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J. PALIS

J.-C. YOCCOZ

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RIGIDITY OF CENTRALIZERS OF DIFFEOMORPHISMS

BY J. PALIS AND J. C. YOCCOZ ⁽¹⁾

ABSTRACT. — We show that a large class of smooth diffeomorphisms in every compact boundaryless manifold have trivial centralizers; *i. e.*, the diffeomorphisms commute only with their own powers.

1. Introduction

Centralizers of diffeomorphisms play a relevant role in several topics in Dynamical Systems. This is the case when for instance we attempt to classify diffeomorphisms up to differentiable conjugacies such as in [3], [10], [17]. The same happens in certain aspects of the study of abelian group actions related to deformations-local connectedness, stability of actions and of suspended foliations. Another question of much interest, such as in dynamical bifurcations of diffeomorphisms and flows, is whether certain diffeomorphisms embed in smooth flows as their time one map. In the affirmative case, we say that such diffeomorphisms have a large centralizer since they commute with many other diffeomorphisms. We refer the reader to the references above and also to [2], [5], [7] for some discussions, results and further bibliography about the points above.

We will show in the present paper that a broad class of C^∞ diffeomorphisms of any compact, connected, boundaryless C^∞ manifold have smallest possible centralizers: they commute only with their own integer powers among all C^∞ diffeomorphisms of the manifold. In this case we say that the diffeomorphisms have trivial centralizers.

We denote the manifold by M and by $\text{Diff}^\infty(M) = \text{Diff}(M)$ its set of C^∞ diffeomorphisms endowed with the C^∞ topology. For $f \in \text{Diff}(M)$, $Z(f)$ denotes the centralizer group of f ; *i. e.*, the set of elements in $\text{Diff}(M)$ that commute with f . A question posed by Smale more than twenty years ago is whether the elements of an open and dense subset of $\text{Diff}(M)$ have trivial centralizers. In this generality, it is very hard to address the question except in the case of the circle (for which it is true [4]). We, however, go here quite a long way in showing that the answer is affirmative when the question is

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restricted to the subset $\mathfrak{A}(M)$ of diffeomorphisms satisfying Axiom A and the (strong) transversality condition.

Let us recall basic concepts before stating our results in a precise way; more details and examples may be found in [6], [8], [14], [15].

A point $x \in M$ is wandering for $f \in \text{Diff}(M)$ if there is a neighborhood U of x such that $f^n U \cap U = \emptyset$ for all integers $n \neq 0$. Otherwise we call the point x nonwandering. The set of nonwandering points for f is denoted by $\Omega(f)$.

We say that f satisfies Axiom A if $\Omega(f)$ is hyperbolic and the set of periodic points $P(f)$ is dense in $\Omega(f)$. Recall that a closed f -invariant set Λ is hyperbolic if for any Riemannian metric on M there are constants $C > 0$ and $0 < \lambda < 1$ and a (continuous) splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle of M restricted to Λ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\begin{aligned} \|Df_x^n v\| &\leq C \lambda^n \|v\| && \text{for } v \in E_x^s \\ \|Df_x^{-n} w\| &\leq C \lambda^n \|w\| && \text{for } w \in E_x^u. \end{aligned}$$

Through each point $x \in \Lambda$ we can define the stable manifold $W^s(x) = \{y; d(f^n x, f^n y) \rightarrow 0 \text{ as } n \rightarrow +\infty\}$, where d is the distance function induced by the metric. We also can define the stable set of Λ , $W^s(\Lambda) = \{y; d(f^n y, \Lambda) \rightarrow 0 \text{ as } n \rightarrow +\infty\}$. From the fact that Λ is hyperbolic we have that $W^s(x)$ is an injective immersion of some Euclidean space. Clearly, $W^s(\Omega(f)) = M$ and, when f satisfies Axiom A, we have that $W^s(\Omega(f))$ is the union of $W^s(x)$ for $x \in \Omega(f)$. Similarity for unstable manifolds and unstable sets. Another relevant fact here is Smale's spectral decomposition of $\Omega(f)$ when f satisfies Axiom A [16]. It states that $\Omega(f) = \Omega_1 \cup \dots \cup \Omega_b$, where each Ω_i is closed, f -invariant and transitive (has a dense orbit). The Ω_i are called basic sets for f and they are attractors or repellers if their stable or unstable sets are open subsets of M .

Suppose f satisfies Axiom A. We say that it also satisfies the (strong) transversality condition if $W^s(x)$ is transverse to $W^u(y)$ for all $x, y \in \Omega(f)$. As before we denote by $\mathfrak{A}(M)$ the subset of elements in $\text{Diff}(M)$ that satisfy Axiom A and the transversality condition.

It is well known that $\mathfrak{A}(M)$ is open. We denote by $\mathfrak{A}_1(M)$ the open subset of $\mathfrak{A}(M)$ formed by diffeomorphisms that exhibit either a sink (periodic attractor) or a source (periodic repeller). An important open subset of $\mathfrak{A}_1(M)$ is the one whose elements, called Morse-Smale diffeomorphisms, have their nonwandering sets made up with finitely many periodic orbits. Morse-Smale diffeomorphisms exist on every manifold. Somewhat in the other extreme, there are diffeomorphisms for which all of the ambient manifold is hyperbolic. These diffeomorphisms are defined on special manifolds, like the torus T^n , and are called Anosov diffeomorphisms. A final point, before stating our results, is that for $f \in \mathfrak{A}(M)$ there can be no cycles on $\Omega(f) = \Omega_1 \cup \dots \cup \Omega_b$, each Ω_i being a basic set. This means that there can be no subset of indices j_1, \dots, j_n such that

$$W^s(\Omega_{j_1}) \cap W^u(\Omega_{j_2}) \neq \emptyset, \dots, W^s(\Omega_{j_{n-1}}) \cap W^u(\Omega_{j_n}) \neq \emptyset$$

and

$$W^s(\Omega_{j_n}) \cap W^u(\Omega_{j_1}) \neq \emptyset.$$

We now list our main results about centralizers. Recall that $Z(f)$ denotes the centralizer of f in $\text{Diff}(M)$.

THEOREM 1 (Rigidity). — *Let $f \in \mathfrak{A}(M)$ and $g_1, g_2 \in Z(f)$. If g_1 and g_2 coincide on a non-empty open set of M , then $g_1 = g_2$.*

THEOREM 2. — *There is an open and dense subset of $\mathfrak{A}_1(M)$ whose elements have trivial centralizers.*

THEOREM 3. — (a) *Let $\dim M = 2$. There is an open and dense subset of $\mathfrak{A}(M)$ whose elements have trivial centralizers.*

(b) *Let $\dim M \geq 3$. For a residual (Baire second category) subset of f s in $\mathfrak{A}(M)$ the centralizers $Z(f)$ are trivial. For an open and dense subset of f s in $\mathfrak{A}(M)$, none of the equations $h^k = f^j$, $j, k \in \mathbb{Z}$, have non trivial solutions $h \in Z(f)$; i. e., such f s have no roots of any order.*

The proof of Theorem 1 is presented in Section 2. In Section 3 we analyse the centralizer of a non-resonant linear contraction in \mathbb{R}^n . The proofs of Theorems 2 and 3 are given in Sections 4 and 5.

In a subsequent paper [9] we deal directly with the interesting case of Anosov diffeomorphisms on tori: we show that the elements of an open and dense subset have trivial centralizers.

Our results generalize, among others, previous work of Kopell [4], Anderson [1] and the first author of this paper [7]. Kopell showed the triviality of the centralizer for an open and dense subset of diffeomorphisms of the circle. In higher dimensions, it was proved in [7] that the elements of an open and dense subset of $\mathfrak{A}(M)$ have discrete centralizer. Before, Anderson had shown this fact restricted to Morse-Smale diffeomorphisms. There is a version of these results for Axiom A flows satisfying the transversality condition: Sad [13] has shown that the elements of an open and dense subset only commute with their constant multiples. We also want to mention that our main results, Theorems 2 and 3 above, are very likely to be true for a bigger open set $\tilde{\mathfrak{A}}(M) \supset \mathfrak{A}(M)$: the set of diffeomorphisms satisfying Axiom A and having no cycles on the nonwandering set. The arguments should be very similar; the main point is to determine that the Rigidity Theorem (Theorem 1), which is valid for any $f \in \mathfrak{A}(M)$, is now true for an open and dense subset of $\tilde{\mathfrak{A}}(M)$.

2. Rigidity Theorem

2.1. ELEMENTARY PROPERTIES OF THE CENTRALIZER. — We collect some elementary properties which must be verified by any $h \in Z(f)$ commuting with $f \in \text{Diff}(M)$. They will be used quite often in the sequel without further comments.

Let $f \in \text{Diff}(M)$, $h \in Z(f)$; then, for each $n \geq 1$, the set of fixed points of f^n is h -invariant. Moreover, if p is a fixed point of f^n , the derivative $T_p h$ conjugates $T_p f^n$ and $T_{h(p)} f^n$. For $x \in M$, let

$$W^s(x) = \{ y \in M, \lim_{n \rightarrow +\infty} d(f^n x, f^n y) = 0 \},$$

$$W^u(x) = \{ y \in M, \lim_{n \rightarrow -\infty} d(f^n x, f^n y) = 0 \}.$$

Then we have

$$h(W^s(x)) = W^s(h(x)), \quad h(W^u(x)) = W^u(h(x)) \quad \text{for any } x \in M.$$

Suppose that f satisfies Axiom A and let $\Omega(f) = \Omega_1 \cup \dots \cup \Omega_l$ be the spectral decomposition of $\Omega(f)$; any $h \in Z(f)$ leaves $\Omega(f)$ invariant. Let p a periodic point in some Ω_i , $O(p)$ its orbit, $h(p) \in \Omega_j$; we must have

$$\begin{aligned} h(O(p)) &= O(h(p)), & h(W^s(O(p))) &= W^s(h(O(p))), \\ h(W^s(\Omega_i)) &= \overline{h(W^s(O(p)))} = \overline{W^s(O(h(p)))} = W^s(\Omega_j), & h(W^u(\Omega_i)) &= W^u(\Omega_j), \end{aligned}$$

and

$$h(\Omega_i) = h(W^s(\Omega_i) \cap W^u(\Omega_i)) = W^s(\Omega_j) \cap W^u(\Omega_j) = \Omega_j;$$

in conclusion, we have a group homomorphism $h \rightarrow \sigma_h$ from $Z(f)$ into the symmetric group S_l such that $h(\Omega_i) = \Omega_{\sigma_h(i)}$, $h(W^s(\Omega_i)) = W^s(\Omega_{\sigma_h(i)})$, $h(W^u(\Omega_i)) = W^u(\Omega_{\sigma_h(i)})$ for all $1 \leq i \leq l$, $h \in Z(f)$.

2.2. PROOF OF THEOREM 1. — We say that a diffeomorphism f of an Euclidean space E is a contraction if f has a unique fixed point x_0 such that $\lim_{n \rightarrow +\infty} f^n(x) = x_0$ for all $x \in E$

and all the eigenvalues of $T_{x_0} f$ have modulus strictly less than one. Anderson [1] has showed that for any contraction f , if two diffeomorphisms $g_1, g_2 \in Z(f) \subset \text{Diff}(E)$ coincide on an open set of E , then $g_1 = g_2$.

Let $f \in \mathfrak{A}(M)$, $g \in Z(f)$, and suppose that for x in an open set U of M we have $g(x) = x$. To prove Theorem 1 it is sufficient to prove that $g = \text{id}_M$.

The open set U intersects the stable manifold of some attractor of f (and the unstable manifold of some repeller of f). Suppose that for some attractor Λ , we have $g \equiv \text{id}_M$ on some open subset U_Λ of $W^s(\Lambda)$. Let p a periodic point in Λ of period k . As $W^s(O(p))$ is dense in $W^s(\Lambda)$, there exists an integer i such that $W^s(f^i(p))$ intersects U_Λ ; as $g(W^s(f^i(p))) = W^s(g(f^i(p)))$, and $g(f^i(p))$ is a periodic point, we must have $g(f^i(p)) = f^i(p)$. The restriction $f^k/W^s(f^i(p))$ is a contraction of the Euclidean space $W^s(f^i(p))$ which commutes with $g/W^s(f^i(p))$, and g coincides with $\text{id}_{W^s(f^i(p))}$ on an open set (in the Euclidean topology) of $W^s(f^i(p))$. By Anderson's result, this implies that $g/W^s(f^i(p)) = \text{id}_{W^s(f^i(p))}$, hence $g/W^s(\Lambda) = \text{id}_{W^s(\Lambda)}$. The same argument holds for a repeller. To finish the proof of Theorem 1 we need the following lemma.

LEMMA. — Let Λ, Λ' be two attractors of $f \in \mathfrak{A}(M)$ such that $\overline{W^s(\Lambda)} \cap \overline{W^s(\Lambda')} \neq \emptyset$. Then there exists a repellor Λ'' such that $W^s(\Lambda) \cap W^u(\Lambda'') \neq \emptyset \neq W^s(\Lambda') \cap W^u(\Lambda'')$.

First we indicate how the lemma implies Theorem 1. In fact, it is sufficient to argue that $g(x) = x$ for x in the stable manifold of any attractor of f ; the open set U intersects the stable manifold of an attractor Λ , hence $g|_{W^s(\Lambda)} = \text{id}_{W^s(\Lambda)}$ as we have seen above.

As the union of the stable manifolds of all attractors is dense in M if Λ' is another attractor we can find attractors $\Lambda_0 = \Lambda, \dots, \Lambda_i, \dots, \Lambda_n = \Lambda'$ such that $\overline{W^s(\Lambda_i)} \cap \overline{W^s(\Lambda_{i+1})} \neq \emptyset$. Applying the lemma, we obtain repellors $\Lambda'_0 \dots \Lambda'_{n-1}$ such that $W^s(\Lambda_i) \cap W^u(\Lambda'_i), W^u(\Lambda'_i) \cap W^s(\Lambda_{i+1})$ are non-empty open sets for $0 \leq i \leq n-1$. We then obtain successively that g coincides with id_M on $W^s(\Lambda_0), W^u(\Lambda'_0), W^s(\Lambda_1), \dots, W^s(\Lambda')$, and this shows that $g = \text{id}_M$.

Proof of the Lemma. — Let $x \in \overline{W^s(\Lambda)} \cap \overline{W^s(\Lambda')}$; x belongs to the stable manifold of some basic set Λ_0 . By the λ -lemma ([6], [8]), $\overline{W^s(\Lambda)}$ intersects $W^u(\Lambda_0)$; then, since f satisfies Axiom A, the stable manifold of some basic set Λ_1 intersects $W^u(\Lambda_0)$ and $\overline{W^s(\Lambda)}$. Iterating this construction gives basic sets Λ_j such that $W^s(\Lambda_j)$ cuts $W^u(\Lambda_{j-1})$ and $\overline{W^s(\Lambda)}$; but, there are no cycles, hence this process must stop, and the only way it can do so is to obtain at some stage $\Lambda_j = \Lambda$. As $W^s(\Lambda_i) \cap W^u(\Lambda_{i-1}) \neq \emptyset$ for $1 \leq i \leq j$, again by the λ -lemma we must have $W^s(\Lambda) \cap W^u(\Lambda_0) \neq \emptyset$ and $W^s(\Lambda_0) \subset \overline{W^s(\Lambda)}$. Similarly $W^s(\Lambda_0) \subset \overline{W^s(\Lambda')}$. By the transversality condition, $W^s(\Lambda_0)$ must intersect the unstable manifold of some repellor Λ'' , which has the properties required by the lemma.

3. Non-resonant Linear Contractions in \mathbb{R}^n

3.1. GENERICITY. — We say that a linear automorphism $A \in GL(n, \mathbb{R})$ is a linear contraction if the eigenvalues $\lambda_1 \dots \lambda_n$ of A have modulus strictly less than 1; A is non-resonant if the λ_i are distinct and:

$$(C) \quad \forall (j_1 \dots j_n) \in \mathbb{N}^n, \text{ s. t. } \sum j_k \geq 2, \quad \forall 1 \leq i \leq n \\ \lambda_i \neq \lambda_1^{j_1} \dots \lambda_n^{j_n}$$

In the set \mathcal{E} of all linear contractions, those who are non-resonant form a set \mathcal{E}' which is, locally in \mathcal{E} , the complement of finitely many submanifolds: for if $A \in \mathcal{E}$, $\lambda = \text{Max} |\lambda_i|$, $\lambda' = \text{min} |\lambda_i|$, and $m \in \mathbb{N}$ is the least integer such that $\lambda^m < \lambda'$, then condition (C) is automatically satisfied in a neighbourhood of A when $\sum j_k \geq m$. In particular, \mathcal{E}' is open and dense in \mathcal{E} .

3.2. RIGIDITY. — Let $A \in \mathcal{E}'$, $\lambda_1 \dots \lambda_r$ the real eigenvalues of A , $\lambda_{r+1}, \bar{\lambda}_{r+1}, \dots, \lambda_{r+s}, \bar{\lambda}_{r+s}$ the complex eigenvalues of A , with $r+2s=n$; one can find in \mathbb{R}^n coordinates $x_1, \dots, x_r \in \mathbb{R}, x_{r+1}, \dots, x_{r+s} \in \mathbb{C}$ such that:

$$A(x_1, \dots, x_{r+s}) = (\lambda_1 x_1, \dots, \lambda_{r+s} x_{r+s}).$$

Kopell [4] has proved that if $h \in \text{Diff}^\infty(\mathbb{R}^n)$ commutes with A , then h must be linear. On the other hand, $M \in GL(n, \mathbb{R})$ commute with A if and only if, for some

$(\mu_1, \dots, \mu_{r+s}) \in (\mathbb{R}^*)^r \times (\mathbb{C}^*)^s$:

$$M(x_1, \dots, x_{r+s}) = (\mu_1 x_1, \dots, \mu_{r+s} x_{r+s}),$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Let D be the set of such elements $M \in GL(n, \mathbb{R})$. It is an abelian Lie group isomorphic to $\mathbb{R}^{r+s} \times (\mathbb{Z}/2\mathbb{Z})^r \times (S^1)^s$. When B belongs to a small neighbourhood of A in D , the centralizer $Z(B)$ of B in $\text{Diff}^\infty(\mathbb{R}^n)$ is still equal to D . For our purpose, we want to have an isomorphism Θ_B of $Z(B) = D$ onto $\mathbb{R}^{r+s} \times (\mathbb{Z}/2\mathbb{Z})^r \times (S^1)^s$ with the property that $\Theta_B(B) = \Theta_A(A)$. This is made explicit in the next paragraph.

3.3. THE CENTRALIZER AS AN ABSTRACT GROUP. — For $r, s \in \mathbb{N}$, let $Z_{r,s}$ be the non-connected abelian Lie group $\mathbb{R}^{r+s} \times (\mathbb{Z}/2\mathbb{Z})^r \times (S^1)^s$. If Σ is the hyperplane in \mathbb{R}^{r+s} determined by $\sum_{i=1}^{r+s} \theta_i = 0$, we define a surjective homomorphism $\chi: Z_{r,s} \rightarrow \Sigma$ by:

$$\chi(\theta_1, \dots, \theta_{r+s}, \varepsilon_1, \dots, \varepsilon_{r+s}) = (\theta'_1, \dots, \theta'_{r+s})$$

with

$$\theta'_i = \theta_i - \frac{1}{r+s} \sum_{j=1}^{r+s} \theta_j.$$

Let

$$\mathcal{L} = \{(\theta_1, \dots, \theta_{r+s}, \varepsilon_1, \dots, \varepsilon_{r+s}) \in Z_{r,s} : \theta_i = 1, \varepsilon_j = 1 \forall 1 \leq i \leq r+s, \forall r < j \leq r+s\};$$

\mathcal{L} is a finite set in a natural one to one correspondence with $(\mathbb{Z}/2\mathbb{Z})^r$. Let $\varepsilon \in \mathcal{L}$; the cyclic subgroup $\langle \varepsilon \rangle$ generated by ε is discrete in $Z_{r,s}$; the quotient $Z_{r,s,\varepsilon} = Z_{r,s}/\langle \varepsilon \rangle$ is a non-connected abelian Lie group.

Observe that \mathcal{L} is contained in the kernel of χ . Actually, for any $\varepsilon \in \mathcal{L}$, the group $Z'_{r,s,\varepsilon} = \text{Ker } \chi/(\varepsilon)$ is the maximal compact subgroup of $Z_{r,s,\varepsilon}$ and the quotient $Z_{r,s,\varepsilon}/Z'_{r,s,\varepsilon}$ is isomorphic to $\Sigma \simeq \mathbb{R}^{r+s-1}$.

Let \mathcal{P}_{r+s} the set of subsets of $\{1, \dots, r+s\}$; we write $\emptyset \in \mathcal{P}_{r+s}$ for the empty subset and 1 for the total subset. For $\varepsilon \in \mathcal{L}$, $z \in Z_{r,s,\varepsilon}$, $\chi(z) = (\theta'_1, \dots, \theta'_{r+s})$, we define $\mathcal{P}(z) \in \mathcal{P}_{r+s}$ by:

$$\mathcal{P}(z) = \{i \in \{1, \dots, r+s\}, \theta'_i = \text{Min}_{1 \leq j \leq r+s} \theta'_j\}.$$

Observe that $\mathcal{P}(z) \neq \emptyset$, and $\mathcal{P}(z) = 1$ if and only if $z \in Z'_{r,s,\varepsilon}$.

3.4. Let $A \in \mathcal{E}'$, r, s, λ_i, x_i defined as in 3.2; we have seen that $Z(A)$ coincides with the subset D of $GL(n, \mathbb{R})$ formed by the automorphisms that are diagonalizable in the coordinates (x_1, \dots, x_{r+s}) . Let \mathcal{V} be a connected neighbourhood of A in D sufficiently small so that for $B \in \mathcal{V}$ with eigenvalues $\lambda'_1, \dots, \lambda'_{r+s}$, we have:

$$\text{sgn } \lambda'_i = \text{sgn } \lambda_i \quad \text{for } 1 \leq i \leq r, \quad \text{sgn } \text{Im } \lambda'_j = \text{sgn } \text{Im } \lambda'_j \quad \text{for } r < j \leq r+s.$$

Define on \mathcal{V} a smooth map $\rho = (\rho_1 \dots \rho_{r+s}) : \mathcal{V} \rightarrow \mathbb{R}^r \times \mathbb{C}^s$ by:

$$\begin{aligned} \exp \rho_i(\mathbf{B}) &= |\lambda'_i(\mathbf{B})|, & 1 \leq i \leq r \\ \exp \rho_i(\mathbf{B}) &= \lambda'_i(\mathbf{B}), & r < i \leq r+s. \end{aligned}$$

Putting

$$\theta_i = \frac{\text{Log} |\mu_i|}{\text{Log} |\lambda'_i(\mathbf{B})|}, \quad \varepsilon_i = \frac{\mu_i}{\exp \theta_i \rho_i(\mathbf{B})},$$

we can define an isomorphism $\Theta_{\mathbf{B}}$ of D onto $Z_{r,s}$ by

$$\Theta_{\mathbf{B}}(\text{diag}(\mu_1, \dots, \mu_{r+s})) = (\theta_1, \dots, \theta_{r+s}, \varepsilon_1, \dots, \varepsilon_{r+s})$$

such that $\Theta_{\mathbf{B}}(\mathbf{B}) = (1, \dots, 1, \text{sgn} \lambda'_1, \dots, \text{sgn} \lambda'_r, 1, \dots, 1) = \Theta_{\mathbf{A}}(\mathbf{A}) \in \mathcal{L}$. Moreover, if \mathcal{V} is small enough, $\mathcal{V} \subset \mathcal{E}'$ and hence D is the centralizer of any $\mathbf{B} \in \mathcal{V}$. The image of any \mathbf{B} in \mathcal{V} by $\Theta_{\mathbf{B}}$ is a fixed element $\varepsilon \in \mathcal{L}$; thus the cyclic subgroup of $Z(\mathbf{B}) = D$ generated by \mathbf{B} is discrete and the quotient is naturally isomorphic to $Z_{r,s,\varepsilon}$.

In conclusion, when we consider a small open set \mathcal{V} in $D \cap \mathcal{E}'$, the integers r, s and the element ε are constant on \mathcal{V} ; we then will write $Z_{r,s} = Z$, $Z_{r,s,\varepsilon} = Z_0$, $Z'_{r,s,\varepsilon} = Z_1$. They are non-connected abelian Lie groups; Z_1 is the maximal compact subgroup of Z_0 ; Z is naturally isomorphic to the centralizer of any $\mathbf{B} \in \mathcal{V}$, Z_0 is canonically isomorphic to the quotient $Z(\mathbf{B})/(\mathbf{B})$ for any $\mathbf{B} \in \mathcal{V}$.

3.5. THE SPACE OF ORBITS OF A CONTRACTION IN $\mathbb{R}^n - \{0\}$. — Let \mathcal{V} be a small open set in $D \cap \mathcal{E}'$. For $\mathbf{A} \in \mathcal{V}$, we define a map $\Phi_{\mathbf{A}}$ of $\mathbb{R} \times (\mathbb{R}^n - \{0\})$ onto \mathbb{R}_*^+ by:

$$\Phi_{\mathbf{A}}(t, x_1, \dots, x_{r+s}) = \sum_{i=1}^{r+s} \frac{|x_i|^2}{|\lambda_i|^{2t}},$$

where λ_i are the eigenvalues of \mathbf{A} . Clearly $\Phi_{\mathbf{A}}$ is smooth; moreover, by the implicit function theorem, the equation $\Phi_{\mathbf{A}}(t(x), x) = 1$ defines a C^∞ map $t : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}$. Clearly $t(\mathbf{A}x) = t(x) + 1$. For any $u \in \mathbb{R}$, the set $F_u = \{x \in \mathbb{R}^n - \{0\}; u \leq t(x) \leq u+1\}$ is a fundamental domain for \mathbf{A} in $\mathbb{R}^n - \{0\}$. By identifying via \mathbf{A} the two boundary components of F_u , we obtain a compact connected manifold which we denote by $S_{\mathbf{A}}$ and call it the space of orbits of \mathbf{A} in $\mathbb{R}^n - \{0\}$.

For \mathbf{B} in \mathcal{V} , we construct a canonical diffeomorphism of $S_{\mathbf{B}}$ onto $S_{\mathbf{A}}$ as follows. Let $\lambda'_1, \dots, \lambda'_{r+s}$ be the eigenvalues of \mathbf{B} , ρ_i the value of the main branch of the logarithm at λ'_i/λ_i ; define a smooth diffeomorphism $H_{\mathbf{B}}$ of $\mathbb{R}^n - \{0\}$ by:

$$H_{\mathbf{B}}(x) = H_{\mathbf{B}}(x_1, \dots, x_{r+s}) = (x_1 \exp t(x) \rho_1, x_2 \exp t(x) \rho_2, \dots)$$

Then, for $x \in \mathbb{R}^n - \{0\}$:

$$H_{\mathbf{B}} \mathbf{A}(x) = H_{\mathbf{B}}(\lambda_1 x_1, \dots, \lambda_{r+s} x_{r+s}) = (\lambda_i x_i \exp(t(x) + 1) \rho_i) = (\lambda'_i x_i \exp t(x) \rho_i) = \mathbf{B} H_{\mathbf{B}}(x).$$

Hence H_B conjugates A and B ; passing to the respective quotients, we obtain a canonical identification of S_A and S_B .

Remark. — In general, if $C \in \mathcal{V}$, $C \neq B$, $H_B^{-1}CH_B$ is not linear. Let S be the space of orbits of $A \in \mathcal{V}$ considered as an abstract manifold; it is also the space of orbits of any $B \in \mathcal{V}$. For any $A \in \mathcal{V}$, the centralizer $Z(A) = D$ acts canonically on S_A , and A acts trivially. This gives an action of the abstract abelian Lie group Z_0 on S (depending on A). This action is smooth and depends continuously on $A \in \mathcal{V}$ in the C^∞ topology on compact subsets of Z_0 .

3.6. Keeping the same notation as above, let W_1 be the x_1 -axis, i.e., the eigenspace associated to λ_1 . For $E \in \mathcal{P}_{r+s}$, let $W_E = \{0\}$ if $E = \emptyset$, $W_E = \bigoplus_{i \in E} W_i$ if $E \neq \emptyset$. Let also

W be defined by

$$W = \mathbb{R}^n - \bigcup_{E \neq 1} W_E = \{(x_i) \in \mathbb{R}^r \times \mathbb{C}^s, \prod x_i \neq 0\}.$$

Then for any A in \mathcal{V} , any $E \in \mathcal{P}_{r+s}$, W_E is A -invariant, hence it determines a submanifold \tilde{W}_E of S_A ; as $H_B(W_E) = W_E$ for $B \in \mathcal{V}$, \tilde{W}_E is well defined as a submanifold of S . The same is true for W , which gives in S the open dense subset $\tilde{W} = S - \bigcup_{E \neq 1} \tilde{W}_E$.

For any $B \in \mathcal{V}$, the action of Z_0 on S induced by $Z(B)$ leaves invariant each submanifold \tilde{W}_E and \tilde{W} , and it is free and transitive on \tilde{W} .

For $E \in \mathcal{P}_{r+s}$, let E' be the complementary subset of E in $\{1, \dots, r+s\}$. Let p_E be the projection in \mathbb{R}^n with kernel $W_{E'}$ and image W_E ; as it commutes with the elements of D , it induces, for any $A \in \mathcal{V}$, a smooth map \tilde{p}_E from $S_A - \tilde{W}_{E'}$ onto \tilde{W}_E , hence a smooth map \tilde{p}_E^A from $S - \tilde{W}_{E'}$ onto \tilde{W}_E . However, as p_E does not commute with H_B for $B \in \mathcal{V}$, we do not have in general $\tilde{p}_E^A = \tilde{p}_E^B$ for A, B in \mathcal{V} . As p_E commutes with the elements of D , \tilde{p}_E^A commutes with the action of Z_0 on S induced by $Z(A)$.

3.7. ACTION OF AN ELEMENT IN THE NON-COMPACT PART OF THE CENTRALIZER. TWO BASIC LEMMAS. — For $A \in \mathcal{V}$, \mathcal{V} a small open set in $\mathcal{E}' \cap D$, $h \in Z_0$, $x \in S$, we denote by $h.x$ or $h_A.x$ the image of x under the action of h associated to A . Recall that we have defined in 3.3 an element $\mathcal{P}(h) \in \mathcal{P}_{r+s}$, and it is not 1 iff $h \in Z_0 - Z_1$. In this case, we write $W_h = \tilde{W}_{\mathcal{P}(h)}$, $W'_h = \tilde{W}_{(\mathcal{P}(h))'}$, $p_h = \tilde{p}_{\mathcal{P}(h)}^A$ to simplify the notation.

LEMMA 1. — Let $A \in \mathcal{V}$, $h \in Z_0 - Z_1$. Then, $\lim_{n \rightarrow +\infty} d(h^n.x, h^n.p_h(x)) = 0$ for any x in $S - W'_h$, and there exists a sequence of integers (n_k) such that $\lim_{k \rightarrow +\infty} h^{n_k}.x = p_h(x)$.

Proof. — The first assertion is easily deduced by calculation from the definition of $\mathcal{P}(h)$. To prove the second assertion, let H be a representative of h in $Z(A)$ and let A_h, H_h be the restriction of A, H to $W_{\mathcal{P}(h)}$. If we repeat for $A_h, Z(A_h), W_{\mathcal{P}(h)}$ the constructions that we have done for $A, Z(A), \mathbb{R}^n$, we obtain abstract Lie groups $Z' \simeq Z(A_h)$, $Z'_0 \simeq Z(A_h)/(A_h)$, Z'_1 the maximal compact subgroup of Z'_0 . Moreover, by definition of $\mathcal{P}(h)$, the image h' of H_h in Z'_0 belongs to Z'_1 ; as this group is compact, there is an

increasing sequence of integers n_k such that $\lim_{k \rightarrow +\infty} (h')^{n_k} = e_{Z'_1}$ (identity map on Z'_1).

Then, as $h'_{A_h} \cdot z = h_A \cdot z$ for $z \in W_h$:

$$\lim_{k \rightarrow +\infty} h'^{n_k} \cdot x = \lim_{k \rightarrow +\infty} h'^{n_k} p_h(x) = \lim_{k \rightarrow +\infty} (h')^{n_k} p_h(x) = p_h(x). \quad \blacksquare$$

The second lemma is similar to the first: it describes the action of elements of $Z_0 - Z_1$ on the tangent space level.

Let $A \in \mathcal{V}$; as the subspaces W_E of \mathbb{R}^n are A -invariant, they define for any $x \in S$ subspaces of the tangent space $T_x S \simeq \mathbb{R}^n$. We say that a subspace V of $T_x S$ is transverse to some W_E if

$$\dim V \cap W_E = \text{Max}(0, \dim V + \dim W_E - n).$$

LEMMA 2. — Let $A \in \mathcal{V}$, $h \in Z_0 - Z_1$, (n_k) a sequence of integers which satisfy the conclusion of the Lemma 1; suppose that for some $x \in S - W'_h$ and some subspace V of $T_x S$ transverse to W'_h , we have

$$\lim_{k \rightarrow +\infty} T_x h^{n_k}(V) = V_0 \subset T_{p_h(x)} S.$$

Then, depending on the respective dimensions, we have $V_0 \subset W_h$ or $V_0 \supset W_h$.

Proof. — Let $k = \min(\dim V, \dim W_h)$. The transversality hypothesis allows us to choose e_1, \dots, e_k in V such that:

- if $\dim W_h \leq \dim V$, $(T_x p_h(e_i))$ is a basis for W_h ;
- if $\dim W_h \geq \dim V$, $(T_x p_h(e_i))$ are linearly independent in W_h hence (e_i) form a basis for V .

By construction, $e_i \notin W'_h$, hence $\lim_{k \rightarrow +\infty} T_x h^{n_k}(e_i) = T_x p_h(e_i)$; the properties of the family $(T_x p_h(e_i))$ then imply the lemma. \blacksquare

4. Proofs of Theorems 2 and 3. Preliminary Set Up

4.1. The set $\mathfrak{A}(M)$ is open in $\text{Diff}(M)$; any of its connected components is open and formed by topologically equivalent diffeomorphisms; the set $\mathfrak{A}_1(M)$ is the union of some of these components.

Let \mathcal{C} be a component of $\mathfrak{A}(M)$; for N an integer ≥ 1 , let \mathcal{C}_N be the set of $f \in \mathcal{C}$ which verify:

(i) for any $k \leq N$, if $f^k(p) = p$ and $T_p M = E^s \oplus E^u$ is the hyperbolic decomposition of $T_p M$ at p , the $T_p f^k/E^s$ and $T_p f^{-k}/E^u$ are non-resonant linear contractions;

(ii) if p, p' are periodic points of the same period $k \leq N$, $T_p f^k$ and $T_{p'} f^k$ are not conjugate in $\text{GL}(n, \mathbb{R})$ unless p, p' belong to the same orbit.

From the transversality theory, and the density of \mathcal{E}' in \mathcal{E} , we see that for every $N \geq 1$, \mathcal{C}_N is an open and dense subset of \mathcal{C} .

As elements of \mathcal{C} are topologically conjugate, we can choose an integer $N = N(\mathcal{C})$ such that all basic sets of any $f \in \mathcal{C}$ contains a periodic point of period less than or equal to N ; we fix such N in the sequel.

4.2. LOCALIZATION. — As Theorems 2 and 3 are local assertions in $\mathfrak{A}(M)$ [resp. $\mathfrak{A}_1(M)$], it is sufficient to prove that, given any component \mathcal{C} of $\mathfrak{A}(M)$ [resp. $\mathfrak{A}_1(M)$] and any $f_0 \in \mathcal{C}_{N(\mathcal{C})}$, these theorems hold in a sufficiently small neighbourhood \mathcal{U} of f_0 in $\mathcal{C}_{N(\mathcal{C})}$. So we fix such an f_0 and take a connected neighbourhood \mathcal{U} of f_0 in $\mathcal{C}_{N(\mathcal{C})}$, which we can assume arbitrarily small.

In particular, by structural stability of these diffeomorphisms ([11], [12]), we can take a continuous map H from \mathcal{U} to $\text{Homeo}(M)$ such that $H(f_0) = \text{id}_M$ and $H(f) \circ f_0 \circ H(f)^{-1} = f$ for $f \in \mathcal{U}$. Let $\Omega = \Omega_1 \cup \dots \cup \Omega_l$ the spectral decomposition of the nonwandering set of f_0 ; choose a periodic point p_i in Ω_i of period $k_i \leq N$. For $f \in \mathcal{U}$, define $p_i(f) = H(f)(p_i)$, $\Omega_i(f) = H(f)(\Omega_i)$; then $\Omega(f) = H(f)(\Omega) = \cup \Omega_i(f)$ is the spectral decomposition of the nonwandering set of f , and $p_i(f)$ is a periodic point in $\Omega_i(f)$ of period $k_i \leq N$ which depends continuously on $f \in \mathcal{U}$. When there is no risk of confusion, we simply denote $p_i = p_i(f)$, $\Omega_i = \Omega_i(f)$ for a fixed $f \in \mathcal{U}$.

LEMMA. — For $f \in \mathcal{U}$, $h \in Z(f)$, $1 \leq i \leq l$, we have:

$$\begin{aligned} h(O(p_i)) &= O(p_i), & h(\Omega_i) &= \Omega_i, & h(W^s(\Omega_i)) &= W^s(\Omega_i) \\ h(W^u(\Omega_i)) &= W^u(\Omega_i), & h(W^s(O(p_i))) &= W^s(O(p_i)), \\ h(W^u(O(p_i))) &= W^u(O(p_i)). \end{aligned}$$

Proof. — This results immediately from (2.1) combined with property (ii) of \mathcal{C}_N stated above in (4.1). ■

4.3. LINEARIZATION OF CONTRACTIONS. APPLICATIONS. — We will use a parametric version of the following result of Sternberg: If f is a contraction of an Euclidean space E such that the derivative of f at the fixed point is a non-resonant linear contraction, then f is C^∞ conjugate to this derivative.

The parametric version, proved in [1], says that the conjugacy can be chosen to depend continuously on f in the C^∞ topology on compact subsets of E . This have the following consequences.

In the case of Theorem 2: Consider a small open set $\mathcal{U} \subset \mathfrak{A}_1(M)$ as above; we can assume that $p_1 = p$ is a sink of period k . For any $f \in \mathcal{U}$, there is an embedding $\mathcal{H}(f)$ of \mathbb{R}^n into M , with the following properties

- (i) $\mathcal{H}(f)(\mathbb{R}^n) = W_f^s(p(f)) = W^s(p)$;
- (ii) $\mathcal{H}(f)$ depends continuously on f in the C^∞ topology on compact subsets of \mathbb{R}^n ;
- (iii) there exists $r, s \in \mathbb{N}$ with $r + 2s = n$, coordinates $x_1, \dots, x_{r+s} \in \mathbb{R}^r \times \mathbb{C}^s$ in \mathbb{R}^n , and a continuous map $\lambda = (\lambda_1, \dots, \lambda_{r+s})$: $\mathcal{U} \rightarrow (\mathbb{R}^*)^r \times (\mathbb{C}^* - \mathbb{R}^*)^s$ such that

$A(f) = \mathcal{H}(f)^{-1} \circ f^k / W^s(p) \circ \mathcal{H}(f)$ is a non-resonant linear contraction of \mathbb{R}^n which is diagonalizable in the coordinates (x_i) with eigenvalues $\lambda_i(f)$.

In the case of Theorem 3: Consider a small open set $\mathcal{U} \subset \mathfrak{U}(M)$; we can assume that Ω_1 is an attractor and $p = p_1$ is a periodic point of period k in Ω_1 . Let n_1, n_2 be the respective dimensions of E_p^s, E_p^u ($n_1 + n_2 = n$).

There exists for $f \in \mathcal{U}$, $i = 1, 2$, an immersion $\mathcal{H}_i(f)$ of \mathbb{R}^{n_i} into M , with the following properties:

(i) $\mathcal{H}_1(f)(\mathbb{R}^{n_1}) = W^s(p)$, $\mathcal{H}_2(f)(\mathbb{R}^{n_2}) = W^u(p)$.

(ii) Continuous dependence on f (as in Case 1).

(iii) There are integers r_i, s_i with $r_i + 2s_i = n_i$, coordinates $(x_1, \dots, x_{r_1+s_1}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{s_1}$ in \mathbb{R}^{n_1} , $(x'_1, \dots, x'_{r_2+s_2}) \in \mathbb{R}^{r_2} \times \mathbb{C}^{s_2}$ in \mathbb{R}^{n_2} , continuous maps $\lambda: \mathcal{U} \rightarrow (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^* - \mathbb{R}^*)^{s_1}$ and $\lambda': \mathcal{U} \rightarrow (\mathbb{R}^*)^{r_2} \times (\mathbb{C}^* - \mathbb{R}^*)^{s_2}$ such that $A_1(f) = \mathcal{H}_1(f)^{-1} \circ f^k / W^s(p) \circ \mathcal{H}_1(f)$, resp. $A_2(f) = \mathcal{H}_2(f)^{-1} \circ f^{-k} / W^u(p) \circ \mathcal{H}_2(f)$, is a non-resonant linear contraction of \mathbb{R}^{n_1} (resp. \mathbb{R}^{n_2}), is diagonalizable in the basis (x_i) , resp. (x'_i) , and have eigenvalues $(\lambda_i(f))$, resp. $(\lambda'_i(f))$.

When f varies in \mathcal{U} , $A(f)$, resp. $A_i(f)$, $i = 1, 2$, depends continuously on f and stays diagonal in a fixed basis: this is the situation that we have studied in Section 3. We call D (resp. D_i) the set of matrices in $GL(n, \mathbb{R})$, resp. $GL(n_i, \mathbb{R})$, which are diagonal in this special basis on \mathbb{R}^n , resp. \mathbb{R}^{n_i} .

In Theorem 2, we call S the abstract space of orbits of $A(f)$ in $\mathbb{R}^n - \{0\}$ for any $f \in \mathcal{U}$; it can also be identified, via $\mathcal{H}(f)$, with the space S_f of orbits of f^k in $W^s(p) - \{p\}$. The centralizer $Z(A(f))$ is identified as in (3.4) with a fixed abstract Lie group Z , $Z(A(f))/(A(f))$ is identified with a fixed quotient $Z_0 = Z/(\epsilon)$ of Z , and we call Z_1 the maximal compact subgroup of Z_0 . For any $f \in \mathcal{U}$, the identification of S_f with S and of $Z(A(f))/(A(f))$ with Z_0 gives an action of Z_0 on S which depends continuously on $f \in \mathcal{U}$. The projection maps $\tilde{p}_E^f = \tilde{p}_E^{A(f)}: S - \tilde{W}_E \rightarrow \tilde{W}_E$ are defined for $f \in \mathcal{U}$ and $E \in \mathcal{P}_{r+s}$ as in (3.6), and they depend continuously on f .

In Theorem 3, our basic space of reference will be the product $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, together with the linear action of $D_1 \times D_2$ on it. For $f \in \mathcal{U}$, the subgroup generated by $(A_1(f), (A_2(f))^{-1})$ in $D_1 \times D_2$ is discrete, and the quotient $D_1 \times D_2 / (A_1(f), (A_2(f))^{-1})$ has a natural identification with a fixed (non-connected) Lie group Z_0 ; we call again Z_1 the maximal compact subgroup of Z_0 .

In any case, suppose that $f \in \mathcal{U}$, $h \in Z(f)$. By the lemma in (4.2), $h(O(p)) = O(p)$; therefore there is a unique $0 \leq i < k$ such that, putting $h' = h \circ f^i$, we have

$$h'(p) = p, \quad h'(W^s(p)) = W^s(p), \quad h'(W^u(p)) = W^u(p).$$

Conjugating h' by $\mathcal{H}(f)$ [resp. $\mathcal{H}_i(f)$], we obtain an element of $Z(A(f))$ [resp. of $Z(A_i(f))$]; we denote \bar{h} the projection of this element in Z_0 .

LEMMA. — h is a power of f if and only if $\bar{h} = 1_{Z_0}$.

Proof. — If $h = f^n$, clearly $\bar{h} = 1_{Z_0}$. Suppose that $\bar{h} = 1_{Z_0}$; this means that h' coincide with some iterate f^t of f on $W^s(p)$, hence on $W^s(\Omega) = \overline{W^s(O(p))}$; by Theorem 1, we must have $h' = f^t$, hence $h = f^{t-i}$.

4.4. A KEY PROPOSITION. PROOF OF THEOREM 3. — In Theorem 3, for $f \in \mathcal{U}$, let $\mathcal{J}(f)$ be the set of homoclinic points related to p ; i. e., $\mathcal{J}(f) = W^s(p) \cap W^u(p) - \{p\}$. Define a map φ from $\mathcal{J}(f)$ into $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ by:

$$\varphi(q) = (\mathcal{H}_1(f)^{-1}(q), \mathcal{H}_2(f)^{-1}(q)).$$

The map φ is injective; the image $\tilde{\mathcal{J}}(f) = \varphi(\mathcal{J}(f))$ of \mathcal{J} in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is discrete and closed because $W^s(p)$ and $W^u(p)$ intersect transversally. As \mathcal{J} is f -invariant, $\tilde{\mathcal{J}}$ is invariant under the linear transformation $(A_1, A_2^{-1}) \in D_1 \times D_2$; hence it makes sense to ask if $\tilde{\mathcal{J}}$ is invariant by some element $h \in Z_0 = D_1 \times D_2 / \{(A_1, A_2^{-1})\}$.

Observe that if $h \in Z(f)$, and h' is defined as in (4.3), we must have $h'(W^s(p) \cap W^u(p)) = W^s(p) \cap W^u(p)$, hence $h'(\mathcal{J}(f)) = \mathcal{J}(f)$ and $\bar{h}(\tilde{\mathcal{J}}(f)) = \tilde{\mathcal{J}}(f)$.

PROPOSITION 1. — (a) For a dense open set of $f \in \mathcal{U}$, no $\bar{h} \in Z_1$, $\bar{h} \neq 1_{Z_1}$, leaves $\tilde{\mathcal{J}}(f)$ invariant.

(b) If $\dim M = 2$, for any $f \in \mathcal{U}$, no $\bar{h} \in Z_0 - Z_1$ leaves $\tilde{\mathcal{J}}(f)$ invariant.

(c) If $\dim M \geq 3$, for a residual set of f 's in \mathcal{U} , no $\bar{h} \in Z_0 - Z_1$ leaves $\tilde{\mathcal{J}}(f)$ invariant.

We give the proof of Proposition 1 in Section 5. It is clear that Proposition 1 and the lemma in (4.3) imply Theorem 3: the only non trivial remark to make is that if $h \in Z(f)$ verifies $h^k = f^j$ for some $k \geq 2$, $j \in \mathbb{Z}$, then \bar{h} [as defined in (4.3)] belongs to Z_1 , and part (a) of Proposition 1 applies.

4.5. ANOTHER KEY PROPOSITION. PROOF OF THEOREM 2. — The proof of Theorem 2 is subdivided in 3 subcases:

Subcase 1. — $W^s(p) - \{p\}$ is not contained in the unstable manifold of a single repellor.

Subcase 2. — $W^s(p) - \{p\}$ is contained in the unstable manifold of a source (periodic orbit).

Subcase 3. — $W^s(p) - \{p\}$ is contained in the unstable manifold of a repellor which is not a periodic orbit.

Clearly, one of the three cases must be true for each $f \in \mathcal{U}$. In Subcase 1, the complement in $W^s(p) - \{p\}$ of the unstable manifolds of the repellers of f is a non empty, nowhere dense, closed, f^k -invariant subset $\mathcal{J}(f)$. It determines in the space S of f^k -orbits a non empty, nowhere dense, closed subset $\tilde{\mathcal{J}}(f)$. If $h \in Z(f)$ and h', \bar{h} are defined as in (4.3), then we know from (2.1) that h' permutes the repellers and their unstable manifolds, and hence leaves $\mathcal{J}(f)$ invariant; consequently $\tilde{\mathcal{J}}(f)$ is \bar{h} -invariant in S .

In Subcase 3, the unstable manifold of the repellor is foliated by the unstable manifolds of the points of the repellor. Call $\mathcal{J}(f)$ the restriction of this foliation to $W^s(p) - \{p\}$; as it is f^k -invariant, it determines a foliation $\tilde{\mathcal{J}}(f)$ of S . If $h \in Z(f)$, by (2.1), h' must preserve $\mathcal{J}(f)$ and hence $\tilde{\mathcal{J}}(f)$ is \bar{h} invariant in S .

In subcase 2, first observe that the inclusion $W^s(p) - \{p\} \subset W^u(q) - \{q\}$ for some source q implies that $W^s(p) - \{p\} = W^u(q) - \{q\}$. In fact, if S , resp. S' , is the space of f^k -orbits in $W^s(p) - \{p\}$, resp. $W^u(q) - \{q\}$, the inclusion implies that S is a compact open subset of the space S' which is connected, hence $S = S'$ and the equality must

hold. Also, we can perform for $f^{-k}/W^u(q)$, $W^u(q)$ the same construction that we have done in (4.3) for $f^k/W^s(p)$, $W^s(p)$: When $f \in \mathcal{U}$, the centralizer $Z(f^{-k}/W^u(q))$ is naturally isomorphic to some fixed (non-connected) Lie group Z' , the quotient of $Z(f^{-k}/W^u(q))$ by the cyclic group generated by f^{-k} is naturally isomorphic to some fixed quotient Z'_0 of Z' , and we call Z'_1 the maximal compact subgroup of Z'_0 . Any $f \in \mathcal{U}$ determines an action of Z'_0 on the space of orbits S (and an action of Z_0 on S). If $f \in \mathcal{U}$, $h \in Z(f)$, and h' are defined as in (4.3), the restrictions of h' determine elements in $Z(f^k/W^s(p))$ and $Z(f^{-k}/W^u(q))$, and hence one element \bar{h} in Z_0 and one element \bar{h}' in Z'_0 , such that the actions of \bar{h} and of \bar{h}' on S are identical.

PROPOSITION 2. — *In any of the subcases above, there is a dense and open subset of f 's in \mathcal{U} such that*

- in Subcases 1 or 3, no element \bar{h} in $Z_0 - \{1_{Z_0}\}$ leaves $\tilde{\mathcal{F}}(f)$ invariant;
- in Subcase 2, if $\bar{h} \in Z_0$, $\bar{h}' \in Z'_0$ have identical actions on S , then $\bar{h} = 1_{Z_0}$ and $\bar{h}' = 1_{Z'_0}$.

In view of the previous discussion and of the lemma in (4.3), Proposition 2 clearly implies Theorem 2.

5. Proof of Propositions 1 and 2

5.1. The proofs of Proposition 1 and the three subcases of Proposition 2 are quite similar. In each, it is necessary to distinguish whether $\bar{h} \in Z_0 - Z_1$, or $\bar{h} \in Z_1 - \{1_{Z_1}\}$.

The proof of the latter case uses two facts about compact (not necessarily connected) Lie groups, which we now recall.

LEMMA 1. — *In a compact (not necessarily connected) Lie group G , any strictly decreasing sequence $G_0 \supset G_1 \supset G_2 \dots$ of closed subgroups is finite.*

Proof. — In fact, at any stage, either the dimension or the number of connected components (which is finite) must strictly decrease. ■

LEMMA 2. — *Let G be a compact (not necessarily connected) abelian Lie group, G_0 a closed subgroup. There is a compact set $K \subset G - G_0$ such that any $h \in G - G_0$, has a power in K .*

Proof. — In G/G_0 , take a small open neighbourhood U of $1_{G/G_0}$ which does not contain non trivial subgroups; the inverse image of $G/G_0 - U$ under the projection $G \rightarrow G/G_0$ has the required property. ■

5.2. For $f \in \mathcal{U}$, we define a group $Z_1(f)$ by

- in Theorem 3, and Theorem 2, Subcase 1:

$$Z_1(f) = \{ \bar{h} \in Z_1; h(\tilde{\mathcal{F}}(f)) \subseteq \tilde{\mathcal{F}}(f) \}.$$

- In Theorem 2, Subcase 2:

$$Z_1(f) = \{ (\bar{h}, \bar{h}') \in Z_1 \times Z'_1; \bar{h} \text{ and } \bar{h}' \text{ have identical actions on } S \}.$$

— In Theorem 2, Subcase 3:

$$Z_1(f) = \{ \bar{h} \in Z_1; \text{ for any } x \in S, T_x \bar{h}(T_x \tilde{\mathcal{F}}(f)) = T_{\bar{h}(x)} \tilde{\mathcal{F}}(f) \}.$$

For notational reasons, in Theorem 2, Subcase 2, we now call Z_1 what we were calling $Z_1 \times Z'_1$; hence, in all cases $Z_1(f)$ is a closed subgroup of Z_1 . For a closed subgroup Z_2 of Z_1 , define:

$$\mathcal{U}_{Z_2} = \{ f \in \mathcal{U}, Z_1(f) \subseteq Z_2 \}.$$

LEMMA. — \mathcal{U}_{Z_2} is open in \mathcal{U} .

Proof. — By Lemma 2, there is a compact $K \subset Z_1 - Z_2$ such that any $h \in Z_1 - Z_2$ has a power in K . In Theorem 2, Subcase 3, the following properties are equivalent:

- (i) $f \in \mathcal{U}_{Z_2}$,
- (ii) no element of $Z_1(f)$ belongs to K ,
- (iii) for any $\bar{h} \in K$, there exists $x \in S$ and a neighbourhood V of \bar{h} such that $T_x g(T_x \tilde{\mathcal{F}}(f)) \neq T_{gx} \tilde{\mathcal{F}}(f)$ for $g \in V$,
- (iv) there are points x_1, \dots, x_N in S and open sets O_1, \dots, O_N in Z_1 such that $\cup O_i \supseteq K$ and $T_{x_i} g(T_{x_i} \tilde{\mathcal{F}}(f)) \neq T_{gx_i}(\tilde{\mathcal{F}}(f))$ for $1 \leq i \leq N, g \in O_i$.

Property (iv) is open. In fact, if f satisfies (iv) for an open cover (O_i) , choose open sets $O'_i \subset \bar{O}'_i \subset O_i$ such that $\cup O'_i \supseteq K$; then diffeomorphisms g near f satisfy (iv) for the open cover O'_i (and the same points x_i corresponding to f and the cover (O_i)).

In the other cases, the proof works similarly, with a suitable version of condition (iv) as follows. In Theorem 2, Subcase 1:

- (iv) there are points x_1, \dots, x_N in $\tilde{\mathcal{F}}(f)$, open sets O_1, \dots, O_N such that

$$\cup O_i \supseteq K \quad \text{and} \quad gx_i \notin \tilde{\mathcal{F}}(f) \quad \text{for } g \in O_i.$$

In Theorem 2, Subcase 2:

- (iv) there are points x_1, \dots, x_N in S , open sets O_1, \dots, O_N such that

$$\cup O_i \supseteq K \quad \text{and for } g = (g_1, g_2) \in O_i, g_1(x_i) \neq g_2(x_i).$$

In Theorem 3:

- (iv) there are points x_1, \dots, x_N in $\tilde{\mathcal{F}}(f)$, open sets O_1, \dots, O_N such that $\cup O_i \supseteq K$ and for $\bar{g} \in O_i, g$ any preimage of \bar{g} in $D_1 \times D_2$, we have $gx_i \notin \tilde{\mathcal{F}}(f)$. ■

5.3. LEMMA. — For any $g \in Z_1 - \{1_{Z_1}\}$, the set of $f \in \mathcal{U}$ such that $g \in Z_1(f)$ is nowhere dense in \mathcal{U} .

We first observe that this lemma implies the “compact part” of Propositions 1 and 2: $\mathcal{U}_{\{1_{Z_1}\}}$ is open and dense in \mathcal{U} . Openness follows from (5.2). To show density let \mathcal{V} be open in \mathcal{U} ; by Lemma 1 of (5.1) there exists $f \in \mathcal{V}$ such that $Z_1(f)$ is minimal (for the inclusion) amongst the $Z_1(g), g \in \mathcal{V}$; then for f' near f in \mathcal{V} , $Z_1(f') \subset Z_1(f)$ by Lemma 5.2, hence $Z_1(f') = Z_1(f)$ by minimality; from Lemma 5.3 we conclude that $Z_1(f) = \{1_{Z_1}\}$. ■

5.4. PROOF OF LEMMA 5.3 IN THE CASE OF THEOREM 3. END OF THE PROOF OF PROPOSITION 1. — Let $f \in \mathcal{U}$, and let $x, y \in \mathcal{J}(f)$ be two homoclinic points in $W^s(p) \cap W^u(p) - \{p\}$ which are not in the same orbit of f . Let $\varphi(x) = (x_1, x_2)$, $\varphi(y) = (y_1, y_2)$, where φ is the map from $\mathcal{J}(f)$ to $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defined in (4.4). We show that for any x'_1 sufficiently near x_1 , there exists a small perturbation f' of f in \mathcal{U} such that x, y are homoclinic points of p for f' , but $\varphi_{f'}(x) = (x'_1, x_2)$, $\varphi_{f'}(y) = (y_1, y_2)$. In fact, select a compact neighbourhood V of $f(x)$ so small that it does not intersect the closed set $\{p\} \cup \{f^n(y) : n \in \mathbb{Z}\} \cup \{f^n(x), n \leq 0\}$. Choosing adequately a diffeomorphism ψ of M with support contained in V and leaving $V \cap W^s(p)$ invariant, the perturbation $f' = \psi \circ f$ has the desired property. Taking $x_1 \neq x'_1$, as $x_1 \neq 0 \neq x_2$, $y_1 \neq 0 \neq y_2$, and as f and f' coincide in a neighbourhood of p , it is not possible to have for some $\bar{h} \in Z_0$, $\bar{h}x = y$ for both f and f' . As $\tilde{\mathcal{J}}(f)$ is discrete, this proves that for any $\bar{h} \in Z_0$, $\bar{h} \neq 1_{Z_0}$, the set of $f \in \mathcal{U}$ such that $\tilde{\mathcal{J}}(f)$ is \bar{h} -invariant is nowhere dense; this implies Lemma 5.3.

A refinement of the above argument implies the conclusion (c) of Proposition 1. In fact, it is clear that if we have three homoclinic points x, y, z for f belonging to distinct orbits, by the same construction as above one can perturb f to f' in such way that x, y, z are homoclinic points for f' , $\varphi_f(x) \neq \varphi_{f'}(x)$, $\varphi_f(y) = \varphi_{f'}(y)$, $\varphi_f(z) = \varphi_{f'}(z)$. Then the system $\bar{h}(\varphi(x)) = \varphi(y)$, $\bar{h}(\varphi(y)) = \varphi(z)$ cannot have a solution in $D_1 \times D_2$ both for $\varphi = \varphi_f$ and for $\varphi = \varphi_{f'}$. This shows that for any three homoclinic points x, y, z (depending continuously on $f \in \mathcal{U}$ by structural stability) which belong to distinct orbits, the set of $f \in \mathcal{U}$ such that there exists $\bar{h} \in D_1 \times D_2$ satisfying $\bar{h}(\varphi_f(x)) = \varphi_f(y)$ and $\bar{h}(\varphi_f(y)) = \varphi_f(z)$ is nowhere dense. But the homoclinic points form a countable set; hence for a residual set of $f \in \mathcal{U}$, the relation $\bar{h}(\tilde{\mathcal{J}}(f)) = \tilde{\mathcal{J}}(f)$, $\bar{h} \in Z_0$ implies $\bar{h} = 1_{Z_0}$. As we have already seen (Lemma 4.3) this is sufficient to conclude part (c) of Proposition 1.

To finish the proof of Proposition 1, in view of Lemma 5.3 it is sufficient to prove that if $\dim M = 2$ no $\bar{h} \in Z_0 - Z_1$ can have $\tilde{\mathcal{J}}(f)$ invariant for any $f \in \mathcal{U}$. In fact, in this case $n_1 = n_2 = 1$; let $A(f) = (A_1(f), A_2(f)^{-1}) = \text{diag}(\lambda_1, \lambda_2)$ and $h = \text{diag}(\mu_1, \mu_2)$ be a representative in $D_1 \times D_2$ of $\bar{h} \in Z_0$; $\bar{h} \notin Z_1$ is equivalent to

$$\frac{\text{Log}|\mu_1|}{\text{Log}|\lambda_1|} \neq \frac{\text{Log}|\mu_2|}{\text{Log}|\lambda_2|},$$

and this implies that for some $k, l \in \mathbb{Z}$, $A(f)^k h^l$ is a linear contraction; but a linear contraction cannot leave invariant the discrete infinite set $\tilde{\mathcal{J}}(f)$. This concludes the proof of Proposition 1 and hence Theorem 3 is now proved completely. ■

5.5. PROOF OF LEMMA 5.3 IN THE CASE OF THEOREM 2. SPECIAL PERTURBATIONS. — Let $f \in \mathcal{U}$. In (3.5) we defined the space of orbits S by identifying the two components of the boundary of a fundamental domain

$$F_u = \left\{ \Sigma \frac{|x_i|^2}{|\lambda_i|^{2u}} \leq 1 \leq \Sigma \frac{|x_i|^2}{|\lambda_i|^{2u+2}} \right\}.$$

Notice that we can always choose u in order to separate ∂F_u from a finite number of points.

Suppose that ψ is a diffeomorphism of M , near the identity, with support contained in $\text{int } F_u$; i.e. $\psi(x)=x$ for x in a neighbourhood of $M-F_u$. The diffeomorphism $f'=\psi \circ f$ is a small perturbation of f , and coincides with f in a neighbourhood of p . Therefore we can say that the action of the abstract reduced centralizer Z_0 on the space of orbits S is the same for f and f' . On the other hand, as $\psi(x)=x$ in a neighbourhood of ∂F_u , ψ induces a diffeomorphism $\tilde{\psi}$ of S which is the identity in a neighbourhood of the codimension one submanifold ∂F , ∂F being the image of ∂F_u in S . As such f -orbit cuts at most once $\text{int } F_u$, we have:

– $\tilde{\mathcal{J}}(f')=\tilde{\psi}(\tilde{\mathcal{J}}(f))$ in Subcases 1 and 3.

– In Subcase 2, the actions of the abstract reduced centralizer Z'_0 of the source q associated to f and f' are conjugate by $\tilde{\psi}$.

Conversely, given any diffeomorphism $\tilde{\psi}$ of S which is near the identity and coincides with the identity in a neighbourhood of ∂F , we can lift $\tilde{\psi}$ to a diffeomorphism ψ of M which is near id_M and coincides with id_M on a neighbourhood of $M-F_u$, and define a perturbation $f'=\psi \circ f$ with the above properties.

5.6. PROOF OF LEMMA 5.3. – Given $f \in \mathcal{U}$, $g \in Z_1(f) - \{1_{Z_1}\}$, we construct in each subcase a perturbation f' of f , of the type described in (5.5) such that $g \notin Z_1(f')$.

Subcase 1. – First we may assume that $\tilde{\mathcal{J}}(f) \cap \tilde{W} \neq \emptyset$ [\tilde{W} has been defined in (3.6)]; indeed, if this is not the case, replace f by a special perturbation $f'=\psi \circ f$ such that $\tilde{\psi}(\tilde{\mathcal{J}}(f)) \cap \tilde{W} \neq \emptyset$. Then, take $x \in \tilde{\mathcal{J}}(f) \cap \tilde{W}$ and choose F_u such that $gx \notin \partial F$. As the action of Z_0 on \tilde{W} is free, $gx \neq x$; as $\tilde{\mathcal{J}}(f)$ is nowhere dense, we can find arbitrarily near id_S a diffeomorphism $\tilde{\psi}$ of S which is the identity on a neighbourhood of $\partial \tilde{F} \cup \{x\}$ but such that $\tilde{\psi}^{-1}(gx) \notin \tilde{\mathcal{J}}(f)$. Then, for the perturbation $f'=\psi \circ f$ associated to $\tilde{\psi}$ as in (5.5), we have $x \in \tilde{\mathcal{J}}(f')$, $gx \notin \tilde{\mathcal{J}}(f')=\tilde{\psi}(\tilde{\mathcal{J}}(f))$ and hence $g \notin Z_1(f')$.

Subcase 2. – Now $g=(g_1, g_2)$, with $g_1 \in Z_1$, $g_2 \in Z'_1$, $(g_1, g_2) \neq (1_{Z_1}, 1_{Z'_1})$; the hypothesis $(g_1, g_2) \in Z_1(f)$ means that $g_1 x = g_2 x$ for all $x \in S$. It is clear that we can find a diffeomorphism $\tilde{\psi}$ of S , equal to id_S near $\partial \tilde{F}$ such that for some x , $g_1 x \neq \tilde{\psi} g_2 \tilde{\psi}^{-1}(x)$. For the special perturbation f' associated to $\tilde{\psi}$ we then have $(g_1, g_2) \notin Z_1(f')$.

Subcase 3. – The proof is similar to Subcase 1: take $x \in \tilde{W}$, F_u such that $gx \notin \partial \tilde{F}$, $\tilde{\psi} \in \text{Diff}(S)$ near the identity such that $\tilde{\psi} \equiv \text{id}$ on a neighbourhood of $\partial \tilde{F} \cup \{x\}$, and $T_{gx} \tilde{\psi}^{-1}(T_{gx} \tilde{\mathcal{J}}(f)) = T_x(\tilde{\psi}^{-1} \circ g)(T_x \tilde{\mathcal{J}}(f)) \neq T_{\tilde{\psi}^{-1}(gx)} \tilde{\mathcal{J}}(f)$. Then, for the associated perturbation $f'=\psi \circ f$, we have that

$$T_{gx} \tilde{\mathcal{J}}(f') = T \tilde{\psi}(T_{\tilde{\psi}^{-1}gx} \tilde{\mathcal{J}}(f)) \neq T_{gx} \tilde{\mathcal{J}}(f) = T_x g(T_x \tilde{\mathcal{J}}(f)),$$

hence $g \notin Z_1(f')$.

5.7. END OF THE PROOF OF PROPOSITION 2. – It remains to show in each subcase the conclusion of Proposition 2 for \bar{h} in $Z_0 - Z_1$ [$(Z_0 \times Z'_0) - (Z_1 \times Z'_1)$ in Subcase 2].

Subcase 1. – Say that $f \in \mathcal{U}$ belongs to \mathcal{U}_1 if and only if there exists $x \in \tilde{\mathcal{J}} \cap \tilde{W}$ such that for any $E \in \mathcal{P}_{r+s}$, $E \neq \emptyset$, $E \neq 1$, the point $\tilde{p}_E^f(x) \in \tilde{W}_E$ as defined in (3.6) does not belong to $\tilde{\mathcal{J}}(f)$. As $\tilde{p}_E^f, \tilde{\mathcal{J}}(f)$ depend continuously on f , the set \mathcal{U}_1 is open. For any

$f \in \mathcal{U}$, we can choose a small special perturbation $f' = \psi \circ f$, associated to $\tilde{\Psi} \in \text{Diff } S$, such that for some $x \in \tilde{\mathcal{J}}(f)$, we have $\tilde{\Psi}(x) \in \tilde{W}$, $\tilde{\Psi}^{-1}(\tilde{p}_E^f(\tilde{\Psi}(x))) \notin \tilde{\mathcal{J}}(f)$ for any proper subset $E \in \mathcal{P}_{r+s}$. Then $f' \in \mathcal{U}_1$, proving that \mathcal{U}_1 is dense.

If $f \in \mathcal{U}_1$ and $\bar{h} \in Z_0 - Z_1$ satisfies $\bar{h}(\tilde{\mathcal{J}}(f)) = \tilde{\mathcal{J}}(f)$, apply Lemma 1 of (3.7) to the point x and f given in the definition of \mathcal{U}_1 . We obtain that $p_{\bar{h}}(x)$ is a limit point of the sequence $\bar{h}^n(x)$ which is in the closed set $\tilde{\mathcal{J}}(f)$; hence $p_{\bar{h}}(x) \in \tilde{\mathcal{J}}(f)$, a contradiction with the choice of x . This concludes the proof of Proposition 2 in Subcase 1.

Subcase 2. — For $f \in \mathcal{U}$, we call \tilde{W}' the open and dense subset of S defined relatively to the source q as \tilde{W} has been defined for the sink p . Define a subset \mathcal{U}_1 of \mathcal{U} as formed by the f 's such that for some $x \in \tilde{W} \cap \tilde{W}'$, and any proper subset $E \in \mathcal{P}_{r+s}$, the point $\tilde{p}_E^f(x)$ belongs to \tilde{W}' . As \tilde{W} , \tilde{W}' and \tilde{p}_E^f depends continuously on f , we conclude that \mathcal{U}_1 is open. For any f in \mathcal{U} , we can construct special perturbations $f' = \psi \circ f$ which are in \mathcal{U}_1 . Indeed, take $x \in \tilde{W} \cap \tilde{W}'$, F_u such that none of the points $\tilde{p}_E^f(x)$ belongs to ∂F , and $\tilde{\Psi} \in \text{Diff } S$ with $\tilde{\Psi} \equiv \text{id}$ on a neighbourhood of $\partial F \cup \{x\}$ and $\tilde{p}_E^f(x) \in \tilde{\Psi}(\tilde{W}')$. Then $f' = \psi \circ f \in \mathcal{U}_1$ because $\tilde{W}'(f') = \tilde{\Psi}(\tilde{W}'(f))$. Therefore, \mathcal{U}_1 is dense.

If $f \in \mathcal{U}_1$, and $(h_1, h_2) \in (Z_0 \times Z_0) - (Z_1 \times Z_1)$ is such that h_1, h_2 have the same action on S , then we must have $h_1 \in Z_0 - Z_1$, $h_2 \in Z_0' - Z_1'$. In fact, if we had for instance $h_1 \in Z_1$, $h_2 \in Z_0' - Z_1'$, for an increasing sequence of integers we would have $h_1^{n_k} \rightarrow \text{id}_S$, and $h_2^{n_k} \rightarrow \text{id}_S$ is impossible in view of Lemma 1 of (3.7). Now, with $h_1 \in Z_0 - Z_1$, $h_2 \in Z_0' - Z_1'$, x as in the definition of \mathcal{U}_1 , by Lemma 1 of (3.7), $p_{h_1}(x)$ is a limit point of $(h_1^n(x))_{n \geq 0}$ and the limit set of $(h_2^n(x))_{n \geq 0}$ is contained in $S - \tilde{W}'$. This is a contradiction with the defining property of x , and concludes the proof of Proposition 2 in Subcase 2.

Subcase 3. — We proceed as in Subcase 1. We define \mathcal{U}_1 as the set of $f \in \mathcal{U}$ such that for some point $x \in \tilde{W}$, and any proper subset $E \in \mathcal{P}_{r+s}$, $T_x \tilde{\mathcal{J}}(f)$ is transverse to W_E , and $T_{\tilde{p}_E^f(x)} \tilde{\mathcal{J}}(f)$ is transverse to W_E . The set \mathcal{U}_1 is clearly open, and is seen to be dense by constructing an appropriate special perturbation. If $f \in \mathcal{U}_1$ and $h \in Z_0 - Z_1$ leaves $\tilde{\mathcal{J}}(f)$ invariant, then applying Lemma 2 of (3.7) at the point x given by the definition of \mathcal{U}_1 , we obtain a contradiction which concludes the proof of the Proposition 2 in Subcase 3.

The proof of Theorem 2 is therefore complete. ■

5.8. *Remark.* — We observe that the proof of part (a) of Theorem 2 given in (5.4) applies in a case slightly more general than $\dim M = 2$, $f \in \mathfrak{A}(M)$. The hypothesis $\dim M = 2$ is only used to insure that if $h \in D_1 \times D_2$ has a projection $\bar{h} \in Z_0 - Z_1$ in $D_1 \times D_2 / (f)$, then for some $(k, l) \in \mathbb{Z}^2$, $f^k h^l$ induces a contraction of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, which leads to a contradiction. To obtain this implication, it is sufficient to have $r_1 + s_1 = 1$, $r_2 + s_2 = 1$, instead of $n_1 = n_2 = 1$ as in the text. Thus, we obtain that in a neighbourhood of $f_0 \in \mathfrak{A}(M)$ the diffeomorphisms f 's with trivial centralizer contain an open and dense subset, in the following additional cases:

(a) $\dim M = 3$. Some periodic point p of some attractor of f_0 has one real and two complex conjugate eigenvalues.

(b) $\dim M = 4$. Some periodic point p of some attractor of f_0 has two pairs of complex conjugate eigenvalues.

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J. PALIS,
Instituto de Matemática Pura
e Aplicada (IMPA),
Estrada Dona Castorina 110,
22460 Jardim Botânico,
Rio de Janeiro, Brasil;
J. C. YOCCOZ,
Centre de Mathématiques,
École Polytechnique,
91128 Palaiseau Cedex, France.