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SUBMERSIONS AND EFFECTIVE DESCENT OF ÉTALE MORPHISMS

BY DAVID RYDH

ABSTRACT. — Using the flatification by blow-up result of Raynaud and Gruson, we obtain new results for submersive and subtrusive morphisms. We show that universally subtrusive morphisms, and in particular universally open morphisms, are morphisms of *effective* descent for the fibered category of étale morphisms. Our results extend and supplement previous treatments on submersive morphisms by Grothendieck, Picavet and Voevodsky. Applications include the universality of geometric quotients and the elimination of noetherian hypotheses in many instances.

RÉSUMÉ (*Submersion et descente effective de morphismes étales*)
On applique le théorème de « platification » de Raynaud et Gruson aux morphismes subtrusifs et obtient le théorème de structure suivant: Tout morphisme universellement subtrusif de présentation finie a un raffinement se factorisant en un recouvrement ouvert suivi d'un morphisme propre. La première application de ce théorème de structure est un théorème de descente *effective.* On montre que tout morphisme universellement subtrusif est un morphisme de descente effective pour la catégorie fibrée des morphismes étales. Ce résultat réduit l'écart entres schémas et espaces algébriques. Par exemple, on peut montrer que des quotients géométriques sont universels dans la catégorie des espaces algébriques. La deuxième application concerne les limites projectives de schémas. On démontre que tout morphisme universellement subtrusif de présentation finie est la limite de morphismes universellement *submersifs* entre schémas noethériens. Il en découle que la classe de morphismes subtrusifs, introduite par Picavet,

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est une extension naturelle de la classe de morphismes submersifs entre schémas noethériens. Avec des méthodes semblables on montre aussi un énoncé analogue pour les morphismes universellement ouverts. De plus, on généralise aux espaces algébriques les propriétés fondamentales des topologies h et qfh introduites par Voevodsky.

Introduction

Submersive morphisms, that is, morphisms inducing the quotient topology on the target, appear naturally in many situations such as when studying quotients, homology, descent and the fundamental group of schemes. Somewhat unexpected, they are also closely related to the integral closure of ideals. Questions related to submersive morphisms of *schemes* can often be resolved by topological methods using the description of schemes as locally ringed spaces. Corresponding questions for *algebraic spaces* are significantly harder as an algebraic space is not fully described as a ringed space. The main result of this paper is an effective descent result which bridges this gap between schemes and algebraic spaces.

The first proper treatment of submersive morphisms seems to be due to Grothendieck [19, Exp. IX] with applications to the fundamental group of a scheme. He shows that submersive morphisms are morphisms of descent for the fibered category of étale morphism. He then proves *effectiveness* for the fibered category of quasi-compact and separated étale morphisms in some special cases, e.g., for finite morphisms and universally open morphisms of finite type between noetherian schemes. Our main result consists of several very general effectiveness results extending those of Grothendieck significantly. For example, we show that any universal submersion of noetherian schemes is a morphism of effective descent for quasi-compact étale morphisms. As an application, these effectiveness results imply that strongly geometric quotients are categorical in the category of algebraic spaces [35].

Later on Picavet singled out a subclass of submersive morphisms in [32]. He termed these morphisms *subtrusive* and undertook a careful study of their main properties. The class of subtrusive morphisms is natural in many respects. For example, over a locally noetherian scheme, every submersive morphism is subtrusive. Picavet has also given an example showing that a finitely presented universally submersive morphism is not necessarily subtrusive. In particular, not every finitely presented universally *submersive* morphism is a limit of finitely presented submersive morphisms of noetherian schemes. We will show that every finitely presented universally *subtrusive* morphism is a limit of finitely presented submersive morphisms of noetherian schemes. This is a key result missing in [32] allowing us to eliminate noetherian hypotheses in questions about universal subtrusions of finite presentation. It also shows that the class of subtrusive morphisms is indeed an important and very natural extension of submersive morphisms of noetherian schemes.

A general observation is that in the noetherian setting it is often useful to describe submersive morphisms using the subtrusive property. For example, there is a valuative criterion for submersions of noetherian schemes [25, Prop. 3.7] which rather describes the essence of the subtrusiveness.

Structure theorem. — An important tool in this article is the structure theorems for universally subtrusive morphisms given in §3: Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a morphism $g : X' \to X$ and a factorization of $f \circ g$

$$X' \xrightarrow{f_1} Y' \xrightarrow{f_2} Y$$

where f_1 is fppf and f_2 is proper, surjective and of finite presentation, cf. Theorem 3.10. This is shown using the flatification result of Raynaud and Gruson [34].

We also show that if f is in addition quasi-finite, then there is a similar factorization as above such that f_1 is an open covering and f_2 is finite, surjective and of finite presentation, cf. Theorem 3.11. Combining these results, we show that every universally subtrusive morphism of finite presentation $f : X \to Y$ has a refinement $X' \to Y$ which factors into an open covering f_1 followed by a surjective and proper morphism of finite presentation f_2.

This structure theorem is a generalization to the non-noetherian case of a result of Voevodsky [43, Thm. 3.1.9]. The proof is somewhat technical and the reader without any interest in non-noetherian questions may prefer to read the proof given by Voevodsky which has a more geometric flavor. Nevertheless, our extension is crucial for the elimination of noetherian hypotheses referred to above.

As a first application, we show in Section 4 that universally subtrusive morphisms of finite presentation are morphisms of effective descent for locally closed subsets. This result is not true for universally *submersive* morphisms despite its topological nature.

Effective descent of étale morphisms. — In Section 5 we use the structure theorems of §3 and the proper base change theorem in étale cohomology to prove that

- Quasi-compact universally *subtrusive* morphisms are morphisms of effective descent for quasi-compact étale morphisms. (Theorem 5.17).
- Universally open and surjective morphisms are morphisms of effective descent for étale morphisms. (Theorem 5.19)

In particular, universal submersions between noetherian schemes are morphisms of effective descent for quasi-compact étale morphisms.

Applications. — The effective descent results of §5 have several applications. One is the study of the algebraic fundamental group using morphisms of effective descent for finite étale covers, cf. [19, Exp. IX, §5]. Another application, also the origin of this paper, is in the theory of quotients of schemes by groups. The effective descent results show that strongly geometric quotients are *categorical* in the category of algebraic spaces [35]. This result is obvious in the category of schemes but requires the results of §5 for the extension to algebraic spaces. The third application in mind is similar to the second. Using the effective descent results we can extend some basic results on the h- and qfh-topologies defined by Voevodsky [43] to the category of algebraic spaces. This is done in §§7–8. The h-topology has been used in singular homology [37], motivic homology theories [44] and when studying families of cycles [38]. The h-topology is also related to the integral closure of ideals [9].

Elimination of noetherian hypotheses. — Let S be an inverse limit of affine schemes S_λ. The situation in mind is as follows. Every ring A is the filtered direct limit of its subrings A_λ which are of finite type over $\mathbb{Z}$. The scheme $S = \operatorname{Spec}(A)$ is the inverse limit of the excellent noetherian schemes $S_\lambda = \operatorname{Spec}(A_\lambda)$.

Let $X \to S$ be a finitely presented morphism. Then $X \to S$ descends to a finitely presented morphism $X_\lambda \to S_\lambda$ for sufficiently large λ [17, Thm. 8.8.2]. By this, we mean that $X \to S$ is the base change of $X_\lambda \to S_\lambda$ along $S \to S_\lambda$. If $X \to S$ is proper (resp. flat, étale, smooth, etc.) then so is $X_\lambda \to S_\lambda$ for sufficiently large λ, cf. [17, Thm. 8.10.5, Thm. 11.2.6, Prop. 17.7.8]. Note that the corresponding result for universally open is missing in [17]. As we have mentioned earlier, the analogous result for universally submersive is false.

In Theorem 6.4 we show that if $X \to S$ is *universally subtrusive* then so is $X_\lambda \to S_\lambda$ for sufficiently large λ. We also show the corresponding result for $X \to S$ *universally open*. An easy application of this result is the elimination of noetherian hypotheses in [17, §§14–15]. In particular, every universally open morphism locally of finite presentation has a locally quasi-finite quasi-section, cf. [17, Prop. 14.5.10].

Appendices. — Some auxiliary results are collected in two appendices. In the first appendix we recall the henselian properties of a scheme which is proper over a complete or henselian local ring. These properties follow from the Stein factorization and Grothendieck's existence theorem and constitute a part of the proper base change theorem in étale cohomology. With algebraic spaces we can express these henselian properties in an appealing form which is used when proving the effective descent results in Section 5.

In the second appendix, we briefly recall the weak subintegral closure of rings and weakly normal extensions. We also introduce the *absolute weak normalization* which we have not found elsewhere. When X is an integral scheme, the absolute weak normalization is the weak subintegral closure in the perfect closure of the function field of X. The absolute weak normalization is used to describe the sheafification of a representable functor in the h-topology.

Terminology and assumptions. — A morphism of schemes or algebraic spaces is called a *nil-immersion* if it is a surjective immersion. Equivalently, it is a closed immersion given by an ideal sheaf which is a nil-ideal, i.e., every section of the ideal sheaf is locally nilpotent.

Given a covering $f : X \to Y$ we say that $f' : X' \to Y$ is a *refinement* of f if f' is covering and factors through f. For general terminology and properties of algebraic spaces, see Knutson [23]. As in [23] we assume that all algebraic spaces are quasi-separated.

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1. Topologies

In addition to the Zariski topology, we will have use of two additional topologies which we recall in this section. The first is the *constructible* topology, cf. [18, §7.2], which also is known as the *patch* topology. The second topology is the S-topology where S stands for specialization. We then define submersive morphisms and give examples of morphisms which are submersive in the constructible topology.

The closed (resp. open) subsets of the constructible topology are the pro-constructible (resp. ind-constructible) subsets. A subset is pro-constructible (resp. ind-constructible) if it locally is an intersection (resp. union) of constructible sets. An important characterization of pro-constructible subsets is given by the following proposition.

PROPOSITION 1.1 ([18, Prop. 7.2.1]). — *Let X be a quasi-compact and quasi-separated scheme. A subset $E \subseteq X$ is pro-constructible if and only if there is an affine scheme X' and a morphism $f : X' \to X$ such that $E = f(X')$.*

If X is a scheme, then we denote by $|X|$ its underlying topological space with the Zariski topology and $|X|^{\mathrm{cons}}$ its underlying topological space with the constructible topology. If $f : X \to Y$ is a morphism of schemes then we let f^{cons} be the underlying map in the constructible topology.

PROPOSITION 1.2 ([18, Prop. 7.2.12]). — *Let X be a scheme.*

(i) *If $f : X \to Y$ is a morphism of schemes, then f^{cons} is continuous.*
(ii) *If $f : X \to Y$ is* quasi-compact, *then f^{cons} is closed.*
(iii) *If $f : X \to Y$ is* locally of finite presentation, *then f^{cons} is open.*
(iv) *If $Z \hookrightarrow X$ is closed, then ${|X|^{\mathrm{cons}}}_{|Z} = |Z|^{\mathrm{cons}}$.*
(v) *If $U \subseteq X$ is open, then ${|X|^{\mathrm{cons}}}_{|U} = |U|^{\mathrm{cons}}$.*
(vi) *If W is a locally closed subscheme of X, then ${|X|^{\mathrm{cons}}}_{|W} = |W|^{\mathrm{cons}}$.*

Proof. — (i)–(iii) are [18, Prop. 7.2.12 (iii)–(v)]. Statements (iv) and (v) are consequences of (ii) and (iii) respectively, as closed immersions are quasi-compact and open immersions are locally of finite presentation. Finally (vi) follows immediately from (iv) and (v). □

The Zariski topology induces a partial ordering on the underlying set of points [18, 2.1.1]. We let $x \leq x'$ if $x \in \overline{\{x'\}}$, i.e., if x is a specialization of x', or equivalently if $\overline{\{x\}} \subseteq \overline{\{x'\}}$. The S-topology is the topology associated to this ordering. A subset is thus closed (resp. open) if and only if it is stable under specialization (resp. generization). We denote by $S(E)$ the closure of E in the S-topology. By $\overline{E}$ we will always mean the closure of E in the Zariski topology. A morphism of schemes $f : X \to Y$ is *generizing* (resp. *specializing*) if it is open (resp. closed) in the S-topology [18, §3.9]. An open (resp. closed) morphism of schemes is generizing (resp. specializing) [18, Prop. 3.9.3].

REMARK 1.3. — For an affine scheme $\mathrm{Spec}(A)$ the partial ordering described above corresponds to *reverse* inclusion of prime ideals and a maximal point corresponds to a minimal ideal. In commutative algebra, it is common to take the ordering on the spectrum corresponding to inclusion of prime ideals, but this is less natural from a geometric viewpoint.

PROPOSITION 1.4 ([18, Thm. 7.3.1]). — *Let X be a scheme. If $E \subseteq X$ is an ind-constructible subset then $x \in \mathrm{int}(E)$ if and only if $\mathrm{Spec}(\mathcal{O}_{X,x}) \subseteq E$. Equivalently, we have that the interior of E in the Zariski topology coincides with the interior of E in the S-topology. If $F \subseteq X$ is a pro-constructible subset then $\overline{F} = S(F)$.*

COROLLARY 1.5. — *Let X be a scheme. A subset $E \subseteq X$ is open (resp. closed) in the Zariski topology if and only if E is open (resp. closed) in both the constructible topology and the S-topology.*

Proof. — As a closed (resp. open) immersion is quasi-compact (resp. locally of finite presentation), it follows that the constructible topology is finer than the Zariski topology. That the S-topology is finer than the Zariski topology is obvious. This shows the "only if" part. The "if" part follows from Proposition 1.4. $\square$

A map of topological spaces $f : X \to Y$ is *submersive* or a submersion if f is surjective and Y has the quotient topology, i.e., $E \subseteq Y$ is open (resp. closed) if and only if $f^{-1}(E)$ is open (resp. closed). We say that a morphism of schemes $f : X \to Y$ is submersive if the underlying morphism of topological spaces is submersive. We say that f is *universally submersive* if $f' : X \times_Y Y' \to Y'$ is submersive for every morphism of schemes $Y' \to Y$.

The composition of two submersive morphisms is submersive and if the composition $g \circ f$ of two morphisms is a submersive morphism then so is g. It follows immediately from Corollary 1.5 that if f is submersive in both the constructible and the S-topology, then f is submersive in the Zariski topology.

Proposition 1.6. — *Let $f : X \to Y$ be a surjective morphism of schemes. Then f^{cons} is submersive in the following cases:*

(i) *f is quasi-compact.*
(ii) *f is locally of finite presentation.*
(iii) *f is open.*

Proof. — If f is quasi-compact (resp. locally of finite presentation) then f^{cons} is closed (resp. open) by Proposition 1.2 and it follows that f^{cons} is submersive.

Assume that f is open. Taking an open covering, we can assume that Y is affine. As f is open there is then a quasi-compact open subset $U \subseteq X$ such that $f|_U$ is surjective. As $f|_U$ is quasi-compact it follows by part (i) that $f^{\mathrm{cons}}|_U$ is submersive. In particular, we have that f^{cons} is submersive. $\square$

Proposition 1.7. — *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes and let $f' : X' \to Y'$ be the pull-back of f along g.*

(i) *Assume that g is submersive. If f' is open (resp. closed, resp. submersive) then so is f.*
(ii) *Assume that g is universally submersive. Then f has one of the properties: universally open, universally closed, universally submersive, separated; if and only if f' has the same property.*
(iii) *Assume that g^{cons} is submersive. Then f is quasi-compact if and only if f' is quasi-compact.*

Proof. — (i) Assume that f' is open (resp. closed) and let $Z \subseteq X$ be an open (resp. closed) subset. Then $g^{-1}(f(Z)) = f'(g'^{-1}(Z))$ is open (resp. closed) and thus so is $f(Z)$ if g is submersive. If f' is submersive then so is $g \circ f' = f \circ g'$ which shows that f is submersive. The first three properties of (ii) follow easily from (i) and if f is separated then so is f'. If f' is separated, then $\Delta_{X'/Y'}$ is universally closed and it follows that $\Delta_{X/Y}$ is universally closed and hence a closed immersion [17, Cor. 18.12.6].

(iii) If f is quasi-compact then f' is quasi-compact. Assume that f' is quasi-compact and that g^{cons} is submersive. Then f'^{cons} is closed by Proposition 1.2 and it follows as in (i) that f^{cons} is closed. Moreover, the fibers of f are quasi-compact as the fibers of f' are quasi-compact. If $y \in Y$ then $(X_y)^{\mathrm{cons}}$ is quasi-compact [18, Prop. 7.2.13 (i)] and so is the image $(X^{\mathrm{cons}})_y$ of $(X_y)^{\mathrm{cons}} \to X^{\mathrm{cons}}$. Thus f^{cons} is proper since it is closed with quasi-compact fibers, and it follows that f is quasi-compact by [18, Prop. 7.2.13 (v)]. □

Remark 1.8. — Let us indicate how to extend the results of this section from schemes to algebraic spaces. Recall that associated to every algebraic space X is an underlying topological space $|X|$ and that a morphism f of algebraic spaces induces a continuous map $|f|$ on the underlying spaces [23, II.6]. By definition, a morphism of algebraic spaces $f : X \to Y$ is *submersive* if $|f|$ is submersive. If U is a scheme and $f : U \to X$ is étale and surjective, then $|f|$ is submersive. A morphism of algebraic spaces $f : X \to Y$ is *universally submersive* if $f' : X \times_Y Y' \to Y'$ is submersive for every morphism $Y' \to Y$ of algebraic spaces. For f to be universally submersive it is sufficient that f' is submersive for every (affine) scheme Y'.

The constructible topology (resp. S-topology) on the set $|X|$ is the quotient topology of the corresponding topology on $|U|$ for an étale presentation $U \to X$. This definition is readily seen to be independent on the choice of presentation. The results 1.1–1.7 then follow by taking étale presentations.

It is also possible to define the constructible topology and the S-topology for a (quasi-separated) algebraic space intrinsically. In fact, the notions of specializations, constructible, pro-constructible and ind-constructible sets are meaningful for any topological space. To see that these two definitions agree, it is enough to show that if U is a scheme and $f : U \to X$ is an étale presentation, then f is submersive in both the constructible topology and the S-topology. That f is submersive in the S-topology follows from [26, Cor. 5.7.1]. That f is submersive in the constructible topology follows from Chevalley's Theorem [26, Thm. 5.9.4] but its proof in *loc. cit.* uses [26, Cor. 5.9.2] which appears to have an incorrect proof as only locally closed subsets and not finite unions of such are considered. We now give a different proof:

As the question is local, we can assume that X is quasi-compact (and quasi-separated). Then X has a finite stratification into locally closed constructible subspaces X_i such that the X_i's are quasi-compact and quasi-separated *schemes* [34, Prop. 5.7.6]. The induced morphism $\coprod_i X_i \to X$ is a universal homeomorphism in the constructible topology and it follows that f^{cons} is submersive from the usual result for schemes.

2. Subtrusive morphisms

In this section, we define and give examples of subtrusive morphisms. We then give two valuative criteria and show that for noetherian schemes every universally submersive morphism is universally subtrusive.

PROPOSITION 2.1. — *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent.*

(i) *Every ordered pair $y \leq y'$ of points in Y lifts to an ordered pair of points $x \leq x'$ in X.*
(ii) *For every point $y \in Y$ we have that $f(S(f^{-1}(y))) = S(y)$.*
(iii) *For every subset $Z \subseteq Y$ we have that $f(S(f^{-1}(Z))) = S(Z)$.*
(iv) *For every pro-constructible subset $Z \subseteq Y$ we have that $f\left(\overline{f^{-1}(Z)}\right) = \overline{Z}$.*

Under these equivalent conditions f is submersive in the S-topology.

Proof. — It is clear that (i) $\iff$ (ii). As specialization commutes with unions, it is clear that (ii) $\iff$ (iii). By Proposition 1.4 it follows that (iii) $\implies$ (iv). As every point of Y is pro-constructible we have that (iv) $\implies$ (ii). Finally it is clear that f is submersive in the S-topology when (iii) is satisfied. □

DEFINITION 2.2. — We call a morphism of schemes S-*subtrusive* if the equivalent conditions of Proposition 2.1 are satisfied. We say that a morphism is *subtrusive* if it is S-subtrusive and submersive in the constructible topology. We say that $f : X \to Y$ is *universally subtrusive* if $f' : X \times_Y Y' \to Y'$ is subtrusive for every morphism $Y' \to Y$.

REMARK 2.3. — Picavet only considers spectral spaces and spectral morphisms, i.e., topological spaces that are the spectra of affine rings and quasi-compact morphisms of such spaces [20]. Surjective spectral morphisms are submersive in the constructible topology. Taking this into account, Picavet's definition of subtrusive morphisms [32, Déf. 2] agrees with Definition 2.2. Instead of "S-subtrusive" Picavet uses either "strongly subtrusive in the S-topology" or "strongly submersive of the first order in the S-topology".

Note that the non-trivial results in this paper deal with *quasi-compact* subtrusive morphism. Nevertheless, we have chosen to give the general definition of subtrusive morphisms as this clarifies the usage of the constructible topology.

Every subtrusive morphism is submersive by Corollary 1.5. It is furthermore clear that the composition of two subtrusive morphisms is subtrusive and that if the composition $g \circ f$ of two morphisms is subtrusive then so is g.

PROPOSITION 2.4. — *Let $f : X \to Y$ be a morphism of schemes.*

(i) *f is submersive (resp. subtrusive) if and only if f_{red} is submersive (resp. subtrusive).*
(ii) *Let $Y = \bigcup Y_i$ be an open covering. Then f is submersive (resp. subtrusive) if and only if $f|_{Y_i}$ is submersive (resp. subtrusive) for every i.*
(iii) *Let W be a locally closed subscheme of Y. If f is submersive (resp. subtrusive) then so is $f|_W$.*

Proof. — It is enough to verify the corresponding statements for: f submersive, f^{cons} submersive and f S-subtrusive. This follows easily from the topological definition of submersive, Proposition 1.2 and the characterization of S-subtrusive given in (i) of Proposition 2.1. □

REMARK 2.5. — Let $f : X \to Y$ be a surjective morphism of schemes. Then f is universally subtrusive in the following cases:

(1) f is universally specializing and f^{cons} is universally submersive.
(1a) f is proper.
(1b) f is integral.
(1c) f is essentially proper, i.e., universally specializing, separated and locally of finite presentation (defined in [17, Rem. 18.10.20] for Y locally noetherian).
(2) f is universally generizing and f^{cons} is universally submersive.
(2a) f is fpqc (faithfully flat and quasi-compact).
(2b) f is fppf (faithfully flat and locally of finite presentation).
(2c) f is universally open.

Recall that quasi-compact morphisms, morphisms locally of finite presentation and universally open morphisms are universally submersive in the constructible topology, cf. Proposition 1.6. Thus (1a)–(1c) are special cases of (1) and as flat and open morphisms are generizing it is clear that (2a)–(2c) are special cases of (2). That f is universally subtrusive in (1) and (2) is obvious.

REMARK 2.6. — Let V be a valuation ring. Then every finitely generated ideal in V is principal and a V-module is flat if and only if it is torsion free [8, Ch. VI, §3, No. 6, Lem. 1]. In particular, if B is a V-algebra and an integral domain, then B is flat if and only if $V \to B$ is injective.

Proposition 2.7 ([34, Part II, Prop. 1.3.1], [32, Prop. 16])

Let V be a valuation ring and $f : X \to \operatorname{Spec}(V)$ a morphism of schemes. The following are equivalent:

(i) *f is universally subtrusive.*
(ii) *f is subtrusive.*
(iii) *f is S-subtrusive.*
(iv) *The closure of the generic fiber $X \times_V \operatorname{Spec}(K)$ in X surjects onto V.*
(v) *The pair $\mathfrak{m} \leq (0)$ in $\operatorname{Spec}(V)$ lifts to $x \leq x'$ in X.*
(vi) *There is a valuation ring W and a morphism $\operatorname{Spec}(W) \to X$ such that the composition $\operatorname{Spec}(W) \to X \to \operatorname{Spec}(V)$ is surjective.*
(vii) *Any chain of points in $\operatorname{Spec}(V)$ lifts to a chain of points in X.*
(viii) *There is a closed subscheme $Z \hookrightarrow X$ such that $f|_Z$ is faithfully flat.*

If V is a discrete valuation ring, then the above conditions are equivalent with the following:

(ix) *f is submersive.*

Proof. — It is clear that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) and that (vi) $\implies$ (vii) $\implies$ (v). We will now show that (v) implies (vi) and (viii). We let $Z = \overline{\{x'\}}$ which is an integral closed subscheme of X dominating $\operatorname{Spec}(V)$. We let W be a valuation ring dominating $\mathcal{O}_{Z,x}$. Then both $\operatorname{Spec}(W) \to \operatorname{Spec}(V)$ and $Z \to \operatorname{Spec}(V)$ are flat by Remark 2.6. As the images in $\operatorname{Spec}(V)$ of $\operatorname{Spec}(W)$ and Z contain the closed point, we have that $\operatorname{Spec}(W) \to \operatorname{Spec}(V)$ and $Z \to \operatorname{Spec}(V)$ are surjective.

If (viii) is satisfied then we let $x \in Z$ be a point over the closed point of $\operatorname{Spec}(V)$. The morphism $\operatorname{Spec}(\mathcal{O}_{Z,x}) \to Z \hookrightarrow X \to \operatorname{Spec}(V)$ is faithfully flat and quasi-compact and hence universally subtrusive by case (2a) in Remark 2.5. In particular, we have that $X \to \operatorname{Spec}(V)$ is universally subtrusive.

Finally (ii) always implies (ix) and if V is a discrete valuation ring, then (ix) implies (iv). □

In the proof of the following theorem we use Corollary 6.3. This corollary is independent of Theorem 2.8 as the results of §§6.1–6.3 only uses the basic properties §§2.1–2.4 of subtrusive morphisms.

Theorem 2.8 ([32, Thm. 29, Thm. 37]). — *Let $f : X \to Y$ be a morphism such that f^{cons} is universally submersive (e.g. f quasi-compact).*

(i) *f is universally subtrusive if and only if, for any valuation ring V and morphism $Y' \to Y$ with $Y' = \operatorname{Spec}(V)$, the pull-back $f' : X' \to Y'$ is subtrusive.*

(ii) *f is universally submersive if and only if, for any valuation ring V and morphism $Y' \to Y$ with $Y' = \mathrm{Spec}(V)$, the pull-back $f' : X' \to Y'$ is submersive.*

If Y is locally noetherian then it is enough to consider discrete valuation rings in (i) and (ii), and f is universally subtrusive if and only if f is universally submersive.

Proof. — The necessity of the conditions is clear. To prove the sufficiency of (i), take any base change $Y' \to Y$ and let $y \leq y'$ be an ordered pair in Y'. We have to show that $y \leq y'$ can be lifted to X'. There is a valuation ring V and a morphism $\mathrm{Spec}(V) \to Y'$ such that the pair $\mathfrak{m} \leq (0)$ in $\mathrm{Spec}(V)$ lifts $y \leq y'$, cf. [8, Ch. VI, §1, No. 2, Cor.]. As $X' \times_{Y'} \mathrm{Spec}(V) \to \mathrm{Spec}(V)$ is subtrusive by assumption we can then lift $\mathfrak{m} \leq (0)$ to an ordered pair in $X' \times_{Y'} \mathrm{Spec}(V)$ which after projection onto X' gives a lifting of $y \leq y'$.

To prove the sufficiency of (ii), assume that f' is submersive whenever Y' is the spectrum of a valuation ring. It is then enough to show that f is submersive. Let $W \subseteq Y$ be a subset such that $f^{-1}(W)$ is closed. Then as f^{cons} is submersive it follows that W is pro-constructible. We will now show that W is closed under specialization. Then W is closed and it follows that f is submersive. Let $y \leq y'$ be an ordered pair in Y with $y' \in W$ and choose a valuation ring V with a morphism $Y' = \mathrm{Spec}(V) \to Y$ such that the pair $\mathfrak{m} \leq (0)$ in Y' lifts $y \leq y'$. Let W' be the inverse image of W along $Y' \to Y$. As $f' : X' \to Y'$ is submersive by assumption, we have that W' is closed. As $(0) \in W'$ it follows that $\mathfrak{m} \in W'$ and thus $y \in W$.

When Y is locally noetherian, it is enough to consider locally noetherian base changes $Y' \to Y$ in (i) by Corollary 6.3. As every ordered pair in a noetherian scheme can be lifted to a discrete valuation ring [15, Prop. 7.1.7], it is thus enough to consider discrete valuation rings in (i). Every universally subtrusive morphism is universally submersive. To show the remaining statements, it is thus enough to show that the valuative criteria in (i) and (ii) are equivalent over discrete valuation rings. This is the equivalence of (ii) and (ix) in Proposition 2.7. □

Corollary 2.9. — *Let $f : X \to Y$ be a morphism such that f^{cons} is universally submersive (e.g. f quasi-compact). Then the following are equivalent:*

(i) *f is universally subtrusive.*

(ii) *For every valuation ring V and diagram of solid arrows*

$$\begin{tikzcd} \mathrm{Spec}(V') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & Y \end{tikzcd}$$

there is a valuation ring V' and morphisms such that the diagram becomes commutative and such that the left vertical morphism is surjective.

Proof. — Follows immediately from Proposition 2.7 and Theorem 2.8. □

Corollary 2.10 ([25, Prop. 3.7], [19, Exp. IX, Rem. 2.6])

Let Y be locally noetherian *and let $f : X \to Y$ be a morphism* locally of finite type*. Then the following are equivalent*

(i) *f is universally submersive.*
(ii) *f is universally subtrusive.*
(iii) *For every discrete valuation ring D and diagram of solid arrows*

$$\begin{tikzcd} \mathrm{Spec}(D') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d] \\ \mathrm{Spec}(D) \arrow[r] & Y \end{tikzcd}$$

there is a discrete valuation ring D' and morphisms making the diagram commutative and such that the left vertical morphism is surjective.

Proof. — Note that each of (i), (ii) and (iii) implies that f is surjective. As f is locally of finite presentation, it is thus universally submersive in the constructible topology by Proposition 1.6 under any of these conditions.

The equivalence of (i) and (ii) follows from Theorem 2.8. If (ii) is satisfied and D is a discrete valuation ring with a morphism to Y then $X \times_Y \mathrm{Spec}(D) \to \mathrm{Spec}(D)$ is subtrusive. We can thus find an ordered pair $x \leq x'$ in $X \times_Y \mathrm{Spec}(D)$ above $\mathfrak{m} \leq (0)$ in $\mathrm{Spec}(D)$. As f is locally of finite type we have that X is locally noetherian and we can find a discrete valuation ring D' with a morphism $\mathrm{Spec}(D') \to X \times_Y \mathrm{Spec}(D)$ with image $\{x, x'\}$. This shows that (ii) implies (iii).

Finally, assume that we have a diagram as in (iii). Then the morphism $\mathrm{Spec}(D') \to \mathrm{Spec}(D)$ is submersive and hence so is $X \times_Y \mathrm{Spec}(D) \to \mathrm{Spec}(D)$. Thus (iii) implies (ii) by Theorem 2.8. □

For completeness we mention the following result which is an immediate consequence of Proposition 2.7 and a result of Kang and Oh [22].

Proposition 2.11 ([11, Thm. 3.26]). — *Let $f : X \to Y$ be a universally subtrusive morphism and let $\{y_\alpha\}$ be a chain of points in Y. Assume that $\{y_\alpha\}$ has a lower bound in Y or equivalently that $\{y_\alpha\}$ is contained in an affine open subset of Y. There is then a chain $\{x_\alpha\}$ of points in X which lifts the chain in Y, i.e., such that $f(x_\alpha) = y_\alpha$ for every α.*

Proof. — We can assume that Y is affine. The closure in the Zariski topology of the chain $\{y_\alpha\}$ is irreducible so we can also assume that Y is integral. By [22] there exists a valuation ring V, a morphism $\operatorname{Spec}(V) \to Y$ and a lifting of the chain $\{y_\alpha\}$ to a chain $\{v_\alpha\}$ in $\operatorname{Spec}(V)$. By Proposition 2.7 there is then a lifting of the chain $\{v_\alpha\}$ to a chain in $X \times_Y \operatorname{Spec}(V)$. The projection of this chain onto X gives a lifting of the chain $\{y_\alpha\}$. □

The results of this section, except possibly Proposition 2.11, readily generalize to algebraic spaces. Proposition 2.11 is at least valid for finite chains as such lift over étale surjective morphisms.

3. Structure theorem for finitely presented subtrusions

In this section, we give a structure theorem for finitely presented universally subtrusive morphisms. This result is an extension of [43, Thm. 3.1.9] to the non-noetherian case. If $f : X \to Y$ is universally subtrusive of finite presentation, then we will show the existence of a refinement $X' \to X \to Y$ of f such that there is a factorization $X' \to Y' \to Y$ where the first morphism is an open covering and the second is a surjective, proper and finitely presented morphism. If in addition f is quasi-finite then there is a similar refinement with a factorization in which the second morphism is finite.

Notation 3.1. — Let X be a scheme. We denote by X_{gen} the set of maximal points of X, i.e., the generic points of the irreducible components of X.

The main tools we will use are the "flatification by blowup"-result of Raynaud and Gruson [34] and the following lemma:

Lemma 3.2. — *Let $f : X \to Y$ be a morphism of schemes and let $U \subseteq Y$ be any subset containing Y_{gen}. Let $V = \overline{f^{-1}(U)}$ be the closure in the Zariski topology. If f is S-subtrusive then $f|_V$ is surjective.*

Proof. — Let $y \in Y$ and choose a generization $y' \in U$. As f is S-subtrusive, the pair $y \leq y'$ lifts to a pair $x \leq x'$. We have that $x \in S(f^{-1}(U)) \subseteq \overline{f^{-1}(U)}$. □

DEFINITION 3.3. — We say that a morphism $p : \widetilde{S} \to S$ is a *blow-up*, if there is a closed subscheme $Z \hookrightarrow S$ given by a finitely generated ideal sheaf, such that $\widetilde{S}$ is the blow-up of S in Z. Then p is proper and an isomorphism over the retrocompact open subset $U = S \setminus Z$. Let $f : X \to S$ be another morphism. The *strict transform* $\widetilde{X}$ of X under p is the schematic closure of $f^{-1}(U)$ in $X \times_S \widetilde{S}$. The strict transform $\widetilde{f}$ of f under p is the composition $\widetilde{f} : \widetilde{X} \hookrightarrow X \times_S \widetilde{S} \to \widetilde{S}$.

REMARK 3.4. — Let $f : X \to Y$ be universally subtrusive and let $p : \widetilde{Y} \to Y$ be a blow-up. By Lemma 3.2 it follows that the strict transform $\widetilde{f}$ of f is surjective.

To begin with, we will need the technical condition that Y_{gen} is quasi-compact. Note that if Y is quasi-separated then Y_{gen} is always Hausdorff, cf. [27, Ch. I, Lem. 2.8]. Rings A such that $\mathrm{Min}(A) = \mathrm{Spec}(A)_{\mathrm{gen}}$ is compact are studied in [30, Ch. II].

LEMMA 3.5. — *Let Y be a reduced quasi-compact and quasi-separated scheme such that Y_{gen} is quasi-compact, e.g., Y is reduced and noetherian. Let $f : X \to Y$ be a finitely presented morphism. There is then an open dense quasi-compact subset $U \subseteq Y$ such that f is flat over U.*

Proof. — As Y is reduced f is flat over Y_{gen} and hence flat over an open subset $V \subseteq Y$ containing Y_{gen}, cf. [17, Cor. 11.3.2]. As Y_{gen} is quasi-compact, there is an open quasi-compact subset $U \subseteq V$ containing Y_{gen}. □

PROPOSITION 3.6. — *Let Y be a* reduced *quasi-compact and quasi-separated scheme such that Y_{gen} is quasi-compact, e.g., Y noetherian. Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a surjective blow-up $\widetilde{Y} \to Y$ of finite type such that the strict transform $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is faithfully flat of finite presentation.*

Proof. — By Lemma 3.5 there is an open quasi-compact dense subset U over which f is flat. By [34, Thm. 5.2.2], there is a blow-up $p : \widetilde{Y} \to Y$ such that p is an isomorphism over U and such that the strict transform $\widetilde{f}$ is flat and finitely presented. By Remark 3.4 the morphism $\widetilde{f}$ is surjective. □

PROPOSITION 3.7. — *Let Y be affine and such that Y_{gen} is quasi-compact, e.g., Y irreducible. Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a refinement $f' : X' \to Y$ of f and a factorization of f' into a faithfully flat morphism $X' \to Y'$ of finite presentation followed by a proper surjective morphism $Y' \to Y$ of finite presentation. If in addition f is universally open, then we may choose f' such that $X' \to X \times_Y Y'$ is a nil-immersion.*

Proof. — Write Y as an inverse limit of noetherian affine schemes Y_λ. By Lemma 3.5 there is an open quasi-compact dense subset U such that f is flat over U_{red}. By [17, Cor. 8.2.11] there is an index λ and an open subset $U_\lambda \subseteq Y_\lambda$ such that $U = U_\lambda \times_{Y_\lambda} Y$. Increasing λ, we may then assume that there is a finitely presented morphism $f_\lambda : X_\lambda \to Y_\lambda$ such that $X \cong X_\lambda \times_{Y_\lambda} Y$ and such that f_λ is flat over $(U_\lambda)_{\mathrm{red}}$ [17, Thm. 11.2.6]. By [34, Thm. 5.2.2], there is a blow-up $p : \widetilde{Y_\lambda} \to (Y_\lambda)_{\mathrm{red}}$ such that p is an isomorphism over $(U_\lambda)_{\mathrm{red}}$ and such that the strict transform $\widetilde{f'_\lambda} : \widetilde{X_\lambda} \to \widetilde{Y_\lambda}$ of $f'_\lambda : X_\lambda \times_{Y_\lambda} (Y_\lambda)_{\mathrm{red}} \to (Y_\lambda)_{\mathrm{red}}$ along p is flat. Note that $\widetilde{f'_\lambda}$ is not necessarily surjective.

Let $f_1 : X' \to Y'$ (resp. $f_2 : Y' \to Y$) be the pull-back of $\widetilde{f'_\lambda} : \widetilde{X_\lambda} \to \widetilde{Y_\lambda}$ (resp. $\widetilde{Y_\lambda} \to (Y_\lambda)_{\mathrm{red}} \hookrightarrow Y_\lambda$) along $Y \to Y_\lambda$. Then f_1 is flat and of finite presentation, and f_2 is proper, surjective and of finite presentation. We will now show that f_1 is surjective. Note that f_2 is an isomorphism over U_{red} and that $X' \hookrightarrow X \times_Y Y'$ is an isomorphism over $X \times_Y (U_{\mathrm{red}})$.

Let $\widetilde{X}$ be the closure of $X \times_Y (U_{\mathrm{red}})$ in $X \times_Y Y'$. We then have a canonical factorization $\widetilde{X} \hookrightarrow X' \hookrightarrow X \times_Y Y'$. As f is universally subtrusive $\widetilde{X} \to Y'$ is surjective by Lemma 3.2. Thus $X' \to Y'$ is surjective. If in addition f is universally open, then $\widetilde{X} \hookrightarrow X \times_Y Y'$ is a nil-immersion and it follows that $X' \hookrightarrow X \times_Y Y'$ is a nil-immersion. □

To treat the case where Y_{gen} is not compact, we use the *total integral closure*.

Definition 3.8. — A scheme X is said to be *totally integrally closed* or TIC if:

(i) X is reduced.
(ii) For every $x \in X$, the closed subscheme $\overline{\{x\}}$ is normal and has an algebraically closed field of fractions.
(iii) The underlying topological space of X is extremal [21, §2].

Properties 3.9. — We briefly list the basic properties of TIC schemes.

(i) X is TIC if and only if X is TIC on an open covering.
(ii) An affine TIC scheme is the spectrum of a totally integrally closed ring [21, Thm. 1].
(iii) If X is TIC, quasi-compact and quasi-separated then X_{gen} is compact [21, Prop. 5].
(iv) If X is TIC then for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a strictly henselian normal domain [21, Prop. 7], [6, Prop. 1.4].
(v) If $f : X' \to X$ is an affine morphism and X' is TIC then the integral closure of X relative to X' is TIC.

(vi) Every reduced ring A has an injective and integral homomorphism into a totally integrally closed ring $\mathrm{TIC}(A)$, cf. [21, p. 769]. If $X = \mathrm{Spec}(A)$ then we denote the corresponding TIC scheme with $\mathrm{TIC}(X) = \mathrm{Spec}(\mathrm{TIC}(A))$. For an arbitrary affine scheme X we let $\mathrm{TIC}(X) = \mathrm{TIC}(X_{\mathrm{red}})$.

(vii) If X has a finite number of irreducible components, e.g., if X is noetherian, then there is a surjective and integral morphism $\mathrm{TIC}(X) \to X$ such that $\mathrm{TIC}(X)$ is totally integrally closed. Concretely, if $x_1, x_2, \dots, x_n$ are the generic points of X then $\mathrm{TIC}(X)$ is the integral closure of X_{red} in $\mathrm{Spec}\big(\prod_i \overline{k(x_i)}\big)$. This is the *absolute integral closure* of X introduced by Artin [6, §1].

(viii) Every monic polynomial with coefficients in a TIC ring factors completely into monic linear factors [21, p. 769].

(ix) If X is an affine TIC scheme and $Z \to X$ is a finite morphism of schemes then there is a finite and finitely presented surjective morphism $Z' \to Z$ such that Z' is a disjoint union of closed subschemes $Z_i \hookrightarrow X$. This follows from (viii).

Note that, as with the algebraic closure of a field, $\mathrm{TIC}(A)$ is only unique up to non-unique isomorphism and thus this construction does not immediately extend to arbitrary schemes. It is possible to show that if X is a quasi-compact and quasi-separated scheme, then there is a TIC scheme X' together with a surjective integral morphism $X' \to X$. However, this construction is slightly awkward and does not yield a unique X'.

THEOREM 3.10. — *Let Y be an affine or noetherian scheme. Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a refinement $f' : X' \to Y$ of f and a factorization of f' into a faithfully flat morphism $X' \to Y'$ of finite presentation followed by a proper surjective morphism $Y' \to Y$ of finite presentation. If in addition f is universally open, then we may choose f' such that $X' \to X \times_Y Y'$ is a nil-immersion.*

Proof. — If Y is noetherian, the theorem follows from Proposition 3.6. If Y is affine, then we have a surjective integral morphism $\mathrm{TIC}(Y) \to Y$ from a TIC scheme. As $\mathrm{TIC}(Y)_{\mathrm{gen}}$ is quasi-compact, we can by Proposition 3.7 find a refinement $X'' \to X \times_Y \mathrm{TIC}(Y) \to \mathrm{TIC}(Y)$ such that there is a factorization $X'' \to Y'' \to \mathrm{TIC}(Y)$ where the first morphism is faithfully flat of finite presentation and the second is proper, surjective and finitely presented. If f is universally open, we may also assume that $X'' \to X \times_Y Y''$ is a nil-immersion.

As the integral morphism $\mathrm{TIC}(Y) \to Y$ is the inverse limit of finite and finitely presented Y-schemes Y_λ [17, Lem. 11.5.5.1], it follows that there is an index λ and morphisms $X''_\lambda \to Y''_\lambda \to Y_\lambda$, $X''_\lambda \to X$ with the same properties as $X'' \to Y'' \to \mathrm{TIC}(Y)$ [17, Thm. 8.10.5 (xii) and Thm. 11.2.6]. If in addition

f is universally open, it follows from [17, Thm. 8.10.5 (ii) and (vi)] that after increasing λ, we can assume that $X''_\lambda \to X \times_Y Y''_\lambda$ is a nil-immersion. Putting $X' = X''_\lambda$ and $Y' = Y''_\lambda$ gives a refinement with the required factorization. □

THEOREM 3.11. — *Let Y be an affine or noetherian scheme. Let $f : X \to Y$ be a* quasi-finite *universally subtrusive morphism of finite presentation. Then there is a refinement $X' \to Y$ of f which is the composition of an open covering $X' \to Y'$ of finite presentation and a finite surjective morphism $Y' \to Y$ of finite presentation.*

Proof. — Replacing X with an open covering, we can assume that f is separated. By Zariski's Main Theorem [17, Thm. 8.12.6] there is then a factorization $X \to Y' \to Y$ where $f_1 : X \to Y'$ is an open immersion and $f_2 : Y' \to Y$ is finite. If Y is noetherian then f_2 is of finite presentation. If Y is affine then by [17, Rem. 8.12.7] we can find a factorization such that f_2 is of finite presentation.

If we can obtain a refinement $X' \to \mathrm{TIC}(Y)$ of $X \times_Y \mathrm{TIC}(Y) \to \mathrm{TIC}(Y)$ with a factorization of the specified form, then by a limit argument there is a similar refinement $X'_\lambda \to Y_\lambda$ of $X \times_Y Y_\lambda \to Y_\lambda$ for some finitely presented finite morphism $Y_\lambda \to Y$. The refinement $X'_\lambda \to Y_\lambda \to Y$ of $X \to Y$ then has a factorization of the requested form. We can thus assume that $Y = \mathrm{TIC}(Y)$ is totally integrally closed.

We will now show that $f = f_2 \circ f_1 : X \to Y' \to Y$ has a refinement $X' \to Y$ which is an open covering. To show this, we can replace Y with an open covering and assume that Y is affine. Now as Y is totally integrally closed and affine, there is a finite and finitely presented surjective morphism $Y'' \to Y'$ such that Y'' is a finite disjoint union of closed subschemes $Y_i \hookrightarrow Y$, cf. Properties 3.9 (ix). Let $X_i = X \times_{Y'} Y_i$. Then $X_i \to Y$ is the composition of an open quasi-compact immersion $X_i \to Y_i$ and a closed immersion $Y_i \to Y$ of finite presentation. We can replace X with $\coprod_i X_i$ and Y' with $\coprod_i Y_i$.

Let $X' = \overline{f^{-1}(Y_{\mathrm{gen}})} \hookrightarrow X$ with the reduced structure. Then $X' \to Y$ is surjective by Lemma 3.2. We will now show that $X' = \coprod_i \mathrm{int}(X_i)$ so that $X' \to Y$ is an open covering. The key observation is that the immersion $X_i \to Y_i \to Y$ is of finite presentation and hence *constructible*. Since every local ring of Y is irreducible, it thus follows from Proposition 1.4 that the interior of X_i coincides with the closure of $Y_{\mathrm{gen}} \cap X_i$ in X_i. □

The following theorem is [43, Thm. 3.1.9] except that we do not require that Y is an excellent noetherian scheme:

THEOREM 3.12. — *Let Y be an affine or noetherian scheme. Let $f : X \to Y$ be a universally subtrusive morphism of finite presentation. Then there is a* *refinement $X' \to Y$ of f which factors as a quasi-compact open covering $X' \to Y'$ followed by a proper surjective morphism $Y' \to Y$ of finite presentation.*

Proof. — By Theorem 3.10 we have a refinement $X' \to Y$ of f together with a factorization $X' \to Y' \to Y$ where $X' \to Y'$ is fppf and $Y' \to Y$ is proper. Taking a quasi-section [17, Cor. 17.16.2] we can in addition assume that $X' \to Y'$ is quasi-finite. If Y is not noetherian but affine, we can write Y as a limit of noetherian schemes and consequently we can assume that Y and Y' are noetherian. By Theorem 3.11 we can now refine $X' \to Y'$ into an open covering followed by a finite morphism. □

Remark 3.13. — Using the limit methods of Thomason and Trobaugh [41, App. C], we can replace the condition that Y is affine or noetherian in Theorems 3.10–3.12 with the condition that Y is quasi-compact and quasi-separated.

Remark 3.14. — If Y is quasi-compact and quasi-separated and $f : X \to Y$ is a quasi-separated morphism of finite type, then there is a finitely presented morphism $f' : X' \to Y$ and a closed immersion $X \hookrightarrow X'$ of Y-schemes. This follows from similar limit methods as in [41, App. C], cf. [10, Thm. 4.3]. Using this fact, analogues of Theorems 3.10–3.12 for universally subtrusive morphisms of *finite type* can be proved, at least if the base scheme has a finite number of irreducible components. In these analogues, the flat and open coverings are of finite presentation but the proper and the finite morphisms need not be. For example, Proposition 3.6 for $f : X \to Y$ of finite type and Y with a finite number of components follows from [34, Thm. 3.4.6].

4. Descent of locally closed subsets

Recall that a subset $E \subseteq X$ is locally closed if every point $x \in E$ admits an open neighborhood U such that $E \cap U$ is closed in U. Equivalently, E is the intersection of an open subset and a closed subset. Recall that a locally closed subset $E \subseteq X$ is retrocompact if and only if E is pro-constructible and if and only if $E \to X$ is quasi-compact.

Let $f : S' \to S$ be a faithfully flat and quasi-compact morphism of schemes and let $E \subseteq S$ be a subset. Then E is locally closed and retrocompact, if and only if $f^{-1}(E) \subseteq S'$ is locally closed and retrocompact [18, Prop. 7.3.7]. In this section, we give generalizations of this result for universally subtrusive morphisms. The proof of Theorem 4.1 only requires the results of §2 whereas Theorem 4.2 depends upon a structure theorem in §3.

THEOREM 4.1. — *Let $f : S' \to S$ be a universally subtrusive morphism of schemes and let $E \subseteq S$ be a subset. Then E is locally closed and constructible if and only if $f^{-1}(E)$ is locally closed and constructible. If f is also quasi-compact, then E is locally closed and retrocompact if and only if $f^{-1}(E)$ is locally closed and retrocompact.*

Proof. — If E is locally closed and constructible (resp. retrocompact) then so is $f^{-1}(E)$. Assume that $E' = f^{-1}(E)$ is locally closed and constructible (resp. retrocompact). Then E is constructible (resp. pro-constructible) since f is submersive in the constructible topology, and we have that $\overline{E} = S(E)$ by Proposition 1.4. The theorem follows if we show that $Z = \overline{E} \setminus E = S(E) \setminus E$ is closed. By Corollary 1.5 it is enough to show that Z is pro-constructible and stable under specialization.

If f is quasi-compact, then $f|_{\overline{E'}} : \overline{E'} \to \overline{E}$ is quasi-compact and surjective since f is S-subtrusive. In particular, we have that $f|_{\overline{E'}}$ is submersive in the constructible topology. It follows that E is ind-constructible in $\overline{E}$ since $f^{-1}(E) = E'$ is open in $\overline{E'}$. Thus, in both cases E is constructible as a subset of $\overline{E}$ and so is its complement Z.

Let $z \in Z$ and let $s \in S$ be a specialization of Z. Then there exists a generization $e \in E$ of z and we obtain the ordered triple $s \leq z \leq e$ in $\overline{E}$. As f is *universally* subtrusive, there exists by Proposition 2.11 a lifting $s' \leq z' \leq e'$ of this chain to S' where $e' \in E'$ and $z' \notin E'$. As E' is locally closed it follows that $s' \notin E'$ and hence $s \in Z = \overline{E} \setminus E$. □

THEOREM 4.2. — *Let $f : S' \to S$ be a morphism of algebraic spaces which is either*

(i) *open and surjective,*
(ii) *closed and surjective,*
(iii) *universally subtrusive of finite presentation.*

Then a subset $E \subseteq S$ is locally closed if and only if $f^{-1}(E) \subseteq S'$ is locally closed.

Proof. — The condition is clearly necessary and the sufficiency when f is as in (i) or (ii) is an easy exercise left to the reader. Let f be as in (iii) and assume that $f^{-1}(E)$ is locally closed. According to (i) the question is local in the étale topology so we can assume that S and S' are affine schemes. By Theorem 3.10 there is a refinement $S'' \to S$ of f which factors as an open surjective morphism followed by a closed surjective morphism. It follows that E is locally closed from the cases (i) and (ii). □

The following example shows that neither theorem is true if we replace universally subtrusive with universally submersive.

EXAMPLE 4.3. — Let S be the spectrum of a valuation ring V of dimension two. Then $\mathrm{Spec}(S) = \{x_0 \leq x_1 \leq x_2\}$. Let $s, t \in V$ be elements such that $\mathrm{Spec}(V/s) = \{x_0, x_1\}$ and $\mathrm{Spec}(V_t) = \{x_1, x_2\}$. Let $S' = \mathrm{Spec}(V/s \times V_t)$ with the natural morphism $f : S' \to S$. Then f is a universally submersive morphism of finite presentation, cf. [32, Cor. 33]. Let $E = \{x_0, x_2\} \subset S$ be the subset consisting of the minimal and the maximal point. Then E is not locally closed but $f^{-1}(E)$ is locally closed and constructible.

5. Effective descent of étale morphisms

In this section, we will show that quasi-compact universally subtrusive morphisms are morphisms of effective descent for the fibered category of quasi-compact and separated étale schemes. We will also show that this holds for the fibered category of quasi-compact, but not necessarily separated, étale *algebraic spaces*.

There is no need to include algebraic spaces when considering separated étale morphisms as any separated locally quasi-finite morphism of algebraic spaces is representable by schemes. On the other hand, starting with a non-separated étale scheme equipped with a descent datum, this can descend to an algebraic space which is not a scheme. We therefore need to extend the basic results about étale morphisms to algebraic spaces and this is done in Appendix A. The methods and results of this section are similar to and extend those of [19, Exp. IX].

NOTATION 5.1. — Let **Sch** be the category of quasi-separated schemes. Let **E** be the following fibered category over **Sch**: The objects of **E** are étale morphisms $X \to S$ where X is an algebraic space. The morphisms of **E** are commutative squares $(X', S') \to (X, S)$. The structure functor $\mathbf{E} \to \mathbf{Sch}$ is the forgetful functor taking an object $X \to S$ to its target S and a morphism $(X', S') \to (X, S)$ to the morphism $S' \to S$. We will also consider the following fibered full subcategories of **E** where the objects are:

$$\begin{aligned} \mathbf{E}_{\mathrm{sep}} &= \{\text{étale and separated morphisms}\} \\ \mathbf{E}_{\mathrm{qc}} &= \{\text{étale and quasi-compact morphisms}\} \\ \mathbf{E}_{\mathrm{sep,qc}} &= \{\text{étale, separated and quasi-compact morphisms}\} \\ \mathbf{E}_{\mathrm{fin}} &= \{\text{étale and finite morphisms}\}. \end{aligned}$$

It follows from Proposition A.1 that the objects of $\mathbf{E}_{\mathrm{fin}} \subseteq \mathbf{E}_{\mathrm{sep,qc}} \subseteq \mathbf{E}_{\mathrm{sep}}$ are morphisms of *schemes*.

Remark 5.2. — We have chosen to use **Sch** as the base category for convenience. We could instead have used the category of affine schemes or the category of algebraic spaces and all results would have remained valid as can be seen from Proposition 5.11. Note that as algebraic spaces are assumed to be quasi-separated, the objects of $\mathbf{E}_{\mathrm{qc}}$ are of finite presentation. In particular, the category $\mathbf{E}_{\mathrm{qc}/S}$ is equivalent to the category of constructible sheaves on S, cf. proof of Proposition A.7.

Proposition 5.3 ([19, Exp. IX, Cor. 3.3]). — *Let $f : S' \to S$ be a universally submersive morphism of schemes. Then f is a morphism of* $\mathbf{E}$*-descent. This means that for étale morphisms $X \to S$ and $Y \to S$ the sequence*

$$\mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'')$$

is exact, where X' and Y' are the pull-backs of X and Y along $S' \to S$, and X'' and Y'' are the pull-backs of X and Y along $S'' = S' \times_S S' \to S$.

Proof. — Follows easily from Corollary A.3. □

Proposition 5.4. — *Let $f : S' \to S$ be a universally* subtrusive *morphism of schemes. Let $X \to S$ be an étale morphism. If $X \times_S S'$ has one of the properties: universally closed, separated, quasi-compact; then so has $X \to S$. In particular, if $X \times_S S' \to S'$ lies in one of the categories: $\mathbf{E}_{\mathrm{sep}}$, $\mathbf{E}_{\mathrm{qc}}$, $\mathbf{E}_{\mathrm{sep,qc}}$, $\mathbf{E}_{\mathrm{fin}}$; then so does $X \to S$.*

Proof. — This follows immediately from Proposition 1.7. For the last statement, recall that the étale morphism $X \to S$ is finite if and only if it is separated, quasi-compact and universally closed [17, Thm. 8.11.1]. □

5.5 (Descent data). — *Let $S' \to S$ be any morphism and let $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S'$. Let $X \to S$ be an étale morphism, $X' = X \times_S S'$ and $X'' = X \times_S S''$. Then X'' is canonically S''-isomorphic with $\pi_1^* X'$ and $\pi_2^* X'$ where $\pi_1, \pi_2 : S'' \to S'$ are the two projections. In particular we have an S''-isomorphism $\varphi : \pi_1^* X' \to \pi_2^* X'$ satisfying the* cocycle condition, *i.e., if $\pi_{ij} : S''' \to S''$ denotes the projection on the i^{th} and j^{th} factors then*

$$\begin{tikzcd}
 & \pi_{12}^*\pi_2^*X' \arrow[r, "\mathrm{can}", "\cong"'] & \pi_{23}^*\pi_1^*X' \arrow[dr, "\pi_{23}^*(\varphi)"] & \\
\pi_{12}^*\pi_1^*X' \arrow[ur, "\pi_{12}^*(\varphi)"] & & \circ & \pi_{23}^*\pi_2^*X' \arrow[dl, "\mathrm{can}", "\cong"'] \\
 & \pi_{31}^*\pi_2^*X' \arrow[ul, "\mathrm{can}"', "\cong"] & \pi_{31}^*\pi_1^*X' \arrow[l, "\pi_{31}^*(\varphi)"'] &
\end{tikzcd}$$

commutes.

Conversely, given an étale morphism $X' \to S'$ we say that an S''-isomorphism $\varphi : \pi_1^* X' \to \pi_2^* X'$ satisfying the cocycle condition is a descent datum *for $X' \to S'$. We say that $(X' \to S', \varphi)$ is* effective *if it is isomorphic to the canonical descent datum associated with an étale morphism $X \to S$ as above. If $S' \to S$ is a morphism of $\mathbf{E}$-descent, e.g., universally submersive, then there is at most one morphism $X \to S$ which descends $(X' \to S', \varphi)$.*

We say that $S' \to S$ is a morphism of effective *$\mathbf{E}$-descent if every object $(X' \to S') \in \mathbf{E}_{/S'}$ equipped with a descent datum is effective. We say that $S' \to S$ is a morphism of* universal *$\mathbf{E}$-descent (resp. universal effective $\mathbf{E}$-descent) if $S' \times_S T \to T$ is a morphism of $\mathbf{E}$-descent (resp. effective $\mathbf{E}$-descent) for any base change $T \to S$.*

We briefly state some useful reduction results.

Proposition 5.6 ([13, Prop. 10.10, Prop. 10.11]). — *Let $\mathbf{F}$ be a category fibered over $\mathbf{Sch}$. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If f and g are morphisms of universal effective $\mathbf{F}$-descent, then so is $g \circ f$. If $g \circ f$ is a morphism of universal effective $\mathbf{F}$-descent, then so is g.*

Proposition 5.7 ([13, Thm. 10.8 (ii)]). — *Let $\mathbf{F}$ be a category fibered over $\mathbf{Sch}$. Let $f : S' \to S$ be a morphism of universal $\mathbf{F}$-descent and $g : T \to S$ be a morphism of universal effective $\mathbf{F}$-descent. Let $f' : T' \to T$ be the pull-back of f along g. Let $x' \in \mathbf{F}_{/S'}$ be an object equipped with a descent datum φ. Let $y' \in \mathbf{F}_{/T'}$ and φ_T be the pull-back of x' and φ along g. If (y', φ_T) is effective, then so is (x', φ). In particular, if f' and g are morphisms of universal effective $\mathbf{F}$-descent, then so is f.*

Proposition 5.8. — *Let $\mathbf{F} \subseteq \mathbf{E}_{\mathrm{qc}}$ be a category fibered over $\mathbf{Sch}$. Let $S =$ $\mathrm{Spec}(A)$ be affine and $S' = \varprojlim_\lambda S'_\lambda$ an inverse limit of affine S-schemes such that $S' \to S$ is universally submersive. Then $S' \to S$ is a morphism of universal effective $\mathbf{F}$-descent if and only if $S'_\lambda \to S$ is a morphism of universal effective $\mathbf{F}$-descent for every λ.*

Proof. — The necessity follows from Proposition 5.6. For sufficiency, we only need to show effectiveness by Proposition 5.3. Effectiveness follows easily from the fact that any object $X' \to S'$ in $\mathbf{F}_{/S'}$ is of finite presentation. In fact, there is an index λ and a étale morphism $X'_\lambda \to S'_\lambda$ such that $X' = X'_\lambda \times_{S'_\lambda} S'$, cf. [17, Thm. 8.8.2, Prop. 17.7.8]. If $X' \to S'$ is equipped with a descent datum, i.e., an $S' \times_S S'$-isomorphism $\varphi : X' \times_S S' \to S' \times_S X'$, then there is $\lambda' \geq \lambda$ such that $X_{\lambda'} = X_\lambda \times_{S_\lambda} S_{\lambda'}$ has a descent datum $\varphi_{\lambda'}$ which coincides with the descent datum φ after the pull-back along $S' \times_S S' \to S'_\lambda \times_S S'_\lambda$. This follows from [17, Cor. 8.8.2.5]. □

Proposition 5.9. — *Let* $\mathbf{F} \subseteq \mathbf{E}_{\mathrm{qc}}$ *be a category fibered over* **Sch**. *Let* $S = \mathrm{Spec}(A)$ *be affine and* $T = \varprojlim_\lambda T_\lambda$ *an inverse limit of affine* S*-schemes. Let* $S' \to S$ *be a morphism of universal* $\mathbf{F}$*-descent and let* $X' \to S'$ *be an element of* $\mathbf{F}$ *together with a descent datum* φ. *We let* $X'_{T_\lambda} \to T'_\lambda = T_\lambda \times_S S'$ *be the pull-back of* $X' \to S'$ *along* $T_\lambda \to S$ *and* φ_{T_λ} *the corresponding descent datum. We define* X'_T *and* φ_T *in the obvious way. If* (X'_T, φ_T) *is effective, then* $(X'_{T_\lambda}, \varphi_{T_\lambda})$ *is effective for some index* λ.

Proof. — As (X'_T, φ_T) is effective there is an étale and quasi-compact morphism $X_T \to T$ together with an isomorphism $X_T \times_T T' \cong X'_T$ compatible with the descent datum. As $X_T \to T$ is of finite presentation, there is an index λ and an étale and quasi-compact scheme $X_{T_\lambda} \to T_\lambda$. After increasing λ, we can assume that there is an isomorphism $X_{T_\lambda} \times_{T_\lambda} T'_\lambda \cong X'_{T_\lambda}$ and that this is compatible with the descent datum φ_{T_λ}. □

The following proposition is an immediate consequence of effective fpqc-descent for quasi-affine schemes [19, Exp. VIII, Cor. 7.9] as separated, quasi-compact and étale morphisms are quasi-affine by Zariski's main theorem [17, Thm. 8.12.6].

Proposition 5.10 ([19, Exp. IX, Prop. 4.1]). — *Let* $f : S' \to S$ *be faithfully flat and quasi-compact. Then* f *is a morphism of universal* effective $\mathbf{E}_{\mathrm{sep,qc}}$*-descent.*

Proposition 5.11. — *Let* $f : S' \to S$ *be faithfully flat and locally of finite presentation. Then* f *is a morphism of universal* effective $\mathbf{E}$*-descent.*

Proof. — This follows from [26, Cor. 10.4.2]. □

Proposition 5.12. — *Let* $f : S' \to S$ *be a universally submersive morphism and let* $X' \to S'$ *be an object in* $\mathbf{E}_{\mathrm{qc}}$ *equipped with a descent datum* φ. *For any morphism* $T \to S$ *we let* $X'_T \to T' = T \times_S S'$ *be the pull-back of* $X' \to S'$ *and* φ_T *the corresponding descent datum. The following are equivalent:*

(i) (X', φ) *is effective.*
(ii) (X'_T, φ_T) *is effective for every* T *such that* $T = \mathrm{Spec}(\mathcal{O}_{S,s})$ *for some* $s \in S$.
(iii) (X'_T, φ_T) *is effective for every* T *such that* $T = \mathrm{Spec}({}^{\mathrm{sh}}\mathcal{O}_{S,s})$ *is the strict henselization of* S *at some point* $s \in S$.

If in addition S *is locally noetherian and* $X' \to S'$ *is in* $\mathbf{E}_{\mathrm{qc,sep}}$ *then these statements are equivalent to the following:*

(iv) (X'_T, φ_T) *is effective for every* T *such that* $T = \mathrm{Spec}(\widehat{\mathcal{O}_{S,s}})$ *is the completion of* S *at some point* $s \in S$.

Proof. — It is clear that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv). Assume that (ii) holds, then it follows from Proposition 5.9 that (X', φ) is effective in an open neighborhood of any point. In particular, there is an open covering $T \to S$ such that (X'_T, φ_T) is effective. By Proposition 5.11 and Proposition 5.7 it follows that (X', φ) is effective. Similarly, if (iii) holds, there is an étale covering over which (X', φ) is effective and we can again conclude that (X', φ) is effective by Proposition 5.11. Finally (iv) $\implies$ (iii) by Proposition 5.10. □

Remark 5.13. — If S is *excellent*, then $\widehat{\mathscr{O}_{S,s}}$ is a direct limit of smooth $\mathscr{O}_{S,s}$-algebras by Popescu's theorem [36, 40]. It thus follows from Propositions 5.9 and 5.11 that (iv) implies (i) also for the fibered category $\mathbf{E}_{\mathrm{qc}}$ when S is excellent. We will not use this fact.

Proposition 5.14 ([19, Exp. IX, Thm. 4.12]). — *Proper surjective morphisms of finite presentation are morphisms of effective* $\mathbf{E}_{\mathrm{fin}}$*-descent.*

Proof. — Let $f : S' \to S$ be a proper and surjective morphism of finite presentation. To show that f is a morphism of effective descent, we can assume that S is affine by Proposition 5.12. As $S' \to S$ and the morphisms of $\mathbf{E}_{\mathrm{fin}}$ are of finite presentation, we can by a limit argument reduce to the case where S is noetherian. By Proposition 5.12 we can further assume that S is the spectrum of a complete noetherian local ring.

Let S_0 be the closed point of S and let S'_0, S''_0 and S'''_0 be the fibers of S', S'' and S''' over S_0. By Theorem A.6, the morphisms $S_0 \hookrightarrow S$, $S'_0 \hookrightarrow S'$, etc., induce equivalences between the category of finite étale covers over the source and the category of finite étale covers over the target. Thus $f : S' \to S$ is a morphism of effective descent for $\mathbf{E}_{\mathrm{fin}}$ if and only if $f_0 : S'_0 \to S_0$ is of effective descent. But f_0 is flat and hence of effective descent by Proposition 5.10. □

Corollary 5.15. — *Proper and surjective morphisms of finite presentation are morphisms of effective descent for* $\mathbf{E}_{\mathrm{sep,qc}}$.

Proof. — Let $f : S' \to S$ be a proper and surjective morphism of finite presentation. To show that f is a morphism of effective descent, we can as in the proof of Proposition 5.14 assume that S is the spectrum of a noetherian local ring. In particular, we can assume that S is noetherian and of finite dimension. We will now prove effectiveness using induction on the dimension of S.

Let $n = \dim(S)$ and assume that every proper surjective morphism of finite presentation $T' \to T$ such that $\dim(T) < n$ is a morphism of effective descent. If $n < 0$ then it is clear that f is effective. By Proposition 5.12, it is enough to show effectiveness for the completion of every local ring of S. We can thus assume that S is a complete local noetherian ring of dimension at most n.

Let $X' \to S'$ be a quasi-compact and separated étale morphism. Let S_0 be the closed point of S. Let S'_0 and X'_0 be the inverse images of S_0. As $S'_0 \to S_0$ is fpqc, there exists $X_0 \to S_0$ such that $X'_0 \to S'_0$ is the pull-back. Clearly $X_0 \to S_0$ is finite and hence $X'_0 \to S'_0$ is finite. Thus $X'_0 \to S_0$, $X''_0 \to S_0$ and $X'''_0 \to S_0$ are proper.

By [16, Cor. 5.5.2] there are thus canonical decompositions into open disjoint subsets $X' = Z' \amalg U'$, $X'' = Z'' \amalg U''$ and $X''' = Z''' \amalg U'''$ such that Z', Z'' and Z''' are proper over S and contain X'_0, X''_0 and X'''_0 respectively. Replacing X' with Z' or U' we can thus assume that either X' is finite over S' or that X'_0 is empty. In the first case it follows that f is effective from Proposition 5.14. In the second case we can replace S with $S \setminus S_0$ which has dimension at most $n-1$. It then follows from the induction hypothesis that f is effective. □

The proof of the following generalization is similar to and independent of Corollary 5.15. As the methods are less standard and involve algebraic spaces, we have chosen to not state Corollary 5.15 as a corollary of 5.16.

Corollary 5.16. — *Proper and surjective morphisms of finite presentation are morphisms of effective descent for* $\mathbf{E}_{\mathrm{qc}}$.

Proof. — Let $f : S' \to S$ be a proper and surjective morphism of finite presentation. As in the proof of Corollary 5.15 we can reduce to S noetherian and of finite dimension and we proceed by induction on the dimension of S.

Let $n = \dim(S)$ and assume that every proper surjective morphism of finite presentation $T' \to T$ such that $\dim(T) < n$ is a morphism of effective descent. If $n < 0$ then it is clear that f is effective. By Proposition 5.12, it is enough to show effectiveness for the strict henselization of every local ring of S. We can thus assume that S is a strictly local noetherian ring of dimension at most n.

Let $X' \to S'$ be a quasi-compact étale morphism. Let S_0 be the closed point of S. Let S'_0 and X'_0 be the inverse images of S_0. As $S'_0 \to S_0$ is fppf, there exists $X_0 \to S_0$ such that $X'_0 \to S'_0$ is the pull-back. As S_0 is the spectrum of a separably closed field, we have that X_0 is a disjoint union of m copies of S_0. Let $s_1, s_2, \dots, s_m$ be the corresponding sections of X_0/S_0 and let s'_i, s''_i, s'''_i be the corresponding sections of X'_0/S'_0 etc.

As (S', S'_0) (resp. (S'', S''_0) etc.) are 0-henselian pairs by Proposition A.12, the sections s'_i, (resp. s''_i etc.) uniquely lift to sections of X'/S' (resp. X''/S'' etc.) by Proposition A.7. Let $Z' = S'^{\amalg m}$ and $U' = X' \times_{S'} S' \setminus S'_0$ and similarly for Z'', U'' etc. From the sections we obtain canonical open coverings $Z' \amalg U' \to X'$ (resp. $Z'' \amalg U'' \to X''$ etc.). By the induction hypothesis it follows that f is a morphism of effective descent for U' and $U' \cap Z'$. That f is a morphism of effective descent for Z' follows from Proposition 5.14. We thus obtain algebraic spaces U and Z together with gluing data on $U \cap Z$. The gluing of U and Z along $U \cap Z$ is an algebraic space X which descends X'. □

THEOREM 5.17. — *Quasi-compact universally subtrusive morphisms are morphisms of effective* $\mathbf{E}_{\mathrm{qc}}$*-descent.*

Proof. — Let $f : S' \to S$ be a universally subtrusive quasi-compact morphism. To show that f is effective we can, replacing S and S' by open covers, assume that S and S' are affine. Proposition 5.8 shows that it is enough to show effectiveness for finitely presented f. Such an f has a refinement which is a composition of a flat morphism followed by a proper morphism by Theorem 3.10. The theorem thus follows from Propositions 5.6, 5.11 and Corollary 5.16. □

As a corollary we answer a question posed by Grothendieck [19, Exp. IX, Comment after Cor. 3.3] affirmatively.

COROLLARY 5.18. — *Universal submersions between noetherian schemes are morphisms of effective descent for* $\mathbf{E}_{\mathrm{fin}}$*,* $\mathbf{E}_{\mathrm{sep,qc}}$ *and* $\mathbf{E}_{\mathrm{qc}}$*.*

As another corollary we have the following result:

THEOREM 5.19. — *The following classes of morphisms are classes of effective* $\mathbf{E}$*-descent.*

(i) *Universally open and surjective morphisms.*
(ii) *Universally closed and surjective morphisms of finite presentation.*
(iii) *Universally subtrusive morphisms of finite presentation.*

Proof. — First note that the morphisms in the first two classes are universally subtrusive, cf. Remark 2.5. Moreover by Theorem 3.12, a morphism in the third class has a refinement which is the composition of a morphism in the first class and a morphism in the second class. Thus, it is enough to prove the effectiveness of the first two classes by Proposition 5.6.

Let $f : S' \to S$ be either a universally open morphism or a universally closed morphism of finite presentation. Let $X' \to S'$ be an étale morphism equipped with a descent datum. By Proposition 5.12, we can assume that S is affine. If f is universally open, there is then an open quasi-compact subset U of S' such that $f|_U$ is surjective. Replacing S' with U we can assume that f is quasi-compact.

Let $x' \in X'$. If f is universally open, let $V \subseteq X'$ be an open quasi-compact neighborhood of x' and let $R(V) = \pi_2(\pi_1^{-1}(V))$ be the saturation of V with respect to the equivalence relation $R = (\pi_1, \pi_2) : X'' \to X' \times_S X'$. As the π_i's are quasi-compact and open, we have that $U = R(V)$ is an open quasi-compact R-stable neighborhood of x'.

If f is universally closed and finitely presented, let $R(x') = \pi_2(\pi_1^{-1}(x')) \subseteq X'$ be the saturation of x'. As π_i is quasi-compact, the subset $R(x')$ is quasi-compact. Let V be an open quasi-compact neighborhood of $R(x')$. As X' is quasi-separated, we have that $V \subseteq X'$ is retrocompact and thus pro-constructible. In particular the complement $X' \setminus V$ is ind-constructible. As π_i is closed and finitely presented, the saturation $R(X' \setminus V)$ is a closed ind-constructible subset of X' disjoint from $R(x')$. Thus $U = X' \setminus R(X' \setminus V)$ is an open R-stable pro-constructible neighborhood of x' contained in V. In particular $U \subseteq V$ is retrocompact and hence U is quasi-compact.

In both cases, we thus have an open covering $\coprod U_{x'} \to X'$, stable under the descent datum, such that each $U_{x'}$ is quasi-compact. By Theorem 5.17 every space $U_{x'}$ descends to an étale quasi-compact space U_x over S. As $U_{x'} \to U_x$ is submersive, the intersection $U_{x'_1} \cap U_{x'_2}$ descends to an open subset of both U_{x_1} and U_{x_2}. Finally as $S' \to S$ is a morphism of $\mathbf{E}$-descent, the gluing datum of the $U_{x'}$'s descends to a gluing datum of the U_x's. Thus the U_x's glue to an algebraic space X étale over S which descends $X' \to S'$. $\square$

Recall that a morphism of algebraic spaces $f : X \to Y$ is a *universal homeomorphism* if $f' : X \times_Y Y' \to Y'$ is a homeomorphism (of topological spaces) for every morphism $Y' \to Y$ of algebraic spaces. As usual, it is enough to consider base changes such that Y' is a scheme or even an affine scheme.

The diagonal of a universally injective morphism is surjective. Thus, any homeomorphism of *schemes* is separated. It then follows from Zariski's main theorem that a finite type morphism of schemes is a universal homeomorphism if and only if it is universally injective, surjective and finite. More generally, a morphism of schemes $f : X \to Y$ is a universal homeomorphism if and only if f is universally injective, surjective and integral [17, Cor. 18.12.11].

The diagonal of a universal homeomorphism of *algebraic spaces* is a surjective monomorphism but not necessarily an immersion. In particular, not every homeomorphism of algebraic spaces is separated. This is demonstrated by the following classical example of an algebraic space which is not locally separated, i.e., its diagonal is not an immersion.

Example 5.20 ([23, Ex. 1, p. 9]). — Let U be the union of two secant affine lines and let R be the equivalence relation on U which identifies the two lines except at the singular point. Then the quotient $X = U/R$ is an algebraic space whose underlying topological space is the affine line. In fact, there is a universal homeomorphism $X \to \mathbb{A}^1$ such that $U \to X \to \mathbb{A}^1$ induces the identity on the two components. The corresponding two sections $\mathbb{A}^1 \to X$ are bijective but not universally closed. The space X looks like the affine line except at a special point where it has two different tangent directions.

The following theorem generalizes [19, Exp. IX, Thm. 4.10]:

Theorem 5.21. — *Let $S' \to S$ be a* separated *universal homeomorphism of algebraic spaces. Then the functor $\mathbf{E}_{/S} \to \mathbf{E}_{/S'}$*

$$\begin{aligned} \{\textit{étale spaces over } S\} &\longrightarrow \{\textit{étale spaces over } S'\} \\ X &\longmapsto X \times_S S' \end{aligned}$$

is an equivalence of categories. In particular, we have induced equivalences of categories $\mathbf{F}_{/S} \to \mathbf{F}_{/S'}$ where $\mathbf{F}$ is one of the fibered categories $\mathbf{E}_{\mathrm{fin}}$, $\mathbf{E}_{\mathrm{sep,qc}}$, $\mathbf{E}_{\mathrm{qc}}$, $\mathbf{E}_{\mathrm{sep}}$.

Proof. — As $S' \to S$ is a separated universal homeomorphism $S' \hookrightarrow S' \times_S S'$ is a nil-immersion. The functor from étale algebraic spaces over $S' \times_S S'$ to étale algebraic spaces over S' is therefore an equivalence by Proposition A.4. In particular, every étale algebraic space over S' comes with a unique descent datum. This shows that the functor in the theorem is fully faithful.

Essential surjectivity for $\mathbf{F} = \mathbf{E}_{\mathrm{qc}}$ follows from Theorem 5.17. For an object X' in the category $\mathbf{E}_{/S'}$ we first choose an open covering $\{U'_\alpha\}$ of X' such that the U'_α's are quasi-compact spaces. These U'_α's come with unique descent data and can be descended to S. As in the last part of the proof of Theorem 5.19 we can glue the descended spaces to an algebraic space which descends X'. □

Corollary 5.22. — *A separated universal homeomorphism of algebraic spaces is representable by schemes and is integral.*

Proof. — Let $S' \to S$ be a separated universal homeomorphism and choose an étale presentation $U' \to S'$ such that U' is a *scheme*. Then by Theorem 5.21 this induces an étale morphism $U \to S$ such that $U' = U \times_S S'$. As $U' \to U$ is a representable universal homeomorphism it is integral by [17, Cor. 18.12.11]. In particular $U' \to U$ is affine and it follows by étale descent that $S' \to S$ is representable and integral. □

Remark 5.23. — The proof of Corollary 5.22 shows that Theorem 5.21 is false for non-separated universal homeomorphisms.

Example 5.24 (Push-outs). — Let $Z \hookrightarrow X$ be a closed immersion of affine schemes and let $Z \to Y$ be any morphism of affine schemes. Then the push-out $X \amalg_Z Y$ exists in the category of schemes and is affine [12, Thm. 5.1]. Furthermore $Z = X \times_{X \amalg_Z Y} Y$ and $f : X \amalg Y \to X \amalg_Z Y$ is universally submersive [12, Thm. 7.1 A)]. Let $E \to X \amalg Y$ be an affine étale morphism equipped with a descent datum with respect to f. The descent datum gives in particular an isomorphism $E|_X \times_X Z \to E|_Y \times_Y Z$. We can then form the push-out $E|_X \amalg_{E|_Z} E|_Y$ which is affine and étale over $X \amalg_Z Y$ and descends E [12, Thm. 2.2 (iv)]. Thus, f is a morphism of effective descent for the category of affine and étale morphisms.

In general, $f : X \amalg Y \to X \amalg_Z Y$ is not *subtrusive*. For example, let $Z \to Y$ be the open immersion $\mathbb{A}^1_x \setminus \{x = 0\} \subseteq \mathbb{A}^1_x$, let $X = \mathbb{A}^2_{x,y} \setminus \{x = 0\}$ and let $Z \hookrightarrow X$ be the hyperplane defined by $y = 0$. Then $Y \hookrightarrow X \amalg_Z Y$ is a closed immersion, $X \to X \amalg_Z Y$ is an open immersion and X and Y intersect along the locally closed subset Z. The ordered pair $\{x = y = 0\} < \xi$ of points on $X \amalg_Z Y$, where ξ is the generic point on Y, cannot be lifted to $X \amalg Y$.

This example motivates the following question:

Question 5.25. — *Are quasi-compact universally* submersive *morphisms of effective* $\mathbf{E}_{\mathrm{qc}}$*-descent?*

6. Passage to the limit

In this section, we first show that subtrusive morphisms are *stable* under inverse limits. This result follows from basic properties of subtrusive morphisms (2.1–2.4). The corresponding stability result for universally open morphisms is proved in [17, Prop. 8.10.1]. We then show that subtrusive morphisms and universally open morphisms *descend* under inverse limits. The proofs of these results are much more difficult and use the structure theorems of Section 3.

Notation 6.1. — We use the following notation, cf. [17, §8]: Let S_0 be a scheme and let S_λ be a filtered inverse system of schemes, affine over S_0. Let $S = \varprojlim_\lambda S_\lambda$ be the inverse limit which is a scheme affine over S_0. Let α be an index and let $f_\alpha : X_\alpha \to Y_\alpha$ be a morphism of S_α-schemes. For every $\lambda \geq \alpha$ we let $f_\lambda : X_\lambda \to Y_\lambda$ be the pull-back of f_α along $S_\lambda \to S_\alpha$ and we let $f : X \to Y$ be the pull-back of f_α along $S \to S_\alpha$. Let $u_\lambda : X \to X_\lambda$ and $v_\lambda : Y \to Y_\lambda$ be the canonical morphisms.

Proposition 6.2 ([32, Part II, Prop. 3]). — *Let f and f_λ be morphisms as in Notation 6.1 and assume that f^{cons} is submersive. If there exists λ such that f_μ is subtrusive (resp. universally subtrusive) for every $\mu \geq \lambda$, then f is subtrusive (resp. universally subtrusive).*

Proof. — If f_λ is universally subtrusive then it follows from the definition that the pull-back f is universally subtrusive. Assume that there is λ such that f_μ is subtrusive for $\mu \geq \lambda$. To prove that f is subtrusive, it is enough to show that if $Z \subseteq Y$ is pro-constructible, then $\overline{Z} = f(\overline{f^{-1}(Z)})$ by Proposition 2.1. Let $Z_\mu = v_\mu(Z)$ which is pro-constructible as v_μ is quasi-compact. Then

$$Z = \bigcap_{\mu \geq \lambda} v_\mu^{-1}(Z_\mu)$$

and as $Y = \varprojlim_\mu Y_\mu$ as topological spaces, it follows that

$$\overline{Z} = \bigcap_{\mu \geq \lambda} v_\mu^{-1}\left(\overline{Z_\mu}\right).$$

Similarly

$$\overline{f^{-1}(Z)} = \bigcap_{\mu \geq \lambda} u_\mu^{-1}\left(\overline{f_\mu^{-1}(Z_\mu)}\right).$$

As f_μ is subtrusive we have that $\overline{Z_\mu} = f_\mu(\overline{f_\mu^{-1}(Z_\mu)})$. It thus follows that

$$\overline{Z} = \bigcap_{\mu \geq \lambda} v_\mu^{-1}\left(f_\mu\left(\overline{f_\mu^{-1}(Z_\mu)}\right)\right) = \bigcap_{\mu \geq \lambda} f\left(u_\mu^{-1}\left(\overline{f_\mu^{-1}(Z_\mu)}\right)\right) = f\left(\overline{f^{-1}(Z)}\right)$$

as the intersections are filtered. □

Corollary 6.3. — *Let $f : X \to Y$ be a morphism of schemes. Then f is universally subtrusive if and only if f^{cons} is universally submersive and $f_n : X \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}} \to Y \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ is subtrusive for every positive integer n.*

Proof. — The condition is necessary by the definition of universally subtrusive. For the sufficiency, assume that f_n is subtrusive for all n. As subtrusiveness is Zariski-local on the base by Proposition 2.4, we can assume that Y is affine. It is also enough to check that $f' : X' \to Y'$ is subtrusive for base changes $Y' \to Y$ such that Y' is affine. First assume that $Y' \to Y$ is of finite type. Then we can factor $Y' \to Y$ through a closed immersion $Y' \hookrightarrow Y \times \mathbb{A}^n$ and it follows by the assumptions on f_n and Proposition 2.4 that f' is subtrusive. For arbitrary affine $Y' \to Y$, we write Y' as a limit of finite type schemes and invoke Proposition 6.2. □

Theorem 6.4. — *Assume that S_0 is quasi-compact and $f_\alpha : X_\alpha \to Y_\alpha$ is of finite presentation with notation as in 6.1. Then $f : X \to Y$ is universally subtrusive if and only if f_λ is universally subtrusive for some $\lambda \geq \alpha$.*

Proof. — The condition is sufficient by definition. To prove the necessity we assume that f is universally subtrusive. As S_0 is quasi-compact there is a finite affine covering of S_0. As subtrusiveness is local on the base by Proposition 2.4 we can therefore assume that S_0 is affine. We can then by Theorem 3.10 find a refinement $f' : X' \to Y$ of $f : X \to Y$ such that f' has a factorization into a finitely presented flat surjective morphism $X' \to Y'$ followed by a finitely presented proper surjective morphism $Y' \to Y$. By [17, Thm. 8.10.5 and Thm. 11.2.6] the morphism f' descends to a morphism $f'_\lambda : X'_\lambda \to Y_\lambda$ with a similar factorization. In particular f'_λ is universally subtrusive and it follows that f_λ is universally subtrusive as well. □

Corollary 6.5. — *Let $S=\operatorname{Spec}(A)$ be an affine scheme and let $f:X\to S$ be a morphism of finite presentation. Then the following are equivalent:*

(i) *f is universally subtrusive.*
(ii) *There exists an affine noetherian scheme $S_0=\operatorname{Spec}(A_0)$, a morphism $f_0:X_0\to S_0$ of finite presentation and a morphism $S\to S_0$ such that $X=X_0\times_{S_0}S$ and f_0 is universally submersive.*
(iii) *There exists a scheme S_0 and a morphism f_0 as in (ii) such that in addition A_0 is a sub-$\mathbb{Z}$-algebra of A of finite type.*

Picavet gives an example [32, Cor. 33] showing that the corresponding result for universally submersive is false. That is, there exists a universally submersive morphism of finite presentation which is not universally subtrusive.

Theorem 6.6. — *Assume that S_0 is quasi-compact and $f_\alpha:X_\alpha\to Y_\alpha$ is of finite presentation with notation as in 6.1. Then $f:X\to Y$ is universally open if and only if f_λ is universally open for some $\lambda\geq\alpha$.*

Proof. — As the condition is clearly sufficient, we assume that f is universally open. As S_0 is quasi-compact we can easily reduce to the case where Y_α is affine. Using [17, Thm. 8.10.5 and Thm. 11.2.6] we can then descend the refinement of f given by Theorem 3.10. There is thus an index λ, a proper surjective morphism $Y'_\lambda\to Y_\lambda$, a faithfully flat morphism of finite presentation $X'_\lambda\to Y'_\lambda$ and a nil-immersion $X'_\lambda\hookrightarrow X_\lambda\times_{Y_\lambda}Y'_\lambda$. As $X'_\lambda\to Y'_\lambda$ is universally open, so is $X_\lambda\times_{Y_\lambda}Y'_\lambda\to Y'_\lambda$. As $Y'_\lambda\to Y_\lambda$ is universally submersive it follows that $X_\lambda\to Y_\lambda$ is universally open. □

7. Weakly normal descent

Let $f:S'\to S$ be faithfully flat and quasi-compact. Then f is a morphism of descent for the fibered category of all morphisms of algebraic spaces, that is, for any algebraic space X we have that

$$\operatorname{Hom}(S,X)\longrightarrow\operatorname{Hom}(S',X)\rightrightarrows\operatorname{Hom}(S'\times_S S',X)$$

is exact [26, Thm. A.4]. In this section, we give a similar descent result for *weakly normal* universally submersive morphisms.

7.1 (Schematic image). — *Let $f:X\to Y$ be a morphism of algebraic spaces. If there exists a smallest closed subspace $Y'\hookrightarrow Y$ such that f factors through $Y'\hookrightarrow Y$, then we say that Y' is the* schematic image *of f* [18, 6.10]. *If X is reduced, then $\overline{f(X)}$ with its reduced structure is the schematic image.*

Let $f:X\to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces. *Then $f_*\mathcal{O}_X$ is a quasi-coherent sheaf (*[23, Prop. II.4.6] *holds for non-separated morphisms) and the schematic image of f is the closed subspace* *of Y defined by the ideal* $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. *The underlying topological space of the image is $\overline{f(X)}$, as can be checked on an étale presentation.*

A morphism $f : X \to Y$ of algebraic spaces is *schematically dominant* if $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective (in the small étale site). This agrees with the usual definition for schemes [17, Déf. 11.10.2, Thm. 11.10.5 (ii)]. If $f : X \to Y$ is schematically dominant, then the schematic image of f exists and equals Y. Conversely, if f is quasi-compact and quasi-separated or X is reduced, then f is schematically dominant if and only if the schematic image of f equals Y.

PROPOSITION 7.2. — *Let $p : S' \to S$ be a schematically dominant universally submersive morphism of algebraic spaces. Then p is an epimorphism in the category of algebraic spaces, i.e.,* $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S', X)$ *is injective for every algebraic space X.*

Proof. — First assume that X is *separated* and let $f : S \to X$ be a morphism. Then the schematic image of $\Gamma_f \circ p = (p, f \circ p) : S' \to S \times X$ exists and equals the graph Γ_f [18, Prop. 6.10.3]. We can thus recover f from $f \circ p$.

For general X, let $f_1, f_2 : S \to X$ be two morphisms such that $f_1 \circ p = f_2 \circ p$. Let $U \to X$ be an étale surjective morphism such that U is a separated scheme. As p is universally submersive, it is a morphism of descent for étale morphisms by Proposition 5.3. Thus, the canonical S'-isomorphism $V' := p^* f_1^* U \cong p^* f_2^* U$ descends to an S-isomorphism $V := f_1^* U \cong f_2^* U$. To conclude, we have a diagram

$$\begin{tikzcd} V' \arrow[r, "q"] \arrow[d] & V \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d] & U \arrow[d] \\ S' \arrow[r, "p"] & S \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \end{tikzcd}$$

where the vertical morphisms are étale, the natural squares are cartesian and $g_1 \circ q = g_2 \circ q$. Note that q is schematically dominant as p is schematically dominant and $V \to S$ is étale. We apply the special case of the proposition to deduce that $g_1 = g_2$ and it follows that $f_1 = f_2$. □

7.3 (Weak normalization). — *Let $f : S' \to S$ be a dominant, quasi-compact and quasi-separated morphism. A* wn-factorization *of f is a factorization $f = f_2 \circ f_1$ such that f_1 is schematically dominant and f_2 is a separated universal homeomorphism. A wn-factorization is* trivial *if f_2 is an isomorphism. We say that f is* weakly normal *(or weakly subintegrally closed) if any wn-factorization of f is trivial. The* weak normalization *(or weak subintegral closure) of S in S', denoted $S^{S'/\mathrm{wn}}$, is the maximal separated universal homeomorphism $S^{S'/\mathrm{wn}} \to S$ such that there exists a wn-factorization $S' \to S^{S'/\mathrm{wn}} \to S$*

of f. There exists a unique weak normalization and it fits into a unique wn-factorization. For more details on weakly normal morphisms and the weak normalization, see Appendix B.

THEOREM 7.4. — *Let $\pi : X \to S$ be a morphism of algebraic spaces and let $p : T' \to T$ be a quasi-compact, quasi-separated, universally submersive and weakly normal morphism of algebraic spaces over S. Assume either that $X \to S$ is locally separated (this is the case if X is a scheme) or that p is universally subtrusive. Then:*

$$\mathrm{Hom}_S(T,X) \longrightarrow \mathrm{Hom}_S(T',X) \rightrightarrows \mathrm{Hom}_S((T'\times_T T')_{\mathrm{red}},X)$$

is exact.

Proof. — As p is weakly normal, p is schematically dominant and it follows from Proposition 7.2 that $\mathrm{Hom}_S(T,X) \to \mathrm{Hom}_S(T',X)$ is injective.

Let $T'' = (T'\times_T T')_{\mathrm{red}}$ and let $\pi_1,\pi_2 : T'' \to T'$ denote the two projections. To show exactness in the middle, let $f' : T' \to X$ be a morphism such that $f'' := f'\circ\pi_1 = f'\circ\pi_2$. Let $s' = (f',\mathrm{id}_{T'}) : T' \to X\times_S T'$ and $s'' = (f'',\mathrm{id}_{T''}) : T'' \to X\times_S T''$ be the induced sections. Denote the set-theoretical images by $\Gamma' = s'(T')$ and $\Gamma'' = s''(T'')$. As $f'' = f'\circ\pi_i$, we have that s'' is the pull-back of s' along either of the two projections $\mathrm{id}_X\times\pi_i$, $i=1,2$. In particular, we have that $\Gamma'' = (\mathrm{id}_X\times\pi_i)^{-1}(\Gamma')$. Let $p_X = \mathrm{id}_X\times p : X\times_S T' \to X\times_S T$ denote the pull-back of p. Let $\Gamma := p_X(\Gamma')$ so that $p_X^{-1}(\Gamma) = \Gamma'$.

First assume that s' is a closed immersion so that Γ' and Γ'' are closed. Then Γ is also closed since p_X is submersive. We let T_1 be the schematic image of the map $(f',p) = p_X\circ s' : T' \to X\times_S T$ so that the underlying set of T_1 is Γ. Let $q : T' \to T_1$ be the induced morphism. Then q is surjective and the graph of q is a nil-immersion $T' \to T_1\times_T T'$ since both source and target are closed subspaces of $X\times_S T'$ with underlying set Γ'. In particular, it follows that $T_1\times_T T' \to T'$ is a separated universal homeomorphism. We now apply Proposition 1.7 to $T_1 \to T$ and $p : T' \to T$ and deduce that $T_1 \to T$ is universally closed, separated, universally injective and surjective, i.e., a separated universal homeomorphism. Since $p : T' \to T_1 \to T$ is weakly normal, we have that $T_1 \to T$ is an isomorphism and the morphism $f : T = T_1 \hookrightarrow X\times_S T \to X$ lifts f'.

Instead assume that $X \to S$ is locally separated, i.e., that the diagonal morphism $\Delta_{X/S} : X \to X\times_S X$ is an immersion. Then the sections s' and s'' are also immersions. The image of an immersion of *algebraic spaces* is locally closed. Indeed, this follows from taking an étale presentation and Theorem 4.2. Thus $\Delta_{X/S}(X)$, Γ' and Γ'' are locally closed subsets. We will now show that Γ is locally closed. If p_X is universally *subtrusive* this follows from Theorem 4.1.

Let $V \subseteq X \times_S X$ be an open neighborhood of $\Delta(X)$ such that $\Delta(X) \subseteq V$ is closed. Consider the morphism $(f' \times \mathrm{id}_X) : T' \times_S X \to X \times_S X$. The composition with either of the two morphism $\pi_i \times \mathrm{id}_X$ is $f'' \times \mathrm{id}_X$. Let $U' = (f' \times \mathrm{id}_X)^{-1}(V)$ and $U'' = (f'' \times \mathrm{id}_X)^{-1}(V)$ so that if we let $U = p_X(U')$ then $U' = p_X^{-1}(U)$. The subset $U \subseteq X \times_S T$ is open since p_X is submersive. Note that the pull-back of $\Delta_{X/S}$ along $(f' \times \mathrm{id}_X)$ is s'. Therefore $\Gamma' \subseteq U'$ is closed and f' factors through U. After replacing X and S with U and T, the section s' becomes a closed immersion so that the previous case applies.

Now, let X be arbitrary and assume that p is universally subtrusive. As the question is local on T, we may assume that T is quasi-compact. After replacing X with a quasi-compact open $U \subseteq X$ through which f' factors, we can also assume that X is quasi-compact. Let $U \to X$ be an étale presentation such that U is a quasi-compact scheme. Let $V' = f'^{-1}(U)$ and $V''_r = f''^{-1}(U)$. By Proposition A.4 there is a unique étale $T' \times_T T'$-scheme V'' which restricts to V''_r on T''. By Theorem 5.17 the étale morphism $V' \to T'$ descends to an étale morphism $V \to T$. As the weak normalization commutes with étale base change, cf. Proposition B.6, we have that $V' \to V$ is weakly normal.

We now apply the first case of the theorem to $V' \to V$ and $V' \to U$ and obtain a morphism $V \to U \to X$ lifting $V' \to X$. Similarly, we obtain a lifting $V \times_T V \to U \times_X U \to X$ of $V' \times_{T'} V' \to U \times_X U \to X$. Finally, we obtain the morphism $f : T \to X$ by étale descent. $\square$

Remark 7.5. — Suppose that we remove the assumption that $T' \to T$ is weakly normal in the theorem. If $X \to S$ is locally separated, then the proof of the theorem shows that there exists a minimal wn-factorization $T' \to T_1 \to T$ such that $f' : T' \to X$ lifts to T_1. If X/S is locally of finite type, then $T_1 \to T$ is of finite type. It can be shown that such a minimal wn-factorization also exists if $X \to S$ is arbitrary and $T' \to T$ is universally subtrusive.

We obtain the following generalization of Lemma B.5:

Corollary 7.6. — *Let $p : S' \to S$ be a quasi-compact and quasi-separated universally submersive morphism. Let $q : (S' \times_S S')_{\mathrm{red}} \to S$ be the structure morphism of the reduced fiber product. Then the sequence*

$$\mathcal{O}_{S^{S'/\mathrm{wn}}} \hookrightarrow p_*\mathcal{O}_{S'} \rightrightarrows q_*\mathcal{O}_{(S' \times_S S')_{\mathrm{red}}}.$$

is exact. In particular, we have that p is weakly normal if and only if

$$\mathcal{O}_S \longrightarrow p_*\mathcal{O}_{S'} \rightrightarrows q_*\mathcal{O}_{(S' \times_S S')_{\mathrm{red}}}$$

is exact.

Proof. — This follows from the fact that $(S' \times_S S')_{\mathrm{red}} = (S' \times_{S^{S'/\mathrm{wn}}} S')_{\mathrm{red}}$ together with Theorem 7.4 applied to $X = \mathbb{A}^1$. $\square$

8. The h-topology

In this section, we look at the h- and qfh-topologies. An easy description of the coverings in these topologies is obtained from the structure theorems of Section 3. In contrast to the Grothendieck topologies usually applied, the h- and qfh-topologies are not sub-canonical, i.e., not every representable functor is a sheaf. It is therefore important to give a description of the associated sheaf to a representable functor [9, 43].

Let X be an algebraic space of finite presentation over a base scheme S. The main result of this section is that the associated sheaf to the functor $\mathrm{Hom}_S(-,X)$ coincides with the functor $T \mapsto \mathrm{Hom}_S(T^{\mathrm{wn}},X)$ where T^{wn} is the absolute weak normalization of T. This has been proved by Voevodsky [43] when S and X are excellent noetherian *schemes*. When S is non-noetherian, it is natural to replace submersive morphisms with subtrusive morphisms. To treat the case when X is a general algebraic space, we use the effective descent results of Section 5 via Theorem 7.4.

Let S be any scheme and let $\mathbf{Sch}_{/S}$ be the category of schemes over S. The following definitions of the h- and qfh-topologies generalize [43, Def. 3.1.2] which is restricted to the category of noetherian schemes.

Definition 8.1. — The h-topology is the minimal Grothendieck topology on $\mathbf{Sch}_{/S}$ such that the following families are coverings

- (i) Open coverings, i.e., families of open immersions $\{p_i : U_i \to T\}$ such that $T = \bigcup p_i(U_i)$.
- (ii) Finite families $\{p_i : U_i \to T\}$ such that $\coprod p_i : \coprod U_i \to T$ is *universally subtrusive* and of finite presentation.

The qfh-topology is the topology generated by the same types of coverings except that all morphisms should be locally quasi-finite.

Remark 8.2. — Restricted to the category of quasi-compact and quasi-separated schemes, the h-topology (resp. qfh-topology) is the Grothendieck topology associated to the pre-topology whose coverings are of the form (ii).

Remark 8.3. — Consider the following types of morphisms:

- (i) Finite surjective morphisms of finite presentation.
- (ii) Faithfully flat morphisms, locally of finite presentation.
- (iii) Proper surjective morphisms of finite presentation.

(i) and (ii) are coverings in the qfh-topology and (i)–(iii) are coverings in the h-topology. Indeed, morphisms of type (ii) have quasi-finite flat quasi-sections [17, Cor. 17.16.2].

The following theorem generalizes [43, Thm. 3.1.9].

THEOREM 8.4. — *Every h-covering (resp. qfh-covering) $\{U_i \to T\}$ has a refinement of the form $\{W_{jk} \to W_j \to V_j \to T\}$ such that*

- *$\{V_j \to T\}$ is an open covering,*
- *$W_j \to V_j$ is a proper (resp. finite) surjective morphism of finite presentation for every j,*
- *$\{W_{jk} \to W_j\}$ is an open quasi-compact covering for every j.*

In particular, the h-topology (resp. qfh-topology) is the minimal Grothendieck topology such that the following families are coverings:

(i) *Families of open immersions $\{p_i : U_i \to T\}$ such that $T = \bigcup p_i(U_i)$.*
(ii) *Families $\{p : U \to T\}$ consisting of a single proper (resp. finite) surjective morphism of finite presentation.*

Proof. — By [1, Exp. IV, Prop. 6.2.1] it follows that there is a refinement of the form $\{W_j' \to V_j \to T\}$ where $W_j' \to V_j$ are h-coverings (resp. qfh-coverings) of affine schemes and $\{V_j \to T\}$ is an open covering. Theorems 3.11 and 3.12 then show that these coverings have a further refinement as in the theorem. □

We will now review the contents of [43, §3.2] and extend the results to algebraic spaces and non-noetherian schemes. We begin by recalling the construction of the sheaf associated to a presheaf, cf. [29, Ch. II, Thm. 2.11].

DEFINITION 8.5. — Let $\mathscr{F}$ be a presheaf on $\mathbf{Sch}_{/S}$ and equip $\mathbf{Sch}_{/S}$ with a Grothendieck topology $\mathscr{T}$. For any $V \in \mathbf{Sch}_{/S}$ we define an equivalence relation $\sim$ on $\mathscr{F}(V)$ where $f \sim g$ if there exist a covering $\{p_i : U_i \to V\} \in \mathscr{T}$ such that $p_i^*(f) = p_i^*(g)$ for every i. We let $\mathscr{F}'$ be the quotient of $\mathscr{F}$ by this equivalence relation. Furthermore we let

$$\widetilde{\mathscr{F}} = \varinjlim_{\mathscr{U}} \check{H}^0(\mathscr{U}, \mathscr{F}')$$

where the limit is taken over all coverings $\mathscr{U} = \{p_i : U_i \to V\} \in \mathscr{T}$ and

$$\check{H}^0(\mathscr{U}, \mathscr{F}') = \ker\big(\textstyle\prod_i \mathscr{F}'(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}'(U_i \times_V U_j)\big)$$

is the Čech cohomology.

REMARK 8.6. — It is easily seen that $\mathscr{F}'$ is a separated presheaf. By [4, Lem. II.1.4 (ii)] it then follows that $\widetilde{\mathscr{F}}$ is the sheafification of $\mathscr{F}$. Moreover, we have that $\mathscr{F}'$ is the image presheaf of $\mathscr{F}$ by the canonical morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$.

DEFINITION 8.7. — Let X be an algebraic space over S and let $h_X = \mathrm{Hom}_S(-, X)$ be the corresponding presheaf on $\mathbf{Sch}_{/S}$. Let $L'(X) = (h_X)'$ and $L(X) = \widetilde{h_X}$ be the separated presheaf and sheaf associated to h_X in the h-topology. We denote the corresponding notions in the qfh-topology by $L'_{qfh}(X)$ and $L_{qfh}(X)$.

Lemma 8.8 ([43, Lem. 3.2.2]). — *Let X be an algebraic space over S and let T be a reduced S-scheme. Then $L'(X)(T) = L'_{qfh}(X)(T) = \mathrm{Hom}_S(T, X)$.*

Proof. — Let $\{U_i \to T\}$ be an h-covering. Then $\coprod_i U_i \to T$ is universally submersive and schematically dominant. It follows that $\mathrm{Hom}_S(T, X) \to \prod_i \mathrm{Hom}_S(U_i, X)$ is injective by Proposition 7.2 □

Lemma 8.9. — *Let X be an algebraic space* locally of finite type *over S and let $T \in \mathbf{Sch}_{/S}$. Then $L'(X)(T) = L'_{qfh}(X)(T)$ coincides with the image of*

$$\mathrm{Hom}_S(T, X) \to \mathrm{Hom}_S(T_{\mathrm{red}}, X).$$

If $T' \to T$ is universally submersive, then $L(X)(T) \to L(X)(T')$ is injective.

Proof. — If two morphisms $f, g : T \to X$ coincide after the composition with an h-covering $\{U_i \to T\}$, then they coincide after composing with $T_{\mathrm{red}} \to T$. Indeed, we have that $\coprod_i (U_i)_{\mathrm{red}} \to T_{\mathrm{red}}$ is an epimorphism by Proposition 7.2. Conversely, we will show that if f and g coincide on T_{red} then they coincide on a qfh-covering $T' \to T$.

Taking an open covering, we can assume that T is affine. Let $\mathcal{N}$ be the sheaf of nilpotent elements of $\mathcal{O}_T$, i.e., the ideal sheaf defining T_{red}. Then $\mathcal{N}$ is the direct limit of its subsheaves of finite type. Thus T_{red} is the inverse limit of finitely presented nil-immersions $T_\lambda \hookrightarrow T$. As $X \to S$ is locally of finite type $\varinjlim_\lambda \mathrm{Hom}_S(T_\lambda, X) \to \mathrm{Hom}_S(T_{\mathrm{red}}, X)$ is injective, cf. [17, Thm. 8.8.2]. Thus f and g coincide on T_λ for some λ.

To show the last statement, it is enough to show that $L'(X)(T) \to L'(X)(T')$ is injective when $T' \to T$ is universally submersive. From the first part of the lemma, it is thus enough to show that $\mathrm{Hom}_S(T_{\mathrm{red}}, X) \to \mathrm{Hom}_S(T'_{\mathrm{red}}, X)$ is injective and this is Proposition 7.2. □

Remark 8.10. — Voevodsky claims that $L'(X)(T) = \mathrm{Hom}_S(T_{\mathrm{red}}, X)$ in the text following [43, Lem. 3.2.2]. This is not true in general as $\mathrm{Hom}_S(T, X) \to \mathrm{Hom}_S(T_{\mathrm{red}}, X)$ need not be surjective. In fact, a counter-example is given by $X = T_{\mathrm{red}}$ for any scheme T such that $T_{\mathrm{red}} \hookrightarrow T$ does not have a retraction.

Proposition 8.11. — *Let X be an algebraic space locally of finite type over S, and let $T \in \mathbf{Sch}_{/S}$. Then $L(X)(T)$ (resp. $L_{qfh}(X)(T)$) is the filtered direct limit of*

$$\ker\Big(\textstyle\prod_i X(U_i) \rightrightarrows \prod_{i,j} X\big((U_i \times_T U_j)_{\mathrm{red}}\big)\Big)$$

where the limit is taken over all h-coverings (resp. qfh-coverings) $\{U_i \to T\}$.

Proof. — It is clear from the definitions of L and L' that $L(X)(T)$ is the limit of

$$\ker\Big(\textstyle\prod_i X(U_i) \rightrightarrows \prod_{i,j} L'(X)\big(U_i \times_T U_j\big)\Big).$$

The proposition thus follows from Lemma 8.9. □

In the remainder of this section, we can work in either the h-topology or the qfh-topology, i.e., all instances of L and L' can be replaced with L_{qfh} and L'_{qfh} respectively.

Definition 8.12. — Let X be an algebraic space over S, and let $T \in \mathbf{Sch}_{/S}$. Let $f \in L(X)(T)$ be a section and let $\{p_i : U_i \to T\}$ be a covering, i.e., a set of morphisms such that $T = \bigcup_i p_i(U_i)$ but not necessarily an h-covering. We say that f is *realized* on the covering $\{p_i\}$ if there are morphisms $\{f_i \in X(U_i)\}$ such that $p_i^*(f) = f_i$ in $L(X)(U_i)$ for every i.

Let $\{p_i : U_i \to T\}$ be a covering and let π_1, π_2 denote the projections of $(U_i \times_T U_j)_{\mathrm{red}}$. If $f \in L(X)(T)$ is realized on $\{p_i\}$ by $\{f_i : U_i \to X\}$ then $f_i \circ \pi_1 = f_j \circ \pi_2$ by Lemma 8.8. Conversely, if X/S is *locally of finite type* and $\{p_i\}$ is an *h-covering*, then morphisms $\{f_i \in X(U_i)\}$ such that $f_i \circ \pi_1 = f_j \circ \pi_2$, determines an element in $L(X)(T)$ by Proposition 8.11.

Lemma 8.13 ([43, Lem. 3.2.6]). — *Let X be an algebraic space over S, and let $T \in \mathbf{Sch}_{/S}$. Let $f \in L(X)(T)$ and assume that f is realized on an étale covering $\{p_i : U_i \to T\}$. Then f is realized on T_{red}.*

Proof. — Let $f_i : U_i \to X$ be a realization of f on the covering $\{p_i\}$. Then f_i and f_j coincide on $(U_i \times_T U_j)_{\mathrm{red}} = (U_i \times_T U_j) \times_T T_{\mathrm{red}}$. The $\{f_i\}$ thus glue to a morphism $T_{\mathrm{red}} \to X$ which realizes f. □

Proposition 8.14. — *Let X be an algebraic space locally of finite type over S and let T be an S-scheme. Let $f \in L(X)(T)$ be a section. Then f is realized on the absolute weak normalization T^{wn}.*

Proof. — We can replace T with T^{wn} and assume that T is weakly normal. The section f is realized on an h-covering of the form $\{W_j \to V_j \to T\}$ where the $W_j \to V_j$ are quasi-compact h-coverings and $\{V_j \to T\}$ is an open covering. By Theorem 7.4, applied to the weakly normal morphism $(W_j)_{\mathrm{red}} \to V_j$, and Lemma 8.9, we have that f is realized on the covering $\{V_j \to T\}$. Lemma 8.13 then shows that f is realized on T. □

Corollary 8.15. — *Let X be an algebraic space locally of finite presentation over S, and let T be a quasi-compact and quasi-separated S-scheme. Let $f \in L(X)(T)$ be a section. Then f is realized on a universal homeomorphism $U \to T$ of finite presentation.*

Proof. — By Proposition 8.14 the section f is realized on T^{wn}. A limit argument shows that there is a finitely presented universal homeomorphism $U \to T$ which realizes f. □

Theorem 8.16. — *Let X be an algebraic space locally of finite presentation over S, and let $T \in \mathbf{Sch}_{/S}$ be quasi-compact and quasi-separated. Then $L(X)(T) = \varinjlim_\lambda \operatorname{Hom}_S(T_\lambda, X) = \operatorname{Hom}_S(T^{\mathrm{wn}}, X)$ where the limit is taken over all finitely presented universal homeomorphisms $T_\lambda \to T$.*

Proof. — If $T_\lambda \to T$ is a universal homeomorphism then $(T_\lambda \times_T T_\lambda)_{\mathrm{red}} = (T_\lambda)_{\mathrm{red}}$. Thus, by Proposition 8.11 we obtain a canonical map

$$\varinjlim_\lambda \operatorname{Hom}_S(T_\lambda, X) \to L(X)(T).$$

The surjectivity of this map follows from Corollary 8.15. To show injectivity, let $f_1, f_2 : T_\lambda \to X$ be two maps coinciding in $L(X)(T)$. Then f_1 and f_2 coincide on $(T_\lambda)_{\mathrm{red}}$ and hence also on T_μ for a finitely presented nil-immersion $T_\mu \hookrightarrow T_\lambda$. Finally, we have that

$$\operatorname{Hom}_S(T^{\mathrm{wn}}, X) = \varinjlim_\lambda \operatorname{Hom}_S\left((T_\lambda)_{\mathrm{red}}, X\right) = \varinjlim_\lambda \operatorname{Hom}_S\left(T_\lambda, X\right). \qquad \square$$

Remark 8.17. — In the non-noetherian case, it may be useful to change the h-topology (resp. qfh-topology) to only require the coverings to be of finite type instead of finite presentation. In particular $X_{\mathrm{red}} \to X$ would always be an h-covering. Then Lemma 8.9 holds without any assumptions on X and we can drop the assumption that X/S is locally of finite type in 8.11–8.14. The main results 8.15–8.16 remain valid for this topology. It is also likely that for this topology Theorem 8.4 holds if we let $W_j \to V_j$ be any proper (resp. finite) surjective morphism, cf. Remark 3.14.

Appendix A
Étale morphisms and henselian pairs

In this section, we first recall some facts about étale morphisms which we state in the category of algebraic spaces. We then consider schemes which are proper over a local henselian scheme. Let S be a henselian local ring with closed point S_0, let $S' \to S$ be a proper morphism and let $S'_0 = S' \times_S S_0$. Then (S', S'_0) is 0-henselian (i.e., a henselian couple) and 1-henselian (i.e., induces an equivalence between finite étale covers). This is the key fact in the proof of the proper base change theorem in étale cohomology for degrees 0 and 1, cf. Theorem A.13. We interpret these henselian properties using algebraic spaces in Proposition A.7. These results are the core of the proof that proper morphisms are morphisms of effective descent for étale morphisms, cf. Proposition 5.14 and Corollary 5.16.

The results A.2–A.4 are well-known for schemes. We indicate how to extend these results to algebraic spaces:

Proposition A.1 ([23, Cor. II.6.17]). — *An étale and separated morphism of algebraic spaces is representable.*

Proposition A.2. — *Let $f : X \to Y$ be an étale morphism of algebraic spaces. Then:*

(i) *Δ_f is an open immersion.*
(ii) *Any section of f is an open immersion.*
(iii) *If f is universally injective, then f is an open immersion.*

Proof. — (i) follows easily from the case where X and Y are schemes. (ii) follows from (i) as any section of f is a pull-back of Δ_f. For (iii) we note that if f is universally injective then Δ_f is surjective. It follows by (i) that f is separated and by Proposition A.1 that f is representable. We can thus assume that X and Y are schemes. □

Corollary A.3. — *Let X and Y be algebraic spaces over S such that $Y \to S$ is étale. There is then a one-to-one correspondence between morphisms $f : X \to Y$ and open subspaces Γ of $X \times_S Y$ such that $\Gamma \to X$ is universally injective and surjective. This correspondence is given by mapping f to its graph Γ_f.*

Proof. — This follows immediately from (ii) and (iii) of Proposition A.2. □

Proposition A.4 ([17, Thm. 18.1.2]). — *Let $S_0 \hookrightarrow S$ be a nil-immersion of schemes, i.e., a surjective closed immersion. Then the functor $X \mapsto X \times_S S_0$ from the category of étale S-spaces (resp. S-schemes) to the category of étale S_0-spaces (resp. S_0-schemes) is an equivalence of categories.*

Proof. — That the functor is fully faithful follows from Corollary A.3. Let us prove essential surjectivity. For the category of schemes, this follows from [17, Thm. 18.1.2]. Let $X_0 \to S_0$ be an étale morphism of algebraic spaces. Let $U_0 \to X_0$ be an étale presentation with a scheme U_0. Then $R_0 = U_0 \times_{X_0} U_0$ is also a scheme. We thus obtain S-schemes R and U and an étale equivalence relation $R \rightrightarrows U$ which restricts to the equivalence relation given by R_0 and U_0. The quotient X of this equivalence relation restricts to X_0. □

We recall two fundamental results for schemes which are proper over a complete local ring.

Proposition A.5. — *Let S be the spectrum of a noetherian complete local ring with closed point S_0. Let $S' \to S$ be a* proper *morphism and $S'_0 = S' \times_S S_0$. The map $W' \mapsto W' \cap S'_0$ is a bijection between the open and closed subsets of S' and the open and closed subsets of S'_0.*

Proof. — This is a special case of [16, Prop. 5.5.1]. □

Theorem A.6 ([17, Thm. 18.3.4]). — *Let S be the spectrum of a noetherian complete local ring with closed point S_0. Let $S' \to S$ be a* proper *morphism and $S'_0 = S' \times_S S_0$. The functor $X' \mapsto X' \times_{S'} S'_0$ from the category of étale and finite S'-schemes to étale and finite S'_0-schemes is then an equivalence of categories.*

Proof. — Let $\widehat{S}$ and $\widehat{S'}$ be the completions of S and S' along S_0 and S'_0 respectively. Grothendieck's existence theorem [16, Thm. 5.1.4] shows that $X' \mapsto X' \times_{S'} \widehat{S'}$ is an equivalence between the categories of finite étale covers of S' and $\widehat{S'}$ respectively. Proposition A.4 then shows that $\widehat{X'} \mapsto \widehat{X'} \times_{\widehat{S'}} S'_0$ is an equivalence between covers of $\widehat{S'}$ and covers of S'_0. For details see [17, Thm. 18.3.4]. □

Using étale cohomology, we get a nice interpretation of the above two results:

Proposition A.7. — *Let S be a quasi-compact and quasi-separated scheme. Let $S_0 \hookrightarrow S$ be a closed subscheme. If F is a sheaf on the small étale site on S, then we let F_0 denote the pull-back of F to S_0. Then*

(i) *The following conditions are equivalent:*
 (a) *For any sheaf of sets F on the small étale site on S, the canonical map*
$$H^0_{ét}(S,F) \to H^0_{ét}(S_0,F_0)$$
 is bijective.
 (a′) *For any* constructible *sheaf of sets F on the small étale site on S, the canonical map*
$$H^0_{ét}(S,F) \to H^0_{ét}(S_0,F_0)$$
 is bijective.
 (b) *For any finite morphism $S' \to S$, the map $W' \mapsto W' \cap (S' \times_S S_0)$ from open and closed subsets of S' to open and closed subsets of $S' \times_S S_0$ is bijective.*
 (c) *For any étale morphism of algebraic spaces $X \to S$ the canonical map*
$$\Gamma(X/S) \to \Gamma(X \times_S S_0/S_0)$$
 is bijective.
 (c′) *For any étale* finitely presented *morphism of algebraic spaces $X \to S$ the canonical map*
$$\Gamma(X/S) \to \Gamma(X \times_S S_0/S_0)$$
 is bijective.

(ii) *The following conditions are equivalent:*

(a) *For any sheaf F of ind-finite groups on S, the canonical map*

$$H^i_{ét}(S,F) \to H^i_{ét}(S_0,F_0)$$

is bijective for $i=0,1$.

(b) *The functor $X \mapsto X \times_S S_0$ from the category of étale and finite S-schemes to étale and finite S_0-schemes is an equivalence of categories.*

Proof. — Every sheaf of sets is the filtered direct limit of constructible sheaves by [3, Exp. IX, Cor. 2.7.2]. As $H^0_{\text{ét}}$ commutes with filtered direct limits [3, Exp. VII, Rem. 5.14], the equivalence between (a) and (a′) follows. The equivalence between (a) and (b) in (i) and (ii) is a special case of [3, Exp. XII, Prop. 6.5]. For the equivalence between (a) and (c) in (i) we recall that there is an equivalence between the category of sheaves on the small étale site on S with the category of algebraic spaces X étale over S, cf. [29, Ch. V, Thm. 1.5] or [7, Ch. VII, §1]. This takes a sheaf to its "espace étalé" and conversely an algebraic space to its sheaf of sections. Furthermore, a sheaf is constructible if and only if its espace étalé is of finite presentation [3, Exp. IX, Cor. 2.7.1].

If $X \to S$ is an étale morphism corresponding to the sheaf F, then $H^0_{\text{ét}}(S,F) = \Gamma(X/S)$. For any morphism $g : S' \to S$, the pull-back g^*F is represented by $X \times_S S'$. This shows that (a) and (c) as well as (a′) and (c′) are equivalent. □

Remark A.8. — If S is not locally noetherian, then an espace étalé need not be quasi-separated. However, do note that any étale morphism is *locally separated* by Proposition A.2 and that finitely presented morphisms are quasi-separated.

Remark A.9. — Part (i) of Proposition A.7 is a generalization of [17, Prop. 18.5.4] which only shows that (b) implies (c) for the category of *separated* étale morphisms $X \to S$. An example of Artin [3, Exp. XII, Rem. 6.13] shows that condition (c) restricted to morphisms of *schemes* does not always imply (a) and (b). It does suffice when S is affine though, cf. [33, Ch. XI, Thm. 1].

Definition A.10. — Let S be a quasi-compact and quasi-separated scheme and $S_0 \hookrightarrow S$ a closed subscheme. We say that the pair (S,S_0) is 0-*henselian* or *henselian* (resp. 1-*henselian*) if (S,S_0) satisfies the equivalent conditions of (i) (resp. (ii)) of Proposition A.7.

We can now rephrase Proposition A.5 and Theorem A.6 as follows:

Theorem A.11. — *Let S be the spectrum of a noetherian complete local ring with closed point S_0. Let $S' \to S$ be a proper morphism and $S'_0 = S' \times_S S_0$. Then (S', S'_0) is 0-henselian and 1-henselian.*

Proposition A.5 is easily extended to noetherian henselian local rings using the connectedness properties of the Stein factorization:

Proposition A.12 ([17, Prop. 18.5.19]). — *Let S be the spectrum of a noetherian* henselian *local ring with closed point S_0. Let $S' \to S$ be a proper morphism and $S'_0 = S' \times_S S_0$. Then (S', S'_0) is 0-henselian.*

It is more difficult to show that (S', S'_0) is 1-henselian under the assumptions of Proposition A.12 (and we will not need this). One possibility is to use Artin's approximation theorem. This is done in [5, Thm. 3.1]. Another possibility is to use Popescu's theorem [36, 40]. As these powerful results were not available at the time, Artin gave an independent proof in [3, Exp. XII]. This result is also slightly more general as it does not require the proper morphism to be finitely presented:

Theorem A.13 ([3, Exp. XII, Cor. 5.5]). — *Let S be the spectrum of a henselian local ring with closed point S_0. Let $S' \to S$ be a proper morphism and $S'_0 = S' \times_S S_0$. Then the pair (S', S'_0) is 0-henselian and 1-henselian.*

Theorem A.13 is only part of the full proper base change theorem in étale cohomology [3, Exp. XII, Thm. 5.1, Cor. 5.5]. A slightly less general but easier proof of this theorem utilizing Artin's approximation theorem and algebraic spaces can be found in [7, Ch. VII].

Appendix B
Absolute weak normalization

In this section, we introduce the *absolute weak normalization*. This is an extension of the *weak normalization*, cf. [2, 28, 45]. The weak normalization (resp. absolute weak normalization) is dominated by the normalization (resp. total integral closure). Recall that a separated universal homeomorphism $X' \to X$ of algebraic spaces is the same thing as an integral, universally injective and surjective morphism, cf. Corollary 5.22 and [17, Cor. 18.12.11].

Definition B.1. — A scheme or algebraic space X is *absolutely weakly normal* if

(i) X is reduced.
(ii) If $\pi : X' \to X$ is a separated universal homeomorphism and X' is reduced, then π is an isomorphism.

If $X' \to X$ is a separated universal homeomorphism such that X' is absolutely weakly normal, then we say that X' is an absolute weak normalization of X.

PROPERTIES B.2. — We briefly list some basic properties of absolutely weakly normal schemes.

(i) If $Y' \to Y$ is a separated universal homeomorphism and X is absolutely weakly normal, then any morphism $X \to Y$ factors uniquely through Y'. In fact, $(X \times_Y Y')_{\mathrm{red}} \to X$ is an isomorphism. In particular, an absolute weak normalization is unique if it exists.
(ii) The spectrum of a perfect field is absolutely weakly normal.
(iii) A TIC scheme, cf. Definition 3.8, is absolutely weakly normal.
(iv) An absolutely flat scheme with perfect residue fields is absolutely weakly normal. Every scheme X has a canonical affine universally bijective morphism $T^{-\infty}(X) \to X$ where $T^{-\infty}(X)$ is absolutely flat with perfect residue fields [31].

We first establish the existence of the absolute weak normalization in the affine case and then show that it localizes.

DEFINITION B.3. — A ring extension $A \hookrightarrow A'$ is called *weakly subintegral* if $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a universal homeomorphism. For an arbitrary extension $A \hookrightarrow B$, the *weak subintegral closure* ${}^*_BA$ of A in B is the largest sub-extension $A \hookrightarrow {}^*_BA$ which is weakly subintegral. A ring A is *absolutely weakly normal* if its spectrum is absolutely weakly normal. If $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is an absolute weak normalization then we say that A' is the absolute weak normalization of A and denote A' with ${}^*A$.

Some comments on the existence of ${}^*_BA$ are due. If $A \hookrightarrow A'_1$ and $A \hookrightarrow A'_2$ are two weakly subintegral sub-extensions of $A \hookrightarrow B$, then the union $A'_1 \cup A'_2 = \mathrm{im}(A'_1 \otimes_A A'_2 \to B)$ is a weakly subintegral sub-extension of $A \hookrightarrow B$. If (A'_i) is a filtered union of weakly subintegral extension, then $A' = \bigcup_i A'_i$ is weakly subintegral [17, Cor. 8.2.10]. The existence of ${}^*_BA$ then follows from Zorn's lemma.

PROPERTIES B.4. — The following properties are readily verified:

(i) The weak subintegral closure is inside the integral closure [17, Cor. 18.12.11].
(ii) If $A \hookrightarrow B$ is an extension and B is absolutely weakly normal then ${}^*_BA$ is absolutely weakly normal.
(iii) If A is an integral domain then the weak subintegral closure of A in a perfect closure of its fraction field is the absolute weak normalization.
(iv) If A is any ring then the weak subintegral closure of A_{red} in $\mathrm{TIC}(A_{\mathrm{red}})$ (or $T^{-\infty}(A_{\mathrm{red}})$) is the absolute weak normalization ${}^*A$.

We have furthermore the following characterization of the weak subintegral closure:

LEMMA B.5 ([28, Thm. (I.6)]). — *Let $A \hookrightarrow B$ be an* integral *extension. Then $b \in B$ is in the weak subintegral closure ${}^*_BA$ if and only if $b \otimes 1 = 1 \otimes b$ in $(B \otimes_A B)_{\mathrm{red}}$.*

Proof. — Let $A' = A[b] \subseteq B$. Then $\mathrm{Spec}(B) \to \mathrm{Spec}(A')$ is surjective and it follows that $(A' \otimes_A A')_{\mathrm{red}} \to (B \otimes_A B)_{\mathrm{red}}$ is injective. Thus $b \otimes 1 = 1 \otimes b$ in $(B \otimes_A B)_{\mathrm{red}}$ if and only if $(A' \otimes_A A')_{\mathrm{red}} \to A'_{\mathrm{red}}$ is an isomorphism. Equivalently, the diagonal $\Delta_{\mathrm{Spec}(A')/\mathrm{Spec}(A)}$ is surjective which by [18, Prop. 3.7.1] is equivalent to $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ being universally injective. As $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is finite and surjective, $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is universally injective if and only if $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a universal homeomorphism. Thus $A \hookrightarrow A[b]$ is weakly subintegral if and only if $b \otimes 1 = 1 \otimes b$. □

PROPOSITION B.6. — *Let $A \hookrightarrow B$ be an extension and let $A \to A'$ be a homomorphism. Assume that $A \to A'$ is a localization or is étale. Let $B' = B \otimes_A A'$. Then:*

(i) *The weak subintegral closure ${}^*_BA$ of A in B commutes with the base change $A \to A'$, i.e., ${}^*_{B'}A' = ({}^*_BA) \otimes_A A'$.*
(ii) *The absolute weak normalization ${}^*A$ of A commutes with the base change $A \to A'$, i.e., ${}^*A' = ({}^*A) \otimes_A A'$.*

Proof. — As the integral closure commutes with étale base change [17, Prop. 18.12.15] and localizations, we can assume that $A \hookrightarrow B$ is integral. By Lemma B.5, the sequence

$$ {}^*_BA \hookrightarrow B \rightrightarrows (B \otimes_A B)_{\mathrm{red}} $$

is exact. As exactness is preserved by flat morphisms and reduced rings are preserved by localization and étale base change, cf. [17, Prop. 17.5.7], it follows that ${}^*_{B'}A' = ({}^*_BA) \otimes_A A'$.

For the second part, let $B = \mathrm{TIC}(A)$ (or $B = T^{-\infty}(A)$). Then ${}^*A = {}^*_BA$ and in order to show that ${}^*A' = ({}^*A) \otimes_A A'$ it is enough to show that B' is absolutely weakly normal as $({}^*A) \otimes_A A' = {}^*_{B'}A'$ by the first part. Furthermore, it suffices to show that $B'_{\mathfrak{p}'}$ is absolutely weakly normal for every prime $\mathfrak{p}' \in \mathrm{Spec}(B')$. Let $\mathfrak{p}$ be the image of $\mathfrak{p}'$ by $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$. Then $B_{\mathfrak{p}} \to B'_{\mathfrak{p}'}$ is essentially étale. But $B_{\mathfrak{p}}$ is strictly henselian, cf. Properties 3.9, and thus $B_{\mathfrak{p}} \to B'_{\mathfrak{p}'}$ is an isomorphism. As $B_{\mathfrak{p}}$ is a TIC ring it is absolutely weakly normal. If we instead use $B = T^{-\infty}(A)$ the last part of the demonstration becomes trivial as $B_{\mathfrak{p}}$ and $B'_{\mathfrak{p}'}$ are perfect fields. □

Let S be a scheme or algebraic space. The proposition implies that given an extension of quasi-coherent algebras $\mathscr{A} \hookrightarrow \mathscr{B}$ on S, there is a unique quasi-coherent sub-algebra ${}^*_{\mathscr{B}}\mathscr{A}$ which restricts to the weak subintegral closure on any affine covering. If $\varphi : \mathscr{A} \to \mathscr{B}$ is not injective but $\mathrm{Spec}(\mathscr{B}) \to \mathrm{Spec}(\mathscr{A})$ is dominant, then we let ${}^*_{\mathscr{B}}\mathscr{A}$ be the weak subintegral closure of $\mathscr{A}/\ker(\varphi)$ in $\mathscr{B}$. Furthermore, there is a quasi-coherent sheaf of algebras ${}^*\mathscr{O}_S = {}^*\mathscr{O}_{S_{\mathrm{red}}}$ and the spectrum of this algebra is the absolute weak normalization of S. In the geometric case we adhere to the notation in [24, Ch. I, 7.2]:

Definition B.7. — Let S be a scheme or algebraic space. The *weak normalization* of S with respect to a quasi-compact and quasi-separated dominant morphism $f : X \to S$ is the spectrum of the weak subintegral closure of $\mathscr{O}_S$ in $f_*\mathscr{O}_X$ and is denoted $S^{X/\mathrm{wn}}$. The *absolute weak normalization* of S is denoted S^{wn}.

Remark B.8. — An integral domain is said to be weakly normal if it is weakly normal in its fraction field. Similarly, a reduced ring with a finite number of irreducible components is weakly normal if it is weakly normal in its total fraction ring [28, 46]. If A is an excellent noetherian ring, then its weak normalization is finite over A and thus noetherian. The absolute weak normalization on the other hand, need not be finite and may well reside outside the category of noetherian rings.

There is also the notions of *subintegral closure* and *semi-normality* [14, 39, 42] which coincide with weak subintegral closure and weak normality in characteristic zero. The difference in positive characteristic is that $A \hookrightarrow B$ is subintegral if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism with *trivial residue field extensions*, while weakly subintegral morphisms may have purely inseparable field extensions. If A is an excellent noetherian ring then its semi-normalization is finite over A. In particular, if A is an excellent noetherian ring of characteristic zero, then the absolute weak normalization, being equal to the semi-normalization, is finite over A.

BIBLIOGRAPHY

[1] *Schémas en groupes. I: Propriétés générales des schémas en groupes* – Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Math., vol. 151, Springer, 1970.

[2] A. Andreotti & E. Bombieri – "Sugli omeomorfismi delle varietà algebriche", *Ann. Scuola Norm. Sup Pisa* **23** (1969), p. 431–450.

[3] M. Artin, A. Grothendieck & J. L. Verdier (eds.) – *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Math., vol. 305, Springer, 1973.

[4] M. Artin – *Grothendieck topologies*, Harvard University, 1962, lecture notes.

[5] ______, "Algebraic approximation of structures over complete local rings", *Publ. Math. I.H.É.S.* **36** (1969), p. 23–58.

[6] ______, "On the joins of Hensel rings", *Advances in Math.* **7** (1971), p. 282–296.

[7] ______, *Théorèmes de représentabilité pour les espaces algébriques*, Les Presses de l'université de Montréal, 1973.

[8] N. Bourbaki – *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations*, Actualités Scientifiques et Industrielles, No. 1308, Hermann, 1964.

[9] H. Brenner – "Grothendieck topologies and ideal closure operations", preprint arXiv:math/0612471.

[10] B. Conrad – "Deligne's notes on Nagata compactifications", *J. Ramanujan Math. Soc.* **22** (2007), p. 205–257.

[11] D. E. Dobbs, M. Fontana & G. Picavet – "Generalized going-down homomorphisms of commutative rings", in *Commutative ring theory and applications (Fez, 2001)*, Lecture Notes in Pure and Appl. Math., vol. 231, Dekker, 2003, p. 143–163.

[12] D. Ferrand – "Conducteur, descente et pincement", *Bull. Soc. Math. France* **131** (2003), p. 553–585.

[13] J. Giraud – "Méthode de la descente", *Bull. Soc. Math. France Mém.* **2** (1964).

[14] S. Greco & C. Traverso – "On seminormal schemes", *Compositio Math.* **40** (1980), p. 325–365.

[15] A. Grothendieck – "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes", *Publ. Math. I.H.É.S.* **8** (1961).

[16] ______, "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents", *Publ. Math. I.H.É.S.* **11, 17** (1961, 1963).

[17] ______, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas", *Publ. Math. I.H.É.S.* **20, 24, 28, 32** (1964-67).

[18] ______, *Éléments de géométrie algébrique. I. Le langage des schémas*, second ed., Grundl. Math. Wiss., vol. 166, Springer, 1971.

[19] ______ (ed.) – *Revêtements étales et groupe fondamental*, Lecture Notes in Math., vol. 224, Springer, 1971.

[20] M. Hochster – "Prime ideal structure in commutative rings", *Trans. Amer. Math. Soc.* **142** (1969), p. 43–60.

[21] ———, "Totally integrally closed rings and extremal spaces", *Pacific J. Math.* **32** (1970), p. 767–779.

[22] B. G. Kang & D. Y. Oh – "Lifting up an infinite chain of prime ideals to a valuation ring", *Proc. Amer. Math. Soc.* **126** (1998), p. 645–646.

[23] D. Knutson – *Algebraic spaces*, Lecture Notes in Math., vol. 203, 1971.

[24] J. Kollár – *Rational curves on algebraic varieties*, Ergebn. der Math. und ihrer Grenzg., vol. 32, Springer, 1996.

[25] ———, "Quotient spaces modulo algebraic groups", *Ann. of Math.* **145** (1997), p. 33–79.

[26] G. Laumon & L. Moret-Bailly – *Champs algébriques*, Ergebn. der Math. und ihrer Grenzg., vol. 39, Springer, 2000.

[27] D. Lazard – "Autour de la platitude", *Bull. Soc. Math. France* **97** (1969), p. 81–128.

[28] M. Manaresi – "Some properties of weakly normal varieties", *Nagoya Math. J.* **77** (1980), p. 61–74.

[29] J. S. Milne – *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton Univ. Press, 1980.

[30] J.-P. Olivier – "Anneaux absolument plats universels et épimorphismes à buts réduit", in *Séminaire d'Algèbre Commutative 1967–1968*, Secrétariat mathématique, 1968, exposé n° 6.

[31] ———, "Le foncteur $T^{-\infty}$. Globalisation du foncteur T", in *Séminaire d'Algèbre Commutative 1967–1968*, Secrétariat mathématique, 1968, exposé n° 9.

[32] G. Picavet – "Submersion et descente", *J. Algebra* **103** (1986), p. 527–591.

[33] M. Raynaud – *Anneaux locaux henséliens*, Lecture Notes in Math., vol. 169, Springer, 1970.

[34] M. Raynaud & L. Gruson – "Critères de platitude et de projectivité. Techniques de "platification" d'un module", *Invent. Math.* **13** (1971), p. 1–89.

[35] D. Rydh – "Existence of quotients by finite groups and coarse moduli spaces", preprint arXiv:0708.3333.

[36] M. Spivakovsky – "A new proof of D. Popescu's theorem on smoothing of ring homomorphisms", *J. Amer. Math. Soc.* **12** (1999), p. 381–444.

[37] A. Suslin & V. Voevodsky – "Singular homology of abstract algebraic varieties", *Invent. Math.* **123** (1996), p. 61–94.

[38] ———, "Relative cycles and Chow sheaves", in *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, 2000, p. 10–86.

[39] R. G. SWAN – “On seminormality”, *J. Algebra* **67** (1980), p. 210–229.

[40] ———, “Néron-Popescu desingularization”, in *Algebra and geometry (Taipei, 1995)*, Lect. Algebra Geom., vol. 2, Int. Press, Cambridge, MA, 1998, p. 135–192.

[41] R. W. THOMASON & T. TROBAUGH – “Higher algebraic K-theory of schemes and of derived categories”, in *The Grothendieck Festschrift, Vol. III*, Progr. Math., vol. 88, Birkhäuser, 1990, p. 247–435.

[42] C. TRAVERSO – “Seminormality and Picard group”, *Ann. Scuola Norm. Sup. Pisa* **24** (1970), p. 585–595.

[43] V. VOEVODSKY – “Homology of schemes”, *Selecta Math. (N.S.)* **2** (1996), p. 111–153.

[44] V. VOEVODSKY, A. SUSLIN & E. M. FRIEDLANDER – *Cycles, transfers, and motivic homology theories*, Annals of Math. Studies, vol. 143, Princeton Univ. Press, 2000.

[45] H. YANAGIHARA – “Some results on weakly normal ring extensions”, *J. Math. Soc. Japan* **35** (1983), p. 649–661.

[46] ———, “On an intrinsic definition of weakly normal rings”, *Kobe J. Math.* **2** (1985), p. 89–98.