  $g_r$ :

$$|g_r(t)| \leq \frac{C_n}{\sqrt{t - \ell r^2}} \quad \text{if } 0 \leq \ell r^2 < t \leq (\ell + 1)r^2 \leq (n + 1)r^2$$

and

$$|g_r(t)| \leq C_n r^{2n} t^{-n-\frac{1}{2}} \quad \text{if } t > (n + 1)r^2.$$

The latter estimate comes from the inequality

$$\left| \sum_{k=0}^n \binom{n}{k} (-1)^k \varphi(t - kr^2) \right| \leq C_n \sup_{u \geq \frac{t}{n+1}} |\varphi^{(n)}(u)| r^{2n},$$

which can be obtained by expanding  $\varphi(t - ks)$  using Taylor’s formula about  $t$  and using the classical relations  $\sum_{k=0}^n \binom{n}{k} (-1)^k k^\ell = 0$  for  $\ell = 0, \dots, n - 1$  (see [40, Problem 16, p. 65]). This yields the following estimates, uniformly with respect to  $r$ :

$$\int_0^\infty |g_r(t)| e^{-\frac{\alpha' 4^i r^2}{t}} \frac{dt}{\sqrt{t}} \leq C_n 4^{-in}.$$

Now, an easy consequence of (D) is that for all  $y \in M, r > 0$ , and  $\theta \geq 1$

$$(3.3) \quad V(y, \theta r) \leq C \theta^\nu V(y, r),$$

for some constants  $C$  and  $\nu > 0$ . Therefore, since  $C_i \subset 2^{i+1}B$ ,

$$\|f\|_{L^2(C_i)} \leq \mu(2^{i+1}B)^{1/2} (\mathcal{M}(|f|^2)(x))^{1/2} \leq \sqrt{C} 2^{(i+1)\nu/2} \mu(B)^{1/2} (\mathcal{M}(|f|^2)(x))^{1/2}.$$

Choosing  $2n > \nu/2$ , we have

$$\|\|\nabla \Delta^{-1/2} (I - e^{-r^2 \Delta})^n f\|\|_{L^2(B)} \leq C' \left( \sum_{i \geq 1} 2^{i(\nu/2 - 2n)} \right) \mu(B)^{1/2} (\mathcal{M}(|f|^2)(x))^{1/2},$$

which proves Lemma 3.1.  $\square$

We now show that (2.2) holds. We begin with a lemma.

LEMMA 3.2. – Assume (D), (P) and  $(G_{p_0})$ . Then the following estimates hold: for every  $p \in (2, p_0)$ , for every ball  $B$  with radius  $r$  and every  $L^2$  function  $f$  supported in  $C_i = 2^{i+1}B \setminus 2^iB$ ,  $i \geq 2$ , or  $C_1 = 4B$ , and every  $k \in \{1, \dots, n\}$ , where  $n$  is chosen as above, one has

$$(3.4) \quad \left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2 \Delta} f|^p d\mu \right)^{1/p} \leq \frac{C e^{-\alpha 4^i}}{r} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu \right)^{1/2}$$

for some constants  $C$  and  $\alpha$  depending only on (D), (P),  $p$  and  $p_0$ .

Proof. – By interpolating  $(G_{p_0})$  with the  $L^2$  Gaffney estimates, we obtain  $L^p$  Gaffney estimates for any  $p \in (2, p_0)$ : for every  $t > 0$ , for every closed sets  $E$  and  $F$  and every function  $f$  supported in  $E$ , one has

$$(3.5) \quad \|\|\sqrt{t} \nabla e^{-t \Delta} f\|\|_{L^p(F)} \leq C e^{-\alpha d(E,F)^2/t} \|f\|_{L^p(E)},$$

with  $C > 0$  and  $\alpha > 0$  depending on  $p, p_0$ , and the constant  $C_{p_0}$  in  $(G_{p_0})$ . Now let  $B$  be a ball with radius  $r$  and let  $f$  be supported in  $C_i$ .

In the following proof, many constants will implicitly depend on  $n$ , which itself only depends on (D).

Let us begin with the case  $i = 1$ . The above estimate (or, directly,  $(G_p)$ ) yields

$$(3.6) \quad \left( \int_B |\nabla e^{-kr^2\Delta} f|^p d\mu \right)^{1/p} \leq \frac{C}{r} \left( \int_M |e^{-(k/2)r^2\Delta} f|^p d\mu \right)^{1/p}.$$

Let  $t = (k/2)r^2$ . Since  $(UE)$  follows from  $(D)$  and  $(P)$ , one has the upper estimate

$$p_t(x, y) \leq \frac{C}{V(y, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right),$$

for all  $x, y \in M$ . Because of the doubling property,

$$V(y, \sqrt{t}) \simeq V(x_B, \sqrt{t}) \simeq \mu(B)$$

where  $x_B$  is the center of  $B$  and  $y \in 4B$ . It follows that

$$(3.7) \quad |e^{-(k/2)r^2\Delta} f(x)| \leq \left( \frac{C}{\mu(B)} \int_{4B} |f|^2 d\mu \right)^{1/2},$$

for all  $x \in M$ . On the other hand, by Gaffney (or  $L^2$  contractivity of the heat semigroup),

$$(3.8) \quad \int_M |e^{-(k/2)r^2\Delta} f|^2 d\mu \leq C \int_{4B} |f|^2 d\mu.$$

Thus, by Hölder,

$$\left( \int_M |e^{-(k/2)r^2\Delta} f|^p d\mu \right)^{1/p} \leq C \mu(B)^{\frac{1}{p} - \frac{1}{2}} \left( \int_{4B} |f|^2 d\mu \right)^{1/2},$$

which, together with (3.6), yields (3.4) in this case.

Next assume that  $i \geq 2$ . Denote by  $\chi_{C_\ell}$  the characteristic function of  $C_\ell$  and write

$$\nabla e^{-kr^2\Delta} f = \sum_{\ell \geq 1} h_\ell, \quad h_\ell = \nabla e^{-(k/2)r^2\Delta} (\chi_{C_\ell}) e^{-(k/2)r^2\Delta} f.$$

From (3.5) we have

$$\left( \frac{1}{\mu(B)} \int_B |h_\ell|^p d\mu \right)^{1/p} \leq \left( \frac{\mu(2^{\ell+1}B)}{\mu(B)} \right)^{1/p} \frac{C e^{-\alpha 4^\ell}}{r} \left( \frac{1}{\mu(2^{\ell+1}B)} \int_{C_\ell} |e^{-(k/2)r^2\Delta} f|^p d\mu \right)^{1/p}$$

and, using (3.3),

$$(3.9) \quad \left( \frac{1}{\mu(B)} \int_B |h_\ell|^p d\mu \right)^{1/p} \leq C' 2^{(\ell+1)\nu/p} \frac{e^{-\alpha 4^\ell}}{r} \times \left( \frac{1}{\mu(2^{\ell+1}B)} \int_{C_\ell} |e^{-(k/2)r^2\Delta} f|^p d\mu \right)^{1/p}.$$

From the  $L^2$  Gaffney estimates for the semigroup, one has

$$\int_{C_\ell} |e^{-(k/2)r^2\Delta} f|^2 d\mu \leq K_{i\ell} \int_{C_i} |f|^2 d\mu$$

with

$$K_{i\ell} \leq \begin{cases} Ce^{-c4^i} & \text{if } \ell \leq i - 2, \\ C & \text{if } i - 1 \leq \ell \leq i + 1, \\ Ce^{-c4^\ell} & \text{if } \ell \geq i + 2. \end{cases}$$

Since, by (3.3),  $K_{i\ell} \frac{\mu(2^{i+1}B)}{\mu(2^{\ell+1}B)} \leq K_{i\ell} \sup(1, C2^{(i-\ell)\nu})$ , and if we still denote by  $K_{i\ell}$  a sequence of the same form with different constants, we may also write

$$(3.10) \quad \frac{1}{\mu(2^{\ell+1}B)} \int_{C_\ell} |e^{-(k/2)r^2\Delta} f|^2 d\mu \leq K_{i\ell} \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu.$$

Next, it easily follows from (UE) and (D) that for all  $x \in C_\ell$ ,

$$\begin{aligned} |e^{-(k/2)r^2\Delta} f(x)| &\leq C \int_{C_i} V(y, r)^{-1} \exp\left(-c \frac{d^2(x, y)}{r^2}\right) |f(y)| d\mu(y) \\ &\leq K_{i\ell} \int_{C_i} V(y, r)^{-1} |f(y)| d\mu(y). \end{aligned}$$

If  $y \in C_i$ ,  $2^{i+1}B \subset B(y, 2^{i+2}r)$ , so that

$$\frac{1}{V(y, r)} \leq \frac{C2^{(i+2)\nu}}{V(y, 2^{i+2}r)} \leq \frac{C2^{(i+2)\nu}}{\mu(2^{i+1}B)},$$

and it follows that

$$(3.11) \quad |e^{-(k/2)r^2\Delta} f(x)| \leq K_{i\ell} 2^{(i+2)\nu} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu \right)^{1/2}.$$

By applying Hölder and using (3.10) and (3.11), one obtains

$$\left( \frac{1}{\mu(2^{\ell+1}B)} \int_{C_\ell} |e^{-(k/2)r^2\Delta} f|^p d\mu \right)^{1/p} \leq K_{i\ell} 2^{(i+2)\nu(1-\frac{2}{p})} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu \right)^{1/2}.$$

Together with (3.9) and summing in  $\ell$ , this yields

$$\begin{aligned} &\left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2\Delta} f|^p d\mu \right)^{1/p} \\ &\leq C \sum_{\ell \geq 1} 2^{(\ell+1)\nu/p} \frac{e^{-c4^\ell}}{r} K_{i\ell} 2^{(i+2)\nu(1-\frac{2}{p})} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu \right)^{1/2} \end{aligned}$$



$$\leq C \frac{e^{-c'4^i}}{r} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f|^2 d\mu \right)^{1/2}.$$

This ends the proof of Lemma 3.2.  $\square$

Equipped with this lemma, we can prove (2.2) for any  $p \in (2, p_0)$ . Fix such a  $p$ . By expanding  $I - (I - e^{-r^2\Delta})^n$  it suffices to show

$$(3.12) \quad \left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2\Delta} f|^p d\mu \right)^{1/p} \leq C (\mathcal{M}(|\nabla f|^2))^{1/2}(y)$$

for  $f$  with  $f, \nabla f$  locally square integrable,  $B$  any ball with  $r = r(B)$ ,  $y \in B$  and  $k = 1, 2, \dots, n$ . Recall that  $n$  is chosen larger than  $\nu/4$  where  $\nu$  is given in (3.3).

Recall that our assumptions ensure that  $M$  satisfies (1.5). In other words,

$$e^{-t\Delta} 1 \equiv 1, \quad \forall t > 0.$$

We may therefore write

$$\nabla e^{-kr^2\Delta} f = \nabla e^{-kr^2\Delta} (f - f_{4B}).$$

Write  $f - f_{4B} = f_1 + f_2 + f_3 + \dots$  where  $f_i = (f - f_{4B})\chi_{C_i}$ . For  $i = 1$ , we use the lemma and (P) to obtain

$$\left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2\Delta} f_1|^p d\mu \right)^{1/p} \leq C \left( \frac{1}{\mu(4B)} \int_{4B} |\nabla f|^2 d\mu \right)^{1/2} \leq C (\mathcal{M}(|\nabla f|^2))^{1/2}(y).$$

For  $i \geq 2$ , we have similarly

$$\left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2\Delta} f_i|^p d\mu \right)^{1/p} \leq \frac{C e^{-\alpha 4^i}}{r} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f_i|^2 d\mu \right)^{1/2}.$$

But

$$\int_{C_i} |f_i|^2 d\mu \leq \int_{2^{i+1}B} |f - f_{4B}|^2 d\mu,$$

$$|f - f_{4B}| \leq |f - f_{2^{i+1}B}| + \sum_{\ell=2}^i |f_{2^\ell B} - f_{2^{\ell+1}B}|$$

and observe that

$$|f_{2^\ell B} - f_{2^{\ell+1}B}|^2 \leq \frac{1}{\mu(2^{\ell+1}B)} \int_{2^{\ell+1}B} |f - f_{2^{\ell+1}B}|^2 d\mu \leq (2^\ell r)^2 \mathcal{M}(|\nabla f|^2)(y).$$

Hence, by the Minkowski inequality, we easily obtain

$$(3.13) \quad \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f_i|^2 d\mu \right)^{1/2} \leq C(2^i r) (\mathcal{M}(|\nabla f|^2))^{1/2}(y).$$

It remains to sum for  $i \geq 2$

$$\begin{aligned} \sum_{i \geq 2} \frac{C e^{-\alpha 4^i}}{r} \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |f_i|^2 d\mu \right)^{1/2} &\leq \sum_{i \geq 2} \frac{C e^{-\alpha 4^i}}{r} (2^i r) (\mathcal{M}(|\nabla f|^2))^{1/2}(y) \\ &\leq C \sum_{i \geq 2} e^{-\alpha 4^i} 2^i (\mathcal{M}(|\nabla f|^2))^{1/2}(y). \end{aligned}$$

This yields (3.12), and the proof of Theorem 1.2 is finished.

*Remark.* –

A slight modification of the proof allows the following improvement of (3.12): for some constants  $c, C > 0$

$$(3.14) \quad \left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2 \Delta} f|^p d\mu \right)^{1/p} \leq C \inf_{x \in B} (e^{-cr^2 \Delta} (|\nabla f|^2))^{1/2}(x)$$

for  $f$  with  $f, \nabla f$  locally square integrable,  $B$  any ball with  $r = r(B)$  and  $k = 1, 2, \dots, n$ , where  $n$  is chosen as above. See Lemma 3.3 in Section 3.2 for the case  $p = \infty$  of this inequality.

The techniques of proof of Theorem 2.1 extend easily to the vector-valued setting. It can be checked that it applies as well to obtain the square function estimate (seen as  $L^p$  boundedness of a vector-valued operator)

$$(3.15) \quad \left\| \left( \int_0^\infty |\nabla e^{-t\Delta} f|^2 dt \right)^{1/2} \right\|_p \leq C \|f\|_p$$

from  $(D)$ ,  $(P)$ , and  $(G_{p_0})$ , for  $2 < p < p_0$ . Compare with [20], Section 3, where it is observed that a related inequality (with the Poisson semigroup instead of the heat semigroup) holds under (1.12) only. In particular,  $(D)$  and  $(P)$  are not used there. This explains further our discussion in Section 1.4 where we question the relevance of  $(D)$  and  $(P)$  for the Riesz transform boundedness.

### 3.2. A simpler situation

In this section, we show a slightly weaker version of Theorem 1.4 by replacing  $(G)$  with the stronger inequality (1.8), that is

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right), \quad \forall x, y \in M, t > 0,$$

as this is a simpler application of Theorem 2.1 and this hypothesis is related to the domination condition (1.12), therefore to the conjecture in [20] explained in Section 1.4.

We set again  $Tf = |\nabla \Delta^{-1/2} f|$  but choose here  $A_r = e^{-r^2 \Delta}$ , and apply Theorem 2.1 in the case  $p_0 = \infty$ .

We begin with the verification of (2.1). Let  $B$  be a ball of radius  $r$ . Let  $f \in L^2(M)$ . Write  $f = f_1 + f_2$  where  $f_1 = f$  on  $2B$  and 0 elsewhere. First, the  $L^2$  boundedness of  $T(I - A_r)$  gives us

$$\int_B |T(I - A_r)f_1|^2 d\mu \leq \int_M |f_1|^2 d\mu \leq \mu(2B) \mathcal{M}(|f|^2)(z)$$

whenever  $z \in B$ . To conclude the proof of (2.1), it remains to obtain the same bound for  $f_2$ . One has

$$\begin{aligned} T(I - A_r)f_2(x) &= |\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})f_2(x)| \\ &= \left| \int_M \tilde{k}_r(x, y)f_2(y) d\mu(y) \right| \\ &\leq \int_M |\tilde{k}_r(x, y)||f_2(y)| d\mu(y), \end{aligned}$$

where

$$\tilde{k}_r(x, y) = \int_0^\infty g_r(s)\nabla_x p_s(x, y) ds$$

and

$$g_r(t) = \frac{1}{\sqrt{t}} - \frac{\chi_{\{t>r^2\}}}{\sqrt{t-r^2}}.$$

Using (1.8) and following the case  $n = 1$  in the proof of Lemma 3.1, one obtains that if  $d(x, y) \geq r$  then

$$|\tilde{k}_r(x, y)| \leq \frac{C}{V(x, r)} \exp\left(-c\frac{d^2(x, y)}{r^2}\right).$$

This estimate and the support property of  $f_2$  ensure

$$\int_M |\tilde{k}_r(x, y)||f_2(y)| d\mu(y) \leq CM(|f_2|)(z) \leq CM(|f|)(z)$$

(see for instance [36, Proposition 2.4]), therefore the pointwise bound

$$|T(I - A_r)f_2(x)| \leq CM(|f|)(z)$$

whenever  $x$  and  $z$  belong to  $B$ , hence the bound in  $L^2$  average.

The next step is the verification of (2.2), which in this case becomes the maximal estimate

$$(3.16) \quad \sup_{y \in B} |\nabla e^{-r^2(B)\Delta} f(y)|^2 \leq C' \inf_{x \in B} \mathcal{M}(|\nabla f|^2)(x),$$

for all balls  $B$  and functions  $f$  with  $f, \nabla f$  square integrable. But, under (D) and (DUE), this follows from (1.8).

Indeed, assume (1.8). Recall from the discussion in Sections 1.2 and 1.4 that in such a case, the full (LY) estimates hold and that (1.8) is equivalent to (1.10), that is

$$|\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}} p_{C't}(x, y)$$

for some constants  $C, C' > 0$ . Also, by Cauchy-Schwarz, (P) implies

$$(3.17) \quad \int_B |f - f_B| d\mu \leq Cr(B)\sqrt{\mu(B)} \left( \int_B |\nabla f|^2 d\mu \right)^{1/2}$$

for any ball  $B$  in  $(M, d)$  with radius  $r(B)$  and any  $f$  with  $f, \nabla f$  square integrable on  $B$  (in fact (P) and (3.17) are equivalent, see [49]).

Fix  $B$  a ball in  $M$ ,  $x, y \in B$ , and set  $t = r^2(B)$ . Using (1.5) again, write

$$(3.18) \quad |\nabla e^{-t\Delta} f(y)| = |\nabla e^{-t\Delta}(f - f(x))(y)| \leq \int_M |\nabla p_t(y, z)| |f(z) - f(x)| d\mu(z).$$

Then recall that (3.17) admits a reformulation in terms of a pointwise estimate (see [48], [50], Theorem 3.2), in particular it implies:

$$(3.19) \quad |f(x) - f(z)| \leq Cd(x, z)(h(x) + h(z)),$$

with

$$h(x) = (\mathcal{M}(|\nabla f|^2)(x))^{1/2}$$

for all  $f$  with  $f, \nabla f$  locally square integrable and  $x, z \in M$ .

Then, plugging (3.19) and (1.8) into (3.18),

$$|\nabla e^{-t\Delta} f(y)| \leq \frac{C}{V(x, \sqrt{t})} \int_M \frac{d(x, z)}{\sqrt{t}} \exp\left(-\frac{d^2(y, z)}{Ct}\right) (h(x) + h(z)) d\mu(z),$$

and since  $x, y \in B$ ,  $\frac{d(x, z)}{\sqrt{t}}$  is comparable up to an additive constant with  $\frac{d(y, z)}{\sqrt{t}}$ , therefore

$$\begin{aligned} |\nabla e^{-t\Delta} f(y)| &\leq \frac{C}{V(x, \sqrt{t})} \int_M \exp\left(-\frac{d^2(y, z)}{C't}\right) (h(x) + h(z)) d\mu(z) \\ &\leq Ch(x) + \frac{C}{V(x, \sqrt{t})} \int_M \exp\left(-\frac{d^2(y, z)}{C't}\right) h(z) d\mu(z) \\ &\leq Ch(x) + C(\mathcal{M}h)(x), \end{aligned}$$

again by [36, Proposition 2.4]. Now, a result of Coifman and Rochberg [14], which is extendable to spaces of homogeneous type, states that for any  $g$  for which  $\mathcal{M}g < \infty$  a.e. and for any positive  $\gamma < 1$ , the weight  $w \equiv (\mathcal{M}g)^\gamma$  belongs to the Muckenhoupt class  $A_1$ , with  $A_1$  constant depending on  $\gamma$  (but not on  $g$ ). Thus,  $\mathcal{M}w \leq C_\gamma w$  a.e., so that, in particular,

$$\mathcal{M}[(\mathcal{M}g)^{1/2}] \leq C_{1/2}(\mathcal{M}g)^{1/2},$$

almost everywhere.

Applying this with  $g = |\nabla f|^2$  in the right-hand side of the above inequality, one obtains (3.16).

For the sake of completeness, we are now going to give a lemma which clarifies the relationship between the pointwise gradient bound (1.8), an integral version of it, the domination condition (1.12), and the maximal estimate (3.16). This lemma also offers a less direct, but more elementary approach to the implication from (1.8) to (3.16).

LEMMA 3.3. – *In presence of (FK), the following three properties are equivalent:*

- (i) *The pointwise heat kernel gradient bound (1.8).*

(ii) *The weighted  $L^2$  heat kernel gradient bound*

$$(3.20) \quad \int_M |\nabla_x p_t(x, y)|^2 e^{\alpha \frac{d^2(x, y)}{t}} d\mu(y) \leq \frac{C}{tV(x, \sqrt{t})},$$

for  $\alpha > 0$  small enough, all  $t > 0, x \in M$ .

(iii) *The domination condition (1.12)*

$$|\nabla e^{-t\Delta} f|^2 \leq C e^{-C't\Delta} (|\nabla f|^2),$$

for some  $C, C' > 0$  and  $f$  with  $f, \nabla f$  square integrable and all  $t > 0$ .

Any of these conditions implies the maximal estimate (3.16).

*Proof.* – It is easy to see by integrating (1.8) and using the doubling property that (i) implies (ii). The converse follows from an argument somewhat similar to [20, p. 14]. Write

$$\nabla_x p_{2t}(x, y) = \int_M \nabla_x p_t(x, z) p_t(z, y) d\mu(z).$$

Using (ii) and the Cauchy–Schwarz inequality,

$$|\nabla_x p_{2t}(x, y)|^2 \leq \frac{C}{tV(x, \sqrt{t})} \int_M |p_t(z, y)|^2 e^{-\alpha \frac{d^2(x, z)}{t}} d\mu(z)$$

for all  $\alpha > 0$  small enough. According to (DUE),

$$|p_t(z, y)|^2 \leq \frac{C}{V^2(y, \sqrt{t})} e^{-c \frac{d^2(z, y)}{t}}$$

for some  $c > 0$ . For  $\beta$  small enough we have

$$e^{-\alpha \frac{d^2(x, z)}{t}} e^{-c \frac{d^2(z, y)}{t}} \leq e^{-\beta \frac{d^2(x, y)}{t}} e^{-c \frac{d^2(z, y)}{2t}},$$

thus

$$|\nabla_x p_{2t}(x, y)|^2 \leq \frac{C}{tV(x, \sqrt{t})V^2(y, \sqrt{t})} e^{-\beta \frac{d^2(x, y)}{t}} \int_M e^{-c \frac{d^2(z, y)}{2t}} d\mu(z).$$

Using (D), it is easy to check that the quantity

$$\frac{1}{V(y, \sqrt{t})} \int_M e^{-c \frac{d^2(z, y)}{2t}} d\mu(z)$$

is uniformly bounded, and (i) follows readily.

Assume next that (i) holds and let us prove (iii). Recall that we may use freely (LY) and (3.17).

Fix  $x \in M$  and  $B \equiv B(x, \sqrt{t})$ . Set  $C_1 = 4B$  and  $C_j = 2^{j+1}B \setminus 2^j B$  for  $j \geq 2$ . Using again (1.5), write

$$\begin{aligned} |\nabla e^{-t\Delta} f(x)| &= |\nabla e^{-t\Delta}(f - f_{4B})(x)| \\ &= \left| \int_M \nabla_x p_t(x, y)(f(y) - f_{4B}) d\mu(y) \right| \\ &\leq \sum_{j \geq 1} \int_{C_j} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y). \end{aligned}$$

It follows from the lower bound in (LY) that, for all  $\varepsilon > 0$ , there exist  $c_\varepsilon, C_\varepsilon > 0$  independent of  $x \in M$  and  $t > 0$  such that

$$(3.21) \quad \frac{c_\varepsilon}{V(x, \sqrt{t})} \exp(-\varepsilon 4^j) \leq p_{C_\varepsilon t}(x, y)$$

for all  $y \in 2^j B, t > 0$ .

Let us treat the first term in the above sum, that is when  $j = 1$ . According to (1.8),

$$\int_{4B} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y) \leq \frac{C}{\sqrt{t}V(x, \sqrt{t})} \int_{4B} |f(y) - f_{4B}| d\mu(y).$$

By (3.17),

$$\int_{4B} |f(y) - f_{4B}| d\mu(y) \leq C\sqrt{t}\sqrt{V(x, \sqrt{t})} \left( \int_{4B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2},$$

hence

$$\int_{4B} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y) \leq C \left( \frac{1}{V(x, \sqrt{t})} \int_{4B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2},$$

and by (3.21),

$$\int_{4B} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y) \leq C'_\varepsilon \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}.$$

Now for the other terms in the sum. By (1.8) again,

$$\begin{aligned} &\int_{C_j} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y) \\ &\leq \frac{C}{\sqrt{t}V(x, \sqrt{t})} \int_{C_j} \exp\left(-\frac{d^2(x, y)}{Ct}\right) |f(y) - f_{4B}| d\mu(y) \\ &\leq \frac{C \exp(-c4^j)}{\sqrt{t}V(x, \sqrt{t})} \int_{C_j} |f(y) - f_{4B}| d\mu(y). \end{aligned}$$

Then

$$\begin{aligned} \int_{C_j} |f(y) - f_{4B}| d\mu(y) &\leq \int_{2^{j+1}B} \left( |f(y) - f_{2^j B}| + \sum_{\ell=2}^j |f_{2^\ell B} - f_{2^{\ell+1}B}| \right) d\mu(y) \\ &= \int_{2^{j+1}B} |f(y) - f_{2^j B}| d\mu(y) + \sum_{\ell=2}^j \int_{2^{\ell+1}B} V(x, 2^j \sqrt{t}) |f_{2^\ell B} - f_{2^{\ell+1}B}| d\mu(y). \end{aligned}$$

Again by (3.17),

$$\int_{2^{j+1}B} |f(y) - f_{2^j B}| d\mu(y) \leq C 2^j \sqrt{t} \sqrt{V(x, 2^j \sqrt{t})} \left( \int_{2^{j+1}B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2},$$

and by (3.21)

$$\int_{2^{j+1}B} |\nabla f(y)|^2 d\mu(y) \leq C_\varepsilon V(x, \sqrt{t}) \exp(\varepsilon 4^j) \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y),$$

thus

$$\begin{aligned} &\int_{2^{j+1}B} |f(y) - f_{2^j B}| d\mu(y) \\ &\leq C_\varepsilon 2^j \sqrt{t} V(x, 2^j \sqrt{t}) \exp(\varepsilon 4^j) \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} |f_{2^\ell B} - f_{2^{\ell+1}B}| &\leq \frac{1}{V(x, 2^\ell \sqrt{t})} \int_{2^{\ell+1}B} |f(y) - f_{2^{\ell+1}B}| d\mu(y) \\ &\leq C 2^\ell \sqrt{t} \left( \frac{1}{V(x, 2^\ell \sqrt{t})} \int_{2^{\ell+1}B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2}, \end{aligned}$$

thus by (3.21) again,

$$|f_{2^\ell B} - f_{2^{\ell+1}B}| \leq C_\varepsilon 2^\ell \sqrt{t} \exp(\varepsilon 4^{\ell+1}) \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}.$$

Hence

$$\begin{aligned} &\int_{C_j} |f(y) - f_{4B}| d\mu(y) \\ &\leq C_\varepsilon \sqrt{t} V(x, 2^j \sqrt{t}) \left( 2^j + \sum_{\ell=2}^j 2^\ell \right) \exp(\varepsilon 4^j) \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2} \\ &\leq C'_\varepsilon 2^j \sqrt{t} V(x, 2^j \sqrt{t}) \exp(\varepsilon 4^j) \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}. \end{aligned}$$

Gathering the above estimates and using the doubling property, one obtains

$$\begin{aligned} & \int_{C_j} |\nabla_x p_t(x, y)| |f(y) - f_{4B}| d\mu(y) \\ & \leq C 2^j \exp(-(c - \varepsilon)4^j) \frac{V(x, 2^j \sqrt{t})}{V(x, \sqrt{t})} \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2} \\ & \leq C' 2^j \exp(-(c - \varepsilon)4^j) 2^{\nu j} \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}. \end{aligned}$$

Choosing  $\varepsilon < c$ , since then  $\sum_{j \geq 2} 2^j \exp(-(c - \varepsilon)4^j) 2^{\nu j} < \infty$ ,

$$|\nabla e^{-t\Delta} f(x)| \leq C \left( \int_M p_{C_\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2},$$

and (iii) is proved.

The converse, that is the implication from (iii) to (i), again follows from a variant of the argument in [20, p. 14]. Using again

$$\nabla_x p_{2t}(x, y) = \int_M \nabla_x p_t(x, z) p_t(z, y) d\mu(z)$$

and (iii), we have

$$|\nabla_x p_{2t}(x, y)|^2 \leq C \int_M p_{C't}(x, z) |\nabla_z p_t(z, y)|^2 d\mu(z).$$

Invoke a weighted  $L^2$  estimate for  $\nabla_x p_t(\cdot, y)$  proved in [45] (see also [18, Lemma 2.4]) and valid under (D) and (DUE): for some  $\gamma > 0$  and all  $y \in M, t > 0$ ,

$$(3.22) \quad \int_M |\nabla_x p_t(x, y)|^2 e^{\gamma \frac{d^2(x, y)}{t}} d\mu(x) \leq \frac{C}{tV(y, \sqrt{t})}$$

(note that contrary to (ii), the integration here is with respect to  $x$ , see the remark below). This yields

$$|\nabla_x p_{2t}(x, y)|^2 \leq \frac{C}{tV(y, \sqrt{t})} \left( \sup_{z \in M} p_{C't}(x, z) e^{-\gamma \frac{d^2(z, y)}{t}} \right).$$

Using (UE), the above supremum can easily be controlled by

$$\frac{C e^{-\beta \frac{d^2(x, y)}{t}}}{V(x, \sqrt{t})}$$

for  $\beta > 0$  small enough, and (i) follows.  $\square$



It remains to deduce the maximal estimate (3.16) from one of the other equivalent conditions. Assume (iii), that is

$$|\nabla e^{-t\Delta} f(y)|^2 \leq C e^{-C't\Delta} (|\nabla f|^2)(y),$$

then, since by (LY),

$$e^{-ct\Delta} (|\nabla f|^2)(y) \leq C e^{-C't\Delta} (|\nabla f|^2)(x)$$

as soon as  $d(x, y) \leq \sqrt{t}$ , one obtains

$$|\nabla e^{-t\Delta} f(y)|^2 \leq C e^{-C't\Delta} (|\nabla f|^2)(x).$$

On the other hand,  $e^{-C't\Delta} (|\nabla f|^2)(x) \leq C \mathcal{M}(|\nabla f|^2)(x)$  by (UE) and [36, Proposition 2.4]. This readily yields (3.16).

*Remarks.* –

- Although not necessary for the main argument developed in this section, we have recorded the equivalence between (i) and (ii) to point out a difference with the case  $1 < p < 2$  (see [18]). In that case, the crucial ingredient in the proof was the weighted estimate of the gradient (3.22), where integration and differentiation are taken with respect to the same variable. As we already said, (3.22) follows from the pointwise estimate of the heat kernel only. In contrast, the estimate (ii), with integration and differentiation with respect to different variables, requires in addition the pointwise estimate (i) of the gradient. All this also explains at a technical level why more assumptions are needed in Theorem 1.4 than in Theorem 1.1, and also why (1.8) holds on manifolds satisfying (FK) and

$$|\nabla_x p_t(x, y)| \leq C |\nabla_y p_t(x, y)|$$

for all  $t > 0, x, y \in M$  ([45, Theorem 1.3]).

- A sufficient condition for (iii) in terms of Ricci curvature follows from [64, Theorem 3.2, (3.7)]. According to the above,  $(R_p)$  holds for all  $p \in (1, \infty)$  on manifolds satisfying this condition plus (FK).
- The  $L^2$  version of Poincaré inequalities (P), which follows from the assumptions of the lemma and (i), is used in the above proof. If instead one has the stronger  $L^1$  version

$$\int_B |f - f_B| d\mu \leq Cr \int_B |\nabla f| d\mu,$$

for any ball  $B$  in  $(M, d)$  with radius  $r$  and any  $f$  with  $f, \nabla f$  locally integrable on  $B$  (which is the case for instance on Lie groups of polynomial volume growth, manifolds with non-negative Ricci curvature and co-compact coverings with polynomial growth deck transformation group), together (D), (DUE), then one can show in the same way the equivalence of (1.8) with the estimates

$$\sup_{y \in B} |\nabla e^{-r^2(B)\Delta} f(y)| \leq C' \inf_{x \in B} \mathcal{M}(|\nabla f|)(x),$$

and

$$|\nabla e^{-t\Delta} f| \leq C e^{-t\Delta} (|\nabla f|),$$

which are stronger than their  $L^2$  counterparts.

**3.3. Proofs of some other results**

In this section,  $M$  satisfies  $(D)$  and  $(DUE)$ .

*Proof of Proposition 1.10.* – First assume  $(G_p)$ . Write

$$\| |\nabla_x p_{2t}(\cdot, y)| \|_p = \| |\nabla e^{-t\Delta}(p_t(\cdot, y))| \|_p \leq \frac{C_p}{\sqrt{t}} \| p_t(\cdot, y) \|_p.$$

The estimate  $(UE)$  easily yields

$$\| p_t(\cdot, y) \|_p \leq \frac{C}{[V(y, \sqrt{t})]^{1-\frac{1}{p}}}.$$

This implies

$$\| |\nabla_x p_{2t}(\cdot, y)| \|_p \leq \frac{C_p}{\sqrt{t}[V(y, \sqrt{t})]^{1-\frac{1}{p}}}.$$

Conversely, assume for a  $p \in (2, p_0)$  for all  $y \in M$  and  $t > 0$ .

$$\| |\nabla_x p_t(\cdot, y)| \|_p \leq \frac{C_p}{\sqrt{t}[V(y, \sqrt{t})]^{1-\frac{1}{p}}}.$$

We shall prove  $(G_q)$  for any  $2 < q < p$ . Using (3.22) and interpolating with the above unweighted  $L^p$  estimate, we have

$$\int_M |\nabla_x p_t(x, y)|^q e^{\gamma' \frac{d^2(x, y)}{t}} d\mu(x) \leq \frac{C}{t^{q/2} [V(y, \sqrt{t})]^{q-1}}.$$

Now let  $f \in L^q(M, \mu)$ . Estimate  $|\nabla e^{-t\Delta} f(x)|$  by

$$\int_M |\nabla_x p_t(x, y)| e^{\frac{\gamma' d^2(x, y)}{qt}} [V(y, \sqrt{t})]^{1/q'} |f(y)| \frac{e^{-\frac{\gamma' d^2(x, y)}{qt}}}{[V(y, \sqrt{t})]^{1/q'}} d\mu(y)$$

which, by Hölder inequality, is controlled by

$$\left( \int_M |\nabla_x p_t(x, y)|^q e^{\gamma' \frac{d^2(x, y)}{t}} [V(y, \sqrt{t})]^{q/q'} |f(y)|^q d\mu(y) \right)^{1/q} \left( \int_M \frac{e^{-\frac{q' \gamma' d^2(x, y)}{qt}}}{V(y, \sqrt{t})} d\mu(y) \right)^{1/q'}.$$

Now, it follows easily from the doubling property that the second integral is bounded uniformly in  $t, x$ . Integrating with respect to  $x$ , by Fubini's theorem and the weighted  $L^q$ -estimate, we obtain

$$\int_M |\nabla e^{-t\Delta} f(x)|^q d\mu(x) \leq \frac{C}{t^{q/2}} \int_M |f(y)|^q d\mu(y)$$

as desired.  $\square$

*Proof of Theorem 1.4.* – We have already observed that the hypotheses in the statement imply (D) and (P). It remains to obtain  $(G_p)$  for all  $p \in (1, \infty)$ . In view of the previous result, it is enough to prove (1.6). But this follows by interpolating  $(G)$  with the  $L^2$  bound

$$\|\|\nabla_x p_t(\cdot, y)\|\|_2 \leq \frac{C_2}{\sqrt{t}[V(y, \sqrt{t})]^{1/2}}.$$

The latter follows from (DUE):

$$\begin{aligned} \|\|\nabla_x p_t(\cdot, y)\|\|_2^2 &= (\Delta p_t(\cdot, y), p_t(\cdot, y)) \\ &\leq \|\Delta p_t(\cdot, y)\|_2 \|p_t(\cdot, y)\|_2. \end{aligned}$$

Now

$$\|\Delta p_t(\cdot, y)\|_2 = \|\Delta e^{-(t/2)\Delta} p_{t/2}(\cdot, y)\|_2 \leq \frac{C}{t} \|p_{t/2}(\cdot, y)\|_2,$$

by analyticity of the heat semigroup on  $L^2$ , and  $\|p_{t/2}(\cdot, y)\|_2^2 = p_t(y, y) \leq \frac{C}{V(y, \sqrt{t})}$  by (DUE). The claim is proved.  $\square$

#### 4. Localization

This section is devoted to the proof of Theorem 1.5. Theorem 1.7 can then be deduced as in the global case and we leave details to the reader.

For  $a > 0$ , we have

$$\nabla(\Delta + a)^{-1/2} = c \int_0^\infty e^{-at} \nabla e^{-t\Delta} \frac{dt}{\sqrt{t}}.$$

Let  $v$  be a  $C^\infty$  function on  $[0, \infty)$  with  $0 \leq v \leq 1$ , which equals 1 on  $[0, 3/4]$  and vanishes on  $t \geq 1$ .

If  $a > \alpha$ , Minkowski’s integral inequality implies that

$$\left\| \int_0^\infty (1 - v(t)) e^{-at} \nabla e^{-t\Delta} f \frac{dt}{\sqrt{t}} \right\|_p \leq C \int_{3/4}^\infty e^{(\alpha-a)t} \frac{dt}{\sqrt{t}} \|f\|_p \leq C' \|f\|_p.$$

It is enough to prove the  $L^p$  boundedness of the sublinear operator  $\tilde{T}$  defined by

$$\tilde{T}f = \left| \int_0^\infty v(t) \nabla e^{-t(\Delta+a)} f \frac{dt}{\sqrt{t}} \right|.$$

Without loss of generality, we may and do replace in what follows  $\Delta + a$  by  $\Delta$  as the value of  $a$  plays no further role.

For this purpose, we begin with the localization technique of [36] as in [18].

Before we start, observe that, as a consequence of (E) and  $(P_{loc})$ ,  $p_t(x, y)$  satisfies the estimates (LY) for small times.

Let  $(x_j)_{j \in J}$  be a maximal 1-separated subset of  $M$ : the collection of balls  $B^j = B(x_j, 1)$ ,  $j \in J$ , covers  $M$ , whereas the balls  $B(x_j, 1/2)$  are pairwise disjoint. It follows from the local

doubling property that there exists  $N \in \mathbb{N}^*$  such that every  $x \in M$  is contained in at most  $N$  balls  $4B^j = B(x_j, 4)$ .

Consider a  $C^\infty$  partition of unity  $\varphi_j, j \in J$ , such that  $\varphi_j \geq 0$  and is supported in  $B^j$ . Let  $\chi_j$  be the characteristic function of the ball  $4B^j$ . For  $f \in C_0^\infty(M)$  and  $x \in M$ , write

$$\tilde{T}f(x) \leq \sum_j \chi_j \tilde{T}(f\varphi_j)(x) + \sum_j (1 - \chi_j) \tilde{T}(f\varphi_j)(x) = \text{I} + \text{II}.$$

Let us first treat the term II. The first observation is that, from the finite overlap property of the balls  $B^j$ , we have

$$\sum_j |(1 - \chi_j)(x)\varphi_j(y)| \leq N\chi_{d(x,y) \geq 3}.$$

Hence

$$\begin{aligned} \text{II} &\leq \int_0^1 \int_M |\nabla_x p_t(x, y)| \sum_j |(1 - \chi_j)(x)\varphi_j(y)| |f(y)| d\mu(y) \frac{dt}{\sqrt{t}} \\ &\leq N \int_0^1 \int_{d(x,y) \geq 3} |\nabla_x p_t(x, y)| |f(y)| d\mu(y) \frac{dt}{\sqrt{t}}. \end{aligned}$$

From there, we follow the argument in Section 3.3 by inserting the Gaussian terms and using the Hölder inequality to estimate the integral on  $d(x, y) \geq 3$ . Since (LY) for small times applies, we have

$$\int_{d(x,y) \geq 3} \frac{e^{-\frac{p'\gamma d^2(x,y)}{pt}}}{V(y, \sqrt{t})} d\mu(y) \leq Ce^{-\frac{c}{t}}$$

for some constants  $C, c > 0$  independent of  $t$  and  $x$ . Therefore,

$$\text{II} \leq C \int_0^1 \left( \int_M |\sqrt{t} \nabla_x p_t(x, y)|^p e^{\gamma \frac{d^2(x,y)}{t}} [V(y, \sqrt{t})]^{p/p'} |f(y)|^p d\mu(y) \right)^{1/p} \frac{e^{-\frac{c}{t}}}{t} dt.$$

It follows from  $(G_{p_0}^{\text{loc}})$  and the argument in Section 3.3 for small times (which is valid under the exponential growth assumption and the Gaussian upper bound for  $p_t(x, y)$  for small times, see [18]) that for some  $\gamma > 0$  and all  $t \in (0, 1)$  and  $y \in M$ ,

$$\int_M |\nabla_x p_t(x, y)|^p e^{\gamma \frac{d^2(x,y)}{t}} d\mu(x) \leq \frac{C}{t^{p/2} [V(y, \sqrt{t})]^{p-1}}.$$

Since the measure  $e^{-c/t}/t dt$  has finite mass, one can use Jensen’s inequality with respect to  $t$ , Fubini’s theorem and the weighted  $L^p$  estimate above to conclude that  $\int_M |\text{II}|^p d\mu(x)$  is bounded by  $C\|f\|_p^p$ .

We now turn to I which is the main term. The uniform overlap of the balls  $4B^j$  implies

$$\sum_j \|g\chi_j\|_r^r \leq C\|g\|_r^r$$

for all  $g$  and  $1 \leq r \leq \infty$ . Hence

$$\int_M |g(x)| \left| \sum_j \chi_j \tilde{T}(f\varphi_j)(x) \right| d\mu(x) \leq C \sum_j \|f\varphi_j\|_p \|g\chi_j\|_{p'} \leq C \|f\|_p \|g\|_q,$$

provided we show that

$$\|\chi_j |\tilde{T}(f\varphi_j)|\|_p \leq C \|f\varphi_j\|_p$$

with a bound uniform in  $j$ . In other words, we want to show that  $\tilde{T}$  maps  $L^p(B^j)$  into  $L^p(4B^j)$ . To this end we apply Theorem 2.4 with  $E_1 = B^j$  and  $E_2 = 4B^j$  since  $4B^j$  has the doubling property by the following lemma, which is implicit in [58], and whose proof we postpone until the end of this section.

LEMMA 4.1. – *The balls  $4B^j$  equipped with the induced distance and measure satisfy the doubling property (D) and the doubling constant may be chosen independently of  $j$ . More precisely, there is a constant  $C \geq 0$  such that for all  $j \in J$ ,*

$$(4.1) \quad \mu(B(x, 2r) \cap 4B^j) \leq C \mu(B(x, r) \cap 4B^j), \quad \forall x \in 4B^j, r > 0,$$

and also

$$(4.2) \quad \mu(B(x, r)) \leq C \mu(B(x, r) \cap 4B^j), \quad \forall x \in 4B^j, 0 < r \leq 8.$$

Define the local maximal function on  $M$  by

$$\mathcal{M}^{\text{loc}} f(x) = \sup_{B \ni x, r(B) \leq 32} \frac{1}{\mu(B)} \int_B |f| d\mu,$$

for  $x \in M$  and  $f$  locally integrable on  $M$ . By local doubling,  $\mathcal{M}^{\text{loc}}$  is bounded on all  $L^p(M)$ ,  $1 < p \leq \infty$ . We have the following estimates for balls  $B$  of  $M$  centered in  $4B^j$  and with radii less than 8,  $x \in B \cap 4B^j$  and functions  $f$  in  $L^2(M)$  with support in  $B^j$ :

1) There is an integer  $n$  depending only on the condition (E) such that

$$(4.3) \quad \frac{1}{\mu(B \cap 4B^j)} \int_{B \cap 4B^j} |\tilde{T}(I - e^{-r^2\Delta})^n f|^2 d\mu \leq C \mathcal{M}^{\text{loc}}(|f|^2)(x).$$

2) If  $2 < p < p_0$ , then for  $I - A_r = (I - e^{-r^2\Delta})^n$

$$(4.4) \quad \left( \frac{1}{\mu(B \cap 4B^j)} \int_{B \cap 4B^j} |\tilde{T}A_r f|^p d\mu \right)^{1/p} \leq C (\mathcal{M}_{4B^j}(|\tilde{T}f|^2) + (Sf)^2)^{1/2}(x)$$

with

$$(4.5) \quad (Sf)^2 = \mathcal{M}^{\text{loc}}(|\tilde{T}f|^2 \chi_{(M \setminus 4B^j)}) + \mathcal{M}^{\text{loc}}(|h|)^2 + \mathcal{M}_{4B^j}(|f|^2)$$

and  $h = \int_0^\infty v(t) e^{-t\Delta} f \frac{dt}{\sqrt{t}}$ .

Admitting these inequalities, it remains to see that  $\|Sf\|_{L^p(4B^j)} \leq C \|f\|_{L^p(B^j)}$ . By the study of II, we have that

$$\|(\mathcal{M}^{\text{loc}}(|\tilde{T}f|^2 \chi_{(M \setminus 4B^j)}))^{1/2}\|_{L^p(4B^j)} \leq C \|\tilde{T}f \chi_{(M \setminus 4B^j)}\|_{L^p(M)} \leq C \|f\|_{L^p(B^j)}.$$

Next, by definition of  $h$  and the contraction property of the heat semigroup,

$$\|h\|_{L^p(M)} \leq c\|f\|_{L^p(B^j)},$$

hence

$$\|\mathcal{M}^{\text{loc}}(|h|)\|_{L^p(4B^j)} \leq C\|f\|_{L^p(B^j)}.$$

Finally, we conclude the argument by invoking the boundedness of  $\mathcal{M}_{4B^j}$  on  $L^{p/2}(4B^j)$  to bound  $\|(\mathcal{M}_{4B^j}(|f|^2))^{1/2}\|_p$ .

*Proof of (4.3).* – Take a ball  $B$  centered in  $4B^j$  with  $r = r(B) < 8$  and let  $x$  be any point in  $B \cap 4B^j$ . By Lemma 4.1 we have  $\mu(B) \leq C\mu(B \cap 4B^j)$  for some  $C$  independent of  $B$  and  $j$ . Hence,

$$\begin{aligned} & \frac{1}{\mu(B \cap 4B^j)} \int_{B \cap 4B^j} |\tilde{T}(I - e^{-r^2\Delta})^n f|^2 d\mu \\ & \leq \frac{C}{\mu(B)} \int_B |\tilde{T}(I - e^{-r^2\Delta})^n f|^2 d\mu \end{aligned}$$

and we follow the calculations of Section 3 with  $\tilde{T}$  replacing  $\nabla\Delta^{-1/2}$ . Introduce  $i_r$  the integer defined by  $2^{i_r}r < 8 \leq 2^{i_r+1}r$ . Denote by  $C_i$  the ring  $2^{i+1}B \setminus 2^iB$  if  $i \geq 2$  and  $C_1 = 4B$ . Decompose  $f$  as  $f_1 + f_2 + f_3 + \dots + f_{i_r}$  with  $f_i = f\chi_{C_i}$ . The decomposition stops since  $f$  is supported in  $B^j$  and  $4B^j \subset 2^iB$  when  $i > i_r$ . By Minkowski’s inequality we have that

$$\|\tilde{T}(I - e^{-r^2\Delta})^n f\|_{L^2(B)} \leq \sum_{i=1}^{i_r} \|\tilde{T}(I - e^{-r^2\Delta})^n f_i\|_{L^2(B)}.$$

For  $i = 1$  we use the  $L^2$  boundedness of  $\tilde{T}(I - e^{-r^2\Delta})^n$ :

$$\|\tilde{T}(I - e^{-r^2\Delta})^n f_1\|_{L^2(B)} \leq \|f\|_{L^2(4B)} \leq \mu(4B)^{1/2}(\mathcal{M}^{\text{loc}}(|f|^2)(x))^{1/2},$$

For  $i \geq 2$  we use the definition of  $\tilde{T}$ :

$$\begin{aligned} \tilde{T}(I - e^{-r^2\Delta})^n f &= \left| \int_0^\infty v(t)\nabla e^{-t\Delta} f(I - e^{-r^2\Delta})^n \frac{dt}{\sqrt{t}} \right| \\ &= \left| \int_0^\infty \tilde{g}_r(t)\nabla e^{-t\Delta} f dt \right|, \end{aligned}$$

where using the usual notation for the binomial coefficient,

$$\tilde{g}_r(t) = \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{v(t - kr^2)}{\sqrt{t - kr^2}} \chi_{\{t > kr^2\}}.$$

By Minkowski’s integral inequality and the Gaffney estimates (3.1) using the support of  $f_i$ , we have that

$$\|\tilde{T}(I - e^{-r^2\Delta})^n f_i\|_{L^2(B)} \leq C \int_0^\infty |\tilde{g}_r(t)| e^{-\frac{\alpha'4^i r^2}{t}} \frac{dt}{\sqrt{t}} \|f\|_{L^2(C_i)}.$$

The latter integral can be estimated as follows. Elementary analysis yields the following estimates for  $\tilde{g}_r(t)$ :

$$\begin{aligned} |\tilde{g}_r(t)| &\leq \frac{C}{\sqrt{t - kr^2}} \quad \text{if } kr^2 < t \leq (k + 1)r^2 \leq (n + 1)r^2, \\ |\tilde{g}_r(t)| &\leq Cr^{2n} \quad \text{if } (n + 1)r^2 < t, \end{aligned}$$

and  $\tilde{g}_r(t) = 0$  for  $t \geq 1 + nr^2$ . Hence

$$\int_0^\infty |\tilde{g}_r(t)| e^{-\frac{\alpha'4^i r^2}{t}} \frac{dt}{\sqrt{t}} \leq C \min\{4^{-in}, r^{2n}\} = C8^{-2n}r^{2n}.$$

Now, an easy consequence of local doubling and  $r(2^i B) \leq 8$  when  $1 \leq i \leq i_r$ , is that

$$\mu(2^{i+1} B) \leq m(2^{i+1})\mu(B)$$

with  $m(\theta) = C(1 + \theta)^\nu$ ,  $C$  and  $\nu$  independent of  $B$  and  $j$ . Therefore, as  $C_i \subset 2^{i+1} B$ ,

$$\|f\|_{L^2(C_i)} \leq \mu(2^{i+1} B)^{1/2} (\mathcal{M}^{\text{loc}}(|f|^2)(x))^{1/2} \leq \sqrt{m(2^{i+1})}\mu(B)^{1/2} (\mathcal{M}^{\text{loc}}(|f|^2)(x))^{1/2}.$$

Choosing  $2n > \nu/2$  and using the definition of  $i_r$  and  $r \leq 8$ , we obtain

$$\begin{aligned} \|\tilde{T}(I - e^{-r^2\Delta})^n f\|_{L^2(B)} &\leq C \sum_{i=1}^{i_r} 2^{i\nu/2} r^{2n} \mu(B)^{1/2} (\mathcal{M}^{\text{loc}}(|f|^2)(x))^{1/2} \\ &\leq C\mu(B)^{1/2} (\mathcal{M}^{\text{loc}}(|f|^2)(x))^{1/2}. \end{aligned}$$

*Proof of (4.4).* – We first establish the analog of Lemma 3.2: for every  $p \in (2, p_0)$ , for every ball  $B$  centered in  $4B^j$  with radius  $r < 8$  and every  $L^2$  function  $g$  supported in  $C_i = 2^{i+1} B \setminus 2^i B$  if  $i \geq 2$  or  $C_1 = 4B$  and every  $k \in \{1, \dots, n\}$ , where  $n$  is chosen as above, one has

$$\begin{aligned} &\left( \frac{1}{\mu(B \cap 4B^j)} \int_{B \cap 4B^j} |\nabla e^{-kr^2\Delta} g|^p d\mu \right)^{1/p} \\ (4.6) \quad &\leq \frac{C e^{-\alpha 4^i}}{r} \left( \frac{1}{\mu(2^{i+1} B)} \int_{C_i} |g|^2 d\mu \right)^{1/2} \end{aligned}$$

for some constants  $C$  and  $\alpha$  depending only on  $(E)$ ,  $(P_{\text{loc}})$ ,  $p$  and  $p_0$ .

The proof given in Section 3 can be copied *in extenso* provided we make three remarks. First, as in the previous argument since  $\mu(B) \leq C\mu(B \cap 4B^j)$  we may replace  $B \cap 4B^j$  by  $B$  in the left-hand side of (4.6). Second, as we already said,  $p_t(x, y)$  satisfies the estimates  $(LY)$  for, say,  $t = kr^2 \leq 64n$ . Third, the polynomial volume growth is replaced by an exponential volume

growth but the estimates still carry out in this case thanks to the Gaussian terms in the sums. Further details are left to the reader.

Next, assume that  $f \in L^2(B^j)$  and take  $h = \int_0^\infty v(t)e^{-t\Delta} f \frac{dt}{\sqrt{t}}$ . Since  $\tilde{T}f = |\nabla h|$  and according to the first remark above, it is enough to control

$$\left( \frac{1}{\mu(B)} \int_B |\nabla e^{-kr^2\Delta} h|^p d\mu \right)^{1/p}.$$

Write  $\nabla e^{-kr^2\Delta} h = \nabla e^{-kr^2\Delta}(h - h_{4B}) = \sum_{i \geq 1} \nabla e^{-kr^2\Delta} g_i$  where  $g_i = (h - h_{4B})\chi_{C_i}$ . Applying (4.6) to each  $g_i$ , we are reduced to estimating  $(\frac{1}{\mu(2^{i+1}B)} \int_{C_i} |g_i|^2 d\mu)^{1/2}$ . We distinguish the two regimes  $i \leq i_r$  and  $i > i_r$  where  $i_r$  is the largest integer satisfying  $2^i r < 8$ . In the first regime, the argument in Section 3 using the local Poincaré inequalities for balls with radii not exceeding 16 can be repeated and (3.13) becomes

$$(4.7) \quad \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |g_i|^2 d\mu \right)^{1/2} \leq C \sum_{\ell=1}^i (2^\ell r) \left( \frac{1}{\mu(2^{\ell+1}B)} \int_{2^{\ell+1}B} |\nabla h|^2 d\mu \right)^{1/2}.$$

Write then

$$\begin{aligned} \frac{1}{\mu(2^{\ell+1}B)} \int_{2^{\ell+1}B} |\nabla h|^2 d\mu &\leq \frac{1}{\mu(2^{\ell+1}B \cap 4B^j)} \int_{2^{\ell+1}B \cap 4B^j} |\nabla h|^2 d\mu \\ &\quad + \frac{1}{\mu(2^{\ell+1}B)} \int_{2^{\ell+1}B} \chi_{(M \setminus 4B^j)} |\nabla h|^2 d\mu \\ &\leq \mathcal{M}_{4B^j}(|\nabla h|^2)(x) + \mathcal{M}^{\text{loc}}(|\nabla h|^2 \chi_{(M \setminus 4B^j)})(x) \end{aligned}$$

where  $x$  is any point in  $B \cap 4B^j$ . Hence the contribution of the terms in the first regime does not exceed

$$\sum_{1 \leq \ell \leq i \leq i_r} \frac{C e^{-\alpha 4^i}}{r} (2^\ell r) (\mathcal{M}_{4B^j}(|\nabla h|^2)(x) + \mathcal{M}^{\text{loc}}(|\nabla h|^2 \chi_{(M \setminus 4B^j)})(x))^{1/2}.$$

For the second regime, we proceed directly by

$$\left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |g_i|^2 d\mu \right)^{1/2} \leq \left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |h|^2 d\mu \right)^{1/2} + |h_{4B}|.$$

First  $|h_{4B}| \leq \mathcal{M}^{\text{loc}}(|h|)(x)$  since  $B$  has radius less than 8. Next,

$$\int_{C_i} |h|^2 d\mu \leq \int_M |h|^2 d\mu \leq C \int_M |f|^2 d\mu = C \int_{4B^j} |f|^2 d\mu$$

since  $f$  is supported in  $B^j \subset 4B^j$ . Also since  $i > i_r$  we have  $2^{i+1}B \supset 4B^j$  and

$$\left( \frac{1}{\mu(2^{i+1}B)} \int_{C_i} |h|^2 d\mu \right)^{1/2} \leq C \left( \frac{1}{\mu(4B^j)} \int_{4B^j} |f|^2 d\mu \right)^{1/2} \leq C (\mathcal{M}_{4B^j}(|f|^2)(x))^{1/2}.$$



The contribution of the terms in the second regime is bounded above by

$$\sum_{i>i_r} \frac{Ce^{-\alpha 4^i}}{r} (\mathcal{M}^{\text{loc}}(|h|)(x) + (\mathcal{M}_{4B^j}(|f|^2)(x))^{1/2})$$

and it remains to recall that  $1/r \leq 2^i/8$  when  $i > i_r$  to conclude the proof of (4.4).

The proof of Theorem 1.5 is complete provided we prove Lemma 4.1 which we do now.

We begin with the first inequality. Fix  $j \in J, x \in 4B^j, r > 0$ . If  $r \geq 8$ , there is nothing to prove. We assume  $r < 8$ . There is a point  $x_*$  such that  $B(x_*, r/8) \subset 4B^j$  and  $d(x, x_*) \leq 3r/8$ . Indeed, if  $d(x, x_j) \leq 3r/8$  then  $B(x, r/8) \subset 4B^j$ , so that one can take  $x_* = x$ . Otherwise, since  $M$  is connected, there is a curve joining  $x$  and  $x_j$  whose length is smaller than, say,  $d(x, x_j) + 2r/8$ . On this curve, one can choose  $x_*$  such that  $d(x, x_*) = 3r/8$ , thus  $d(x_*, x_j) \leq d(x, x_j) - r/8$ . The point  $x_*$  satisfies the required properties. Then one may write

$$\begin{aligned} \mu(B(x, 2r) \cap 4B^j) &\leq V(x, 2r) \leq V(x_*, 19r/8) \\ &\leq CV(x_*, r/8) \leq C\mu(B(x, r) \cap 4B^j), \end{aligned}$$

where one uses the local doubling property for balls with radii not exceeding 19. This proves the first inequality of the lemma and the proof of the second one is contained in the argument.

### 5. $L^p$ Hodge decomposition for non-compact manifolds

The Hodge decomposition, which associates to a form its exact, co-exact and harmonic parts, is well-known to be bounded on  $L^2$  on any complete Riemannian manifold (see for instance [29, Theorem 24, p. 165], and the recent survey [11]). The question of the  $L^p$  Hodge decomposition has been mainly examined on closed manifolds (see [86]) and domains therein [57]. On non-compact manifolds, the connection with the Riesz transform was established in the case of 1-forms and an example was treated in [92]. The unpublished manuscript [66] contains results for forms of all degrees in the case of manifolds with positive bottom of the spectrum; see also [67]. In the case of degree one, one can deduce more general results from Theorems 1.2 and 1.8. We denote by  $L^p T^*M$  the usual  $L^p$  space of 1-forms.

**THEOREM 5.1.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying either (D), (P) and  $(G_{p_0})$  for some  $p_0 \in (2, \infty]$ , or the assumptions of Theorem 1.8. Then the Hodge projector from 1-forms onto exact forms is bounded on  $L^p T^*M$ , for all  $p \in (q_0, p_0)$  where  $q_0$  is the conjugate exponent to  $p_0$ .*

*Proof.* – The projector on exact forms is

$$d\Delta^{-1}\delta = d\Delta^{-1/2}(d\Delta^{-1/2})^*.$$

Now  $d\Delta^{-1/2}$  is bounded from  $L^p(M, \mu)$  to  $L^p T^*M$  for all  $p \in (q_0, p_0)$  by the results in [18] and Theorem 1.2 or Theorem 1.8, depending on the assumptions. By duality,  $(d\Delta^{-1/2})^*$  is bounded from  $L^p T^*M$  to  $L^p(M, \mu)$ , for  $p$  in the same range, and the claim follows.

**COROLLARY 5.2.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying either (D), (P) and  $(G_{p_0})$  for some  $p_0 \in (2, \infty]$ , or the assumptions of Theorem 1.8. Let  $p \in (q_0, p_0)$ . Then the Hodge decomposition on 1-forms extends to  $L^p T^*M$  if one of the following additional assumptions holds:*

- (a)  $M$  is a surface.
- (b) There are no  $L^p$  harmonic 1-forms.
- (c) The heat semigroup on 1-forms,  $e^{-t\bar{\Delta}}$ , is bounded on  $L^p T^*M$  uniformly in  $t > 0$ .

*Proof.* – Let  $p \in (q_0, p_0)$ . The projector on exact forms is bounded on  $L^p T^*M$  by Theorem 5.1. In case (a), we use an argument from [92]: in dimension 2, the projector on co-exact forms is nothing but  $d\Delta^{-1}\delta$  conjugated by the Hodge star operator, therefore it is also bounded on  $L^p T^*M$ . In cases (b) and (c), the projection on harmonic forms extends boundedly to  $L^p T^*M$ : In case (b), this projector is trivial by assumption, and in case (c), it is bounded because it is the limit of  $e^{-t\bar{\Delta}}$  as  $t \rightarrow \infty$ . In all three cases, two out of the three projectors in the Hodge decomposition extend boundedly to  $L^p T^*M$ , therefore the whole decomposition is an  $L^p$  decomposition.

Since, on manifolds with non-negative Ricci curvature, the Riesz transform is bounded on  $L^p$  for all  $p \in (1, \infty)$ , and for all  $t > 0$  and  $\omega \in C^\infty T^*M$ ,

$$|e^{-t\bar{\Delta}}\omega| \leq e^{-t\Delta}|\omega|,$$

the proofs of Theorem 5.1 and Corollary 5.2 show in particular that the Hodge decomposition extends to  $L^p$  on such manifolds. This fact seems known to experts, although we could not find it stated in the literature.

More generally, if the heat semigroup on forms on  $M$  is dominated by the heat semigroup on functions in the way discussed in Section 1.4:

$$(5.1) \quad |e^{-t\bar{\Delta}}\omega| \leq C e^{-ct\Delta}|\omega|,$$

for some  $C, c > 0$  and all  $t > 0$  and  $\omega \in C^\infty T^*M$ , then both assumptions (b) (because there are no  $L^p$  harmonic functions on a complete non-compact Riemannian manifold, see [100]) and (c) are satisfied for all  $p \in (1, \infty)$ . It was proved in [20] (and it also follows from Theorem 1.4) that (D), (DUE) and (5.1) imply the boundedness of the Riesz transform for all  $p \in (1, \infty)$ . Therefore, we can state the following.

**COROLLARY 5.3.** – *Let  $M$  be a complete non-compact Riemannian manifold satisfying (D), (DUE) and (5.1). Then the Hodge decomposition on 1-forms extends to  $L^p T^*M$  for all  $1 < p < \infty$ .*

A similar statement could be formulated, with suitable assumptions, for manifolds where the bottom of the spectrum is positive, by using Theorem 1.9.

Finally, recall that the  $L^p$  boundedness of the Hodge decomposition may be useful, together with other ingredients, to show that, in certain situations,  $e^{-t\bar{\Delta}}$  cannot be contractive on  $L^p T^*M$  for all  $p$  (see [92, p. 77]; this problem was also considered in [93]). We will not pursue this here.

### 6. Final remarks

Our method should apply without major difficulties to some settings which had already been treated in the range  $1 < p < 2$  by the use of the method in [18]: graphs [82] and vector bundles [97]. In the first direction, some results already exist in the range  $1 < p < \infty$  in the group case, see [33,55]. It should also be possible to cover the general setting of [88].

Other directions are left open by the present work. Since we work in the framework of singular integrals theory (and, in particular, use a volume growth assumption), the estimates we obtain do

depend on the dimension, contrary to the ones in [6,7]. To obtain dimension-free estimates, or to work in an infinite dimensional setting, is a subject in itself, and was achieved so far only in rather specific situations: see [90,73,78,23,30,77], and the references therein. Let us point out that this is an advantage of the approach to Riesz transform boundedness via the Littlewood–Paley theory, as in [6,7]. See also [20]. We do not see for the time being how to cumulate the advantages of both methods, which would yield a proof of the conjecture in [20].

We do not touch either the question of higher order Riesz transforms, whose boundedness would require much more regularity; recall that already for Lie groups with polynomial growth, even the  $L^2$  boundedness of second order Riesz transforms only takes place on nilpotent groups and their compact extensions (see [1,38]).

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