

# MÉMOIRES DE LA S. M. F.

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*Mémoires de la S. M. F.*, tome 46 (1976), p. 121-130

[http://www.numdam.org/item?id=MSMF\\_1976\\_\\_46\\_\\_121\\_0](http://www.numdam.org/item?id=MSMF_1976__46__121_0)

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NONLINEAR STRUCTURES DETERMINED BY MEASURES ON BANACH SPACES

By K. David ELWORTHY

0. INTRODUCTION.

A. A Gaussian measure  $\gamma$  on a separable Banach space  $E$ , together with the topological vector space structure of  $E$ , determines a continuous linear injection  $i : H \rightarrow E$ , of a Hilbert space  $H$ , such that  $\gamma$  is induced by the canonical cylinder set measure of  $H$ . Although the image of  $H$  has measure zero, nevertheless  $H$  plays a dominant role in both linear and nonlinear analysis involving  $\gamma$ , [8], [9], [10]. The most direct approach to obtaining measures on a Banach manifold  $M$ , related to its differential structure, requires a lot of extra structure on the manifold : for example a linear map  $i_x : H \rightarrow T_x M$  for each  $x$  in  $M$ , and even a subset  $M_H$  of  $M$  which has the structure of a Hilbert manifold, [6], [7]. In the manifold case it has not been clear how much of this additional structure is really required ; or, slightly reformulated : do certain measures on an infinite dimensional manifold  $M$ , together with the differential structure of  $M$ , determine any such additional structures ? As a special case of this we can ask whether every diffeomorphism of  $E$  which preserves the Gaussian measure  $\gamma$  necessarily maps  $i(H)$  to itself, or has derivatives which preserve  $i(H)$ .

Along similar lines, it is plausible that the Hilbert manifold  $L_{x_0}^{2,1}(x)$  of  $L^{2,1}$  paths starting at  $x_0$  on a Riemannian manifold  $X$  may play a central role for the Wiener measure on the manifold  $C_{x_0}(X)$  of continuous paths in  $X$ , [6], [7]. If so it should be possible to characterise  $L_{x_0}^{2,1}(X)$  in terms of that measure and the differentiable structure of  $C_{x_0}(X)$ .

Although we do not answer these questions, we show here that any strictly positive Radon measure on a smooth manifold determines some structure : namely a partition of  $M$  into subsets invariant under measure preserving diffeomorphisms, and subspaces in the tangent spaces to  $M$  invariant under the derivatives of such diffeomorphisms. For infinite dimensional  $M$  these are shown to be non-trivial in a class of important cases : and the partition may well be non trivial in general, in infinite dimensions. A concrete consequence is obtained in Corollary 4 A : *the group of diffeomorphisms of an infinite dimensional separable Banach space  $E$  preserving a given Gaussian measure does not act transitively on  $E$ .* This is false for the group of measure class preserving diffeomorphisms : Theorem 1B. Another consequence, concerning group actions, is given in § 3C.

The precise definitions of the invariants may seem rather unnatural : they have been chosen from a wide range of similar definitions simply in order to make the theorems true and to show that non-trivial invariants exist, not because of any obvious intrinsic geometric meaning. A particularly interesting point is that the interpolation K-functors, as described by PEETRE in [13], play an important role in several different places : especially Corollary 2 C and Proposition 4 B. Full proofs and a more detailed discussion will be made available elsewhere.

B. We are concerned with measures on topological spaces : by which we mean positive Borel measures, usually locally finite (or even finite) and strictly positive ; so every point has a neighbourhood of finite measure, and each open set has non-zero measure. Moreover in order that our constructions are non-trivial we shall often have to assume that the measures  $\mu$  are *tight* i.e. for each Borel set B

$$\mu(B) = \sup \{ \mu(K) : K \subset B, K \text{ compact} \}.$$

This follows automatically, when  $\mu$  is finite, if the space is separable and admits a complete metric, see [12], [15]. Recall that a Borel measure is a *Radon* measure if it is locally finite and tight.

Two measures  $\lambda, \mu$  on X are *equivalent*,  $\lambda \approx \mu$ , if they have the same sets of measure zero. If so the Radon-Nikodym derivatives  $\frac{d\lambda}{d\mu}$ ,  $\frac{d\mu}{d\lambda}$  are defined, almost everywhere, as measurable functions. This relation between measures seems to be too weak for our purposes (see Theorem 1B) : so for  $x$  in X we define  $\lambda$  and  $\mu$  to be *pointwise equivalent at x*

$$\lambda \approx \mu \text{ pointwise at } x$$

if for all neighbourhood bases  $\mathcal{U}$  at  $x$ , directed by inclusion, and all Borel sets B

$$\liminf_{U \in \mathcal{U}} \frac{\mu(U \cap B)}{\lambda(U \cap B)} > 0 \quad \text{and} \quad \liminf_{U \in \mathcal{U}} \frac{\lambda(U \cap B)}{\mu(U \cap B)} > 0$$

where, in the computation of the lower limits, we replace  $\frac{0}{0}$  by 1 and  $\frac{r}{0}$  by  $\infty$ , if  $r > 0$ .

For strictly positive measures  $\lambda, \mu$ , we see that if  $\lambda, \mu$  are orthogonal they are not pointwise equivalent at any point, while, for X first countable, if  $\lambda$  and  $\mu$  are equivalent they are pointwise equivalent at  $x$  iff both Radon-Nikodym derivatives are essentially bounded in some neighbourhood of  $x$ .

### 1. GAUSSIAN MEASURES.

A. Since Gaussian measures furnish our main test bed we quickly give the defini-

tion and most relevant properties. For simplicity we consider only strictly positive, mean zero measures.

Let  $E$  be a separable real Banach space. A measure  $\gamma$  on  $E$  is *Gaussian* if for all continuous linear surjections with finite dimensional range :

$$T : E \rightarrow F_T$$

the induced measure  $T(\gamma)$  on  $F_T$

$$T(\gamma)(B) = \gamma T^{-1}(B) \quad B \in \text{Borel}(F_T)$$

is given by

$$T(\gamma)(B) = (2\pi)^{-n/2} \int_B \exp\left(-\frac{|x|_T^2}{2}\right) dx$$

where  $n = \dim F_T$ , and the Lebesgue measure and norm come from some inner product  $\langle \cdot, \cdot \rangle_T$  on  $F_T$ .

For such a measure  $\gamma$  :

G1 : [4], there is a compact linear injective map  $i : H \rightarrow E$ , of a Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  into  $E$  such that the inner product  $\langle \cdot, \cdot \rangle_T$  in the definition is the quotient inner product under the map  $T \circ i : H \rightarrow F_T$ .

G2 : the image of  $H$ ,  $i(H)$ , has  $\gamma$ -measure 0.

G3 : translation,  $T_z : E \rightarrow E$ , by an element  $z$  of  $E$  preserves sets of measure zero, i.e.  $T_z(\gamma) \approx \gamma$ , iff  $z$  lies in the image of  $H$ .

G4 : if  $j : E^* \rightarrow H$  denotes the adjoint of  $i$  then  $T_z(\gamma) \approx \gamma$  pointwise at some point iff  $z$  lies in the image of  $E^*$  by  $j \circ i$ , in which case  $T_z(\gamma) \approx \gamma$  pointwise at every point of  $E$ .

G5 : the image of  $H$  in  $E$  is the intersection of all measurable linear subspaces of  $E$  with non-zero measure. Such subspaces have measure 1 (see [2] for a short proof).

B. Given a Gaussian measure  $\gamma$  on  $E$  let  $\Phi : U \rightarrow V$  be a  $C^1$  diffeomorphism of open subsets of  $E$ , having the form  $\Phi(x) = x + i \circ j \circ \alpha(x)$  where  $\alpha : U \rightarrow E^*$  is  $C^1$ . Then H-H. KUO [10], proved that  $\Phi$  preserves sets of measure zero and its "jacobian"

$$\frac{d\Phi^{-1}(\gamma)}{d\gamma} = U \rightarrow \mathbb{R}$$

is given by  $x \mapsto |\det D\Phi(x)| \exp\left\{\frac{1}{2}[-2\alpha(x)(x) - |j \circ \alpha(x)|^2]\right\}$  (the determinant

refers to  $D\varphi(x)|_H : H \rightarrow H$ , and is proved to exist).

It follows that

$$\varphi(\gamma|U) \approx \gamma|V \text{ pointwise on } V.$$

Ramer [14] has a stronger version of Kuo's theorem.

When  $E$ , and hence  $H$ , is infinite dimensional we can follow the construction used by BESSAGA [1], or [3], to show that  $H - \{0\}$  is diffeomorphic to  $H$ , and for any  $v$  in  $E$  we can obtain a  $C^\infty$  diffeomorphism  $\varphi : E \rightarrow E$  with  $\varphi(0) = v$  and such that

$$\varphi|_{E - \{0\}} : E - \{0\} \rightarrow E - \{v\}$$

satisfies the conditions of Kuo's theorem. It follows that still  $\varphi(\gamma) \approx \gamma$  : although now the pointwise equivalence at  $v$  will be lost in general. This proves :

THEOREM 1B. - *Let  $\gamma$  be a strictly positive Gaussian measure on a separable Banach space  $E$ . Then the group of  $C^\infty$  diffeomorphisms preserving  $\gamma$  up to equivalence, acts transitively on  $E$ .*

We show below, Corollary 4A, that the theorem is false for infinite dimensional  $E$  when equivalence is replaced by pointwise equivalence at all points of  $E$ . In any case the theorem does not necessarily imply that measure class preserving diffeomorphisms can behave in a completely abandoned way : for example

*Problem.* - With the notation of the theorem : does there exist a diffeomorphism  $\varphi : E \rightarrow E$  with

$$\begin{aligned} \varphi(\gamma) &\approx \gamma \\ \text{and } \varphi(i(H)) \cap i(H) &= \emptyset ? \end{aligned}$$

## 2. TANGENT CONES AND INTERPOLATION FUNCTORS.

A.† Let  $A$  be a subset of the real Banach space  $E$ .

For a point  $a$  in the closure  $\bar{A}$  of  $A$  we shall define the *tangent cone*  $TC_a(A)$  to  $A$  at  $a$  by

$$TC_a(A) = \{v \in E \text{ s.t. } d(a + sv, A) = O(s^2) \text{ as } s \rightarrow 0\}.$$

Note that the more natural definition would have  $O(s)$  instead of our  $O(s^2)$ . This would have the advantage of being invariant under  $C^1$  diffeomorphisms, but Corollary 2C below would not hold with that definition. Our construction is easily seen to

have the following properties.

TC(i) : If  $v \in TC_a(A)$  and  $\lambda > 0$  then  $\lambda v \in TC_a(A)$ .

TC(ii) : If  $A$  is convex then so is  $TC_a(A)$ .

TC(iii) : If  $\varphi : U \rightarrow V$  is a  $C^2$  map of open sets of Banach spaces, and  $\bar{A} \subset U$ , then  $D\varphi(a)(TC_a(A)) \subset TC_{\varphi(a)}(\varphi(A))$ .

From TC(iii) it follows that tangent cones are defined for subsets  $A$  of  $C^2$  Banach manifolds  $M$ . They then lie in the tangent spaces :

$$TC_a(A) \subset T_a M.$$

B. Let  $\vec{E}$  denote a pair of Banach spaces  $(E_1, E)$  with a given continuous linear injection  $i : E_1 \rightarrow E$ . As in PEETRE [13], for  $0 < t < \infty$  and  $v \in E$  define

$$K(t, v) = \inf \{ \|v - v_1\| + t \|v_1\|_1 : v_1 \in E_1 \},$$

where  $\| \cdot \|$  and  $\| \cdot \|_1$  denote the norms of  $E$  and  $E_1$  respectively, and elements of  $E_1$  are identified with their image in  $E$ .

For  $0 < \theta < 1$  define

$$\vec{E}_{\theta, \infty} = \{ v \in E : K(t, v) = O(t^\theta) \}$$

and set  $\|v\|_{\theta, \infty} = \sup_{t > 0} \frac{K(t, v)}{t^\theta}$  if  $v \in \vec{E}_{\theta, \infty}$ .

This is a special case of the more general construction of  $K$ -functors described in [13]. The properties we need are

K1 :  $\vec{E}_{\theta, \infty}$  with  $\| \cdot \|_{\theta, \infty}$  is a Banach space.

K2 : The map  $i$  factorizes by continuous linear maps

$$E_1 \xrightarrow{\alpha} \vec{E}_{\theta, \infty} \xrightarrow{\beta} E.$$

K3 : If  $i$  was compact then so are both  $\alpha$  and  $\beta$ .

C. For  $\vec{E}$  as above, let  $B_1(x; r)$  denote the closed ball

$$\{ y : \|x - y\|_1 \leq r \}$$

about  $x$ , radius  $r$ , in  $E_1$ .

Define the *contact space*  $\tau(E_1, E)$  of  $\vec{E}$  by

$$\tau(E_1, E) = TC_0(i[B_1(0; 1)]).$$

PROPOSITION 2C. - As subsets of  $E$

$$\tau(E_1, E) = E_1 \left( \frac{1}{2}, \infty \right).$$

From property K3 and the fact that any compact subset of  $E$  lies in the image of the unit ball of Banach space mapped into  $E$  by a compact linear map, the proposition yields :

COROLLARY 2C. - If  $K$  is a compact subset of the infinite dimensional Banach space  $E$  then for all  $a \in K$

$$\text{Linear span } TC_a(K) \neq E.$$

[However  $TC_a(K)$  can certainly be dense in  $E$ .]

### 3. INFINITESIMAL PROPERTIES OF MEASURES.

A. Let  $\mu$  be a strictly positive, locally finite, measure on a metric space  $(M, d)$ . We say that the Borel subset  $A$  of  $M$  *infinitesimally supports*  $\mu$  at the point  $a$  of  $M$ ,  $A \in \text{Supp}(\mu ; a)$ , if for all  $r > 0$   $\frac{\mu(B(a ; t) - A)}{\mu(B(a ; rt^2))} \rightarrow 0$  as  $t \rightarrow 0$  where  $B(a ; t)$

denotes the closed ball about  $a$ , radius  $t$ . It is easy to see that this definition depends only on the local Lipschitz class of the metric  $d$ , and on the pointwise equivalence class at  $a$  of  $\mu$ .

PROPOSITION 3A. - If  $\mu$  is a Radon measure, (e.g. if  $(M, d)$  is complete and if  $A \in \text{Supp}(\mu ; a)$ ) then there is a compact set  $K$  with

$$K \subset A \cup \{a\}$$

and  $K \in \text{Supp} \{ \mu ; a \}.$

Now suppose that  $M$  is a separable  $C^2$  Banach manifold and that the metric  $d$  is in the local Lipschitz class determined by the differentiable structure. For  $a$  in  $M$  define the *tangent cone*,  $\tau_c(\mu) = \bigcap \{ TC_a(A) : A \in \text{Supp}(\mu ; a) \}.$

Thus  $\tau_c(\mu) \subset T_a M.$

THEOREM 3A :

(i) For every strictly positive Radon measure  $\mu$  on the infinite dimensional metrizable  $C^2$  Banach manifold  $M$  the tangent cone to  $\mu$  at a general point  $a$  satisfies.

$$\text{Linear span } \tau_c(\mu) \neq T_a M.$$

(ii) Let  $\varphi : M \rightarrow M$  be a  $C^2$  diffeomorphism such that

$\varphi(\mu) \approx \mu$  pointwise at  $\varphi(a)$ . Then

$$T_a \varphi [\tau_c(\mu)] = \tau_c \varphi(a)(\mu).$$

Part (i) follows from Corollary 2C and Proposition 3A, and (ii) is straightforward.

B. We have yet to show that  $\tau_c(\mu)$  can ever be larger than  $\{0\}$  when  $E$  is infinite dimensional. This can be done using a more geometric differential invariant of measures: For a point  $a$  of a  $C^2$  manifold  $M$  and a strictly positive measure  $\mu$  on  $M$  define

$$Q_a(\mu) \subset T_a M$$

to consist of those tangent vectors  $v$  for which there exists a  $C^2$  vector field  $\xi$  on  $M$  with  $\xi(a) = v$  such that there is a neighbourhood  $V$  of  $a$  and positive constants  $\epsilon, \alpha$  satisfying

(i) the flow  $\sigma : V \times (-\epsilon, \epsilon) \rightarrow M$  of  $\xi$  is defined.

(ii) there is a base  $\mathcal{B}$  for the neighbourhood system of  $a$  in  $V$  with

$$\mu(\sigma_t(B)) \geq \alpha \mu(B) \text{ for all } 0 < t < \epsilon, B \in \mathcal{B}.$$

THEOREM 3B. -  $Q_a(\mu) \subset \tau_c(\mu)$ .

COROLLARY 3B1. -  $Q_a(\mu)$  does not span  $T_a M$ , if  $M$  is infinite dimensional and separable.

COROLLARY 3B2. - For a Gaussian measure  $\gamma$  on a separable Banach space  $E$ ,

$$\tau_c(\gamma) \neq \{0\} \quad \text{all } a \in E.$$

In 3B2 we have  $i_* j(E^*) \subset Q_a(\gamma) \subset \tau_c(\gamma)$ , each  $a \in E$ .

C. As an application of the above: if  $G \times M \rightarrow M$  is a  $C^2$  action of a Banach Lie group  $G$  on a metrizable Banach manifold  $M$  which preserves, up to pointwise equivalence, some strictly positive, Radon measure on  $M$ , then for each  $a$  in  $M$  the derivative map at the identity

$$T_e G \rightarrow T_a M$$

obtained from  $g \mapsto g.a$  is compact.

However it seems likely that the above is true for group actions which only preserve the measure up to equivalence. For the linear case with  $G$  a group of translations see [16].



## 4. ORDERING INDUCED BY A MEASURE.

A. Let  $\mu$  be a strictly positive measure on a metric space  $(M, d)$ . For  $x$  and  $y$  in  $M$  write

$$x < y, \text{ if } \lim_{s \rightarrow 0} \frac{\mu(B(y; s))}{\mu(B(x, rs))} = 0, \text{ for all } r > 0.$$

If neither  $x < y$  nor  $y < x$  write  $x \sim y$ . It is easy to see that this defines an equivalence relation on  $M$ .

PROPOSITION 4A. - Let  $f : M \rightarrow M$  be a homeomorphism which is locally bi-Lipschitz and satisfies  $f(\mu) \approx \mu$  pointwise on  $M$ . Then  $x \sim f(x)$  all  $x \in M$ .

From the proposition, in order to show that the group of such homeomorphisms  $f$  of  $M$  does not act transitively on  $M$  it suffices to show that the equivalence relation  $\sim$  is non-trivial. Possibly this is true for a general class of measures on  $M$  when  $M$  is infinite dimensional. The proof of the following theorem depends on the fact that Gaussian measures are convex in the sense of BORELL [2] : in fact the theorem is true for arbitrary convex measures.

THEOREM 4A. - For a Gaussian measure  $\gamma$  on the Banach space  $E$  ; if  $\bar{0}$  denotes the equivalence class of  $0$ , we have

$$\bar{0} \subset \tau_{c_0}(\gamma).$$

This combines with Theorem 3A. to give the required non-triviality, whence :

COROLLARY 4A. - Let  $E$  be a separable infinite dimensional Banach space and  $\gamma$  a Gaussian measure on  $E$ . Then the group of locally bi-Lipschitz homeomorphisms of  $E$  with  $f(\gamma) \approx \gamma$  pointwise on  $E$  does not act transitively on  $E$ .

B. The problem remains of characterizing the orbits of  $0$  under the group of diffeomorphisms of Corollary 4A, (or under the group of measure preserving diffeomorphisms) or perhaps a simpler problem is to characterize the equivalence class  $\bar{0}$  of Theorem 4A. The following is suggestive, at least of the type of characterizations which may be true.

PROPOSITION 4B. - For a Gaussian measure  $\gamma$  on  $E$ , with corresponding maps

$$E^* \xrightarrow{j} H \xrightarrow{i} E \text{ we have}$$

$$\tau(E^*, H) \subset \bar{0}$$

In fact for all  $z \in H$  and  $x \in \tau(E^*, H)$  we have  $z \sim z+x$ .

[We identify points of  $E^*$  and  $H$  with their images in  $E$ ].

*Proof.* - Let  $\| \cdot \|$  denote the norm of  $H$ , and  $\| \cdot \|$  that of  $E$ .

Since  $x \in \tau(E^*, H)$  there is a function  $e : (0, 1) \rightarrow E^*$  and a constant  $k$  with

$$|x - e(s)| < \frac{1}{2} s$$

and 
$$\|e(s)\|_{E^*} \leq k/s.$$

By the change of variable formula, § 1B, for  $e = e(s)$

$$\begin{aligned} \gamma(B(z; s)) &= \int_{B(z+e; s)} \exp(e(y) - \frac{1}{2} |e|^2) d\gamma(y) \\ &= \exp(e(z) + \frac{1}{2} |e|^2) \int_{B(z+e; s)} \exp(e(y-e-z)) d\gamma(y) \\ &\geq \exp(e(z) + \frac{1}{2} |e|^2) \exp(-s \|e\|_{E^*}) \gamma(B(z+e; s)) \\ &\geq \exp(\langle x, z \rangle - \langle x-e, z \rangle + \frac{1}{2} |e|^2 - k) \gamma(B(z+e; s)) \end{aligned}$$

Now  $B(z+x; \frac{1}{2} s) \subset B(z+e; s)$ , so we have

$$\lim_{s \rightarrow 0} \frac{\gamma(B(z; s))}{\gamma(B(z+x; \frac{1}{2} s))} \geq \exp(\langle x, z \rangle - k) > 0$$

whence  $z \leq z + x$ .

substitution shows that also

$$z + x \leq (z+x) - x$$

giving

$$z \sim z+x \text{ as required.}$$

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