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ON A THEOREM OF SAEKI CONCERNING CONVOLUTION SQUARES OF SINGULAR MEASURES

BY THOMAS KÖRNER

ABSTRACT. — If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu$ is a Lipschitz function of class $\alpha - \frac{1}{2}$.

RÉSUMÉ (*Carrés de convolution des mesures singulières*). — Si $1 > \alpha > 1/2$, alors il existe une mesure de probabilité μ avec support de dimension d'Hausdorff α tel que $\mu * \mu$ est une fonction Lipschitz de classe $\alpha - \frac{1}{2}$.

1. Introduction

We work on on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write τ for the Lebesgue measure on $\mathbb{T}$ and $|E| = \tau(E)$. We say that a function $f : \mathbb{T} \to \mathbb{C}$ is Lipschitz β if

$$\sup_{t\in\mathbb{T}} \sup_{h\neq 0} |h|^{-\beta}|f(t+h) - f(t)| < \infty$$

for some $1 \geq \beta > 0$.

In a famous paper [11], Wiener and Wintner constructed a singular measure μ such that $\mu * \mu \in L^p(T)$ for all $p \geq 1$ and other authors have given further examples along these lines (see Chapter 6 of [2] and [4]). The strongest result is due to Saeki [10] who constructs a singular measure μ with support of Lebesgue

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measure zero such that $\mu * \mu = f\tau$ where f has a uniformly convergent Fourier series. He also remarks that this can be improved to give $\mu * \mu = f\tau$ with f Lipschitz β provided that $\beta < 1/2$, but leaves the proof as an exercise to the reader. This paper may be considered as an extension to that exercise, although I believe that the results obtained are more precise and that the method used is not that envisaged by Saeki.

The object of this paper is to prove the following theorem.

THEOREM 1. — *If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu = f\tau$ where f is Lipschitz $\alpha - \frac{1}{2}$.*

At the two extremes $\alpha = 1$ and $\alpha = 1/2$ we get the following versions of Theorem 1.

THEOREM 2. — *There exists a probability measure μ such that the support of μ has Lebesgue measure 0 and $\mu * \mu = f\tau$ where f is Lipschitz β for all $\beta < 1/2$.*

THEOREM 3. — *There exists a probability measure μ such that the Hausdorff dimension of the support of μ is $1/2$ and $\mu * \mu = f\tau$ where f is continuous with uniformly convergent Fourier series.*

We shall see, in Corollary 23, how to show that all these results remain true if we replace $\mathbb{T}$ by $\mathbb{R}$. We note a consequence of Corollary 23 here.

LEMMA 4. — *Suppose $G : \mathbb{R} \to \mathbb{R}$ is a positive continuous function of bounded support. Then, given any $\epsilon > 0$, we can find a positive measure σ with support of Lebesgue measure zero such that $\sigma * \sigma = F\tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is Lebesgue measure and F is continuous with $\|F - G * G\|_\infty < \epsilon$.*

This indicates that the any numerical method for finding the approximate 'convolution square root' must overcome substantial difficulties.

The next lemma shows that, at the relatively coarse level of Hausdorff dimension and Lipschitz coefficients, these results must be close to best possible.

LEMMA 5. — (i) *If μ is a measure whose support has Hausdorff dimension α and $\mu * \mu = f\tau$ where f is Lipschitz β, then $\alpha - \frac{1}{2} \geq \beta$.*

(ii) *If μ is a measure whose support has Hausdorff dimension α and $\mu * \mu = f\tau$ where f is continuous, then $\alpha \geq \frac{1}{2}$.*

Proof. — (i) Since f is Lipschitz β, it follows that (see, for example, [1] Chapter II §3)

$$\sum_{n\le|k|\le 2n-1} |\hat{f}(k)| \le C_1 n^{(1-2\beta)/2}$$

for some constant C_1 depending on f. Since $|\hat{f}(k)| = |\hat{\mu}(k)|^2$, we have

$$\sum_{n\le|k|\le 2n-1} |\hat{\mu}(k)|^2 \le C_1 n^{(1-2\beta)/2}$$

and so, if $\eta > 0$,

$$\sum_{k=n}^{2n-1} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \le C_2 n^{-(1+2\beta-2\eta)/2}$$

for all $n \ge 1$ and some constant C_2. By Cauchy's condensation test,

$$\sum_{k\ne 0} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \text{ converges}$$

whenever $(1+2\beta)/2 > \eta$.

We know (by Theorems I and V of Chapter 3 in [7]), that if σ is a non-zero measure with

$$\sum_{k\ne 0} \frac{|\hat{\sigma}(k)|^2}{|k|^{1-\eta}} \text{ convergent}$$

for some $0 < \eta < 1$, it follows that the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least η. Thus the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least η for each η with $(1+2\beta)/2 > \eta$. We conclude that the Hausdorff dimension of $\operatorname{supp}\mu$ must be at least $(1+2\beta)/2$.

(ii) This follows the proof of (i) with $\beta = 0$. □

Our method of proof gives two slightly stronger versions of the theorems announced above which we state as Theorems 7 and 8. We need a preliminary pair of definitions.

DEFINITION 6. — *Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$.*

(i) *We say that a set $E \subseteq \mathbb{T}$ has Hausdorff ψ measure zero if, given $\epsilon > 0$, we can find a countable collection $\mathcal{I}$ of closed intervals such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \text{ and } \sum_{I\in\mathcal{I}} \psi(|I|) < \epsilon.$$

(ii) *We say that function $f : \mathbb{T} \to \mathbb{C}$ lies in Λ_ψ if*

$$\sup_{t,h\in\mathbb{T},h\ne 0} \psi(|h|)^{-1} |f(t+h) - f(t)| < \infty.$$

THEOREM 7. — *Let $1 > \alpha \geq 1/2$. Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $t^{-\beta}\psi(t) \to 0$ as $t \to 0+$ whenever $\beta < \alpha - \frac{1}{2}$. Then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $\mu * \mu = f\tau$ with $f \in \Lambda_\psi$.*

If $1 > \alpha > 1/2$ and $\psi(t) = t^{\alpha-1/2}$, we recover Theorem 1. If $\alpha = 1/2$ and $\psi(t) = (\log t^{-1})^{-2}$ for $0 < t < 1/2$, we recover Theorem 3 since the Dini–Lipschitz test tells us that any function in Λ_ψ must have uniformly convergent Fourier series. (See, for example, [1] Chapter IV §4.)

THEOREM 8. — *Let $1 \geq \alpha > 1/2$. Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $t^{-\beta}\psi(t) \to 0$ as $t \to 0+$ whenever $\beta < \alpha - \frac{1}{2}$. Then there exists a probability measure μ such that* $\operatorname{supp}\mu$ *has Hausdorff ψ measure zero and $\mu * \mu = f\tau$ with f Lipschitz β for all $\beta < \alpha - \frac{1}{2}$.*

If $\alpha = 1$ and $\psi(t) = t$, we recover Theorem 2.

Section 2 is devoted to the proof of the key Lemma 9. The proof is probabilistic and, as in several other papers, I acknowledge the influence of Kaufman's elegant note [8]. In the Section 3 we smooth the result of Lemma 9 to obtain Lemma 17 which we later use in Section 5 to prove a Baire category version of Theorem 7. In the final section we sketch the very similar proof of a Baire category version of Theorem 8.

In my opinion, the main ideas of the paper are to be found in the proofs of Lemmas 10 and 26.

2. The basic construction

The key to our construction is the following lemma.

LEMMA 9. — *Suppose $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n) \to \infty$ as $n \to \infty$. If $1 > \gamma > 0$ and $\epsilon > 0$, there exist an $M(\gamma)$ and $n_0(\phi, \gamma, \epsilon) \geq 1$ with the following property. If $n \geq n_0(\phi, \gamma, \epsilon)$, n is odd and $n^\gamma \geq N$ we can find N points*

$$x_j \in \{r/n : r \in \mathbb{Z}\}$$

(not necessarily distinct) such that, writing

$$\mu = N^{-1} \sum_{j=1}^{N} \delta_{x_j},$$

we have

$$|\mu * \mu(\{k/n\}) - n^{-1}| \leq \epsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}}$$

and

$$\mu(\{k/n\}) \leq \frac{M(\gamma)}{N}$$

for all $1 \leq k \leq n$.

Since any event with positive probability must have at least one instance, Lemma 9 follows from its probabilistic version.

Lemma 10. — *Suppose* $\phi : \mathbb{N} \to \mathbb{R}$ *is a sequence with* $\phi(n) \to \infty$ *as* $n \to \infty$. *If* $1 > \gamma > 0$ *and* $\epsilon > 0$, *there exists a* $M(\gamma)$ *and an* $n_0(\phi, \gamma, \epsilon) \geq 1$ *with the following property. Suppose that* $n \geq n_0(\kappa, \gamma, \epsilon)$, n *is odd,* $n^\gamma \geq N$ *and* X_1, X_2, ..., X_N *are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n \in \mathbb{T} : 1 \leq r \leq n\}.$$

Then, if we write $\sigma = N^{-1} \sum_{j=1}^{N} \delta_{X_j}$, *we have*

$$|\sigma * \sigma(\{k/n\}) - n^{-1}| \leq \epsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}},$$

and

$$\sigma(\{k/n\}) \leq \frac{M(\gamma)}{N}$$

for all $1 \leq k \leq n$, *with probability at least* $1/2$.

(The condition n odd can be removed but has the advantage that $2X_j$ is also uniformly distributed on Γ_n.)

We start our proof of Lemma 10 with a simple observation.

Lemma 11. — *Suppose that* $0 < Np \leq 1$ *and* $m \geq 2$. *Then, if* Y_1, Y_2, ..., Y_N *are independent random variables with*

$$\Pr(Y_j = 1) = p, \ \Pr(Y_j = 0) = 1 - p,$$

it follows that

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) \leq \frac{2(Np)^m}{m!}.$$

Proof. — If we set

$$u_k = \binom{N}{k} p^k$$

then

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) = \sum_{k=m}^{N} \Pr\left(\sum_{j=1}^{n} Y_j = k\right) \leq \sum_{k=m}^{N} u_k.$$

But

$$\frac{u_{k+1}}{u_k} = \frac{(N-k)p}{k} \leq \frac{1}{k} \leq \frac{1}{2}$$

for all $N \geq k \geq m$ and so

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) \leq \sum_{k=m}^{N} u_k \leq 2u_m \leq \frac{2(Np)^m}{m!}.$$

□

Lemma 12. — *If* $1 > \gamma > 0$ *and* $\epsilon > 0$*, there exists an* $M(\gamma,\epsilon) \geq 1$ *the following property. Suppose that* $n \geq 2$*,* $n^\gamma \geq N$ *and* $X_1, X_2, \dots, X_N$ *are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n : 1 \leq r \leq n\}.$$

Then, with probability at least $1 - \epsilon/n$*,*

$$\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \leq M(\gamma,\epsilon).$$

for all $1 \leq r \leq n$*.*

Proof. — Since $1 - \gamma > 0$ we can find an integer $M(\gamma,\epsilon)$ such that $n^{2-M(\gamma,\epsilon)(1-\gamma)} < \epsilon/2$ for all $n \geq 2$. Fix r for the time being and set

$$Y_j = \delta_{X_j}(\{r/n\}).$$

We observe that $Y_1, Y_2, \dots, Y_N$ are independent random variables with

$$\Pr(Y_j = 1) = n^{-1},\ \Pr(Y_j = 0) = 1 - n^{-1}.$$

By Lemma 11, it follows that

$$\begin{aligned}\Pr\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\epsilon)\right) &= \Pr\left(\sum_{j=1}^{N} Y_j \geq M(\gamma,\epsilon)\right)\\ &\leq \frac{2(Nn^{-1})^{M(\gamma,\epsilon)}}{M(\gamma,\epsilon)!} \leq 2n^{-(1-\gamma)M(\gamma,\epsilon)} < \frac{\epsilon}{n^2}.\end{aligned}$$

Allowing r to vary, we see at once that

$$\begin{aligned}&\Pr\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\epsilon) \text{ for some } 1 \leq r \leq n\right)\\ &\qquad \leq \sum_{r=1}^{n} \Pr\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\epsilon)\right) < \frac{\epsilon}{n}\end{aligned}$$

as required. □

We now make use of a technique that goes back to Bernstein but is best known through its use to prove Hoeffding's inequality (see for example §12.2 of [3] or Hoeffding's paper [6]). Recall the following definition.

DEFINITION 13. — *A sequence W_r is said to be a martingale with respect to a sequence X_r of random variables if*

(i) $\mathbb{E}|W_r| < \infty$,
(ii) $\mathbb{E}(W_{r+1}|X_0, X_1, \dots X_r) = W_r$.

The next lemma is standard.

LEMMA 14. — *Let $\delta > 0$ and let W_r be a martingale with respect to a sequence X_r of random variables. Write $Y_{r+1} = W_{r+1} - W_r$. Suppose that*

$$\mathbb{E}(e^{\lambda Y_{r+1}}|X_0, X_1, \dots X_r) \leq e^{a_{r+1}\lambda^2/2}$$

for all $|\lambda| < \delta$ and some $a_{r+1} \geq 0$. Suppose further that $A \geq \sum_{r=1}^{N} a_r$. Then, provided that $0 \leq x < A\delta$, we have

$$\Pr\left(|W_N - W_0| \geq x\right) \leq 2\exp\left(\frac{-x^2}{2A}\right).$$

Proof. — Observe that, if $0 \leq \lambda < \delta$,

$$\mathbb{E}e^{\lambda(W_N - W_0)} = \mathbb{E}e^{\lambda(Y_1+Y_2+\cdots+Y_N)} = \mathbb{E}\prod_{r=1}^{N} e^{\lambda Y_r} \leq \prod_{r=1}^{N} e^{a_r\lambda^2/2} \leq e^{A\lambda^2/2}$$

where $A = \sum_{r=1}^{N} a_r$. Thus

$$\Pr(W_N - W_0 \geq x)e^{\lambda x} \leq e^{A\lambda^2/2}$$

and so

$$\Pr(W_N - W_0 \geq x) \leq e^{A(\lambda^2/2)-\lambda x}.$$

Setting $\lambda = xA^{-1}$ we get

$$\Pr(W_N - W_0 \geq x) \leq e^{-x^2/(2A)}.$$

Exactly the same argument applies to the martingale $-W_r$ so

$$\Pr(W_0 - W_N \geq x) \leq e^{-x^2/(2A)}$$

and combining the last two inequalities gives the required result. □

Proof of Lemma 10. — Let $M(\gamma,1/4)$ be as in Lemma 12 and set $M(\gamma) = M(\gamma,1/4)$. Fix r for the time being and define Y_1, Y_2, …, Y_N as follows. If $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\gamma)$ for all u with $1\leq u\leq n$, set

$$Y_j = -\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1}\delta_{X_v+X_j}(\{r/n\}).$$

Otherwise set $Y_j = 0$. We take $W_0 = 0$ and

$$W_j = \sum_{m=1}^{j} Y_m.$$

We now find $\mathbb{E}Y_j$ given that the values of X_1, X_2, $\dots X_{j-1}$ are known. To this end, observe that, if s is a fixed integer, $X_j + s/n$ and $2X_j$ are uniformly distributed over Γ_n. Thus, if $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\gamma)$ for all u with $1\leq u\leq n$, then

$$\begin{aligned}\mathbb{E}Y_j &= -\frac{2j-1}{n} + \mathbb{E}\delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1}\mathbb{E}\delta_{X_v+X_j}(\{r/n\})\\ &= -\frac{2j-1}{n} + \frac{1}{n} + 2\sum_{v=1}^{j-1}\frac{1}{n} = 0.\end{aligned}$$

If it is not true that $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\gamma)$ for all u, then, automatically $\mathbb{E}Y_j = \mathbb{E}0 = 0$. Thus the sequence W_j is a martingale.

In order to apply Lemma 14 we must estimate $\mathbb{E}(e^{\lambda Y_j})$, given that the values of X_1, X_2, $\dots X_{j-1}$ are known.

First suppose that

$$\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\gamma)$$

for all u with $1\leq u\leq n$. Observe that $Y_j\leq 2M(\gamma)$ and if we set $Z_j = Y_j + (2j-1)/n$, then $\Pr(Z_j\neq 0)\leq j/n$. Thus, provided only that n is large enough and $0\leq\lambda\leq 1/\bigl(2M(\gamma)\bigr)$,

$$\begin{aligned}\mathbb{E}(e^{\lambda Y_j}) &= \sum_{k=0}^{\infty}\frac{\lambda\mathbb{E}Y_j^k}{k!} = 1 + \sum_{k=2}^{\infty}\frac{\lambda^k\mathbb{E}Y_j^k}{k!} \leq 1 + \sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}|Y_j|^k}{k!}\\ &= 1 + \Pr(Z_j = 0)\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}(|Y_j|^k|Z_j = 0)}{k!}\\ &\qquad + \Pr(Z_j\neq 0)\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}(|Y_j|^k|Z_j\neq 0)}{k!}\end{aligned}$$

$$\leq 1+\sum_{k=2}^{\infty}\frac{\big(|\lambda|(2j-1)/n\big)^k}{k!}+\frac{j}{n}\sum_{k=2}^{\infty}\frac{\big(2|\lambda|M(\gamma)\big)^k}{k!}$$
$$\leq 1+\sum_{k=2}^{\infty}\frac{\big(|\lambda|(2N-1)/n\big)^k}{k!}+\frac{N}{n}\sum_{k=2}^{\infty}\frac{\big(2|\lambda|M(\gamma)\big)^k}{k!}$$
$$\leq 1+\frac{\big(\lambda(2N-1)/n\big)^2}{2}\sum_{k=2}^{\infty}2^{2-k}+\frac{N}{n}\frac{\big(2\lambda M(\gamma)\big)^2}{2}\sum_{k=2}^{\infty}2^{2-k}$$
$$=1+\left(\left(\frac{2N-1}{N}\right)^2+\frac{4N}{n}M(\gamma)^2\right)\lambda^2$$
$$\leq 1+(4M(\gamma)^2+1)\frac{N}{n}\lambda^2\leq \exp\left(8(M(\gamma)^2+1)\frac{N}{n}\lambda^2\right).$$

If it is not true that $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\})<M(\gamma)$ for all u, then, automatically

$$\mathbb{E}(e^{\lambda Y_j})=\mathbb{E}(1)=1\leq \exp\left(8(M(\gamma)^2+1)\frac{N}{n}\lambda^2\right).$$

Combining the two cases we obtain

$$\mathbb{E}(e^{\lambda Y_r}|X_0,X_1,\dots X_{r-1})\leq \exp\left((8M(\gamma)^2+1)\frac{N}{n}\lambda^2\right).$$

Applying Lemma 14 with

$$A=16(M(\gamma)^2+1)\frac{N^2}{n}\ \text{and}\ x=\epsilon\frac{N\phi(n)(\log n)^{1/2}}{n^{1/2}}$$

we see that (since $\phi(n)\to\infty$ as $n\to\infty$) we can choose $n_0(\phi,\gamma,\epsilon)$ so that

$$\Pr(|W_N-W_0|\geq \phi(n)(\log n)^{1/2}Nn^{-1/2})\leq 1/(4n)$$

for all $n\geq n_0(\phi,\gamma,\epsilon)$. For the rest of the proof we assume that n satisfies this condition.

By Lemma 12 we know that with probability at least $1-1/(4n)$,

$$\sum_{v=1}^{N}\delta_{X_v}(\{r/n\})\leq M(\gamma)$$

for all $1\leq r\leq n$ whence

$$\sum_{v=1}^{j-1}\delta_{X_v}(\{r/n\})\leq M(\gamma)$$

for all r with $1\leq r\leq n$ and all $1\leq j\leq n$ so that

$$Y_j=-\frac{2j-1}{n}+\delta_{2X_j}(\{r/n\})+2\sum_{v=1}^{j-1}\delta_{X_v+X_j}(\{r/n\})$$

for all $1 \leq j \leq n$ and so

$$\begin{aligned} W_N - W_0 &= \sum_{j=1}^{N}\left(-\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1}\delta_{X_v+X_j}(\{r/n\})\right) \\ &= -\frac{N^2}{n} + \sum_{v=1}^{n}\sum_{j=1}^{n}\delta_{X_v+X_j}(\{r/n\}). \end{aligned}$$

Combining the results of the last two paragraphs, we see that, with probability at least $1 - 1/(2n)$, we have

$$\sum_{v=1}^{j-1}\delta_{X_v}(\{r/n\}) \leq M(\gamma)$$

for all r with $1 \leq r \leq n$ and

$$\left|\sum_{v=1}^{n}\sum_{j=1}^{n}\delta_{X_v+X_j}(\{r/n\}) - \frac{N^2}{n}\right| < \phi(n)(\log n)^{1/2} N n^{-1/2}.$$

If we write $\sigma = N^{-1}\sum_{j=1}^{N}\delta_{X_j}$, then this inequality can be written

$$|\sigma * \sigma(\{r/n\}) - n^{-1}| \leq \epsilon\frac{(\log n)^{1/2}}{N n^{1/2}}.$$

If we now allow r to take the values 1 to n, the result follows. □

3. Point masses to smooth functions

In this section we convert Lemma 9 to a more usable form.

We write $\mathbb{I}_A$ for the indicator function of the set A (so that $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise).

LEMMA 15. — *Suppose $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n) \to \infty$ as $n \to \infty$. If $1 > \gamma > 0$ and $\epsilon > 0$, there exist an $M(\gamma)$ and an $n_0(\phi, \gamma, \epsilon) \geq 1$ with the following property. If $n \geq n_0(\phi, \gamma, \epsilon)$, n is odd and $n^\gamma \geq N$ we can find*

$$x_j \in \{r/n : r \in \mathbb{Z}\}$$

(not necessarily distinct) such that, writing

$$g = \frac{n}{N}\sum_{j=1}^{N}\mathbb{I}_{[x_j-(2n)^{-1}, x_j+(2n)^{-1})}$$

we have $g * g$ continuous and

(i) $\|g * g - 1\|_\infty \leq \epsilon\frac{\phi(n)(\log n)^{1/2}}{n^{1/2}N}$.

(ii) $|h|^{-1}|g * g(t+h) - g * g(t)| \leq 2\epsilon\frac{\phi(n)n^{1/2}(\log n)^{1/2}}{N}$ *for all $t, h \in \mathbb{T}$, $h \neq 0$.*

(iii) $|g(t)| \leq \frac{nM(\gamma)}{N}$ *for all* $t \in \mathbb{T}$.

Proof. — Let x_j and μ be as in Lemma 9. Then $g = \mu * n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})}$ and so

$$g * g = \mu * \mu * \left(n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})} * n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})}\right) = \mu * \mu * n^2\triangle_n$$

where

$$\triangle_n = \max(0, 1 - n|x|).$$

Thus $g * g$ is the simplest piecewise linear function with

$$g * g(r/n) = n\mu * \mu(\{r/n\}).$$

By inspection, $g * g$ is continuous everywhere, linear on each interval $[r/n, (r+1)/n]$, and

$$|g * g(r/n) - 1| \leq \epsilon \frac{\phi(n)(\log n)^{1/2}}{Nn^{1/2}}.$$

Conclusions (i) to (iii) follow at once. □

We now smooth g by convolving with a suitable function.

Lemma 16. — *Suppose $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n) \to \infty$ as $n \to \infty$. If $1 > \gamma > 0$ and $\epsilon > 0$, then there exists an $M(\gamma)$ and an $n_0(\phi, \gamma, \epsilon) \geq 1$ with the following property. If $n \geq n_0(\phi, \gamma, \epsilon)$ and $n^\gamma \geq N$, we can find a positive infinitely differentiable function f such that*

(i) $\|f * f - 1\|_\infty \leq \epsilon \frac{\phi(n)(\log n)^{1/2}}{n^{1/2}N}$.

(ii) $\|(f * f)'\|_\infty \leq 2\epsilon \frac{\phi(n)n^{1/2}(\log n)^{1/2}}{N}$.

(iii) $\|f\|_\infty \leq \frac{nM(\gamma)}{N}$.

(iv) $\|f'\|_\infty \leq 10n^2M(\gamma)N$.

(v) $\int_{\mathbb{T}} f(t)dt = 1$.

(vi) $\operatorname{supp} f$ *can be covered by N intervals of length* $2/n$.

Proof. — The nature of the result is such that it suffices to prove it when n is odd. Let $K : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable positive function such that $K(x) = 0$ for $|x| \geq 1/4$, $|K'(x)| \leq 10$ for all x and $\int_{\mathbb{R}} K(x)dx = 1$. We define $K_n : \mathbb{T} \to \mathbb{R}$ by $K_n(t) = nK(nt)$ for $-1/2 \leq t < 1/2$.

If we take g as in Lemma 15 and set $f = g * K_n$, then the results of this lemma follow at once from the corresponding parts of Lemma 16. □

Our method of proof for Theorems 7 and 8 is not sufficiently delicate to make use of the careful estimates we have made so far, so we shall be rather more lax in our bounds from now on.

LEMMA 17. — *Suspose that k is an integer with $k \geq 1$, and $\alpha - \frac{1}{2} > \beta > 0$ If $\epsilon > 0$ there exists an $m_1(k, \alpha, \beta, \epsilon) \geq 1$ with the following property. If $m > m_1(k, \alpha, \beta, \epsilon)$ we can find a positive infinitely differentiable function F which is periodic with period $1/m$ such that*

(i) $\|F * F - 1\|_\infty \leq \epsilon$.

(ii) $\|(F * F)'\| \leq m^{1-k\beta}$

(iii) $\|F\|_\infty \leq m^k$,

(iv) $\|F'\|_\infty \leq m^{2k+1}$

(v) $\int_{\mathbb{T}} F(t)dt = 1$.

(vi) *We can find a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I \in \mathcal{I}} I \supseteq \operatorname{supp} F \text{ and } \sum_{I \in \mathcal{I}} |I|^{(1+k\alpha)/(k+1)} < \epsilon.$$

(vii) $|h|^{(1-k\beta)/(k+1)} |F * F(t+h) - F * F(t)| \leq \epsilon$ *for all* $t, h \in \mathbb{T}$ *with* $h \neq 0$.

(viii) $|\hat{F}(r)| \leq \epsilon$ *for all* $r \neq 0$.

Proof. — Set $\kappa = (\alpha + \beta)/2$, $\phi(r) = \log(r+1)$, $n = m^k$ and take $N = [n^\kappa]$. Provided that m is large enough, Lemma 16, tells us that we can find a positive infinitely differentiable function f such that

(i′) $\|f * f - 1\|_\infty \leq \epsilon^2$.

(ii′) $\|(f * f)'\| \leq m^{-k\beta}$.

(iii′) $\|f\|_\infty \leq m^k$.

(iv′) $\|f'\|_\infty \leq m^{2k}$.

(v′) $\int_{\mathbb{T}} f(t)dt = 1$.

(vi′) $\operatorname{supp} f$ can be covered by $[m^{k\kappa}]$ intervals of length $2/m^k$.

By the mean value theorem, condition (ii′) gives

(ii″) $|h|^{-1} |f * f(t+h) - f * f(t)| \leq m^{-k\beta}$ for all $t, h \in \mathbb{T}$ with $h \neq 0$.

If we set $F(t) = f(mt)$, we see, at once, that F is positive infinitely differentiable function such that

(i″) $\|F * F - 1\|_\infty \leq \epsilon^2$.

(ii) $\|(F * F)'\| \leq m^{1-k\beta}$

(iii) $\|F\|_\infty \leq m^k$.

(iv) $\|F'\|_\infty \leq m^{2k+1}$.

(v) $\int_{\mathbb{T}} F(t)dt = 1$.

(vi″) We can find a collection $\mathcal{I}$ of $m \times [m^{k\kappa}]$ intervals of length $2/m^{k+1}$ which cover $\operatorname{supp} F$.

Looking at (vi$''$) we see that, provided only that m is large enough,

$$\sum_{I\in\mathcal{I}} |I|^{(k\alpha+1)/(k+1)} \leq m^{1+k\kappa} \times 2^{(k\alpha+1)/(k+1)} \times m^{-(k+1)(1+k\alpha)/(k+1)} \leq \epsilon.$$

Thus condition (vi) holds for m large enough.

Next we observe that, if $r \neq 0$,

$$|\hat{F}(r)|^2 = |\widehat{F*F}(r)| = |(F*F-1)\hat{}(r)| \leq \|F*F-1\|_\infty \leq \epsilon^2$$

and so condition (viii) holds.

To obtain condition (vii) we note first that, by condition (ii) and the mean value theorem

$$|h|^{-1}|F*F(t+h) - F*F(t)| \leq m^{1-k\beta}$$

for all $t, h \in \mathbb{T}$ with $m^{-1}/2 > |h| \neq 0$.

Thus, if $m^{-(k+1)} \geq |h| > 0$,

$$\begin{aligned}
|h|^{-(1-k\beta)/(k+1)}&|F*F(t+h) - F*F(t)| \\
&= |h|^{(1-\beta)k/(k+1)}|h|^{-1}|F*F(t+h) - F*F(t)| \\
&\leq \epsilon|h|^{(1-\beta)k/(k+1)}m^{1-k\beta} \leq \epsilon m^{-(k+1)(1-\beta)k/(k+1)}m^{1-k\beta} \leq \epsilon
\end{aligned}$$

for all $0 < |h| \leq m^{-(k+1)}$. But, if $|h| \geq m^{-(k+1)}$, condition (i$'$) tells us that

$$\begin{aligned}
|h|^{-(1-k\beta)/(k+1)}&|F*F(t+h) - F*F(t)| \\
&\leq m^{(k+1)(1-k\beta)/(k+1)}m^{-k(1+\beta)} \leq \epsilon m^{-1} \leq \epsilon
\end{aligned}$$

so condition (vi) holds for all $h \neq 0$ and we are done. □

4. A Baire category context

The object of this section is to state a Baire category version of Theorem 7 and take some preliminary steps towards its proof.

In order to use Baire category we need an appropriate complete metric space.

LEMMA 18. — (i) *Consider the space $\mathcal{F}$ of non-empty closed subsets of $\mathbb{T}$. If we set*

$$d_{\mathcal{F}}(E,F) = \sup_{e\in E}\inf_{f\in F}|e-f| + \sup_{f\in F}\inf_{e\in E}|e-f|,$$

then $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space.

(ii) *Consider the space $\mathcal{E}$ consisting of ordered pairs (E,μ) where $E \in \mathcal{F}$ and μ is a probability measure with* $\operatorname{supp}\mu \subseteq E$ *and $\hat{\mu}(r) \to 0$ as $|r| \to \infty$. If we take*

$$d_{\mathcal{E}}\big((E,\mu),(F,\sigma)\big) = d_{\mathcal{F}}(E,F) + \sup_{r\in\mathbb{Z}}|\hat{\mu}(r) - \hat{\sigma}(r)|,$$

then $(\mathcal{E}, d_{\mathcal{E}})$ is a complete metric space.

(iii) *Consider the space* $\mathcal{G}$ *consisting of those* $(E,\mu)\in\mathcal{E}$ *such that* $\mu*\mu=f_\mu\tau$ *with* f_μ *continuous. If we take*

$$d_{\mathcal{G}}\big((E,\mu),(F,\sigma)\big)=d_{\mathcal{E}}\big((E,\mu),(F,\sigma)\big)+\|f_\mu-f_\tau\|_\infty,$$

then $(\mathcal{G},d_{\mathcal{G}})$ *is a complete metric space.*

Proof. — (i) This space was introduced by Hausdorff in [5] and is discussed, with proofs, in [9] (see Chapter II §21 VII and Chapter III §33 IV).

(ii) Suppose that (E_j,μ_j) is a Cauchy sequence in $(\mathcal{E},d_{\mathcal{E}})$. Then E_j is Cauchy in $(\mathcal{F},d_{\mathcal{F}})$ so we can find an $E\in\mathcal{F}$ such that $d_{\mathcal{F}}(E_j,E)\to 0$ as $j\to\infty$.

By the weak compactness of the set of probability measures, we can find $j(k)\to\infty$ and μ a probability measure such that $\mu_{j(k)}\to\mu$ weakly. Since $\hat\mu_{j(k)}(r)\to 0$ as $|r|\to\infty$ and

$$\sup_{r\in\mathbb{Z}}|\hat\mu_{j(k)}(r)-\hat\mu_{j(l)}(r)|\to 0 \text{ as } k,l\to\infty$$

we have $\hat\mu(r)\to 0$ as $|r|\to\infty$. Since $E_{j(k)}\supseteq\operatorname{supp}\mu$ and $d_{\mathcal{F}}(E_{j(k)},E)\to 0$ as $k\to\infty$, we have $E\supseteq\operatorname{supp}\mu$, so $(E,\mu)\in\mathcal{E}$. Weak convergence tells us that $\hat\mu_{j(k)}(r)\to\hat\mu(r)$ for each fixed r and so

$$d_{\mathcal{E}}\big((E_{j(k)},\mu_{j(k)}),(E,\mu)\big)\to 0$$

as $k\to\infty$. Since a Cauchy sequence with a convergent subsequence must converge to the same limit as the subsequence,

$$d_{\mathcal{E}}\big((E_j,\mu_j),(E,\mu)\big)\to 0$$

and we are done.

(iii) If (E_j,μ_j) is a Cauchy sequence in $(\mathcal{G},d_{\mathcal{G}})$, then it is Cauchy in $(\mathcal{E},d_{\mathcal{E}})$. Thus we can find an $(E,\mu)\in\mathcal{E}$ with

$$d_{\mathcal{E}}\big((E_j,\mu_j),(E,\mu)\big)\to 0.$$

Since f_{μ_j} is Cauchy in the uniform norm, f_{μ_j} converges uniformly to some continuous function f. By looking at Fourier coefficients or using properties of weak convergence, $\mu*\mu=f\tau$, so $(E,\mu)\in\mathcal{G}$ and

$$d_{\mathcal{G}}\big((E_j,\mu_j),(E,\mu)\big)\to 0.$$

□

LEMMA 19. — *Let* $1>\alpha\geq 1/2$ *and suppose that* $\psi:\mathbb{R}\to\mathbb{R}^+$ *is a strictly increasing continuous function with* $\psi(0)=0$ *and* $t^{-\beta}\psi(t)\to 0$ *as* $t\to 0+$ *whenever* $\beta<\alpha-\frac{1}{2}$. *If* $f\in\Lambda_\psi$, *let us write*

$$\omega_\psi(f)=\sup_{t,h\in\mathbb{T},h\neq 0}\psi(|h|)^{-1}|f(t+h)-f(t)|.$$

Consider the space $\mathcal{L}_\psi$ consisting of those $(E,\mu)\in\mathcal{G}$ such that $\mu*\mu=f_\mu\tau$ with $f_\mu\in\Lambda_\psi$. If we take

$$d_\psi\big((E,\mu),(F,\sigma)\big)=d_{\mathcal{G}}(E,F)+\omega_\psi(f_\mu-f_\sigma),$$

then $(\mathcal{L}_\psi,d_\psi)$ is a complete metric space.

Proof. — This follows a standard pattern. It is easy to check that $(\mathcal{L}_\psi,d_\psi)$ is a metric space. If (E_n,μ_n) forms a Cauchy sequence in $(\mathcal{L}_\psi,d_\psi)$, then it forms a Cauchy sequence in $(\mathcal{G},d_{\mathcal{G}})$. Thus there exists a $(E,\mu)\in\mathcal{G}$ such that

$$d_{\mathcal{G}}\big((E_n,\mu_n),(E,\mu)\big)\to 0$$

as $n\to\infty$.

Let us write $\mu_n*\mu_n=f_n\tau$ and $\mu_n*\mu_n=f\tau$ with f_n and f continuous. If $m\geq n$, we have

$$\begin{aligned}
&\psi(|h|)^{-1}|(f-f_n)(t+h)-(f-f_n)(t)|\\
&\qquad\leq\psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)|\\
&\qquad\qquad+\psi(|h|)^{-1}|(f_n-f_m)(t+h)-(f_n-f_m)(t)|\\
&\qquad\leq\psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)|+d_\psi\big((E_n,\mu_n),(E_m,\mu_m)\big)\\
&\qquad\leq\psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)|+\sup_{p,q\geq n}d_\psi\big((E_p,\mu_p),(E_q,\mu_q)
\end{aligned}$$

Allowing $m\to\infty$, we deduce that

$$\psi(|h|)^{-1}|(f-f_n)(t+h)-(f-f_n)(t)|\leq\sup_{p,q\geq n}d_\psi\big((E_p,\mu_p),(E_q,\mu_q)\big)$$

for all $t,h\in\mathbb{T}$, $h\neq 0$. Thus $f-f_n\in\Lambda_\psi$ and

$$\omega_\psi(f-f_n)\leq\sup_{p,q\geq n}d_\psi\big((E_p,\mu_p),(E_q,\mu_q)\big).$$

It follows that $f\in\Lambda_\psi$, $(E,\mu)\in\mathcal{L}_\psi$ and

$$d_\psi\big((E_n,\mu_n),(E,\mu)\big)\to 0$$

as $n\to\infty$. □

We recall some well known properties of ω_ψ which we will use later.

LEMMA 20. — *Let $\psi:\mathbb{R}\to\mathbb{R}^+$ be a strictly increasing continuous function with $\psi(0)=0$ and $t\geq\psi(t)$ for all $t\geq 0$. Let Λ_ψ be the set of continuous functions $f:\mathbb{T}\to\mathbb{C}$ with*

$$\sup_{t,h\in\mathbb{T},h\neq 0}\psi(|h|)^{-1}|f(t+h)-f(t)|<\infty$$

and let us write

$$\omega_\psi(f)=\sup_{t,h\in\mathbb{T},h\neq 0}\psi(|h|)^{-1}|f(t+h)-f(t)|$$

for all $f \in \Lambda_\psi$. Then

(i) *If $f, g \in \Lambda_\psi$, then $f+g \in \Lambda_\psi$ and*

$$\omega_\psi(f+g) \le \omega_\psi(f) + \omega_\psi(g).$$

(ii) *If $f, g \in \Lambda_\psi$, then $f+g \in \Lambda_\psi$ and*

$$\omega_\psi(fg) \le \omega_\psi(f)\|g\|_\infty + \omega_\psi(g)\|f\|_\infty.$$

(iii) *If $f \in \Lambda_\psi$ and $g \in L^1(\mathbb{T})$, then $f * g \in \Lambda_\psi$ and*

$$\omega_\psi(f * g) \le \omega_\psi(f)\|g\|_1.$$

(iv) *If $f : \mathbb{T} \to \mathbb{C}$ has continuous derivative, then $f \in \Lambda_\psi$ and*

$$\omega_\psi(f) \le \|f'\|_\infty.$$

Proof. — The proofs are much the same as those for the corresponding results on differentiation. □

We can now state our new version of Theorem 7.

THEOREM 21. — *Suppose α, ψ and $(\mathcal{L}_\psi, d_\psi)$ are as in Lemma 19. The complement of the set*

$$\mathcal{H} = \{(E, \mu) \in \mathcal{L}_\psi : E \text{ has Hausdorff dimension } \alpha\}$$

is of first category in $(\mathcal{L}_\psi, d_\psi)$.

The use of Baire category is, of course, merely a book keeping convenience and the reader who prefers a more direct construction will be able to extract one from our proof of Theorem 21.

We note two nice corollaries.

COROLLARY 22. — *Given $\epsilon > 0$ we can find a measure μ with support of Hausdorff dimension $1/2$ and a continuous function $f : \mathbb{T} \to \mathbb{R}$ such that $\mu * \mu = f\tau$ and*

$$\|f - 1\|_\infty < \epsilon.$$

Proof. — We apply Theorem 21 with $\alpha = 1/2$ and $\psi(t) = (\log t^{-1})^{-2}$ for $0 < t < 1/2$. Since $(\mathbb{T}, \tau) \in \mathcal{L}_\psi$, this tells us that we can find $(E, \mu) \in \mathcal{H}$ with $d_\psi\big((\mathbb{T}, \tau), (E, \mu)\big) < \epsilon$. □

The statement of the next corollary involves measures on $\mathbb{R}$ rather than $\mathbb{T}$. We denote Lebesgue measure on $\mathbb{R}$ by $\tau_{\mathbb{R}}$.

COROLLARY 23. — *Suppose $G : \mathbb{R} \to \mathbb{R}$ is a positive continuous function of bounded support. Then, given any $\epsilon > 0$, we can find a positive measure σ with support of Hausdorff dimension $1/2$ such that $\sigma * \sigma = F\tau_{\mathbb{R}}$ with F continuous and $\|F - G\|_\infty < \epsilon$.*

Proof. — We suppose $\epsilon < 1/16$. Since the result is invariant under multiplication by a constant, dilation and translation, we may assume that $\operatorname{supp} G \subseteq [-1/16, 1/16]$ and $\int_{\mathbb{R}} G(t)dt = 1$. Define $g : \mathbb{T} \to \mathbb{R}$ by $g(t) = G(t)$ for $|t| \leq 1/16$, $g(t) = 0$ otherwise. We apply Theorem 21 with $\alpha = 1/2$ and $\psi(t) = (\log t^{-1})^{-2}$ for $0 < t < 1/2$. Since $(\operatorname{supp} g, g\tau) \in \mathcal{L}_\psi$, this tells us that we can find $(E, \mu) \in \mathcal{H}$ with

$$d_\psi\big((\operatorname{supp} g, g\tau), (E, \mu)\big) < \epsilon,$$

and so in particular $\mu * \mu = f\tau$ with $\|f - g * g\|_\infty < \epsilon$. We observe that $E \subseteq [-1/8, 1/8]$ and so

$$\operatorname{supp} \mu * \mu \subseteq [-1/4, 1/4].$$

Thus we can choose a probability measure σ on $\mathbb{R}$ corresponding to μ in a natural way so that $\sigma * \sigma = F\tau_{\mathbb{R}}$ with $F(t) = f(t)$ for $|t| \leq 1/4$, $F(t) = 0$ otherwise. The result follows. □

Theorem 21 follows from the following simpler result.

LEMMA 24. — *Suppose α, ψ and $(\mathcal{L}_\psi, d_\psi)$ are as in Lemma 19. Let $\mathcal{H}_n$ be the subset of consisting of those $(E, \mu) \in \mathcal{L}_\psi$ such that we can find a finite collection of intervals $\mathcal{I}$ with*

$$\bigcup_{I \in \mathcal{I}} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha + 1/n} < 1/n.$$

Then $\mathcal{H}_n$ is dense in $(\mathcal{L}_\psi, d_\psi)$.

Proof of Theorem 21 from Lemma 24. — We first prove that $\mathcal{H}_n$ is open in $(\mathcal{L}_\psi, d_\psi)$. Suppose that $(E, \mu) \in \mathcal{H}_n$. By definition, we can find a finite collection of closed intervals $\mathcal{I}$ with

$$\bigcup_{I \in \mathcal{I}} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha + 1/n} < 1/n.$$

Since $\mathcal{I}$ is finite we can find an $\eta > 0$ such that

$$\sum_{I \in \mathcal{I}} (|I| + 2\eta)^{\alpha + 1/n} < 1/n.$$

Let $\tilde{\mathcal{I}}$ consist of intervals $[a - \eta, b + \eta]$ with $[a, b] \in \mathcal{I}$. If $(F, \sigma) \in \mathcal{L}_\psi$ with

$$d_\psi\big((E, \mu), (F, \sigma)\big) < \eta,$$

then, automatically,

$$\bigcup_{I \in \tilde{\mathcal{I}}} I \supseteq F \text{ and } \sum_{I \in \tilde{\mathcal{I}}} |I|^{(n-1)/(2n)} < 1/n$$

and $(F, \sigma) \in \mathcal{H}_n$.

Lemma 24 tells us that $\mathcal{H}_n$ is dense in $(\mathcal{L}_\psi, d_\psi)$ so it follows that the complement of $\mathcal{H}_n$ is of first category. Thus the complement of $\mathcal{H} = \bigcap_{n=1}^\infty \mathcal{H}_n$ is of first category. If $(E,\mu) \in \mathcal{H}$ then we can find finite collections $\mathcal{I}(n)$ of closed intervals such that

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \text{ and } \sum_{I\in\mathcal{I}} |I|^{\alpha+1/n} < 1/n$$

for all $n \geq 1$ and so E, and therefore $\operatorname{supp}\mu$, has Hausdorff dimension at least α and we are done. □

The proof of Lemma 24 is simplified by the following observation.

LEMMA 25. — *Suppose α, ψ and $(\mathcal{L}_\psi, d_\psi)$ are as in Lemma 19. If $(E,\mu) \in \mathcal{L}_\psi$ and $\epsilon > 0$, we can find an infinitely differentiable function $g : \mathbb{T} \to \mathbb{R}$ and a closed set H such that $(H, g\tau) \in \mathcal{L}_\psi$ and*

$$d_\psi\big((E,\mu),(H,g\tau)\big) < \epsilon.$$

Proof. — This is just a smoothing argument. If we take K_n as in the first paragraph of the proof of Lemma 16 and set

$$g = K_n * \mu,\ \ H = E + [-n^{-1}, n^{-1}],$$

then the required result holds, provided only that n is large enough. □

5. The main construction

Lemma 25 shows that Lemma 24 will follow at once from the following smoothed result.

LEMMA 26. — *Suppose that $n \geq 1$, $g : \mathbb{T} \to \mathbb{R}$ is an infinitely differentiable positive function with*

$$\int_{\mathbb{T}} g(t)dt = 1$$

and H is closed set with $H \supseteq \operatorname{supp} g$. Then, given $\epsilon > 0$, we can find an infinitely differentiable positive function $f : \mathbb{T} \to \mathbb{R}$ with

$$\int_{\mathbb{T}} f(t)dt = 1$$

and a closed set $E \supseteq \operatorname{supp} f$ such that $(E, f\tau) \in \mathcal{H}_n$ and

$$d_\psi\big((E,f\tau),(H,g\tau)\big) < \epsilon.$$

Proof. — Since $\mathcal{H}_n \supseteq \mathcal{H}_{n+1}$, we may restrict ourselves to the case when $\alpha + 4/n < 1$. Set $\alpha' = \alpha + 4/n$ and $\beta' = \alpha + 3/n$. By looking at what happens when $k \to \infty$, we see that we can choose k sufficiently large that

$$\frac{k\alpha' + 1}{k+1}, \ \frac{k\beta' - 1}{k+1} > \alpha + \frac{2}{n}.$$

Since g is infinitely differentiable, repeated integration by parts shows that there exists a constant C_1 such that

$$|\hat{g}(r)| \leq C_1 |r|^{-(2k+4)}$$

for $r \neq 0$ and so there exists a constant C such that

$$\bigstar \qquad \sum_{|r| \geq m} |r||\hat{g}(r)| \leq C|m|^{-(2k+2)}$$

for all $m \geq 1$.

Let m be a positive integer and let $\eta > 0$. Provided that m is sufficiently large, Lemma 17 (with α replaced by α' and β by β') tells us that we can find a positive infinitely differentiable function F_m which is periodic with period $1/(2m+1)$ such that

- (i)$_m$ $\|F_m * F_m - 1\|_\infty \leq \eta$.
- (ii)$_m$ $\|(F_m * F_m)'\| \leq m^{(1-k\alpha)}$.
- (iii)$_m$ $\|F_m\|_\infty \leq 4^{2k} m^{2k}$.
- (iv)$_m$ $\|F_m'\|_\infty \leq 4^{2k+1} m^{2k+1}$.
- (v)$_m$ $\int_{\mathbb{T}} F_m(t)dt = 1$.
- (vi)$_m$ We can find a finite collection of intervals $\mathcal{I}_m$ such that

$$\bigcup_{I \in \mathcal{I}_m} I \supseteq \operatorname{supp} F_m \text{ and } \sum_{I \in \mathcal{I}} |I|^{(1+k\alpha')/(k+1)} < \epsilon$$

and so

$$\sum_{I \in \mathcal{I}_m} |I|^{\alpha + 1/n} < \frac{1}{n}.$$

- (vii)$_m$ $\omega_\psi(F_m * F_m) \leq \eta$.
- (viii)$_m$ $|\hat{F}_m(r)| \leq \eta$ for all $r \neq 0$.

If we set $G_m(t) = g(t)F_m(t)$ and

$$f(t) = \left(\int_{\mathbb{T}} G_m(s)ds\right)^{-1} G_m(t),$$

then, automatically,

$$\operatorname{supp} f \subseteq E \cap \operatorname{supp} F_m.$$

Thus, by choosing an appropriate finite set A and setting $H = A \cup \operatorname{supp} f$, we can ensure that $(E, f\tau) \in \mathcal{L}_\psi$,

$$d_{\mathcal{G}}\big((E, f\tau), (H, g\tau)\big) < \epsilon/4.$$

and we can find a finite collection of intervals $\mathcal{I}$ such that

$$\bigcup_{I \in \mathcal{I}_m} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}_m} |I|^{\alpha + 1/n} < \frac{1}{n}.$$

We have shown that $(E, f\tau) \in \mathcal{H}_n$ and all we need to do is to show that, for appropriate choices of η and m, we have

$$\sup_{r \in \mathbb{Z}} |\hat{f}(r) - \hat{g}(r)| < \epsilon/4, \ \ \|f * f - g * g\|_\infty < \epsilon/4$$

and $\omega_\psi(f * f - g * g) < \epsilon/4$. Without loss of generality we may suppose $\epsilon < 1$, so simple calculations show that it is sufficient to prove

$$\sup_{r \in \mathbb{Z}} |\hat{g}(r) - \hat{G}_m(r)| < \epsilon/8, \ \ \|g * g - G_m * G_m\|_\infty < \epsilon/8$$

and $\omega_\psi(g * g - G_m * G_m) < \epsilon/8$.

Using $(\mathrm{v})_m$ we have

$$\begin{aligned}
\left|\hat{g}(r) - \hat{G}_m(r)\right| &= \left|\hat{g}(r) - \sum_{j=-\infty}^{\infty} \hat{g}(r-j)\hat{F}_m(j)\right| \\
&= \left|\sum_{u \neq 0} \hat{g}(r-u)\hat{F}_m(u)\right| \leq \sum_{u \neq 0} |\hat{g}(r-u)||\hat{F}_m(u)| \\
&\leq \sum_{u \neq 0} |\hat{g}(r-u)|\eta \leq \eta \sum_{j=-\infty}^{\infty} |\hat{g}(j)| < \epsilon/8
\end{aligned}$$

for all r, provided only that η is small enough. We now fix η once and for all so that the inequality just stated holds and

$$\eta < \big(\|g\|_\infty + \omega_\psi(g * g) + 1\big)^{-1} \epsilon/24$$

but we leave m free.

We have now arrived at the core of the proof which lies in showing that

$$\|g * g - G_m * G_m\|_\infty < \epsilon/8 \text{ and } \omega_\psi(g * g - G_m * G_m) < \epsilon/8$$

provided only that m is large enough. The proofs of the two inequalities are similar. We start with the first which is slightly easier.

We write

$$P_m(t) = \sum_{|r| \leq m} \hat{g}(r) \exp(irt).$$

By ★ we see that, if $m \geq 1$,

$$\bigstar\bigstar \qquad \|g - P_m\|_\infty, \|(g - P_m)'\|_\infty \leq C|m|^{-(2k+2)}.$$

We shall take m sufficiently large that $m \geq 1$ and

$$\|g - P_m\|_\infty, \|(g - P_m)'\|_\infty \leq 1.$$

Now, since F_m is periodic with period $1/(2m+1)$ and P_m is a trigonometric polynomial of degree at most m,

$$\widehat{P_m F_m}\big((2m+1)u + v\big) = \hat{F}_m((2m+1)u\big)\hat{P}_j(v)$$

for all u and v so that

$$\begin{aligned}\big((P_m F_m) * (P_m F_m)\big)\hat{}\,((2m+1)u + v\big) &= \big(\hat{F}_m((2m+1)u\big)\hat{P}_j(v)\big)^2 \\ &= \Big((P_m F_m)\hat{}\,((2m+1)u + v\big)\Big)^2 \\ &= \big((P_m * P_m)(F_m * F_m)\big)\hat{}\,((2m+1)u + v\big)\end{aligned}$$

and $$(P_m F_m) * (P_m F_m)(t) = (P_m * P_m)(t)(F_m * F_m)(t).$$

Using this equality, we obtain

$$\begin{aligned}\|g * g - G_m * G_m\|_\infty &= \|g * g - (gF_m) * (gF_m)\|_\infty \\ &\leq \|g * g - P_m * P_m\|_\infty + \|P_m * P_m - (P_m F_m) * (P_m F_m)\|_\infty \\ &\qquad + \|(P_m F_m) * (P_m F_m) - (gF_m) * (gF_m)\|_\infty \\ &= \|g * g - P_m * P_m\|_\infty + \|P_m * P_m - (P_m * P_m)(F_m * F_m)\|_\infty \\ &\qquad + \|(P_m F_m) * (P_m F_m) - (gF_m) * (gF_m)\|_\infty.\end{aligned}$$

We estimate the three terms separately.

First we observe that

$$\begin{aligned}\|g * g - P_m * P_m\|_\infty &= \|(g - P_m) * (g - P_m) + 2(g - P_m) * P_m\|_\infty \\ &\leq \|(g - P_m) * (g - P_m)\|_\infty + 2\|(g - P_m) * P_m\|_\infty \\ &\leq \|g - P_m\|_\infty^2 + 2\|g - P_m\|_\infty \|P_m\|_\infty \\ &\leq \|g - P_m\|_\infty^2 + 2\|g - P_m\|_\infty (1 + \|g\|_\infty) < \epsilon/12.\end{aligned}$$

provided only that m is large enough.

Next we observe that

$$\begin{aligned}&\|(P_m F_m) * (P_m F_m) - (gF_m) * (gF_m)\|_\infty \\ &\quad \leq \big\|\big((g - P_m)F_m\big) * \big((g - P_m)F_m\big)\|_\infty + 2\big\|\big((g - P_m)F_m\big) * (gP_m)\|_\infty\end{aligned}$$

$$\begin{aligned}
&\le \|(g-P_m)F_m\|_\infty^2 + 2\|(g-P_m)F_m\|_\infty \|gP_m\|_\infty \\
&\le (\|(g-P_m)\|_\infty \|F_m\|_\infty)^2 + 2\|g-P_m\|_\infty \|F_m\|_\infty \|g\|_\infty \|P_m\|_\infty \\
&\le (\|(g-P_m)\|_\infty \|F_m\|_\infty)^2 + 2|g-P_m\|_\infty \|F_m\|_\infty \|g\|_\infty (1+\|g\|_\infty) \\
&\le \left(\frac{C}{m^{2k+2}}4^{2k}m^{2k}\right)^2 + \frac{2C}{m^{2k+2}}4^{2k}m^{2k}\|g\|_\infty(1+\|g\|_\infty) < \epsilon/12
\end{aligned}$$

provided only that m is large enough.

Finally we note that

$$\begin{aligned}
\|P_m * P_m - (P_m * P_m)(F_m * F_m)\|_\infty &= \|(P_m * P_m)(1 - F_m * F_m)\|_\infty \\
&= \|P_m * P_m\|_\infty \|1 - F_m * F_m\|_\infty \\
&\le \|P_m\|_\infty^2 \|1 - F_m * F_m\|_\infty \\
&\le (1+\|g\|_\infty)^2 \eta < \epsilon/12
\end{aligned}$$

Combining our estimates we obtain $\|g * g - G_m * G_m\|_\infty < \epsilon/4$ as required.

We turn now to the second inequality. Much as before,

$$\begin{aligned}
\omega_\psi(g * g - G_m * G_m) = \omega_\psi(g * g - P_m * P_m)& \\
+ \omega_\psi\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big)& \\
+ \omega_\psi\big((P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\big)&.
\end{aligned}$$

We bound the first term.

$$\begin{aligned}
\omega_\psi(g * g - P_m * P_m) &\le \omega_\psi\big((g-P_m) * (g-P_m)\big) + 2\omega_\psi\big((g-P_m) * P_m\big) \\
&\le \big\|\big((g-P_m) * (g-P_m)\big)'\big\|_\infty + 2\big\|\big((g-P_m) * P_m\big)'\big\|_\infty \\
&= \|(g-P_m)' * (g-P_m)\|_\infty + 2\|(g-P_m)' * P_m\|_\infty \\
&\le \|(g-P_m)'\|_\infty \|g-P_m\|_\infty + 2\|(g-P_m)'\|_\infty \|P_m\|_\infty \\
&\le \|(g-P_m)'\|_\infty \|g-P_m\|_\infty + 2\|(g-P_m)'\|_\infty (1+\|g\|_\infty) < \epsilon/12
\end{aligned}$$

provided only that m is large enough.

Next we bound the third term.

$$\begin{aligned}
&\omega_\psi\big((P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\big) \\
&\le \omega_\psi\Big(\big((g-P_m)F_m\big) * \big((g-P_m)F_m\big)\Big) + 2\omega_\psi\Big(\big((g-P_m)F_m\big) * (gP_m)\Big) \\
&\le \Big\|\Big(\big((g-P_m)F_m\big) * \big((g-P_m)F_m\big)\Big)'\Big\|_\infty + 2\Big\|\Big(\big((g-P_m)F_m\big) * (gP_m)\Big)'\Big\|_\infty
\end{aligned}$$

$$\begin{aligned}
&= \big\|\big((g-P_m)F_m\big)' * \big((g-P_m)F_m\big)\big\| + 2\big\|\big((g-P_m)F_m\big)' * (gP_m)\big\|_\infty \\
&\le \|((g-P_m)F_m)'\|_\infty \|(g-P_m)F_m\|_\infty + 2\|((g-P_m)F_m)'\|_\infty \|gP_m\|_\infty \\
&\le (\|(g-P_m)'\|_\infty \|F_m\|_\infty + \|g-P_m\|_\infty \|F'_m\|_\infty)\|g-P_m\|_\infty \|F_m\|_\infty \\
&\qquad + 2(\|(g-P_m)'\|_\infty \|F_m\|_\infty + \|g-P_m\|_\infty \|F'_m\|_\infty)\|g\|_\infty \|P_m\|_\infty \\
&\le \left(\frac{C}{m^{2k+2}}4^{2k}m^{2k} + \frac{C}{m^{2k+2}}4^{2k+1}m^{2k+1}\right)\frac{C}{m^{2k+2}}4^{2k}m^{2k} \\
&\qquad\qquad + 2\left(\frac{C}{m^{2k+2}}4^{2k}m^{2k} + \frac{C}{m^{2k+2}}4^{2k+1}m^{2k+1}\right)\|g\|_\infty(1+\|g\|_\infty) \\
&< \frac{\epsilon}{12}
\end{aligned}$$

provided only that m is large enough.

Finally we estimate the second term

$$\begin{aligned}
\omega_\psi\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big) &= \omega_\psi\big((P_m * P_m)(1 - F_m * F_m)\big) \\
&\le \|P_m * P_m\|_\infty \omega_\psi(1 - F_m * F_m) + \omega_\psi(P_m * P_m)\|1 - F_m * F_m\|_\infty \\
&= \|P_m * P_m\|_\infty \omega_\psi(F_m * F_m) + \omega_\psi(P_m * P_m)\|1 - F_m * F_m\|_\infty \\
&\le \|P_m * P_m\|_\infty \eta + \omega_\psi(P_m * P_m)\eta.
\end{aligned}$$

Estimates of a familiar kind show that

$$\begin{aligned}
\omega_\psi(P_m * P_m) &\le \omega_\psi(g * g) + \omega_\psi(P_m * P_m - g * g) \\
&\le \omega_\psi(g * g) + \omega_\psi\big((P_m - g) * (P_m - g)\big) + 2\omega_\psi\big((P_m - g) * g\big) \\
&\le \omega_\psi(g * g) + \|(g-P_m)'\|_\infty \|g - P_m\|_\infty + 2\|(g-P_m)'\|_\infty \|g\|_\infty \\
&\le \omega_\psi(g * g) + 1
\end{aligned}$$

and, similarly,

$$\|P_m * P_m\|_\infty \le \|g * g\|_\infty + 1 \le \|g\|_\infty^2 + 1$$

provided only that m is large enough. Thus

$$\omega_\psi\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big) \quad \le (\omega_\psi(g * g) + \|g\|_\infty^2 + 2)\eta < \epsilon/12$$

provided only that m is large enough.

Combining our estimates we obtain

$$\omega_\psi(g * g - G_m * G_m) < \epsilon/4$$

and this completes the proof. □

6. The second theorem

The proof of Theorem 8 follows much the same path as that of Theorem 7, so we shall sketch it leaving the proofs to the reader.

We use a Baire category version in an appropriate metric space.

LEMMA 27. — *Let $1 \geq \alpha > 1/2$. As before, let us write Λ_β for the set of continuous function with*

$$\sup_{t,h\in\mathbb{T},h\neq 0} |h|^{-\beta}|f(t+h)-f(t)| < \infty$$

and set

$$\omega_\beta(f) = \sup_{t,h\in\mathbb{T},h\neq 0} |h|^{-\beta}|f(t+h)-f(t)|$$

whenever $f \in \Lambda_\beta$. Chose β_j so that

$$0 < \beta_j < \beta_{j+1} < \alpha - \tfrac{1}{2}$$

and $\beta_j \to \alpha - \frac{1}{2}$ as $j \to \infty$.

Now consider the metric space $(\mathcal{G}, d_{\mathcal{G}})$ introduced in Lemma 18. Let $\mathcal{M}$ be the space of those (E,μ) such that $\mu * \mu = f_\mu \tau$ with $f_\mu \in \bigcap_{\beta<1/2-\alpha}\Lambda_\beta$. If we take

$$d_{\mathcal{M}}\big((E,\mu),(F,\sigma)\big) = d_{\mathcal{G}}(E,F) + \sum_{j=1}^{\infty} 2^{-j}\min\big(1, \omega_{\beta_j}(f_\mu - f_\sigma)\big),$$

then $(\mathcal{M}, d_{\mathcal{M}})$ is a complete metric space.

We can now give a Baire version of Theorem 8.

THEOREM 28. — *Let $1 \geq \alpha > 1/2$. Suppose that $\psi : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $t^{-\beta}\psi(t) \to 0$ as $t \to 0+$ whenever $\beta < \alpha - \frac{1}{2}$. The complement of the set*

$$\mathcal{H} = \{(E,\mu) : E \text{ has Hausdorff } \psi \text{ measure zero}\}$$

is of first category in $(\mathcal{M}, d_{\mathcal{M}})$.

Theorem 28 follows from the following simpler result.

LEMMA 29. — *Suppose α, ψ and $(\mathcal{L}_\psi, d_\psi)$ are as in Lemma 27. Let $\mathcal{H}_n$ be the subset of consisting of those $(E,\mu) \in \mathcal{L}_\psi$ such that we can find a finite collection of intervals $\mathcal{I}$ with*

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \text{ and } \sum_{I\in\mathcal{I}} \psi(|I|) < 1/n.$$

Then $\mathcal{K}_n$ is dense in $(\mathcal{M}, d_{\mathcal{M}})$.

A smoothing argument shows that it is sufficient to prove the following form of Lemma 29.

LEMMA 30. — *Suppose that $n \geq 1$, $g : \mathbb{T} \to \mathbb{R}$ is an infinitely differentiable positive function with*

$$\int_{\mathbb{T}} g(t)dt = 1$$

and H is closed set with $H \supseteq \operatorname{supp} g$. Then, given $\epsilon > 0$, we can find an infinitely differentiable positive function $f : \mathbb{T} \to \mathbb{R}$ with

$$\int_{\mathbb{T}} f(t)dt = 1$$

and a closed set $E \supseteq \operatorname{supp} f$ such that $(E, f\tau) \in \mathcal{K}_n$ and

$$d_{\mathcal{K}}\big((E, f\tau), (H, g\tau)\big) < \epsilon.$$

The form of the metric $d_{\mathcal{M}}$ allows one further simplification. We observe that Lemma 30 follows at once from the following version.

LEMMA 31. — *Suppose that $g : \mathbb{T} \to \mathbb{R}$ is an infinitely positive differentiable function with*

$$\int_{\mathbb{T}} g(t)dt = 1$$

and H is closed set with $H \supseteq \operatorname{supp} g$. Then, given $\epsilon > 0$ and $N \geq 1$, we can find an infinitely differentiable positive function $f : \mathbb{T} \to \mathbb{R}$ with

$$\int_{\mathbb{T}} f(t)dt = 1$$

and a closed set $E \supseteq \operatorname{supp} f$ such that $(E, f\tau) \in \mathcal{K}_N$,

$$d_{\mathcal{G}}(E, F) < \epsilon$$

and

$$\omega_{\beta_N}(f * f - g * g) < \epsilon.$$

The proof of Lemma 31 is essentially the same as that of Lemma 26.

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