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Philipp Habegger

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A BOGOMOLOV PROPERTY FOR CURVES MODULO ALGEBRAIC SUBGROUPS

by Philipp Habegger

Abstract. — Generalizing a result of Bombieri, Masser, and Zannier we show that on a curve in the algebraic torus which is not contained in any proper coset only finitely many points are close to an algebraic subgroup of codimension at least 2. The notion of close is defined using the Weil height. We also deduce some cardinality bounds and further finiteness statements.

Résumé (*Une propriété de Bogomolov pour des courbes modulo des sous-groupes algébriques*)

En généralisant un résultat de Bombieri, Masser, et Zannier on montre qu'une courbe plongée dans le tore algébrique qui n'est pas contenue dans un translaté d'un sous-groupe algébrique strict n'a qu'un nombre fini de points proches d'un sous-groupe algébrique de codimension au moins 2. La notion de proximité est définie en utilisant la hauteur de Weil. On déduit également des bornes pour la cardinalité et d'autres énoncés de finitude.

1. Introduction

Let X be an irreducible algebraic curve embedded in the algebraic torus $\mathbf{G}_m^n$ and defined over $\overline{\mathbf{Q}}$, an algebraic closure of $\mathbf{Q}$. Bombieri, Masser, and Zannier [5] showed that if X is not contained in the translate of a proper algebraic subgroup, then only finitely many points in X are contained in an algebraic

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Philipp Habegger, ETH Zürich, Departement Mathematik, Rämistrasse 101, 8092 Zürich, Switzerland • *E-mail :* `habegger@math.ethz.ch`

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subgroup of $\mathbf{G}_m^n$ of dimension $n-2$. The subgroup dimension $n-2$ is best-possible. Their result is related to several general conjectures stated in the mean time by those three authors [7], Pink [19], and Zilber [27].

In this paper we show that only finitely many points in X are close to an algebraic subgroup of dimension $n-2$, where the notion of close is defined with respect to the Weil height. We also give some finiteness results and cardinality bounds for higher dimensional varieties.

All varieties in this paper are defined over $\overline{\mathbf{Q}}$ and will be identified with their set of algebraic points. By irreducible we will always mean geometrically irreducible. For brevity we call the translate of an algebraic subgroup of $\mathbf{G}_m^n$ a *coset* and the translate of an algebraic subgroup of $\mathbf{G}_m^n$ by a torsion point a *torsion coset*. For an integer m with $0 \leq m \leq n$ we define $\mathcal{H}_m$ to be the set of points in $\mathbf{G}_m^n$ that are contained in an algebraic subgroup of dimension at most m; if $m < 0$ we set $\mathcal{H}_m = \varnothing$. With this notation and with X as in the first paragraph, Bombieri, Masser, and Zannier's Theorem states that $X \cap \mathcal{H}_{n-2}$ is finite.

Let $h(\cdot)$ denote the absolute logarithmic Weil height on $\mathbf{G}_m^n$; the precise definition is given in section 2. This height has the important property, usually called Kronecker's Theorem, that it vanishes precisely on the torsion points of $\mathbf{G}_m^n$. For any subset $H \subset \mathbf{G}_m^n$ and any $\epsilon \in \mathbf{R}$ we define the "truncated cone" around H as

$$C(H,\epsilon) = \{ab; a \in H, b \in \mathbf{G}_m^n, h(b) \leq \epsilon(1+h(a))\}.$$

Kronecker's Theorem implies $C(\mathcal{H}_m, 0) = \mathcal{H}_m$.

This definition showed up in the work of Evertse [12]. A special case of his Theorem 5(i) implies that if $X \subset \mathbf{G}_m^n$ is an irreducible curve not equal to a coset and if Γ is the division closure of a finitely generated subgroup of $\mathbf{G}_m^n$, then $X \cap C(\Gamma, \epsilon)$ is finite for $\epsilon > 0$ small enough. Actually Evertse proved a result for X of any dimension. Earlier, Poonen [20] proved a related result in the context of semi-abelian varieties which was then generalized by Rémond [21]. We will study the intersection of subvarieties of $\mathbf{G}_m^n$ with $C(\mathcal{H}_m, \epsilon)$ for small $\epsilon > 0$.

THEOREM 1.1. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed algebraic curve defined over $\overline{\mathbf{Q}}$. If X is not contained in a proper coset there exists $\epsilon > 0$ effective and depending only on $h(X)$, $\deg(X)$, and n such that $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ is finite with cardinality bounded effectively in terms of $h(X)$, $\deg(X)$, and n.*

A quite explicit bound for the cardinality is given by (61).

The height $h(X)$ of any irreducible subvariety X of $\mathbf{G}_m^n$ used in this article is the height $h_{\iota|X}$ defined by Philippon on page 346 [18] where ι is the embedding of $\mathbf{G}_m^n$ into projective space $\mathbf{P}^n$ defined in section 2. This height was also used by the author in [15]. The definition of $\deg(X)$, the degree of X, is recalled in section 2.

Theorem 1.1 generalizes Bombieri, Masser, and Zannier's Theorem and also generalizes the Bogomolov property for our curve X. The Bogomolov property (for curves in $\mathbf{G}_m^n$) actually holds more generally and states that if an irreducible curve in $\mathbf{G}_m^n$ is not a torsion coset, then all but finitely many points on this curve have height bounded below by a positive constant. In Theorem 6.2, Zhang [26] proved this and also a higher dimensional analogue. If $n = 2$, Theorem 1.1 actually follows from the Bogomolov property since $C(\mathcal{H}_0, \epsilon)$ is precisely the set of points in $\mathbf{G}_m^n$ with height $\leq \epsilon$. Theorem 1.1 can be viewed as a sort of Bogomolov property for curves modulo subgroups of dimension $n-2$. We remark that no new proof of the Bogomolov property is given in this article since Theorem 1.1 itself depends on a quantitative version of this property by Amoroso and David [3].

Theorem 1.1 is proved in two steps. First, we apply a Theorem proved by the author [15], see Theorem 7.1 further down, which uniformly bounds the height of points in the intersection $X \cap C(\mathcal{H}_{n-1}, \epsilon)$ if $\epsilon > 0$ is small enough. The second step, done below in Theorem 1.2, consists in showing that a subset of $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ of bounded height is finite if $\epsilon > 0$ is small enough. Theorem 1.1 follows since we already know that $X \cap C(\mathcal{H}_{n-2}, \epsilon) \subset X \cap C(\mathcal{H}_{n-1}, \epsilon)$ has bounded height for small ϵ.

In Theorem 1.2 below we prove a finiteness statement which holds not only for curves but for any irreducible closed subvariety $X \subset \mathbf{G}_m^n$. This is the main technical result of the article, but before we state it we need some definitions.

The set X^{oa} is obtained by removing from X all anomalous subvarieties and X^{ta} is obtained by removing from X all torsion-anomalous subvarieties; see section 2 for the definition of anomalous and torsion-anomalous subvarieties. The sets X^{oa} and X^{ta} were defined by Bombieri, Masser, and Zannier [7] who showed that X^{oa} is Zariski open in X.

For r and n real numbers with $1 \leq r \leq n$, we define

$$\mathfrak{m}(r,n) = n - 2r + 2^{-d}(r(d+2) - n) \text{ with } d = \left[\frac{n-1}{r}\right], \tag{1}$$

here $[x]$ denotes the greatest integer less or equal to x.

THEOREM 1.2. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety of dimension $r \geq 1$ defined over $\overline{\mathbf{Q}}$. Let $B \geq 1$ and let m be an integer with $m < \mathfrak{m}(r,n)$.*

(i) *If X is not contained in a proper coset there exists $\epsilon > 0$ effective and depending only on B,* $\deg(X)$, *and n such that*

$$\{p \in X \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$

is not Zariski dense in X.

(ii) *For unrestricted X let $\Delta = (B^r \deg(X))^{(n+6)^{4r}2^{nr}}$. There exists $c(n) > 0$ effective and depending only on n such that if $\epsilon \leq (c(n)\Delta)^{-1}$ then*

$$\{p \in X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$

is finite of cardinality at most $c(n)\Delta$.

A possible choice for ϵ in part (i) is the right-hand side of (48) with s replaced by n. We by no means claim that the hypothesis on ϵ or the cardinality bound in part (ii) are best-possible with respect to any of the involved quantities like B or $\deg(X)$. We do remark that Δ and $c(n)$ are independent of a field of definition or height of X. This uniformity can be used to obtain the following uniform cardinality bound for a simple family of curves:

COROLLARY 1.3. — *Let $\tau \in \overline{\mathbf{Q}}$ and let $X_\tau \subset \mathbf{G}_m^3$ be the curve defined by $(x+1, x+\tau, x-\tau)$ where $x \neq -1, \pm\tau$. There exist $\epsilon > 0$ and an integer N such that $X_\tau \cap C(\mathcal{H}_1, \epsilon)$ is finite with cardinality bounded by N for all $\tau \in \overline{\mathbf{Q}}\backslash\{0, \pm 1\}$.*

Although the corollary could possibly be generalized to more complicated families of curves, our method cannot handle other simple examples such as $(x, x-1, x-\tau)$.

Corollary 1.3 motivates the following two questions. In Theorem 1.1, can ϵ be chosen depending only on $\deg(X)$ and n? In the same theorem, can the cardinality be bounded in function only of $\deg(X)$ and n?

By definition we have $\mathfrak{m}(1, n) = n - 2 + 2^{-(n-1)} > n - 2$ and thus Theorem 1.2 is optimal with respect to the subgroup dimension if X is a curve. But it is likely that the somewhat unnatural function $\mathfrak{m}(r, n)$ does not lead to optimal results if $2 \leq r \leq n - 2$. In fact we conjecture that Theorem 1.2(i) holds with $\mathfrak{m}(r, n)$ replaced by $n - r$. If $1 \leq r \leq n$ and if d is as in (1), then $d > \frac{n-1}{r} - 1$, hence $r(d + 2) > n - 1 + r$. We conclude $\mathfrak{m}(r, n) > n - 2r$. Therefore one may always take $m = n - 2r$ in Theorem 1.2. Of course this choice is only interesting if $r \leq n/2$. Further down, in Lemma 6.2 we will see that $\mathfrak{m}(r, n) \geq (n - r)/2$ holds for all $1 \leq r \leq n - 1$.

Statements related to the ones in Theorem 1.2 were known earlier with $\epsilon = 0$. Work was done in the multiplicative case by Bombieri, Masser, and Zannier (Lemma 8.1 [8]) and in the abelian case by Rémond (Theorem 2.1 [22]). For example by Lemma 8.1 [8] the set of $p \in X^{\mathrm{ta}} \cap \mathcal{H}_{n-r-1}$ with $h(p) \leq B$ is finite. In this result the subgroup dimension $n - r - 1$ is best-possible for any r and finiteness is obtained for X^{ta} instead of the possibly smaller X^{oa}. These earlier finiteness results involved Lehmer-type height lower bounds. In the multiplicative case such a bound gives a positive lower bound for $h(p)$ if $p \in \mathbf{G}_m^n$ is not contained in any proper algebraic subgroup of $\mathbf{G}_m^n$. Typical lower bounds depend essentially on the degree of p over a fixed field such as $\mathbf{Q}$ or its maximal abelian extension.

Unfortunately, the methods from [8] and [22] using Lehmer-type lower bounds do not generalize well to the situation of Theorem 1.2 where a positive ϵ is involved. Rather than using a Lehmer-type height bound we use a Bogomolov-type height lower bound. Given an irreducible closed subvariety X of $\mathbf{G}_m^n$ not equal to a torsion coset, such a bound supplies a Zariski closed proper $Z \subsetneq X$ and a positive height lower bound on $X \backslash Z$. To prove Theorem 1.2 we will apply a bound by Amoroso and David [3] (cf. Theorem 5.1). If $X \neq \mathbf{G}_m^n$ is not contained in a proper coset then their lower bound depends only on $\deg(X)$ and n. Furthermore, the dependency in $\deg(X)$ is essentially best-possible, a point which is crucial for our application.

The hypothesis of Amoroso and David's theorem is reflected in the hypothesis of Theorem 1.2(i). Under the weaker hypothesis that $X \neq \mathbf{G}_m^n$ is not contained in a proper torsion coset, Amoroso and David used their Lehmer-type height lower bound [1] to obtain a positive lower bound for the height on $X \backslash Z$ for some Zariski closed proper $Z \subsetneq X$ (cf. Theorem 5.2). This lower bound has a similar form as the one from Theorem 5.1 but with $\deg(X)$ replaced by $[K : \mathbf{Q}]\deg(X)$ for K a field of definition of X. It is not difficult to adapt the proof of Theorem 1.2 to use this lower bound and obtain:

THEOREM 1.4. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety of dimension $r \geq 1$ defined over $\overline{\mathbf{Q}}$. Let $B \geq 1$ and let m be an integer with $m < \mathfrak{m}(r, n)$.*

(i) *If X is not contained in a proper torsion coset there exists $\epsilon > 0$ effective such that*
$$\{p \in X \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$
is not Zariski dense in X.

(ii) *For unrestricted X there exists $\epsilon > 0$ effective such that*
$$\{p \in X^{\mathrm{ta}} \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$$
is finite.

We note that the hypothesis on X in part (i) is weaker than in Theorem 1.2(i). On the other hand, ϵ may now depend on a field of definition of X. Moreover, in part (ii) we prove finiteness for X^{ta} instead of the possibly smaller X^{oa} but at the same time we can no longer expect to find an ϵ or a bound for the cardinality which is independent of a field of definition of X.

The proof of Theorem 1.1 on curves relies on Theorems 1.2 and 7.1. But these two latter theorems also give results for subvarieties of $\mathbf{G}_m^n$ of arbitrary dimension. Although the dimension of the subgroups involved may not be optimal in either one, we state the consequences.

Theorem 1.5. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety of dimension $r \geq 1$ defined over $\overline{\mathbf{Q}}$. If m is an integer with $m < \min\{n/r, \mathfrak{m}(r,n)\}$ there exists $\epsilon > 0$ effective and depending only on $h(X)$, $\deg(X)$, and n such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)$ is finite with cardinality bounded effectively in terms of $h(X)$, $\deg(X)$, and n.*

In [15] the author conjectured that one can replace $m < \min\{n/r, \mathfrak{m}(r,n)\}$ by $m < n - r$ in Theorem 1.5 and still obtain finiteness.

Using the cardinality bound in Theorem 1.2(ii) and the explicit height bound from Theorem 7.1 one can bound the cardinality in Theorem 1.5 polynomially in $h(X)$ and $\deg(X)$, cf. (60). A similar remark also holds for the following corollary of Theorem 1.5 which implies finiteness if the algebraic subgroups have dimension 1.

Corollary 1.6. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. If $\dim X \leq n - 3$ there exists $\epsilon > 0$ effective and depending only on $h(X)$, $\deg(X)$, and n such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_1, \epsilon)$ is finite with cardinality bounded effectively in terms of $h(X)$, $\deg(X)$, and n.*

We show some not completely immediate consequences of Theorems 1.1 and 1.5 as well as Corollary 1.6.

Corollary 1.7. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$.*

(i) *If X is a curve there exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-2}, \epsilon)$ is finite and equal to $X^{\mathrm{oa}} \cap \mathcal{H}_{n-2}$.*
(ii) *If $r = \dim X \geq 1$ and if m is an integer with $m < \min\{n/r, \mathfrak{m}(r,n)\}$ there exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)$ is finite and equal to $X^{\mathrm{oa}} \cap \mathcal{H}_m$.*
(iii) *If $\dim X \leq n - 3$ there exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_1, \epsilon)$ is finite and equal to $X^{\mathrm{oa}} \cap \mathcal{H}_1$.*

So, a curve which is not contained in a proper coset contains no points close to an algebraic subgroup of dimension $n - 2$ which do not already lie on such a subgroup.

For any subset $H \subset \mathbf{G}_m^n$ and any $\epsilon \in \mathbf{R}$ we define the "tube" around H as

$$T(H, \epsilon) = \{ab; a \in H, b \in \mathbf{G}_m^n, h(b) \leq \epsilon\}.$$

Theorem 1.1 motivates the following definition: let $n \geq 2$ and let $X \subset \mathbf{G}_m^n$ be an irreducible algebraic curve, we define

$$\hat{\mu}_C^{\mathrm{ess}}(X) = \sup\{\epsilon \geq 0; X \cap C(\mathcal{H}_{n-2}, \epsilon) \text{ finite}\}$$

where by convention $\sup \varnothing = -\infty$. We also define

$$\hat{\mu}_T^{\mathrm{ess}}(X) = \sup\{\epsilon \geq 0; X \cap T(\mathcal{H}_{n-2}, \epsilon) \text{ finite}\}.$$

Then clearly $\hat{\mu}_C^{\mathrm{ess}}(X) \leq \hat{\mu}_T^{\mathrm{ess}}(X) \leq \hat{\mu}^{\mathrm{ess}}(X)$ where $\hat{\mu}^{\mathrm{ess}}(X)$ is the essential minimum of X, see (10) for a definition. We are interested in bounding $\hat{\mu}_C^{\mathrm{ess}}(X)$ and $\hat{\mu}_T^{\mathrm{ess}}(X)$ from below.

Let us study $\hat{\mu}_C^{\mathrm{ess}}(X)$ if the curve X is not contained in a proper coset. Then Theorem 1.1 implies $\hat{\mu}_C^{\mathrm{ess}}(X) > 0$. Moreover, it even states that $\hat{\mu}_C^{\mathrm{ess}}(X)$ can be bounded below in terms of $h(X)$, $\deg(X)$, and n only. If we consider again the case $n = 2$, Amoroso and David's Theorem 5.1 implies that $\hat{\mu}_C^{\mathrm{ess}}(X) = \hat{\mu}^{\mathrm{ess}}(X)$ is bounded below in terms of $\deg(X)$. Also, if $\tau \in \overline{\mathbf{Q}}\backslash\{0, \pm 1\}$ and if X_τ is the family of curves in Corollary 1.3, then $\hat{\mu}_C^{\mathrm{ess}}(X_\tau)$ can be bounded from below by a positive number independent of τ. The first question posed below Corollary 1.3 amounts to asking if $\hat{\mu}_C^{\mathrm{ess}}(X)$ can be bounded below solely in terms of $\deg(X)$ and n.

What can be said about $\hat{\mu}_C^{\mathrm{ess}}(X)$ and $\hat{\mu}_T^{\mathrm{ess}}(X)$ if we only demand that X is not contained in a proper algebraic subgroup of $\mathbf{G}_m^n$? Conjecture A of Bombieri, Masser, and Zannier [6] expects that $X \cap \mathcal{H}_{n-2}$ is finite, or in other words $\hat{\mu}_C^{\mathrm{ess}}(X) \geq 0$. This conjecture was proved recently by Maurin [17]. But strict inequality $\hat{\mu}_C^{\mathrm{ess}}(X) > 0$ cannot hold in general as is demonstrated by the following example:

Assume the curve $X \subset \mathbf{G}_m^n$ is contained in a coset of codimension 2 (but not in a proper algebraic subgroup). After an automorphism of $\mathbf{G}_m^n$ we may suppose $X = \{(\gamma_1, \gamma_2)\} \times X'$ where $X' \subset \mathbf{G}_m^{n-2}$ is a curve. Let ϵ be a positive real number, then any $p \in X$ can be written as $p = ab$ with $a = (1, 1, p') \in \mathcal{H}_{n-2}$ and $b = (\gamma_1, \gamma_2, 1, \ldots, 1)$. Hence if $h(p)$ is large with respect to ϵ and $h(b)$ then $h(p') = h(a)$ will be large with respect to ϵ and $h(b)$. Therefore $p \in C(\mathcal{H}_{n-2}, \epsilon)$ if $h(p)$ is large. So $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ is infinite for all positive ϵ, hence $\hat{\mu}_C^{\mathrm{ess}}(X) \leq 0$. This argument does not imply $\hat{\mu}_T^{\mathrm{ess}}(X) \leq 0$. Indeed since we are assuming that X is not contained in a proper algebraic subgroup neither γ_1 nor γ_2 can be a root of unity. So $h(b) = h(\gamma_1, \gamma_2) > 0$ by Kronecker's Theorem.

We state the following conjecture for varieties of arbitrary dimension:

CONJECTURE 1.8. — *Let X be an irreducible closed subvariety of $\mathbf{G}_m^n$ of dimension r defined over $\overline{\mathbf{Q}}$. There exists an $\epsilon > 0$ such that $X^{\mathrm{ta}} \cap T(\mathcal{H}_{n-r-1}, \epsilon)$ is finite.*

If X is a curve, then $X^{\mathrm{ta}} = X$ if and only if X is not contained in a proper algebraic subgroup. Hence the conjecture expects $\hat{\mu}_T^{\mathrm{ess}}(X) > 0$ if X is not contained in a proper algebraic subgroup.

We discuss the abelian situation. More specifically we replace $\mathbf{G}_m^n$ by E^n where E is an elliptic curve. The set $\mathcal{H}_m$ also makes sense in this setting, as do $T(\cdot, \cdot)$ and $C(\cdot, \cdot)$ when using for example the Néron-Tate height associated to an ample and symmetric line bundle. Let $X \subset E^n$ be an irreducible curve. Intersections of X with $\mathcal{H}_m$ were studied by Viada [24] and Rémond and Viada [23].

Say X is not contained in the translate of a proper algebraic subgroup of E^n. If E does not have complex multiplication Viada proved that $X \cap \mathcal{H}_m$ is finite if $m \leq n/2-2$. If E has complex multiplication, she showed that one has finiteness for the optimal $m = n-2$. Her proof used a height upper bound for curves analog to Theorem 7.1 and, among other things, a lower bound for the Néron-Tate height on powers of elliptic curves. The reason for the seemingly non-optimal $n/2-2$ in the non-complex multiplication case comes from the fact that sufficiently strong Lehmer-type height lower bounds are not available at the moment here. Also, an analogue to Amoroso and David's Theorem 5.2 for E^n seems to be out of reach if E does not have complex multiplication.

But there is hope that a Bogomolov-type height lower bound of the same quality as the one in Amoroso and David's Theorem 5.1 will soon be available in the case where the algebraic torus is replaced by a power of an elliptic curve or possibly more general abelian varieties. Galateau in this recent Ph.D. thesis [14] has proved such a result for subvarieties of products of elliptic curves with codimension at most 2.

Let $X \subset E^n$ be a curve which is not contained in the translate of a proper algebraic subgroup. An appropriate version of Amoroso and David's Theorem for E^n together with the methods presented in this article and [15] should provide a proof for the finiteness of $X \cap C(\mathcal{H}_{n-2}, \epsilon)$ for some $\epsilon > 0$ regardless if E has complex multiplication or not.

Let $X \subset E^n$ be a curve which is not contained in the translate of a proper algebraic subgroup by a torsion point. In an unpublished manuscript from 2007 Viada proved the finiteness of $X \cap T(\mathcal{H}_{n-3}, \epsilon)$ for small ϵ assuming Galateau's result holds without restriction on the codimension. Under the same hypothesis and just prior to the submission this article, she [25] announced a proof that $X \cap T(\mathcal{H}_{n-2}, \epsilon)$ is finite for some $\epsilon > 0$. Hence she obtains finiteness with the optimal subgroup dimension. Moreover, there is hope that her approach also gives non-density results for higher dimensional subvarieties with the correct subgroup size.

The paper is organized as follows. In section 2 we define the height function and fix some notation. In sections 3 and 4 we prove some auxiliary lemmas. In section 5 we prove Proposition 5.5, a lower bound for the product of heights inspired by Theorem 1.6 of Amoroso and David's paper [1]. In section 6 we prove Theorems 1.2 and 1.4. In section 7 we then prove Theorems 1.1 and 1.5 and the three corollaries.

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2. Heights and further notation

We use notation which was also used in [15], it eases calculations in $\mathbf{G}_m^n$. Let $p = (p_1, \ldots, p_n) \in \mathbf{G}_m^n$ with p_i non-zero elements of some field K and $u = (\alpha_1, \ldots, \alpha_n)^{\mathrm{t}} \in \mathbf{Z}^n$ where ${}^{\mathrm{t}}$ means transpose, we set $p^u = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. If U is an $n \times m$ matrix with columns $u_1, \ldots, u_m \in \mathbf{Z}^n$ we set $p^U = (p^{u_1}, \ldots, p^{u_m})$. If q is also a vector with n non-zero entries in K we have $(pq)^U = p^U q^U$. If V is a matrix with m rows and integer coefficients, then $(p^U)^V = p^{UV}$. If $K = \mathbf{R}$ and all p_i are positive we will also allow exponent vectors or matrices with rational numbers as entries. We define the morphism of algebraic groups $\varphi_{(u_1,\ldots,u_m)} : \mathbf{G}_m^n \to \mathbf{G}_m^m$ by sending p to $(p^{u_1}, \ldots, p^{u_m})$.

For $u \in \mathbf{R}^n$ let $|u|$ be the euclidean norm of u. Furthermore, if $L \in \mathbf{R}[X_1, \ldots, X_n]$ is a linear form, then $|L|$ will denote the euclidean norm of the coefficient vector of L.

Let K be a number field. A place of K is an absolute value whose restriction to $\mathbf{Q}$ is a p-adic absolute value or the standard complex absolute value. If v is a place of K, then K_v denotes the completion of K with respect to v. By abuse of notation we also use the symbol v to denote the restriction of v to any subfield of K.

We now define the *absolute logarithmic Weil height*, or short the *height*, of $p \in \mathbf{G}_m^n$ as follows: let K be a number field which contains the coordinates p_i of p, we define

$$h(p) = \frac{1}{[K : \mathbf{Q}]} \sum_v [K_v : \mathbf{Q}_v] \log \max\{1, |p_1|_v, \ldots, |p_n|_v\}$$

where the sum runs over all places of K. It is well-known that this sum is well-defined and that $h(p)$ does not depend on the field K containing the p_i. This height function equals the one used in [15].

The height function satisfies several nice properties in connection with the group structure of $\mathbf{G}_m^n$. For example if $q \in \mathbf{G}_m^n$ one has $h(pq) \le h(p) + h(q)$, $h(p^k) = kh(p)$ if $k \ge 1$ is an integer, $h(p^{-1}) \le nh(p)$, and $h(p) \le h(p_1) + \cdots + h(p_n)$.

Let $\iota : \mathbf{G}_m^n \to \mathbf{P}^n$ denote the morphism which sends $(p_1, \ldots, p_n) \in \mathbf{G}_m^n$ to $[1 : p_1 : \cdots : p_n]$. Given a Zariski closed subset $X \subset \mathbf{G}_m^n$ we define its degree $\deg(X)$ as the degree of the Zariski closure Z of $\iota(X)$ in $\mathbf{P}^n$, i.e. the number of points in the intersection of Z with a linear subvariety of $\mathbf{P}^n$ of dimension $n - \dim X$ lying in general position. The degree of X only measures the irreducible components of X of maximal dimension. It will be useful to take the sum over the degrees of all irreducible components of X and call this number $\deg^0(X)$. If Y is another Zariski closed subset of $\mathbf{G}_m^n$, then Bézout's Theorem (example 8.4.6 [13]) says $\deg^0(X \cap Y) \leq \deg^0(X)\deg^0(Y)$.

Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety and let L be a subfield of $\overline{\mathbf{Q}}$. We say that X is defined over L if it is stable under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/L)$. Furthermore, for a field $R \subset \overline{\mathbf{Q}}$ we define

$$\deg_R(X) = \min\{[L : R]\deg(X);\ L \subset \overline{\mathbf{Q}} \text{ is a finite extension of } R \text{ and } X \text{ is defined over } L\}.$$

For the rest of the article we assume $R = \overline{\mathbf{Q}}$ or $R = \mathbf{Q}$. In the former case $\deg_R(X) = \deg(X)$.

As promised in the introduction we define anomalous and torsion-anomalous subvarieties of an irreducible closed subvariety $X \subset \mathbf{G}_m^n$. An irreducible closed subvariety $Y \subset X$ is called *anomalous* if $\dim Y \geq 1$ and if there exists a coset $H \subset \mathbf{G}_m^n$ containing Y such that $\dim Y \geq \dim X + \dim H - n + 1$. Moreover, we call Y *torsion-anomalous* if H is a torsion coset.

Throughout the paper and unless stated otherwise the symbols $c_1, c_2, \ldots$ denote positive constants which depend only on n.

3. Geometry of numbers

We recall two lemmas proved in [15] which will also be used in this article.

LEMMA 3.1. — *Let $1 \leq m \leq n$ and let $a \in \mathcal{H}_m$. There exist linear forms $L_1, \ldots, L_m \in \mathbf{R}[X_1, \ldots, X_n]$ such that $|L_j| \leq 1$ and*

$$h(a^u) \leq c_1 \max_{1 \leq j \leq m}\{|L_j(u)|\} h(a)$$

for all $u \in \mathbf{Z}^n$.

Proof. — This is Lemma 1 from [15]. □

The second lemma wraps up all the geometry of numbers we will use.

LEMMA 3.2. — *Let $1 \leq m \leq n$ and let $L_1, \ldots, L_m \in \mathbf{R}[X_1, \ldots, X_n]$ be linear forms with $|L_j| \leq 1$. If $\rho \geq 1$, there exist $\lambda_1, \ldots, \lambda_n$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and linearly independent $u_1, \ldots, u_n \in \mathbf{Z}^n$ such that*

$$|u_k| \leq \lambda_k, \quad |L_j(u_k)| \leq \rho^{-1}\lambda_k, \quad \textit{and} \quad \lambda_1 \cdots \lambda_n \leq c_2 \rho^m.$$

Proof. — This is Lemma 2 from [15]. □

We recall Dobrowolski's Theorem: if $\alpha \in \overline{\mathbf{Q}}\backslash\{0\}$ is not a root of unity and $D = [\mathbf{Q}(\alpha) : \mathbf{Q}]$, then

$$h(\alpha) \geq c_3 \frac{1}{D} \left(\frac{\log \log 3D}{\log 2D} \right)^3 \tag{2}$$

where $c_3 > 0$ is an absolute constant. For a proof see Theorem 1 [11].

The geometry of numbers machinery and Dobrowolski's Theorem give the following lemma which is the main ingredient in the proof of Corollary 1.7. It can be regarded as a Dobrowolski type result modulo algebraic subgroups.

Lemma 3.3. — *Let $1 \leq m \leq n$ be an integer, let $\delta > 0$, and let $B \geq 1$. If $p \in C(\mathcal{H}_m, \epsilon)$ with $h(p) \leq B$, $[\mathbf{Q}(p) : \mathbf{Q}] \leq D$, and*

$$\epsilon^{-1} \geq c_7 B^{m+1} D^{m+1+\delta}. \tag{3}$$

then $p \in \mathcal{H}_m$. Here $c_7 > 0$ depends only on n and δ.

Proof. — The symbols c_4, c_5, c_6 denote positive constants which depend only on n and δ. We may assume $c_7 \geq 2n$; we will see how to choose c_7 appropriately further down. We define $\rho \geq 1$ to be the right-hand side of (3).

Say $p = ab$ with $a \in \mathcal{H}_m$ and $h(b) \leq \epsilon(1 + h(a))$. By height properties described in section 2 we have $h(a) = h(pb^{-1}) \leq h(p) + h(b^{-1}) \leq h(p) + nh(b)$. So $h(a) \leq h(p) + n\epsilon(1 + h(a)) \leq h(p) + \frac{1}{2}(1 + h(a))$. We conclude that

$$h(a) \leq 1 + 2h(p) \quad \text{and} \quad h(b) \leq 2\epsilon(1 + h(p)). \tag{4}$$

By (4) we have $h(a) \leq 3B$ and $h(b) \leq 4\epsilon B$. Let $L_1, \ldots, L_m$ be the linear forms from Lemma 3.1 and let λ_k and u_k be from Lemma 3.2 applied to the L_j. We deduce

$$h(a^{u_k}) \leq c_1 \lambda_k \rho^{-1} h(a) \leq 3c_1 \lambda_k \rho^{-1} B. \tag{5}$$

Furthermore, by elementary height inequalities we have

$$h(b^{u_k}) \leq \sqrt{n} |u_k| h(b) \leq 4\sqrt{n} \lambda_k \epsilon B.$$

We combine this inequality with (5) and use $\epsilon \leq \rho^{-1}$ to get

$$h(p^{u_k}) \leq h(a^{u_k}) + h(b^{u_k}) \leq c_4 \lambda_k (\rho^{-1} + \epsilon) B \leq 2c_4 \lambda_k \rho^{-1} B. \tag{6}$$

Say $1 \leq k \leq n - m$. By Lemma 3.2 we have $1 \leq |u_k| \leq \lambda_k$ and

$$\lambda_k \leq (\lambda_{n-m} \cdots \lambda_n)^{\frac{1}{m+1}} \leq (\lambda_1 \cdots \lambda_n)^{\frac{1}{m+1}} \leq c_5 \rho^{\frac{m}{m+1}}.$$

We apply this inequality to (6) and use the definition of ρ to get

$$h(p^{u_k}) \leq c_6 \rho^{-\frac{1}{m+1}} B = c_6 c_7^{-\frac{1}{m+1}} D^{-1-\frac{\delta}{m+1}},$$

this inequality holds for all $1 \leq k \leq n - m$.

Now $[\mathbf{Q}(p^{u_k}) : \mathbf{Q}] \leq D$, so if c_7 is large enough with respect to c_6 and δ, Dobrowolski's Theorem implies that $p^{u_1}, \ldots, p^{u_{n-m}}$ are roots of unity. Since

$u_1, \ldots, u_{n-m}$ are linearly independent we conclude $p \in \mathcal{H}_m$ by Proposition 3.2.7 [4]. $\square$

Let D be an integer, by Northcott's Theorem there exists $\mu_D > 0$ such that if α is algebraic of degree at most D over $\mathbf{Q}$ then $h(x) = 0$ or $h(x) \geq \mu_D$. A variant of the previous lemma can be proved using only this statement instead of Dobrowolski's Theorem after replacing the right-hand side of (3) by a positive constant depending on B, μ_D, and n.

4. Push-forwards and pull-backs

In this section we prove two lemmas on bounds for degrees of push-forwards and pull-backs of varieties by a homomorphism of algebraic groups.

Unless stated otherwise, let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety throughout this section.

Lemma 4.1. — *Let $t \geq 1$ be an integer.*

(i) *Let $u_1, \ldots, u_t \in \mathbf{Z}^n \backslash \{0\}$ and let $0 < \lambda_1 \leq \cdots \leq \lambda_t$ with $|u_k| \leq \lambda_k$. If $\varphi = \varphi_{(u_1,\ldots,u_t)} : \mathbf{G}_m^n \to \mathbf{G}_m^t$ and $q = \dim \varphi(X)$, then*

$$\deg_R(\overline{\varphi(X)}) \leq c_8 \lambda_{t-q+1} \cdots \lambda_t \deg_R(X). \tag{7}$$

(ii) *If $\varphi : \mathbf{G}_m^n \to \mathbf{G}_m^t$ is the projection onto any set of t coordinates, then $\deg_R(\overline{\varphi(X)}) \leq \deg_R(X)$.*

(iii) *If $X \subset \mathbf{G}_m^n$ is Zariski closed and $R = \overline{\mathbf{Q}}$, then (7) holds with $\deg_R(\cdot)$ replaced by $\deg^0(\cdot)$ on both sides.*

Proof. — After permuting coordinates on $\mathbf{G}_m^t$ we may assume $|u_1| \leq \cdots \leq |u_t|$. For brevity we set $Y = \overline{\varphi(X)}$. There exists $U \subset Y$ Zariski open and dense with $U \subset \varphi(X)$ (theorem on page 219 [9]). For the moment we identify with $\overline{\mathbf{Q}}^{q(t+1)}$ the set of q-tuples of polynomials in $X_1, \ldots, X_t$, of degree at most 1, and with coefficients in $\overline{\mathbf{Q}}$. By a Bertini type argument there is a Zariski open and dense subset of $\overline{\mathbf{Q}}^{q(t+1)}$ such that the set of common zeros on Y of each tuple in this set is contained in U and has cardinality $\deg(Y)$. The determinant of the $q \times q$ matrix whose rows correspond to the coefficients of $X_{t-q+1}, \ldots, X_t$ of an element of $\overline{\mathbf{Q}}^{q(t+1)}$ does not vanish identically on this set. It follows that we can even find $(l_1, \ldots, l_q) \in \overline{\mathbf{Q}}^{q(t+1)}$ with $l_i - X_{t-q+i} \in \overline{\mathbf{Q}}[X_1, \ldots, X_{t-q}]$ such that

$$T = \{y \in Y; l_1(y) = \cdots = l_q(y) = 0\}$$

is contained in U and has cardinality $\deg(Y)$. We define $N = \varphi|_X^{-1}(T)$. Then $N \subset X$ is Zariski closed and has at least $\deg(Y)$ irreducible components. On the other hand we have $N = \{x \in X; l_1(\varphi(x)) = \cdots = l_q(\varphi(x)) = 0\}$. The exponent vectors in $l_i \circ \varphi$ have norm bounded by $|u_{t-q+i}|$. Hence Bézout's Theorem implies that the number of irreducible components of N is bounded above by $c_8|u_{t-q+1}|\cdots|u_t|\deg(X)$. Thus $\deg(Y) \leq c_8|u_{t-q+1}|\cdots|u_t|\deg(X)$.

Let $L \subset \overline{\mathbf{Q}}$ be a field over which X is defined, which contains R, and such that $\deg_R(X) = [L:R]\deg(X)$. Then Y is defined over L since the morphism φ is defined by monomials in integer coefficients. The proof of (i) follows.

A simple modification of this proof also leads to a proof of (ii). Part (iii) follows from (7) and on taking the sum over all irreducible components of X. □

then inequality (7)

Lemma 4.2. — *Let $u_1, \ldots, u_t \in \mathbf{Z}^n$ be linearly independent and $\varphi = \varphi_{(u_1,\ldots,u_t)} : \mathbf{G}_m^n \to \mathbf{G}_m^t$.*

(i) *If $Y \subset \mathbf{G}_m^t$ is an irreducible closed subvariety, then*

$$\deg^0(\varphi^{-1}(Y)) \leq c_9|u_1|\cdots|u_t|\deg(Y). \tag{8}$$

(ii) *If $t = n$, then φ is a proper morphism. Moreover, if W is an irreducible component of $\varphi^{-1}(Y)$ with Y as in (i), then* $\dim W = \dim Y$, *W maps surjectively onto Y, and* $\deg_R(W) \leq c_9|u_1|\cdots|u_n|\deg_R(Y)$.

(iii) *If $Y \subset \mathbf{G}_m^n$ is Zariski closed then (8) holds if* $\deg(\cdot)$ *is replaced by* $\deg^0(\cdot)$ *on the right-hand side.*

Proof. — We begin by proving the upper bound for $\deg^0(\varphi^{-1}(Y))$. By Bézout's Theorem the sum over the degrees of the irreducible components of

$$\{(q,p) \in \mathbf{G}_m^n \times \mathbf{G}_m^t; \varphi(q) = p\} \cap (\mathbf{G}_m^n \times Y) \tag{9}$$

is bounded by $c_9|u_1|\cdots|u_t|\deg(Y)$. Now $\varphi^{-1}(Y)$ is just the projection of the set above onto the first factor of $\mathbf{G}_m^n \times \mathbf{G}_m^t$. Part (i) follows from Lemma 4.1(ii) with $R = \overline{\mathbf{Q}}$ applied to the irreducible components of (9)

We prove part (ii), so let us assume $t = n$. We note that $\ker\varphi$, the kernel of φ, is finite and φ is surjective. The properness of φ follows from the Valuative Criterion of Properness (Theorem 4.7, page 101 [16]). In particular, φ is a closed map. Since φ is surjective and closed $\varphi(W_0) = Y$ for some irreducible component W_0 of $\varphi^{-1}(Y)$. By the Theorem on the Dimension of the Fibres (first theorem on page 228 [9]) and since $\varphi|_{W_0}$ has finite fibres we have $\dim W_0 = \dim Y$. Let W be a further irreducible component of $\varphi^{-1}(Y)$. If $w \in W$, then $\varphi(w) = \varphi(w_0)$ for some $w_0 \in W_0$, hence $w \in w_0 \ker\varphi$. We just showed that W equals the finite union

$$\bigcup_{\zeta \in \ker\varphi} W \cap W_0\zeta.$$

Since W is irreducible $W = W_0\zeta$ for some $\zeta \in \ker\varphi$. In particular $\dim W = \dim W_0 = \dim Y$ and $\varphi(W) = \varphi(W_0) = Y$.

To show (ii) it remains to prove the bound for $\deg_R(W)$. Let $L \subset \overline{\mathbf{Q}}$ be a field over which Y is defined, which contains R, and such that $\deg_R(Y) = [L : R]\deg(Y)$. If $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$, then W^σ is contained in $\varphi^{-1}(Y)$ since φ is defined by monomials with integer coefficients. Hence W^σ is an irreducible component of $\varphi^{-1}(Y)$ and in particular the orbit of W is finite. So the subgroup G of all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ which stabilize W is open and has index bounded by l, the number of irreducible components of $\varphi^{-1}(Y)$. Let F be the fixed field of G. Then W is defined over F and $[F : L] \leq l$. Therefore $\deg_R(W) \leq [F : L]\deg(W) = [F : L][L : R]\deg(W) \leq l[L : R]\deg(W)$. Now $l\deg(W) = \deg^0(\varphi^{-1}(Y))$ since each irreducible component of $\varphi^{-1}(Y)$ is the translate of W by a torsion point. This implies $\deg_R(W) \leq [L : R]\deg^0(\varphi^{-1}(Y))$. The proof of (ii) follows from (8).

Part (iii) follows from (8) and on taking the sum over all irreducible components of Y. □

5. A lower bound for the product of heights

In this section let X be an irreducible closed subvariety of $\mathbf{G}_m^n$ with $r = \dim X$.

The *essential minimum* of X is

$$\hat{\mu}^{\mathrm{ess}}(X) = \sup_{Z \subsetneq X} \inf\{h(p); p \in X \backslash Z\}, \tag{10}$$

where Z runs over all Zariski closed and proper subsets of X.

By Zhang's Theorem 6.2 [26] the essential minimum of X vanishes if and only if X is the translate of an algebraic subgroup of $\mathbf{G}_m^n$ by a torsion point. More recently, several authors have obtained lower bounds for the essential minimum if this value is positive. In general such lower bounds depend on $\deg(X)$, n, and a field of definition of X. But if we assume that X is not contained in a proper coset, then a Theorem of Amoroso and David implies that the essential minimum can be bounded below in terms of $\deg(X)$ and n only. Furthermore, the dependency of their lower bound in $\deg(X)$ is essentially optimal:

THEOREM 5.1 (Amoroso, David). — *Let $X \subsetneq \mathbf{G}_m^n$ be a proper irreducible closed subvariety of codimension k defined over $\overline{\mathbf{Q}}$. If X is not contained in a proper coset, then*

$$\hat{\mu}^{\mathrm{ess}}(X) \geq \frac{c_{10}}{(\deg(X))^{1/k}}(\log(3\deg(X)))^{-\lambda(k)} \tag{11}$$

with $\lambda(k) = (9(3k)^{k+1})^k$ and where $c_{10} > 0$ depends only on n.

Proof. — Amoroso and David's Theorem 1.4 [3] is formulated in a more precise way using the *obstruction index* instead of the degree. We will no define this quantity here. Inequality (11) is a direct consequence of Amoroso and David's Theorem together with Chardin's inequality (2) [3] which bounds the obstruction index from above in terms of $\deg(X)$. □

The "arithmetic" analogue to Theorem 5.1 is the following Theorem also by Amoroso and David:

THEOREM 5.2 (Amoroso, David). — *Let $X \subsetneq \mathbf{G}_m^n$ be a proper irreducible closed subvariety of codimension k defined over $\overline{\mathbf{Q}}$. If X is not contained in a proper torsion coset, then*

$$\hat{\mu}^{\mathrm{ess}}(X) \geq \frac{c_{11}}{(\deg_{\mathbf{Q}}(X))^{1/k}}(\log(3\deg_{\mathbf{Q}}(X)))^{-\kappa(n)}$$

where $\kappa(n)$ and $c_{11} > 0$ depend only on n.

Proof. — This is Corollary 1.3 [2]. □

In order to bound the cardinality as in Theorem 1.2(ii) we need to obtain quantitative statements on a Zariski closed subset of X containing points of small height. The following proposition follows from a Theorem of David and Philippon and a Theorem of Zhang.

PROPOSITION 5.3. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. If $R = \overline{\mathbf{Q}}$ we assume that X is not a coset. There exist a constant $0 < c_{12} < 1$ which depends as usual only on n and a Zariski closed proper subset $Z \subsetneq X$ which contains all points of X with height strictly less than $c_{12}\hat{\mu}^{\mathrm{ess}}(X)$. Furthermore, if $R = \overline{\mathbf{Q}}$ then we can choose Z such that*

$$\deg^0(Z) \leq c_{13}\deg(X)^5. \tag{12}$$

Proof. — If $R = \mathbf{Q}$ the proposition follows immediately from the definition of $\hat{\mu}^{\mathrm{ess}}(X)$ with $c_{12} = 1/2$.

Let us assume $R = \overline{\mathbf{Q}}$. The proof is an application of Proposition 5.4(i) [10]. For $p = (p_1, \ldots, p_n) \in \mathbf{G}_m^n$ we note that the height function used in Proposition 5.4 is $h(p_1) + \cdots + h(p_n)$ instead of $h(p)$. But this deviation is harmless since $h(p) \leq h(p_1) + \cdots + h(p_n) \leq nh(p)$. Let $\hat{\mu}^{\mathrm{ess}}(X)$ be the essential minimum of X taken with respect to the height function used in [10]. The normalized height of X occurring in David and Philippon's proposition is bounded from below by $\deg(X)\hat{\mu}^{\mathrm{ess}}(X)$ by a normalized version of Zhang's Theorem 5.2 [26]. □

The following proposition was inspired by Theorem 1.6 [1].

Proposition 5.4. — *If $R = \overline{\mathbf{Q}}$ we assume that X is not contained in a coset and if $R = \mathbf{Q}$ we assume that X is not contained in a torsion coset. There exists $Z \subsetneq X$ Zariski closed and proper such that for each $(p_1, \ldots, p_n) \in X \backslash Z$ there is a subset $\Sigma \subset \{1, \ldots, n\}$ with $|\Sigma| \geq n - r$ and*

$$\prod_{k \in \Sigma} h(p_k) \geq \frac{c_{30}}{\deg_R(X)(\log(3\deg_R(X)))^{c_{29}}} \tag{13}$$

where $c_{30} > 0$. Furthermore, if $R = \overline{\mathbf{Q}}$ then we may choose Z such that $\deg^0(Z) \leq c_{26}\deg(X)^{(n-r+2)(n-r+6)}$.

Proof. — We may assume $X \neq \mathbf{G}_m^n$, otherwise the statement of the proposition is empty. We define

$$V' = \{(p_1, \ldots, p_n) \in X; \max_{|\Sigma| \geq n-r} \prod_{k \in \Sigma} h(p_k) \leq 1\},$$

where Σ runs over the subsets of $\{1, \ldots, n\}$ of cardinality at least $n - r$.

The first statement in the proposition clearly holds for all points $p \in X \backslash V'$ since we may assume $c_{30} \leq 1$ and $c_{29} \geq 0$. Our proof of the statement around (13) follows the following strategy: for each $p \in V'$ we show that either inequality (13) is satisfied for some Σ as specified or that p is contained in the union of two Zariski closed and proper subsets Z_1 and Z_2 of X. In fact by permuting coordinates, it suffices to prove this statement with V' replaced by

$$V = \{(p_1, \ldots, p_n) \in V'; h(p_1) \leq \cdots \leq h(p_n)\}.$$

We start by constructing Z_1. Let $\pi : \mathbf{G}_m^n \to \mathbf{G}_m^{r+1}$ denote the projection onto the first $r + 1$ coordinates and let $W = \overline{\pi(X)} \subset \mathbf{G}_m^{r+1}$. Then $\deg_R(W) \leq \deg_R(X)$ by Lemma 4.1(ii). Also $W \neq \mathbf{G}_m^{r+1}$ by a dimension counting argument. Moreover if $R = \overline{\mathbf{Q}}$, then W is not contained in a proper coset since X is not and if $R = \mathbf{Q}$ then W is not contained in a proper torsion for a similar reason. In particular, we can apply Proposition 5.3 to W. By abuse of notation let c_{12} be the constant from this proposition. There exists a Zariski closed and proper $Y \subsetneq W$ as in Proposition 5.3 such that on $W \backslash Y$ the height is bounded below by $c_{12}\hat{\mu}^{\mathrm{ess}}(W)$. We define

$$Z_1 = \pi^{-1}(Y) \cap X,$$

so Z_1 is Zariski closed and $Z_1 \subsetneq X$.

Assume for the moment $R = \overline{\mathbf{Q}}$, we will bound $\deg^0(Z_1)$. By Lemma 4.2(iii) and inequality (12) we see $\deg^0(\pi^{-1}(Y)) \leq c_{14}\deg(W)^5 \leq c_{14}\deg(X)^5$. Bézout's Theorem now implies

$$\deg^0(Z_1) \leq c_{14}\deg(X)^6. \tag{14}$$

We begin the construction of Z_2. Let $p \in V \backslash Z_1$, then

(15)
$$(r+1)h(p_{r+1}) = (r+1)\max\{h(p_1),\ldots,h(p_{r+1})\} \geq h(\pi(p)) \geq c_{12}\hat{\mu}^{\mathrm{ess}}(W),$$

the first inequality follows from elementary properties of the height. Now $\operatorname{codim} W \geq 1$, thus in (15) we can bound $\hat{\mu}^{\mathrm{ess}}(W)$ from below using Theorem 5.1 if $R = \overline{\mathbf{Q}}$ and Theorem 5.2 if $R = \mathbf{Q}$. In both cases we obtain

(16)
$$h(p_{r+1}) \geq \frac{c_{15}}{\deg_R(W)(\log(3\deg_R(W)))^{c_{16}}} \geq \frac{c_{15}}{\deg_R(X)(\log(3\deg_R(X)))^{c_{16}}}.$$

In particular, $h(p_{r+1}) > 0$ and we may define

$$k_1 = \cdots = k_r = 1 \quad \text{and} \quad k_j = \left[\frac{h(p_j)}{h(p_{r+1})}\right] \geq 1 \quad \text{for} \quad r+1 \leq j \leq n. \tag{17}$$

With (16) and since $p \in V$ we can bound

$$k_1 \cdots k_n = k_{r+1} \cdots k_n \leq \frac{h(p_{r+1}) \cdots h(p_n)}{h(p_{r+1})^{n-r}} \leq c_{17}\deg_R(X)^{n-r}(\log(3\deg_R(X)))^{c_{18}}.$$

So, although the product $k_1 \cdots k_n$ depends on the point p, it can be bounded above independently of p. We define the finite set

(18)
$$\begin{aligned}\mathcal{K} = \{\varphi : \mathbf{G}_m^n \to \mathbf{G}_m^n;\ \varphi(q_1,\ldots,q_n) = (q_1^{k_1},\ldots,q_n^{k_n}),&\\ k_1 = \cdots = k_r = 1 \leq k_{r+1} \leq \cdots \leq k_n \text{ integers with}&\\ k_1 \cdots k_n \leq c_{17}\deg_R(X)^{n-r}(\log(3\deg_R(X)))^{c_{18}}\}.&\end{aligned}$$

Say $\varphi \in \mathcal{K}$ and let $k_1,\ldots,k_n$ be the associated exponents. We apply Lemma 4.2(ii) and choose an irreducible component W^φ of $\varphi^{-1}(X)$, then $\dim W^\varphi = r$, $\varphi(W^\varphi) = X$, and

$$\deg_R(W^\varphi) \leq c_{19}k_1 \cdots k_n \deg_R(X) = c_{19}k_{r+1} \cdots k_n \deg_R(X). \tag{19}$$

If $R = \overline{\mathbf{Q}}$ then W^φ is not contained in a proper coset since X itself is not and if $R = \mathbf{Q}$ then W^φ is not contained in a proper torsion coset for a similar reason. Furthermore we have $W^\varphi \neq \mathbf{G}_m^n$ because $X \neq \mathbf{G}_m^n$. We choose $Z^\varphi \subset W^\varphi$ as in Proposition 5.3, so Z^φ is Zariski closed and proper such that the height is bounded below by $c_{12}\hat{\mu}^{\mathrm{ess}}(W^\varphi)$ on $W^\varphi \backslash Z^\varphi$. If $R = \overline{\mathbf{Q}}$ we apply Theorem 5.1 to bound $\hat{\mu}^{\mathrm{ess}}(W^\varphi)$ from below, similarly if $R = \mathbf{Q}$ we apply Theorem 5.2 instead. In both cases (19) implies that

(20)
$$\begin{aligned}\text{if } q \in W^\varphi \backslash Z^\varphi \text{ then } h(q) &\geq c_{12}\hat{\mu}^{\mathrm{ess}}(W^\varphi)\\ &\geq \frac{c_{20}}{(k_{r+1} \cdots k_n \deg_R(X))^{\frac{1}{n-r}}(\log(3k_{r+1} \cdots k_n \deg_R(X)))^{c_{21}}}.\end{aligned}$$

We define

$$Z_2 = \bigcup_{\varphi\in\mathcal{K}} \varphi(Z^\varphi). \tag{21}$$

Then Z_2 is Zariski closed in X since φ is proper by Lemma 4.2(ii) and since $\mathcal{K}$ is finite. Furthermore, $Z_2 \neq X$ by construction.

Assume for the moment $R = \overline{\mathbf{Q}}$, then bounding $\deg^0(Z_2)$ is not difficult. Indeed if $\varphi \in \mathcal{K}$ then by Proposition 5.3 we have $\deg^0(Z^\varphi) \leq c_{13}\deg(W^\varphi)^5$. The bound (19) and the definition (18) give $\deg^0(Z^\varphi) \leq c_{22}\deg(X)^{5(n-r+2)}$. Lemma 4.1(iii) and again (18) imply $\deg^0(\varphi(Z^\varphi)) \leq c_{23}\deg(X)^{6(n-r+2)}$. Hence $\deg^0(Z_2) \leq c_{23}|\mathcal{K}|\deg(X)^{6(n-r+2)}$, so it remains to bound the cardinality of $\mathcal{K}$. From (18) we deduce the rather crude bound $c_{24}\deg(X)^{(n-r)(n-r+1)}$ for $|\mathcal{K}|$, hence

$$\deg^0(Z_2) \leq c_{25}\deg(X)^{(n-r+6)(n-r+2)}. \tag{22}$$

Now let $Z = Z_1 \cup Z_2$ and $p \in V\backslash Z$. We will show that (13) holds with $\Sigma = \{r+1,\ldots,n\}$. This completes the proof, indeed if $R = \overline{\mathbf{Q}}$ the degree bound for Z holds with $c_{26} = c_{14} + c_{25}$ because of (14) and (22). Since $p \in V\backslash Z_1$ there exists $\varphi \in \mathcal{K}$ with associated $k_1,\ldots,k_n$ as in (17). Let us pick $q = (q_1,\ldots,q_n) \in W^\varphi$ with $\varphi(q) = p$. Then $h(p_j) = k_j h(q_j)$, so

$$h(p_{r+1})\cdots h(p_n) = k_{r+1}\cdots k_n h(q_{r+1})\cdots h(q_n).$$

If $r+1 \leq j \leq n$, then by (17) all $h(q_j)$ are essentially of the same size. More precisely,

$$\frac{1}{2}\frac{h(p_j)}{h(p_{r+1})} \leq k_j \leq \frac{h(p_j)}{h(p_{r+1})}, \quad \text{so} \quad h(p_{r+1}) \leq h(q_j) \leq 2h(p_{r+1}). \tag{23}$$

Hence

$$h(p_{r+1})\cdots h(p_n) \geq k_{r+1}\cdots k_n h(p_{r+1})^{n-r}. \tag{24}$$

Let $q_{j'}$ be a coordinate of q with maximal height, then the height function properties discussed in section 2 imply $h(q) \leq nh(q_{j'})$. Now $h(q_{r+1}) = h(p_{r+1}) \geq h(p_j) = h(q_j)$ for $1 \leq j \leq r$, hence we may assume $j' \geq r+1$. We insert the upper bound for $h(q_j)$ from (23) with $j = j'$ into (24) and use $h(q_{j'}) \geq \frac{1}{n}h(q)$ to derive

$$h(p_{r+1})\cdots h(p_n) \geq 2^{-(n-r)}k_{r+1}\cdots k_n h(q_{j'})^{n-r} \geq c_{27}k_{r+1}\cdots k_n h(q)^{n-r}. \tag{25}$$

Finally, we use the fact that $q \notin Z^{\varphi}$ which follows from $p \notin Z_2$ and (21). So we may apply (20) to (25) and conclude

$$h(p_{r+1})\cdots h(p_n) \geq \frac{c_{28} k_{r+1}\cdots k_n}{k_{r+1}\cdots k_n \deg_R(X)(\log(3k_{r+1}\cdots k_n \deg_R(X)))^{c_{29}}}$$
$$\geq \frac{c_{28}}{\deg_R(X)(\log(3k_{r+1}\cdots k_n \deg_R(X)))^{c_{29}}}.$$

Up to the the product $k_{r+1}\cdots k_n$ in the logarithm, this is already (13) with $\Sigma = \{r+1,\ldots,n\}$. But this remaining product is harmless since it equals $k_1\cdots k_n$ and therefore is bounded polynomially in $\deg_R(X)$ by (18). □

We define

$$\mathrm{s}^{\mathrm{o}}(X) = \min\{\dim H; H \subset \mathbf{G}_m^n \text{ a coset with } X \subset H\} \text{ and}$$
$$\mathrm{s}^{*}(X) = \min\{\dim H; H \subset \mathbf{G}_m^n \text{ a torsion coset with } X \subset H\}.$$

Proposition 5.4 only holds if $\mathrm{s}^{\mathrm{o}}(X) = n$, $\mathrm{s}^{*}(X) = n$ for $R = \overline{\mathbf{Q}}$, $R = \mathbf{Q}$ respectively. A simple projection argument shows that in general we have:

PROPOSITION 5.5. — *If $R = \overline{\mathbf{Q}}$ then let $s = \mathrm{s}^{\mathrm{o}}(X)$ and if $R = \mathbf{Q}$ then let $s = \mathrm{s}^{*}(X)$. There exists $Z \subsetneq X$ Zariski closed and proper such that for each $(p_1,\ldots,p_n) \in X\backslash Z$ there is a subset $\Sigma \subset \{1,\ldots,n\}$ with $|\Sigma| \geq s-r$ and*

$$\prod_{k\in\Sigma} h(p_k) \geq \frac{c_{34}}{\deg_R(X)(\log(3\deg_R(X)))^{c_{35}}} \tag{26}$$

where $c_{34} > 0$. Furthermore, if $R = \overline{\mathbf{Q}}$ then

$$\deg^0(Z) \leq c_{33}\deg(X)^{(s+3)(s+6)}. \tag{27}$$

Proof. — We prove the proposition if $R = \overline{\mathbf{Q}}$, the case $R = \mathbf{Q}$ is similar.

Let H be a coset with $X \subset H$ and $s = \dim H = n-h$. By Proposition 3.2.7 and Corollary 3.2.15 [4] there are linearly independent $u_1,\ldots,u_h \in \mathbf{Z}^n$ such that x^{u_i} is constant on H. After possibly permuting coordinates we may assume that the $h\times h$ matrix whose ith column consists of the first h entries of u_i is non-singular. In this case the projection $\pi : \mathbf{G}_m^n \to \mathbf{G}_m^{n-h}$ onto the last $n-h$ coordinates has finite fibres when restricted to H. Therefore $\pi|_X$ has finite fibres too. By the Theorem on the Dimension of the Fibres we have $\dim W = r$ with $W = \overline{\pi(X)} \subset \mathbf{G}_m^s$. Lemma 4.1(ii) implies $\deg(W) \leq \deg(X)$. Furthermore, W is not contained in a proper coset, indeed otherwise X would be contained in a coset of dimension strictly less than $n-h$. We apply Proposition 5.4 to W and obtain $Z' \subsetneq W$ Zariski closed and proper with $\deg^0(Z') \leq c_{31}\deg(W)^{(s+2)(s+6)} \leq c_{31}\deg(X)^{(s+2)(s+6)}$.

Say $p \in X$, inequality (26) now holds if $p \notin Z = \pi^{-1}(Z')\cap X$ for some $\Sigma \subset \{h+1,\ldots,n\}$ with $|\Sigma| \geq s - \dim W \geq s-r$. By Lemma 4.2(iii) we have

$\deg^0(\pi^{-1}(Z')) \leq c_{32}\deg^0(Z') \leq c_{33}\deg(X)^{(s+2)(s+6)}$; by Bézout's Theorem we obtain

$$\deg^0(Z) \leq c_{33}\deg(X)^{(s+2)(s+6)+1} \leq c_{33}\deg(X)^{(s+3)(s+6)}. \qquad \square$$

If for example $R = \overline{\mathbf{Q}}$ and X is itself a coset, then $\mathrm{s}^{\mathrm{o}}(X) = r$; in this case Proposition 5.5 is an empty statement.

6. Proof of Theorems 1.2 and 1.4

In this section, if not stated otherwise, $X \subset \mathbf{G}_m^n$ is an irreducible closed subvariety of dimension r with $1 \leq r \leq n-1$.

We start off with an auxiliary lemma related to linear programming. We recall that $\mathfrak{m}(r,s)$ was defined in (1).

Lemma 6.1. — *Let s be an integer and $1 \leq r < s \leq n$, let $M = (m_{ij})$ be the $n \times (s-r)$ matrix defined by*

$$m_{ij} = \begin{cases} 1 \text{ if } & i+j \leq n-r+1, \\ 2 \text{ if } & n-r+2 \leq i+j \leq n+1, \\ 0 \text{ else,} & \end{cases}$$

and let $w = (w_1, \ldots, w_{s-r})^{\mathrm{t}} \in \mathbf{R}^{s-r}$ be the column vector with

$$w_j = \begin{cases} 2^{-(\frac{j-1}{r}+1)} \text{ if } & r|(j-1), \\ 0 & \text{else.} \end{cases}$$

Then $w_1 + \cdots + w_{s-r} < 1$. Moreover, $v = (v_1, \ldots, v_n)^{\mathrm{t}} = Mw$ satisfies $v_i \leq 1$ and

$$\sum_{j=1}^{s-r} (s-r-j+1)w_j = \mathfrak{m}(r,s). \tag{28}$$

Proof. — The vector w looks like

$$w = (\frac{1}{2}, 0, \ldots, 0, \frac{1}{4}, 0, \ldots, 0, \frac{1}{8}, 0, \ldots)^{\mathrm{t}} \tag{29}$$

with $r-1$ zeros between consecutive negative powers of 2 (there are no zeros if $r=1$). The inequality $w_1 + \cdots + w_{s-r} < 1$ is immediate. When $1 \leq i \leq n$, the ith row of M starts off with a certain number (possibly zero) of consecutive ones followed by say N consecutive twos and finally consecutive zeros. By definition we have $N \leq r$, hence by (29) there is at most one j with $m_{ij} = 2$ and $w_j \neq 0$. Let N' be the number of j with $m_{ij} = 1$ and $w_j \neq 0$, then

$$v_i = \sum_{j, m_{ij}=1} w_j + 2\sum_{j, m_{ij}=2} w_j \leq (\frac{1}{2} + \cdots + \frac{1}{2^{N'}}) + 2\frac{1}{2^{N'+1}} = 1.$$

So $v_i \leq 1$, as desired. The final assertion, equality (28), follows from an elementary calculation. $\square$

We need two simple lemmas on $\mathfrak{m}(r, n)$.

LEMMA 6.2. — *Let n, n', r, r' be integers with $1 \leq r \leq n-1$, $1 \leq r' \leq n'-1$, $r' \leq r$, and $n - r \leq n' - r'$, then $\mathfrak{m}(r, n) \leq \mathfrak{m}(r', n')$. Moreover, $\mathfrak{m}(r, n) \geq (n-r)/2$.*

Proof. — From (28) and taking $j = kr + 1$ we get

$$\mathfrak{m}(r, n) = \sum_{k=0}^{\infty} \max\{0, \frac{n-(k+1)r}{2^{k+1}}\}. \tag{30}$$

By omitting all terms with $k > 0$ we obtain $\mathfrak{m}(r, n) \geq (n-r)/2$, the second assertion of the lemma. Moreover, as

$$n - (k+1)r = (n-r) - kr \leq (n'-r') - kr' = n' - (k+1)r'$$

for non-negative k, the first assertion follows at once. $\square$

For example, Lemma 6.2 applied with $r' = 1$ and $n' = n$ gives

$$\mathfrak{m}(r, n) \leq \mathfrak{m}(1, n) = n - 2 + 2^{-(n-1)} \leq n - 1. \tag{31}$$

LEMMA 6.3. — *Let m be an integer with $\mathfrak{m}(r, n) > m$, then $\mathfrak{m}(r, n) - m \geq 2^{-(n-1)}$.*

Proof. — By the definition (1) of $\mathfrak{m}(r, n)$, the difference $\mathfrak{m}(r, n) - m$ is a rational with denominator bounded by $2^{(n-1)/r} \leq 2^{n-1}$. The proof follows since this difference is positive by hypothesis. $\square$

Recall that $R = \overline{\mathbf{Q}}$ or $R = \mathbf{Q}$. The following proposition will imply part (i) of Theorems 1.2 and 1.4.

PROPOSITION 6.4. — *If $R = \overline{\mathbf{Q}}$ we set $s = \mathrm{s}^{\mathrm{o}}(X)$ and if $R = \mathbf{Q}$ we set $s = \mathrm{s}^{*}(X)$. Assume $s \geq r + 1$, let m be an integer with $0 \leq m < \mathfrak{m}(r, s)$, and let $B \geq 1$. There exist $c_{51} > 0$, c_{42}, and a Zariski closed and proper subset $Z \subsetneq X$ such that if*

$$0 \leq \epsilon \leq c_{51}(B^{m+1}\deg_R(X)^2)^{-\frac{1}{\mathfrak{m}(r,s)-m}} \tag{32}$$

then $\{p \in X \cap T(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset Z$. Furthermore, if $R = \overline{\mathbf{Q}}$, then

$$\deg^0(Z) \leq c_{42}(B\deg(X))^{(n+6)^4 2^n}. \tag{33}$$

We will prove Proposition 6.4 further down and start introducing some notation. Until the end of the proof of this proposition we assume X, R, s, m, and B are as in the hypothesis. We consider $Q = c_{51}^{-1} \geq n$ as fixed and depending only on n; we will see how to choose it later on. We also define $\rho \in \mathbf{R}$ such that $B\rho^{-1}$ is equal to the right-hand side of (32), i.e.

$$\rho = Q(B^{\mathfrak{m}(r,s)+1}\deg_R(X)^2)^{\frac{1}{\mathfrak{m}(r,s)-m}} \geq 1. \tag{34}$$

The following upper bound for ρ will be useful later on:

Lemma 6.5. — *We have $\rho \leq QB^{n2^{n-1}}\deg_R(X)^{2^n}$.*

Proof. — By Lemma 6.3 we may bound $(\mathfrak{m}(r,s) - m)^{-1} \leq 2^{s-1}$, furthermore inequality (31) gives $\mathfrak{m}(r,s) \leq s-1$. Hence, $\rho \leq Q(B^s\deg(X)^2)^{2^{s-1}}$ and the proof follows since $s \leq n$. □

Recall that c_2 is the constant from Lemma 3.2. We set

$$t_0 = n - s + r + 1 \geq r + 1$$

and define the finite set

$$\begin{aligned} \Phi = \{\varphi_{(u_1,\ldots,u_t)} : \mathbf{G}_m^n \to \mathbf{G}_m^t;\ & t_0 \leq t \leq n, \\ & u_1,\ldots,u_t \in \mathbf{Z}^n \text{ linearly independent}, \\ & |u_1|\cdots|u_t| \leq c_2\rho^m\}. \end{aligned} \tag{35}$$

All elements of Φ are surjective homomorphisms of algebraic groups.

The next lemma controls the push-forward of X by an element of Φ.

Lemma 6.6. — *Let $t_0 \leq t \leq n$, $\varphi = \varphi_{(u_1,\ldots,u_t)} \in \Phi$, and $X_\varphi = \overline{\varphi(X)} \subset \mathbf{G}_m^t$. Then $\dim X_\varphi \leq r \leq t-1$. If $R = \overline{\mathbf{Q}}$ then $\mathrm{s}^{\mathrm{o}}(X_\varphi) \geq s+t-n \geq r+1$ and if $R = \mathbf{Q}$ then $\mathrm{s}^*(X_\varphi) \geq s+t-n \geq r+1$.*

Proof. — We restrict ourselves to the case $R = \overline{\mathbf{Q}}$, the case $R = \mathbf{Q}$ is similar.

Certainly, $\dim X_\varphi \leq \dim X = r \leq t_0 - 1 \leq t-1$ since $\varphi|_X : X \to X_\varphi$ is dominant.

We continue by bounding $\mathrm{s}^{\mathrm{o}}(X_\varphi)$ from below. To do this let $H \subset \mathbf{G}_m^t$ be a coset of dimension $\mathrm{s}^{\mathrm{o}}(X_\varphi)$ that contains X_φ. Then $\dim\varphi^{-1}(H) = \dim H + (n-t)$ and $\varphi^{-1}(H)$ is a coset containing X. Hence

$$\mathrm{s}^{\mathrm{o}}(X_\varphi) = \dim\varphi^{-1}(H) - (n-t) \geq s+t-n \geq s+t_0-n = r+1. \qquad \square$$

Let $\varphi \in \Phi$ and $X_\varphi = \overline{\varphi(X)} \subset \mathbf{G}_m^t$ for some $t_0 \le t \le n$. In view of the previous lemma we apply Proposition 5.5 to X_φ and obtain a Zariski closed and proper $Z_\varphi \subsetneq X_\varphi$ satisfying the stated properties. We define

$$Z = \bigcup_{\varphi \in \Phi} \varphi^{-1}(Z_\varphi) \cap X. \tag{36}$$

This will be the set in the assertion of Proposition 6.4. It is Zariski closed since Φ is a finite set and it is also proper since each $\varphi^{-1}(Z_\varphi)$ does not contain X. Indeed, $\varphi|_X : X \to X_\varphi$ is a dominant map. In the next lemma we bound $\deg^0(Z)$:

Lemma 6.7. — *If $R = \overline{\mathbf{Q}}$ we have* $\deg^0(Z) \le c_{41} Q^{2m(n+6)^4} (B\deg(X))^{(n+6)^4 2^n}$.

Proof. — Let $\varphi = \varphi_{(u_1,\ldots,u_t)} \in \Phi$. Proposition 5.5 implies $\deg^0(Z_\varphi) \le c_{35} \deg(X_\varphi)^{(t+3)(t+6)}$. After possibly permuting coordinates Lemma 4.1(i) and (35) let us bound

$$\deg(X_\varphi) \le c_8 |u_1| \cdots |u_t| \deg(X) \le c_{36} \rho^m \deg(X),$$

hence

$$\deg^0(Z_\varphi) \le c_{37} \rho^{m(t+3)(t+6)} \deg(X)^{(t+3)(t+6)}.$$

Lemma 4.2(iii) and again (35) imply

$$\deg^0(\varphi^{-1}(Z_\varphi)) \le c_{38} |u_1| \cdots |u_t| \deg^0(Z_\varphi) \le c_{39} \rho^{m(t+3)(t+6)+m} \deg(X)^{(t+3)(t+6)},$$

while Bézout's Theorem gives

$$\begin{aligned}\deg^0(\varphi^{-1}(Z_\varphi) \cap X) &\le c_{39} \rho^{m(t+3)(t+6)+m} \deg(X)^{(t+3)(t+6)+1} \\ &\le c_{39} \rho^{m(n+6)^2} \deg(X)^{(n+6)^2},\end{aligned}$$

the last inequality uses $t \le n$. So, in order to bound $\deg^0(Z)$ it remains to control the cardinality of Φ. A crude estimate which follows from the definition (35) is $|\Phi| \le c_{40} \rho^{mn^2}$. We obtain

$$\deg^0(Z) \le c_{41} \rho^{2m(n+6)^2} \deg(X)^{(n+6)^2}.$$

We now apply Lemma 6.5 to show that this last bound for $\deg^0(Z)$ implies (33) with $c_{42} = c_{41} Q^{2m(n+6)^2}$. Indeed, we have

$$\deg^0(Z) \le c_{41} Q^{2m(n+6)^2} B^{mn(n+6)^2 2^n} \deg(X)^{m(n+6)^2 2^{n+1} + (n+6)^2}.$$

Recall that by (31) (with s instead of n) we have $m < \mathfrak{m}(r,s) \le s-1 \le n-1$, so

$$\begin{aligned}\deg^0(Z) &\le c_{41} Q^{2m(n+6)^2} B^{n^2(n+6)^2 2^n} \deg(X)^{n(n+6)^2 2^{n+1}} \\ &\le c_{41} Q^{2m(n+6)^2} (B\deg(X))^{(n+6)^4 2^n}.\end{aligned}$$

□

We now prove Proposition 6.4. Say $p \in X \cap T(\mathcal{H}_m, \epsilon)$ and $h(p) \leq B$, we will show that $p \in Z$.

By definition we may write $p = ab$ with $a \in \mathcal{H}_m$ and $h(b) \leq \epsilon$. Elementary height properties and $\epsilon \leq Q^{-1} \leq 1/n$ imply $h(a) \leq h(p) + h(b^{-1}) \leq h(p) + nh(b) \leq h(p) + n\epsilon \leq 2B$. Let us assume for the moment that $m \geq 1$. By Lemmas 3.1 and 3.2 there exist $\lambda_1, \ldots, \lambda_n$ with $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and linearly independent $u_1, \ldots, u_n \in \mathbf{Z}^n$ such that for $1 \leq k \leq n$

(37)
$$|u_k| \leq \lambda_k, \quad h(a^{u_k}) \leq c_1 h(a)\rho^{-1}\lambda_k \leq 2c_1 B\rho^{-1}\lambda_k, \text{ and } \lambda_1 \cdots \lambda_n \leq c_2\rho^m.$$

Here c_1 is the constant from Lemma 3.1 and we may assume $c_2 \geq 1$. In the case $m = 0$ the statements in (37) also hold if we take $\lambda_k = 1$ and u_k the standard basis elements of $\mathbf{R}^n$. Indeed if $m = 0$, then a is a torsion point and thus has height 0.

Elementary height inequalities give $h(b^{u_k}) \leq \sqrt{n}h(b)|u_k| \leq \sqrt{n}\epsilon\lambda_k$, hence

$$h(p^{u_k}) \leq h(a^{u_k}) + h(b^{u_k}) \leq c_{43}(B\rho^{-1} + \epsilon)\lambda_k \leq 2c_{43}B\rho^{-1}\lambda_k, \tag{38}$$

here we used the bound $\epsilon \leq B\rho^{-1}$ in the last inequality.

For $t_0 \leq t \leq n$ we set $\varphi_t = \varphi_{(u_1,\ldots,u_t)} : \mathbf{G}_m^n \to \mathbf{G}_m^t$ and for brevity let X_t denote $X_{\varphi_t} = \overline{\varphi_t(X)}$. It is important to note that φ_t and so also X_t depend on p. On the other hand, by (37) φ_t is contained in the finite set Φ which is independent of p.

Let $c_{34} > 0$ and c_{35} be the constants from Proposition 5.5 applied to $X_t \subset \mathbf{G}_m^t$; they actually depend on t but since $t_0 \leq t \leq n$ we may assume that they only depend on n. Let us assume for the moment that there exists an integer t with $t_0 \leq t \leq n$ such that for all $\Sigma \subset \{1, \ldots, t\}$ with $|\Sigma| \geq \mathrm{s}^{\mathrm{o}}(X_t) - \dim X_t$ (respectively $|\Sigma| \geq \mathrm{s}^*(X_t) - \dim X_t$ if $R = \mathbf{Q}$) we have the inequality

$$c_{34}^{-1}\left(\prod_{k\in\Sigma} h(p^{u_k})\right)\deg_R(X_t)(\log(3\deg_R(X_t)))^{c_{35}} < 1. \tag{39}$$

The product in (39) is actually a product over heights of certain coordinates of the point $\varphi_t(p) \in X_t$. From (39) and Proposition 5.5 we conclude that $\varphi_t(p)$ is contained in Z_{φ_t}, a set chosen above (36). In particular, if (39) holds, then $p \in Z$. The proposition follows in this case since Z is a Zariski closed and proper subset of X. In the case $R = \overline{\mathbf{Q}}$ the degree bound for Z follows from Lemma 6.7.

What if the statement around (39) does not hold? Then we will deduce a contradiction and this will complete the proof of the proposition.

Let $t_0 \leq t \leq n$ and let $\Sigma \subset \{1, \ldots, t\}$, we define $f_{n-t+1}(\Sigma) \in \mathbf{R}$ to be the expression on the left-hand side of (39). We are assuming that for all

$t_0 \leq t \leq n$ there exists a subset $\Sigma(t) \subset \{1, \ldots, t\}$ with $|\Sigma(t)| \geq \mathrm{s}^{\mathrm{o}}(X_t) - \dim X_t$ (respectively $|\Sigma(t)| \geq \mathrm{s}^*(X_t) - \dim X_t$ if $R = \mathbf{Q}$) and

$$f_{n-t+1}(\Sigma(t)) \geq 1. \tag{40}$$

For brevity we set $f_{n-t+1} = f_{n-t+1}(\Sigma(t))$.

We proceed by bounding f_{n-t+1} from above. To do this we apply (38) to the definition of f_{n-t+1} and get

$$f_{n-t+1} \leq c_{34}^{-1} \left(\prod_{k \in \Sigma(t)} (c_{44} B \rho^{-1} \lambda_k) \right) \deg_R(X_t) (\log(3 \deg_R(X_t)))^{c_{35}}.$$

Next we bound $\deg_R(X_t)$ from above using Lemma 4.1(i) and $\lambda_{t-r+1} \cdots \lambda_t \leq \lambda_t^r$ to see that

$$f_{n-t+1} \leq c_{45} \left(\prod_{k \in \Sigma(t)} \lambda_k \right) \lambda_{t-r+1} \cdots \lambda_t (B\rho^{-1})^{|\Sigma(t)|} \deg_R(X) (\log(3\lambda_t \deg_R(X)))^{c_{35}}.$$

By Lemma 6.6 we have $|\Sigma(t)| \geq \mathrm{s}^{\mathrm{o}}(X_t) - \dim X_t \geq s + t - n - r \geq 1$ if $R = \overline{\mathbf{Q}}$ and $|\Sigma(t)| \geq \mathrm{s}^*(X_t) - \dim X_t \geq s + t - n - r \geq 1$ if $R = \mathbf{Q}$; here we used $\dim X_t \leq r$. This bound for $|\Sigma(t)|$ and $\rho \geq B$ lead to

$$f_{n-t+1} \leq c_{45} \left(\prod_{k \in \Sigma(t)} \lambda_k \right) \lambda_{t-r+1} \cdots \lambda_t (B\rho^{-1})^{s+t-n-r} \deg_R(X) (\log(3\lambda_t \deg_R(X)))^{c_{35}}.$$

Because $\lambda_k \geq 1$ we may replace the product over $\Sigma(t)$ by the product over $\{1, \ldots, t\}$:

$$\begin{aligned} f_{n-t+1} \leq & \\ & c_{45} \lambda_1 \cdots \lambda_{t-r} (\lambda_{t-r+1} \cdots \lambda_t)^2 (B\rho^{-1})^{s+t-n-r} \deg_R(X) (\log(3\lambda_t \deg_R(X)))^{c_{35}}. \end{aligned} \tag{41}$$

This inequality holds for all $t_0 \leq t \leq n$.

Let M, v, w be the matrix respectively vectors from Lemma 6.1. Using notation introduced in section 2 we define

$$\begin{aligned} \Lambda &= (\lambda_1, \ldots, \lambda_n)^M \\ &= (\lambda_1 \cdots \lambda_{n-r} (\lambda_{n-r+1} \cdots \lambda_n)^2, \ldots, \lambda_1 \cdots \lambda_{n-s+1} (\lambda_{n-s+2} \cdots \lambda_{n+r-s+1})^2). \end{aligned}$$

That is, the jth entry of Λ is the main contribution of the λ_k's to the bound for f_j in (41). By Lemma 6.1 and $\lambda_k \geq 1$ we have

$$\Lambda^w = (\lambda_1, \ldots, \lambda_n)^{Mw} = \lambda_1^{v_1} \cdots \lambda_n^{v_n} \leq \lambda_1 \cdots \lambda_n. \tag{42}$$

We define the product

$$f = (f_1, \ldots, f_{s-r})^w = f_1^{w_1} \cdots f_{s-r}^{w_{s-r}}. \tag{43}$$

By (40) and since $w_j \geq 0$ we conclude

$$f \geq 1. \tag{44}$$

We bound f from above using the definition (43) together with the help of (41) and (42)

$$\begin{aligned} f &\leq c_{46}\Lambda^w (B\rho^{-1})^{\sum_{j=1}^{s-r}(s-r-j+1)w_j} (\deg_R(X)(\log(3\lambda_n \deg_R(X)))^{c_{35}})^{w_1+\cdots+w_{s-r}} \\ &\leq c_{46}\lambda_1\cdots\lambda_n (B\rho^{-1})^{\sum_{j=1}^{s-r}(s-r-j+1)w_j} \deg_R(X)(\log(3\lambda_n\deg_R(X)))^{c_{35}}, \end{aligned}$$

here we also used $w_1+\cdots+w_{s-r} < 1$ from Lemma 6.1. By the same lemma the exponent of $B\rho^{-1}$ equals $\mathfrak{m}(r,s)$. We recall (37) and use $\lambda_n \leq \lambda_1\cdots\lambda_n \leq c_2\rho^m$ to bound

$$f \leq c_{47}\rho^{m-\mathfrak{m}(r,s)} B^{\mathfrak{m}(r,s)} \deg_R(X)(\log(3\rho\deg_R(X)))^{c_{35}}. \tag{45}$$

By the choice of ρ made in (34) we have

$$\rho^{m-\mathfrak{m}(r,s)} B^{\mathfrak{m}(r,s)+1}\deg_R(X)^2 = Q^{-(\mathfrak{m}(r,s)-m)}$$

This equality and (45) imply

$$f \leq c_{47} Q^{-(\mathfrak{m}(r,s)-m)} \frac{(\log(3\rho\deg_R(X)))^{c_{35}}}{B\deg_R(X)}.$$

By Lemma 6.5 the ρ in the logarithm is bounded above by $QB^{n2^{n-1}}\deg_R(X)^{2^n}$. Together with elementary inequalities we have

$$\begin{aligned} f &\leq c_{48} Q^{-(\mathfrak{m}(r,s)-m)} \frac{(\log(3QB\deg_R(X)))^{c_{35}}}{B\deg_R(X)} \\ &\leq c_{49} Q^{-(\mathfrak{m}(r,s)-m)} \frac{\max\{\log Q, \log(3B\deg_R(X))\}^{c_{35}}}{B\deg_R(X)} \\ &\leq c_{50} Q^{-(\mathfrak{m}(r,s)-m)} \frac{\max\{(\log Q)^{c_{35}}, B\deg_R(X)\}}{B\deg_R(X)} \\ &\leq c_{50} Q^{-(\mathfrak{m}(r,s)-m)} \max\{(\log Q)^{c_{35}}, 1\}. \end{aligned} \tag{46}$$

Because $\mathfrak{m}(r,s) - m \geq 2^{-(s-1)} \geq 2^{-(n-1)}$ by Lemma 6.3 we may choose $Q \geq n$ depending only on n such that (46) implies $f < 1$. But this contradicts (44).□

For $B \geq 1$ we have the following inclusions

$$\begin{aligned} &\{p \in T(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset \\ &\{p \in C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset \{p \in T(\mathcal{H}_m, 4\epsilon B); h(p) \leq B\}, \end{aligned} \tag{47}$$

if $\epsilon \leq \frac{1}{2n}$. The first inclusion is trivial and holds for unrestricted ϵ, the second one follows easily using arguments around (4). Therefore Proposition 6.4 can be reformulated with $T(\cdot,\cdot)$ replaced by $C(\cdot,\cdot)$ and after choosing a possibly smaller ϵ.

Proposition 6.8. — *Let $R, s, m,$ and B be as in Proposition 6.4 and let us assume $s \geq r+1$. There exist $c_{52} > 0$ and a Zariski closed and proper subset $Z \subsetneq X$ such that if*

$$0 \leq \epsilon \leq c_{52}(B^{\mathfrak{m}(r,s)+1}\deg_R(X)^2)^{-\frac{1}{\mathfrak{m}(r,s)-m}} \tag{48}$$

then $\{p \in X \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset Z$. Furthermore, if $R = \overline{\mathbf{Q}}$ then (33) holds.

Proof. — The proof follows immediately from the inclusion (47) and Proposition 6.4. □

In the notation of the previous proposition, we have $\mathfrak{m}(r,s) \leq s-1 \leq n-1$ by (31). Also, by Lemma 6.3 the lower bound $\mathfrak{m}(r,s) - m \geq 2^{-(s-1)} \geq 2^{-(n-1)}$ holds. Combining these two facts we note that

$$\epsilon \leq c_{52}(B^n \deg_R(X)^2)^{-2^{n-1}} = c_{52}B^{-n2^{n-1}}\deg_R(X)^{-2^n} \tag{49}$$

implies the hypothesis on ϵ in (48).

Lemma 6.9. — *Let m be an integer with $0 \leq m < \mathfrak{m}(r,n)$ and let $B \geq 1$. Say $X' \subset X$ is a irreducible closed subvariety of positive dimension with $X' \cap X^{\mathrm{oa}} \neq \varnothing$ if $R = \overline{\mathbf{Q}}$ or with $X' \cap X^{\mathrm{ta}} \neq \varnothing$ if $R = \mathbf{Q}$. If $0 \leq \epsilon \leq c_{52}B^{-n2^{n-1}}\deg_R(X')^{-2^n}$, there exists $Z \subsetneq X'$ Zariski closed and proper with*

$$\{p \in X' \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset Z.$$

Furthermore, if $R = \overline{\mathbf{Q}}$, then

$$\deg^0(Z) \leq c_{42}(B\deg(X'))^{(n+6)^4 2^n} \tag{50}$$

where c_{42} is the constant from Proposition 6.8.

Proof. — We only prove the case $R = \overline{\mathbf{Q}}$, the proof for $R = \mathbf{Q}$ is similar.

Say $H \subset \mathbf{G}_m^n$ is a coset containing X' with $\mathrm{s}^{\mathrm{o}}(X') = \dim H$. Now X' cannot be an anomalous subvariety of X because $X' \cap X^{\mathrm{oa}} \neq \varnothing$. Thus we have $\dim X' \leq r + \dim H - n$ or

$$\mathrm{s}^{\mathrm{o}}(X') - \dim X' \geq n - r \tag{51}$$

and in particular $\mathrm{s}^{\mathrm{o}}(X') \geq \dim X' + 1$ since $X \neq \mathbf{G}_m^n$ by the assumption made at the beginning of this section. Inequalities (51), $\dim X' \leq \dim X$, and Lemma 6.2 imply $\mathfrak{m}(\dim X', \mathrm{s}^{\mathrm{o}}(X')) \geq \mathfrak{m}(r,n) > m$. The lemma follows by applying Proposition 6.8 to X' and the comment around (49). □

The proof of Theorem 1.2 is by induction over the dimension and by applying the previous lemma. To control certain quantities in the induction step we define

$$\Delta'(r',\delta') = (c_{42}B^{r'}\delta')^{(n+6)^{4r'}2^{nr'}} \quad \text{for } \delta'\geq 1, r'\geq 0 \tag{52}$$

here $B\geq 1$ is considered to be fixed. If $\delta''\geq 1$, then

$$\Delta'(r',\delta')+\Delta'(r',\delta'')\leq \Delta'(r',\delta'+\delta'') \tag{53}$$

since the exponent $(n+6)^{4r'}2^{nr'}$ in (52) is not less than 1. We assume, as we may, that the constant c_{42} from Proposition 6.4 and 6.8 is at least 1.

Lemma 6.10. — *Let $\delta',\delta''\geq 1$ and $0\leq r''\leq r'-1$ with $\delta''\leq c_{42}(B\delta')^{(n+6)^4 2^n}$, then $\Delta'(r'',\delta'')\leq\Delta'(r',\delta')$.*

Proof. — It suffices to prove the inequality with $r''=r'-1$. The bound on δ'' implies

$$\Delta'(r'',\delta'')\leq (c_{42}^2 B^{r'-1+(n+6)^4 2^n}\delta'^{(n+6)^4 2^n})^{(n+6)^{4(r'-1)}2^{n(r'-1)}}.$$

To prove the lemma it is enough to show the inequality in

$$c_{42}^2 B^{r'-1+(n+6)^4 2^n}\delta'^{(n+6)^4 2^n}\leq \Delta'(r',\delta')^{(n+6)^{-4(r'-1)}2^{-n(r'-1)}} = (c_{42}B^{r'}\delta')^{(n+6)^4 2^n}.$$

The exponent of c_{42} on the far left is clearly at most the exponent of c_{42} on the far right and $c_{42}\geq 1$, furthermore the exponents of δ' are equal. The lemma is established if we can show $r'-1+(n+6)^4 2^n\leq r'(n+6)^4 2^n$. But this inequality holds if r' is substituted by 1 and so it must hold for all $r'\geq 1$ because both sides are linear in r'. $\square$

Proof of Theorem 1.2: Let X, r, B, and m be as in the hypothesis. We may assume $m\geq 0$, thus $X\neq \mathbf{G}_m^n$ since $\mathfrak{m}(n,n)=0$. Theorem 1.2(i) follows from Proposition 6.8 applied with $R=\overline{\mathbf{Q}}$ since $\mathrm{s}^{\mathrm{o}}(X)=n$.

It remains to prove part (ii) of the theorem. To do this we show the following statement.

Let $X'\subset X$ be an irreducible closed subvariety of dimension r', and

$$0\leq\epsilon\leq c_{52}\Delta'(r',\deg_R(X'))^{-1} \tag{54}$$

then

$$\{p\in X^{\mathrm{oa}}\cap X'\cap C(\mathcal{H}_m,\epsilon); h(p)\leq B\} \tag{55}$$

is finite with cardinality $\mathrm{N}(X')$ bounded by $\Delta'(r',\deg(X'))$.

The theorem follows by taking $X'=X$ in the statement above since $\Delta'(r,\deg(X))$ equals Δ up to a factor depending only on n. We prove the statement by induction on r'. The case $r'=0$ being trivial we assume $r'\geq 1$ and also $X^{\mathrm{oa}}\cap X'\neq\varnothing$. As $c_{42}\geq 1$, reviewing (54) and the definition (52) of Δ'

shows that ϵ satisfies $\epsilon \leq c_{52} B^{-n2^{n-1}} \deg(X')^{-2^n}$; so ϵ satisfies the hypothesis of Lemma 6.9. Hence

$$\{p \in X^{\mathrm{oa}} \cap X' \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset Z = Z_1 \cup \cdots \cup Z_l$$

where $Z_i \subsetneq X'$ are the irreducible components of Z. Let $1 \leq i \leq l$, we note

$$\deg(Z_i) \leq \deg^0(Z) \leq c_{42}(B\deg(X'))^{(n+6)^4 2^n} \tag{56}$$

and $r'' = \dim Z_i \leq r' - 1$ so by Lemma 6.10 with $\delta' = \deg(X')$, $\delta'' = \deg(Z_i)$ and (54) we have $\epsilon \leq c_{52} \Delta'(\dim Z_i, \deg(Z_i))^{-1}$. By induction we conclude that $\{p \in X^{\mathrm{oa}} \cap Z_i \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\}$ is finite for each i and thus obtain the finiteness statement made around (55). To bound the cardinality we apply the induction hypothesis to bound $\mathrm{N}(Z_i)$ in

$$\mathrm{N}(X') \leq \sum_{i=1}^{l} \mathrm{N}(Z_i) \leq \sum_{i=1}^{l} \Delta'(\dim Z_i, \deg(Z_i)) \leq \sum_{i=1}^{l} \Delta'(r' - 1, \deg(Z_i)),$$

the last inequality holds since Δ' is increasing in the first argument. By (53) we deduce

$$\mathrm{N}(X') \leq \Delta'(r' - 1, \sum_{i=1}^{l} \deg(Z_i)) = \Delta'(r' - 1, \deg^0(Z)).$$

Finally, Lemma 6.10 applied this time with $r'' = r' - 1$, $\delta'' = \deg^0(Z)$ and the bound (56) imply $\mathrm{N}(X') \leq \Delta'(r', \deg(X'))$; the proof follows. $\square$

The proof of Theorem 1.4 is simpler since we do not give an explicit cardinality bound.

Let X, r, B, and m be as in the hypothesis. We may assume $m \geq 0$, thus $X \neq \mathbf{G}_m^n$ since $\mathfrak{m}(n, n) = 0$. Part (i) follows from Proposition 6.8 applied with $R = \mathbf{Q}$ since $\mathrm{s}^*(X) = n$.

To prove (ii) we show the following statement:

Let $X' \subset X$ be an irreducible closed subvariety of dimension r', there exists $\epsilon > 0$ such that

$$\{p \in X^{\mathrm{ta}} \cap X' \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \tag{57}$$

is finite.

The theorem follows by taking $X' = X$ in the statement. We prove the statement by induction on r'. The case $r' = 0$ being trivial we assume $r' \geq 1$ and also $X^{\mathrm{ta}} \cap X' \neq \varnothing$. By Lemma 6.9 with $R = \mathbf{Q}$ there exists an $\epsilon > 0$ such that

$$\{p \in X^{\mathrm{ta}} \cap X' \cap C(\mathcal{H}_m, \epsilon); h(p) \leq B\} \subset Z = Z_1 \cup \cdots \cup Z_l. \tag{58}$$

where $Z_i \subsetneq X'$ are the irreducible components of Z. As $\dim Z_i \leq \dim X' - 1$ we reduce ϵ if necessary and apply the induction hypothesis to conclude that

$\{p \in X^{\mathrm{ta}} \cap Z_i \cap C(\mathcal{H}_m, \epsilon);\ h(p) \leq B\}$ is finite for each i. The finiteness of (57) now follows from (58). □

7. Proof of Theorems 1.1 and 1.5 and the corollaries

Before we can prove the announced results we state in somewhat simplified terms a theorem on boundedness of height proved by the author, see Theorem 1 [15].

THEOREM 7.1. — *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ of dimension r and let m be an integer with $m \cdot r < n$. There exists an effective constant $c'(n) > 0$ which depends only on n such that if*

$$\epsilon \leq (c'(n)\deg(X)^{\frac{n}{n-mr}})^{-1}$$

and $p \in X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)$ then

$$h(p) \leq c'(n)\deg(X)^{\frac{mr}{n-mr}}(\deg(X) + h(X)). \tag{59}$$

Proof of Theorem 1.5: Let X and m be as in the hypothesis, by Theorem 7.1 we may choose an $\epsilon > 0$ depending only on $\deg(X)$ and n such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)$ has height bounded by the right-hand side of (59). We conclude the finiteness of $X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)$ from Theorem 1.2(ii) after choosing a possibly smaller $\epsilon > 0$ which only depends on $h(X)$, $\deg(X)$, and n. Using the cardinality bound given in Theorem 1.2(ii) it is not hard to see that

(60)
$$|X^{\mathrm{oa}} \cap C(\mathcal{H}_m, \epsilon)| \leq c''(n)\deg(X)^{\frac{nr+n-mr}{n-mr}(n+6)^{4r}2^{nr}} \max\{1, h(X)\}^{r(n+6)^{4r}2^{nr}}$$

where $c''(n)$ depends only on n. □

Theorem 1.1 follows from Theorem 1.5 because $X^{\mathrm{oa}} = X$ if X is a curve which is not contained in a proper coset and because $\mathfrak{m}(1, n) = n-2+2^{-(n-1)} > n-2$. Using (60) with $r = 1$ and $m = n-2$ we get

$$|X \cap C(\mathcal{H}_{n-2}, \epsilon)| \leq c''(n)\deg(X)^{\frac{n+2}{2}(n+6)^4 2^n} \max\{1, h(X)\}^{(n+6)^4 2^n} \tag{61}$$

for $\epsilon > 0$ small enough.

Proof of Corollary 1.3: If x_1, x_2, x_3 are the coordinates on $\mathbf{G}_m^3$ we set P to be the plane defined by $2x_1 - x_2 - x_3 - 2 = 0$; clearly $X_\tau \subset P$ for all τ. All coefficients in this defining equation for P are non-zero and so $P^{\mathrm{oa}} \neq \varnothing$ by Theorem 1.3 [8]. Because P^{oa} is Zariski open [7] we may write $P^{\mathrm{oa}} = P \backslash Z$ with $Z \subsetneq P$ Zariski closed and of dimension at most 1.

We claim $X_\tau \not\subset Z$ for all $\tau \in \overline{\mathbf{Q}}$, even for $\tau \in \{0, \pm 1\}$. Indeed, let us assume $X_\tau \subset Z$, we will derive a contradiction. By dimension reasons X_τ is an irreducible component of Z and so we may fix $p \in X_\tau$ not contained in any other irreducible component. Since $p \notin P^{\mathrm{oa}}$ it is contained in an anomalous subvariety $Y \subset Z$. Now P has dimension 2 and we deduce immediately that Y is a coset. Reasoning again with dimension we see that Y is also an irreducible component of Z and since $p \in Y$ we conclude $Y = X_\tau$. Hence X_τ is a 1-dimensional coset and this is a contradiction to its definition.

Theorem 7.1 tells us that the height is bounded from above by some $B \geq 1$ on $(P \backslash Z) \cap C(\mathcal{H}_1, \epsilon)$ for a positive ϵ. Throughout the proof $\epsilon > 0$ is to be understood as independent of τ.

Let $\tau \in \overline{\mathbf{Q}} \backslash \{0, \pm 1\}$. We note

$$X_\tau \cap C(\mathcal{H}_1, \epsilon) \subset (X_\tau \cap Z) \cup ((X_\tau \backslash Z) \cap C(\mathcal{H}_1, \epsilon)),$$

and complete the proof by bounding the cardinality of $X_\tau \cap Z$ and $(X_\tau \backslash Z) \cap C(\mathcal{H}_1, \epsilon)$ separately.

As we have seen above $X_\tau \not\subset Z$. So $X_\tau \cap Z$ is a finite set. By Bézout's Theorem its cardinality is at most $\deg(X_\tau)\deg^0(Z) = \deg^0(Z)$ since $\deg(X_\tau) = 1$; this bound is clearly independent of τ.

We have ${X_\tau}^{\mathrm{oa}} = X_\tau$ because $\tau \neq 0, \pm 1$. Using again $\deg(X_\tau) = 1$, the quantity Δ defined in Theorem 1.2(ii) depends only on B and is thus independent of τ; hence we may assume $\epsilon \leq (c(n)\Delta)^{-1}$. This theorem implies that $\{p \in X_\tau \cap C(\mathcal{H}_1, \epsilon); h(p) \leq B\}$ has cardinality bounded by $c(n)\Delta$, so in particular independently of τ. But we know that B bounds the height on $(X_\tau \backslash Z) \cap C(\mathcal{H}_1, \epsilon)$ since $X_\tau \subset P$. This completes the proof. □

Proof of Corollary 1.6: We may assume $r = \dim X \geq 1$. Lemma 6.2 gives $\mathfrak{m}(r, n) \geq (n - r)/2 > 1$, so $\mathfrak{m}(r, n) > 1$ since $r \leq n - 3$ by hypothesis. The proof follows from Theorem 1.5 with $m = 1$. □

Proof of Corollary 1.7: We start by proving part (i). By Theorem 1.1 there exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-2}, \epsilon)$ is finite. Hence the points of this set have height bounded by some fixed B and degree bounded by some fixed D. The corollary now follows easily from Lemma 3.3 after adjusting ϵ if necessary.

The proofs of parts (ii) and (iii) are similar. □

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