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NEW EVIDENCE FOR GREEN'S CONJECTURE ON SYZYGIES OF CANONICAL CURVES

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ABSTRACT. – What we call the *generic* Green's conjecture predicts what are the numbers of syzygies of the generic canonical curve of genus g. Green and Lazarsfeld have observed that curves with nonmaximal Clifford index have extra syzygies and we call *specific* Green's conjecture the stronger prediction that the curves which have the numbers of syzygies expected for generic curves are precisely those with maximal Clifford index. In this note, we prove that, as stated above, the generic and specific Green's conjectures for canonical curves are equivalent at least when g is odd. © Elsevier, Paris

RÉSUMÉ. – Ce que nous appelons la conjecture de Green *générique* prédit les nombres de syzygies de la courbe canonique générique de genre g. Green et Lazarsfeld ont observé que les courbes avec indice de Clifford non maximal présentaient un excédent de syzygies; et ce que nous appelons conjecture de Green *spécifique* prédit plus précisément que les courbes ayant les nombres de syzygies auxquels on s'attend pour les courbes génériques sont exactement celles ayant l'indice de Clifford maximum. Dans cette note, nous prouvons que ces deux conjectures de Green, la générique et la spécifique, sont équivalentes dans le cas où le genre g est impair. © Elsevier, Paris

Introduction

Some twelve years ago, Mark Green [G] made a few conjectures regarding the behaviour of syzygies of a curve C imbedded in $\mathbb{P}^n$ by a complete linear system. The so-called *generic* Green conjecture on canonical curves pertains to this question when the linear system is the canonical one and the curve is generic in the moduli, and predicts what are the numbers of syzygies in that case. Green and Lazarsfeld [GL] have observed that curves with nonmaximal Clifford index have extra syzygies and we will call *specific* Green conjecture on canonical curves the stronger prediction that the curves which have the numbers of syzygies expected for generic curves are precisely those with maximal Clifford index ($[(g-1)/2]$). (As a matter of fact, the full Green conjecture on canonical curves relates more closely the Clifford number with the existence of extra syzygies.) Many attempts have been made to settle this question, and some nice results have been obtained ([Sch][V]).

In this note, we work over an algebraically closed field of arbitrary characteristic and prove that, as stated above, the generic and specific Green conjectures for canonical curves are equivalent at least when the genus g is odd. Let C be a curve canonically imbedded in $\mathbb{P}^{g-1}$ with ideal sheaf $\mathcal{I}_C$. We denote by Q the universal quotient on $\mathbb{P}^{g-1}$, so that $Q(1)$ is the tangent bundle, and by Q_C its restriction to C. It is generally known (see [P-R] for example) that extra syzygies appear when, for some $i \leq [(g-1)/2]$, the natural map

$\Lambda^i(\Gamma(C, Q_C)) \to \Gamma(C, \Lambda^i Q_C)$ is not surjective. It is easy to see that the relevant quotient of $\Gamma(C, \Lambda^i Q_C)$ by $\Lambda^i(\Gamma(C, Q_C))$ is isomorphic to $\Gamma(\Lambda^{i+1}Q \otimes \mathcal{I}_C(1))$ (*cf.* 2.1).

THEOREM 1.1. – *Let $g = 2k - 1 \geq 5$ be an odd integer. If the generic curve C of genus g has the expected number of syzygies* (i.e.. $\Gamma(\Lambda^k Q \otimes \mathcal{I}_C(1)) = 0$)*, then so does any curve of genus g with maximal Clifford index, namely $k - 1$.*

To prove this, we compute a virtual (divisor) class v for the locus (in the moduli) of curves C for which the cohomology group $\Gamma(\Lambda^k Q \otimes \mathcal{I}_C(1))$ does not vanish. Once v is computed, we compare it with the class c of the locus of k-gonal curves (these are curves with non-maximal Clifford index in our case), which, thanks to Harris and Mumford [HM], is already known, and we find that $v = (k-1)c$. We conclude by proving that the generic k-gonal curve has at least $k-1$ extra syzygies, which implies that the k-gonal locus occurs with multiplicity $k - 1$ in v, leaving no room for another component. Our proof gives another consequence of the generic Green's conjecture, namely that the number of extra syzygies (more precisely $h^0(\Lambda^k Q \otimes \mathcal{I}_C(1))$) is exactly $k-1$ for any k-gonal curve C in the smooth part of the k-gonal locus. Finally, we observe that our argument fails completely in the case of even genus, where the expected codimension of our jump locus is no more one.

2. Preliminaries on syzygies

We collect here a few useful remarks on syzygies.

PROPOSITION 2.1. – *Let S be a linearly normal subscheme of $\mathbb{P}^n$* (i.e.. $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(S, \mathcal{O}(1))$ *is an isomorphism) with ideal sheaf $\mathcal{I}_S$. Then the cokernel of*

$$\Lambda^i\Gamma(\mathbb{P}^n, Q) = \Gamma(\mathbb{P}^n, \Lambda^i Q) \to \Gamma(S, \Lambda^i Q_S)$$

is canonically isomorphic to

$$\Gamma(\mathbb{P}^n, \Lambda^{i+1}Q \otimes \mathcal{I}_S(1)).$$

Proof. – Consider the following exact sequence of sheaves on $P(V) = \mathbb{P}^n$:

$$0 \to \mathcal{O}(-1) \to V_P \to Q \to 0$$

By taking the exterior $(i+1)$-th power and tensoring with $\mathcal{O}(1)$, we get the exact sequence:

$$0 \to \Lambda^i Q \to \Lambda^{i+1}V_P(1) \to \Lambda^{i+1}Q(1) \to 0.$$

This exact sequence of vector bundles remains exact on tensorisation by $\mathcal{I}_S$ as well as $\mathcal{O}_S$. Thus we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^i Q \otimes \mathcal{I}_S & \to & \Lambda^{i+1}V_P \otimes \mathcal{I}_S(1) & \to & \Lambda^{i+1}Q \otimes \mathcal{I}_S(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^i Q & \to & \Lambda^{i+1}V_P(1) & \to & \Lambda^{i+1}Q(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^i Q_S & \to & \Lambda^{i+1}V_S(1) & \to & \Lambda^{i+1}Q_S(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now apply the section functor Γ : the middle row remains exact. Thus we may apply the snake lemma to the two lower rows. This yields the desired isomorphism because under our assumption, $\Gamma(\Lambda^{i+1}V_P(1)) \to \Gamma(\Lambda^{i+1}V_S(1))$ is an isomorphism as well. □

REMARK 2.2. – *Thus we will think of* $\Gamma(\Lambda^j Q \otimes \mathcal{I}_C(1))$ *as the space of extra syzygies. From this point of view, extra syzygies behave in a monotonic way with respect to the degree* j *and the subvariety* C*:*

a) *If* $C \subset S$ *are two subvarieties of* $\mathbb{P}^{g-1}$, *then* $h^0(\Lambda^j Q \otimes \mathcal{I}_C(1)) \geq h^0(\Lambda^j Q \otimes \mathcal{I}_S(1))$. *We will estimate syzygies of our canonical curves by using a scroll* S *containing them.*

b) *If* $i < j$, *then* $h^0(\Lambda^j Q \otimes \mathcal{I}_C(1)) \geq h^0(\Lambda^i Q \otimes \mathcal{I}_C(1))$.

The above proposition is applicable with our canonical curve: $S = C$. *Also, the Clifford index of the generic curve of genus* g, *is well-known to be* $[(g-1)/2]$. *Finally, if* $i < j$ *then* $\Gamma(\Lambda^j Q \otimes \mathcal{I}_C(1)) = 0$ *implies* $\Gamma(\Lambda^i Q \otimes \mathcal{I}_C(1)) = 0$ *so that we have an equivalent formulation of the specific conjecture of Green:*

SPECIFIC GREEN'S CONJECTURE. – *Let* C *be a canonically imbedded curve with maximum Clifford index* $[(g-1)/2]$. *Then* $\Gamma(\Lambda^j Q \otimes \mathcal{I}_C(1))$ *is zero for* $j = [(g+1)/2]$.

We will use in Section 4 the following semi-continuity statement:

PROPOSITION 2.3. – *Let* $p : W \longrightarrow T$ *be a smooth family of projective varieties parametrized by the spectrum* T *of a discrete valuation ring. We suppose that* W *is endowed with a line bundle* $\mathcal{L}$, *that* $h^0(W_t, \mathcal{L}_t)$ *is constant and that for each point* $t \in T$, $\mathcal{L}_t$ *is generated by global sections. This yields a* T*-morphism* m *from* W *to* $\mathbb{P}(p_*\mathcal{L})$. *We denote by* I_t *the ideal sheaf of* $m(W_t)$ *and by* Q_t *the tautological quotient bundle on the fibre* $\mathbb{P}_t = \mathbb{P}(H^0(W_t, \mathcal{L}_t))$. *Then for any* i, *the dimension* $h^0(\mathbb{P}_t, \Lambda^i Q_t \otimes I_t(1))$ *is upper-semi-continuous.*

Proof. – By properness of the Hilbert scheme, there exists a T-flat subscheme $\bar{W}$ of $\mathbb{P}(p_*\mathcal{L})$ with the property that its fibre over the general point t_1 of T is $m(W_{t_1})$. By continuity, its special fibre $\bar{W}_{t_0}$ contains $m(W_{t_0})$ (indeed, they are equal, but we don't need this). Thus, by inclusion, we have

$$h^0(\mathbb{P}_{t_0}, \Lambda^i Q_{t_0} \otimes I_{t_0}(1)) \geq h^0(\mathbb{P}_{t_0}, \Lambda^i Q_{t_0} \otimes I_{\bar{W}_{t_0}}(1)),$$

and by semi-continuity,

$$h^0(\mathbb{P}_{t_0}, \Lambda^i Q_{t_0} \otimes I_{\bar{W}_{t_0}}(1)) \geq h^0(\mathbb{P}_{t_1}, \Lambda^i Q_{t_1} \otimes I_{\bar{W}_{t_1}}(1)),$$

which altogether prove our claim. □

3. The syzygy locus in the case of odd genus

In this section, we write $\mathcal{M} = \mathcal{M}_g^o$ for the open subvariety of $\mathcal{M}_g$ consisting of points that represent isomorphism classes of smooth curves with trivial automorphism group. What we need to know of $\mathcal{M}$ is that an effective divisor on it which is rationally equivalent to zero is indeed zero: this is for instance because $\mathcal{M}$ has a projective compactification with two-codimensional boundary (*cf.* e.g. [A]).

Let x be a point in $\mathcal{M}$ and C the corresponding curve. We consider the canonical imbedding of C in $\mathbb{P}^{g-1}$ (C is not hyperelliptic); we denote by $\mathcal{I}_C$ the ideal sheaf of C and by Q the tautological quotient bundle of rank $g-1$ on $\mathbb{P}^{g-1}$. Finally, we denote by S_g the locus in $\mathcal{M}$ of (points corresponding to) curves C satisfying $\Gamma(\Lambda^k Q \otimes \mathcal{I}_C(1)) \neq 0$. As a jump locus, S_g has a natural Cartier divisor structure (see e.g. the proof of the next proposition) and we compare its class in the Picard group of $\mathcal{M}$ with the class c of the k-gonal locus (*cf.* [HM]).

PROPOSITION 3.1. – *Let $g = 2k - 1 \geq 5$ be an odd integer such that the generic curve C of genus g satisfies $\Gamma(\Lambda^k Q \otimes \mathcal{I}_C(1)) = 0$. Then, in the Picard group of $\mathcal{M}$, the rational class v of S_g is $(k-1)c$.*

Proof. – There exists a universal curve $\mathcal{C}$ over $\mathcal{M}$, that is to say a smooth variety $\mathcal{C}$ and a smooth projective morphism $\pi : \mathcal{C} \to \mathcal{M}$ such that for any $x \in \mathcal{M}$ the fibre of π over x is the curve of genus g whose isomorphism class is given by the point x. Let $\omega = \omega_\pi$ be the cotangent bundle along the fibres and E its direct image on $\mathcal{M}$ by π. Then π factors through the natural canonical imbedding of $\mathcal{C}$ in the projective bundle $p : \mathbb{P} = \mathbb{P}(E) \to \mathcal{M}$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{C}$ in $\mathbb{P}$. The relatively ample (hyperplane) line bundle along the fibres of $\mathbb{P}$ will be denoted as usual by $\mathcal{O}_p(1)$. Finally we write again Q for the vector bundle on $\mathbb{P}$ given by the exact sequence

$$0 \to \mathcal{O}_p(-1) \to p^*(E)^* \to Q \to 0.$$

Observe that $p_*(\Lambda^l Q(1))$ is a vector bundle of rank $\binom{g}{l}g - \binom{g}{l-1}$. Similarly, on each fibre $\mathcal{C}_x$, $\Lambda^l Q_{\mathcal{C}_x} \otimes \omega_{\mathcal{C}_x}$ is semi-stable (*cf.* [PR]) of slope $2l + 2g - 2$, thus non-special, and $p_*(\Lambda^l Q(1) \otimes \mathcal{O}_{\mathcal{C}})$ is also a vector bundle, of rank $\binom{g-1}{l}(2l + g - 1)$, for each $l > 0$.

Substituting k for l, the restriction from $\mathbb{P}$ to the universal curve yields a morphism r from $p_*(\Lambda^k Q(1))$ to $p_*(\Lambda^k Q(1) \otimes \mathcal{O}_{\mathcal{C}})$, and our assumption means that this morphism is injective (at the generic point). We observe that the two vector bundles have the same rank, namely $\binom{2k-2}{k}(4k-2) = \binom{2k-1}{k-1}(2k-2)$. Thus the above map defines a (degeneracy) divisor in $\mathcal{M}$ and this is S_g, by definition. Its (virtual) rational class is $v = c_1(p_*(\Lambda^k Q(1) \otimes \mathcal{O}_{\mathcal{C}})) - c_1(p_*(\Lambda^k Q(1)))$.

We will compute this class in $Pic(\mathcal{M})$ as a multiple of $\lambda = c_1(E)$. We will start with the following computation in the appropriate Grothendieck group K. Let t be an indeterminate and for any vector bundle V, let $\lambda_t(V)$ denote the element $\sum t^i \Lambda^i(V)$ in $K[[t]]$. This extends to a homomorphism of K into the multiplicative group consisting of power series with constant term 1 in $K[[t]]$, and this map is still denoted by λ_t. Consider now $x = p_!(\lambda_t(Q).\mathcal{I}(1)) = p_!(\lambda_t(Q).(\mathcal{O}_p(1) - \mathcal{O}_{\mathcal{C}}(1)))$. Substitute $Q = p^*(E^*) - \mathcal{O}_p(-1)$. Then we obtain

$$\begin{aligned} x &= p_!(\frac{\lambda_t(p^*(E^*))}{\lambda_t(\mathcal{O}(-1))}(\mathcal{O}_p(1) - \mathcal{O}_{\mathcal{C}}(1)) \\ &= \lambda_t(E^*)p_!\left(\frac{\mathcal{O}_p(1) - \mathcal{O}_{\mathcal{C}}(1)}{1 + t\mathcal{O}_p(-1)}\right) \\ &= \lambda_t(E^*)\sum_{j=0}^{j=\infty}(-1)^j t^j (p_!(\mathcal{O}_p(1-j)) - p_!(\mathcal{O}_{\mathcal{C}}(1-j))). \end{aligned}$$

Our class v is the coefficient of t^k in the first Chern class of $-x$. Observe that $p_!(\mathcal{O}_p(1-j)) = 0$, whenever $2 \leq j \leq g$. Also we have $p_!(\mathcal{O}) = 1$ and $p_!(\mathcal{O}_p(1)) = E$. On the other hand, $p_!(\mathcal{O}_{\mathcal{C}}(1-j))$ can be seen to be $E^* - 1$ for $j = 0$ and to be $1 - E$ for $j = 1$. The first Chern class of the direct images for $j \geq 2$ can be computed by the Grothendieck-Riemann-Roch theorem to be $(1 - 6(1-j) + (1-j)^2)\lambda$ (see [M]). The first Chern class of $\lambda_t(E^*)$ is clearly equal to $-t(1+t)^{g-1}\lambda$. Also the rank of $\lambda_t(E)$ is

$(1+t)^g$, while the rank of $p_!(\mathcal{O}_C(n))$ is $(g-1)(2n-1)$. Thus v is equal to $N\lambda$ where $-N$ is the coefficient of t^k in

$$(1+t)^g\{1-\sum_{i=0}^{i=\infty}(-1)^i(1+6i+6i^2)t^i\}-t(1+t)^{g-1}\{g-t-(g-1)\sum_{i=0}^{i=\infty}(-1)^it^i(1-2i)\}.$$

On the one hand, we have

$$\begin{aligned}
&(1+t)^g(1-\sum_{i=0}^{i=\infty}(-1)^i(1-6i+6i^2)t^i)\\
&=(1+t)^g(1-\sum_{i=0}^{i=\infty}(-1)^i(6(i+1)(i+2)-24(i+1)+13)t^i)\\
&=(1+t)^g(1-\frac{12}{(1+t)^3}+\frac{24}{(1+t)^2}-\frac{13}{1+t})\\
&=(1+t)^{g-3}((1+t)^3-13(1+t)^2+24(1+t)-12)\\
&=t(1+t)^{g-3}(t^2-10t+1),
\end{aligned}$$

and on the other,

$$\begin{aligned}
&t(1+t)^{g-1}(g-t-(g-1)\sum_{i=0}^{i=\infty}(-1)^i(3-2(i+1))t^i)\\
&=t(1+t)^{g-1}(g-t-(g-1)(\frac{3}{1+t}-\frac{2}{(1+t)^2}))\\
&=t(1+t)^{g-3}((1+t)^2(g-t)-(g-1)3(1+t)-2(g-1))\\
&=t(1+t)^{g-3}(-t^3+(g-2)t^2+(-g+2)t+1).
\end{aligned}$$

This leads to the determination of N to be the coefficient of t^k in

$$t^2(1+t)^{2k-4}(-t^2+(2k-4)t-(2k-13)),$$

namely

$$-\binom{2k-4}{k-4}+(2k-4)\binom{2k-4}{k-3}-(2k-13)\binom{2k-4}{k-2}$$

and this simplifies to

$$6(k+1)(k-1)\frac{(2k-4)!}{(k-2)!k!}.$$

Now Harris and Mumford [HM] have studied the locus of k-gonal curves in $\mathcal{M}$ and have shown that this variety is a divisor whose class is $6(k+1)\frac{(2k-4)!}{(k-2)!k!}\lambda$, which proves our claim. □

4. Syzygies of scrolls

Extra syzygies of k-gonal curves arise because they lie on scrolls. So we start with estimating some syzygies of scrolls.

PROPOSITION 4.1. – *Let W be a vector bundle on $\mathbb{P}^1$ of rank $k-1$ and degree k. We suppose W to be globally generated. We denote by I_W the ideal of the image of the natural morphism from $\mathbb{P}(W)$ into $\mathbb{P}\Gamma(W)$ and by Q the tautological quotient bundle on this projective space. Then the dimension $h^0(\mathbb{P}\Gamma(W), \Lambda^k Q \otimes I_W(1))$ is at least $k-1$.*

Proof. – By 2.3, we may suppose that W is generic namely $W = \mathcal{O}(1)^{\oplus k-2} \oplus \mathcal{O}(2)$. In this case, the natural morphism $\mathbb{P}(W) \to \mathbb{P}\Gamma(W)$ is an imbedding. We will use freely the identification (2.1). Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and the variety $Y = \mathbb{P}^1 \times \mathbb{P}(W)$. Let us denote as usual by p_1 and p_2 the two projections (in both cases) and by π the fibration $\mathbb{P}(W) \to \mathbb{P}^1$, as well as the morphism $Y \to X$ given by $I \times \pi$. Let Δ be the diagonal divisor in X and D its inverse image in Y. Let Q be the universal quotient bundle on $\mathbb{P}\Gamma(W)$ and its restriction to $\mathbb{P}(W)$. Now consider on Y the bundle homomorphism $p_1^*(W)^* \to p_2^*(Q)$ obtained as the composition of the pull back by p_1 of the natural inclusion $W^* \to \Gamma(W)^* \otimes \mathcal{O}$ and the pull-back by p_2 of the tautological map $\Gamma(W)^* \otimes \mathcal{O} \to Q$. This homomorphism is injective as a sheaf morphism but has one-dimensional kernel on the fibres over points of D. Thus we obtain an inclusion of $\mathcal{L} := p_1^*(\Lambda^{k-1}W^*) \otimes \mathcal{O}(D)$ into $p_2^*(\Lambda^{k-1}Q)$. Note that $\mathcal{O}(D)$ is isomorphic to $p_1^*\mathcal{O}(1) \otimes p_2^*\pi^*\mathcal{O}(1)$ so that $\mathcal{L}$ is isomorphic to $p_1^*(\mathcal{O}(-k+1)) \otimes p_2^*(\pi^*(\mathcal{O}(1)))$. Taking direct image by p_1 we get a homomorphism of $\mathcal{O}(-k+1) \otimes \Gamma(\mathbb{P}^1, \mathcal{O}(1))$ into $\Gamma(\Lambda^{k-1}Q)$. This fits in the following commutative diagram

$$\begin{array}{ccc}
\mathcal{O}(-k) & \longrightarrow & \Lambda^{k-1}\Gamma(Q) \otimes \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O}(-k+1) \otimes \Gamma(\mathbb{P}^1, \mathcal{O}(1)) & \longrightarrow & (\Lambda^{k-1}Q) \otimes \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O}(-k+2) & \longrightarrow & \operatorname{coker}\Lambda^{k-1}\Gamma(Q) \otimes \mathcal{O} \to \Gamma(\Lambda^{k-1}Q) \otimes \mathcal{O}.
\end{array}$$

We wish to make two remarks here. Firstly the lower horizontal arrow is nonzero. In fact, for any point x of $\mathbb{P}^1$, the middle horizontal arrow gives a two-dimensional space of sections of $\Lambda^{k-1}Q$. This is obtained as follows. Consider the sheaf inclusion of the trivial subbundle W_x^* in $\Gamma(W)^*$ on $\mathbb{P}(W)$ and compose it with the natural homomorphism of the trivial bundle with fibre $\Gamma(W)^*$ into Q. Take the $(k-1)$-th exterior power of this map. This becomes an inclusion of $\mathcal{O}(x) = \mathcal{O}(1)$ in $\Lambda^{k-1}Q$. Thus at the Γ-level this gives the two-dimensional space of sections required. Clearly the sections of the trivial bundle $\Lambda^{k-1}(W_x^*)$ give a one-dimensional subspace of this. This is the top horizontal arrow in our diagram. Conversely let s be an element of $\Lambda^{k-1}\Gamma(Q) = \Lambda^{k-1}(\Gamma W)^*$, then its exterior product with any element of W_x^* gives an element of $\Lambda^k\Gamma(Q) = \Lambda^k(\Gamma W)^*$. If s is actually a section of the sub-bundle generated by W_x, then this exterior product should be zero at the generic point and hence 0. This implies that s belongs to $\Lambda^{k-1}(W_x)^* = \mathcal{O}(-k)$.

Secondly, since all our constructions are canonical and W is a homogeneous bundle, it follows that the lower horizontal arrow is $SL(2)$-equivariant. Now the proposition is a consequence of the following claim: If $\mathcal{O}(-n)$ admits a non-zero map into a trivial bundle, which is equivariant for the natural $SL(2)$-actions, then the rank of the trivial bundle is at least $n+1$. To prove it, use the dual map of the trivial bundle into $\mathcal{O}(n)$ and use the fact that $\Gamma(\mathcal{O}(n))$ is an irreducible $SL(2)$-module. This implies that the induced map at the Γ-level, which is nonzero by assumption, is actually injective. $\square$

5. Extra syzygies of gonal curves

We say that a curve of genus $g = 2k - 1$ is k-gonal if it carries a line bundle L of degree k whose linear system has no base points and thus yields a k-sheeted morphism π onto $\mathbb{P}^1$. In this paragraph, we prove that k-gonal curves of genus $2k - 1$ have at least $k - 1$ extra syzygies.

PROPOSITION 5.1. – *Let C be a nonhyperelliptic k-gonal curve of genus $2k - 1$, with L the special line bundle of degree k and Q_C the restriction of the tautological quotient bundle on $\mathbb{P}^{g-1}$ to the canonically imbedded curve C. Then the dimension of $H^0(C, \Lambda^{k-1}Q_C)$ is at least $\binom{g}{k-1} + k - 1$.*

Proof. – Consider the direct image V of the canonical line bundle K of C by π. The so-called trace map gives a homomorphism of V onto $K_{\mathbb{P}^1} = \mathcal{O}(-2)$. Let W be its kernel. Thus we have an exact sequence

$$0 \to W \to V \to \mathcal{O}(-2) \to 0$$

Since $\mathcal{O}(-2)$ has no nonzero sections it follows that $\Gamma(W) = \Gamma(V) = \Gamma(C, K)$. Moreover, the kernel of the evaluation map $\Gamma(C, K) \to W_p$ at any point $p \in \mathbb{P}^1$ is simply the set of sections vanishing on $\pi^{-1}(p)$, that is to say $s\Gamma(K \otimes L^{-1})$ where s is a nonzero section of L vanishing on this fibre. On computing the dimension of this space to be k by Riemann-Roch, we find that the evaluation map from $\Gamma(C, K)_{\mathbb{P}^1}$ to V is actually onto W. Thus, W is generated by global sections and we get a morphism of $\mathbb{P}(W)$ into $\mathbb{P}\Gamma(C, K)$. Finally the pull-back of V to C, namely $\pi^*\pi_*(K)$ comes with a natural homomorphism onto K. Indeed the natural surjection of $\Gamma(K)$ onto K factors through this map, which can be thought of as 'evaluation along fibres'. Thus we have a morphism from C to $\mathbb{P}(W)$ the composition of which with the above mentioned morphism from $\mathbb{P}(W) \to \mathbb{P}\Gamma(K)$ to C is the canonical imbedding. Thus our claim follows from Section 4. □

6. Proof of the theorem

In this section, we give the proof of our theorem.

We start with the

LEMMA 6.1. – *Let S be a smooth variety and E and F two vector bundles of the same rank n. Let $f : E \to F$ be a homomorphism which is generically an isomorphism, and D a subvariety of codimension 1 in S on which f has kernel of rank $\geq r$, then the degeneracy divisor of f contains D as a component of multiplicity at least r.*

Proof. – Note that the question is local and localising at the generic point of D, we may assume that S is a discrete valuation ring with maximal ideal $\mathfrak{M}$ and that f is a square matrix of nonzero determinant. Then by a proper choice of basis we may assume f to be diagonal of the form $\delta_{i,j}t^{m_i}, 0 \leq i, j \leq n$, where t is a generating parameter. Our assumption ensures $m_i > 0$ for at least r indices. Then clearly $det(f)$ is in $\mathfrak{M}^r$. Since the degeneracy locus is defined by $det(f)$, this proves our assertion. □

Before turning to the proof, we state again our

THEOREM 6.2. – *Let $g = 2k - 1 \geq 5$ be an odd integer. If the generic curve C of genus g has no extra syzygies (*i.e.. $\Gamma(\Lambda^k Q \otimes \mathcal{I}_C(1)) = 0$*), then so does any curve of genus g with maximal Clifford index, namely $k - 1$.*

Proof. – We have shown (see Section 3) that the syzygy divisor S_g is the degeneracy locus of a homomorphism of a vector bundle into another of same rank, and (see Section 5) that at the generic k-gonal curve this homomorphism has kernel of dimension at least $k-1$. Thanks to the previous lemma, this implies that the locus of k-gonal curves is contained in S_g with multiplicity at least $k-1$. By our computation in Section 3, the residual divisor has rational class zero, thus is the zero divisor (this is what we need to know about $\mathcal{M}$). Thus in $\mathcal{M}$, curves with extra syzygies are in the k-gonal divisor. Now even around curves with automorphisms, we can see by going to a covering where a universal curve exists, that the locus of curves with extra syzygies is a divisor. Since the locus of curves with automorphisms is of codimension at least two in $\mathcal{M}$, we get that even in $\mathcal{M}_g$, curves with extra syzygies are in the k-gonal divisor, thus have nonmaximal Clifford index. This proves our theorem. □

REMARK 6.3. – *We may even conclude that curves with nonmaximal Clifford index (which have extra syzygies by [GL]) are all in the k-gonal divisor. Note that this result is true (without our assumption on the generic curve), cf.* [ELMS].

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