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STABILIZATION AND CONTROL FOR THE SUBCRITICAL SEMILINEAR WAVE EQUATION

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ABSTRACT. – In this paper, we analyze the exponential decay property of solutions of the semilinear wave equation in $\mathbf{R}^3$ with a damping term which is effective on the exterior of a ball. Under suitable and natural assumptions on the nonlinearity we prove that the exponential decay holds locally uniformly for finite energy solutions provided the nonlinearity is subcritical at infinity. Subcriticality means, roughly speaking, that the nonlinearity grows at infinity at most as a power $p<5$. The method of proof combines classical energy estimates for the linear wave equation allowing to estimate the total energy of solutions in terms of the energy localized in the exterior of a ball, Strichartz's estimates and results by P. Gérard on microlocal defect measures and linearizable sequences. We also give an application to the stabilization and controllability of the semilinear wave equation in a bounded domain under the same growth condition on the nonlinearity but provided the nonlinearity has been cut-off away from the boundary.

RÉSUMÉ. – Nous étudions dans cet article la décroissance exponentielle de l'énergie pour une équation d'ondes semi-linéaire dans $\mathbf{R}^3$, avec un terme d'amortissement effectif à l'extérieur d'une boule. En supposant la non linéarité sous critique et vérifiant certaines conditions naturelles, nous obtenons un résultat de stabilisation locale, c'est-à-dire une décroissance exponentielle de l'énergie, uniforme sur les boules de l'espace d'énergie où sont choisies les données initiales. La démonstration repose sur des inégalités d'énergie classiques qui estiment l'énergie totale en fonction de l'énergie localisée à l'extérieur d'une boule. Elle utilise aussi les estimations de Strichartz et les résultats de P. Gérard sur les mesures de défaut microlocales et les suites linéarisables. Nous donnons aussi, en application, un résultat de stabilisation et de contrôle pour l'équation des ondes semi-linéaire sur un ouvert borné, avec une non linéarité sous critique, tronquée loin du bord.

1. Introduction

This paper is devoted to the study of the following damped semilinear wave equation on $\mathbf{R}^3$

$$\begin{cases} \Box u + f(u) + a(x)\partial_t u = 0 \quad \text{in }]0,+\infty[\times \mathbf{R}^3, \\ u(0,x) = u^0(x) \in H^1(\mathbf{R}^3), \quad \partial_t u(0,x) = u^1(x) \in L^2(\mathbf{R}^3). \end{cases} \tag{1.1}$$

Here and in the sequel $\Box$ denotes the wave operator: $\Box = (\partial_t^2 - \Delta_x)$.

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The nonlinearity f is a function from $\mathbf{R}$ to $\mathbf{R}$, of class C^3, satisfying the following conditions:

$$f(0)=0, \tag{1.2}$$

$$\left|f^{(j)}(s)\right| \leqslant C\big(1+|s|\big)^{p-j}, \quad \text{for } j=1,2,3 \tag{1.3}$$

with $C>0$ where p is a real number such that

$$1 \leqslant p < 5,$$

and

$$sf(s) \geqslant cs^2 \quad \forall s \in \mathbf{R} \tag{1.4}$$

for a positive constant $c>0$.

The techniques and results we develop here can be easily adapted to any space dimension $N \geqslant 1$. Of course, the critical range of exponents is then $p<(N+2)/(N-2)$ (any finite $p \geqslant 1$ is allowed when $N=1,2$). However, for simplicity, we shall focus on the case of dimension $N=3$.

The damping potential $a=a(x)$ is assumed to be in $L^\infty(\mathbf{R}^3)$, almost everywhere non-negative, and such that it satisfies for some $R>0$ and $c_0>0$,

$$a(x) \geqslant c_0 > 0 \quad \text{for } |x| \geqslant R. \tag{1.5}$$

This means that the damping term is effective at infinity and, more precisely, in the exterior of the ball of radius R.

It is well known that for every initial data $(u^0,u^1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, system (1.1) admits a unique solution $u(t,x)$ in the space $C^0([0,+\infty[, H^1(\mathbf{R}^3)) \cap C^1([0,+\infty[, L^2(\mathbf{R}^3))$ (see Jörgens [11] and Ginibre and Velo [7] for the subcritical case $p<5$). Existence and uniqueness are well known by now in the critical case $p=5$ too, see Grillakis [8,9] and Shatah and Struwe [21].

However, the critical case $p=5$ will not be considered here. Indeed, the methods we develop in this paper use in an essential manner the fact that the nonlinearity is subcritical, i.e. the fact that $p<5$. Our method fails for the critical case $p=5$ for two reasons:

(a) The boot-strap argument we employ to improve the regularity of solutions vanishing outside a bounded domain so that the existing results on unique continuation apply, does not work for this critical exponent.

(b) We can not use the linearizability results by P. Gérard [6] to deduce that the microlocal defect measures for the nonlinear problem propagate as in the linear case.

Thus, extending the results of this paper to this critical exponent case is an interesting open problem.

The energy of u at time t is defined by

$$E_u(t) = \frac{1}{2}\int_{\mathbf{R}^3} \left[\left|\partial_t u(t,x)\right|^2 + \left|\nabla_x u(t,x)\right|\right]^2 dx + \int_{\mathbf{R}^3} F\big(u(t,x)\big)\, dx \tag{1.6}$$

where

$$F(u) = \int_0^u f(s)\, ds. \tag{1.7}$$

The following energy dissipation law holds:

$$E_u(t_2) - E_u(t_1) = -\int_{t_1}^{t_2}\int_{\mathbf{R}^3} a(x)\left|\partial_t u(t,x)\right|^2 dt\,dx.$$

This can be easily seen formally multiplying the equation by u_t and integrating in $\mathbf{R}^3 \times (t_2, t_1)$.

According to the energy identity above, E_u is decreasing in time and system (1.1) is dissipative.

The first main result of this paper guarantees that the energy decays exponentially. More precisely, we have the following:

THEOREM 1. – *Under the assumptions above, for every $E_0 > 0$, there exist $C > 0$ and $\gamma > 0$ such that inequality*

$$E_u(t) \leqslant C\,\mathrm{e}^{-\gamma t} E_u(0) \quad t > 0 \tag{1.8}$$

holds for every solution u of system (1.1) *with the initial data (u^0, u^1) satisfying*

$$E_u(0) = \frac{1}{2}\int_{\mathbf{R}^3}\left[\left|u^1(x)\right|^2 + \left|\nabla_x u^0(x)\right|\right]^2 dx + \int_{\mathbf{R}^3} F\big(u^0(x)\big)\,dx \leqslant E_0. \tag{1.9}$$

This theorem is a local stabilization result. Indeed, the constants C and γ are uniform on every ball of the energy space but the theorem does not guarantee that the decay rate is global, i.e. whether (1.8) holds with constants C, γ which are independent of the initial data. This is by now well known to hold when $p \leqslant 3$ (as it is the case in the linear case) and under further qualitative properties of the nonlinearity (see [25]). Under this extra qualitative property, the stabilization property is global in this case too as the following result shows.

THEOREM 2. – *Assume that the conditions above are satisfied. Assume also that*

$$f(s) = cs + g(s)$$

with $g = g(s)$ such that there exists $\delta > 0$ so that

$$g(s)s \geqslant (2+\delta)G(s), \quad \forall s \in \mathbf{R}$$

with

$$G(s) = \int_0^s g(z)\,dz.$$

Then, there exist $C > 0$ and $\gamma > 0$ such that inequality (1.8) *holds for every solution of* (1.1).

Remarks. – (1) These theorems show that the behavior of the semilinear subcritical wave equation ($p < 5$), in what concerns the property of stabilization, is, to some extent, analogous to the one of linear waves. This fact was already well established in the work [6] of P. Gérard through the notion of “linearizable sequences”.

(2) There is a large literature on the problem of stabilization of wave equations. J. Rauch and M. Taylor in [17] and C. Bardos, G. Lebeau and J. Rauch [2] introduced and developed the Geometric Control Condition (GCC). This condition that asserts, roughly speaking, that every

ray of Geometric Optics enters the region where the damping term is effective in a uniform time, turns out to be almost necessary and sufficient for the uniform exponential decay of linear waves. Obviously, this condition is satisfied in the whole space when the damping term is effective in the exterior of a ball. In the nonlinear framework, in addition to [25] mentioned above, we refer to the works by A. Haraux [10] and those of the first and third author [4] and [27] and M. Nakao [16]. However, all these papers treat the case of nonlinearities of order at most $p \leqslant 3$ (for bounded domains in $\mathbf{R}^3$) in which the nonlinearity can be treated as a locally Lipschitz perturbation of the linear wave equation by means of energy estimates. To our knowledge, the present paper is the first one dealing with the case $p > 3$.

(3) Assumptions (1.2) and (1.3) on the nonlinearity f are the natural ones guaranteeing the global well posedness of problem (1.1). By condition (1.4) the energy provides estimates of solutions (u, u_t) in $H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$.

(4) Whether the structural conditions of Theorem 2 on the nonlinearity are necessary for the global stabilization property to hold is an open problem.

(5) Our proof relies on properties of microlocal defect measures introduced by P. Gérard in [5] and more precisely on the localization of their support and its propagation.

Strichartz inequalities are another main ingredient in the proof of Theorem 1. They hold outside convex obstacles (Smith and Sogge [20]). Thus, Theorem 1 can be extended to this case.

(6) As mentioned above, the methods developed in this article fail for the critical exponent $p = 5$. Extending the results of this article to this case is an interesting open problem.

Let us now discuss the main difficulty that occurs in the proof of Theorem 1. Actually, one of the crucial points in this proof (which is, in general, an essential step for all stabilization results), is the use of a unique continuation argument. The situation is the following: In a contradiction argument strategy, one obtains, after normalizing and passing to the limit, a function u in the energy space, say $C^0([0,T], H^1(\mathbf{R}^3)) \cap C^1([0,T], L^2(\mathbf{R}^3))$, solution of

$$\begin{cases} \Box u + f(u) = 0 &]0,T[\times \mathbf{R}^3, \\ \partial_t u = 0 &]0,T[\times \{|x| \geqslant R\}. \end{cases} \tag{1.10}$$

Note that the condition $\partial_t u = 0$ is obtained precisely in the subdomain $\{|x| > R\}$ in which the damping term is effective. It is then necessary to prove that the unique solution of (1.10) is the trivial one $u = 0$. This fact expresses that the only undamped solution of system (1.1) is the trivial one. It is then natural to take the time derivative of the equation and to consider $w = \partial_t u$ as new unknown function. One then gets

$$\begin{cases} \Box w + f'(u)w = 0 &]0,T[\times \mathbf{R}^3, \\ w = 0 &]0,T[\times \{|x| \geqslant R\}. \end{cases} \tag{1.11}$$

The goal is then to apply one of the existing results on unique continuation for solutions of the wave equation with a lower order potential ($f'(u)$ in this case) to deduce that $w \equiv 0$. This would imply that $u = u(x)$ and consequently, u would be a solution of a semilinear elliptic problem for which the unique solution is the trivial one because of the good sign assumption (1.4) on the nonlinearity.

There are various unique continuation results in the literature [19,23,24]. But none of them applies in the present situation because of the mild assumptions we do on the nonlinearity ($p < 5$). Indeed, under that assumption we can only guarantee that $f'(u) \in L^\infty(0,T; L^q(\mathbf{R}^3))$ for some $q > 3/2$ which is not sufficient to apply the existing results that require greater integrability properties on the potential. At this point we introduce a new argument that consists in proving that the nonlinear term $f(u)$ is, actually, more regular than it might seem. Indeed, taking into account that the nonlinearity is subcritical we can prove that $f(u) \in L^1_{\mathrm{loc}}(H^{\varepsilon}_{\mathrm{loc}})$, for some $\varepsilon > 0$. To do this, we prove a refined version of the multiplier lemma by Y. Meyer (see [1] or [15]) (we will refer to it as the L^q-version of that lemma), and we make use, in a crucial way, of Strichartz's estimates, which are fulfilled by solutions of subcritical wave equations. After that, the ellipticity of system (1.12) on the domain $]0,T[\times\{|x|>R\}$ and the propagation of singularities property yield, by boot-strap, to a good regularity of u and a bounded potential $f'(u)$. One can then apply the existing results on unique continuation mentioned above.

In the second part of this article, as a consequence of the stabilization result of Theorem 1, we establish an exact controllability result for a semilinear subcritical wave equation on a bounded open domain of $\mathbf{R}^3$.

More precisely, let Ω be a bounded smooth open set of $\mathbf{R}^3$ and ω a neighbourhood of its boundary $\partial\Omega$, i.e. the intersection of Ω with a neighbourhood of $\partial\Omega$ in $\mathbf{R}^3$. Furthermore, let $f:\mathbb{R}\to\mathbb{R}$ be a function of class C^3, satisfying (1.2), (1.3) and

$$sf(s) \geqslant 0. \tag{1.12}$$

And finally, let $\theta(x)$ be a non-negative function in $C_0^\infty(\Omega)$.

We prove the following theorem:

THEOREM 3. – *Under the assumptions above, for every given $E_0 > 0$, there exists a time $T > 0$ such that for every data (u^0,u^1) and (y^0,y^1) in $H_0^1(\Omega)\times L^2(\Omega)$, satisfying*

$$\big\|(u^0,u^1)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant E_0 \quad \textit{and} \quad \big\|(y^0,y^1)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant E_0,$$

there exists $g\in L^1([0,T], L^2(\Omega))$ with support in $[0,T]\times\omega$, and there exists a unique solution $u(t,x)$ in $C^0([0,+\infty[, H_0^1(\Omega))\cap C^1([0,+\infty[, L^2(\Omega))$ solution of the system

$$\begin{cases} \Box u + \theta(x) f(u) = g(t,x) \quad \textit{in }]0,+\infty[\times\Omega, \\ u = 0 \quad \textit{on }]0,+\infty[\times\partial\Omega, \\ u(0) = u^0, \quad \partial_t u(0) = u^1 \quad \textit{in } \Omega, \end{cases} \tag{1.13}$$

satisfying $u(T,.)\equiv y^0$ and $\partial_t u(T,.) = y^1$.

As an immediate consequence the following holds:

COROLLARY 4. – *Let us consider the system with $\theta\equiv 1$, i.e. without cutting off the nonlinearity. Then, under the assumptions above, the same result as in Theorem 3 above holds with controls g in $L^1(0,T;L^2_{\mathrm{loc}}(\Omega))\cap L^\infty(0,T;L^{6/5}(\Omega))$, except for the uniqueness of the solution.*

Remarks. – (1) These are exact controllability results, since the solution u is driven from the initial state (u^0,u^1) to the final one (y^0,y^1), by means of an internal control localized in ω, i.e. near the boundary of Ω. This result improves those in [26] which are valid under the more restrictive assumption on the nonlinearity $p\leqslant 3$ but without the restriction of cutting off the nonlinear term in a neighborhood of the boundary. Note however that the result is of local nature since the control time T depends on the geometry of Ω and ω, as it does in the controllability result for the linear control problem but it also depends on E_0, the radius of the ball in $H_0^1(\Omega)\times L^2(\Omega)$, in which we choose the initial and final data to be controlled. Whether the control time can be taken independently of the size of the initial data is an open problem. There are very few results in this direction. We refer to [28] for a proof of the exact controllability in uniform time for the $1-d$ wave equation with a nonlinear term that grows at infinity in a slight superlinear way.

(2) Observe that, as stated in Corollary 4, the nonlinearity has to be cut-off only with the purpose of guaranteeing the uniqueness of the solution. Indeed, the existence of the control and of the controlled solution can be easily obtained when $\theta \equiv 1$ too. For, it is sufficient to consider the controlled system (1.13) in which the solution u is unique and to take $\tilde{g} = g + (1-\theta) f(u)$ as new control. The solution u of (1.13) turns out to remain a solution with $\theta \equiv 1$ for this new control $\tilde{g}$ and, obviously, the controllability requirements at $t = 0$ and $t = T$ remain the same.

(3) In what concerns the well posedness of system (1.13), notice that the growth condition on the nonlinearity ($p < 5$) prevents us from applying the classical uniqueness results for the solutions of the mixed problem (1.13) obtained by means of the standard energy identity. Indeed, for bounded domains, uniqueness of finite-energy solutions is only known to hold for $p \leqslant 3$, which is the range in which the energy method applies without difficulty. The exponent $p = 3$ is the critical one for the energy method since it is the largest one for which the nonlinear term lies in $L^2(\Omega)$ whenever u is in $H_0^1(\Omega)$. However, we are able to prove uniqueness for all $p < 5$ because of the fact that the nonlinear term has been cut-off away of the boundary. This guarantees that, Strichartz inequalities, that hold locally in the interior of Ω, can be applied.

(4) The same exact controllability result, with a similar proof, can be obtained for the equation in the whole space by means of controls with support in the exterior of a ball. In this case, of course, the nonlinearity does not need to be cut-off.

The proof of this exact controllability result, which is based in the stabilization result of Theorem 1, is, roughly, as follows. First of all, we show by means of a perturbation argument that, due to the exact controllability property of the linear wave equation in the geometric setting of Theorem 2, small data are controllable for the nonlinear equation too, i.e. given sufficiently small initial and final data the solution can be driven from the initial state to the final one. Then, we adapt the proof of Theorem 1 to the case of the bounded open set noting that, due to the cut-off function $\theta(x)$, the boundary $\partial\Omega$ has no effect on the nonlinearity. Therefore, given the initial data (u^0, u^1) to be controlled, by means of the damping term $-a(x)\partial_t u$ supported in ω near the boundary, i.e. by solving system (1.1), we drive it to a small state in a sufficiently large time. We do the same with the final state solving the system backwards in time. This produces two states which are small enough so that the local controllability result for small data applies. The control function $g(t,x)$ is then as follows: In a first time interval it coincides with the damping term $-a(x)\partial_t u$ obtained when solving (1.1), in a second time interval, g is the control corresponding to the small data and, in the last one, it is the damping term obtained when applying the dissipativity property backwards in time starting from the final state (y^0, y^1).

The rest of this article is organized as follows.

Without loss of generality, in the sequel we assume that $4 \leqslant p < 5$. The other cases $1 \leqslant p < 4$ can in fact be treated in a simpler way following the same arguments.

2. Strichartz estimates

First of all we recall some basic estimates; the so called Strichartz's inequalities for the linear wave equation, which will play a crucial role in the whole of the proof. The interested reader can find them, for example in [7,22] or [6].

Let us consider the linear wave equation

$$\begin{cases} \Box u = F \in L^1([0,+\infty[, L^2(\mathbf{R}^3)), \\ (u(0), \partial_t u(0)) \in \dot{H}^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3). \end{cases} \tag{2.1}$$

Here and in the sequel $\dot{H}^1(\mathbf{R}^3)$ denotes the homogeneous Sobolev space of order one: The closure with respect to the norm $\|\nabla u\|_{L^2(\mathbf{R}^3)}$ of the space of smooth compactly supported test functions.

The following result is by now well known:

LEMMA 5. – *Let $r \in [2,+\infty[$ and q given by $1/q + 1/r = 1/2$. Then, there exists $C > 0$ such that for every $T > 0$ and every solution u of* (2.1), *one has*:

$$\|u\|_{L^q([0,T],L^{3r}(\mathbf{R}^3))} \leqslant C\big[\|F\|_{L^1([0,T],L^2(\mathbf{R}^3))} + \big\|\partial_t u(0)\big\|_{L^2(\mathbf{R}^3)} + \big\|\nabla_x u(0)\big\|_{L^2(\mathbf{R}^3)}\big]. \tag{2.2}$$

We also have the following estimate for solutions of the subcritical semilinear wave equation with $f:\mathbb{R} \to \mathbb{R}$, of class C^1, satisfying (1.2), (1.3) and (1.12) for $j = 1$ (see [7]).

LEMMA 6. – *For every $T > 0, r \in [2,+\infty[$, and $E_0 > 0$, there exists $C(T,r,E_0)$, such that every solution u of the system*

$$\begin{cases} \Box u + f(u) = 0 \quad \text{in }]0,+\infty[\times \mathbf{R}^3, \\ \|u(0)\|_{H^1} + \|\partial_t u(0)\|_{L^2} \leqslant E_0, \end{cases} \tag{2.3}$$

satisfies

$$\|u\|_{L^q([0,T],L^{3r}(\mathbf{R}^3))} \leqslant C(T,r,E_0) \tag{2.4}$$

with $1/q + 1/r = 1/2$.

Henceforth, the first norm in the left hand side of the inequality (2.2) and (2.4) will be called Strichartz norm of u. In particular, we shall say that u has finite Strichartz norms when the norm of u is finite in $L^q([0,T], L^{3r}(\mathbf{R}^3))$ for all $r \geqslant 2$ with q such that $1/q + 1/r = 1/2$.

3. Regularity of the composition

Let us first introduce some notations.

For a tempered distribution u, we denote by $(u_q)_{q\geqslant -1}$its dyadic decomposition

$$u = u_{-1} + \sum_{q\geqslant 0} u_q.$$

It is the usual Littlewood–Paley decomposition.

We will use without more specification the properties of u_q: their regularity, integrability, sensibility to derivation, etc. The key of all these estimates is, naturally, Bernstein's lemma. The interested reader can find a good exposition of this decomposition, for example, in [1,15] or [3].

3.1. Meyer's multipliers

Here, we give an abstract multipliers lemma that will be the basis of the composition theorem below.

LEMMA 7. – *Let $\alpha > 3/2$, and let $(m_q)_{\{q\geqslant -1\}}$ be a sequence of C^∞ functions verifying for every $l \in \mathbb{N}$, $\sum_{|\mu|=l} \|\partial^\mu m_q\|_{L^{2\alpha}} \leqslant C_l 2^{ql}$. Then, for every $r \geqslant 1$, the operator*

$$M : u = \sum_{q\geqslant -1} u_q \mapsto Mu = \sum m_q u_q \tag{3.1.1}$$

is continuous from $H^r(\mathbf{R}^3)$ to $H^{r-t}(\mathbf{R}^3)$ with $t = 3/(2\alpha)$, $0 < t < 1$. More precisely,

$$\|Mu\|_{H^{r-t}} \leqslant C\|u\|_{H^r} \quad \textit{with } C \leqslant \text{Const} \sum_{l\leqslant [r]+1} C_l. \tag{3.1.2}$$

Here and in the sequel $[r]$ denotes the integer part of r.

Remarks. – (1) This is a L^q version of Meyer's multiplier lemma. To our knowledge, the proof of the present version is nowhere written. These multipliers have also a pseudo-differential interpretation.

(2) For a given finite r, it is not necessary to assume the multipliers m_q to be of class C^∞. Indeed, it is sufficient for them to be in the class $C^{[r]+1}$.

(3) The proof we present here is adapted from the one developed in [1, pp. 102–103].

Proof of the lemma. – We follow closely the proof of Meyer's Multipliers Lemma in Lemma 2.2, p. 102 of [1].

The spectrum of u_q is contained in the ring $2^{q-1} \leqslant |\xi| \leqslant 2^{q+1}$. On the other hand, we decompose m_q as $m_q = m_{q,-1} + \sum_{k\geqslant 0} m_{q,k}$, where the spectrum of $m_{q,-1}$ is contained in a ball of radius 2^q, and those of $m_{q,k}$ for $k \geqslant 0$ are contained in rings of order 2^{q+k}.

We set $M_k u = \sum_{q\geqslant -1} m_{q,k} u_q$, $k \geqslant -1$. We will show that each M_k is continuous from H^r to H^{r-t} and that the corresponding operator series converges in norm.

The terms in $M_{-1}u$ have their supports in balls of the order of 2^q. Moreover,

$$||m_{q,-1}u_q||_{L^2} \leqslant ||m_{q,-1}||_{L^{2\alpha}}||u_q||_{L^{2\beta}} \leqslant C||m_q||_{L^{2\alpha}}||u_q||_{L^{2\beta}} \leqslant C||u_q||_{L^{2\beta}}, \tag{3.1.3}$$

by assumption, with $1/\alpha + 1/\beta = 1$.

Taking into account that $t = 3/(2\alpha)$ we choose $\beta = 3/(3-2t)$, so that $1/\alpha + 1/\beta = 1$ and $H^t \hookrightarrow L^{2\beta}$.

Then,

$$||u_q||_{L^{2\beta}} \leqslant C||u_q||_{H^t}.$$

But

$$\|u_q\|_{H^t} = \left\|D^t u_q\right\|_{L^2} = \left\|\left(D^t u\right)_q\right\|_{L^2} \leqslant C c_q 2^{-q(r-t)}\left\|D^t u\right\|_{H^{r-t}}$$

with $\sum c_q^2 < \infty$ (see Proposition 1.2, p. 94 in [1]). That is

$$\|u_q\|_{L^{2\beta}} \leqslant C c_q 2^{-q(r-t)} \|u\|_{H^r}. \tag{3.1.4}$$

Here and in the sequel D^t denotes the pseudodifferential operator of symbol $(1+|\xi|^2)^{t/2}$. (Note that it is a simple fractional derivative that commutes with the spectral localization property.)

Combining (3.1.3) and (3.1.4) we deduce that

$$||m_{q,-1} u_q||_{L^2} \leqslant C c_q 2^{-q(r-t)} \|u\|_{H^r}, \tag{3.1.5}$$

with

$$\sum c_q^2 < \infty. \tag{3.1.6}$$

Then, the synthesis Lemma 2.1 in [1] guarantees that M_{-1} is a bounded operator from H^r into H^{r-t}.

For $k \geqslant 0$, the terms in $M_k u$ have their spectra in annulae of the order of 2^{q+k}. To estimate the terms $m_{q,k} u_q$ we argue as before but, this time, we use the fact that,

$$\|m_{q,k}\|_{L^{2\alpha}} \leqslant C_l 2^{-kl}. \tag{3.1.7}$$

This is true, indeed since, by hypothesis,

$$\|m_{q,k}\|_{L^{2\alpha}} \leqslant C \sum_{|\mu|=l} \left\|\partial^\mu m_q\right\|_{L^{2\alpha}} 2^{-(q+k)l} \leqslant C_l 2^{-kl}.$$

Thus,

$$\|m_{q,k} u_q\|_{L^2} \leqslant C_l 2^{-kl} c_q 2^{-q(r-t)} \|u\|_{H^r} \leqslant C_l 2^{-k(l-(r-t))} c_q 2^{-(q+k)(r-t)} \|u\|_{H^r}.$$

Applying the synthesis lemma again we deduce that $M_k : H^r \to H^{r-t}$ is continuous, with a norm of the order of $C_l 2^{-k(l-(r-t))}$.

Finally, taking $l > r-t$, the operator series $M = \sum M_k$ converges normally in the space of continuous operators from H^r to H^{r-t}, and satisfies clearly estimate (3.1.2). □

3.2. The regularity theorem

Now, we study the regularity of the composed function $f(v)$.

THEOREM 8. – *Let v be a function in $L^\infty(]0,+\infty[, H^r(\mathbf{R}^3))$, $1 \leqslant r < 2$, with finite Strichartz norms*; *and take a function f satisfying conditions* (1.2) *and* (1.3). *Then for every $T > 0$, and every function $\chi(x) \in C_0^\infty(\mathbf{R}^3)$, $\chi(x) f(v) \in L^1([0,T], H^{r-t}(\mathbf{R}^3))$, with $0 < t < 1$, and $1 - t \leqslant (5-p)/2$.*

Remarks. – (1) Analyzing carefully the proof of this theorem, one can easily see that it is possible to replace in the conclusion the space L^1 in time by L^α provided $\alpha > 1$ is such that $t \geqslant \frac{p-1}{2} - \frac{1}{\alpha}$.

(2) The proof of the theorem provides a more precise information. In fact, we establish the following estimate

$$\left\|\chi(x) f(v)\right\|_{L^1([0,T],H^{r-t})} \leqslant C \sup_{s\in[0,T]} \left\|v(s)\right\|_{H^r} \tag{3.2.1}$$

where C depends on χ, on the nonlinearity f and on v through its Strichartz norms.

(3) In the proof of Theorems 1 and 2 this property will be used only with $1 \leqslant r < 2$.

Proof of the regularity theorem. – Following [15] and [1] we write

$$f(v) = f(S_0 v) + f(S_1 v) - f(S_0 v) + \cdots + f(S_{q+1} v) - f(S_q v) + \cdots$$

where

$$S_q v = v_{-1} + v_0 + \cdots + v_q.$$

These are the dyadic blocks of v, and they are spectrally supported in balls of radius 2^{q+1}.

For convenience, here and in the sequel we denote simply by f the function χf.

First, we will work with a fixed s in $[0,T]$. And after, we will examine the integrability in time of the H^{r-t}-norms.

(a) The first term $f(S_0 v)$ has the regularity of f. Indeed, since $v \in L^\infty(H^r)$, then $S_0 v$ is in $L^\infty(C^\infty)$. In particular and more precisely, $S_0 v$ is bounded together with all its space derivatives. Moreover, f has compact support in x. So $f(S_0 v)$ is easy to treat. In particular, inequality (3.2.1) holds for this component of v.

(b) For $q \geqslant 0$, we write $f(S_{q+1} v) - f(S_q v) = m_q v_q$, with $m_q = \int_0^1 f'(S_q v + t v_q)\, dt$.

We will show that the m_q's are Meyer's multipliers in the sense of the previous lemma. More precisely, we establish the following estimate

$$\sum_{|\mu|=l} \left\| \partial^\mu m_q \right\|_{L^{2\alpha}} \leqslant C\left(1 + \|v\|_{L^b}^{p-1}\right) 2^{ql} \quad \text{for } l \leqslant 2 \tag{3.2.2}$$

with $\alpha = 3/(2t)$, and $b = 3(p-1)/t$.

For that, it suffices to consider the quantity $G(S_q v)$, with $G = \chi f'$.

Since $r < 2$, we can take $l = 2$ in the multipliers lemma. Then for any μ with $|\mu| = 2$, $\partial^\mu G(S_q v)$ may be written as a linear combination of terms of the form:

(a) $\frac{\partial^2 G}{\partial x^2}(S_q v)$.

(b) $\frac{\partial^2 G}{\partial x \partial v}(S_q v) \partial^\theta (S_q v)$, $|\theta| = 1$.

(c) $\frac{\partial G}{\partial v}(S_q v) \partial^\theta (S_q v)$, $|\theta| = 2$.

(d) $\frac{\partial^2 G}{\partial v^2}(S_q v) \partial^\theta (S_q v) \partial^\nu (S_q v)$, $|\theta| = |\nu| = 1$.

First, we treat the terms of the form (b).

Taking in account hypothesis (1.3), with $1/a + 1/b = 1/(2\alpha)$ we have:

$$\begin{aligned}
\left\| \frac{\partial^2 G}{\partial x \partial v}(S_q v) \partial^\theta (S_q v) \right\|_{L^{2\alpha}} &\leqslant \left\| \frac{\partial^2 G}{\partial x \partial v}(S_q v) \right\|_{L^a} \left\| \partial^\theta (S_q v) \right\|_{L^b} \\
&\leqslant C \left\| 1 + |S_q v|^{p-2} \right\|_{L^a} 2^q \|S_q v\|_{L^b} \\
&\leqslant C 2^q \|S_q v\|_{L^b} + C 2^q \|S_q v\|_{L^{a(p-2)}}^{p-2} \|S_q v\|_{L^b}.
\end{aligned}$$

Note that $\|S_q v\|_{L^{a(p-2)}}^{p-2} \leqslant \|v\|_{L^{a(p-2)}}^{p-2}$ and $\|S_q v\|_{L^b} \leqslant \|v\|_{L^b}$.

Then we obtain:

$$\left\| \frac{\partial^2 G}{\partial x \partial v}(S_q v) \partial^\theta (S_q v) \right\|_{L^{2\alpha}} \leqslant C 2^q \left[\|v\|_{L^b} + \|v\|_{L^{a(p-2)}}^{p-2} \|v\|_{L^b} \right].$$

We choose a and b such that $a(p-2)=b$, which gives $a=2\alpha(p-1)/(p-2)$, and $b=3(p-1)/t$. Hence:

$$\left\|\frac{\partial^2 G}{\partial x\partial v}(S_q v)\partial^\theta(S_q v)\right\|_{L^{2\alpha}} \leqslant C2^q\big(\|v\|_{L^b}+\|v\|_{L^b}^{p-1}\big)\leqslant C2^q\big(1+\|v\|_{L^b}^{p-1}\big).$$

Now, concerning the term (a), we have directly:

$$\left|\frac{\partial^2 G}{\partial x^2}(S_q v)\right| \leqslant C\big(1+|S_q v|^{p-1}\big).$$

It is then sufficient to take $L^{2\alpha}$-norms and this term is estimated immediately.

On the other hand, replacing 2^q by 2^{2q}, we can treat (c) exactly in the same way (recall that $l=2$).

Finally, for the last term (d), using again Hölder's inequality, we obtain

$$\begin{aligned}\left\|\frac{\partial^2 G}{\partial v^2}(S_q v)\partial^\theta(S_q v)\partial^\nu(S_q v)\right\|_{L^{2\alpha}} &\leqslant \left\|\frac{\partial^2 G}{\partial v^2}(S_q v)\right\|_{L^c}\big\|\partial^\theta(S_q v)\big\|_{L^{2d}}\big\|\partial^\nu(S_q v)\big\|_{L^{2d}}\\ &\leqslant C2^{2q}\big\|1+|S_q v|^{p-3}\big\|_{L^c}\|S_q v\|_{L^{2d}}^2\end{aligned}$$

with $1/c+1/d=1/(2\alpha)$.

We take $c(p-3)=2d$ and then $d=\alpha(p-1)$. Then

$$\left\|\frac{\partial^2 G}{\partial v^2}(S_q v)\partial^\theta(S_q v)\partial^\nu(S_q v)\right\|_{L^{2\alpha}} \leqslant C2^{2q}\big(\|v\|_{L^{2\alpha(p-1)}}^2+\|v\|_{L^{2\alpha(p-1)}}^{p-1}\big)$$

and the arguments we use when estimating the term (b) apply.

Now, to complete the proof, it remains to prove that

$$\int_0^T \big\|v(s,.)\big\|_{L^b}^{p-1}\,ds<\infty,$$

that is $v\in L^{p-1}([0,T],L^b)$.

Since the Strichartz norms of v are finite, i.e. $v\in L^q([0,T],L^b)$, for $1/q=1/2-3/b$, it is sufficient to check that

$$\frac{1}{p-1}\geqslant\frac{1}{2}-\frac{t}{p-1},$$

which is true since it is equivalent to $1-t\leqslant(5-p)/2$. □

4. Proof of Theorems 1 and 2

4.1. Proof of Theorem 1

The solutions of (1.1) satisfy the semigroup property. Thus it is enough to prove the estimate

$$E_u(0)\leqslant c\int_0^T\int_{\mathbf{R}^3} a(x)|\partial_t u|^2\,dt\,dx \tag{4.1}$$

for some $c>0$ and $T>0$, and for every solution u such that $E_u(0)\leqslant E_0$.

To prove that we take $T\geqslant 2R+2$ (in fact, any $T>2R$ would work) and argue by contradiction: we suppose the existence of a sequence (u_n), of solutions of (1.1) such that

$$E_{u_n}(0)\leqslant E_0, \tag{4.2}$$

$$\int_0^T\int_{\mathbf{R}^3} a(x)|\partial_t u_n|^2\,dt\,dx\leqslant \frac{E_{u_n}(0)}{n}. \tag{4.3}$$

Denote $\alpha_n=(E_{u_n}(0))^{1/2}$ and $v_n=u_n/\alpha_n$. Due to (4.2), the sequence α_n is bounded. Moreover v_n satisfies

$$\Box v_n+a(x)\partial_t v_n+\frac{1}{\alpha_n}f(\alpha_n v_n)=0, \tag{4.4}$$

$$\int_0^T\int_{\mathbf{R}^3} a(x)|\partial_t v_n|^2\,dt\,dx\leqslant \frac{1}{n}, \tag{4.5}$$

$$1/C\leqslant E_{v_n}(0)\leqslant C, \tag{4.6}$$

for some finite $C>0$. The classical energy estimate allows us to show that the sequence v_n is bounded in $C^0([0,T],H^1(\mathbf{R}^3))\cap C^1([0,T],L^2(\mathbf{R}^3))$. Then, it admits a subsequence, still denoted v_n, that weakly-* converges in $L^\infty(0,T;H^1(\mathbf{R}^3))\cap W^{1,\infty}(0,\infty;L^2(\mathbf{R}^3))$. In this way, $v_n\rightharpoonup v$ in $H^1([0,T]\times\mathbf{R}^3)$. We can also suppose that $\alpha_n\to\alpha\in[0,E_0]$.

We will distinguish the two cases $\alpha>0$ or $\alpha=0$.

First case: $\alpha_n\to\alpha>0$.

We have $u_n\rightharpoonup u=\alpha v$, and $E_{u_n}(0)\to\alpha^2>0$. Passing to the limit in the equation satisfied by u_n we obtain

$$\begin{cases}\Box u+f(u)=0 \quad \text{in }]0,T[\times\mathbf{R}^3,\\ \partial_t u=0 \quad \text{for }]0,T[\times\{|x|\geqslant R\}.\end{cases} \tag{4.7}$$

Moreover, $u\in C^0([0,T],H^1(\mathbf{R}^3))\cap C^1([0,T],L^2(\mathbf{R}^3))$. We have the following unique continuation result:

LEMMA 9. – *The only solution of system* (4.7) *in the class* $C^0([0,T],H^1(\mathbf{R}^3))\cap C^1([0,T],L^2(\mathbf{R}^3))$ *is the trivial one* $u=0$.

We postpone the proof of this lemma and use it to prove the fact that the convergence of u_n to $u=0$ holds in the strong topology of $H^1([0,T]\times\mathbf{R}^3)$. Of course, this will be in contradiction with the fact that α_n converges to a positive constant. This will allow us to exclude the first case. We will then concentrate on the second one in which α_n tends to zero.

Let us now return to the first case under consideration. We have $u_n\rightharpoonup 0$ in $H^1([0,T]\times\mathbf{R}^3)$ and $\Box u_n+f(u_n)\to 0$ in $L^2([0,T]\times\mathbf{R}^3)$, due to (4.2) and (4.3). Furthermore, the nonlinearity f, by hypothesis, is subcritical. Then, u_n is a linearizable sequence, according to the terminology of P. Gérard [6]. In other words, if y_n is the sequence of solutions of the linear wave equation with the same initial data,

$$\begin{cases}\Box y_n=0,\\ y_n(0)=u_n^0,\quad \partial_t y_n(0)=u_n^1,\end{cases}$$

one has

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbf{R}^3}\left\{\left|\partial_t(u_n-y_n)(t,x)\right|^2+\left|\nabla_x(u_n-y_n)\right|^2(t,x)\right\}dx\to 0,\quad n\to\infty$$

which means in particular, that $(u_n-y_n)\to 0$ in $H^1_{\mathrm{loc}}(]0,T[\times\mathbf{R}^3)$.

Let μ be a microlocal defect measure (m.d.m.) associated to u_n in $H^1(]0,T[\times\mathbf{R}^3)$ (see [5] for the definition of these measures and their properties).

From this “linearizability” property we deduce two facts:

(a) The support of μ is contained in the characteristic set of the wave operator $\{\tau^2=|\xi|^2\}$ (this is known as the elliptic regularity theorem for the m.d.m., [5], Proposition 2.1 and Corollary 2.2).

(b) μ propagates along the bicharacteristic flow of this operator, which means in particular, that if some point $\omega_0=(t_0,x_0;\tau_0,\xi_0)$ is not in $\mathrm{supp}(\mu)$, the whole bicharacteristic issued from ω_0 is out of $\mathrm{supp}(\mu)$.

Now, (4.3) gives $\partial_t u_n\to 0$ in $L^2([0,T]\times(|x|\geqslant R))$ and the convergence holds in the strong topology. So, the elliptic regularity theorem implies that outside the cylinder $[0,T]\times(|x|\leqslant R)$, $\mathrm{supp}(\mu)$ is contained in the set $\{\tau=0\}$.

Hence $\mu=0$ for $|x|>R$. On the other hand, since $T\geqslant 2R+2$, every bicharacteristic ray enters the region $(|x|>R)$ before the time T. We then obtain by propagation that $\mu=0$ everywhere. Hence $u_n\to 0$ in $H^1([0,T]\times(|x|\leqslant A))$, for every $A>0$. Since, on the other hand, $\partial_t u_n\to 0$ in $L^2([0,T]\times(|x|\geqslant R))$, we get $\partial_t u_n\to 0$ in $L^2([0,T]\times\mathbf{R}^3)$.

It is then easy to show that, $E_{u_n}(0)$ converges to zero, which is in contradiction with the assumption that $E_{u_n}(0)\to\alpha^2>0$. This can be easily done using the classical identity guaranteeing the equipartition of energy. Indeed, we multiply the equation

$$\Box u_n+a(x)\partial_t u_n+f(u_n)=0$$

by $\varphi(t)u_n$, with $\varphi\in C_0^\infty(]0,T[)$, $\varphi=1$ on $]\varepsilon,T-\varepsilon[$, $\varphi\geqslant 0$, and we integrate. This gives:

$$-\int_0^T\int_{\mathbf{R}^3}\varphi'(t)u_n\partial_t u_n\,dt\,dx-\int_0^T\int_{\mathbf{R}^3}\varphi|\partial_t u_n|^2\,dt\,dx+\int_0^T\int_{\mathbf{R}^3}\varphi|\nabla_x u_n|^2\,dt\,dx$$
$$+\int_0^T\int_{\mathbf{R}^3}\varphi u_n a(x)\partial_t u_n+\int_0^T\int_{\mathbf{R}^3}\varphi u_n f(u_n)\,dt\,dx=0.$$

The second term goes to 0 if $n\to\infty$. Moreover, assumption (1.4) implies that the norm of u_n in $L^2([0,T]\times\mathbf{R}^3)$ is bounded by the initial energy. This gives that the first and the 4th terms above go also to 0 if $n\to\infty$. Finally, the positivity of the last term yields

$$\int_0^T\int_{\mathbf{R}^3}\varphi|\nabla_x u_n|^2\,dt\,dx\to 0\quad\text{and}\quad\int_0^T\int_{\mathbf{R}^3}\varphi u_n f(u_n)\,dt\,dx\to 0,\tag{4.8}$$

as we wanted to prove.

Thus, it remains to prove the unique continuation result of Lemma 7. We recall that u solves

$$\begin{cases}\Box u+f(u)=0 & \text{in }]0,+\infty[\times\mathbf{R}^3,\\ \partial_t u=0 & \text{for } |x|\geqslant R,\end{cases}\tag{4.9}$$

and the solution u is in the class $C^0([0,T],H^1(\mathbf{R}^3))\cap C^1([0,T],L^2(\mathbf{R}^3))$. Furthermore, the function $w=\partial_t u$ satisfies

$$\begin{cases}\Box w+f'(u)w=0 & \text{in }]0,T[\times\mathbf{R}^3,\\ w=0 & \text{for }]0,T[\times(|x|\geqslant R).\end{cases}$$

We seek to obtain $w\equiv 0$, which would give us $u\equiv 0$. Indeed, if $\partial_t u=0$, u satisfies

$$-\Delta u+f(u)=0,\quad u\in H^1\big(\mathbf{R}^3\big)$$

so

$$\int_{\mathbf{R}^3}|\nabla u|^2\,dx+\int_{\mathbf{R}^3}uf(u)\,dx=0,$$

and we get $u\equiv 0$ in view of (1.4).

In order to apply one of the available unique continuation results, (see for example [19,23,24]), it suffices to show that

$$f'(u)\in L^\infty\big([0,T],L^3(|x|\leqslant R+1)\big).$$

Obviously this does not hold from Sobolev's embedding thanks to the fact that u is of finite energy. This argument applies for exponents in the nonlinearity up to $p=3$. However, we are dealing with a larger range $p<5$.

We are now going to develop a boot-strap argument showing that this regularity property holds. In fact, we will prove that u is actually smooth.

We have $\Box u=-f(u)$, but, taking into account that $\partial_t u$ vanishes for $|x|>R$, we deduce that, in that set: $u\in H^1(\{|x|>R\})$, $\Delta u=f(u)$. The fact that f is subcritical allows to use elliptic regularity results and a boot-strap argument showing that $u=u(x)$ is in fact of class $C^{4,\alpha}$ in that set.

In fact more regularity on u could also be obtained if we were assuming more regularity on the nonlinearity than $f\in C^3$.

The goal then is to prove that u is also smooth enough in the cylinder $[0,T]\times(|x|\leqslant R)$, so that the existing unique continuation results might be applied. Since the values of the nonlinear term $f(u)$ away from the cylinder $[0,T]\times(|x|\leqslant R+T)$, do not affect the regularity of u on the cylinder $[0,T]\times(|x|\leqslant R)$, without loss of generality, we can suppose that f is compactly supported in x, say $f=\chi f$, with support of χ in $\{|x|\leqslant R+T+1\}$, and $\chi=1$ in $\{|x|\leqslant R+T\}$.

The following holds:

PROPOSITION 10. – *Every solution u of system* (4.9) *above satisfies*

$$u\in L^\infty\big(0,T;H^k(|x|\leqslant R+1)\big),$$

for all $k<2$. In particular, $u\in L^\infty([0,T]\times(|x|\leqslant R+1))$.

Proof. – We know that all the Strichartz norms of the function u are finite.

Define for $\nu\geqslant 0$ the vector space of functions:

$$V_\nu=C^0\big([0,T],H^{1+\nu}_{\mathrm{loc}}\big)\cap C^1\big([0,T],H^{\nu}_{\mathrm{loc}}\big)\cap L^q\big([0,T],L^{3r}_{\mathrm{loc}}\big),\tag{4.10}$$

where the last intersection is over all the couples (q,r), such that $r\geqslant 2$ and $1/q=1/2-1/r$.

We know that the solution u of (4.9) belongs to V_0.

We start the argument by applying the regularity theorem (Theorem 8) to $f(u)$ with $r=1$. Denoting $\varepsilon=(5-p)/2$, we then obtain $f(u)\in L^1([0,T],H^{\varepsilon}(|x|\leqslant R+T+1))$.

The theorem of propagation of singularities for traces (see for example [1, p. 115, Prop. 2, and p. 117, §1.3]) hence gives $u(0,.)\in H^{1+\varepsilon}$, and $\partial_t u(0,.)\in H^{\varepsilon}$; so $u\in V_{\varepsilon}$.

This implies that $f(u)\in L^1([0,T],H^{2\varepsilon}(|x|\leqslant R+T+1))$, hence $u\in V_{2\varepsilon}$. Then we iterate this process to obtain $u\in V_{m\varepsilon}\subset L^{\infty}([0,T],H^{1+m\varepsilon}_{\mathrm{loc}})$, $m\in\mathbb{N}$, large enough, and reach $L^{\infty}([0,T],H^k(|x|\leqslant R+1))$ for any desired real $k<2$.

This completes the proof of the proposition. □

This completes the discussion of the case $\alpha_n\to\alpha>0$ that has to be excluded.

Remark. – By considering a more regular function f, one can obviously improve the regularity of the composition $f(u)$.

We continue now the proof of Theorem 1, and we consider the 2nd case.

Second case: $\alpha_n\to 0$.

We write $f(u_n)=f'(0)u_n+R(u_n)$ where R verifies

$$\big|R(s)\big|\leqslant C\big(|s|^2+|s|^p\big). \tag{4.11}$$

Eq. (4.4) becomes then

$$\Box v_n+a(x)\partial_t v_n+f'(0)v_n+\frac{1}{\alpha_n}R(\alpha_n v_n)=0. \tag{4.12}$$

Passing to the limit $\alpha_n\to 0$ we obtain

$$\Box v+f'(0)v=0 \quad \text{in } D'(]0,T[\times\mathbf{R}^3), \tag{4.13}$$

$$\partial_t v=0 \quad \text{in }]0,T[\times(|x|\geqslant R). \tag{4.14}$$

Here the unique continuation result is obvious. Indeed, the equation has constant coefficients, so we can apply Holmgren uniqueness theorem to the system

$$\Box w+f'(0)w=0 \quad \text{in }]0,T[\times\mathbf{R}^3, \tag{4.15}$$

$$w=0 \quad \text{in }]0,T[\times(|x|\geqslant R) \tag{4.16}$$

with $w=\partial_t v$ and deduce that $w\equiv 0$. Then $v\equiv 0$.

Let us now prove that $v_n\to 0$ strongly in $H^1(]0,T[\times\mathbf{R}^3)$. Obviously this will be in contradiction with (4.6) and will complete the proof of the inequality (4.1) and that of Theorem 1.

First of all, we have to prove that the nonlinear term in Eq. (4.12) goes to 0 in $L^1([0,T],L^2_{\mathrm{loc}}(\mathbf{R}^3))$, when $n\to\infty$. For that, we proceed as follows. We write

$$R(\alpha_n v_n)=R(u_n),$$

and we estimate (with uniform constants in n):

$$\begin{aligned}\big\|\chi(x)R(u_n)\big\|_{L^1([0,T],L^2)}&\leqslant C\big(\big\|\chi u_n^2\big\|_{L^1([0,T],L^2)}+\big\|\chi|u_n|^p\big\|_{L^1([0,T],L^2)}\big)\\ &\leqslant C\int_0^T\big\|u_n(s)\big\|^2_{L^6}\,ds+C\int_0^T\big\|u_n(s)\big\|^{3/2}_{L^6}\bigg(\int|u_n|^{2(2p-3)}\bigg)^{1/4}ds.\end{aligned} \tag{4.17}$$

Due to the injection $H^1 \hookrightarrow L^6$, and taking in account that the energy of u_n is decreasing, we obtain

$$\big\|\chi(x)R(u_n)\big\|_{L^1([0,T],L^2)} \leqslant C\alpha_n^2 + C\alpha_n^{3/2}\int_0^T \left(\int |u_n|^{2(2p-3)}\right)^{1/4} ds \tag{4.18}$$

where the last integral is bounded by a Strichartz norm of u_n (take $(q,r) = (\frac{2p-3}{p-3}, \frac{2(2p-3)}{3}))$, and then uniformly bounded in n. Finally

$$\left\|\frac{1}{\alpha_n}\chi(x)R(u_n)\right\|_{L^1([0,T],L^2)} \leqslant C\alpha_n^{1/2} \tag{4.19}$$

which yields to the desired result.

Starting at this point, we may argue as in the first case. Indeed, we first prove that every m.d.m. μ associated to the sequence v_n vanishes. This guarantees the strong convergence to zero of $v_{n,t}$ in $L^2(]0,T[\times \mathbf{R}^3)$. Then, multiplying by $\varphi(t)v_n$ and integrating (i.e. using the equipartition of energy), one deduces that $v_n \to 0$ in $H^1([0,T]\times \mathbf{R}^3)$. This contradicts (4.6).

The proof of Theorem 1 is now complete.

4.2. Proof of Theorem 2

The proof of Theorem 2 uses some of the tools developed in the proof of Theorem 1. However, we need to employ also the multiplier techniques in [25,27].

We argue as follows.

Using the multiplier techniques in [27] one can easily deduce the existence of $T>0$ and $C>0$ such that

$$E_u(T) \leqslant C\left[\int_{\mathbf{R}^3}\int_0^T a(x)|u_t|^2\,dx\,dt + \|u\|^2_{L^2(]0,T[\times B_{4R})}\right]. \tag{4.20}$$

This is in fact the statement of Lemma 5 in [27] whose proof applies in all the range $p \leqslant 5$.

We emphasize that this inequality holds under the further qualitative property on the nonlinearity of Theorem 2 and that it is of global nature in the sense that T and the constant C are independent of the solution.

It is then sufficient to prove the existence of a constant $C>0$ such that

$$\|u\|^2_{L^2(]0,T[\times B_{4R})} \leqslant C\int_{\mathbf{R}^3}\int_0^T a(x)|u_t|^2\,dx\,dt, \tag{4.21}$$

for every solution.

To prove this we argue by contradiction as in Lemma 6 in [27]. We suppose there exists a sequence of solutions $\{u_k\}$ such that

$$\|u_k\|^2_{L^2(]0,T[\times B_{4R})}\Big/\int_{\mathbf{R}^3}\int_0^T a(x)|u_{k,t}|^2\,dx\,dt \to \infty, \tag{4.22}$$

as $k\to\infty$.

The key point is to observe that $\lambda_k = \|u_k\|_{L^2(]0,T[\times B_{4R})}$ is necessarily bounded. This is so since the time T and the constant C in inequality (4.21) remain bounded when the nonlinearity f is replaced by the rescaled family $h_\lambda(s) = f(\lambda s)/\lambda$ with $\lambda > 0$. This is so precisely because of the qualitative assumption on the nonlinearity we introduce in Theorem 2.

Once $\lambda_k = \|u_k\|_{L^2(]0,T[\times B_{4R})}$ is known to be bounded, the rest of the proof holds exactly as in the proof of Theorem 1 above.

5. The subcritical wave equation in a bounded domain

In this section we consider the subcritical nonlinear wave equation in a bounded domain of $\mathbf{R}^3$, with a nonlinear term that has been cut-off away from the boundary. First, in Section 5.1, we prove a global existence and uniqueness result. At this respect it is important to note that the mixed problem is in general well posed for at most cubic semilinearities (cf. [14]). However, we can deal with subcritical nonlinearities $(p < 5)$, because it has been cut-off away from the boundary and this allows using local Strichartz's inequalities. In Section 5.2 we prove a stabilization result. In Section 5.3 we prove the controllability results in Theorem 3 and Corollary 4 in a non-uniform time guaranteeing that every initial state can be driven to any final state if the time is large enough, depending on the size of the data to be controlled.

5.1. Global existence and uniqueness

Let Ω be a smooth, open bounded set of $\mathbf{R}^3$. Consider also a nonlinear function $f:\mathbf{R}\to\mathbf{R}$ verifying conditions (1.2), (1.3) and (1.12). Let finally $\theta(x) \in C_0^\infty(\Omega)$ be a non-negative function.

We have the following result

THEOREM 11. – *For every function $g \in L^1([0,+\infty[, L^2(\Omega))$, and every pair of initial data $(u^0,u^1) \in H_0^1(\Omega)\times L^2(\Omega)$, system*

$$\begin{cases} \Box u + \theta(x) f(u) = g \quad \textit{in}\ ]0,+\infty[\times\Omega, \\ u = 0 \quad \textit{on}\ ]0,+\infty[\times\partial\Omega, \\ u(0) = u^0, \partial_t u(0) = u^1 \quad \textit{in}\ \Omega \end{cases} \tag{5.1.1}$$

has a unique solution u in the space $C^0([0,+\infty[, H_0^1(\Omega)) \cap C^1([0,+\infty[, L^2(\Omega))$.

This solution satisfies moreover the following Strichartz estimates: For every finite $T > 0$, $r \geqslant 2$, q given by $1/q = 1/2 - 1/r$, and $\chi \in C_0^\infty(\Omega)$, there exists a constant $C > 0$ such that

$$\big\|\chi(x)u\big\|_{L^q([0,T];L^{3r}(\Omega))} \leqslant C\big(\|g\|_{L^1([0,T];L^2(\Omega))}, E_u(0)\big) \tag{5.1.2}$$

for every g and every initial data as before.

Here and in the sequel E_u stands for the energy of solutions of this system, i.e.

$$E(t) = \frac{1}{2}\int_\Omega \big[|\nabla u|^2 + |u_t|^2\big]\,dx + \int_\Omega \theta(x)F(u)\,dx. \tag{5.1.3}$$

Proof. – We proceed in three steps.

Step 1. *Existence*. We decouple system (5.1.1), by cutting off the initial data (u^0,u^1) and the right hand side term g. Let $\mathcal{V}$ be a neighbourhood of the compact set $\operatorname{supp}(\theta)$ such that $\overline{\mathcal{V}} \subset \Omega$;

and let $\psi \in C_0^\infty(\Omega)$, be such that $\psi = 1$ on $\mathcal{V}$. Define $g_1 = \psi g$ and $g_2 = (1-\psi)g$, in such a manner that $g_2 = 0$ on $\operatorname{supp}(\theta)$. Consider the two following systems

$$\begin{cases} \Box v + \theta(x) f(v) = g_1 & \text{in }]0,+\infty[\times\Omega, \\ v = 0 & \text{on }]0,+\infty[\times\partial\Omega, \\ (v(0), \partial_t v(0)) = \psi(x)(u^0, u^1) & \text{in } \Omega, \end{cases} \tag{5.1.4}$$

$$\begin{cases} \Box w = g_2 & \text{in }]0,+\infty[\times\Omega, \\ w = 0 & \text{on }]0,+\infty[\times\partial\Omega, \\ (w(0), \partial_t w(0)) = (1-\psi)(u^0, u^1) & \text{in } \Omega. \end{cases} \tag{5.1.5}$$

Let $T_0 = \min(d_1, d_2)$, where

$$d_1 = \operatorname{distance}\big(\operatorname{supp}(\psi), \partial\Omega\big), \quad \text{and} \quad d_2 = \operatorname{distance}\big(\operatorname{supp}(1-\psi), \operatorname{supp}(\theta)\big).$$

Then we solve the two systems above (5.1.4) and (5.1.5) on the time interval $[0, T_0]$.

Because of the finite speed propagation of waves ($= 1$ in the present model), it is clear that:

(i) For this time interval, the solution of (5.1.4) coincides, in the support of ψ, with that of the Cauchy problem in the free space $\mathbf{R}^3$. Indeed, the solution of the latter vanishes on the boundary because of the fact that the initial data and the right hand side have been confined to $\operatorname{supp}(\psi)$ and $T_0 \leqslant d_1$.

(ii) For $0 \leqslant t \leqslant T_0$, $\operatorname{supp}(w) \subset \Omega \setminus \operatorname{supp}(\theta)$, i.e. $w = 0$ on $\operatorname{supp}(\theta)$. This clearly gives $\theta(x) f(v) = \theta(x) f(v + w)$.

The function $u = v + w$, constructed as above, belongs to

$$C^0\big([0, T_0], H_0^1(\Omega)\big) \cap C^1\big([0, T_0], L^2(\Omega)\big)$$

and solves (5.1.1) for $0 \leqslant t \leqslant T_0$. Indeed, the fact that it solves the equation above is an obvious consequence of the previous discussion. The continuity in time of the solution is a consequence of the fact that both components v and w are indeed continuous. The continuity in time of v is consequence of the fact that it coincides, in the time interval $0 \leqslant t \leqslant T_0$, with the solution of the Cauchy problem in the whole space, that it is known to be continuous in time with values in the energy space.

Step 2. *Energy and Strichartz estimates*. Adding the classical energy estimate for each of the wave equations above one obtains that

$$E_u(t) \leqslant C\big[\|g\|_{L^1(L^2)} + E_u(0)\big] \quad \text{for } 0 \leqslant t \leqslant T_0.$$

Taking into account that the time T_0 depends only on the geometry of the problem (i.e. ω and the supports of ψ and θ), it is clear that one may iterate this process to obtain a global in time solution.

On the other hand, let $\chi(x)$ be a cut-off function and assume, to simplify the notation, that $\chi \equiv 1$ in the support of θ. The function $\widetilde{u} = \chi(x) u$ solves the (free) system

$$\begin{cases} \Box\tilde{u} + \theta(x) f(\tilde{u}) = \chi g + [\Box, \chi] u \in L^1_{\mathrm{loc}}([0, +\infty[, L^2(\mathbf{R}^3)), \\ (\tilde{u}(0), \partial_t \tilde{u}(0)) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3), \end{cases} \tag{5.1.6}$$

which, combined with (2.2), and the energy estimate above provides the Strichartz estimates (5.1.2).

Step 3. *Uniqueness*. We prove now the uniqueness of the solution. For that, we need the following lemma.

LEMMA 12. – *Let* u *and* v *be two solutions of* (5.1.1). *Then for every* $T>0$ *and* $1<\alpha\leqslant 2/(p-3)$, *there exists* $C>0$, *satisfying*

$$\bigl\|\theta(x)\bigl(f(u)-f(v)\bigr)\bigr\|_{L^\alpha([0,T],L^2(\Omega))}\leqslant C\|u-v\|_{L^\infty(0,T;H^1(\Omega))}. \tag{5.1.7}$$

Assuming for the moment that this lemma holds, let us show the uniqueness of the solution. Let u and v be two solutions of (5.1.1). The function $u-v$ solves the system

$$\begin{cases}\Box(u-v)+\theta(f(u)-f(v))=0 & \text{in }]0,+\infty[\times\Omega,\\ u-v=0 & \text{on }]0,+\infty[\times\partial\Omega,\\ (u-v)(0)=\partial_t(u-v)(0)=0 & \text{in } \Omega.\end{cases} \tag{5.1.8}$$

The energy inequality guarantees that

$$\begin{aligned}\|u-v\|_{L^\infty(0,T;H^1(\Omega))}&\leqslant C\bigl\|\theta(x)\bigl(f(u)-f(v)\bigr)\bigr\|_{L^1([0,T],L^2(\Omega))}\\ &\leqslant CT^{1/\beta}\bigl\|\theta(x)\bigl(f(u)-f(v)\bigr)\bigr\|_{L^\alpha([0,T],L^2(\Omega))}\end{aligned} \tag{5.1.9}$$

with $1/\alpha+1/\beta=1$, thanks to Hölder's inequality. Using (5.1.7), we obtain

$$\|u-v\|_{L^\infty(0,T;H^1(\Omega))}\leqslant CT^{1/\beta}\|u-v\|_{L^\infty(0,T;H^1(\Omega))} \tag{5.1.10}$$

which yields to the result $u\equiv v$, by taking T such that $CT^{1/\beta}<1$.

Now we come back to the proof of the lemma.

By hypothesis (1.3), one can write $f(u)-f(v)=(u-v)G(u,v)$ where $G(u,v)$ verifies

$$\bigl|G(u,v)\bigr|\leqslant C\bigl(1+|u|^{p-1}+|v|^{p-1}\bigr). \tag{5.1.11}$$

So, by Hölder's inequality it follows that

$$\begin{aligned}&\int_0^T\bigl\|\theta(x)\bigl(f(u)-f(v)\bigr)(t)\bigr\|_{L^2(\Omega)}^\alpha\,dt\\ &\quad\leqslant\int_0^T\bigl\|(u-v)(t)\bigr\|_{L^6(\Omega)}^\alpha\biggl(\int_\Omega|\theta G|^3dx\biggr)^{\alpha/3}dt\\ &\quad\leqslant\bigl[\bigl\|(u-v)(t)\bigr\|_{L^\infty(0,T;H^1(\Omega))}\bigr]^\alpha\int_0^T\biggl(\int_\Omega|\theta G|^3\,dx\biggr)^{\alpha/3}dt.\end{aligned}$$

To complete the proof of the lemma, it is sufficient to get a suitable upper bound on the last integral. Obviously, the last integral can be bounded above in terms of the $L^\alpha(0,T;L^3(\Omega))$-norm of $\theta G(u,v)$ which may be estimated in terms of the $L^{\alpha(p-1)}(0,T;L^{3(p-1)}(\mathrm{supp}(\theta))$-norms of u and v. These two norms can be easily estimated in terms of the Strichartz-norms in (5.1.2). Indeed, it is sufficient to set $r=(p-1)$ which does verify the condition $r\geqslant 2$. Then the exponent $q=2r/(r-2)$ corresponding to this choice of r in the Strichatrz norm (5.1.2) coincides with $q=2(p-1)/(p-3)$, which is greater that $\alpha(p-1)$ provided $2/(p-3)\geqslant\alpha$. This is precisely the range of exponents in the statement of the lemma. □

5.2. Stabilization

The stabilization result in the case of bounded domains is as follows.

PROPOSITION 13. – *Assume that the hypotheses of the previous theorem are satisfied. Let the set $\omega = \{x \in \Omega, a(x) \geqslant c_0 > 0\}$ be a neighbourhood of the boundary $\partial\Omega$, i.e. the intersection with Ω of a neighbourhood of $\partial\Omega$ in $\mathbf{R}^3$. Then the local stabilization property holds for system*

$$\begin{cases} \Box u + a(x)\partial_t u + \theta(x) f(u) = 0 \quad \textit{in }]0,+\infty[\times\Omega, \\ u = 0 \quad \textit{on }]0,+\infty[\times\partial\Omega, \\ (u(0), \partial_t u(0)) \in H_0^1(\Omega)\times L^2(\Omega). \end{cases} \tag{5.2.1}$$

More precisely, for every $E_0 > 0$, there exist $C > 0$ and $\gamma > 0$ such that inequality (1.8) *holds for the energy E_u in* (5.1.3) *provided $E_u(0) \leqslant E_0$.*

Remarks. – (a) Note that, in this proposition, the assumption (1.4) on the nonlinearity may be relaxed to (1.12).

(b) It would be interesting to see if a global stabilization result as that in Theorem 2 is true in this case.

We follow the same approach of the proof of Theorem 1.

As usually, we seek for an estimate of type (4.1) for every solution u of (5.2.1) verifying $E_u(0) \leqslant E_0$. A contradiction argument provides a sequence u_n which contradicts (4.1) and a sequence v_n such that

$$\begin{cases} \Box v_n + a(x)\partial_t v_n + \frac{1}{\alpha_n}\theta(x) f(\alpha_n v_n) = 0 \quad \text{in }]0,+\infty[\times\Omega, \\ v_n = 0 \quad \text{on }]0,+\infty[\times\partial\Omega, \\ \int_0^T\int_\Omega a(x)|\partial_t v_n|^2\,dt\,dx \to 0, \\ 1/C \leqslant Ev_n(0) \leqslant C, \\ \alpha_n = (Eu_n(0))^{1/2} \to \alpha. \end{cases} \tag{5.2}$$

We examine, again, separately the cases $\alpha > 0$ and $\alpha = 0$. We denote by v the weak limit of the sequence $\{v_n\}$.

First case: $\alpha > 0$. We come back to the equation in u_n and we pass to the limit. We then obtain

$$\begin{cases} \Box u + \theta(x) f(u) = 0 \quad \text{in }]0,T[\times\Omega, \\ \partial_t u = 0 \quad \text{on }]0,T[\times\omega, \\ u \in L^\infty([0,T], H_0^1(\Omega)), \quad \partial_t u \in L^\infty([0,T], L^2(\Omega)). \end{cases}$$

Let $\chi(x) \in C_0^\infty(\Omega)$ be such that $\chi = 1$ on $\operatorname{supp}(\theta)$, and $\operatorname{supp}(\nabla\chi) \subset \omega$. The function $\tilde u = \chi u$ verifies

$$\begin{cases} \Box\tilde u + \theta(x) f(\tilde u) = \nabla\chi.\nabla u + (\Delta\chi)u \in L^1([0,T], L^2(\mathbf{R}^3)), \\ \partial_t\tilde u = 0 \quad \text{in }]0,T[\times(\mathbf{R}^3\backslash\Omega), \\ \tilde u \in L^\infty([0,T], H^1(\mathbf{R}^3)), \quad \partial_t\tilde u \in L^\infty([0,T], L^2(\mathbf{R}^3)). \end{cases}$$

The right hand member is in $L^1([0,T], L^2(\mathbf{R}^3))$, so $\tilde u$ has bounded Strichartz norms. Applying then the regularity theorem (Theorem 8), we obtain that $\tilde u$ is bounded as well as $f'(\tilde u)$. Then,

$w = \partial_t \tilde{u}$ satisfies

$$\begin{cases} \Box w + \theta(x) f'(\tilde{u}) w = 0 & \text{in }]0,T[\times \mathbf{R}^3, \\ w = 0 & \text{in }]0,T[\times (|x| > R) \end{cases}$$

where R is large enough. By unique continuation we deduce that $w \equiv 0$. Thus $u = u(x) \in H_0^1(\Omega)$ for $t \in]0,T[$, and it satisfies $-\Delta u + \theta f(u) = 0$, in Ω. Multiplying this equation by u and integrating over Ω, we obtain

$$\int_\Omega \bigl(|\nabla u|^2 + \theta u f(u)\bigr)\, dx = 0,$$

and this implies $u \equiv 0$, because of the good-sign assumption (1.4) on f.

Consequently, $u_n \rightharpoonup 0$ in $H^1(]0,T[\times\Omega)$. Here, we use again an argument based on microlocal defect measures. Let μ be a m.d.m. associated to u_n in $H^1(]0,T[\times\Omega)$. It is easy to see from (5.2.2) that $\mu = 0$ in $]0,T[\times\omega$. To complete the argument, we use the propagation property of the m.d.m. in Ω (away from the boundary). This gives $\mu = 0$ everywhere; hence $u_n \to 0$ in $H^1(]0,T[\times\Omega)$, which contradicts the fact that $\alpha > 0$.

Second case: $\alpha = 0$. Letting $n \to \infty$, we obtain that the limit v of the sequence $\{v_n\}$ satisfies

$$\begin{cases} \Box v + \theta(x) f'(0) v = 0 & \text{in }]0,T[\times\Omega, \\ \partial_t v = 0 & \text{on }]0,T[\times\omega, \\ v \in L^\infty([0,T], H_0^1(\Omega)). \end{cases}$$

The existing results on unique continuation applied to this system after derivation in time allow to show that $v = 0$. The rest of the proof is very close to the corresponding one of Theorem 1. Indeed, a suitable set of truncature functions replaces the problem under consideration by a global one in the whole space and the same arguments apply.

The proof of the stabilization result on the domain Ω is now complete.

5.3. Exact controllability in a non-uniform time

In this section we give the proof of Theorem 3 and Corollary 4.

We first prove Theorem 3 and then indicate how Corollary 4 may be obtained.

Proof of Theorem 3. – Recall that, according to the results of the previous section and, in particular, in view of the stabilization result of Proposition 13, we can assume the initial and the final data to be controlled to be small. Indeed, it is sufficient to solve the dissipative system (5.2.1) with the initial data to be controlled. Then, the solution can be made as small as we wish by taking the time sufficiently large. The same can be done backwards in time, taking into account that the system under consideration is time-independent, starting from the final data to be controlled.

Thus it is enough to prove the exact controllability to zero for small data. We will use a nonlinear variant of Lions' H.U.M. (Hilbert Uniqueness Method) (see [13]), following closely the proof developed in [26].

Let us first consider the linearized system

$$\begin{cases} \Box u + \theta(x) f'(0) u = g(t,x) 1_\omega & \text{in }]0,+\infty[\times\Omega, \\ u = 0 & \text{on }]0,+\infty[\times\partial\Omega, \\ u(0) = u^0, \quad \partial_t u(0) = u^1 & \text{in } \Omega. \end{cases} \tag{5.3.1}$$

Here and in the sequel ω denotes the neighborhood of the boundary where the control is supported and 1_ω its characteristic function.

This system is exactly controllable in time $T > 2R$. Indeed, for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists g in $L^2(0,T; L^2(\omega))$ such that the solution of (5.3.1) satisfies

$$u(T) \equiv u_t(T) \equiv 0.$$

Moreover, the control g of minimal norm is unique and depends continuously on the initial data (u^0, u^1) in the corresponding norms. More precisely, the control g is the restriction to $[0,T] \times \omega$ of a solution Φ of

$$\begin{cases} \Box\Phi + \theta(x) f'(0)\Phi = 0 \quad \text{in }]0,+\infty[\times \Omega, \\ \Phi = 0 \quad \text{on }]0,+\infty[\times \partial\Omega, \\ \Phi(0) = \Phi^0 \in L^2(\Omega), \quad \partial_t\Phi(0) = \Phi^1 \in H^{-1}(\Omega). \end{cases} \tag{5.3.2}$$

One can identify the solution Φ of (5.3.2) associated with the data (u^0, u^1) to be controlled as follows. For any $(\Phi^0, \Phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a unique solution

$$\Phi \in C\big([0,T]; L^2(\Omega)\big) \cap C^1\big([0,T]; H^{-1}(\Omega)\big).$$

We then solve

$$\begin{cases} \Box\Psi + \theta(x) f'(0)\Psi = 1_\omega \Phi \quad \text{in }]0,+\infty[\times \Omega, \\ \Psi = 0 \quad \text{on }]0,+\infty[\times \partial\Omega, \\ \Psi(T) \equiv \partial_t\Psi(T) \equiv 0. \end{cases} \tag{5.3.3}$$

Clearly

$$\Psi \in C^0\big([0,T], H_0^1(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big).$$

The operator

$$\Lambda : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega)$$

such that $\Lambda(\Phi^0, \Phi^1) = (-\partial_t\Psi(0), \Psi(0))$ is an isomorphism. Indeed,

$$\big\langle \Lambda\big(\Phi^0, \Phi^1\big), \big(\Phi^0, \Phi^1\big)\big\rangle = \int_0^T \int_\omega |\Phi|^2 \, dx \, dt,$$

and, on the other hand, taking into account that $T > 2R$, one can prove the existence of a positive constant $C > 0$ such that

$$\big\|\big(\Phi^0, \Phi^1\big)\big\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leqslant C \int_0^T \int_\omega |\Phi|^2 \, dx \, dt,$$

for every solution Φ of (5.3.2). This can be done using multiplier methods (see [13]) or the arguments in the previous section dealing with microlocal defect measures.

Thus, given any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists (Φ^0, Φ^1) in $L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$\Lambda\big(\Phi^0, \Phi^1\big) = \big(-u^1, u^0\big),$$

and this is precisely equivalent to saying that the solution u of (5.3.1) with control Φ coincides with Ψ and therefore, in particular, fulfills the requirement $u(T) \equiv u_t(T) \equiv 0$.

Now, in what concerns the nonlinear system, after solving Eq. (5.3.2) for Φ, we solve

$$\begin{cases} \Box u + \theta(x) f(u) = \Phi 1_\omega & \text{in }]0,+\infty[\, \times \Omega, \\ u = 0 & \text{on }]0,+\infty[\, \times \partial\Omega, \\ u(T) = \partial_t u(T) = 0 & \text{in } \Omega. \end{cases} \tag{5.3.4}$$

The problem is then to show that the operator A defined on $L^2(\Omega) \times H^{-1}(\Omega)$, with values in its dual $L^2(\Omega) \times H_0^1(\Omega)$ by

$$A(\Phi^0, \Phi^1) = \big(-\partial_t u(0), u(0)\big),$$

is onto on a small neighbourhood of the origin.

Note that the function $v = u - \Psi$, where Ψ is the solution of the corresponding linear problem (5.3.3), belongs to $C^0([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ (in fact, both u and Ψ do belong to this space). Moreover, it satisfies

$$\begin{cases} \Box v + \theta(x) f'(0) v = -\theta(x) R(u) & \text{in }]0,+\infty[\, \times \Omega, \\ v = 0 & \text{on }]0,+\infty[\, \times \Omega, \\ v(T) = \partial_t v(T) = 0 & \text{in } \Omega, \end{cases} \tag{5.3.5}$$

where

$$R(u) = f(u) - f'(0)u.$$

We have $u = \Psi + v$ and therefore

$$A(\Phi^0, \Phi^1) = \Lambda(\Phi^0, \Phi^1) + K(\Phi^0, \Phi^1)$$

where

$$K(\Phi^0, \Phi^1) = \big(-\partial_t v(0), v(0)\big).$$

Taking into account that

$$\Lambda : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega)$$

is an isomorphism, solving the equation

$$A(\Phi^0, \Phi^1) = \big(-u^1, u^0\big), \tag{5.3.6}$$

which is equivalent to finding the control for the data (u^0, u^1), is also equivalent to solving

$$B(\Phi^0, \Phi^1) = -\Lambda^{-1} K(\Phi^0, \Phi^1) + \Lambda^{-1}(-u^1, u^0) = (\Phi^0, \Phi^1). \tag{5.3.7}$$

Therefore, the problem is to find a fixed point for the operator B, defined from $L^2(\Omega) \times H^{-1}(\Omega)$ into itself.

We claim that the operator B is compact. For, it is sufficient to check that K, as operator from $L^2(\Omega) \times H^{-1}(\Omega)$ into its dual, is compact. To show this fact we observe that, by Theorem 11, the Strichartz norms of u are bounded on the support of θ by the norm of

(Φ^0, Φ^1) in $L^2(\Omega) \times H^{-1}(\Omega)$. Applying the regularity theorem (Theorem 8), we obtain that $\theta(x)R(u) \in L^1([0,T], H^\varepsilon(\Omega))$, for some $\varepsilon > 0$, small enough; which leads to

$$v \in C^0\big([0,T], H_0^{1+\varepsilon}(\Omega)\big) \cap C^1\big([0,T], H^\varepsilon(\Omega)\big),$$

with a bound on v in that space in terms of $\|(\Phi^0, \Phi^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}$. This completes the proof of the compactness property.

Therefore, in order to obtain a fixed point, we may apply the Schauder fixed point theorem. To do that it suffices to find a constant $\rho > 0$, such that

$$\text{(5.3.8)} \qquad \begin{cases} \|B(\Phi^0, \Phi^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leqslant \rho, \\ \forall (\Phi^0, \Phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)\colon\ \|(\Phi^0, \Phi^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leqslant \rho. \end{cases}$$

We are going to show that this $\rho > 0$ exists provided (u^0, u^1) is sufficiently small in $H_0^1(\Omega) \times L^2(\Omega)$. In view of the structure of B it is sufficient to show that there exists $\rho > 0$ such that

$$\text{(5.3.9)} \qquad \begin{cases} \|K(\Phi^0, \Phi^1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant \rho/2, \\ \forall (\Phi^0, \Phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)\colon\ \|(\Phi^0, \Phi^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leqslant \rho. \end{cases}$$

For that, we write an energy inequality for system (5.3.5). Define

$$E_v(t) = \frac{1}{2}\int_\Omega \big(|\partial_t v|^2 + |\nabla_x v|^2\big)\,dx + \frac{1}{2}\int_\Omega \theta(x) f'(0) v^2\,dx.$$

Multiplying the equation of (5.3.5) by $\partial_t v$ and integrating, we obtain

$$\text{(5.3.10)} \qquad \frac{dE_v(t)}{dt} = -\int_\Omega \partial_t v(t)\theta(x)R(u)(t)\,dx.$$

But

$$\begin{aligned} \big\|\partial_t v(t)\theta(x)R(u)(t)\big\|_{L^1(\Omega)} &\leqslant \big\|\partial_t v(t)\big\|_{L^2(\Omega)}\big\|\theta(x)R(u)(t)\big\|_{L^2(\Omega)} \\ &\leqslant C\big(Ev(t)\big)^{1/2}\big\|\theta(x)R(u)(t)\big\|_{L^2(\Omega)}. \end{aligned}$$

Hence

$$\frac{dE_v(t)}{dt} \geqslant -C\big(E_v(t)\big)^{1/2}\big\|\theta(x)R(u)(t)\big\|_{L^2(\Omega)}.$$

Integrating on the time interval $[t, T]$, and taking in account that $v(T) = \partial_t v(T) = 0$, we obtain

$$\text{(5.3.11)} \qquad \begin{aligned} \big\|\big(v(t), \partial_t v(t)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} &\leqslant C\big(E_v(t)\big)^{1/2} \\ &\leqslant C\big\|\theta(x)R(u)\big\|_{L^1(0,T;L^2(\Omega))}. \end{aligned}$$

Now, arguing as in (4.18), we have

$$\begin{aligned} \big\|\theta(x)R(u)\big\|_{L^2(\Omega)} &\leqslant C\big\|\theta|u|^2 + \theta|u|^p\big\|_{L^2(\Omega)} \\ &\leqslant C\left[\|u\|_{L^6(\Omega)}^2 + \|u\|_{L^6(\Omega)}^{3/2}\left(\int_\Omega \theta^2|u|^{2(2p-3)}dx\right)^{1/4}\right]. \end{aligned}$$

Hence

$$\begin{aligned}\big\|\big(v(t),\partial_t v(t)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant C \sup_{0\leqslant t\leqslant T}\|u(t)\|_{L^6(\Omega)}^{3/2}\Bigg[\sup_{0\leqslant t\leqslant T}\|u(t)\|_{L^6(\Omega)}^{1/2}\\ +\int_0^T\bigg(\int_\Omega \theta^2|u|^{2(2p-3)}\,dx\bigg)^{1/4}dt\Bigg]\end{aligned} \tag{5.3.12}$$

where the last integral satisfies

$$\int_0^T\bigg(\int_\Omega \theta^2|u|^{2(2p-3)}\,dx\bigg)^{1/4}dt \leqslant C\big(T,\|\Phi\|_{L^1(0,T;L^2(\Omega))}\big)$$

by comparison with Strichartz norms of u. On the other hand, applying the same energy inequality to u, we have

$$\begin{aligned}\big(E_u(t)\big)^{1/2} &\leqslant C\|\Phi\|_{L^1(0,T;L^2(\Omega))}\\ &\leqslant C\big\|\big(\Phi^0,\Phi^1\big)\big\|_{L^2(\Omega)\times H^{-1}(\Omega)} \quad \forall t\in[0,T].\end{aligned} \tag{5.3.13}$$

Using the embedding from $H_0^1(\Omega)$ into $L^6(\Omega)$, we obtain by combining (5.3.12) and (5.3.13)

$$\sup_{0\leqslant t\leqslant T}\big\|\big(v(t),\partial_t v(t)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant C\big\|\big(\Phi^0,\Phi^1\big)\big\|_{L^2(\Omega)\times H^{-1}(\Omega)}^{3/2}.$$

This nonlinear estimate immediately yields (5.3.9).

The proof of Theorem 3 is now complete. □

Proof of Corollary 4. – Let us consider any initial and final data

$$\big(u^0,u^1\big),\big(y^0,y^1\big)\in H_0^1(\Omega)\times L^2(\Omega).$$

According to Theorem 3 there exist $T>0$ and a control $g\in L^2(]0,T[\times\Omega)$ with support in $[0,T]\times\omega$ such that the unique solution u of (1.13) satisfies

$$u(T)\equiv y^0,\quad u_t(T)\equiv y^1.$$

We now introduce the control $\tilde g = g+(1-\theta)f(u)$. This control has also its support in $[0,T]\times\omega$. This can be guaranteed by taking the cut-off function θ so that $\theta\equiv 1$ in $\Omega\setminus\omega$. On the other hand, the solution u of (1.13) satisfies also

$$\begin{cases}\Box u + f(u) = \tilde g & \text{in }]0,+\infty[\times\Omega,\\ u=0 & \text{on }]0,+\infty[\times\partial\Omega,\\ u(0)=u^0,\quad \partial_t u(0)=u^1 & \text{in } \Omega.\end{cases} \tag{5.3.14}$$

Note that we are not in conditions to guarantee that the finite energy solution u of (5.1) is unique since we do not know whether Strichartz inequalities hold in the domain Ω up to the boundary. But the existence is guaranteed. In fact, u solution of (1.13) solves (5.3.14) too.

In order to conclude the proof of corollary it is sufficient to analyze the regularity of $\tilde g$. We know that $g\in L^2(]0,T[\times\Omega)$. Thus, it is sufficient to analyze the regularity of

$(1-\theta)f(u)$. The function u has finite Strichartz norms in the interior of Ω. In particular, $u \in L^5(0,T;L^{10}_{\text{loc}}(\Omega))$ (take $q=5$ and $r=10/3$ in the Strichartz norms). Consequently, $(1-\theta)f(u) \in L^1(0,T;L^2_{\text{loc}}(\Omega))$. On the other hand, taking into account that u has finite energy it is easy to see that $(1-\theta)f(u) \in L^\infty(0,T;L^{6/5}(\Omega))$.This concludes the proof of Corollary 4. □

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