ANNALES SCIENTIFIQUES DE L'É.N.S.

MOGENS FLENSTED-JENSEN
DAVID L. RAGOZIN

Spherical functions are Fourier transforms of L_1-functions

Annales scientifiques de l'É.N.S. 4e série, tome 6, nº 4 (1973), p. 457-458
<http://www.numdam.org/item?id=ASENS_1973_4_6_4_457_0>

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1973, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (http://www.elsevier.com/locate/ansens) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. scient. Éc. Norm. Sup.,
4e série, t. 6, 1973, p. 457 à 458.

SPHERICAL FUNCTIONS ARE FOURIER TRANSFORMS OF L_1-FUNCTIONS

By Mogens FLENSTED-JENSEN and David L. RAGOZIN *

In this brief note we apply a result of Kostant, (4.1) in [2], to prove the following. (All notation is as in Kostant's paper).

Theorem 1. — *Let* (G, K) *be an irreducible Riemannian symmetric pair of non-compact type. Fix an Iwasawa decomposition* $G = KAN$. *For each* $b \in A$, $b \neq 1$, *let* $\mu_b \in M^1(\mathfrak{a})$ *be the finite measure on* $\mathfrak{a}$ *such that*

$$\int_K f(\log a(bv))\, dv = \int_{\mathfrak{a}} f(x)\, d\mu_b(x) \qquad f \in \mathcal{K}(\mathfrak{a}).$$

Then $\mu_b \in L_1(\mathfrak{a})$ *and* supp μ_b *is the compact set* $\mathfrak{a}(\log b)$.

Remark 2. — T. H. Koornwinder has proved this in the rank 1 case by explicitly computing μ_b (*see* [1]).

This result has an immediate application to the spherical functions on G. If we write $\hat{\mu}(\tau) = \int_{\mathfrak{a}} e^{-i\tau(x)}\, d\mu(x)$, $\tau \in \mathfrak{a}^*$, for the Fourier Stieltjes transform on $\mathfrak{a}$, then we have

Corollary 3. — *For* $b \neq 1$, $b \in A$ *and* $\nu = \sigma - i\tau \in \mathfrak{a}^* + i\,\mathfrak{a}^*$, *the spherical function* $\varphi_\nu(b) = \int_K e^{\langle \nu, \log a(bv)\rangle}\, dv$ *is, as a function of* τ, *the Fourier transform of the compactly supported measure* $e^\sigma \mu_b \in L_1(\mathfrak{a})$. *Hence, for any tube* $T = C + i\,\mathfrak{a}^*$ *with* C *compact in* $\mathfrak{a}^*$, $\varphi_\nu(b) \to 0$ *as* $\nu \to \infty$ *in* T.

Remark 4. — The second sentence generalizes (3.13) in [3].

Proof of Theorem 1. — The map $g_b : K \to \mathfrak{a}$ with $g_b(v) = \log a(bv)$, $v \in K$, is real analytic and, for $S \subseteq \mathfrak{a}$, $\mu_b(S) = m_K(g_b^{-1}(S))$ where m_K is Haar measure on K. We must show $\mu_b(S) = 0$ when S has Lebesgue measure zero. We claim that it suffices to show that g_b has rank equal to dim $\mathfrak{a}$ at some point of K. For if this is so then g_b has rank equal to dim $\mathfrak{a}$ except on a proper real analytic subvariety U of K since K is

(*) Research supported by NSF Grant GP-32840 X.

connected. But then $\dim U < \dim K$ and hence $m_K(U) = 0$. Now, on $K - U$, g_b, in appropriate coordinates, is just an orthogonal projection between Euclidean spaces. So since m_K is equivalent to Lebesgue measure in any coordinate patch, Fubini's theorem shows

$$m_K(g_b^{-1}(S)) = m_K(g_b^{-1}(S) \cap K - U) = 0,$$

when S has Lebesgue mesure zero.

Now to see g_b has rank equal to $\dim \mathfrak{a}$ at some point, it suffices by Sard's theorem (or the theorem on functional dependence) to show that the range of g_b has interior points in $\mathfrak{a}$. Now Kostant shows in (4.1) of [2] that $g_b(K) = \mathfrak{a}(\log b) = \text{co}(W \cdot \log b)$, in particular, $\mathfrak{a}(\log b)$ is a non-trivial convex W-invariant set. So by the irreducibility of the action of W on $\mathfrak{a}$, $0 \in \mathfrak{a}(\log b)$ and $\text{span}(\mathfrak{a}(\log b)) = \mathfrak{a}$. Thus $\mathfrak{a}(\log b)$ must have interior.

It is clear that $\text{supp}\ \mu_b = g_b(K) = \mathfrak{a}(\log b)$ and so is compact. □

REMARK 5. — The same proof holds for non-irreducible (G, K) provided $\mathfrak{a}(\log b)$ has interior in $\mathfrak{a}$. For instance if b is regular or more generally if b has non-zero coordinate in each irreducible factor.

Proof of Corollary 3. — The first statement follows from the definition of μ_b. For the second note that if $C = \{\sigma\}$, then the Riemann-Lebesgue lemma says $\varphi_{\sigma+i\tau}(b) = (e^\sigma \mu_b)^\wedge(\tau) \to 0$ as $\tau \to \infty$. In general, $\sigma \to e^\sigma \mu_b$ is a continuous function from $\mathfrak{a}^*$ to $L_1(\mathfrak{a})$ since μ_b has compact support. So it is uniformly continuous on the compact set C from which the result follows as

$$|\varphi_{\sigma+i\tau}(b) - \varphi_{\sigma'+i\tau}(b)| \leq \| e^\sigma \mu_b - e^{\sigma'} \mu_b \|_{L_1(\mathfrak{a})}. \quad \square$$

One would like to have more precise asymptotic information on φ_ν as $\nu \to \infty$, but that does not seem to be obtainable by our simple methods.

REFERENCES

[1] T. H. KOORNWINDER, *A short proof of a Paley-Wiener type theorem for the Jacobi transform* (Preprint).

[2] B. KOSTANT, *On convexity, the Weyl group and the Iwasawa decomposition* (*Ann. scient. Éc. Norm. Sup.*, t. 6, 1973, p. 413-455).

[3] D. L. RAGOZIN, *Zonal measure algebras on isotropy irreducible homogeneous spaces* (Preprint).

(Manuscrit reçu le 1er octobre 1973.)

Mogens FLENSTED-JENSEN,
University of Copenhagen,
Universitetsparken 5,
DK-2100 Copenhagen, Denmark.
and David L. RAGOZIN,
University of Washington,
Seattle, Washington 98195.