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QUANTUM VARIANCE FOR HECKE EIGENFORMS ☆

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ABSTRACT. – We calculate the quantum variance for the modular surface. This variance, introduced by S. Zelditch, describes the fluctuations of a quantum observable. The resulting quadratic form is then compared with the classical variance. The expectation that these two coincide only becomes true after inserting certain subtle arithmetic factors, specifically the central values of corresponding L-functions. It is the off-diagonal terms in the analysis that are responsible for the rich arithmetic structure arising from the diagonalization of the quantum variance.

RÉSUMÉ. – Nous calculons la variance quantique pour la surface modulaire. Cette variance, introduite par S. Zelditch, décrit les fluctuations d'une observable quantique. La forme quadratique ainsi obtenue est comparée avec la variance classique. On s'attend à ce que toutes les deux coïncident, mais cela ne se passe qu'après inclusion de certains facteurs arithmétiques subtils, précisément les valeurs centrales des fonctions L appropriées. Les termes non diagonaux apparaissant dans l'analyse de la diagonalisation de la variance quantique sont responsables de la riche structure arithmétique.

1. Introduction

This is the third paper of the series [23,24] dealing with the equidistribution of mass of automorphic forms on $X = \Gamma\backslash\mathbf{H}$ with $\Gamma = SL(2,\mathbf{Z})$ and $\mathbf{H}$ the upper half plane. We realize $\mathbf{H}$ as $SL(2,\mathbf{R})/SO(2,\mathbf{R})$ with its hyperbolic metric and $Y = \Gamma\backslash SL(2,\mathbf{R})$ as the unit cotangent space to X. Functions on X can be thought of as $SO(2,\mathbf{R})$ invariant functions on Y and we will often do so. In this way the Casimir element ω in the universal enveloping algebra of $sl(2,\mathbf{R})$ restricts to the Laplace–Beltrami operator Δ when acting on functions on X.

There are two types of automorphic forms which we study. The first are the Maass–Hecke cusp forms ϕ on X (see [28]). They satisfy

$$\Delta\phi + \lambda\phi = 0, \qquad T_n\phi = \lambda_\phi(n)\phi, \tag{1}$$

where for $n \geqslant 1$, T_n is the normalized Hecke operator (see [11]). We normalize these cusp forms so that

$$\|\phi\|_2^2 = \int_X \left|\phi(z)\right|^2 \frac{dx\,dy}{y^2} = 1.$$

If we order the ϕ's by their eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$, and correspondingly $\phi_1, \phi_2, \ldots$, we obtain an orthonormal basis for the cuspidal subspace $L^2_{cusp}(X)$ of $L^2(X)$. It is known (after Selberg)

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that these eigenvalues satisfy a Weyl law

$$N(\lambda)=\sum_{\lambda_j\leqslant\lambda}1\sim\frac{\text{area}(X)}{4\pi}\lambda=\frac{\lambda}{12},$$

as $\lambda\to\infty$.

The other automorphic forms which we consider are the holomorphic cusp forms in $S_k(\Gamma)$ of even integral weight k for Γ (see [30]). $S_k(\Gamma)$ is a vector space with the Petersson inner product. Let H_k be the orthonormal basis of Hecke eigenforms for $S_k(\Gamma)$. According to the Riemann–Roch theorem we have

$$\dim S_k(\Gamma)=\#H_k\sim\frac{k}{12},$$

as $k\to\infty$.

Our interest is in the distribution of the probability measures on X, $\mu_\phi=|\phi(z)|^2\frac{dx\,dy}{y^2}$ for ϕ in (1) and $\mu_f=y^k|f(z)|^2\frac{dx\,dy}{y^2}$ for $f\in H_k$, as well as their behavior as λ or k goes to infinity (that is, in the semi-classic limit).

To explain what to expect, we recall some conjectures (or suggestions) from the physics literature. The motion by geodesics on X gives rise to a Hamiltonian flow $\mathcal{G}_t$ on Y given by

$$\Gamma g\to\Gamma g\begin{pmatrix}e^{t/2}&0\\0&e^{-t/2}\end{pmatrix},\quad t\in\mathbf{R}.$$

This flow preserves normalized Haar measure dg on Y and is ergodic. It has positive entropy as well as all other characteristics of a chaotic Hamiltonian. Let $C_0^\infty(Y)$ denote the space of smooth functions on Y which decay rapidly in the cusp and similarly we define the space $C_0^\infty(X)$. Thus if $\psi\in C_0^\infty(X)$ and for any $A>0$ there is a constant $C=C(A,\psi)$ such that $|\psi(z)|\leqslant C(A,\psi)y^{-A}$ for $y=\Im(z)\geqslant\sqrt{3}/2$ and similarly for the derivatives of ψ. Let $C_{0,0}^\infty(X)$ (respectively $C_{0,0}^\infty(Y)$) be the subspace of $C_0^\infty(X)$ consisting of functions with mean zero (i.e. $\int_X\psi(z)\frac{dx\,dy}{y^2}=0$) and whose zeroth Fourier coefficient $\int_0^1\psi(z)\,dx$ is zero for y large enough (depending on ψ). Thus $C_{0,0}^\infty(X)$ contains the space $C_{c,0}^\infty(X)$ of smooth functions on X with compact support and mean zero, as well as $C_{cusp}^\infty(X)$, the space of smooth rapidly decaying functions on X which are cuspidal. The last is spanned by the Hecke–Maass cusp forms. It is known [25] that if $\psi\in C_{0,0}^\infty(Y)$, then its fluctuations along a generic orbit of the geodesic flow obey a central limit theorem. Precisely $\frac{1}{\sqrt{T}}\int_0^T\psi(\mathcal{G}_t(g))\,dt$ become Gaussian with mean 0 and variance $V(\psi)$ given by the following non-negative Hermitian form on $C_{0,0}^\infty(Y)$:

$$V(\psi_1,\psi_2)=\int_{-\infty}^{\infty}\int_{\Gamma\backslash SL(2,\mathbf{R})}\psi_1\left(g\begin{pmatrix}e^{t/2}&0\\0&e^{-t/2}\end{pmatrix}\right)\overline{\psi_2(g)}\,dg\,dt.\tag{2}$$

The t-integral in (2) converges absolutely in view of the exponential decay of the correlations for the flow $\mathcal{G}_t$ [26]. We call the variance $V(\psi)$ of the 'classical observable' ψ, the classical variance. Since ω commutes with the regular representation, it follows from (2) and integration by parts that

$$V(\omega\psi_1,\psi_2)=V(\psi_1,\omega\psi_2),\quad\text{and}\quad V(R_{a_1}\psi_1,R_{a_2}\psi_2)=V(\psi_1,\psi_2),\tag{3}$$

where R is the regular representation given by

$$R_{g_1}\psi(\Gamma g)=\psi(\Gamma g g_1),$$

and

$$a_1=\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_1^{-1}\end{pmatrix},\qquad a_2=\begin{pmatrix}\alpha_2 & 0\\ 0 & \alpha_2^{-1}\end{pmatrix}.$$

In particular V is diagonalized by the invariant subspaces for the regular action of $SL(2,\mathbf{R})$ on $L_0^2(Y)$. Restricting V to $C_{0,0}^\infty(X)$ we see from (3) that

$$V(\psi_1,\Delta\psi_2)=V(\Delta\psi_1,\psi_2).$$

Thus on $C_{0,0}^\infty(X)$, V is diagonalized by the Maass cusp forms ϕ_j (and corresponding unitary Eisenstein series). In Appendix A we compute the eigenvalue of V on ϕ_j, it is given by

$$V(\phi_j)=\frac{|\Gamma(\frac{1}{4}-\frac{it_j}{2})|^4}{2\pi|\Gamma(\frac{1}{2}-it_j)|^2}, \tag{4}$$

where $\lambda_j=\frac{1}{4}+t_j^2$.

The eigenvalue problem (1) gives the eigenstates for the quantization of the Hamilton flow $\mathcal{G}_t$. Quantization also provides a self-adjoint operator $Op(\psi)$ on $L^2(X)$, for any real valued ψ in $C_0^\infty(Y)$. In this case a 'canonical' quantization is given by Zelditch [32]. $Op(\psi)$ is the quantum observable corresponding to the classical observable ψ and $\langle Op(\psi)\phi_j,\phi_j\rangle$ gives the value of this observable in state ϕ_j. Note that if $\psi\in C_0^\infty(X)$, then $Op(\psi)$ is simply the multiplication operator $(Op(\psi)h)(z)=\psi(z)h(z)$ and $\langle Op(\psi)\phi_j,\phi_j\rangle=\mu_{\phi_j}(\psi)$.

As mentioned before our interest is in the relation between the classical observable $\psi(\mathcal{G}_t(g))$ as $t\to\infty$ and the quantum observables $\langle Op(\psi)\phi_j,\phi_j\rangle$ as $\lambda_j\to\infty$. It is known [32] that their means agree. For $\psi\in C_0^\infty(Y)$,

$$\lim_{\lambda\to\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\leqslant\lambda}\langle Op(\psi)\phi_j,\phi_j\rangle=\int_{\Gamma\backslash SL(2,\mathbf{R})}\psi(g)\,dg. \tag{5}$$

In studying the fluctuations we will assume that $\psi\in C_{0,0}^\infty(Y)$. In [6] and [5] it is proposed that for such classically chaotic Hamiltonians, the variance of the quantum observables $\langle Op(\psi)\phi_j,\phi_j\rangle$ corresponds to the classical variance $V(\psi)$ and that the distribution of these numbers becomes Gaussian after normalization by the square root of the variance. More precisely the proposed quantum variance is

$$S_\psi(\lambda):=\sum_{\lambda_j\leqslant\lambda}\left|\langle Op(\psi)\phi_j,\phi_j\rangle\right|^2\sim V(\psi)N(\lambda)^{1/2}, \tag{6}$$

as $\lambda\to\infty$.

Zelditch [33] introduced these quantum variance sums in his treatment of the quantum ergodicity for this surface. He established the non-trivial bound $S_\psi(\lambda)=\mathrm{O}_\psi(\lambda/\log\lambda)$. In [23] we showed that for $\psi\in C_{0,0}^\infty(X)$ and any $\epsilon>0$, $S_\psi(\lambda)=\mathrm{O}_\psi(\lambda^{1/2+\epsilon})$, and Jakobson [16] extended this bound to all $\psi\in C_{0,0}^\infty(Y)$. The analysis leading to these $\mathrm{O}(\lambda^{1/2+\epsilon})$ bounds involves off-diagonal terms coming from an application of Kuznetsov's trace formula (see the outline below). These were handled using the large sieve inequalities of Deshouillers and Iwaniec [3]. In order to get rid of the ϵ and obtain an asymptotic for $S_\psi(\lambda)$, one cannot afford to just estimate these off-diagonal terms. In fact as shown below, these terms contribute to the main term in the asymptotics.

As is clear from the later sections of this paper, the analysis of these quantum variance sums is rather delicate. We will follow our strategy in [24], to examine first the quantum variance for the very similar problem with ϕ_j replaced by $f \in H_k$. That is, for $\psi \in C_{0,0}^\infty(X)$, set

$$\langle Op(\psi)f, f\rangle := \mu_f(\psi) = \int_X y^k |f(z)|^2 \psi(z) \frac{dx\,dy}{y^2}. \tag{7}$$

The corresponding quantum variance sums are

$$\sum_{k\leqslant K, 2|k} \sum_{f\in H_k} |\mu_f(\psi)|^2.$$

Note that k plays the role of $\sqrt{\lambda}$. The only difference between our treatment of (7) and $S_\psi(\lambda)$ of (6) is that for the holomorphic case one uses the Petersson formula (see [12]) in place of the Kuznetsov formula [21]. This simplifies the analysis especially as far as the special functions are involved. We leave the details of the analysis of the asymptotics of $S_\psi(\lambda)$ to a later paper, though we will record below the leading term in that case for the purpose of comparison.

We can now state the main result of this paper. In view of the Petersson formula it is convenient to consider a weighted version of the quantum variance sums. The weights are mildly varying arithmetic weights given by special values at $s = 1$ of L-functions. With a little more effort (see [13]) these weights can be removed, and they have no effect on the final asymptotics. For $f \in H_k$ or ϕ a Maass–Hecke cusp form, let $L(s, f)$ and $L(s, \phi)$ be the corresponding standard L-functions (finite part), see [14], for example, for a description of the L-functions that we need. The completed L-functions $\Lambda(s, f)$ and $\Lambda(s, \phi)$ are entire and satisfy functional equations. Let $\mathrm{sym}^2(f)$ and $\mathrm{sym}^2(\phi)$ be the symmetric square lifts of f and ϕ respectively to cusp forms on $GL_3(\mathbf{A}_Q)$ (see [7]). The corresponding L-functions, which are Euler products of degree 3, are denoted by $L(s, \mathrm{sym}^2(f))$ and $L(s, \mathrm{sym}^2(\phi))$. Their completed L-functions $\Lambda(s, \mathrm{sym}^2(f))$ and $\Lambda(s, \mathrm{sym}^2(\phi))$ are entire and satisfy a functional equation relating the values at s and $1 - s$. We will also have the occasion to use the Rankin–Selberg L-functions $L(s, \mathrm{sym}^2(f) \otimes \phi)$ of degree 6 and their completion $\Lambda(s, \mathrm{sym}^2(f) \otimes \phi)$. The weights in question are $L(1, \mathrm{sym}^2(f))$. Being special values at $s = 1$, they satisfy the bounds (see [10])

$$k^{-\epsilon} \ll_\epsilon L\big(1, \mathrm{sym}^2(f)\big) \ll_\epsilon k^\epsilon,$$

for any $\epsilon > 0$.

THEOREM 1. – *Fix $u \in C_0^\infty(0, \infty)$.*

(A) *There is a non-negative Hermitian form B_ω defined on $C_{0,0}^\infty(X)$ such that for $\psi \in C_{0,0}^\infty(X)$ and $\epsilon > 0$,*

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \sum_{f\in H_k} L\big(1, \mathrm{sym}^2(f)\big)|\mu_f(\psi)|^2$$

$$= B_\omega(\psi)\left(\int_0^\infty u(t)\,dt\right)K + \mathrm{O}_{\epsilon,\psi}(K^{1/2+\epsilon}), \tag{8}$$

as $K \to \infty$.

(B) *B_ω satisfies the symmetries*

$$B_\omega(\Delta\psi_1,\psi_2)=B_\omega(\psi_1,\Delta\psi_2),$$

and for $n\geqslant 1$,

$$B_\omega(T_n\psi_1,\psi_2)=B_\omega(\psi_1,T_n\psi_2).$$

(C) *Restricting B_ω to $L^2_{cusp}(X)$, B_ω is diagonalized by the orthonormal basis $\{\phi_j\}$ of Maass–Hecke cusp forms and the eigenvalues of B_ω at ϕ_j is $\frac{\pi}{2}L(1/2,\phi_j)$.*

Remarks. –

(1) A simple approximation argument in (A) allows us to take u to be the characteristic function of an interval. Hence as $K\to\infty$,

$$\sum_{k\leqslant K, 2|k}\sum_{f\in H_k} L\big(1,\operatorname{sym}^2(f)\big)\big|\mu_f(\psi)\big|^2\sim B_\omega(\psi)K.$$

Also

$$\sum_{k\leqslant K, 2|k}\sum_{f\in H_k} L\big(1,\operatorname{sym}^2(f)\big)\sim\frac{\zeta^2(2)}{48}K^2.$$

Thus we obtain the analogue for the μ_f's of the asymptotics of $S_\psi(\lambda)$. As mentioned earlier the methods of the proof of Theorem 1 apply to $S_\psi(\lambda)$ and yield

$$S_\psi(\lambda)\sim B(\psi)\sqrt{\lambda}$$

as $\lambda\to\infty$. The Hermitian form B on $C^\infty_{0,0}(X)$ satisfies the same symmetry relation (B) of Theorem 1. The only difference is that the eigenvalue of B at ϕ_j is given by $B(\phi_j)=\frac{1}{2}L(1/2,\phi_j)V(\phi_j)$. Hence both the forms B and V are diagonalized by the ϕ_j's and the proposed quantum variance (6) is correct if one inserts the subtle arithmetic factor $L(1/2,\phi_j)$ to the eigenvalues of V.

(2) The numbers $L(1/2,\phi_j)$, which are essentially the eigenvalues of the non-negative Hermitian form B_ω, must satisfy $L(1/2,\phi_j)\geqslant 0$. This non-obvious fact is quite deep and useful (see [14]). It was first established in [17]. The present eigenvalue proof is interesting from various points of view. There is a lot of evidence that the zeros of an L-function are spectral in their nature (see [18]). Here we have the numbers $L(1/2,\phi_j)$, as ϕ_j varies over the family of Maass–Hecke eigenforms, being the eigenvalues of a non-negative operator.

(3) It is known [15] that at least 50% of the even ϕ_j's, i.e. those satisfying $\phi_j(-\bar{z})=\phi_j(z)$, have $L(1/2,\phi_j)\neq 0$. For the odd ϕ_j, $L(1/2,\phi_j)=0$ in view of the sign of the functional equation of $\Lambda(1/2,\phi_j)$, and also $\mu_f(\phi_j)=0$ since any f in H_k is real on $y=0$ and so $\overline{f(-\bar{z})}=f(z)$. One can show that (see [22])

$$\sum_{k\leqslant K, 2|k}\sum_{f\in H_k} L^2\big(1,\operatorname{sym}^2(f)\big)\sim\frac{\zeta(3)\zeta^5(2)}{48\cdot\zeta(6)}K^2.$$

Combining this with Theorem 1 and Cauchy's inequality, we see that for ϕ with $L(1/2,\phi)\neq 0$,

$$\mu_f(\phi)=\Omega(k^{-1/2})$$

as $k \to \infty$. This shows that the rate of equidistribution for the $\mu_f(\psi)$'s in the QUE problem, i.e. for any $\epsilon > 0$,

$$\mu_f(\psi) = \mathrm{O}_{\epsilon,\psi}(k^{-1/2+\epsilon}),$$

as predicted by Watson's formula (see below) and the GRH, is essentially sharp.

We end the introduction with an outline of the paper. As in [23] and [24], we establish (A) using the Poincaré series $P_{m,h}$ (see Section 2). These form a dense subspace of $C_{0,0}^{\infty}(X)$, and they allow us to analyze the quantity $\mu_f(P_{m,h})$ in terms of sums over Fourier coefficients of f. This in turn allows us to exploit the multiplicativity of these coefficients, which comes from the fact that f is a Hecke eigenform (a crucial ingredient). We then average over H_k, using Petersson's formula (see Section 2). This introduces diagonal and non-diagonal terms. The off-diagonal terms involve standard Kloosterman sums. Next we execute the smooth sum over k using Poisson summation. An application of Lemmas 4.1 and 4.2 from [24] introduces an arithmetic twisting of the Kloosterman sums which become Salié sums. In this way the main term (as $K \to \infty$) is identified and it contains an infinite series of exponential sums $S_c(\gamma)$ discussed in Appendix B. These non-diagonal terms appear as part of the rather complicated main term that is given in Theorem 2 in Section 2. In this form the Hermitian form B_ω is given in terms of its values at Poincaré series. In Section 3 we analyze B_ω. Using the series expression as obtained in Theorem 2, we verify directly the symmetry properties (B) of Theorem 1 when ψ_1, ψ_2 are Poincaré series. In Section 4 the symmetry is extended to $C_{0,0}^{\infty}(X)$. With this and the fact that ϕ_j is uniquely determined by the eigenvalues λ_j and $\lambda_j(n), n \geqslant 1$, we infer easily that B_ω is diagonalized by the ϕ_j's. In Section 5 we compute the eigenvalues of B_ω at ϕ_j. To do so we go back to the original asymptotics in Theorem 1 with $\psi = \phi$, an even Maass–Hecke eigenform. For such a ϕ we use Watson's identity [29]

$$|\mu_f(\phi)|^2 = \left|\int_X y^k |f(z)|^2 \phi(z) \frac{dx\,dy}{y^2}\right|^2 = \frac{\Lambda(1/2, \mathrm{sym}^2(f) \otimes \phi)\Lambda(1/2, \phi)}{\Lambda(1, \mathrm{sym}^2(f))^2 \Lambda(1, \mathrm{sym}^2(\phi))}. \tag{9}$$

Thus the quantum variance sum over f boils down, after an analysis of the archimedean factors on the r.h.s. of (9), to averaging $L(1/2, \mathrm{sym}^2(f) \otimes \phi)$ over f. Using Rankin–Selberg theory for $GL(3) \times GL(2)$, we can express these values in a suitable series (see Section 5), after which the averaging over $f \in H_k$ and over even k can be carried out. Unlike the case of the general ψ in Theorem 1, in this analysis for ϕ only the diagonal terms contribute to the main term in the variance sum. This leads to the eigenvalue, i.e. $B_\omega(\phi)$, taking the simple form as stated in part (C) of Theorem 1.

To conclude the introduction we comment on the proposed Gaussian behavior of either $\mu_f(\phi)$ as f varies, or $\mu_{\phi_j}(\phi)$ as j varies, with ϕ a fixed even Maass–Hecke form. According to (9) and an analysis of the archimedean factors in (9), this amounts to the distribution of the numbers $L(1/2, \mathrm{sym}^2(f) \otimes \phi)$ as f varies. This family of L-functions, $L(s, \mathrm{sym}^2(f) \otimes \phi)$ with $f \in H_k$, $k \to \infty$, is an $SO(\text{even})$ family according to [18]. This is shown in [4] which examines the distribution of the low-lying zeros for this family (note the signs of the functional equations for this self-dual family are all 1, yet the family has an orthogonal rather than symplectic symmetry). Hence according to the conjectures of Keating and Snaith [19, (77)] the moments of these special values should satisfy

$$\frac{48}{K^2} \sum_{k \leqslant K, 2|k} \sum_{f \in H_k} L^m\big(1/2, \mathrm{sym}^2(f) \otimes \phi\big) \sim (\log K^2)^{m(m-1)/2} a(m) f_{SO(even)}(m),$$

where

$$f_{SO(even)}(m) = \frac{2^m}{\prod_{j=1}^{m-1}(2j-1)!!},$$

and $a(m)$ is a product over primes specific to this family, which can be computed for any given m, but for which we don't have a simple closed formula.

Thus if these conjectures are true, then the distribution of the numbers $L(1/2, \mathrm{sym}^2(f) \otimes \phi)$ and hence $|\mu_f(\phi)|^2$ is clearly not Gaussian, at least in the sense of convergence of moments.

To conclude we point the reader to the recent preprint [20] where a similar anomaly for the quantum variance is found for the cat map.

2. Poincaré series

We use the same notations as in [24]. For $h(x) \in C_0^\infty(0,\infty)$, the incomplete Poincaré series ($m \in \mathbf{Z}, m \neq 0$) is defined as

$$P_{h,m}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h\big(y(\gamma z)\big) e\big(mx(\gamma z)\big),$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ n \in \mathbf{Z} \right\}.$$

A cusp form $f \in H_k$ has a Fourier expansion

$$f(z) = \sum_{r \geqslant 1} a_f(r) e(rz),$$

and we define

$$\lambda_f(r) = \frac{a_f(r) r^{(-k+1)/2}}{a_f(1)}.$$

Denote by $L(s, \mathrm{sym}^2(f))$ the symmetric square L-function associated to f:

$$L\big(s, \mathrm{sym}^2(f)\big) = \zeta(2s) \sum_{n=1}^\infty \frac{\lambda_f(n^2)}{n^s},$$

and recall that, for any $\epsilon > 0$, the following bounds hold:

$$k^{-\epsilon} \ll L\big(1, \mathrm{sym}^2(f)\big) \ll k^\epsilon. \tag{10}$$

Since $\langle f, f\rangle = 1$ we have the relation

$$\big|a_f(1)\big|^2 = \frac{(4\pi)^{k-1}}{\Gamma(k)} \frac{2\pi^2}{L(1, \mathrm{sym}^2(f))}. \tag{11}$$

Let $m_1, m_2 \in \mathbf{Z}$, $m_1 m_2 \neq 0$ and $h_1, h_2 \in C_0^\infty(0,\infty)$. Recall if $m_i > 0$, we have (see Proposition 2.1 in [24])

$$\langle \mu_f,\, P_{h_i,m_i}(z)\rangle = \frac{2\pi^2}{(k-1)L(1,\mathrm{sym}^2(f))}\sum_{r\geqslant 1}\lambda_f(r)\lambda_f(r+m_i)h_i\left(\frac{k-1}{4\pi(r+m_i/2)}\right) + \mathrm{O}(k^{-1+\epsilon}).$$

Without loss of generality we may assume $m_1>0, m_2>0$, since $\langle \mu_f, P_{h,m}(z)\rangle = \langle \mu_f, P_{h,-m}(z)\rangle$. Thus, by the above formula, (11) and the multiplicativity of Hecke eigenvalues,

$$\begin{aligned}
&\langle \mu_f, P_{h_1,m_1}(z)\rangle\overline{\langle \mu_f,\, P_{h_2,m_2}(z)\rangle}\\
&\quad= \frac{2\pi^2}{(k-1)L(1,\mathrm{sym}^2(f))}\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\\
&\qquad\times\sum_{d_1|m_1,d_2|m_2}\sum_{r_1,r_2}\frac{a_f(r_1(r_1+m_1/d_1))\overline{a_f(r_2(r_2+m_2/d_2))}}{(r_1(r_1+m_1/d_1))^{(k-1)/2}(r_2(r_2+m_2/d_2))^{(k-1)/2}}\\
&\qquad\times h_1\left(\frac{k-1}{4\pi d_1(r_1+m_1/(2d_1))}\right)\overline{h_2}\left(\frac{k-1}{4\pi d_2(r_2+m_2/(2d_2))}\right)+\mathrm{O}(k^{-1+\epsilon}).
\end{aligned}$$

Fix $u\in C_0^\infty(0,\infty)$. From the above formula and by the Petersson formula (see [11]):

$$\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{\lambda_f(m)\lambda_f(n)}{L(1,\mathrm{sym}^2(f))} = \delta_{m,n}+2\pi(-1)^{k/2}\sum_{c\geqslant 1}\frac{S(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

we have

$$\begin{aligned}
&\sum_{k\geqslant 1,2|k}u\left(\frac{k-1}{K}\right)\sum_{f\in H_k}L\big(1,\mathrm{sym}^2(f)\big)\langle \mu_f, P_{h_1,m_1}(z)\rangle\overline{\langle \mu_f, P_{h_2,m_2}(z)\rangle}\\
&\quad=\sum_{k\geqslant 1,2|k}\frac{2\pi^2}{k-1}u\left(\frac{k-1}{K}\right)\sum_{d_1|m_1,d_2|m_2}\ \sum_{r_1(r_1+m_1/d_1)=r_2(r_2+m_2/d_2)}\\
&\qquad\times h_1\left(\frac{k-1}{4\pi d_1(r_1+m_1/(2d_1))}\right)\overline{h_2}\left(\frac{k-1}{4\pi d_2(r_2+m_2/(2d_2))}\right)\\
&\qquad-\sum_{d_1|m_1,d_2|m_2}\ \sum_{r_1,r_2\geqslant 1}\ \sum_{c\geqslant 1}\frac{S(r_1(r_1+m_1/d_1),r_2(r_2+m_2/d_2);c)}{c}\\
&\qquad\times\sum_{k\geqslant 1,2|k}2\pi(-1)^{k/2}u\left(\frac{k-1}{K}\right)\frac{2\pi^2}{k-1}\\
&\qquad\times h_1\left(\frac{k-1}{4\pi d_1(r_1+m_1/(2d_1))}\right)\overline{h_2}\left(\frac{k-1}{4\pi d_2(r_2+m_2/(2d_2))}\right)\\
&\qquad\times J_{k-1}\left(\frac{4\pi\sqrt{r_1r_2(r_1+m_1/d_1)(r_2+m_2/d_2)}}{c}\right)+\mathrm{O}(K^{1/2+\epsilon}).
\end{aligned}$$

We evaluate the diagonal terms by means of the Poisson summation formula as

$$\frac{K\pi}{4}\int_0^\infty u(\xi)\,d\xi\sum_{d_1|m_1,d_2|m_2;m_1/d_1=m_2/d_2}\frac{1}{d_1d_2}\int_0^\infty h_1(d_2\eta)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2}+\mathrm{O}(1), \tag{12}$$

since $r_1(r_1+m_1/d_1)=r_2(r_2+m_2/d_2)$ has at most finitely many solutions if $m_1/d_1\neq m_2/d_2$, while the integer solutions to $r_1(r_1+m_1/d_1)=r_2(r_2+m_1/d_1)$ are only $r_1=r_2$.

Applying Lemmas 4.1 and 4.2 from [24], we deduce that the non-diagonal terms are equal to

$$\begin{aligned}
&\frac{-2\pi^{5/2}}{K}\sum_{d_1|m_1,d_2|m_2}\sum_{r_1,r_2\geqslant 1}\sum_{c\geqslant 1}\frac{S(r_1(r_1+m_1/d_1),r_2(r_2+m_2/d_2);c)}{c}\\
&\quad\times\int_0^\infty u\left(\frac{\sqrt{\Delta c^{-1}y}}{K}\right)\frac{K}{\sqrt{\Delta c^{-1}y}}h_1\left(\frac{\sqrt{\Delta c^{-1}y}}{4\pi d_1(r_1+m_1/(2d_1))}\right)\overline{h_2}\left(\frac{\sqrt{\Delta c^{-1}y}}{4\pi d_2(r_2+m_2/(2d_2))}\right)\\
&\quad\times\sin(\Delta c^{-1}/2+y-\pi/4)\frac{dy}{\sqrt{y}}+\mathrm{O}(1)\\
&\quad=\sum_{d_1|m_1,d_2|m_2}\sum_{r_1,r_2\geqslant 1}\sum_{c\geqslant 1}\frac{S(r_1(r_1+m_1/d_1),r_2(r_2+m_2/d_2);c)}{c}J_{r_1,r_2,c}+\mathrm{O}(1),
\end{aligned}$$

say, where

$$\Delta=8\pi\sqrt{r_1r_2(r_1+m_1/d_1)(r_2+m_2/d_2)}.$$

The terms with $c\gg K^\epsilon$ contribute $\mathrm{O}(1)$, by partial integration. Making the change of variable $t=\frac{\sqrt{\Delta c^{-1}y}}{K}$, we see $J_{r_1,r_2,c}$ is

$$\begin{aligned}
&-4\pi^{5/2}\frac{\sqrt{c}}{\sqrt{\Delta}}\int_0^\infty\frac{u(t)}{t}\sin\bigl(\Delta c^{-1}/2+(tK)^2c/\Delta-\pi/4\bigr)\\
&\quad\times h_1\left(\frac{tK}{4\pi d_1(r_1+m_1/(2d_1))}\right)\overline{h_2}\left(\frac{tK}{4\pi d_2(r_2+m_2/(2d_2))}\right)dt.
\end{aligned}$$

Note

$$\begin{aligned}
\frac{\Delta i}{2c}&=\frac{4\pi i}{c}\sqrt{r_1r_2(r_1+m_1/d_1)(r_2+m_2/d_2)}\\
&=\frac{2\pi i}{c}\left(2r_1r_2+\frac{m_2r_1}{d_2}+\frac{m_1r_2}{d_1}+\frac{1}{2}\frac{m_1}{d_1}\frac{m_2}{d_2}-\frac{1}{4}\left(\frac{m_1}{d_1}\right)^2\frac{r_2}{r_1}-\frac{1}{4}\left(\frac{m_2}{d_2}\right)^2\frac{r_1}{r_2}+\cdots\right).
\end{aligned}$$

We first assume the test functions u,h_1,h_2 are all real-valued and write for simplicity

$$J_{r_1,r_2,c}=\Im\left\{e_c\left(2r_1r_2+\frac{m_2r_1}{d_2}+\frac{m_1r_2}{d_1}\right)f_c(r_1,r_2)\right\},$$

say, where $e_c(x)=\exp(2\pi ix/c)$.

Reducing the summation over r_1,r_2 into congruence classes $\bmod c$, we have,

$$\begin{aligned}
&\sum_{r_1,r_2\geqslant 1}S\bigl(r_1(r_1+m_1/d_1),r_2(r_2+m_2/d_2);c\bigr)e_c\left(2r_1r_2+\frac{m_2r_1}{d_2}+\frac{m_1r_2}{d_1}\right)f_c(r_1,r_2)\\
&\quad=\sum_{a,b(\mathrm{mod}\ c)}S\bigl(a(a+m_1/d_1),b(b+m_2/d_2);c\bigr)e_c\left(2ab+\frac{m_2a}{d_2}+\frac{m_1b}{d_1}\right)\\
&\qquad\times\sum_{r_1\equiv a(\mathrm{mod}\ c),r_2\equiv b(\mathrm{mod}\ c)}f_c(r_1,r_2)
\end{aligned}$$

$$= \frac{1}{c^2} \sum_{u (\mathrm{mod}\ c)} \sum_{v (\mathrm{mod}\ c)} \left(\sum_{a,b (\mathrm{mod}\ c)} S\big(a(a+m_1/d_1), b(b+m_2/d_2); c\big) \right.$$
$$\left. \times e_c\left(2ab + \left(\frac{m_2}{d_2}+u\right)a + \left(\frac{m_1}{d_1}+v\right)b\right)\right)\left(\sum_{r_1,r_2} f_c(r_1,r_2) e_c(-ur_1 - vr_2)\right).$$

Next we apply the Poisson summation formula for the sum in r_1, r_2 and obtain

$$\sum_{r_1,r_2} f_c(r_1,r_2) e_c(-ur_1 - vr_2) = \sum_{l_1,l_2} B(l_1,l_2),$$

where

$$B(l_1,l_2) = \iint_{\mathbf{R}^2} f_c(r_1,r_2) e_c(-ur_1 - vr_2) e(l_1 r_1 + l_2 r_2)\, dr_1\, dr_2$$
$$= \iint_{\mathbf{R}^2} f_c(r_1,r_2) e\big((l_1 - u/c) r_1 + (l_2 - v/c) r_2\big)\, dr_1\, dr_2.$$

We can assume $|u| \leqslant c/2, |v| \leqslant c/2$, and by partial integration sufficiently many times, we see that

$$\sum_{l_1,l_2} B(l_1,l_2) = B(0,0) + \mathrm{O}(K^{-A}),$$

for any $A > 1$. Thus,

$$\sum_{r_1,r_2} f_c(r_1,r_2) e_c(-ur_1 - vr_2) = \iint_{\mathbf{R}^2} f_c(r_1,r_2) e_c(-ur_1 - vr_2)\, dr_1\, dr_2 + \mathrm{O}(K^{-A}).$$

For $(u,v) \neq (0,0)$, by partial integration sufficiently many times, we infer that (recall $c \ll K^{\epsilon}$)

$$\iint_{\mathbf{R}^2} f_c(r_1,r_2) e_c(-ur_1 - vr_2)\, dr_1\, dr_2 \ll K^{-A},$$

for any $A > 0$. Thus only $(u,v) = (0,0)$ contributes. Moreover we can allow $c \gg K^{\epsilon}$ in the c-summation since

$$\iint_{\mathbf{R}^2} f_c(r_1,r_2)\, dr_1\, dr_2 \ll c^{-A} K^2,$$

for any $A > 0$.

For fixed $d_i, m_i (i = 1,2)$ and integer γ, denote

$$S_c(\gamma) = \sum_{a,b (\mathrm{mod}\ c)} S\big(a(\gamma a + m_1/d_1), b(\gamma b + m_2/d_2); c\big) e_c\left(2\gamma ab + \left(\frac{m_2}{d_2}\right)a + \left(\frac{m_1}{d_1}\right)b\right),$$

and $S_c = S_c(1)$. We also write $S_{c,m_1/d_1,m_2/d_2}$ for S_c if we need to indicate the dependence on other parameters. Obviously $S_{c,m_1/d_1,m_2/d_2} = S_{c,m_2/d_2,m_1/d_1}$.

Thus, the non-diagonal contribution is

$$\begin{aligned}
&\sum_{d_1|m_1,d_2|m_2}\sum_{c\geqslant 1}\frac{S_c}{c^2}\iint\limits_{\mathbf{R}^2} f_c(r_1,r_2)\,dr_1\,dr_2+\mathrm{O}(1)\\
&=-4\pi^{5/2}\sum_{d_1|m_1,d_2|m_2}\sum_{c\geqslant 1}\Im\Bigg\{\frac{S_c}{c^{5/2}}\overline{\zeta_8}\int\limits_0^\infty\frac{u(t)}{t}\\
&\quad\times\iint\limits_{\mathbf{R}^2}\frac{1}{\sqrt{\Delta}}e_c\left(\frac{1}{2}\frac{m_1}{d_1}\frac{m_2}{d_2}-\frac{1}{4}\left(\frac{m_1}{d_1}\right)^2\frac{r_2}{r_1}-\frac{1}{4}\left(\frac{m_2}{d_2}\right)^2\frac{r_1}{r_2}\right)e^{i(tK)^2c/\Delta}\\
&\quad\times h_1\left(\frac{tK}{4\pi d_1(r_1+m_1/(2d_1))}\right)h_2\left(\frac{tK}{4\pi d_2(r_2+m_2/(2d_2))}\right)dr_1\,dr_2\,dt\Bigg\}+\mathrm{O}(1)\\
&=\frac{-2\pi^2}{\sqrt{2}}\sum_{d_1|m_1,d_2|m_2}\sum_{c\geqslant 1}\Im\Bigg\{\frac{S_c}{c^{5/2}}\overline{\zeta_8}\int\limits_0^\infty\frac{u(t)}{t}\\
&\quad\times\iint\limits_{\mathbf{R}^2}\frac{1}{\sqrt{r_1r_2}}e_c\left(\frac{1}{2}\frac{m_1}{d_1}\frac{m_2}{d_2}-\frac{1}{4}\left(\frac{m_1}{d_1}\right)^2\frac{r_2}{r_1}-\frac{1}{4}\left(\frac{m_2}{d_2}\right)^2\frac{r_1}{r_2}\right)e^{i(tK)^2c/\Delta}\\
&\quad\times h_1\left(\frac{tK}{4\pi d_1r_1}\right)h_2\left(\frac{tK}{4\pi d_2r_2}\right)dr_1\,dr_2\,dt\Bigg\}+\mathrm{O}(1)\\
&=\frac{-K\pi}{2\sqrt{2}}\left(\int\limits_0^\infty u(t)\,dt\right)\sum_{d_1|m_1,d_2|m_2}\sum_{c\geqslant 1}\Im\Bigg\{\frac{S_c}{c^{5/2}}\overline{\zeta_8}e\left(\frac{1}{2c}\frac{m_1}{d_1}\frac{m_2}{d_2}\right)\\
&\quad\times\iint\limits_{\mathbf{R}^2}e\left(-\frac{1}{4c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{1}{4c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+\xi\eta c\right)\\
&\quad\times h_1\left(\frac{\xi}{d_1}\right)h_2\left(\frac{\eta}{d_2}\right)\frac{d\xi\,d\eta}{(\xi\eta)^{3/2}}\Bigg\}+\mathrm{O}(1)\\
&=\frac{-K\pi}{2\sqrt{2}}\left(\int\limits_0^\infty u(t)\,dt\right)\sum_{d_1|m_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\Im\Bigg\{\frac{S_c}{c^{5/2}}\overline{\zeta_8}e\left(\frac{1}{2c}\frac{m_1}{d_1}\frac{m_2}{d_2}\right)\\
&\quad\times\iint\limits_{\mathbf{R}^2}e\left(-\frac{1}{4c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{1}{4c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+(d_1d_2)^2\xi\eta c\right)\\
&\quad\times\frac{h_1(d_2\xi)}{\sqrt{\xi}}\frac{h_2(d_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}\Bigg\}+\mathrm{O}(1).
\end{aligned}$$

By the multiplicativity of $S_c(\gamma)$:

$$S_{c_1c_2,m_1/d_1,m_2/d_2}(\gamma)=S_{c_1,m_1/d_1,m_2/d_2}(\gamma c_2)S_{c_2,m_1/d_1,m_2/d_2}(\gamma c_1),\quad\text{for } (c_1,c_2)=1,$$

and using the fact $\overline{2c_2}/c_1+\overline{c_1}/(2c_2)\equiv 1/(2c_1c_2)(\mathrm{mod}\ 1)$ for $(c_1,2c_2)=1$ as well as the result in Appendix A, one can check that

$$S_ce\left(\frac{1}{2c}\frac{m_1}{d_1}\frac{m_2}{d_2}\right)\in\mathbf{R},$$

and hence we obtain, under the assumption that the test functions u, h_1, h_2 are all real-valued, the following theorem. Before stating it we need a little extra notation. For A a non-negative integer define $\|\cdot\|_A$ on $C_c^\infty(0,\infty)$ by

$$\|h\|_A = \max_{0\leqslant i\leqslant A, |j|\leqslant A, x\in(0,\infty)} \left|\frac{h^i(x)}{x^j}\right|. \tag{13}$$

THEOREM 2. – *For $m_1, m_2 \in \mathbf{Z}, m_1m_2 \neq 0, u, h_1, h_2 \in C_c^\infty(0,\infty)$ and $\epsilon > 0$, we have*

$$\sum_{k\geqslant 1, 2|k} u\left(\frac{k-1}{K}\right) \sum_{f\in H_k} L\big(1, \operatorname{sym}^2(f)\big)\langle \mu_f, P_{h_1,m_1}(z)\rangle \overline{\langle \mu_f, P_{h_2,m_2}(z)\rangle}$$
$$= B_\omega(P_{h_1,m_1}, P_{h_2,m_2}) K\left(\int_0^\infty u(t)\,dt\right) + \mathrm{O}(K^{1/2+\epsilon}), \tag{14}$$

where

$$\begin{aligned} B_\omega(P_{h_1,m_1}, P_{h_2,m_2}) = {}& \frac{\pi}{4} \sum_{d_1|m_1, d_2|m_2; |m_1|/d_1 = |m_2|/d_2} \frac{1}{d_1d_2} \int_0^\infty h_1(d_2\eta)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2} \\ & - \frac{\pi}{2\sqrt{2}} \sum_{d_1|m_1, d_2|m_2} \frac{1}{d_1d_2} \sum_{c\geqslant 1} \frac{S_{c,|m_1|/d_1,|m_2|/d_2}}{c^{5/2}} e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2|}{d_2}\right) \\ & \times \iint_{\mathbf{R}^2} \sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta} - \frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi} + 2\pi(d_1d_2)^2\xi\eta c\right) \\ & \times \frac{h_1(d_2\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}. \end{aligned} \tag{15}$$

Moreover there is an absolute constant A and $C(=C(\epsilon))$ such that the implicit constant in (14) *is at most*

$$C\big(\big(|m_1|+1\big)\big(|m_2|+1\big)\big)^A\|h_1\|_A\cdot\|h_2\|_A, \tag{16}$$

and the series defining the B_ω converges absolutely and satisfies

$$\big|B_\omega(P_{h_1,m_1}, P_{h_2,m_2})\big| \leqslant C\big(\big(|m_1|+1\big)\big(|m_2|+1\big)\big)^A\|h_1\|_A\cdot\|h_2\|_A. \tag{17}$$

(16) is proven by keeping track of the dependence on h_1 and h_2 in the derivation of (14). (17) follows from integrating by parts in the double integral in (15), and then directly estimating the terms.

A closer inspection of the proof actually shows that if any incomplete Poincaré series P_{h_i,m_i} in the Theorem 2 is replaced by incomplete Eisenstein series (i.e. $m_i = 0$) with zero mean $\int_X P_{h_i,m_i}\tilde{\nu} = 0$ (i.e. $\int_0^\infty h_i(y)y^{-2}\,dy = 0$), then the Theorem 2 is still valid except for the case $m_1 = m_2 = 0$. For the case $m_1 = m_2 = 0$, (14) and (15) of the Theorem 2 continue to hold as long as the term

$$\frac{\pi}{4} \sum_{d_1|m_1, d_2|m_2; m_1/d_1 = m_2/d_2} \frac{1}{d_1d_2} \int_0^\infty h_1(d_2\eta)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2}$$

is replaced by

$$\frac{\pi}{4}\int_{1/A}^{\infty}\left(\int_0^{\infty}\int_0^{\infty} b_2(\xi)b_2(\eta)H_1\left(\frac{r}{\xi}\right)H_2\left(\frac{r}{\eta}\right)\frac{d\xi\,d\eta}{(\xi\eta)^2}\right)\frac{dr}{r^2},$$

where $H_i(\xi)=h_i''(\xi)\xi^2+2h_i'(\xi)\xi=(h_i'(\xi)\xi^2)'$; $h_i(\xi)=0$ for $0<\xi\leqslant 1/A, i=1,2$; $2b_2(x)=B_2(x-[x])$, and $B_2(x)=x^2-x+1/6$ is the Bernoulli polynomial of degree 2. This follows by Euler–MacLaurin summation formula from

$$\sum_{d_i\geqslant 1} h_i\left(\frac{k-1}{4\pi d_i r}\right)=-\int_0^{\infty} b_2(\xi)H_i\left(\frac{k-1}{4\pi\xi r}\right)\frac{d\xi}{\xi^2},\quad \text{for } r\geqslant 1,$$

which vanishes if $r\geqslant A\frac{k-1}{4\pi}$.

It is in turn equal to

$$\begin{aligned}&\frac{\pi}{4}\int_0^{\infty}\left(\int_0^{\infty}\int_0^{\infty} b_2(\xi)b_2(\eta)H_1\left(\frac{r}{\xi}\right)H_2\left(\frac{r}{\eta}\right)\frac{d\xi\,d\eta}{(\xi\eta)^2}\right)\frac{dr}{r^2}\\ &\quad=\frac{\pi}{4}\int_0^{\infty}\left(\int_0^{\infty}\int_0^{\infty} b_2(\xi)b_2(\eta)\frac{d^2}{d\xi^2}h_1\left(\frac{r}{\xi}\right)\frac{d^2}{d\eta^2}h_2\left(\frac{r}{\eta}\right)d\xi\,d\eta\right)\frac{dr}{r^2}.\end{aligned}$$

3. Symmetry properties of B_ω

Let $L_m=L_m^x$ be the differential operator on $C_0^\infty(0,\infty)$ given by

$$L_m h(x)=\left(x^2\frac{d^2}{dx^2}-4\pi^2m^2x^2\right)h(x). \tag{18}$$

If we define the inner product on $C_0^\infty(0,\infty)$ by

$$(h_1,h_2)=\int_0^{\infty} h_1(x)\overline{h_2(x)}\frac{dx}{x^2}, \tag{19}$$

then L_m is symmetric with respect to $(,)$, i.e.

$$(L_m h_1,h_2)=(h_1,L_m h_2).$$

We have

$$\Delta\big(h(y)e(mx)\big)=(L_m h)(y)e(mx), \tag{20}$$

and hence

$$\Delta P_{h,m}=P_{L_m h,m}, \tag{21}$$

where Δ is the hyperbolic Laplacian. Moreover, we have (see the proof of Theorem 6.9 in [12])

(22) $$T_n P_{h,m}(z) = \sum_{d|(m,n)} \left(\frac{d^2}{n}\right)^{1/2} P_{h(\frac{ny}{d^2}),\frac{mn}{d^2}}(z),$$

where T_n is the nth Hecke operator (see Section 8.5 in [12]). It turns out that the bilinear form $B_\omega(\cdot,\cdot)$, defined on the space P spanned by all $P_{h,m}$'s, is self-adjoint with respect to the Laplacian Δ and the Hecke operators $T_n, n \geqslant 1$:

(23) $$B_\omega(\Delta P_{h_1,m_1}, P_{h_2,m_2}) = B_\omega(P_{h_1,m_1}, \Delta P_{h_2,m_2}),$$

(24) $$B_\omega(T_n P_{h_1,m_1}, P_{h_2,m_2}) = B_\omega(P_{h_1,m_1}, T_n P_{h_2,m_2}).$$

The verification of (23) is straightforward by (21), by change of variables (when $m_1 m_2 \neq 0$, to symmetrize the integral kernel)

$$\xi \to \frac{m_2}{d_2}\xi, \qquad \eta \to \frac{m_1}{d_1}\eta,$$

and in view of the fact

$$\left(\xi^2 \frac{d^2}{d\xi^2} - 4\pi^2 m_1^2 m_2^2 \xi^2\right) K_{m_1,m_2,d_1,d_2}(\xi,\eta) = \left(\eta^2 \frac{d^2}{d\eta^2} - 4\pi^2 m_1^2 m_2^2 \eta^2\right) K_{m_1,m_2,d_1,d_2}(\xi,\eta),$$

i.e.

$$\left(\xi^2 \frac{d^2}{d\xi^2} - 4\pi^2 m_1^2 m_2^2 \xi^2\right) K_{m_1,m_2,d_1,d_2}(\xi,\eta) \text{ is a symmetric function in } \xi,\eta,$$

where

$$K_{m_1,m_2,d_1,d_2}(\xi,\eta) = \sqrt{\xi\eta}\sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\frac{m_1 m_2}{d_1 d_2}\frac{\xi}{\eta} - \frac{\pi}{2c}\frac{m_1 m_2}{d_1 d_2}\frac{\eta}{\xi} + 2\pi(d_1 d_2)^2 \frac{m_1 m_2}{d_1 d_2}\xi\eta c\right).$$

If $m_1 m_2 = 0$, we then desymmetrize the integral kernel and use the continuity argument.

In order to prove (24), it suffices to check it for each $T_p(p$ is a prime$)$, which can be verified by a tedious computation, using (22) and the explicit evaluation of $S_{c,m_1/d_1,m_2/d_2}(\gamma)$ in Appendix A. For the details, see Appendix A.3.

4. Extension of B_ω and diagonalization

Let $P:\mathbf{H} \to X$ be the usual projection map, and let $\{D_{0j}\}_{0\leqslant i\leqslant 3}\bigcup\{D_{k1}\cup D_{k2}\}_{k\geqslant 1}$ be a system of open sets with compact closures in $\mathbf{H}$ whose projections to X form a locally finite open covering of X (see [9]), such that the restriction $P|_{D_{kj}}$ is injective except for D_{00} or D_{01}; D_{00} (resp. D_{01}) is a neighborhood of i (resp. $\rho = e^{2\pi i/3}$) and the restriction $P|_{D_{00}}$ (resp. $P|_{D_{01}}$) is two to one (resp. three to one) map except at i (resp. except at ρ). We choose ($k \geqslant 1$)

$$D_{02} = \{z,\ \Im(z) < 2,\ |\Re(z)| < 1/2,\ |z| > 1\},$$

$$\begin{aligned} D_{03} = &\{z,\ \Im(z) < 2,\ -1/2 \leqslant \Re(z) < 0,\ |z| > 1\} \\ &\cup \{z,\ \Im(z) < 2,\ -1 < \Re(z) \leqslant -1/2,\ |z+1| > 1\}, \end{aligned}$$

$$D_{k1} = \{z,\ 3^k/2 < \Im(z) < 2 \cdot 3^k,\ -1 < \Re(z) < 0\},$$

$$D_{k2} = \{z,\ 3^k/2 < \Im(z) < 2 \cdot 3^k,\ -1/2 < \Re(z) < 1/2\}.$$

Let $\{f_{kj}\}_{k \geqslant 0}$ be the partition of unity subordinate to the above covering of X (see [9]). Each f_{kj} can be regarded as an automorphic function with respect to Γ. The restriction $f_{kj}|_{D_{kj}}$ has compact support in D_{kj} and we extend it to a smooth Γ_∞ periodic function $\tilde{f}_{kj}$ on $\mathbf{H}$. There exists $y_0 > 0$ so that $\tilde{f}_{kj}$ are all supported in the half-plane $y \geqslant y_0$, on which $\tilde{f}_{kj}(z) = f_{kj}(z)$, except when $k = 0$ and $j = 2$ or 3.

Let ψ be a fixed element in $C^\infty_{0,0}(X)$. We have

$$\psi(z) = \sum_{k,j} f_{kj}(z)\psi(z),$$

$$f_{kj}(z)\psi(z) = \frac{1}{n_{kj}} \sum_{\gamma \in \mathbf{\Gamma}_\infty \backslash \mathbf{\Gamma}} \tilde{f}_{kj}(\gamma z)\psi(\gamma z),$$

where

$$n_{kj} = \begin{cases} 2, & \text{if } k = 0,\ j = 0; \\ 3, & \text{if } k = 0,\ j = 1; \\ 1, & \text{if otherwise.} \end{cases}$$

Expanding $\tilde{f}_{kj}(z)\psi(z)$ into its Fourier series in x gives

$$\tilde{f}_{kj}(z)\psi(z) = \sum_{m \in \mathbf{Z}} h_{kjm}(y)e(mx).$$

$h_{kjm}(y)$ are smooth with compact support, and since $\psi \in C^\infty_{0,0}(X)$ the h's satisfy

$$h_{kjm}(y) \ll_A y^{-A}\big(|m| + 1\big)^{-A} \tag{25}$$

for any $A > 0$. Hence

$$\begin{aligned}
\psi(z) &= \sum_{k,j} \sum_{m \in \mathbf{Z}} \frac{1}{n_{kj}} P_{h_{kjm},m}(z) \\
&= \sum_{k,j} \sum_{m \neq 0} \frac{1}{n_{kj}} P_{h_{kjm},m}(z) + \sum_{k,j} P_{h_{kj0},0}(z) - \frac{1}{2} P_{h_{000},0}(z) - \frac{2}{3} P_{h_{010},0}(z) \\
&= \sum_{k,j} \sum_{m \neq 0} \frac{1}{n_{kj}} P_{h_{kjm},m}(z) + P_{H,0}(z) - \frac{1}{2} P_{h_{000},0}(z) - \frac{2}{3} P_{h_{010},0}(z),
\end{aligned}$$

say, where $H(y) \in C^\infty_c(0,\infty)$ (recall that for y large enough the zeroth coefficient of ψ is zero) is defined as

$$H(y) = \sum_{k,j} h_{kj0}(y).$$

This follows from the fact that $\sum_{k,j} f_{kj}(z) = 1$, and $\tilde{f}_{kj}$ are all supported in the half-plane $y \geqslant y_0$, on which we have

$$\sum_{k,j} h_{kj0}(y) = \int_0^1 \biggl(\sum_{k,j} \tilde{f}_{kj}(z)\biggr)\psi(z)\,dx$$
$$= \int_0^1 \biggl(\sum_{k,j} f_{kj}(z) + \bigl(\tilde{f}_{02}(z) - f_{02}(z)\bigr) + \bigl(\tilde{f}_{03}(z) - f_{03}(z)\bigr)\biggr)\psi(z)\,dx$$
$$= \int_0^1 \bigl(\bigl(\tilde{f}_{02}(z) - f_{02}(z)\bigr) + \bigl(\tilde{f}_{03}(z) - f_{03}(z)\bigr)\bigr)\psi(z)\,dx.$$

Moreover we have

$$\int_X \biggl(P_{H,0}(z) - \frac{1}{2}P_{h_{000},0}(z) - \frac{2}{3}P_{h_{010},0}(z)\biggr)\tilde{\nu} = 0.$$

Write

$$P_{H,0}(z) - \frac{1}{2}P_{h_{000},0}(z) - \frac{2}{3}P_{h_{010},0}(z) = P_{h,0}(z),$$

with

$$h = H - \frac{1}{2}h_{000} - \frac{2}{3}h_{010}.$$

We then have

$$\psi(z) = \sum_{k,j}\sum_{m\neq 0} \frac{1}{n_{kj}} P_{h_{kjm},m}(z) + P_{h,0}(z), \tag{26}$$

with

$$\int_X P_{h,0}(z)\tilde{\nu} = 0.$$

It follows from Theorem 2 and the comments following it, together with (25) and (26), that for ψ and ϕ in $C_{0,0}^{\infty}(X)$ we have

$$\sum_{k\geqslant 1,2|k} u\biggl(\frac{k-1}{K}\biggr) \sum_{f\in H_k} L\bigl(1,\mathrm{sym}^2(f)\bigr)\langle\mu_f,\psi\rangle\overline{\langle\mu_f,\phi\rangle}$$
$$= B_\omega(\psi,\phi)K\biggl(\int_0^\infty u(t)\,dt\biggr) + \mathrm{O}_{\psi,\phi,\epsilon}(K^{1/2+\epsilon}), \tag{27}$$

where

$$B_\omega(\psi,\phi) = \sum_{k_1,j_1,n_1\neq 0;k_2,j_2,n_2\neq 0} \frac{1}{n_{k_1j_1}n_{k_2j_2}} B_\omega(P_{h^{(\psi)}_{k_1j_1m_1},m_1}, P_{h^{(\phi)}_{k_2j_2m_2},m_2})$$
$$+ \sum_{k_1,j_1,n_1\neq 0} \frac{1}{n_{k_1j_1}} B_\omega(P_{h^{(\psi)}_{k_1j_1m_1,m_1}}, P_{h^{(\phi)},0})$$
$$+ \sum_{k_2,j_2,n_2\neq 0} \frac{1}{n_{k_2j_2}} B_\omega(P_{h^{(\psi)},0}, P_{h^{(\phi)}_{k_2j_2m_2,m_2}})$$
$$+ B_\omega(P_{h^{(\psi)},0}, P_{h^{(\phi)},0}). \tag{28}$$

In view of (25) for ϕ and ψ respectively and (17), we see that the series (28) converges absolutely. From (23), (24), (26) and (28) it follows that the bilinear form $B_\omega(\psi_1, \psi_2)$ now defined on $C^\infty_{0,0}(X) \times C^\infty_{0,0}(X)$ satisfies

$$B_\omega(\Delta\psi_1, \psi_2) = B_\omega(\psi_1, \Delta\psi_2), \tag{29}$$

and for $n \geqslant 1$,

$$B_\omega(T_n\psi_1, \psi_2) = B_\omega(\psi_1, T_n\psi_2). \tag{30}$$

This completes the proof of parts (A) and (B) of Theorem 1.

Now restrict B_ω to the subspace

$$C^\infty_{cusp}(X) = C^\infty_{0,0}(X) \cap L^2_{cusp}(X).$$

If ϕ_1, ϕ_2 are distinct Hecke–Maass eigenforms in $C^\infty_{cusp}(X)$, then for $n \geqslant 1$

$$B_\omega(T_n\phi_1, \phi_2) = B_\omega(\phi_1, T_n\phi_2)$$

implies

$$\lambda_n(\phi_1)B_\omega(\phi_1, \phi_2) = \lambda_n(\phi_2)B_\omega(\phi_1, \phi_2).$$

According to the theory of Hecke operators and Fourier coefficients there is an n such that $\lambda_n(\phi_1) \neq \lambda_n(\phi_2)$. It follows that if ϕ_1 and ϕ_2 are distinct Hecke–Maass cusp forms then

$$B_\omega(\phi_1, \phi_2) = 0. \tag{31}$$

Thus we have shown that B_ω is diagonalized by the orthonormal basis of Hecke–Maass cusp forms in $L^2_{cusp}(X)$. In the next section we compute the value of B_ω on such a ϕ.

5. Eigenvalues of B_ω

Let $\phi(z)$ be an even Maass–Hecke cuspidal eigenform for the modular group Γ, with the Laplacian eigenvalue $\lambda_\phi = \frac{1}{4} + it_\phi$, and let $L(s, \phi)$ be the associated standard L-function, which is well known to admit analytic continuation to the whole complex plane and satisfies the functional equation:

$$\Lambda_\phi(s) = \Lambda_\phi(1-s),$$

where

$$\Lambda_\phi(s) = \pi^{-s}\Gamma\left(\frac{s+it_\phi}{2}\right)\Gamma\left(\frac{s-it_\phi}{2}\right)L(s,\phi).$$

We assume $\phi(z)$ is normalized so that its first Fourier coefficient $a_\phi(1) = 1$. From the works of Watson [29], we have

$$\begin{aligned}
\left|\langle \mu_f, \phi\rangle\right|^2 &= \frac{\pi^2}{2\cosh(\pi t_\phi)} \frac{|\Gamma(k - \frac{1}{2} + it_\phi)|^2}{(4\pi)^k\Gamma(k)} L^{-1}\big(1, \mathrm{sym}^2(f)\big)\left|a_f(1)\right|^2 L(1/2, \phi\otimes f\otimes f) \\
&= \frac{\Gamma(k-1)}{(4\pi)^k} \frac{\pi^2}{2\cosh(\pi t_\phi)} L^{-1}\big(1, \mathrm{sym}^2(f)\big)\left|a_f(1)\right|^2 L(1/2, \phi\otimes f\otimes f)\big(1 + \mathrm{O}(k^{-1})\big),
\end{aligned}$$

in view of the fact that for any vertical strip $0 < a \leqslant \Re(s) \leqslant b$, we have that

$$\frac{\Gamma(s+k-1)}{\Gamma(k-1)} = (k-1)^s\bigl(1+\mathrm{O}_{a,b}\bigl(\bigl(|s|+1\bigr)^2k^{-1}\bigr)\bigr), \tag{32}$$

by Stirling's formula. Note that

$$L(1/2,\phi\otimes f\otimes f) = L(1/2,\phi)L\bigl(1/2,\phi\otimes \mathrm{sym}^2(f)\bigr).$$

Thus,

$$\begin{aligned}
&\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\sum_{f\in H_k} L\bigl(1,\mathrm{sym}^2(f)\bigr)\bigl|\langle\mu_f,\phi\rangle\bigr|^2\\
&\quad= \frac{\pi}{8}\bigl(1+\mathrm{O}(k^{-1})\bigr)\frac{L(1/2,\phi)}{\cosh(\pi t_\phi)}\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\\
&\quad\quad\times\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\bigl|a_f(1)\bigr|^2 L\bigl(1/2,\phi\otimes\mathrm{sym}^2(f)\bigr).
\end{aligned} \tag{33}$$

Define

$$\begin{aligned}
\Lambda_{\phi,f}(s) = \pi^{-3s}&\Gamma\left(\frac{s+k-1+it_\phi}{2}\right)\Gamma\left(\frac{s+k-1-it_\phi}{2}\right)\Gamma\left(\frac{s+k+it_\phi}{2}\right)\Gamma\left(\frac{s+k-it_\phi}{2}\right)\\
&\times\Gamma\left(\frac{s+1+it_\phi}{2}\right)\Gamma\left(\frac{s+1-it_\phi}{2}\right)L\bigl(s,\phi\otimes\mathrm{sym}^2(f)\bigr),
\end{aligned}$$

then $\Lambda_{\phi,f}(s)$ admits analytic continuation to $\mathbf{C}$ as an entire function and satisfies the functional equation

$$\Lambda_{\phi,f}(s) = \Lambda_{\phi,f}(1-s).$$

Let F be the cuspidal automorphic form on $GL(3)$ which is the Gelbart–Jacquet lift of the cusp form f, with the Fourier coefficients $a_F(m_1,m_2)$, where

$$a_F(m_1,m_2) = \sum_{d|(m_1,m_2)} \lambda_F(m_1/d,1)\lambda_F(m_2/d,1)\mu(d),$$

and

$$\lambda_F(r,1) = \sum_{s^2t=r}\lambda_f(t^2).$$

The Rankin–Selberg convolution $L(s,\phi\otimes\mathrm{sym}^2(f))$ is represented by the Dirichlet series (see [1,2]),

$$L\bigl(s,\phi\otimes\mathrm{sym}^2(f)\bigr) = \sum_{m_1,m_2\geqslant 1}\lambda_\phi(m_1)a_F(m_1,m_2)(m_1m_2^2)^{-s}, \tag{34}$$

where $\lambda_\phi(r)$ is the rth Hecke eigenvalue of ϕ.

We have

$$\Lambda_{\phi,f}(1/2) = \frac{2}{2\pi i}\int_{(2)}\Lambda_{\phi,f}(s+1/2)\frac{ds}{s}.$$

Hence,

$$L\big(1/2,\phi\otimes\mathrm{sym}^2(f)\big)=2\sum_{m_1,m_2\geqslant 1}\lambda_\phi(m_1)a_F(m_1,m_2)(m_1m_2^2)^{-1/2}G_k(\pi^3m_1m_2^2), \tag{35}$$

where

$$\begin{aligned}G_k(\xi)&=\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma\big(\frac{(s+1/2)+k-1+it_\phi}{2}\big)\Gamma\big(\frac{(s+1/2)+k+it_\phi}{2}\big)}{\Gamma\big(\frac{1/2+k-1+it_\phi}{2}\big)\Gamma\big(\frac{1/2+k+it_\phi}{2}\big)}\frac{\Gamma\big(\frac{(s+1/2)+k-1-it_\phi}{2}\big)\Gamma\big(\frac{(s+1/2)+k-it_\phi}{2}\big)}{\Gamma\big(\frac{1/2+k-1-it_\phi}{2}\big)\Gamma\big(\frac{1/2+k-it_\phi}{2}\big)}\\&\quad\times\frac{\Gamma\big(\frac{(s+1/2)+1+it_\phi}{2}\big)\Gamma\big(\frac{(s+1/2)+1-it_\phi}{2}\big)}{\Gamma\big(\frac{1/2+1+it_\phi}{2}\big)\Gamma\big(\frac{1/2+1-it_\phi}{2}\big)}\xi^{-s}\frac{ds}{s}\\&=\frac{1}{2\pi i}\int_{(2)}\big(1+T_k(s)\big)\frac{\Gamma\big(\frac{(s+1/2)+1+it_\phi}{2}\big)\Gamma\big(\frac{(s+1/2)+1-it_\phi}{2}\big)}{\Gamma\big(\frac{1/2+1+it_\phi}{2}\big)\Gamma\big(\frac{1/2+1-it_\phi}{2}\big)}\left(\frac{4\xi}{(k-1)^2}\right)^{-s}\frac{ds}{s},\end{aligned}$$

where

$$T_k(s)=\sum_{1\leqslant r\leqslant 6}\frac{p_{r+1}(s)}{(k-1)^r}+\mathrm{O}\left(\frac{(|s|+1)^8}{(k-1)^7}\right)$$

is an analytic function in $\Re s\geqslant -2$, in view of Stirling's formula. Here $p_{r+1}(s)$ is a polynomial of degree at most $r+1$. Denote

$$U_k(s)=\big(1+T_k(s)\big)\frac{\Gamma\big(\frac{(s+1/2)+1+it_\phi}{2}\big)\Gamma\big(\frac{(s+1/2)+1-it_\phi}{2}\big)}{\Gamma\big(\frac{1/2+1+it_\phi}{2}\big)\Gamma\big(\frac{1/2+1-it_\phi}{2}\big)}.$$

We have

$$\begin{aligned}&\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\big|a_f(1)\big|^2L\big(1/2,\phi\otimes\mathrm{sym}^2(f)\big)\\&\quad=\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\big|a_f(1)\big|^2\\&\qquad\times 2\sum_{m_1,m_2\geqslant 1}\lambda_\phi(m_1)a_F(m_1,m_2)(m_1m_2^2)^{-1/2}G_k(\pi^3m_1m_2^2)\\&\quad=2\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{n_1,n_2\geqslant 1}\lambda_\phi(dn_1)G_k(\pi^3d^3n_1n_2^2)(n_1n_2^2)^{-1/2}\\&\qquad\times\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\big|a_f(1)\big|^2\lambda_F(n_1,1)\lambda_F(n_2,1)\\&\quad=2\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{s_1,s_2,t_1,t_2\geqslant 1}\lambda_\phi(ds_1^2t_1)G_k(\pi^3d^3s_1^2t_1s_2^4t_2^2)(s_1^2t_1s_2^4t_2^2)^{-1/2}\\&\qquad\times\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\big|a_f(1)\big|^2\lambda_f(t_1^2)\lambda_f(t_2^2).\end{aligned}$$

By the Petersson formula, we deduce that

$$\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in H_k}\left|a_f(1)\right|^2 L\big(1/2,\phi\otimes \mathrm{sym}^2(f)\big)$$
$$= 2\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{s_1,s_2,t_1\geqslant 1}\lambda_\phi(ds_1^2t_1)G_k(\pi^3d^3s_1^2s_2^4t_1^3)(s_1^2s_2^4t_1^3)^{-1/2}$$
$$+ 2\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{s_1,s_2,t_1,t_2\geqslant 1}\lambda_\phi(ds_1^2t_1)(s_1^2t_1s_2^4t_2^2)^{-1/2}\sum_{c\geqslant 1}\frac{S(t_1^2,t_2^2;c)}{c}$$
$$\times\sum_{k\geqslant 1,2|k} 2\pi(-1)^{k/2}u\left(\frac{k-1}{K}\right)G_k(\pi^3d^3s_1^2t_1s_2^4t_2^2)J_{k-1}\left(\frac{4\pi t_1t_2}{c}\right).$$

The diagonal term is (writing $r = dt_1$)

$$2\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\sum_{r\geqslant 1} r^{-3/2}\sum_{d|r}\mu(d)\sum_{s_2\geqslant 1}s_2^{-2}\sum_{s_1\geqslant 1}s_1^{-1}\lambda_\phi(rs_1^2)G_k(\pi^3s_1^2s_2^4r^3)$$
$$= 2\sum_{k\geqslant 1,2|k} u\left(\frac{k-1}{K}\right)\sum_{s_2\geqslant 1}s_2^{-2}\sum_{s_1\geqslant 1}s_1^{-1}\lambda_\phi(s_1^2)G_k(\pi^3s_1^2s_2^4).$$

Now

$$\sum_{s_1\geqslant 1}s_1^{-1}\lambda_\phi(s_1^2)G_k(\pi^3s_1^2s_2^4) = \frac{1}{2\pi i}\int_{(2)}\sum_{s_1\geqslant 1}\frac{\lambda_\phi(s_1^2)}{s_1^{2s+1}}U_k(s)\left(\frac{4\pi^3s_2^4}{(k-1)^2}\right)^{-s}\frac{ds}{s}. \tag{36}$$

We have

$$\sum_{s_1\geqslant 1}\frac{\lambda_\phi(s_1^2)}{s_1^s} = \frac{1}{\zeta(2s)}L\big(s,\mathrm{sym}^2(\phi)\big).$$

Moving the line of integration in (18) to $\Re(s) = -1/4+\epsilon$, we obtain

$$\sum_{s_1\geqslant 1}s_1^{-1}\lambda_\phi(s_1^2)G_k(\pi^3s_1^2s_2^4)$$
$$= \frac{1}{\zeta(2)}L\big(1,\mathrm{sym}^2(\phi)\big)U_k(0)$$
$$+\int_{(-1/4+\epsilon)}\frac{1}{\zeta(2+4s)}L\big(1+2s,\mathrm{sym}^2(\phi)\big)U_k(s)\left(\frac{4\pi^3s_2^4}{(k-1)^2}\right)^{-s}\frac{ds}{s}$$
$$= \frac{1}{\zeta(2)}L\big(1,\mathrm{sym}^2(\phi)\big)U_k(0) + \mathrm{O}(K^{-1/2+\epsilon}).$$

Thus, the diagonal terms contribute

$$\frac{2K}{\zeta(2)}L\big(1,\mathrm{sym}^2(\phi)\big)\int_0^\infty u(\xi)\,d\xi + \mathrm{O}(K^{1/2+\epsilon}).$$

Since

$$G_k(\xi) = \frac{1}{2\pi i}\int_{(2)}U_k(s)\left(\frac{4\xi}{(k-1)^2}\right)^{-s}\frac{ds}{s},$$

we can write

$$G_k(\xi) = H\left(\frac{4\xi}{(k-1)^2}\right) + \sum_{1\leqslant r\leqslant 6} \frac{1}{(k-1)^r} H_r\left(\frac{4\xi}{(k-1)^2}\right) + \mathrm{O}\left(\frac{1}{(k-1)^7}\right).$$

Applying Lemmas 4.1 and 4.2 from [24], we deduce that the non-diagonal terms are equal to

$$\begin{aligned}
&-2\pi^{1/2}\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{s_1,s_2,t_1,t_2\geqslant 1}\lambda_\phi(ds_1^2t_1)(s_1^2t_1s_2^4t_2^2)^{-1/2}\sum_{c\geqslant 1}\frac{S(t_1^2,t_2^2;c)}{c}\\
&\times\int_0^\infty u\left(\frac{\sqrt{8\pi t_1t_2c^{-1}y}}{K}\right)H\left(\frac{4\pi^3d^3s_1^2t_1s_2^4t_2^2}{8\pi t_1t_2c^{-1}y}\right)\\
&\times\sin(8\pi t_1t_2c^{-1}/2+y-\pi/4)\frac{dy}{\sqrt{y}}+\mathrm{O}(1)\\
&= -2\pi^{1/2}\sum_{d\geqslant 1}\frac{\mu(d)}{d^{3/2}}\sum_{s_1,s_2,t_1,t_2\geqslant 1}\lambda_\phi(ds_1^2t_1)(s_1^2t_1s_2^4t_2^2)^{-1/2}\\
&\times\sum_{c\geqslant 1}\frac{S(t_1^2,t_2^2;c)}{c}J_{c,d,s_1,s_2,t_1,t_2}+\mathrm{O}(1),
\end{aligned}$$

say.

We can assume $d^3s_1^2t_1s_2^4t_2^2 \ll K^{2+\epsilon}$ since $H(\xi)$ has exponential decay as $\xi\to\infty$. The terms with $c \gg K^{2\epsilon}$ as well as the terms with $t_1t_2 \ll K^{2-\epsilon}$ contribute $\mathrm{O}(1)$, by partial integration. So we can assume $c \ll K^{2\epsilon}$ and $t_1t_2 \gg K^{2-\epsilon}$. Moreover from $t_1t_2^2 \ll K^{2+\epsilon}$ and $t_1t_2 \gg K^{2-\epsilon}$ we deduce that $t_2 \ll K^{2\epsilon}$. Making the change of variable $t = \frac{\sqrt{8\pi t_1t_2c^{-1}y}}{K}$, we see J_{c,d,s_1,s_2,t_1,t_2} is

$$2K\frac{\sqrt{c}}{\sqrt{8\pi t_1t_2}}\int_0^\infty u(t)\sin\bigl(8\pi t_1t_2c^{-1}/2+(tK)^2c/(8\pi t_1t_2)-\pi/4\bigr)H\left(\frac{4\pi^3d^3s_1^2t_1s_2^4t_2^2}{(tK)^2}\right)dt.$$

From Hecke's bound [12, Theorem 8.1]

$$\sum_{r\leqslant R}\lambda_\phi(r)e(r\alpha)r^{-1/2} \ll_\epsilon R^\epsilon,$$

where $\alpha\in\mathbf{R}$; the Hecke relation

$$\lambda_\phi(r_1r_2) = \sum_{d|(r_1,r_2)}\mu(d)\lambda_\phi(r_1/d)\lambda_\phi(r_2/d);$$

and partial summation, we infer that the contribution from the non-diagonal terms is $\mathrm{O}(K^{4\epsilon})$. We conclude that

$$\begin{aligned}
&\sum_{k\geqslant 1, 2|k} u\left(\frac{k-1}{K}\right)\sum_{f\in H_k} L\bigl(1,\mathrm{sym}^2(f)\bigr)\bigl|\langle\mu_f,\phi\rangle\bigr|^2\\
&= \frac{\pi K}{4}\int_0^\infty u(\xi)\,d\xi\,\frac{L(1/2,\phi)}{\cosh(\pi t_\phi)}L\bigl(1,\mathrm{sym}^2(\phi)\bigr)+\mathrm{O}(K^{1/2+\epsilon}).
\end{aligned}\tag{37}$$

This completes the proof of Theorem 1, in view of the fact

$$L\bigl(1, \mathrm{sym}^2(\phi)\bigr) = 2\langle\phi,\phi\rangle\cosh(\pi t_\phi).$$

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Appendix A

Appendix A.1. Classical variance

We evaluate the classical variance given by (4). This evaluation is general and applies to any $Y = \Gamma\backslash SL(2,\mathbf{R})$, where Γ is a lattice (not necessarily arithmetic). Assume that $C_{0,0}(Y)$ consists of functions on Y of mean zero. The classical variance V is given by the symmetric bilinear form

$$V(\psi_1,\psi_2) = \int_{-\infty}^{\infty}\int_{Y}\psi_1(g)\psi_2\left(g\begin{pmatrix} e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix}\right)dg\,dt. \tag{A.1}$$

From this it is clear that V is diagonalized by the irreducible subspaces in the decomposition of the right regular representation of $SL(2,\mathbf{R})$ on $L^2_{0,0}(Y)$. If $\psi(g)$ is an element in $L^2_{0,0}(Y)$ which is $SO(2)$ invariant on the right, then ψ is a Maass form on $\mathbf{H} = SL(2,\mathbf{R})/SO(2)$ with eigenvalue $\lambda = \frac{1}{4}+t^2 > 0$. We evaluate the matrix coefficient

$$F(g) := \int_{\Gamma\backslash SL(2,\mathbf{R})}\psi(g_1)\psi(g_1 g)\,dg_1. \tag{A.2}$$

As a function on $SL(2,\mathbf{R})$, F satisfies
(i) $F(k_1 g k_2) = F(g)$, for $k_1, k_2 \in SO(2)$;
(ii) $\omega F = \lambda F$;
(iii) $F(e) = 1$. (We are normalizing ψ so that $\int_Y |\psi(g)|^2\,dg = 1$.)

According to the theory of spherical functions these determine F uniquely. Specifically F is given explicitly (see [31, p. 143]) by

$$F\left(\begin{pmatrix} e^{r/2} & 0\\ 0 & e^{-r/2}\end{pmatrix}\right) = P_{-\frac{1}{2}+it}(\cosh r),$$

where P_s is the associated Legendre function. Hence

$$V(\psi,\psi) = \int_{-\infty}^{\infty} P_{-\frac{1}{2}+it}(\cosh r)\,dr. \tag{A.3}$$

This integral may be computed [8, p. 810] and yields

$$V(\psi,\psi) = \frac{\bigl|\Gamma\bigl(\frac{1}{4}+\frac{it}{2}\bigr)\bigr|^4}{2\pi\bigl|\Gamma\bigl(\frac{1}{2}+it\bigr)\bigr|^2}.$$

Appendix A.2. Evaluation of the sum $S_c(\gamma)$

For $2 \nmid c$, we have

$$S_c(\gamma) = \epsilon_c c^{3/2}\left(\frac{\gamma}{c}\right) T\left(-\overline{4\gamma}\left(\frac{m_1}{d_1}\right)^2, -\overline{4\gamma}\left(\frac{m_2}{d_2}\right)^2; c\right) e_c\left(-\overline{2\gamma}\frac{m_1 m_2}{d_1 d_2}\right),$$

where

$$\epsilon_c = \begin{cases} 1 & \text{if } c \equiv 1 (\mathrm{mod}\ 4); \\ i & \text{if } c \equiv -1 (\mathrm{mod}\ 4), \end{cases}$$

is the sign of the Gauss sum, and

$$T(m,n;c) = \sum_{d (\mathrm{mod}\ c)} \left(\frac{d}{c}\right) e\left(\frac{m\overline{d} + nd}{c}\right) \tag{A.4}$$

is the Salié sum [27].

If $(c, 2n) = 1$, we know (see [11, Lemma 4.9])

$$T(m,n;c) = \left(\frac{n}{c}\right)\epsilon_c c^{1/2} \sum_{y^2 \equiv mn (\mathrm{mod}\ c)} e\left(\frac{2y}{c}\right).$$

Hence if $(p, 2n) = 1, 2 \nmid c$, then $T(pm, n; p^2 c) = 0$.

If $c = p^k$ and $k \geqslant 2t \geqslant 2$, we write $d = l + rp^{k-t},\ l (\mathrm{mod}\ p^{k-t}),\ (p,l) = 1,\ r (\mathrm{mod}\ p^t)$, then $\overline{d} \equiv \overline{l} - r\overline{l}^2 p^{k-t} (\mathrm{mod}\ p^k)$, and hence

$$T(m,n;p^k) = \sum_{l (\mathrm{mod}\ p^{k-t})} \left(\frac{l}{p^k}\right) e\left(\frac{m\overline{l} + nl}{p^k}\right) \sum_{r (\mathrm{mod}\ p^t)} e\left(\frac{(n - m\overline{l}^2) r}{p^t}\right). \tag{A.5}$$

For $c = 2^l$ and $m_1/d_1 \not\equiv m_2/d_2 (\mathrm{mod}\ 2)$, we have

$$S_c(\gamma) = \begin{cases} 0, & l \geqslant 2; \\ 4, & l = 1, \end{cases}$$

since for $2 \nmid AB$,

$$\sum_{a (\mathrm{mod}\ c)} e_c(Aa^2 + Ba) = \begin{cases} 0, & l \geqslant 2; \\ 2, & l = 1. \end{cases}$$

On the other hand, for $c = 2^l$ and $m_1/d_1 \equiv m_2/d_2 (\mathrm{mod}\ 2)$, we have

$$S_c(\gamma) = \frac{1}{4} \sum_{x (\mathrm{mod}\ 4c), (2,x)=1} cG(\gamma x, c) e_{4c}\left(-\overline{\gamma}\left(\left(\frac{m_1}{d_1}\right)^2 x + \left(\frac{m_2}{d_2}\right)^2 \overline{x}\right)\right) e_{2c}\left(-\overline{\gamma}\frac{m_1 m_2}{d_1 d_2}\right),$$

where

$$G(n, 2^l) = \sum_{t (\mathrm{mod}\ c)} e\left(\frac{nt^2}{2^l}\right) = \begin{cases} 0, & l = 1; \\ (1 + i^n) 2^{l/2}, & 2 | l; \\ 2^{(l+1)/2} e^{\pi i n/4}, & 2 \nmid l > 1. \end{cases}$$

Let

$$2^s=\left(\left(\frac{m_1}{d_1}\right)^2,\left(\frac{m_2}{d_2}\right)^2,4c\right)$$

and we assume without loss of generality that $2^s\leqslant c$, and $2^s\|(\frac{m_2}{d_2})^2$. We distinguish two cases (note $2^{-s}(m_2/d_2)^2\equiv 1(\bmod 8)$):

(a) $2|l$.

$$e_{2c}\left(\overline{\gamma}\frac{m_1m_2}{d_1d_2}\right)S_c(\gamma)=\frac{2^sc^{3/2}}{4}\left\{S\left(\overline{\gamma}^2\left(\frac{m_1m_2}{d_1d_2 2^s}\right)^2,1;\frac{4c}{2^s}\right)\right.$$
$$\left.+S\left(-\frac{c}{2^s}+\overline{\gamma}^2\left(\frac{m_1m_2}{d_1d_2 2^s}\right)^2,1;\frac{4c}{2^s}\right)\right\}.$$

(b) $2\nmid l>1$.

$$e_{2c}\left(\overline{\gamma}\frac{m_1m_2}{d_1d_2}\right)S_c(\gamma)=\frac{2^sc^{3/2}}{2\sqrt{2}}S\left(-\frac{c}{2^{s+1}}+\overline{\gamma}^2\left(\frac{m_1m_2}{d_1d_2 2^s}\right)^2,1;\frac{4c}{2^s}\right).$$

Here $S(m,n;c)$ is the usual Kloosterman sum.

Appendix A.3. Self-adjointness of Hecke operators for B_ω

We write

$$B_\omega(P_{h_1,m_1},P_{h_2,m_2})=B_\infty(P_{h_1,m_1},P_{h_2,m_2})+B_f(P_{h_1,m_1},P_{h_2,m_2}),$$

where

$$B_\infty(P_{h_1,m_1},P_{h_2,m_2})=\frac{\pi}{4}\sum_{d_1|m_1,d_2|m_2;|m_1|/d_1=|m_2|/d_2}\frac{1}{d_1d_2}\int_0^\infty h_1(d_2\eta)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2};$$

and

$$B_f(P_{h_1,m_1},P_{h_2,m_2})=-\frac{\pi}{2\sqrt{2}}\sum_{d_1|m_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2|}{d_2}\right)$$
$$\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(d_1d_2)^2\xi\eta c\right)$$
$$\times\frac{h_1(d_2\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}.$$

We first consider the special case $p\nmid m_1m_2$ in details. The general cases, as we see later, can be treated similarly by induction. We have, by (22), that

$$T_pP_{h,m}(z)=p^{-1/2}P_{h(p\cdot),pm}(z).$$

Thus, since the conditions $d_1|pm_1,d_2|m_2;|m_1|p/d_1=|m_2|/d_2$ implies $p|d_1$, we infer that

$$\begin{aligned}
&B_\infty(T_pP_{h_1,m_1},P_{h_2,m_2})=p^{-1/2}B_\infty(P_{h_1(p\cdot),pm_1},P_{h_2,m_2})\\
&=\frac{\pi}{4}p^{-1/2}\sum_{d_1|m_1,d_2|m_2;|m_1|/d_1=|m_2|/d_2}\frac{1}{pd_1d_2}\int_0^\infty h_1(pd_2\eta)\overline{h_2}(pd_1\eta)\frac{d\eta}{\eta^2}\\
&=p^{-1/2}B_\infty(P_{h_1,m_1},P_{h_2(p\cdot),pm_2})\\
&=B_\infty(P_{h_1,m_1},T_pP_{h_2,m_2}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&B_f(T_pP_{h_1,m_1},P_{h_2,m_2})=p^{-1/2}B_f(P_{h_1(p\cdot),pm_1},P_{h_2,m_2})\\
&=-\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|pm_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1p|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1p|}{d_1}\frac{|m_2|}{d_2}\right)\\
&\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1p}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(d_1d_2)^2\xi\eta c\right)\\
&\quad\times\frac{h_1(d_2p\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}\\
&=-\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|m_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1p|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1p|}{d_1}\frac{|m_2|}{d_2}\right)\\
&\quad\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1p}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(d_1d_2)^2\xi\eta c\right)\\
&\quad\times\frac{h_1(d_2p\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}\\
&\quad-\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|m_1,d_2|m_2}\frac{1}{pd_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2|}{d_2}\right)\\
&\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(pd_1d_2)^2\xi\eta c\right)\\
&\quad\times\frac{h_1(d_2p\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(pd_1\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}\\
&=\sum\nolimits_1+\sum\nolimits_2,
\end{aligned}$$

say, where $\sum_1,\sum_2$ correspond to the conditions $p\nmid d_1$ and $p|d_1$ respectively in the initial sum. Making the change of variables $\xi\to\xi/p,\eta\to p\eta$ in $\sum_1$, we see that

$$\begin{aligned}
\sum\nolimits_1&=-\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|m_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1p|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1p|}{d_1}\frac{|m_2|}{d_2}\right)\\
&\quad\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2p}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(d_1d_2)^2\xi\eta c\right)\\
&\quad\times\frac{h_1(d_2\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1p\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}.
\end{aligned}$$

Similarly,

$$B_f(P_{h_1,m_1}, T_pP_{h_2,m_2}) = p^{-1/2}B_f(P_{h_1,m_1}, P_{h_2(p\cdot),pm_2}) = \sum\nolimits_1' + \sum\nolimits_2,$$

where

$$\begin{aligned}\sum\nolimits_1' = &-\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|m_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1|/d_1,|m_2p|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2p|}{d_2}\right)\\ &\times\iint_{\mathbf{R}^2}\sin\left(-\frac{\pi}{4}-\frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2\frac{\xi}{\eta}-\frac{\pi}{2c}\left(\frac{m_2p}{d_2}\right)^2\frac{\eta}{\xi}+2\pi(d_1d_2)^2\xi\eta c\right)\\ &\times\frac{h_1(d_2\xi)}{\sqrt{\xi}}\frac{\overline{h_2}(d_1p\eta)}{\sqrt{\eta}}\frac{d\xi\,d\eta}{\xi\eta}.\end{aligned}$$

However, in view of Appendix A.2, we have

$$S_{c,|m_1p|/d_1,|m_2|/d_2} = S_{c,|m_1|/d_1,|m_2p|/d_2}. \tag{A.6}$$

To see this, recall the multiplicativity of $S_c(\gamma)$:

$$S_{c_1c_2,m_1/d_1,m_2/d_2}(\gamma) = S_{c_1,m_1/d_1,m_2/d_2}(\gamma c_2)\cdot S_{c_2,m_1/d_1,m_2/d_2}(\gamma c_1),\quad \text{for}(c_1,c_2)=1.$$

We write $c=c_1c_2c_3$ if $p>2$, where $c_1|p^\infty, c_2|2^\infty$, and $(c_3,2p)=1$; $c=c_1c_2$ if $p=2$, where $c_1|p^\infty,(c_2,2)=1$. Then

$$\begin{aligned}S_{c_1c_2c_3,|pm_1|/d_1,|m_2|/d_2} = {}&S_{c_1,|pm_1|/d_1,|m_2|/d_2}(c_2c_3)\cdot S_{c_2,|pm_1|/d_1,|m_2|/d_2}(c_1c_3)\\ &\cdot S_{c_3,|pm_1|/d_1,|m_2|/d_2}(c_1c_2),\end{aligned}$$

if $p>2$;

$$S_{c_1c_2,pm_1/d_1,m_2/d_2} = S_{c_1,|pm_1|/d_1,|m_2|/d_2}(c_2)\cdot S_{c_2,|pm_1|/d_1,|m_2|/d_2}(c_1)$$

if $p=2$. We also decompose $S_{c,|m_1|/d_1,|m_2p|/d_2}$ in the same way. From the evaluation in Appendix A.2 (for $S_{c_3,|pm_1|/d_1,|m_2|/d_2}(c_1c_2)$, making the change of variable $d\to p^2d$ inside the sum $T(\cdot,\cdot;c_3)$), formula (A.6) follows.

Thus we see that $\sum_1 = \sum_1'$, and consequently

$$B_f(T_pP_{h_1,m_1}, P_{h_2,m_2}) = B_f(P_{h_1,m_1}, T_pP_{h_2,m_2}).$$

Let us consider the general case by induction on a with $p^a\|(m_1,m_2)$. Since

$$T_pP_{h,m}(z) = p^{-1/2}P_{h(p\cdot),pm}(z) + p^{1/2}P_{h(\cdot/p),m/p}(z),$$

where if $p\nmid m$, we understand that $P_{h(\cdot/p),m/p}(z)=0$, we have

$$\begin{aligned}&B_f(T_pP_{h_1,m_1}, P_{h_2,m_2})\\ &\quad= p^{-1/2}B_f(P_{h_1(p\cdot),pm_1}, P_{h_2,m_2}) + p^{1/2}B_f(P_{h_1(\cdot/p),m_1/p}, P_{h_2,m_2})\\ &\quad= -\frac{\pi}{2\sqrt{2}}p^{-1/2}\sum_{d_1|pm_1,d_2|m_2}\frac{1}{d_1d_2}\sum_{c\geqslant 1}\frac{S_{c,|m_1p|/d_1,|m_2|/d_2}}{c^{5/2}}e\left(\frac{1}{2c}\frac{|m_1p|}{d_1}\frac{|m_2|}{d_2}\right)\end{aligned}$$

$$\begin{aligned}
&\times \iint_{\mathbf{R}^2} \sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\left(\frac{m_1 p}{d_1}\right)^2 \frac{\xi}{\eta} - \frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2 \frac{\eta}{\xi} + 2\pi (d_1 d_2)^2 \xi\eta c\right)\\
&\times \frac{h_1(d_2 p\xi)}{\sqrt{\xi}} \frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}} \frac{d\xi\, d\eta}{\xi\eta}\\
&-\frac{\pi}{2\sqrt{2}} p^{1/2} \sum_{d_1|m_1/p, d_2|m_2} \frac{1}{d_1 d_2} \sum_{c\geqslant 1} \frac{S_{c,|m_1|/pd_1,|m_2|/d_2}}{c^{5/2}} e\left(\frac{1}{2c}\frac{|m_1|}{pd_1}\frac{|m_2|}{d_2}\right)\\
&\times \iint_{\mathbf{R}^2} \sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\left(\frac{m_1}{pd_1}\right)^2 \frac{\xi}{\eta} - \frac{\pi}{2c}\left(\frac{m_2}{d_2}\right)^2 \frac{\eta}{\xi} + 2\pi (d_1 d_2)^2 \xi\eta c\right)\\
&\times \frac{h_1(d_2 \xi/p)}{\sqrt{\xi}} \frac{\overline{h_2}(d_1\eta)}{\sqrt{\eta}} \frac{d\xi\, d\eta}{\xi\eta}\\
=\,& I_A + I_B,
\end{aligned}$$

say.

Similarly

$$\begin{aligned}
&B_f(P_{h_1,m_1}, T_p P_{h_2,m_2})\\
&= p^{-1/2} B_f(P_{h_1,m_1}, P_{h_2(p\cdot),pm_2}) + p^{1/2} B_f(P_{h_1,m_1}, P_{h_2(\cdot/p),m_2/p})\\
&= -\frac{\pi}{2\sqrt{2}} p^{-1/2} \sum_{d_1|m_1, d_2|pm_2} \frac{1}{d_1 d_2} \sum_{c\geqslant 1} \frac{S_{c,|m_1|/d_1,|m_2 p|/d_2}}{c^{5/2}} e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2 p|}{d_2}\right)\\
&\quad\times \iint_{\mathbf{R}^2} \sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2 \frac{\xi}{\eta} - \frac{\pi}{2c}\left(\frac{m_2 p}{d_2}\right)^2 \frac{\eta}{\xi} + 2\pi (d_1 d_2)^2 \xi\eta c\right)\\
&\quad\times \frac{h_1(d_2 \xi)}{\sqrt{\xi}} \frac{\overline{h_2}(d_1 p\eta)}{\sqrt{\eta}} \frac{d\xi\, d\eta}{\xi\eta}\\
&\quad-\frac{\pi}{2\sqrt{2}} p^{1/2} \sum_{d_1|m_1, d_2|m_2/p} \frac{1}{d_1 d_2} \sum_{c\geqslant 1} \frac{S_{c,|m_1|/d_1,|m_2|/pd_2}}{c^{5/2}} e\left(\frac{1}{2c}\frac{|m_1|}{d_1}\frac{|m_2|}{pd_2}\right)\\
&\quad\times \iint_{\mathbf{R}^2} \sin\left(-\frac{\pi}{4} - \frac{\pi}{2c}\left(\frac{m_1}{d_1}\right)^2 \frac{\xi}{\eta} - \frac{\pi}{2c}\left(\frac{m_2}{pd_2}\right)^2 \frac{\eta}{\xi} + 2\pi (d_1 d_2)^2 \xi\eta c\right)\\
&\quad\times \frac{h_1(d_2 \xi)}{\sqrt{\xi}} \frac{\overline{h_2}(d_1\eta/p)}{\sqrt{\eta}} \frac{d\xi\, d\eta}{\xi\eta}\\
&= II_A + II_B,
\end{aligned}$$

say.

According to whether or not $p|(c,*,*)$ in $S_{c,*,*}$, we further decompose the sums I_A, I_B, II_A, II_B into

$$I_A = I_{A1} + I_{A2}, \qquad I_B = I_{B1} + I_{B2}, \qquad II_A = II_{A1} + II_{A2}, \qquad II_B = I_{B1} + I_{B2}.$$

Note if $p|(c,*,*), S_{c,*,*} = 0$ unless $p^2|c$. Write $c = p^2 c_1$, we have

$$S_{c,|m_1 p|/d_1,|m_2|/d_2} = S_{c_1,|m_1|/d_1,|m_2|/pd_2} p^2 \left(1 - \frac{\delta(p,c_1)}{p}\right),$$

where

$$\delta(p,c_1)=\begin{cases}0, & \text{if } p|c_1,\\ 1, & \text{if } p\nmid c_1,\end{cases}$$

and write correspondingly $I_{A1}=I'_{A1}-I''_{A1}$;

$$S_{c,|m_1|/pd_1,|m_2|/d_2}=S_{c_1,|m_1|/p^2d_1,|m_2|/pd_2}p^2\left(1-\frac{\delta(p,c_1)}{p}\right),$$

and $I_{B1}=I'_{B1}-I''_{B1}$;

$$S_{c,|m_1|/d_1,|m_2p|/d_2}=S_{c_1,|m_1|/pd_1,|m_2|/d_2}p^2\left(1-\frac{\delta(p,c_1)}{p}\right),$$

and $II_{A1}=II'_{A1}-II''_{A1}$;

$$S_{c,|m_1|/d_1,|m_2|/pd_2}=S_{c_1,|m_1|/pd_1,|m_2|/p^2d_2}p^2\left(1-\frac{\delta(p,c_1)}{p}\right),$$

and $II_{A1}=II'_{B1}-II''_{B1}$.

We see, by the induction hypothesis on $(m_1/p,m_2/p)$, that $I'_{A1}+I'_{B1}=II'_{A1}+II'_{B1}$.

Moreover note that if $p\nmid bc$, we have $S_{cp,ap,b}=p^2S_{c,a,b}$ and $S_{tp^2,ap,b}=0$. Using this, together with the evaluation of $S_{c,*,*}$ in Appendix A.2 one can readily verify that (where $I_{A2}(p|d_1)$, for example, denotes the partial sum of I_{A2} in which $p|d_1$)

$$I_{A2}(p|d_1)=II_{A2}(p|d_2);\qquad I_{A2}(p\nmid d_1,p\nmid d_2,p\nmid c)=II_{A2}(p\nmid d_2,p\nmid d_1,p\nmid c);$$

$$I_{A2}(p\nmid d_1,p\|d_2,p\nmid c)=II_{A2}(p\nmid d_2,p\|d_1,p\nmid c);$$

$$I_{A2}(p\nmid d_1,p^2|d_2,p\nmid c)=I''_{A1}(p\nmid d_1,p^2|m_2/d_2);$$

$$I_{A2}(p\nmid d_1,p|c)=I''_{A1}(p\nmid d_1,p\|m_2/d_2);$$

$$II''_{A1}(p\nmid d_2,p^2|m_1/d_1)=II_{A2}(p\nmid d_2,p^2|d_1,p\nmid c);$$

$$II''_{A1}(p\nmid d_2,p\|m_1/d_1)=II_{A2}(p\nmid d_2,p\nmid c);$$

$$I''_{A1}(p|d_1)=II''_{A1}(p|d_2);\qquad I_{B2}(p|d_2)=II_{B2}(p|d_1);$$

$$I_{B2}(p\nmid d_2,p\nmid d_1,p\nmid c)=II_{B2}(p\nmid d_1,p\nmid d_2,p\nmid c);$$

$$I_{B2}(p\nmid d_2,p\|d_1,p\nmid c)=II_{B2}(p\nmid d_1,p\|d_2,p\nmid c);$$

$$I_{B2}(p\nmid d_2,p^2|d_1,p\nmid c)=I''_{B1}(p\nmid d_2,p^3|m_1/d_1);$$

$$I_{B2}(p\nmid d_2, p|c) = I''_{B1}(p\nmid d_2, p^2\|m_1/d_1);$$

$$II''_{B1}(p|d_1) = I''_{B1}(p|d_2);$$

$$II_{B2}(p\nmid d_1, p^2|d_2, p\nmid c) = II''_{B1}(p\nmid d_1, p^3|m_2/d_2);$$

$$II_{B2}(p\nmid d_1, p|c) = II''_{B1}(p\nmid d_1, p^2\|m_2/d_2).$$

We deduce from the above that

$$B_f(T_pP_{h_1,m_1}, P_{h_2,m_2}) = B_f(P_{h_1,m_1}, T_pP_{h_2,m_2}).$$

On the other hand, we have

$$\begin{aligned}
&B_\infty(T_pP_{h_1,m_1}, P_{h_2,m_2})\\
&\quad = p^{-1/2}B_\infty(P_{h_1(p\cdot),pm_1}, P_{h_2,m_2}) + p^{1/2}B_\infty(P_{h_1(\cdot/p),m_1/p}, P_{h_2,m_2})\\
&\quad = \frac{\pi}{4}p^{-1/2}\sum_{d_1|m_1p,d_2|m_2;|m_1|p/d_1=|m_2|/d_2}\frac{1}{d_1d_2}\int_0^\infty h_1(pd_2\eta)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2}\\
&\qquad + \frac{\pi}{4}p^{1/2}\sum_{d_1|m_1/p,d_2|m_2;|m_1|/pd_1=|m_2|/d_2}\frac{1}{d_1d_2}\int_0^\infty h_1(d_2\eta/p)\overline{h_2}(d_1\eta)\frac{d\eta}{\eta^2}\\
&\quad = A+B,
\end{aligned}$$

say. Similarly

$$\begin{aligned}
&B_\infty(P_{h_1,m_1}, T_pP_{h_2,m_2})\\
&\quad = p^{-1/2}B_\infty(P_{h_1,m_1}, P_{h_2(p\cdot),pm_2}) + p^{1/2}B_\infty(P_{h_1,m_1}, P_{h_2(\cdot/p),m_2/p})\\
&\quad = \frac{\pi}{4}p^{-1/2}\sum_{d_1|m_1,d_2|m_2p;|m_1|/d_1=|m_2|p/d_2}\frac{1}{d_1d_2}\int_0^\infty h_1(d_2\eta)\overline{h_2}(pd_1\eta)\frac{d\eta}{\eta^2}\\
&\qquad + \frac{\pi}{4}p^{1/2}\sum_{d_1|m_1,d_2|m_2/p;|m_1|/d_1=|m_2|/pd_2}\frac{1}{d_1d_2}\int_0^\infty h_1(d_2\eta)\overline{h_2}(d_1\eta/p)\frac{d\eta}{\eta^2}\\
&\quad = A'+B',
\end{aligned}$$

say. One can easily check that

$$\begin{aligned}
A(p|d_1) &= A'(p|d_2); &\qquad A(p\nmid d_1) &= B'(p\nmid d_1);\\
B(p\nmid d_2) &= A'(p\nmid d_2); &\qquad B(p|d_2) &= B'(p|d_1).
\end{aligned}$$

Thus,

$$B_\infty(T_pP_{h_1,m_1}, P_{h_2,m_2}) = B_\infty(P_{h_1,m_1}, T_pP_{h_2,m_2}).$$

This completes the proof that

$$B_\omega(T_pP_{h_1,m_1}, P_{h_2,m_2}) = B_\omega(P_{h_1,m_1}, T_pP_{h_2,m_2}).$$

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