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ACTION OF ALGEBRAIC GROUPS OF AUTOMORPHISMS ON THE DUAL OF A CLASS OF TYPE I GROUPS [(1) (2)]

By L. PUKANSZKY

1. Let G be a real linear algebraic group operating on the finite dimensional real vector space V. If G_0 is the connected component of the neutral element in G, by virtue of a known theorem of C. Chevalley (*cf.* [4], p. 316 and [6], p. 183, bottom) the orbit space V/G_0 is countably separated. Using this, in the present paper we are going to prove the the following :

THEOREM. — *Suppose, that* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are algebraic Lie algebras of endomorphisms of* V, *such that* $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$. *Let* G *be a connected and simply connected Lie group belonging to* $\mathfrak{g}$, *and* H *the analytic subgroup of* G *determined* $\mathfrak{h}$. *Then the natural action of* G *on the dual* $\hat{H}$ *of* H *is countably separated.*

An assertion of the indicated type plays an important role in J. Dixmier's discussion of the semifiniteness of the left ring of an arbitrary separable locally compact connected group (*cf.* [7], 4.6. Lemme, p. 432 and Théorème, p. 423 resp., and Remark 1.2 below) (3). Our proof depends in an essential fashion on various reasonings, to be specified later, of the papers [4] and [7] of this author.

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(2) An outline of this paper was submitted to the *International Congress on Harmonic Analysis*, University of Maryland, November 8-12, 1971.

(3) One of the motivations of the present research is to complete the proof given *loc. cit.* Our Theorem easily implies 4.6. Lemme in [7]. The proof of this, however, as given *loc. cit.* is incomplete, since it is not shown, that the maps Φ and Ψ (*cf.* p. 433) are bijective, as claimed there.

Remark 1.1. — We recall, that by virtue of 2.1. Proposition of [7] (p. 425), in particular, any real connected Lie group, which is locally isomorphic to a linear algebraic group, is of type I. Thus H, as in the Theorem, is of type I.

Remark 1.2. — Let us observe, that in [1] only the following special case is needed ([4]). Assume, that $\mathfrak{a}$ is a Lie algebra of endomorphisms of V, such that the greatest nilpotent ideal $\mathfrak{n}$ (= nilradical of $\mathfrak{a}$) consists of nilpotent endomorphisms. Then take for $\mathfrak{g}$ the smallest algebraic Lie algebra containing $\mathfrak{a}$ and set $\mathfrak{h} = \mathfrak{n} + [\mathfrak{a}, \mathfrak{a}]$.

Remark 1.3. — Let us add, that statements, similar to our Theorem, can be established in other situations. For instance, assume, that $\mathfrak{g}$ is a real solvable exponential Lie algebra, G a corresponding connected and simply connected group, $\mathfrak{d}$ an algebraic Lie algebra of derivations of $\mathfrak{g}$ such that $\mathfrak{d} \supset \mathrm{ad}(\mathfrak{g})$ and D the corresponding connected subgroup of Aut (G). *Then* $\hat{G}/D$ *is countably separated.* We only sketch the proof. We infer first from the result, quoted at the start, of Chevalley, that $\mathfrak{g}'/D$ ($\mathfrak{g}'$ = dual of the underlying space of $\mathfrak{g}$) is countably separated. On the other hand it is known, that there is a canonical bijection from $\mathfrak{g}'/(\mathrm{Ad}(G))'$ onto $\hat{G}$, which is equivariant with respect to Aut (G) and continuous (*cf.* [12], Proposition 2, p. 89 and the references at the end of *loc. cit.*). Since G is of type I, $\hat{G}$ is countably separated, and thus the said map is a Borel isomorphism (*cf.* e. g. [1], Proposition 2.11, p. 9); whence the result.

2. Some notational conventions. — Given a group G operating on the set X as a group of transformations, if x is some element in X, G_x will stand for the stabilizer of x in G. Given a group G or a Lie algebra $\mathfrak{g}$, we shall write $G^{\natural}$ ($\mathfrak{g}^{\natural}$ resp.) for its center. All Lie groups and Lie algebras occurring in the sequel will be assumed to be defined over the real field. Given a Lie algebra $\mathfrak{g}$, we shall write exp ($\mathfrak{g}$) for a corresponding connected and simply connected Lie group. Let G be any connected Lie group belonging to $\mathfrak{g}$, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$; then exp ($\mathfrak{h}$; G) will denote the analytic subgroup, determined by $\mathfrak{h}$, of G. ad ( ) and Ad ( ) will indicate the adjoint representation of $\mathfrak{g}$ and G resp. Given unitary representations U and V of a group G, we write $U \sim V$, if they are unitarily equivalent, and $U \approx V$, if they are quasi-equivalent (*cf.* e. g. [5], 13.1.4, p. 250). If G is invariant in K, for any $a \in K$, $a\,U$ will stand for the

([4]) *Cf.* footnote ([3]).

unitary representation defined by $(a\,\mathrm{U})(g) \equiv \mathrm{U}(a^{-1}ga)$ $(g \in \mathrm{G})$. *Observe, that we shall use the same symbol for a unitary representation and the corresponding unitary equivalence class, whenever convenient.*

3. Let H be a separable locally compact group, and N a closed, invariant and type I subgroup of H. Given a factor representation λ of H, $\lambda \mid \mathrm{N}$ is quasi-equivalent to a continuous direct sum, formed by aid of a Borel measure μ_λ on $\hat{\mathrm{N}}$, of irreducible representations of N, the equivalence class (= quasi-orbit) of μ_λ being uniquely determined by λ. Following [7], 4.1. Definition (p. 429), given a subset A of $\hat{\mathrm{N}}$, we shall write $\hat{\mathrm{H}}_{\mathrm{A}}$ for the set of all those elements λ in $\hat{\mathrm{H}}$, for which μ_λ is carried by A. Let ω be a fixed element of $\hat{\mathrm{N}}$. Then $\mathrm{S} = \mathrm{H}_\omega$ is a closed subgroup, containing N, of H. We denote by $\hat{\mathrm{S}}_\omega$ the subset of all those elements in $\hat{\mathrm{S}}$ which, when restricted to N, are multiples of ω. Let us assume, that G is a separable locally compact group containing H and N as closed invariant subgroups. ω being as before, we set $\Omega = \mathrm{G}\,\omega \subset \hat{\mathrm{N}}$; Ω is a Borel subset of $\hat{\mathrm{N}}$, and $\hat{\mathrm{H}}_\Omega$ is invariant under the action of G on $\hat{\mathrm{H}}$ (*cf.* the end of 2 above). Similarly G_ω transforms $\hat{\mathrm{S}}_\omega$ into itself.

LEMMA 3.1. — *With the above notations let us assume, that* H *is of type* I, *and that each* λ *in* $\hat{\mathrm{H}}_\Omega$ *gives rise to a transitive quasi-orbit on* $\hat{\mathrm{N}}$. *Then if one of the spaces* $\hat{\mathrm{H}}_\Omega/\mathrm{G}$ *or* $\hat{\mathrm{S}}_\omega/\mathrm{G}_\omega$ *is countably separated, so is the other, and then the two are Borel isomorphic.*

Proof. — *a.* We start by observing, that in virtue of our assumptions, $\hat{\mathrm{S}}_\omega$ is standard. In fact, we denote by ω' a $(\iota(\alpha))^{-1}$ extension of ω to S (*cf.* [1], p. 59-61) and put $\Lambda = (\mathrm{S/N}, \alpha)\hat{}$. Since H is of type I, $(\mathrm{S/N}, \alpha)$, too, is of type I (*cf.* [1], Proposition 10.4, p. 63) and thus there is a Borel cross section ψ from Λ into the space of concrete α representations of S/N. We set $\varphi(\lambda) = \omega' \otimes \iota(\psi(\lambda))$ $(\lambda \in \Lambda)$. It is known, that φ is a Borel injection from Λ into the standard Borel space Irr (S) (*cf.* [5], p. 323) and, if π is the canonical projection from Irr (S) onto $\hat{\mathrm{S}}$, $\pi \circ \varphi$ is a bijection between Λ and $\hat{\mathrm{S}}_\omega \subset \hat{\mathrm{S}}$. From this we conclude first, using the theorem of Souslin (*cf.* e. g. [1], Proposition 2.5, p. 7) that $\varphi(\Lambda) = \mathrm{E} \subset \mathrm{Irr}(\mathrm{S})$ is Borel. Next we observe, that the canonical map from $\mathrm{E} \subset \mathrm{Irr}(\mathrm{S}) \subset \mathrm{Fac}(\mathrm{S})$ onto its image in $\hat{\mathrm{S}}_{\mathrm{I}} \subset \hat{\mathrm{S}}$ is a Borel isomorphism (*cf.* [5], 7.2.3. Proposition, p. 136). This image, however, corresponds to $\hat{\mathrm{S}}_\omega$ under the canonical isomorphism between $\hat{\mathrm{S}}_{\mathrm{I}}$ and $\hat{\mathrm{S}}$ (*cf.* [5], 18.6.2, p. 324).

b. Let us put $O = H\omega \subset \hat{N}$. Our next objective is to establish the existence of a G_ω equivariant Borel isomorphism from $\hat{S}_\omega$ onto $\hat{H}_0$. Let γ be a Borel cross section from $\hat{S}_\omega$ into its inverse image in Irr (S). Then the map $\eta \mapsto \delta(\eta) = \operatorname{ind}_{S \uparrow H} \gamma(\eta)$ $\left(\eta \in \hat{S}_\omega\right)$ is a Borel map from $\hat{S}_\omega$ into Irr (H) (*cf.* [11], Theorem 10.1, p. 123) such that, if σ stands for the canonical projection from Irr (H) onto $\hat{H}$, $\Phi = \sigma \circ \delta$ is a Borel bijection from $\hat{S}_\omega$ onto $\hat{H}_0$. Since H is of type I, the Borel structure of $\hat{H}$ is standard and hence, by Souslin's theorem, Φ is a Borel isomorphism. Its G_ω equivariance is clear.

We conclude from the preceding discussion, that to establish our Lemma, it suffices to prove the assertion arising by replacing *loc. cit.* $\hat{S}_\omega / G_\omega$ through $\hat{H}_0 / G_\omega$.

c. In the sequel we are going to make a repeated use of the following result (*cf.* [1], Proposition 2.11, p. 9). Assume, that E and F are Borel spaces such that E is analytic and F countably separated. Let *f* be a Borel map from E into F, and let us also write *f* for the equivalence relation on E defined by the condition, that $x \sim y$ $(x, y \in E)$ if and only if $f(x) = f(y)$. Then $f(E) \subset F$ is analytic, and the natural map from E/f onto $f(E)$ is a Borel isomorphism. Thus E/f, too, is analytic.

From this we conclude first, thar $\hat{H}_\Omega$ is analytic. In fact, since any λ in $\hat{H}_\Omega$ gives rise to a transitive quasi orbit on $\hat{N}$, we have $\hat{H}_\Omega = \cup_{a \in G} a \hat{H}_0$. Let us denote by F the map, from $G \times \hat{H}_0$ into $\hat{H}$, defined by $F(a, \lambda) = a\lambda$; we have $F\left(G \times \hat{H}_0\right) = \hat{H}_\Omega$. Since $\hat{H}_0$ is standard [*cf.* (*a*) above], so is $G \times \hat{H}_0$. It is known, that the map $(a, \eta) \mapsto a\eta$ from $G \times \hat{H}$ onto $\hat{H}$ is continuous, and thus F is Borel. Finally, H being of type I, $\hat{H}$ is countably separated. In this fashion, by the assertion quoted above, $F\left(G \times \hat{H}_0\right) = \hat{H}_\Omega \subset \hat{H}$ is analytic.

d. Let us suppose now, that $\hat{H}_0 / G_\omega$ is countably separated. We are going to show, that then so is $\hat{H}_\Omega / G$ and that the two are Borel isomorphic. In fact : (*d* 1) Let us denote by ρ the canonical projection from $\hat{H}_0$ onto $\hat{H}_0 / G_\omega$. It is easy to see, that if we have $a_1 \lambda_1 = a_2 \lambda_2$ $\left(a_1, a_2 \in G;\ \lambda_1, \lambda_2 \in \hat{H}_0\right)$, then $\rho(\lambda_1) = \rho(\lambda_2)$. In fact, since λ_k $(k = 1, 2)$ restricts on N to $O = H\omega$, we obtain at once $a_1 H \omega = a_2 H \omega$ and thus $a_2^{-1} a_1$ is of the form bh ($b \in G_\omega$, $h \in H$) implying $\lambda_2 = b \lambda_1$ or $\rho(\lambda_1) = \rho(\lambda_2)$. Since $\hat{H}_\Omega = G\hat{H}_0$, we conclude from this, that there is a map Φ from $\hat{H}_\Omega$ onto $\hat{H}_0/G_\omega$ such that $\Phi(a\lambda) = \rho(\lambda)$ $(a \in G,\ \lambda \in \hat{H}_0)$. (*d* 2) We show next, that Φ is Borel. We set $\varphi = \Phi \circ F$ and observe, that since $\varphi((a, \lambda)) = \rho(\lambda)$, φ is a Borel map from $G \times \hat{H}_0$ onto $\hat{H}_0/G_\omega$. Let B be a Borel subset of the last space and set $D = \overset{-1}{\Phi}(B)$; we have to prove, that $D \subset \hat{H}_\Omega$ is Borel. By what we have just seen, $\overset{-1}{F}(D)$ is Borel in $G \times \hat{H}_0$, from where the desired conclusion follows through an easy application of the assertion, quoted at the start of (*c*). (*d* 3) By the same token, setting $E = \hat{H}_\Omega$ [= analytic by (*c*)], $F = \hat{H}_0/G_\omega$ (= countably separated by assumption) and $f = \Phi$ we get, that $\hat{H}_\Omega/\Phi$ is Borel isomorphic to $\hat{H}_0/G_\omega$. From here to complete our proof it is enough to observe, that on $\hat{H}_\Omega$ the Φ-fibers and G orbits clearly coincide.

e. In order to prove our Lemma, it suffices now to show, that if $\hat{H}_\Omega/G$ is countably separated, then it is Borel isomorphic to $\hat{H}_0/G_\omega$. Let I be the identity map from $\hat{H}_0$ onto $\hat{H}_\Omega$ $(\supset \hat{H}_0)$, π the canonical projection from the latter onto $\hat{H}_\Omega/G$, and let us put $\Psi = \pi \circ I$. Then Ψ is a Borel map from $\hat{H}_0$ onto $\hat{H}_\Omega/G$. From here we complete our proof proceeding analogously as in (*d*) above by observing, that on $\hat{H}_0$ the families of Ψ fibers and G_ω orbits resp. coincide. Q. E. D.

4. Proposition 4.1. — *Let* G, H *and* N *be as above. We assume, that each* λ *in* $\hat{H}$ *gives rise to a transitive quasi orbit on* $\hat{N}$, *and that* $\hat{N}/G$ *is countably separated. Then* $\hat{H}/G$ *is countably separated if and only if, for any choice of* ω *in* $\hat{N}$, $\hat{S}_\omega/G_\omega$ (*cf.* 3) *is countably separated.*

Proof. — *a.* Let us suppose first, that $\hat{H}/G$ is countably separated. If ω is any fixed element of $\hat{N}$ and $\Omega = G\omega$, then $\hat{H}_\Omega/G$, too, is countably separated. Thus, by the previous lemma, so is $\hat{S}_\omega/G_\omega$.

b. Our proof of the opposite implication follows closely a reasoning of J. Dixmier (*cf.* [7], p. 430-431). To start we recall (*cf.* [9], Theorem 1, p. 124) that if G is a locally compact group acting as transformation group on the locally quasi-compact and almost Hausdorff space M, G and M both being assumed to satisfy the second axiom of countability then, in particular, the following conditions are equivalent : (1) M/G is countably separated; (2) For any m in M, G m is locally closed [*cf.* (3) and (1), *loc. cit.*]. Instances, to be considered in the sequel, of this situation are provided by a separable locally compact group acting on the dual of a type I closed invariant subgroup. In this fashion to establish our point it is enough to show that, for any λ in $\hat{H}$, $G\lambda$ is locally closed in $\hat{H}$. Let us suppose, that λ restricts to $H\omega \subset \hat{N}\,(\omega \in \hat{N})$. Since, by hypothesis, $\hat{N}/G$ is countably separated, by virtue of the result just quoted, $\Omega = G\omega$ is locally closed in $\hat{N}$. From this, using 4.2.Lemme (iii) in [7] we conclude, that $\hat{H}_\Omega$ is locally closed in $\hat{H}$. In this fashion, to arrive at the desired conclusion it suffices to show, that $G\lambda$ is locally closed in $\hat{H}_\Omega$. In fact, in this case there are closed sets F, F_1 and open sets O, O_1 in $\hat{H}$, such that $\hat{H}_\Omega = F \cap O$ and $G\lambda = \hat{H}_\Omega \cap F_1 \cap O_1$, and thus $G\lambda = (F \cap F_1) \cap (O \cap O_1)$, proving our point. Since, by assumption, $\hat{S}_\omega / G_\omega$ is countably separated, the Lemma of Section 3 above implies the same for $\hat{H}_\Omega / G$. $\hat{H}_\Omega$, being locally closed in $\hat{H}$, it is, in the induced topology, locally quasi compact and almost Hausdorff. Therefore another application of the result quoted at the start of this section yields, that $G\lambda$ is locally closed in $\hat{H}_\Omega$, completing the proof of our Proposition. Q. E. D.

5. We insert here another proof for one of the assertions in the demonstration of Lemma 3.1 (Section 3) which, though operating under somewhat more restrictive assumptions, throws light from a different angle on the situation studied there. Let G, H, N, ω and Ω be as *loc. cit.*, and *assume, that* $\Omega \subset \hat{N}$ *is locally closed and* $G_\omega H$ *is closed in* G. (Observe, that the last two conditions are satisfied, if $\hat{N}/G$ and $\hat{N}/H$ are countably separated.) *Then, if* $\hat{H}_0 / G_\omega$ *is countably separated, so is* $\hat{H}_\Omega / G$ [*cf.* (*d*), *loc. cit.*]. We start by observing, that Ω/H is standard. In fact, since $\hat{N}$ is standard, the map ψ, from G/G_ω onto $\Omega = G\omega \subset \hat{N}$, defined by $\psi(a\,G_\omega) = a\,\omega$ $(a \in G)$ is a Borel isomorphism between these spaces (*cf.* [1], Proposition 3.7, p. 16). Since ψ is clearly equivariant with respect to H, to obtain the desired conclusion it is enough to observe that, $G_\omega.H$ being closed, $G/G_\omega.H$ is standard and Borel isomorphic to Ω/H. From this we infer, that any λ in $\hat{H}_\Omega$, when restricted to N, gives rise to a transitive H orbit $0(\lambda)$.

We define a map φ from $\hat{H}_\Omega$ onto $D = \Omega/H$ by setting $\varphi(\lambda) = 0(\lambda)$ and claim, that φ is continuous. To show this it is enough to establish, that if $\mathcal{O}$ is any H invariant open set in $\hat{N}$, then $\hat{H}_{\Omega\cap\mathcal{O}}$ is open in $\hat{H}_\Omega$. This, however, follows from the easily verifiable fact, that $\hat{H}_{\Omega\cap\mathcal{O}} = \hat{H}_\Omega \cap \hat{H}_\mathcal{O}$ along with the observation, that $\hat{H}_\mathcal{O}$ is open in H (*cf.* [7], 4.2. Lemme, (ii)). In this fashion, in particular, φ is Borel, and is evidently G equivariant. Next we recall the following (*cf.* [1], p. 70-75). Let E be an analytic G space, D a transitive analytic G space, and φ a G-equivariant Borel map from E onto D. If μ is some Borel measure on E, and ν its direct image on D, then for each y in D there is a Borel measure μ_y, carried by $E(y) = \overset{-1}{\varphi}(y)$, on E, such that for any Borel set σ in E,

$$\mu(\sigma) = \int_D \mu_y(\sigma)\, d\nu(y).$$

Let us assume furthermore, that μ is quasi-invariant, ergodic and non-transitive under G. Then, as shown by C. C. Moore, we can assume that, for each a in G and y in E, μ_{ay} and $a\,\mu_y$ are equivalent, and that μ_y is non transitive-ergodic, with respect to G_y, on $E(y)$ (*cf.* Theorem 1, p. 75, *loc. cit.*). Bearing this in mind, we complete our proof in the following fashion. Since Ω, by assumption, is locally closed, so is $\hat{H}_\Omega$ (*cf.* 4 above), and thus it is locally quasi compact and almost Hausdorff. If $\hat{H}_\Omega/G$ is not countably separated, there is a Borel measure μ, which is quasi-invariant, ergodic and non-transitive with respect to G, on $\hat{H}_\Omega$ (*cf.* [9], Theorem 1, (3) and (4), p. 124). We apply the result just quoted with $E = \hat{H}_\Omega$, $D = \Omega/H$ and φ defined as above. Let y be the image, in Ω/H, of $\omega \in \Omega$; then $G_y = G_\omega . H$ and $E(y) = \hat{H}_0$. Thus there is a Borel measure μ', quasi-invariant, ergodic and non-transitive with respect to G_ω, on $\hat{H}_0$. But this contradicts out assumption that $\hat{H}_0/G_\omega$ is countably separated.

Q. E. D.

6. We turn now the proof of our Theorem and employ the notations explained in Section 1. We proceed by induction, assuming the validity of the assertion to be established for dimensions, smaller than that of $\mathfrak{g}$. The method of reduction, to be used in the sequel, is partly inspired by a reasoning of J. Dixmier (*cf.* [4], p. 326-328).

We denote by $\mathfrak{m}$ the greatest ideal of nilpotency (*cf.* [2], Définition 2, p. 60) of the identical representation of $\mathfrak{h}$. Let $\mathfrak{r}$ be the radical of $\mathfrak{h}$ and $\mathfrak{s}$ the set of all nilpotent elements of $\mathfrak{r}$; then $\mathfrak{m} = \mathfrak{s}$. In fact $\mathfrak{m}$, being composed of nilpotent endomorphisms, is a nilpotent ideal of $\mathfrak{h}$, and therefore $\mathfrak{m} \subseteq \mathfrak{r}$ and hence also $\mathfrak{m} \subseteq \mathfrak{r}$. The converse is implied by Corollaire 6 in [2] (p. 67). In the following we shall distinguish notions, already defined for $\mathfrak{h}$, with respect to $\mathfrak{g}$, by an index zero. This being so we observe, that $\mathfrak{m} = \mathfrak{m}_0 \cap \mathfrak{h}$, and thus, in particular, $\mathfrak{m}$ is an ideal in $\mathfrak{g}$. To show this all what we have to establish is that $\mathfrak{m} \subseteq \mathfrak{m}_0$. But $\mathfrak{m} \subseteq \mathfrak{r} = \mathfrak{r}_0 \cap \mathfrak{h} \subseteq \mathfrak{r}_0$, and hence the desired conclusion is implied by the result of [2] just quoted.

We can assume from the beginning, that $\mathfrak{m}$ is nontrivial. In fact, otherwise the nilpotent radical $[\mathfrak{h}, \mathfrak{r}]$ of $\mathfrak{h}$, too, is trivial (*cf.* [2], Définition 3 and Remarque 2, p. 64, and Proposition 6, p. 81 resp.) In this case, however, $\mathfrak{h}$ is reductive and hence of the form $\mathfrak{h}_1 \times \mathfrak{h}_2$, where $\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}]$ and $\mathfrak{h}_2 = \mathfrak{h}^{\natural}$ (*cf.* [2], Proposition 5, p. 79). Putting $\mathrm{H}_k = \exp(h_k; \mathrm{H})$ ($k = 1, 2$) (*cf.* 2) we have also $\mathrm{H} = \mathrm{H}_1 \times \mathrm{H}_2$ and $\hat{\mathrm{H}} = \hat{\mathrm{H}}_1 \times \hat{\mathrm{H}}_2$ (product of Lie groups and of topological spaces resp.). Since $\mathfrak{h}_1$ is semi-simple, we have $\mathrm{Ad}(\mathrm{G}) \mid \mathfrak{h}_1 \subseteq \mathrm{Ad}(\mathrm{H}_1)$ and thus $\hat{\mathrm{H}}/\mathrm{G} = \hat{\mathrm{H}}_1 \times \left(\hat{\mathrm{H}}_2/\mathrm{G}\right)$. Hence it suffices to show, that the last factor is countably separated. Since, by assumption, the Lie algebra $\mathfrak{h}$ of G is algebraic, $\mathrm{Ad}(\mathrm{G})$ $[\subset \mathrm{GL}(\mathfrak{g})]$, and thus also $\mathrm{Ad}(\mathrm{G}) \mid \mathfrak{h}_2\, [\subset \mathrm{GL}(\mathfrak{h}_2)]$ is the connected component of a real linear algebraic group. Hence the same is valid for the contragredient group operating on $\mathfrak{h}_2'$, whence the desired conclusion follows by an easy application of the result of Chevalley quoted at the start of Section 1.

We recall [*cf.* 4 (*b*)], that $\hat{\mathrm{H}}/\mathrm{G}$ is countably separated if and only if, for each λ in $\hat{\mathrm{H}}$, $\mathrm{G}\lambda$ is locally closed in $\hat{\mathrm{H}}$. Let us choose and keep fixed a λ of the said sort. Since $\mathfrak{m}$ is nontrivial and nilpotent, it contains a nonzero abelian $\mathfrak{g}$ ideal $\mathfrak{a}$ (e. g. $\mathfrak{m}^{\natural}$). We set $\mathrm{A} = \exp(\mathfrak{a}; \mathrm{H}) \subset \mathrm{H}$. A is closed, simply connected and invariant under G. By virtue of a reasoning just given, $\hat{\mathrm{A}}/\mathrm{G}$ and $\hat{\mathrm{A}}/\mathrm{H}$ are countably separated. From this we conclude first, that λ on A restricts to a transitive quasi-orbit $\mathrm{H}\,\omega$ (say). Next, writing $\Omega = \mathrm{G}\,\omega \subset \hat{\mathrm{A}}$, we remark, that Ω is locally closed and hence so is $\hat{\mathrm{H}}_\Omega$ [*cf.* 4 (*b*)] in $\hat{\mathrm{H}}$. In this fashion we shall attain our goal by establishing, that $\mathrm{G}\lambda$ is locally closed in $\hat{\mathrm{H}}_\Omega$ (cf. *loc. cit.*). Let us set $\mathrm{S} = \mathrm{H}_\omega$.

LEMMA 6.1. — S *is of type* I.

Proof. — We recall (*cf.* 1), that $\mathfrak{h}$ is given as an algebraic subalgebra of $\mathfrak{gl}(\mathrm{V})$. Let H_1 be the irreducible algebraic group, $\subset \mathrm{GL}(\mathrm{V})$, determined by $\mathfrak{h}$. For each fixed h in H_1 the map $a \mapsto h a h^{-1}$ ($a \in \mathfrak{a}$) transforms $\mathfrak{a}$

into itself, and thus H_1 operates on A and $\hat{A}$. Let us put $d\omega = if$ ($f \in \mathfrak{a}'$). We have $(H_1)_\omega = (H_1)_f$, the right hand side being taken with respect to the action, contragredient to that just described, of H_1 on $\mathfrak{a}'$. Hence $(H_1)_\omega$ is algebraic. Let us denote by $\overline{H}$ the connected component of the identity in H_1 and by τ the canonical homomorphism from $H\,[= \exp(\mathfrak{h})]$ onto $\overline{H}$. We have $S = \tau^{-1}((\overline{H})_\omega)$ and hence S_0 is locally isomorphic to $(\overline{H})_\omega$ and thus also to $(H_1)_\omega$ implying (*cf.* Remark 1.1), that S_0 is of type I. Let us denote by S_1 the complete inverse image, in H, of $((\overline{H})_\omega)_0$. We have $\tau(S_1) = \tau(S_0)$ and hence $S_1 = KS_0$, where $K = \ker(\tau) \subset H^{\natural}$. From this we conclude at once, that S_1, too, is of type I. In fact, writing R (T) for the von Neumann algebra generated by the unitary representation T, we have clearly $R(U) = R(U \mid S_0)$ for any factor representation U of S. From here we complete our proof by observing that S/S_1, being isomorphic to a subgroup of $(H_1)_\omega/((H_1)_\omega)_0$, is finite, implying, that S is of type I (*cf.* [4], Lemme 3, p. 319).

Q. E. D.

We shall distinguish two cases, according to whether : (A) There is an abelian $\mathfrak{g}$ ideal $\mathfrak{a}$ in $\mathfrak{m}$, such that either $\dim(\mathfrak{a}) > 1$, or if $\dim(\mathfrak{a}) = 1$, $\mathfrak{a} \not\subset \mathfrak{g}^{\natural}$; (B) Condition (A) can not be fulfilled. Case (B) will be discussed in the next section.

(Ad A) Here we shall consider two subcases : (A 1) $\dim(\Omega) > 0$; (A 2) $\dim(\Omega) = 0$.

(A 1) To show, that $G\lambda$ is locally closed it is enough to establish [*cf.* 4 (*b*) above], that $\hat{H}_\Omega/G$ is countably separated. Since A is abelian and $\hat{A}/H$ countably separated we can employ Lemma 3.1 by substituting A in place of N *loc. cit.* and conclude that, to attain our objective, it suffices to show, that $\hat{S}_\omega/G_\omega$ is countably separated. This, however, is implied by the following :

LEMMA 6.2. — *If* $\dim(G_\omega) < \dim(G)$, $\hat{S}/G_\omega$ *is countably separated.*

Proof. — (*a*) Let us put $F = (G_\omega)_0$ and *show first, that* $\hat{S}_0/F$ *is countably separated.* We denote by G_1 the irreducible algebraic group, determined by $\mathfrak{g}$, in GL (V). H_1 (*cf.* Lemma 6.1) is invariant in G_1. We see as *loc. cit.*, that G_1 operates on A and $\hat{A}$, and that $(G_1)_\omega$ is algebraic, containing $(H_1)_\omega$ as an invariant subgroup. Let us denote by $\mathfrak{g}_\omega$ and $\mathfrak{h}_\omega$ resp. the Lie algebras of these groups. They are algebraic Lie algebras in $\mathfrak{gl}(V)$, $\mathfrak{h}_\omega$ is an ideal in $\mathfrak{g}_\omega$ and hence they satisfy the conditions of our Theorem

(Section 1). We denote by $\mathfrak{F}$ the universal covering of F; we have $\mathfrak{F} = \exp(\mathfrak{g}_\omega)$, Let us write $\mathfrak{H} = \exp(\mathfrak{h}_\omega;\ \mathfrak{F})$. Since

$$\dim(\mathfrak{g}_\omega) = \dim(G_\omega) < \dim(G) = \dim(\mathfrak{g})$$

we are in position to employ the assumption of our inductive procedure (*cf.* the start of this section) and conclude, that $\hat{\mathfrak{H}}/\mathfrak{F}$ is countably separated. From here, to obtain the desired conclusion, it is enough to remark, that $\hat{S}_0$ is identifiable to a closed, $\mathfrak{F}$ invariant subset of $\hat{\mathfrak{H}}$ and that clearly $\hat{S}_0/F = \hat{S}_0/\mathfrak{F}$.

(*b*) Let us write $\bar{G}$ for $(G_1)_0$, and σ for the canonical homomorphism from G onto $\bar{G}$. Its kernel L is discrete in $G^\natural$ and we show, as in the proof of the previous lemma (with G in place of H *loc. cit.*), that L. F is of a finite index in G_ω. Hence the latter operates on $\hat{S}_0/F$ as a finite group and thus we can conclude, that $\hat{S}_0/G_\omega$ *is countably separated.*

(*c*) We recall, that $S_1 = KS_0$, where $K = L \cap H \subset L \subset G^\natural$; thus S_1 is invariant in G_ω. Next we show, that $\hat{S}_1/G_\omega$ *is countably separated.* This can be obtained, for instance, through a simple application of Proposition 4.1. In fact, upon replacing *loc. cit.* G, H and N by G_ω, S_1 and S_0 resp., all conditions are fulfilled since, besides what we saw in the proof of the previous lemma, by virtue of (*b*) $\hat{S}_0/G_\omega$ is countably separated and S_1 acts trivially on $\hat{S}_0$. Therefore $\hat{S}_1/G_\omega$ is countably separated since, for each η in $\hat{S}_0$, its stabilizer in G_ω acts trivially on $(\hat{S}_1)_\eta$.

(*d*) From here we complete the proof of our lemma through another application of Proposition 4.1. We substitute *loc. cit.* in place of G, H and N, G_ω, S and S_1 resp. Since S/S_1 is finite, by virtue of what we saw above all conditions are met. Therefore to conclude, that $\hat{S}/G_\omega$ is countably separated it is enough to remark that, for each η in $\hat{S}_1$, putting $K = S_\eta$, $\hat{K}_\eta$ is finite. Q. E. D.

Summing up, in this fashion we have completed the proof, that if $\dim(\Omega) > 0$ [case (A 1)] $G\lambda$ is locally closed in $\hat{H}$.

(A 2) Here we assume, that $\dim(\Omega) = 0$. This implies immediately, that the restriction of λ to A is a multiple of a character ω of A. Furthermore, ω is G invariant, hence its kernel J is a closed invariant subgroup, contained in A, of G. We remark, that its dimension is positive. This is evident, if dim $(\mathfrak{a}) > 1$. If, however, dim $(\mathfrak{a}) = 1$, ω is the unit character and thus $A = J$. In fact, by assumption, in this case $\mathfrak{a}$ is not contained in $\mathfrak{g}^{\natural}$. Hence, if l_0 is a nonzero element in $\mathfrak{a}$, we have

$$\mathrm{Ad}\,(a)\, l_0 \equiv \mu\,(a)\, l_0 \;(a \in G),$$

where μ is not identically one. In this fashion we have for all a in G and real t :

$$(a\,\omega)\,(\exp\,(tl_0)) = \omega\,(\exp\,(t\,\mathrm{Ad}\,(a^{-1})\,l_0)) = \omega\,(\exp\,([t/\mu\,(a)]\,l_0)) = \omega\,(\exp\,(tl_0)),$$

proving our statement. Let us denote by $\mathfrak{j}$ the Lie algebra of J. By what we have just seen $\mathfrak{j}$ is a nontrivial $\mathfrak{g}$ ideal contained in $\mathfrak{a} \subseteq \mathfrak{m}$. Hence, being composed of nilpotent endomorphisms, it is an algebraic subalgebra of $\mathfrak{gl}$ (V). We recall, that a linear representation ρ of $\mathfrak{g}$ is called rational, if it is the differential of a rational representation of G_1 (*cf.* [3], p. 47). In this case $\rho\,(\mathfrak{h})$ is algebraic for any algebraic subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (*cf.* [3], Corollaire 1, p. 48). Finally we conclude from Proposition 11 in [3] (p. 119), that there is a rational representation σ of $\mathfrak{g}$, such that its kernel is equal to $\mathfrak{j}$. With σ so chosen, the pair $\sigma\,(\mathfrak{g}) \supset \sigma\,(\mathfrak{h})$ satisfies the conditions of our Theorem. Let $\mathfrak{G}$ be a simply connected group belonging to $\rho\,(\mathfrak{g})$ and let us put $\mathfrak{H} = \exp\,(\sigma\,(\mathfrak{h});\, \mathfrak{G})$. Since dim $(\mathfrak{G}) <$ dim (G) we are in position to apply the hypothesis of our induction and conclude, that $\hat{\mathfrak{H}}/\mathfrak{G}$ is countably separated. We set $J = \exp\,(\mathfrak{j};\, H)$. $\mathfrak{H}$ is isomorphic to H/J and hence $\hat{\mathfrak{H}}$ identifies to a closed subsed of $\hat{H}$. J is contained in the kernel of λ and thus, if φ is the canonical homomorphism from H onto $\mathfrak{H}$, there is a $\tilde{\lambda}$ in $\hat{\mathfrak{H}}$, such that $\lambda = \tilde{\lambda} \circ \varphi$. Finally one sees at once, that $\mathfrak{G}\,\tilde{\lambda} = G\,\lambda$ under the above identification. Since the left hand side is locally closed in $\hat{\mathfrak{H}} \subset \hat{H}$, so is the right hand side in $\hat{H}$, which is the desired conclusion.

7. (Ad B) We recall (*cf.* 6 above), that to complete the proof of our Theorem it suffices to discuss the situation arising when any abelian $\mathfrak{g}$ ideal $\mathfrak{a}$ in $\mathfrak{m}$ is of dimension one and contained in $\mathfrak{g}^{\natural}$ [case (B), *loc. cit.*]. In the following we shall assume, that $\mathfrak{m}$ itself is not abelian, and leave the modifications, necessary to settle the remaining case to the reader.

We recall, that a nilpotent algebra $\mathfrak{n}$ is called a Heisenberg algebra, if $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}^{\natural}$, and dim $(\mathfrak{n}^{\natural}) = 1$. Let us show, that $\mathfrak{m}$ is a Heisenberg algebra. It is clear, that dim $(\mathfrak{m}^{\natural}) = 1$ and $[\mathfrak{m}, \mathfrak{m}] \supset \mathfrak{m}^{\natural}$; therefore it is enough to establish, that dim $([\mathfrak{m}, \mathfrak{m}]) = 1$. To prove this we proceed as in Lemma 10 of [4] (p. 325). If dim $([\mathfrak{m}, \mathfrak{m}]) > 1$, there are ideals

$\mathfrak{m}_k$ ($k = 1, 2$) of $\mathfrak{m}$, such that

$$\dim(\mathfrak{m}_k) = k \quad (k = 1, 2) \qquad \text{and} \qquad [\mathfrak{m}, \mathfrak{m}] \supseteq \mathfrak{m}_2 \supset \mathfrak{m}_1 = \mathfrak{m}^\natural.$$

But then $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}_2] \subseteq [\mathfrak{m}, \mathfrak{m}_1] = 0$ and thus the center of $[\mathfrak{m}, \mathfrak{m}]$, which is an abelian $\mathfrak{g}$ ideal in $\mathfrak{m}$, is of a dimension larger than one, contradicting the assumption of (B).

Our objective is to show that, for any λ in $\hat{H}$, $G\lambda$ is locally closed in $\hat{H}$. Let us put $M = \exp(\mathfrak{m}; H)$. Since we have $M^\natural \subseteq H^\natural$, $\lambda \mid M^\natural$ is multiple of some character η of $M^\natural$. We can assume, that η is not identically one; in fact otherwise, to attain our goal, we could adopt the reasoning of (A 2) in 6. Since M is a Heisenberg group, it admits a uniquely determined irreducible representation ω which, on $M^\natural$, restricts to a multiple of η. Since $M^\natural \subset G^\natural$, ω is G invariant and clearly $\lambda \mid M$ is a multiple of ω, or $\lambda \in \hat{S}_\omega$ (*cf.* the start of 3). Since $\{\omega\}$ is closed in $\hat{M}$, so is $\hat{S}_\omega$ in $\hat{H}$ (*cf.* e. g. [7], 4.2. Lemme, (i)). In this fashion to show, that $G\lambda$ is locally closed in $\hat{H}$, it suffices to prove the same with $\hat{S}_\omega$ in place of $\hat{H}$. In order to establish this, it is enough to show, that $\hat{S}_\omega/G$ is countably separated [*cf.* 4 (*b*)].

The subsequent construction is inspired by 4.4 and 4.5 in [7] (p. 431).

LEMMA 7.1. — *Let* $\mathfrak{g}$ *be some Lie algebra,* $\mathfrak{h}$ *an ideal in* $\mathfrak{g}$ *and* $\mathfrak{r}_0$ *the radical of* $\mathfrak{g}$. *There is an ideal* $\mathfrak{g}_1$ *of* $\mathfrak{g}$, *such that* $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{h}$ (sum of subspaces) *and* $[\mathfrak{g}_1, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{r}_0] \cap \mathfrak{h}$.

Proof. — We write $\mathfrak{r}$ for the radical of $\mathfrak{h}$; we have $\mathfrak{r} = \mathfrak{r}_0 \cap \mathfrak{h}$. Let $\mathfrak{d}_2$ and $\mathfrak{d}$ be Levi subalgebras of $\mathfrak{h}$ and $\mathfrak{g}$ resp., such that $\mathfrak{d}_2 \subseteq \mathfrak{d}$ (*cf.* [2], Définition 7, p. 88 and Corollaire 1 (*b*), p. 91). We have $\mathfrak{d}_2 = \mathfrak{d} \cap \mathfrak{h}$ and hence there is an ideal $\mathfrak{d}_1$ in $\mathfrak{d}$, such that $\mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2$ and $[\mathfrak{d}_1, \mathfrak{d}_2] = 0$. We set $\mathfrak{g}_1 = \mathfrak{d}_1 + \mathfrak{r}_0$ and claim, that it meets the requirements of our lemma. First $\mathfrak{g}_1$ is an ideal, since $[\mathfrak{g}_1, \mathfrak{g}] \subseteq [\mathfrak{d}_1, \mathfrak{d}] + \mathfrak{r}_0 = \mathfrak{d}_1 + \mathfrak{r}_0 = \mathfrak{g}_1$. We have evidently $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{h}$. Finally, since $[\mathfrak{g}_1, \mathfrak{h}] \subseteq \mathfrak{h}$, to complete our proof it is enough to observe that, by virtue of $[\mathfrak{d}_1, \mathfrak{d}_2] = 0$,

$$[\mathfrak{g}_1, \mathfrak{h}] = [\mathfrak{d}_1 + \mathfrak{r}_0, \mathfrak{d}_2 + \mathfrak{r}] \subseteq [\mathfrak{d}_1, \mathfrak{r}] + [\mathfrak{d}_2, \mathfrak{r}_0] + [\mathfrak{r}_0, \mathfrak{r}] \subseteq [\mathfrak{g}, \mathfrak{r}_0].$$

Q. E. D.

COROLLARY 7.1. — *Suppose, that* $\mathfrak{g}$, $\mathfrak{h}$ *and* $\mathfrak{m}$ *are as prior to* Lemma 7.1. *Then there is an ideal* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *such that, putting* $G_1 = \exp(\mathfrak{g}_1; G)$, *we have* $G = G_1 . H$ *and, for any* r *in* G_1 *and* h *in* H, $rhr^{-1} \in hM$ [$M = \exp(\mathfrak{m}; G)$].

Proof. — The first assertion is clear from $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{h}$ along with the connectedness of G. To prove the second we observe first, that $\mathfrak{m}$ being the greatest ideal of nilpotency of the identical representation of $\mathfrak{h} \subset \mathfrak{gl}(V)$, we have $[\mathfrak{g}, \mathfrak{r}_0] \cap \mathfrak{h} \subset \mathfrak{m}$, and hence also $[\mathfrak{g}_1, \mathfrak{h}] \subset \mathfrak{m}$. From here, by virtue of the formula of Hausdorff-Campbell we infer, that if r and h are close to the unity in G_1 and H resp., we have $rh.r^{-1}h^{-1} \in M$ and thus $rhr^{-1} \subset h\,M$. Using the connectedness of G_1 and H, and the invariance of M in G the validity of the same relation, for any r and h as just indicated, follows easily. Q. E. D.

Let f be an element of $\mathfrak{m}'$ (= dual of the underlying space of $\mathfrak{m}$), such that ω (as above) belongs to the orbit $M f \subset \mathfrak{m}'$ in the sense of Kirillov. All we shall use from this circumstance in the sequel will be that : (1) We have then $\eta(\exp(l)) = \exp[i(l, f)]$ $(l \in \mathfrak{m})$; hence, since $\eta \not\equiv 1$, f is not orthogonal to $\mathfrak{m}^{\natural}$. (2) If M (as above) is a closed, invariant subgroup of the Lie group A, and A_f is the stabilizer of f with respect to the representation which is contragredient to $A \ni a \mapsto \mathrm{Ad}(a) \mid \mathfrak{m}$, we have $A_\omega = A_f.M$. Thus, in particular, $G = G_\omega = G_f.M$.

Let ω' be an $(\iota(\alpha))^{-1}$ extension of ω from M to $H_\omega = H$ [$\alpha \in Z^2(H/M, T)$; *cf.* for all this [1], Section 4, p. 18]. We set $K = H/M$ and, given some group R, we write X (R) for the group of characters of R. We recall (cf. *loc. cit.*). That if ω'' is another extension of the indicated kind, there is an element χ of X (K) such that $\omega'' = \iota(\chi)\,\omega'$, where $\iota(\chi)$ stands for χ lifted to H. Let a be some element of G_1. We infer from Corollary 7.1, that $a\,\omega'$ is a $(\iota(\alpha))^{-1}$ extension of $a\,\omega$; hence we have, with some χ in X (K), $a\ \omega' = \iota(\chi)\,\omega'$ in the sense of unitary equivalence.

Lemma 7.2. — *There is a continuous homomorphism* χ *from* $(G_1)_f$ *into* X (K), *such that, for each* a *in* $(G_1)_f$, $a\,\omega' = \iota(\chi(a))\,\omega'$ *in the sense of unitary equivalence.*

We shall give the precise form of χ below (*cf.* Remark 7.1).

Proof. — (*a*) We denote by H (ω) the space of ω and show, that there is a continuous unitary representation T of G_f on H (ω) such that $T(a)\,\omega(m)\,T(a^{-1}) = \omega(ama^{-1})$ for all a in G_f and m in M. In the following we shall write al for $\mathrm{Ad}(a)\,l$ $(a \in G,\ l \in \mathfrak{m})$.

(*a* 1) Let c be a nonzero element of $\mathfrak{m}^{\natural}$. Since $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}^{\natural}$ $(= R\,c)$, there is a bilinear form B_1 on $\mathfrak{m} \times \mathfrak{m}$ such that $[x, y] = B_1(x, y)\,c$ $(x, y \in \mathfrak{m})$. By $\mathfrak{m}^{\natural} \subset \mathfrak{g}^{\natural}$, we see at once, that $B_1(ax, ay) \equiv B_1(x, y)$ for all a in G. We denote by $\mathfrak{m}_0$ the annihilator of f in $\mathfrak{m}$; we have clearly $G_f\,\mathfrak{m}_0 \subseteq \mathfrak{m}_0$. Since f is not orthogonal to $\mathfrak{m}^{\natural}$, the restriction of B_1 to $\mathfrak{m}_0 \times \mathfrak{m}_0$ is nondegenerate. Let $\mathfrak{S}$ be the group of linear transformations on $\mathfrak{m}_0$ leaving B invariant. Summing up, putting $\psi(a) = \mathrm{Ad}(a) \mid \mathfrak{m}_0$ $(a \in G_f)$, we obtain a representation of G_f in $\mathfrak{S}$.

(a 2) We denote by $\mathfrak{R}$ the universal covering of $\mathfrak{S}$, and by p the canonical projection from $\mathfrak{R}$ onto $\mathfrak{S}$. We claim, that there is a homomorphism ψ' of G_f into $\mathfrak{R}$, such that the diagramm

$$\begin{array}{ccc} G_f & & \\ {\scriptstyle\psi}\downarrow & \searrow{\scriptstyle\psi'} & \\ \mathfrak{S} & \xleftarrow[p]{} & \mathfrak{R} \end{array}$$

be commutative. To this end it suffices to show, that G_f is simply connected. We have $M_f = M^\sharp$, hence M_f is connected and simply connected. In this fashion it is enough to establish that G_f/M_f, too, is connected and simply connected. But since $G = G_\omega = G_f.M$ we have :

$$G/M = G_f.M/M = G_f/G_f \cap M = G_f/M_f,$$

proving our statement.

(a 3) We have $\mathfrak{m} = \mathfrak{m}_0 + R\,c$. One verifies easily that, for any fixed s in $\mathfrak{S}$, the map $l + tc \mapsto sl + tc$ ($l \in \mathfrak{m}_0$, $t \in R$) defines an automorphism of $\mathfrak{m}$. Let us denote by $m \mapsto m^s$ ($m \in M$) the corresponding automorphism of M. One shows easily, that $m^{\psi(a)} = ama^{-1}$ ($m \in M$; $a \in G_f$).

(a 4) It is known (*cf.* [10], Lemma 3, p.39), that there is a representation W of $\mathfrak{R}$ on H (ω), such that $W(b)\,\omega(m)\,W(b^{-1}) = \omega(m^{p(b)})$ ($m \in M$, $b \in \mathfrak{R}$). Hence, to obtain T with the properties specified above, it is enough to put $T(a) = W(\psi'(a))$ ($a \in G_f$).

(b) Let γ be a Borel cross section from H_f/M_f into H_f, and let us set $S = \gamma(H_f/M_f)$. We have then $H_f = SM_f$. Since $H = H_\omega = H_f.M$, any a in H admits a unique representation sm ($s \in S$, $m \in M$); sometimes we shall write $s(a)$ and $m(a)$ in place of s and m resp. This being so, let us define $\omega'(a) = T(s(a))\,\omega(m(a))$. We claim, that

$$\omega'(a)\,\omega'(b) = \beta(a, b)\,\omega'(ab), \qquad \text{where} \quad \beta(a, b) = \overline{\eta(m(s(a).s(b)))} \qquad (a, b \in H).$$

In fact, assuming $a = rm$, $b = tn$ (t, $t \in S$; m, $n \in M$) we have

$$\omega'(a).\omega'(b) = T(r)\,\omega(m)\,T(t)\,\omega(n) = T(rt)\,\omega(t^{-1}mt.n) = T(s(rt)).\omega(t^{-1}mt.n)$$

by (a), bearing in mind, that $T \mid M_f \equiv I$. On the other hand,

$$ab = rmtn = s(rt)\,[m(rt)\,t^{-1}mt.n]$$

and thus since $\omega \mid M^\sharp$ is a multiple of η,

$$\omega'(ab) = \eta(m(rt))\,T(s(rt))\,\omega(t^{-1}mtn)$$

proving our statement. Since $\beta(am, bn) \equiv \beta(a, b)$ $(a, b \in H,\ m, n \in M)$, there is a $\alpha \in Z^2(H/M, T)$ such that $\beta = (\iota(\alpha))^{-1}$ and hence ω' is a $(\iota(\alpha))^{-1}$ extension of ω from M to H.

(*c*) Given elements a, b of G, we shall write $[a, b] = aba^{-1}b^{-1}$. Let c be a fixed element of $(G_1)_f$. We are going to show, that with T and ω' as in (*a*) and (*b*) above, we have $(c\,\omega')(a) \equiv \varphi_c(a)(T(c))^{-1}\,\omega'(a)\,T(c)$, where $\varphi_c(a) \equiv \eta([r^{-1}, c^{-1}])$ $[r = s(a),\ a \in H]$. In fact putting $c^{-1}rc = r.r_c$, by Corollary 7.1, r_c belongs to $M_f = M^{\xi}$. Hence, writing m in place of $m(a)$, we have by definition

$$(c\,\omega')(a) = \omega'(c^{-1}ac) = T(r).\omega(r_c.c^{-1}mc) = \eta(r_c)\,T(r).\omega(c^{-1}mc).$$

Since, by (*a*), $\omega(c^{-1}mc) = T(c^{-1}).\omega(m)\,T(c)$, and T is a representation of G_f, to complete our proof, it is enough to establish, that $T(crc^{-1}) = T(r)$. This, however, is implied by $T \mid M^{\xi} \equiv I$.

(*d*) Next we show the existence of a continuous homomorphism χ from $(G_1)_f$ into X (K), such that $\varphi_c(a) \equiv \iota(\chi(c))(a)$ $[c \in (G_1)_f)\ a \in H]$. This results from the following series of observations, the verification of which we leave to the reader. (*d* 1) Setting, for a fixed $c \in (G_1)_f$ and $a \in H_f$, $\psi_c(a) = \eta([c, a])$, the map $a \mapsto \psi_c(a)$ is in $X(H_f)$; (*d* 2) Evidently $\psi_c \mid M_f \equiv 1$ and hence, since $M_f = H_f \cap M$, there is an element ψ'_c of X (H), uniquely determined by the conditions $\psi'_c \mid H_f \equiv \psi_c \mid M \equiv 1$; (*d* 3) The map $(G_1)_f \ni c \mapsto \psi'_c$ is a continuous homomorphism of $(G_1)_f$ in X (H); (*d* 4) We have $\overline{\varphi_c(a)} \equiv \psi'_c(a)$ $(a \in H)$. Hence, finally, there is a ψ, as in our lemma, with $\varphi_c(b) \equiv \iota(\chi(c))(b)$ $[a \in (G_1)_f,\ b \in H]$.

(*e*) We have thus established the claim of our lemma for a particular choice, described in (*b*) above, of ω'. From here to prove the general case it is enough to recall, that if ω'' is another projective extension of ω from M to H, we have $\omega'' \equiv \iota(h)\,\omega'$, where h is some mesurable function from $K = H/M$ into the circle group. Q. E. D.

Remark 7.1. — Observe that, by what we have just seen, χ can be chosen such, that we have $\iota(\eta(a))(b) \equiv \chi([a^{-1}, b]) \equiv \eta([a, b^{-1}])$ $[a \in (G_1)_f,\ b \in H_f]$. Since $H = H_f.M$, these conditions determine χ uniquely.

Remark 7.2. — Using the results of [8] and Lemma 7.1 above, one can prove analogous results for the following situation. Suppose, that G is a connected Lie group, H, N are closed, connected, invariant subgroups of G and $H \supseteq N$. We denote by $\mathfrak{n}$ the Lie algebra of N and assume, that $\mathfrak{n}$ is nilpotent. Let ω be an irreducible representation of N corresponding, in the sense of Kirillov, to the orbit $Nf \subset \mathfrak{n}'$, and such that $G_\omega = G$. All what one has to do is to replace above η by χ_f, the latter being the character of N_f, uniquely determined by the condition that $d\chi_f = i\,(f \mid n_f)$ ($\mathfrak{n}_f$ = Lie algebra of N_f).

We recall (*cf.* the start of this section), that to complete the proof of our Theorem it suffices to show, that $\hat{S}_\omega/G$ is countably separated $\left(\hat{S}_\omega = \left\{\xi;\ \xi \in \hat{H},\ \xi \mid M \approx \omega\right\}\right)$. Since $G = G_1 H$, and $G_1 = (G_1)_\omega = (G_1)_f M$ we have $G = (G_1)_f H$ and thus it is enough to consider $\hat{S}_\omega/(G_1)_f$. Let again ω' be an $(\iota(\alpha))^{-1}$ extension of ω to H, and let us put $\Lambda = (K, \alpha)^\wedge$. Since H is of type I, Λ is standard. As in (*a*) of the proof of Lemma 3.1 (Section 3) we show, that there is a Borel isomorphism φ from Λ onto $\hat{S}_\omega$, such that

$$\varphi(\lambda) = \omega' \otimes \iota(\lambda) \qquad (\lambda \in \Lambda).$$

If a is in $(G_1)_f$, we have $a.\varphi(\lambda) = \varphi(\chi(a)\lambda)$; hence to attain our goal, it suffices to prove, that the action of $(G_1)_f$ on Λ, consisting of multiplication by χ, is countably separated. We remark, that $K = H/M$ is simply connected and reductive. The last observation follows from $[\mathfrak{h}, \mathfrak{r}_0] \subset \mathfrak{m}$ ([2], Remarque 2, p. 64 and Proposition 6, (*a*), p. 81). Let us denote by $a \in H^2(K, T)$ the cohomology class of $\alpha \in Z^2(K, T)$, and by K_a the corresponding central extension, by the circle group, of K. Hence K_a contains a well-determined central 1-torus T, such that K_a/T is isomorphic to K. A class one representation of K_a is an element of $\hat{K}_a$ which, when restricted to T, coincides with the identity map of the circle group onto itself. We denote by $\overset{A}{K}_a$ the set of all unitary equivalence classes of class one representations and recall, that there is a canonical Borel isomorphism between Λ and $\overset{A}{K}_a$. Let us write χ' [$\in$ Hom (K_a, T)] for χ (*cf.* Lemma 7.2) lifted to K_a. Then to the action of $b \in (G_1)_f$ on Λ, it is the multiplication by $\chi'(b) \in X(K_a)$ which corresponds on $\overset{A}{K}_a$. By what we saw above concerning K, K_a is a direct priduct $D \times A_a \times A$, where D is semi-simple, A_a central extension, by T, of a vector group, such that A_a is also factor group of a Heisenberg group according to a discrete subgroup of its center, and A is a vector group (*cf.* the proof of Lemma 9, p. 324 in [4]). It is known, that $\hat{A}_a$ contains a uniquely determined class one representation. Let ψ and φ be the restriction of χ' onto D and A resp. Since D is semi-simple, ψ is trivial. In this fashion we conclude, that there is a Borel isomorphism between $\overset{A}{K}_a$, and hence also between $\hat{S}_\omega$, and $\hat{D} \times \hat{A}$, such that the action of $a \in (G_1)_f$ on the former goes over into multiplication by $\varphi(a) \in X(A)$ of the $\hat{A}$ component of the latter. Since $(G_1)_f$ is connected, $\varphi((G_1)_f) = L$ is a closed subgroup of A. Hence, finally, $\hat{S}_\omega/G = \hat{S}_\omega/(G_1)_f$ is Borel isomorphic to $\hat{D} \times (\hat{A}/L)$ and thus, in particular, countably separated, completing the proof of our main theorem.

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