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CONVEX FUNCTIONS ON COMPLETE NONCOMPACT MANIFOLDS: DIFFERENTIABLE STRUCTURE

BY R. E. GREENE AND K. SHIOHAMA

The determination of the structure of a manifold from the analysis of a function on the manifold by the Morse theory of critical point behavior depends in the theory's standard version upon the function's being of differentiability class C^2 at least. Most functions which arise naturally from the geometry of the manifold are constructed from the Riemannian distance function and are not in general even of class C^1, but rather are locally Lipschitz continuous only. It is natural to try to remedy this discrepancy by constructing approximating functions which have the essential behavior of the geometrically arising functions but have at the same time a higher degree of differentiability. A typical instance of this situation is in the study of geodesically convex functions on Riemannian manifolds. A procedure is known ([6], [9]) for approximation of geodesically convex functions by C^∞ functions which in a suitable sense are almost convex; and the approximations of strongly convex functions can be taken to be strongly convex. But it remains unknown whether an arbitrary convex function on a Riemannian manifold can be approximated by C^2 convex functions.

The purpose of the present paper is to show how, in spite of the possible absence of smooth convex approximations, a complete Morse-theoretic analysis of a convex function can be obtained even though the function might not have C^2 regularity. In the authors' previous work [5] (*see* also their preliminary announcement in the last section of [17]), topological versions of the present paper's differentiable structure Theorems were established; the present paper's emphasis is thus specifically on differentiable structure. To some extent, the results on differentiable structure could be deduced from the topological versions combined with general results on differential topology. For instance, under the hypotheses in Theorem A of this paper, it was shown in [5] that the manifold is homeomorphic to a product of a topological manifold with $\mathbb{R}$; it follows then from general results ([13], *cf.* also [3], [12]) that the manifold is diffeomorphic to a product of a differentiable manifold with $\mathbb{R}$ if the manifold has dimension at least five (*cf.* the remarks on the application of the *h*-cobordism Theorem in [4]). The authors are indebted to S. Morita for helpful conversations concerning this point. To achieve the most general possible results in all dimensions as well

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as to obtain a self-contained treatment, a different approach to the differentiable structure questions is used here; the approach here is to use the smooth approximation results of [6] (*see* also [7], [8]; in the case of nonnegative curvature, similar techniques were applied in [16] and [18]).

The analysis of the Morse theory of convex functions given here does not depend directly upon the convexity of the functions; rather it depends primarily upon the local Lipschitz continuity of the functions [6] and upon the existence, on the complement of the global minimum set of the function, of continuous vector fields along which the rate of change of the function is locally bounded from above by a negative constant [5]. This second property is the nonsmooth analogue of noncriticality. In the presence of these properties, but without further appeal to convexity, the smooth approximation technique of [6] and [9] yield C^∞ approximations which are without critical points (away from the minimum set). Although the main goal is the investigation of the convex function case, some attempt has been made to separate the arguments which depend explicitly on convexity from those which apply in more general circumstances.

A critical point of a C^∞ convex function on a complete Riemannian manifold is necessarily a global minimum. As noted, a possible nonsmooth convex function is also, in a suitably generalized sense, noncritical except at its global minimum set. Associated to this property is a manifold structure Theorem:

THEOREM A. — *If* $\varphi : M \to \mathbb{R}$ *is a convex function on a complete* C^∞ *Riemannian manifold* M *and if* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ *is empty, then there is a differentiable manifold* N *such that* M *is diffeomorphic to* $N \times \mathbb{R}$.

In fact, N is homeomorphic to any level set $\{ x \in M \mid \varphi(x) = a \}$, $a \in M$, all these sets being homeomorphic to each other according to [5].

In Theorem A, much more structural determination is obtained than is obtainable simply from the existence of a C^∞ function on M without critical points, since such a nowhere critical function exists on any noncompact C^∞ manifold [15]. The convexity of φ forces a certain focusing of the analogues of the negative gradient integral curves of the function and makes the situation resemble the Morse theory of smooth functions without critical points but with compact level sets, even though φ nay have noncompact level sets. (In the case where φ is C^∞ and has compact levels, a product structure was obtained in [1], using the techniques initiated in [2].) A similar phenomenon occurs in the situation where the global minimum set is nonempty. The global minimum set is, if nonempty, a totally geodesic submanifold with perhaps nonempty and not necessarily smooth boundary (*cf.* [4]).

THEOREM B. — *If* $\varphi : M \to \mathbb{R}$ *is a convex function on a complete Riemannian manifold* M *and if* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ *is nonempty then:*

(*a*) *there is a differentiable manifold* N *and a continuous mapping* $F : N \times [0, +\infty) \to M$ *such that* F *is a proper mapping and* $F \mid N \times (0, +\infty)$ *is a* C^∞ *diffeomorphism of* $N \times (0, +\infty)$ *onto* $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$;

(*b*) *if* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ *is a smooth (totally geodesic) submanifold of* M *without boundary then* M *is diffeomorphic to the total space of the normal bundle in* M *of* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$.

As in Theorem A, the manifold N in part (*a*) of Theorem B is homeomorphic to $\{x \in M \mid \varphi(x) = a\}$, $a > \inf_M \varphi$, these sets being all homeomorphic to each other. If φ has a disconnected level set, then the minimum set $\{x \in M \mid \varphi(x) = \inf_M \varphi\}$ is a (connected) codimension one C^∞ submanifold of M with trivial normal bundle [5]; in this case, M is diffeomorphic to the minimum set $\times \mathbb{R}$, according to Theorem B (*b*).

If the minimum set is allowed to have nonempty interior in M, then Theorem B imposes no restriction, other than noncompactness, on M. In fact, on any noncompact manifold, there is a nonconstant C^∞ function which is convex relative to a suitable complete Riemannian metric [5]. M clearly has the homotopy type of the minimum set of any convex function on M, if the minimum set is nonempty; hence, if the interior of the minimum set is required to have empty interior, then considerable restrictions on the topology (and differential topology) of M do occur: For example, M has then the homotopy type of a lower dimensional manifold-without-boundary (in the case of empty minimum set) or of a lower dimensional submanifold (possibly) with nonempty, nonsmooth boundary in case of nonempty minimum set, since in that case the minimum set is a totally geodesic lower dimensional submanifold with possibly nonempty, possibly nonsmooth boundary. Also, if the minimum set has empty interior, then M has at most two ends [5].

1. Vector fields and Lipschitz continuous functions

A locally Lipschitz continuous function on a Riemannian manifold is almost everywhere differentiable. Even so, the appropriate idea of a point of the manifold being a noncritical point of such a function cannot be defined directly in terms of the almost everywhere existing derivative because by its very nature critical point behavior is typically concentrated on sets of measure zero. To escape this difficulty, it is useful to introduce the idea of one-sided upper and lower derivatives along a vector field: Suppose $\varphi : M \to \mathbb{R}$ is a locally Lipschitz continuous function on a Riemannian manifold M (*see*, e. g., [6] for detailed definition and basic properties). If $p \in M$ and $X \in TM_p$ (= the tangent space of M at p), then the quantity:

$$\limsup_{t \to 0^+} t^{-1}(\varphi(C(t)) - \varphi(C(0))),$$

is finite and is the same for all C^1 curves C with $C(0) = p$ and $C'(0) = X$; this fact is an immediate consequence of the agreement to second order of any two such curves and of the Lipschitz continuity of φ. Set $X_R^+ \varphi =$ the value of this lim sup; similarly set $X_R^- = \liminf_{t \to 0^+} t^{-1}(\varphi(C(t)) - \varphi(C(0)))$, C as before. And set $X_L^+ = -[(-X)_R^-(\varphi)]$, $X_L^-(\varphi) = -[(-X)_R^+(\varphi)]$. Of course if φ is differentiable at p, all four of the $X_R^+(\varphi)$, $X_R^-(\varphi)$, $X_L^+(\varphi)$ and $X_L^-(\varphi)$ are equal, being all $\langle X, \operatorname{grad} \varphi \rangle$.

DEFINITION. – A vector field X is *transversal* to a locally Lipschitz continuous function φ at a point p if there are neighborhood U of p and $\varepsilon > 0$ such that, at every $q \in U$, $X_R^+(\varphi) < -\varepsilon$. A point p is *noncritical* for φ if there is a C^∞ vector field X defined in some neighborhood of p such that X is transversal to φ at p.

If $B>0$ is a local Lipschitz constant for φ near p and if $X, Y \in TM_p$, then $|X_R^+(\varphi)-Y_R^+(\varphi)| \leqq \|X-Y\|.B$, as one sees from elementary considerations. It follows from standard approximation-by-C^∞ results that if there is a continuous vector field Y defined near p and transversal to φ at p, then there is a C^∞ vector field near p also transversal to φ and p is thus a noncritical point for φ.

If φ is a geodesically convex function on a Riemannian manifold M (φ is then automatically locally Lipschitz continuous: [6]) and if a point p is not a local minimum of φ, then p is a noncritical point of φ in the sense just introduced. This fact can be verified in the following way: Choose a number b less than $\varphi(p)$ but very close to $\varphi(p)$. Then there will be points of the set $M^b(\varphi)=\{x\in M \mid \varphi(x)\leqq b\}$ near p, since p is not a local minimum. Every point q close enough to p has a unique shortest (geodesic) connection to $M^b(\varphi)$ and the direction of this connection depends continuously on q (*cf.* [19]). The vector field $q \to$ the unit tangent vector to the shortest (geodesic) connection from q to $M^b(\varphi)$ is transversal to φ at p so that, by the C^∞ approximation observation of the previous paragraph, p is a noncritical point of φ. On a complete Riemannian manifold, a local minimum of a convex function is necessarily a global minimum; thus the critical points of a convex function $\varphi : M \to \mathbb{R}$ on a complete Riemannian manifold M are exactly the points p (if any) such that $\varphi(p)=\inf_M \varphi$ (*see* [5] for detailed discussion of the properties of convex functions just asserted).

If X is a C^∞ vector field and if φ is a locally Lipschitz continuous function, the restriction $\varphi(C(t))$ of φ to an integral surve $C(t)$ of X is a locally Lipschitz continuous function of t. Thus:

$$\varphi(C(t_2))-\varphi(C(t_1))=\int_{t_1}^{t_2}\frac{d}{dt}[\varphi(C(t))],$$

the derivative existing almost everywhere. It follows that if X is transversal to φ, then $t \to \varphi(C(t))$ is a strictly decreasing function of t.

PROPOSITION 1.1. – *If $\varphi : M \to \mathbb{R}$ is a locally Lipschitz continuous function on a Riemannian manifold* M, *if* X *is a* C^∞ *vector field on* M *everywhere transversal to* φ, *and if each maximal integral curve* C *on* X *has the property that the range of* $\varphi \circ C =$ *the range* $\varphi(M)$ *of* φ *on* M, *then:*

(1) *for any numbers* $a, b \in \varphi(M)$ *the sets* $\{x\in M \mid \varphi(x)=a\}$ *and* $\{y\in M \mid \varphi(y)=b\}$ *are homeomorphic and* M *is homeomorphic to* $\{x\in M \mid \varphi(x)=a\} \times \mathbb{R}$;

(2) *there exists a differentiable manifold* N, *homeomorphic to* $\{x\in M \mid \varphi(x)=a\}$ *for any* $a\in\varphi(M)$, *such that* M *is diffeomorphic* $N \times \mathbb{R}$.

Proof. – As noted, the function $t \to \varphi(C(t))$, C an integral curve of X, is a strictly monotone decreasing function. For each $b\in\varphi(M)$, there is by hypothesis a value t_0 such that $\varphi(C(t_0))=a$ and hence there is exactly one such value. The function from $\{x\in M \mid \varphi(x)=a\}$ to $\{y\in M \mid \varphi(y)=b\}$ defined by $x \to C(t_b)$, where C = the integral curve of X through x and t_b = the unique t_b such that $\varphi(C(t_b))=b$, is a homeomorphism. Similarly, one obtains a topological product structure on M, i.e., a homeomorphism $\{x\in M \mid \varphi(x)=a\} \times \varphi(M) \to M$ by $(x, b) \to C(t_b)$. Since $\varphi \circ C$ being strictly monotone

decreasing implies that $\varphi(M)$ = the range of $\varphi \circ C$ is a connected open subset of $\mathbb{R}$, $\varphi(M)$ is homeomorphic to $\mathbb{R}$ and (1) follows.

To prove (2), some smooth approximation procedure is needed. In outline, N will be obtained by fixing a number $a \in \varphi(M)$, approximating φ near $\{x \in M \mid \varphi(x) = a\}$ by a suitable C^∞ function, and letting N be a level set of the approximating function. The vector field X will be transversal to N (in the usual sense) and each maximal integral curve of X will intersect N in exactly one point. Since M is a union of these integral curves, the required diffeomorphism $M \to N \times \mathbb{R}$ can be constructed.

For the construction of the smooth approximations, the process of smoothing by integration against a C^∞ kernel will be used. Specifically, let $\varkappa : \mathbb{R} \to \mathbb{R}$ be a nonnegative even C^∞ function, constant on some interval around 0, with support in $[-1, 1]$ and with the property that $\int_{v \in \mathbb{R}^n} \varkappa(\|v\|) = 1$, n = dimension M. For any function $f : M \to \mathbb{R}$ define $f_\delta : M \to \mathbb{R}$ by:

$$f_\delta(p) = \delta^{-n} \int_{v \in TM_p} f(\exp_p v) . \varkappa(\|v\|/\delta)\, dv,$$

where dv is the Riemannian volume measure on TM_p. Then for any compact set K in M, the functions f_δ are C^∞ in a neighborhood of K for all sufficiently small positive δ, and as $\delta \to 0^+$ the functions $f_\delta \mid K$ converge uniformly to $f \mid K$ [6]. Also if a C^∞ vector field X is transversal to φ at p with $X_R^+ \varphi < -\varepsilon < 0$ on a neighborhood of p, then there is a neighborhood of p and a $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, $X\varphi_\delta < -\varepsilon/2$ on the neighborhood (this is a special case of a property proved in [7]).

If $\varphi(p) = a$, then, for all sufficiently small $\delta > 0$, the function φ_δ attains the value a on the integral curve of X through p at exactly one point near p: this follows from the facts that φ_δ is near φ, uniformly in a neighborhood of p, and that φ_δ is strictly monotone decreasing at least rate $\varepsilon/2$ along the integral curve. Similarly, if $\varphi_\delta(q) = a$ for some q near p then there is exactly one point q_1 near p on the integral curve of X through q with $\varphi(q_1) = a$. Thus locally near p for $\delta > 0$ sufficiently small there is a bijection from the a-level of φ_δ to the a-level of φ. This bijection is bicontinuous. Also, the a-level of φ_δ is a C^∞ submanifold of codimension 1 since $X\varphi_\delta = X_R^+ \varphi_\delta \neq 0$ near p. Thus locally the construction of N is accomplished.

To construct N globally, it is necessary to use different values of $\delta > 0$ near different points p (a uniform choice of δ can be made in general only if $\{x \in M \mid \varphi(x) = a\}$ is compact. It is then necessary to combine the resulting φ_δ functions to obtain an approximation of φ in a neighborhood of $\{x \in M \mid \varphi(x) = a\}$. The construction proceeds in a standard partition-of-unity fashion, using the following easily calculated estimates on variable-coefficient finite convex combinations of approximations:

(*a*) if $\sum \lambda_1 = 1$ and $\lambda_i \geqq 0$, all i, then:

$$|\varphi(p) - \sum \lambda_i \varphi_{\delta_i}(p)| \leqq \sum \lambda_i |\varphi(p) - \varphi_{\delta_i}(p)|;$$

(*b*) if $\sum \lambda_i = 1$ and $\lambda_i \geqq 0$, all i, then:

$$X(\sum \lambda_i \varphi_{\delta_i}|_p) = \sum (X\lambda_i)\varphi + \sum (X\lambda_i)(\varphi_{\delta_i} - \varphi) + \sum \lambda_i (X\varphi_{\delta_i})$$

$$\leqq \sum \lambda_i (X\varphi_\delta)|_p + \sum_i |(X\lambda_i)|_p \| \varphi(p) - \varphi_{\delta_i}(p) \|,$$

since $\sum (X\lambda_i) = 0$.

The first estimate shows that combining approximations with a partition of unity yields an approximation no worse at p than the worst among the individual approximations having nonzero coefficient at p. The second estimate shows that if, for instance, $X\varphi_{\delta_i} \leqq -\varepsilon/2$ at p then $X(\sum \lambda_i \varphi_{\delta_i}) \leqq -\varepsilon/3$ at p, provided that each φ_{δ_i} is a sufficiently good (uniform) approximation of φ at p. Thus transversality can be preserved under the partition-of-unity process.

It follows by the standard partition-of-unity arguments that there is an open set U containing $\{x \in M \mid \varphi(x) = a\}$ and a C^∞ function $\psi : U \to \mathbb{R}$ such that $X\psi < 0$ on U, and every maximal integral curve of X intersects $\{x \in M \mid \psi(x) = a\}$ exactly once.

Since each maximal integral curve of X also intersects $\{x \in M \mid \varphi(x) = a\}$ exactly once, it follows that $\{x \in M \mid \varphi(x) = a\}$ and $\{x \in M \mid \psi(x) = a\}$ are homeomorphic. Also, because $X\psi < 0$ on U, the set $\{x \in M \mid \psi(x) = a\}$ is a C^∞ codimension one submanifold of M.

Now choose a positive continuous function $\gamma : \varphi(M) \to \mathbb{R}^+$ such that the interval $(\alpha - \gamma(\alpha), \alpha + \gamma(\alpha)) \subset \varphi(M)$ and $\gamma(\alpha) < 1/2$ for all $\alpha \in \varphi(M)$. The function ψ could have been and will now be assumed to have been chosen so that $|\psi(p) - \varphi(p)| < (1/2)\gamma(\varphi(p))$ for all $p \in U$. Again by applying the local approximation process and the partition-of-unity construction, a C^∞ function $\psi_1 : M \to \mathbb{R}$ can be obtained such that $\psi_1(p) = a$ if $\psi(p) = a$; $|\psi_1(p) - \varphi(p)| < \gamma(\varphi(p))$ for all $p \in M$; and $X\psi_1 < 0$. Clearly the range of $\psi_1 \circ C = \varphi(M)$ for any maximal integral curve C of X. The map of $\{p \in M \mid \psi(p) = a\} \times \varphi(M) \to M$ defined by $(p, t) \to$ the unique point on the maximal integral of X through p at which $\psi_1 = t$ is a diffeomorphism. Thus, as required, M is diffeomorphic to an $(n-1)$-manifold $\times \mathbb{R}$, the $(n-1)$-manifold being homeomorphic to $\{x \in M \mid \varphi(x) = a\}$. □

An alternative formulation of noncriticality can be given: Let $\varphi : M \to \mathbb{R}$ be a locally Lipschitz continuous function, and set, for each $p \in M$, Lim grad $\varphi(p) = \{v \in TM_p \mid v =$ the limit of grad $\varphi(p_i)$ for some sequence $\{p_i\}$ converging to p with φ differentiable at each $p_i\}$. This set is closed in TM_p and the union of the sets Lim grad $\varphi(p)$ over $p \in M$ is closed in the tangent bundle of M. It is easy to check that a point $p \in M$ is noncritical for φ in the sense already introduced if and only if Lim grad $\varphi(p)$ is contained in an open half-space in TM_p. This property of Lim grad $\varphi(p)$ at noncritical points p can be used also to approximate φ on a compact set every point of which is noncritical by smooth functions with gradients nonvanishing on a neighborhood of the compact set [11].

2. The differentiable structure theorems

To apply the results of the first section to prove Theorems A and B, one needs a C^∞ vector field which is transversal to the convex function φ (away from the minimum set of φ) and which has integral curves each meeting every (non-minimum) level of φ. Vector fields

transversal to φ are constructed in [5], but they are not generally C^∞. The finding of suitable transversal C^∞ vector fields will make use of the following Lemmas. These are stated in the form suitable for the proof of Theorem A. The easy modifications to suit the proof of Theorem B will be summarized briefly at the end of the section.

LEMMA 2.1. — *If* X *is a locally Lipschitz continuous vector field on* M *which is transversal to a locally Lipschitz continuous function* φ *on* M *and if, for every maximal integral curve* C *of* X, *the range of* $\varphi \circ C = \varphi(M)$, *then there is a* ($C^0$ *fine*) *neighborhood* $\mathcal{U}$ *of* X *in the* C^0 *fine topology on continuous vector fields on* M *such that: if* Y *is a* C^∞ *vector field in* $\mathcal{U}$ *then* Y *is transversal to* φ *on* M *and, for any maximal integral curve* C_Y *of* Y_i, *range* $\varphi \circ C_Y = \varphi(M)$. *In particular, a* C^∞ *vector field* Y *exists which is transversal to* φ *and has range* $\varphi \circ C_Y = \varphi(M)$ *for all maximal integral curves* C_Y.

Proof of Lemma 2.1. — The second statement follows from the first together with the density of C^∞ in C^0 vector fields relative to the C^0 fine topology (*see*, e. g., [14], which see also for the detailed definition of the C^0 fine topology).

To prove the first statement, recall from the previous section that transversality is a stable property, i. e., if X is transversal to φ at p then there is an $\varepsilon > 0$ such that if $\| X - Y \| < \varepsilon$ in some neighborhood of p then Y is transversal to φ at p. It follows that if X is a continuous vector field transversal to φ on M then there is a neighborhood $\mathcal{U}$ of X in the C^0 fine topology on the continuous vector fields of M such that if $Y \in \mathcal{U}$ then Y is transversal to φ on M. The standard results on stability of integral curves of vector fields imply easily (because of the hypotheses on the integral curves of X) that the C^0 fine neighborhood $\mathcal{U}$ can be chosen to have the further property that, if $Y \in \mathcal{U}$, then the maximal integral curves C_Y of Y have range $(\varphi \circ C_Y) = \varphi(M)$. More precisely, the integral curves of Y can be forced (by suitable choice of $\mathcal{U}$) to remain close to those of $\mathcal{U}$ so that the monotone strictly decreasing function $t \to (\varphi \circ C_Y)(t)$ reaches a given value [in $\varphi(M)$] at a t-value near the t-value at which φ along an X integral curve does: in particular, $\varphi \circ C_Y$ will then assume every value in $\varphi(M)$. The straightforward details of this point are left to the reader. □

The obvious analogue of Lemma 2.1 for X such that $-X$ is transversal to φ also holds: in this case, φ is monotone strictly increasing along integral curves of X and nearby integral curves of approximating vector fields reach a level above a given level at approximately the same parameter value as the original integral curve of X. (Incidentally, the role of local Lipschitz continuity of X in all this discussion is simply to make the existence-and-continuous-variation Theorem for integral curves apply.)

In the first statement of Lemma 2.1 and its $-$ X analogue, that Y be C^∞ is not essential to the conclusions; but it is the second, existence assertion of Lemma 2.1 that is to be applied later to produce C^∞ vector fields of a specific sort, so the other assertions have been formulated for the C^∞ case only, also.

Now suppose that M is a complete Riemannian manifold and that $\varphi : M \to \mathbb{R}$ is a convex function (φ is then automatically locally Lipschitz continuous, as noted earlier, even if M were not complete). If $p \in M$ and $\varphi(p) > \inf_M \varphi$, then there is a locally Lipschitz continuous vector field X defined near p and transversal to φ at p. Such a vector field can be constructed as follows (for details of this and related constructions to be used later, *see* [5]). Pick a

number b less than $\varphi(p)$ but close to $\varphi(p)$. Set $M^b(\varphi) = \{ x \in M \mid \varphi(p) \leqq b \}$. If b is close enough to $\varphi(p)$ then there is, for each q near p, a unique shortest (geodesic) connection from q to $M^b(\varphi)$. The unit direction vectors of these connections form a locally Lipschitz continuous vector field [19]; more precisely, the value of the vector field at q is the unit tangent vector at q, to the shortest connection from q to $M^b(\varphi)$. This vector field is transversal to φ at p. This construction can be extended to find a vector field defined on M (or $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$), which vector field is transversal to φ and has a further desirable geometric property which will be called piecewise local Lipschitz continuity.

DEFINITION. – If $\varphi : M \to \mathbb{R}$ is a locally Lipschitz continuous function and X is a vector field on M transversal to φ then X is said to be *piecewise locally Lipschitz continuous* if it has the following property: For each compact set K in M, there exist a finite set of numbers $a_1, \ldots, a_k$ and locally Lipschitz vector fields X_i, $i = 1, \ldots, k$ defined, respectively, in a neighborhood of $K \cap \{ x \in M \mid a_i \leqq \varphi(x) \leqq a_{i+1} \}$ such that:

(*a*) $a_1 \leqq a_2 \leqq \ldots \leqq a_k$;

(*b*) $a_1 < \inf_K \varphi \leqq \sup_K \varphi \leqq a_k$;

(*c*) $X(y) = X_i(y)$ if $y \in K \cap \{ x \in M \mid a_i < \varphi(x) \leqq a_{i+1} \}$, $i = 1, 2, \ldots, k$.

The intuitive meaning of piecewise local Lipschitz continuity of a vector field is that the vector field between the a-levels of φ is locally Lipschitz continuous but that X may jump as an a-level is crossed.

Piecewise locally Lipschitz vector fields have integral curves: indeed a unique maximal integral curve passes through each point. These are simply the integral curves of the X_i between a-levels attached continuously across a-levels; the curves are piecewise C^1 (*see* [5] for details of this point).

The following result was proved in [5] by a partition-of-unity construction applied to the geodesic connections to sublevels already described here. (Note, however, that the terminology of piecewise local Lipschitz continuity was not used in [5].)

LEMMA 2.2. – *If* $\varphi : M \to \mathbb{R}$ *is a convex function on a complete Riemannian manifold with* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \} = \emptyset$, *then there exists on* M *a piecewise locally Lipschitz continuous vector field* X, *the maximal integral curves* C_X *of which each have the property that the range of* $\varphi \circ C_X = \varphi(M)$.

To be able to apply Proposition 1 combined with Lemma 2.1 to the convex function situations (Theorem A), it is necessary to be able to modify the vector field X to obtain an everywhere locally Lipschitz continuous vector field on M that has still full-φ-range maximal integral curves:

LEMMA 2.3. – *If* $\varphi : M \to \mathbb{R}$ *is a convex function on a complete Riemannian manifold* M *with* $\{ x \in M \mid \varphi(x) = \inf_M \varphi \} = \emptyset$, *then there exists on* M *a locally Lipschitz continuous vector field* Y *the maximal integral curves* C_Y *of which each have the property that the range of* $\varphi \circ C_Y = \varphi(M)$.

Proof of Lemma 2.3. – Let $\{ U_i \}$ be a locally finite covering of $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ such that each $\overline{U}_i$ has compact closure and let $\rho_i : U_i \to \mathbb{R}$ be a partition-of-unity subordinate to $\{ U_i \}$. Then, with X the vector field of Lemma 2.2, write

$X=\sum_i \rho_i X$. The required new global vector field Y will be expressed in the form $Y=\sum_i \rho_i \tilde{X}_i$, where each $\tilde{X}_i$ is a modification of $X \mid U_i$. Of course, each $\tilde{X}_i$ need only be defined on the corresponding U_i. For fixed i, note that, because $\bar{U}_i$ is compact and from the piecewise local Lipschitz continuity of X, there exist an $\varepsilon>0$, a finite set of numbers $b_1, \ldots, b_k, b_1<\ldots<b_k$, and a neighborhood V_i (also with compact closure) of $\bar{U}_i$ such that $\bar{V}_i \subset \{x \in M \mid b_1<\varphi(X)<b_k\}$, such that X is locally Lipschitz continuous on the sets:

$$\{x \in V_i \mid b_j<\varphi(x) \leqq b_{j+1}\}, \qquad j=1, \ldots, k-1,$$

and in fact such that, for each of these sets, X restricted to the set is the restriction to the set of a locally Lipschitz continuous vector field defined on a neighborhood of the closure of the set. $\tilde{X}$ is then constructed by patching together these locally Lipschitz continuous vector fields. The situation will be clear if the first stage of the patching is described: Suppose Z_1 is a locally Lipschitz continuous vector field defined in a neighborhood of the closure of:

$$\{x \in V_i \mid b_i<\varphi(x) \leqq b_2\} \qquad \text{with} \quad Z_1 \mid \{x \in V_i \mid b_1<\varphi(x) \leqq b_2\} = X \mid \{x \in V_i \mid b_1<\varphi(x)<b_2\};$$

and Z_2 is locally Lipschitz continuous in a neighborhood of the closure of:

$$\{x \in V_i \mid b_2<\varphi(x) \leqq b_3\}$$

and

$$Z_2 \mid \{x \in V_i \mid b_2<\varphi(x) \leqq b_3\} = X \mid \{x \in V_i \mid b_2<\varphi(x) \leqq b_3\}.$$

Let ρ be a C^∞ function in V_i such that $\rho=1$ on $\{x \in V_i \mid b_1<\varphi(x)<b_2\}$ and $\rho=0$ on $\{x \in V_i \mid b_2<\varphi(x) \leqq b_3\}$ except (in both cases) in a very small neighborhood W_1 of $\{x \in V_i \mid \varphi(x)=b_2\}$, in which neighborhood ρ goes rapidly from 0 to 1 (ρ is to take values $\in [0, 1]$ everywhere). The vector field $\rho Z_1+(1-\rho)Z_2$ then $=X$ on $\{x \in V_i \mid b_1<\varphi(x) \leqq b_2\}$ except very near $\{x \in V_i \mid \varphi(x)=b_2\}$, where it changes rapidly from Z_1 to Z_2. On $\{x \in V_i \mid b_1<\varphi(x) \leqq b_2\}$, $\rho Z_1+(1-\rho)Z_2$ is locally Lipschitz continuous and its integral curves differ (in C^0) arbitrarily little from those of X if W_1 is a sufficiently small neighborhood of $\{x \in V_i \mid \varphi(x)=b_1\}$. And as before $\rho Z_1+(1-\rho)Z_2$ is transversal to φ everywhere. Continuing this patching construction in an obvious way yields the required $\tilde{X}_i$.

It is now straightforward to verify (using local finiteness of $\{U_i\}$) that if for each i the W's are all chosen sufficiently small then the vector field $\sum_i \rho_i \tilde{X}_i$ (which will be locally Lipschitz continuous and transversal to φ) will again have maximal integral curves C such that the range $\varphi \circ C=\varphi(M)$. □

Proof of Theorem A *completed.* — According to Lemma 2.1, the vector field Y from Lemma 2.3 can be replaced by a C^∞ vector field still transversal to φ and with maximal curves again each having range of φ along the curve equal to $\varphi(M)$. Then Theorem A follows immediately from Proposition 1. □

Essentially the same arguments used to prove Theorem A are used to prove Theorem B, but some modifications are necessary. They are as follows: The vector field of Lemma 2.2 is replaced by another piecewise locally Lipschitz continuous vector field X such that X on $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ is transversal to φ there, such that the integral curves of X on $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ each reach a unique limit in $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ in finite parameter value, while φ on the integral curves of $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ has range $= (\inf_M \varphi, +\infty)$, and such that, for any $a \in (\inf_M \varphi, +\infty)$, the map $\{ x \in M \mid \varphi(x) = a \} \to \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ obtained by running along integral curves of X to their limits in $\{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ is a continuous proper map; such a vector field was produced in [5], again by shortest geodesic connections. Using again the argument already used to prove Lemma 2.3 allows one to show that there is a locally Lipschitz vector field (not just a piecewise locally Lipschitz one) on $M - \{ x \in M \mid \varphi(x) = \inf_M \varphi \}$ which otherwise has all the properties just required of X. A suitable C^∞ fine approximation (as in Lemma 2.1) of this locally Lipschitz vector field will still retain the integral curve properties of X as indicated. Then the proof of Theorem B is completed by applying a modification of Proposition 1.1, obtained by essentially the same method as is Proposition 1.1 itself.

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