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ON THE STRUCTURE OF TRIANGULATED CATEGORIES WITH FINITELY MANY INDECOMPOSABLES

BY CLAIRE AMIOT

ABSTRACT. — We study the problem of classifying triangulated categories with finite-dimensional morphism spaces and finitely many indecomposables over an algebraically closed field k. We obtain a new proof of the following result due to Xiao and Zhu: the Auslander-Reiten quiver of such a category $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$ where Δ is a disjoint union of simply-laced Dynkin diagrams and G a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$. Then we prove that for 'most' groups G, the category $\mathcal{T}$ is standard, *i.e.* k-linearly equivalent to an orbit category $\mathcal{D}^b(\mod k\Delta)/\Phi$. This happens in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Moreover, if $\mathcal{T}$ is standard and algebraic, we can even construct a triangle equivalence between $\mathcal{T}$ and the corresponding orbit category. Finally we give a sufficient condition for the category of projectives of a Frobenius category to be triangulated. This allows us to construct non standard 1-Calabi-Yau categories using deformed preprojective algebras of generalized Dynkin type.

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Résumé (*Sur la structure des catégories triangulées*). — Cet article traite du problème de classification des catégories triangulées sur un corps algébriquement clos k dont les espaces de morphismes sont de dimension finie et avec un nombre fini d'indécomposables. Nous obtenons une nouvelle preuve du résultat suivant dû à Xiao et Zhu : le carquois d'Auslander-Reiten d'une telle catégorie $\mathcal{T}$ est de la forme $\mathbb{Z}\Delta/G$ où Δ est une union disjointe de diagrammes de Dynkin simplement lacés et G est un groupe d'automorphismes de $\mathbb{Z}\Delta$ faiblement admissible. Nous montrons ensuite que pour 'presque' tous groupes G, la catégorie $\mathcal{T}$ est standard, c'est-à-dire k-linéairement équivalente à une catégorie d'orbites $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$. C'est en particulier le cas lorsque $\mathcal{T}$ est maximale d-Calabi-Yau avec $d \geq 2$. De plus, si $\mathcal{T}$ est standard et algébrique, nous pouvons même construire une équivalence triangulée entre $\mathcal{T}$ et la catégorie d'orbites correspondante. Nous donnons finalemant une condition suffisante pour que la catégorie de projectifs d'une catégorie de Frobenius soit triangulée. Cela nous permet de construire des catégories 1-Calabi-Yau non standard en utilisant les algèbres préprojectives déformées de type Dynkin généralisé.

Introduction

Let k be an algebraically closed field and $\mathcal{T}$ a small Krull-Remak-Schmidt k-linear triangulated category (see [47]). We assume that

a) $\mathcal{T}$ is Hom-*finite*, *i.e.* the space $\operatorname{Hom}_{\mathcal{T}}(X,Y)$ is finite-dimensional for all objects X, Y of $\mathcal{T}$.

It follows that indecomposable objects of $\mathcal{T}$ have local endomorphism rings and that each object of $\mathcal{T}$ decomposes into a finite direct sum of indecomposables [17, 3.3]. We assume moreover that

b) $\mathcal{T}$ is *locally finite*, *i.e.* for each indecomposable X of $\mathcal{T}$, there are at most finitely many isoclasses of indecomposables Y such that $\operatorname{Hom}_{\mathcal{T}}(X,Y)\neq 0$.

It was shown in [48] that condition *b*) implies its dual. Condition *b*) holds in particular if we have

b′) $\mathcal{T}$ is *additively finite*, *i.e.* there are only finitely many isomorphism classes of indecomposables in $\mathcal{T}$.

The study of particular classes of such triangulated categories $\mathcal{T}$ has a long history. Let us briefly recall some of its highlights:

1) If A is a representation-finite selfinjective algebra, then the stable category $\mathcal{T}$ of finite-dimensional (right) A-modules satisfies our assumptions and is additively finite. The structure of the underlying k-linear category of $\mathcal{T}$ was determined by C. Riedtmann in [39], [40], [41] and [42].

2) In [21], D. Happel showed that the bounded derived category of the category of finite-dimensional representations of a representation-finite quiver is locally finite and described its underlying k-linear category.

3) The stable category $\underline{\mathrm{CM}}(R)$ of Cohen-Macaulay modules over a commutative complete local Gorenstein isolated singularity R of dimension d is a Hom-finite triangulated category which is $(d-1)$-Calabi-Yau (cf. for example [28] and [50]). In [4], M. Auslander and I. Reiten showed that if the dimension of R is 1, then the category $\underline{\mathrm{CM}}(R)$ is additively finite and computed the shape of the components of its Auslander-Reiten quiver.

4) The cluster category $\mathcal{C}_Q$ of a finite quiver Q without oriented cycles was introduced in [12] if Q is an orientation of a Dynkin diagram of type $\mathbb{A}$ and in [11] in the general case. The category $\mathcal{C}_Q$ is triangulated [30] and, if Q is representation-finite, satisfies a) and b$'$).

In a recent article [48], J. Xiao and B. Zhu determined the structure of the Auslander-Reiten quiver of a locally finite triangulated category. In this paper, we obtain the same result with a new proof in Section 4, namely that each connected component of the Auslander-Reiten quiver of the category $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$, where Δ is a simply-laced Dynkin diagram and G is trivial or a weakly admissible group of automorphisms. Contrary to J. Xiao and B. Zhu, we do not discuss separately the case where the Auslander-Reiten contains a loop.

We are interested in the k-linear structure of $\mathcal{T}$. If the Auslander-Reiten quiver of $\mathcal{T}$ is of the form $\mathbb{Z}\Delta$, we show that the category $\mathcal{T}$ is standard, *i.e.* it is equivalent to the mesh category $k(\mathbb{Z}\Delta)$. Then in Section 6, we prove that $\mathcal{T}$ is standard if the number of vertices of $\Gamma = \mathbb{Z}\Delta/G$ is strictly greater than the number of isoclasses of indecomposables of $\mathrm{mod}\, k\Delta$. In the last section, using [8] we construct examples of non standard triangulated categories such that $\Gamma = \mathbb{Z}\Delta/\tau$.

Finally, in the standard cases, we are interested in the triangulated structure of $\mathcal{T}$. For this, we need to make additional assumptions on $\mathcal{T}$. If the Auslander-Reiten quiver is of the form $\mathbb{Z}\Delta$, and if $\mathcal{T}$ is the base of a tower of triangulated categories [29], we show that there is a triangle equivalence between $\mathcal{T}$ and the derived category $\mathcal{D}^b(\mathrm{mod}\, k\Delta)$. For the additively finite cases, we have to assume that $\mathcal{T}$ is standard and algebraic in the sense of [31]. We then show that $\mathcal{T}$ is (algebraically) triangle equivalent to the orbit category of $\mathcal{D}^b(\mathrm{mod}\, k\Delta)$ under the action of a weakly admissible group of automorphisms. In particular, for each $d \geq 2$, the algebraic triangulated categories with finitely many indecomposables which are maximal Calabi-Yau of CY-dimension d are parametrized by the simply-laced Dynkin diagrams.

Our results apply in particular to many stable categories $\underline{\mathrm{mod}}\, A$ of representation-finite selfinjective algebras A. These algebras were classified up to stable equivalence by C. Riedtmann [40], [42] and H. Asashiba [1]. In [9], J. Białkowski and A. Skowroński give a necessary and sufficient condition on these algebras so that their stable categories $\underline{\mathrm{mod}}\,A$ are Calabi-Yau. In [26] and [27], T. Holm and P. Jørgensen prove that certain stable categories $\underline{\mathrm{mod}}\,A$ are in fact d-cluster categories. These results can also be proved using our Corollary 7.3.

This paper is organized as follows: In Section 1, we prove that $\mathcal{T}$ has Auslander-Reiten triangles. Section 2 is dedicated to definitions about stable valued translation quivers and admissible automorphisms groups [23], [24], [14]. We show in Section 3 that the Auslander-Reiten quiver of $\mathcal{T}$ is a stable valued quiver and in Section 4, we reprove the result of J. Xiao and B. Zhu [48]: The Auslander-Reiten quiver is a disjoint union of quivers $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$, and G a weakly admissible group of automorphisms. In Section 5, we construct a covering functor $\mathcal{D}^b(\mathrm{mod}\, k\Delta) \to \mathcal{T}$ using Riedtmann's method [39]. Then, in Section 6, we exhibit some combinatorial cases in which $\mathcal{T}$ has to be standard, in particular when $\mathcal{T}$ is maximal d-Calabi-Yau with $d \geq 2$. Section 7 is dedicated to the algebraic case. If $\mathcal{T}$ is algebraic and standard, we can construct a triangle equivalence between $\mathcal{T}$ and an orbit category. If $\mathcal{P}$ is a k-category such that $\mathrm{mod}\,\mathcal{P}$ is a Frobenius category satisfying certain conditions, we will prove in Section 8 that $\mathcal{P}$ has naturally a triangulated structure. This allows us to deduce in Section 9 that the category $\mathrm{proj}\, P^f(\Delta)$ of the projective modules over a deformed preprojective algebra of generalized Dynkin type [8] is naturally triangulated and to reduce the classification of the additively finite triangulated categories which are 1-Calabi-Yau to that of the deformed preprojective algebras in the sense of [8]. In particular, thanks to [8], we obtain the existence of non standard 1-Calabi-Yau categories in characteritic 2. Using our results and an extension of those of [8], Białkowski and Skowroński have recently proved [10] the existence of non standard 1-Calabi-Yau categories in characteristic 3. This is noteworthy since in characteristic different from 2, additively finite *module categories* are always standard [6].

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Notation and terminology

We work over an algebraically closed field k. By a *triangulated category*, we mean a k-linear triangulated category $\mathcal{T}$. We write S for the suspension functor of $\mathcal{T}$ and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ for a distinguished triangle. We say that $\mathcal{T}$ is Hom-*finite* if for each pair X, Y of objects in $\mathcal{T}$, the space

$\mathrm{Hom}_{\mathcal{T}}(X,Y)$ is finite-dimensional over k. The category $\mathcal{T}$ will be called a *Krull-Remak-Schmidt* category if each object is isomorphic to a finite direct sum of indecomposable objects with unicity (up to reordering) of this decomposition, and if the endomorphism ring of an indecomposable object is a local ring. This implies that idempotents of $\mathcal{T}$ split, *i.e.* if e is an idempotent of X, then $e = \sigma\rho$ where σ is a section and ρ is a retraction [22, I, 3.2]. The category $\mathcal{T}$ will be called *locally finite* if for each indecomposable X of $\mathcal{T}$, there are only finitely many isoclasses of indecomposables Y such that $\mathrm{Hom}_{\mathcal{T}}(X,Y) \neq 0$. This property is selfdual by [48, prop. 1.1].

The *Serre functor* will be denoted by ν (see definition in Section 1). The *Auslander-Reiten translation* will always be denoted by τ (Section 1).

Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories. An *S-functor* (F,ϕ) is given by a k-linear functor $F : \mathcal{T} \to \mathcal{T}'$ and a functor isomorphism ϕ between the functors $F \circ S$ and $S' \circ F$, where S is the suspension of $\mathcal{T}$ and S' the suspension of $\mathcal{T}'$. The notion of ν-functor, or τ-functor is then clear. A *triangle functor* is an S-functor (F,ϕ) such that for each triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$ of $\mathcal{T}$, the sequence $FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{\phi_U \circ Fw} S'FU$ is a triangle of $\mathcal{T}'$.

The category $\mathcal{T}$ is *Calabi-Yau* if there exists an integer $d > 0$ such that we have a triangle functor isomorphism between S^d and ν. We say that $\mathcal{T}$ is *maximal d-Calabi-Yau* if $\mathcal{T}$ is d-Calabi-Yau and if for each covering functor $\mathcal{T}' \to \mathcal{T}$ with $\mathcal{T}'$ d-Calabi-Yau, we have a k-linear equivalence between $\mathcal{T}$ and $\mathcal{T}'$.

For an additive k-category $\mathcal{E}$, we write $\mathrm{mod}\,\mathcal{E}$ for the category of contravariant finitely presented functors from $\mathcal{E}$ to $\mathrm{mod}\,k$ (Section 8), and if the projectives of $\mathrm{mod}\,\mathcal{E}$ coincide with the injectives, $\underline{\mathrm{mod}}\,\mathcal{E}$ will be the *stable category.*

1. Serre duality and Auslander-Reiten triangles

1.1. Serre duality. — Recall from [38] that a *Serre functor* for $\mathcal{T}$ is an autoequivalence $\nu : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\,\mathrm{Hom}_{\mathcal{T}}(X,?) \simeq \mathrm{Hom}_{\mathcal{T}}(?,\nu X)$ for each $X \in \mathcal{T}$, where D is the duality $\mathrm{Hom}_k(?,k)$.

THEOREM 1.1. — *Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Then $\mathcal{T}$ has a Serre functor ν.*

Proof. — Let X be an object of $\mathcal{T}$. We write $X^\wedge$ for the functor $\mathrm{Hom}_{\mathcal{T}}(?,X)$ and F for the functor $D\,\mathrm{Hom}_{\mathcal{T}}(X,?)$. Using the lemma [38, I.1.6] we just have to show that F is representable. Indeed, the category $\mathcal{T}^{\mathrm{op}}$ is locally finite as well. The proof is in two steps.

Step 1: The functor F is finitely presented. — Let $Y_1,\ldots,Y_r$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$ such that FY_i is not zero. The space $\mathrm{Hom}(Y_i^\wedge, F)$ is finite-dimensional over k. Indeed it is isomorphic to FY_i by the Yoneda lemma. Therefore, the functor $\mathrm{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge$ is representable. We get an epimorphism from a representable functor to F:

$$\bigoplus_{i=1}^{r} \mathrm{Hom}(Y_i^\wedge, F) \otimes_k Y_i^\wedge \longrightarrow F.$$

By applying the same argument to its kernel we get a projective presentation of F of the form $U^\wedge \to V^\wedge \to F \to 0$, with U and V in $\mathcal{T}$.

Step 2: A cohomological functor $H : \mathcal{T}^{\mathrm{op}} \to \mathrm{mod}\, k$ *is representable if and only if it is finitely presented.* — Let

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{\phi} H \to 0$$

be a presentation of H. We form a triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} SU$. We get an exact sequence

$$U^\wedge \xrightarrow{u^\wedge} V^\wedge \xrightarrow{v^\wedge} W^\wedge \xrightarrow{w^\wedge} (SU)^\wedge.$$

Since the composition of ϕ with $u^\wedge$ is zero and H is cohomological, the morphism ϕ factors through $v^\wedge$. But H is the cokernel of $u^\wedge$, so $v^\wedge$ factors through ϕ. We obtain a commutative diagram

$$\begin{array}{ccccccc} U^\wedge & \xrightarrow{u^\wedge} & V^\wedge & \xrightarrow{v^\wedge} & W^\wedge & \xrightarrow{w^\wedge} & SU. \\ & & {\scriptstyle\phi}\downarrow & \overset{i}{\nearrow}\swarrow_{\phi'} & & & \\ & & H & & & & \end{array}$$

The equality $\phi' \circ i \circ \phi = \phi' \circ v^\wedge = \phi$ implies that $\phi' \circ i$ is the identity of H because ϕ is an epimorphism. We deduce that H is a direct factor of $W^\wedge$. The composition $i \circ \phi' = e^\wedge$ is an idempotent. Then $e \in \mathrm{End}(W)$ splits and we get $H = W'^\wedge$ for a direct factor W' of W. □

1.2. Auslander-Reiten triangles

DEFINITION 1.2.1 (see [21]). — A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $\mathcal{T}$ is called an *Auslander-Reiten triangle* or *AR-triangle* if the following conditions are satisfied:

(AR1) X and Z are indecomposable objects;

(AR2) $w \neq 0$;

(AR3) if $f : W \to Z$ is not a retraction, there exists $f' : W \to Y$ such that $vf' = f$;

(AR3′) if $g : X \to V$ is not a section, there exists $g' : Y \to V$ such that $g'u = g$.

Let us recall that, if (AR1) and (AR2) hold, the conditions (AR3) and (AR3′) are equivalent. We say that a triangulated category $\mathcal{T}$ *has Auslander-Reiten triangles* if, for any indecomposable object Z of $\mathcal{T}$, there exists an AR-triangle ending at Z: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$. In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of Z.

The following proposition is proved in [38, Prop. I.2.3].

PROPOSITION 1.2. — *Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Then the category $\mathcal{T}$ has Auslander-Reiten triangles.*

The composition $\tau = S^{-1}\nu$ is called the *Auslander-Reiten translation*. An AR-triangle of $\mathcal{T}$ ending at Z has the form

$$\tau Z \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \nu Z.$$

2. Valued translation quivers and automorphism groups

2.1. Translation quivers. — We recall some definitions and notations concerning quivers [14]. A quiver $Q = (Q_0, Q_1, s, t)$ is given by the set Q_0 of its vertices, the set Q_1 of its arrows, a source map s and a tail map t. If $x \in Q_0$ is a vertex, we denote by x^+ the set of direct successors of x, and by x^- the set of its direct predecessors. We say that Q is *locally finite* if for each vertex $x \in Q_0$, there are finitely many arrows ending at x and starting at x (in this case, x^+ and x^- are finite sets). The quiver Q is said to be *without double arrows*, if two different arrows cannot have the same tail and source.

DEFINITION 2.1.1. — A *stable translation quiver* (Q, τ) is a locally finite quiver without double arrows with a bijection $\tau : Q_0 \to Q_0$ such that

$$(\tau x)^+ = x^- \quad \text{for each vertex } x.$$

For each arrow $\alpha : x \to y$, let $\sigma\alpha$ be the unique arrow $\tau y \to x$.

Note that a stable translation quiver can have loops.

DEFINITION 2.1.2. — A *valued translation quiver* (Q, τ, a) is a stable translation quiver (Q, τ) with a map $a : Q_1 \to \mathbb{N}$ such that

$$a(\alpha) = a(\sigma\alpha) \quad \text{for each arrow } \alpha.$$

If α is an arrow from x to y, we write a_{xy} instead of $a(\alpha)$.

DEFINITION 2.1.3. — Let Δ be an oriented tree. The *repetition of* Δ is the quiver $\mathbb{Z}\Delta$ defined as follows:

- $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$,

- $(\mathbb{Z}\Delta)_1 = \mathbb{Z} \times \Delta_1 \cup \sigma(\mathbb{Z} \times \Delta_1)$ with arrows
$$(n, \alpha) : (n, x) \longrightarrow (n, y) \quad \text{and} \quad \sigma(n, \alpha) : (n-1, y) \to (n, x)$$
for each arrow $\alpha : x \to y$ of Δ.

The quiver $\mathbb{Z}\Delta$ with the translation $\tau(n, x) = (n-1, x)$ is clearly a stable translation quiver which does not depend (up to isomorphism) on the orientation of Δ (see [39]).

2.2. Groups of weakly admissible automorphisms

Definition 2.2.1. — An automorphism group G of a quiver is said to be *admissible* [39] if no orbit of G intersects a set of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one point. It said to be *weakly admissible* [14] if, for each $g \in G - \{1\}$ and for each $x \in Q_0$, we have $x^+ \cap (gx)^+ = \varnothing$.

Note that an admissible automorphism group is a weakly admissible automorphism group. Let us fix a numbering and an orientation of the simply-laced Dynkin trees.

$$\mathbb{A}_n : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

$$\mathbb{D}_n : \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-2 \begin{array}{l} \swarrow n-1 \\ \\ \nwarrow n \end{array}$$

$$\begin{array}{lccccccccc} & & & & 4 & & & & & \\ & & & & \uparrow & & & & & \\ \mathbb{E}_n : & 1 & \longleftarrow & 2 & \longleftarrow 3 & \longrightarrow & 5 & \longrightarrow & \cdots \longrightarrow & n \end{array}$$

Let Δ be a Dynkin tree. We define an automorphism S of $\mathbb{Z}\Delta$ as follows:

— if $\Delta = \mathbb{A}_n$, then $S(p, q) = (p+q, n+1-q)$;
— if $\Delta = \mathbb{D}_n$ with n even, then $S = \tau^{-n+1}$;
— if $\Delta = \mathbb{D}_n$ with n odd, then $S = \tau^{-n+1}\phi$ where ϕ is the automorphism of $\mathbb{D}_n$ which exchanges n and $n-1$;
— if $\Delta = \mathbb{E}_6$, then $S = \phi\tau^{-6}$ where ϕ is the automorphism of $\mathbb{E}_6$ which exchanges 2 and 5, and 1 and 6;
— if $\Delta = \mathbb{E}_7$, then $S = \tau^{-9}$;
— and if $\Delta = \mathbb{E}_8$, then $S = \tau^{-15}$.

In [39, Anhang 2], Riedtmann describes all admissible automorphism groups of Dynkin diagrams. Here is a more precise result in which we describe all weakly admissible automorphism groups of Dynkin diagrams.

Theorem 2.1. — *Let Δ be a Dynkin tree and G a non trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. Then G is isomorphic to $\mathbb{Z}$, and here is a list of its possible generators:*

- *if $\Delta = \mathbb{A}_n$ with n odd, possible generators are τ^r and $\phi\tau^r$ with $r \geq 1$, where $\phi = \tau^{\frac{1}{2}(n+1)}S$ is an automorphism of $\mathbb{Z}\Delta$ of order 2;*
- *if $\Delta = \mathbb{A}_n$ with n even, then possible generators are ρ^r, where $r \geq 1$ and where $\rho = \tau^{\frac{1}{2}n}S$ (since $\rho^2 = \tau^{-1}$, τ^r is a possible generator);*
- *if $\Delta = \mathbb{D}_n$ with $n \geq 5$, then possible generators are τ^r and $\tau^r\phi$, where $r \geq 1$ and where $\phi = (n-1, n)$ is the automorphism of $\mathbb{D}_n$ exchanging n and $n-1$;*
- *if $\Delta = \mathbb{D}_4$, then possible generators are $\phi\tau^r$, where $r \geq 1$ and where ϕ belongs to $\mathfrak{S}_3$ the permutation group on three elements seen as subgroup of automorphisms of $\mathbb{D}_4$;*
- *if $\Delta = \mathbb{E}_6$, then possible generators are τ^r and $\phi\tau^r$, where $r \geq 1$ and where ϕ is the automorphism of $\mathbb{E}_6$ exchanging 2 and 5, and 1 and 6;*
- *if $\Delta = \mathbb{E}_n$ with $n = 7, 8$, possible generators are τ^r, where $r \geq 1$.*

The unique weakly admissible automorphism group which is not admissible exists for $\mathbb{A}_n$, n even, and is generated by ρ.

3. Property of the Auslander-Reiten translation

We define the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of the category $\mathcal{T}$ as a valued quiver (Γ, a). The vertices are the isoclasses of indecomposable objects. Given two indecomposable objects X and Y of $\mathcal{T}$, we draw one arrow from $x = [X]$ to $y = [Y]$ if the vector space $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ is not zero, where $\mathcal{R}(?,?)$ is the radical of the bifunctor $\mathrm{Hom}_{\mathcal{T}}(?,?)$. A morphism of $\mathcal{R}(X,Y)$ which does not vanish in the quotient $\mathcal{R}(X,Y)/\mathcal{R}^2(X,Y)$ will be called *irreducible*. Then we put

$$a_{xy} = \dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y).$$

Remark that the fact that $\mathcal{T}$ is locally finite implies that its AR-quiver is locally finite. The aim of this section is to show that $\Gamma_{\mathcal{T}}$ with the translation τ defined in the first part is a valued translation quiver. In other words, we want to show the proposition:

Proposition 3.1. — *If X and Y are indecomposable objects of $\mathcal{T}$, we have*

$$\dim_k \mathcal{R}(X,Y)/\mathcal{R}^2(X,Y) = \dim_k \mathcal{R}(\tau Y, X)/\mathcal{R}^2(\tau Y, X).$$

Let us recall some definitions [22].

Definition 3.0.2. — A morphism $g : Y \to Z$ is called *sink morphism* if the following hold

1) g is not a retraction;
2) if $h : M \to Z$ is not a retraction, then h factors through g;
3) if u is an endomorphism of Y which satisfies $gu = g$, then u is an automorphism.

Dually, a morphism $f : X \to Y$ is called *source morphism* if the following hold:

1) f is not a section;
2) if $h : X \to M$ is not a section, then h factors through f;
3) if u is an endomorphism of Y which satisfies $uf = f$, then u is an automorphism.

These conditions imply that X and Z are indecomposable. Obviously, if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ is an AR-triangle, then u is a source morphism and v is a sink morphism. Conversely, if $v \in \mathrm{Hom}_{\mathcal{T}}(Y, Z)$ is a sink morphism (or if $u \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ is a source morphism), then there exists an AR-triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ (see [22, I, 4.5]).

The following lemma (and the dual statement) is proved in [43, 2.2.5].

LEMMA 3.2. — *Let g be a morphism from Y to Z, where Z is indecomposable and $Y = \bigoplus_{i=1}^{r} Y_i^{n_i}$ is the decomposition of Y into indecomposables. Then the morphism g is a sink morphism if and only if the following hold:*

1) *For each $i = 1, \dots, r$ and $j = 1, \dots, n_i$, the restriction $g_{i,j}$ of g to the j-th component of the i-th isotopic part of Y belongs to the radical $\mathcal{R}(Y_i, Z)$.*
2) *For each $i = 1, \dots, r$, the family $(\bar{g}_{i,j})_{j=1,\dots,n_i}$ forms a k-basis of the space $\mathcal{R}(Y_i, Z)/\mathcal{R}^2(Y_i, Z)$.*
3) *If $h \in \mathrm{Hom}_{\mathcal{T}}(Y', Z)$ is irreducible and Y' indecomposable, then h factors through g and Y' is isomorphic to Y_i for some i.*

Using this lemma, it is easy to see that Proposition 3.1 holds. Thus, the Auslander-Reiten quiver $\Gamma_{\mathcal{T}} = (\Gamma, \tau, a)$ of the category $\mathcal{T}$ is a valued translation quiver.

4. Structure of the Auslander-Reiten quiver

This section is dedicated to another proof of a theorem due to J. Xiao and B. Zhu:

THEOREM 4.1 (see [49]). — *Let $\mathcal{T}$ be a Krull-Remak-Schmidt, locally finite triangulated category. Let Γ be a connected component of the AR-quiver of $\mathcal{T}$. Then there exists a Dynkin tree Δ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, a weakly admissible automorphism group G of $\mathbb{Z}\Delta$ and an isomorphism of valued translation quivers*

$$\theta : \Gamma \xrightarrow{\sim} \mathbb{Z}\Delta/G.$$

The underlying graph of the tree Δ *is unique up to isomorphism (it is called the* type of Γ*), and the group* G *is unique up to conjugacy in* $\mathrm{Aut}(\mathbb{Z}\Delta)$*. In particular, if* $\mathcal{T}$ *has an infinite number of isoclasses of indecomposable objects, then* G *is trivial, and* Γ *is the repetition quiver* $\mathbb{Z}\Delta$*.*

4.1. Auslander-Reiten quivers with a loop. — In this section, we suppose that the Auslander-Reiten quiver of $\mathcal{T}$ contains a loop, *i.e.* there exists an arrow with same tail and source. Thus, we suppose that there exists an indecomposable X of $\mathcal{T}$ such that

$$\dim_k \mathcal{R}(X,X)/\mathcal{R}^2(X,X) \geq 1.$$

PROPOSITION 4.2. — *Let* X *be an indecomposable object of* $\mathcal{T}$*. Suppose that we have* $\dim_k \mathcal{R}(X,X)/\mathcal{R}^2(X,X) \geq 1$*. Then* τX *is isomorphic to* X*.*

To prove this, we need a lemma.

LEMMA 4.3. — *Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1}$$

be a sequence of irreducible morphisms between indecomposable objects with $n \geq 2$*. If the composition* $f_n \circ f_{n-1} \circ \cdots \circ f_1$ *is zero, then there exists an* i *such that* $\tau^{-1}X_i$ *is isomorphic to* X_{i+2}*.*

Proof. — The proof proceeds by induction on n. Let us show the assertion for $n = 2$. Suppose $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence such that $f_2 \circ f_1 = 0$. We can then construct an AR-triangle:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{(f_1,f)^T} & X_2 \oplus X & \xrightarrow{(g_1,g_2)} & \tau^{-1}X_1 & \longrightarrow & SX_1 \\ & & \downarrow{\scriptstyle (f_2,0)} & \swarrow{\scriptstyle \beta} & & & \\ & & X_3 & & & & \end{array}$$

The composition $f_2 \circ f_1$ is zero, thus the morphism f_2 factors through g_1. As the morphisms g_1 and f_2 are irreducible, we conclude that β is a retraction, and X_3 a direct summand of $\tau^{-1}X_1$. But X_1 is indecomposable, so β is an isomorphism between X_3 and $\tau^{-1}X_1$.

Now suppose that the property holds for an integer $n-1$ and that we have $f_n \circ f_{n-1} \circ \cdots \circ f_1 = 0$. If the composition $f_{n-1} \circ \cdots \circ f_1$ is zero, the proposition holds by induction. So we can suppose that for $i \leq n-2$, the objects $\tau^{-1}X_i$ and X_{i+2} are not isomorphic. We show now by induction on i that for each $i \leq n-1$, there exists a map $\beta_i : \tau^{-1}X_i \to X_{n+1}$ such that $f_n \circ \cdots \circ f_{i+1} = \beta_i g_i$ where $g_i : X_{i+1} \to \tau^{-1}X_i$ is an irreducible morphism. For $i = 1$, we construct an AR-triangle:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{(f_1, f_1')^T} & X_2 \oplus X_1' & \xrightarrow{(g_1, g_1')} & \tau^{-1}X_1 & \longrightarrow & SX_1 \\ & & \Big\downarrow{\scriptstyle (f_n \circ \cdots \circ f_2, 0)} & \swarrow{\scriptstyle \beta_1} & & & \\ & & X_{n+1} & & & & \end{array}$$

As the composition $f_n \circ \cdots \circ f_1$ is zero, we have the factorization $f_n \circ \cdots \circ f_2 = \beta_1 g_1$.

Now for i, as $\tau^{-1}X_{i-1}$ is not isomorphic to X_{i+1}, there exists an AR-triangle of the form

$$\begin{array}{ccccccc} X_i & \xrightarrow{(g_{i-1}, f_i, f_i')^T} & \tau^{-1}X_{i-1} \oplus X_{i+1} \oplus X_i' & \xrightarrow{(g_i'', g_i, g_i')} & \tau^{-1}X_i & \longrightarrow & SX_i \\ & & \Big\downarrow{\scriptstyle (-\beta_{i-1}, f_n \circ \cdots \circ f_{i+1}, 0)} & \swarrow{\scriptstyle \beta_i} & & & \\ & & X_{n+1} & & & & \end{array}$$

By induction, $-\beta_{i-1} g_{i-1} + f_n \circ \cdots \circ f_{i+1} f_i$ is zero, thus $f_n \circ \cdots \circ f_{i+1}$ factors through g_i. This property is true for $i = n-1$, so we have a map $\beta_{n-1} : \tau^{-1}X_{n-1} \to X_{n+1}$ such that $\beta_{n-1} g_{n-1} = f_n$. As g_{n-1} and f_n are irreducible, we conclude that β_{n-1} is an isomorphism between X_{n+1} and $\tau^{-1}X_{n-1}$. □

Now we are able to prove Proposition 4.2. There exists an irreducible map $f : X \to X$. Suppose that X and τX are not isomorphic. Then from the previous lemma, the endomorphism f^n is non zero for each n. But since $\mathcal{T}$ is a Krull-Remak-Schmidt, locally finite category, a power of the radical $\mathcal{R}(X, X)$ vanishes. This is a contradiction.

4.2. Proof of Theorem 4.1.

— Let $\widetilde{\Gamma} = (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \tilde{a})$ be the valued quiver obtained from Γ by removing the loops, *i.e.* we have

$$\widetilde{\Gamma}_0 = \Gamma_0, \quad \widetilde{\Gamma}_1 = \{\alpha \in \Gamma_1 \text{ such that } s(\alpha) \neq t(\alpha)\}, \quad \text{and} \quad \tilde{a} = a_{|\widetilde{\Gamma}_1}.$$

Lemma 4.4. — *The quiver $\widetilde{\Gamma} = (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \tilde{a})$ with the translation τ is a valued translation quiver without loop.*

Proof. — We have to check that the map σ is well-defined. But from Proposition 4.2, if α is a loop on a vertex x, $\sigma(\alpha)$ is the unique arrow from $\tau x = x$ to x, *i.e.* $\sigma(\alpha) = \alpha$. Thus $\widetilde{\Gamma}$ is obtained from Γ by removing some σ-orbits and it keeps the structure of stable valued translation quiver. □

Now, we can apply Riedtmann's Struktursatz [39] and the result of Happel-Preiser-Ringel [24]. There exist a tree Δ and an admissible automorphism group G (which may be trivial) of $\mathbb{Z}\Delta$ such that $\widetilde{\Gamma}$ is isomorphic to $\mathbb{Z}\Delta/G$ as a valued translation quiver. The underlying graph of the tree Δ is then unique up to isomorphism and the group G is unique up to conjugacy in $\mathrm{Aut}(\mathbb{Z}\Delta)$. Let x be a vertex of Δ. We write $\overline{x}$ for the image of x by the map:

$$\Delta \to \mathbb{Z}\Delta \xrightarrow{\pi} \mathbb{Z}\Delta/G \simeq \widetilde{\Gamma} \hookrightarrow \Gamma.$$

Let $C : \Delta_0 \times \Delta_0 \to \mathbb{Z}$ be the matrix defined as follows:

- $C(x,y) = -a_{\overline{x}\,\overline{y}}$ (resp. $-a_{\overline{y}\,\overline{x}}$) if there exists an arrow from x to y (resp. from y to x) in Δ,
- $C(x,x) = 2 - a_{\overline{x}\,\overline{x}}$,
- $C(x,y) = 0$ otherwise.

The matrix C is symmetric; it is a 'generalized Cartan matrix' in the sense of [23]. If we remove the loops from the 'underlying graph of C' (in the sense of [23]), we get the underlying graph of Δ.

In order to apply the result of Happel-Preiser-Ringel [23, Section 2], we have to show:

Lemma 4.5. — *The set Δ_0 of vertices of Δ is finite.*

Proof. — Riedtmann's construction of Δ is the following. We fix a vertex x_0 in $\widetilde{\Gamma}_0$. Then the vertices of Δ are the paths of $\widetilde{\Gamma}$ beginning on x_0 and which do not contain subpaths of the form $\alpha\sigma(\alpha)$, where α is in $\widetilde{\Gamma}_1$. Now suppose that Δ_0 is an infinite set. Then for each n, there exists a sequence

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1} \xrightarrow{\alpha_n} x_n$$

such that $\tau x_{i+2} \neq x_i$. Then there exist some indecomposables $X_0, \ldots, X_n$ such that the vector space $\mathcal{R}(X_{i-1}, X_i)/\mathcal{R}^2(X_{i-1}, X_i)$ is not zero. Thus from Lemma 4.3, there exists irreducible morphisms $f_i : X_{i-1} \to X_i$ such that the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$ does not vanish. But the functor $\mathrm{Hom}_{\mathcal{T}}(X_0, ?)$ has finite support. Thus there is an indecomposable Y which appears an infinite number of times in the sequence $(X_i)_i$. But since $\mathcal{R}^N(Y,Y)$ vanishes for an N, we have a contradiction. □

Let $\mathcal{S}$ a system of representatives of isoclasses of indecomposables of $\mathcal{T}$. For an indecomposable Y of $\mathcal{T}$, we put

$$\ell(Y) = \sum_{M \in \mathcal{S}} \dim_k \mathrm{Hom}_{\mathcal{T}}(M, Y).$$

This sum is finite since $\mathcal{T}$ is locally finite.

Lemma 4.6. — *For x in Δ_0, we write $d_x = \ell(\overline{x})$. Then for each $x \in \Delta_0$, we have*

$$\sum_{y\in\Delta_0} d_y C_{xy} = 2.$$

Proof. — Let X and U be indecomposables of $\mathcal{T}$. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

be an AR-triangle. We write $(U,?)$ for the cohomological functor $\operatorname{Hom}_{\mathcal{T}}(U,?)$. Thus, we have a long exact sequence

$$(U,S^{-1}Z) \xrightarrow{S^{-1}w_*} (U,X) \xrightarrow{u_*} (U,Y) \xrightarrow{v_*} (U,Z) \xrightarrow{w_*} (U,SX).$$

Let $S_Z(U)$ be the image of the map w_*. We have the exact sequence:

$$0 \to S_{S^{-1}Z}(U) \to (U,X) \xrightarrow{u_*} (U,Y) \xrightarrow{v_*} (U,Z) \xrightarrow{w_*} S_Z(U) \to 0.$$

Thus we have the equality

$$\dim_k S_Z(U) + \dim_k S_{S^{-1}Z}(U) + \dim_k(U,Y) = \dim_k(U,X) + \dim_k(U,Z).$$

If U is not isomorphic to Z, each map from U to Z is radical, thus $S_Z(U)$ is zero. If U is isomorphic to Z, the map w_* factors through the radical of $\operatorname{End}(Z)$, so $S_Z(Z)$ is isomorphic to k. Then summing the previous equality when U runs over $\mathcal{S}$, we get

$$\ell(X) + \ell(Z) = \ell(Y) + 2.$$

Clearly ℓ is τ-invariant, thus $\ell(Z)$ equals $\ell(X)$. If the decomposition of Y is of the form $\bigoplus_{i=1}^r Y_i^{n_i}$, we get

$$\ell(Y) = \sum_i n_i \ell(Y_i) = \sum_{i, X\to Y_i \in \tilde{\Gamma}} a_{XY_i}\ell(Y_i) + a_{XX}\ell(X).$$

We deduce the formula

$$2 = (2 - a_{XX})\ell(X) - \sum_{i, X\to Y_i\in\tilde{\Gamma}} a_{XY_i}\ell(Y_i).$$

Let x be a vertex of the tree Δ and $\overline{x}$ its image in $\widetilde{\Gamma}$. Then an arrow $\overline{x} \to Y$ in $\widetilde{\Gamma}$ comes from an arrow $(x,0) \to (y,0)$ in $\mathbb{Z}\Delta$ or from an arrow $(x,0) \to (y,-1)$ in $\mathbb{Z}\Delta$, *i.e.* from an arrow $(y,0) \to (x,0)$. Indeed the projection $\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$ is a covering. From this we deduce the equality

$$2 = (2 - a_{\overline{x}\,\overline{x}})d_x - \sum_{y, x\to y\in\Delta} a_{\overline{x}\,\overline{y}} d_y - \sum_{y, y\to x\in\Delta} a_{\overline{y}\,\overline{x}} d_y = \sum_{y\in\Delta_0} d_y C_{xy}. \qquad \square$$

Now we can prove Theorem 4.1. The matrix C is a 'generalized Cartan matrix'. The previous lemma gives us a subadditive function which is not additive. Thus by [23], the underlying graph of C is of 'generalized Dynkin type'. As C is symmetric, the graph is necessarily of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, or $\mathbb{L}$. But this graph is the graph Δ with the valuation a. We are done in the cases $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.

The case $\mathbb{L}_n$ occurs when the AR-quiver contains at least one loop. We can see $\mathbb{L}_n$ as $\mathbb{A}_n$ with valuations on the vertices with a loop. Then, it is obvious that the automorphism groups of $\mathbb{Z}\mathbb{L}_n$ are generated by τ^r for an $r \geq 1$. But Proposition 4.2 tell us that a vertex x with a loop satisfies $\tau x = x$. Thus G is generated by τ and the AR-quiver has the following form:

This quiver is isomorphic to the quiver $\mathbb{Z}\mathbb{A}_{2n}/G$ where G is the group generated by the automorphism $\tau^n S = \rho$.

The suspension functor S sends the indecomposables on indecomposables, thus it can be seen as an automorphism of the AR-quiver. It is exactly the automorphism S defined in Section 2.2.

As shown in [49], it follows from the results of [30] that for each Dynkin tree Δ and for each weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$, there exists a locally finite triangulated category $\mathcal{T}$ such that $\Gamma_{\mathcal{T}} \simeq \mathbb{Z}\Delta/G$. This category is of the form $\mathcal{T} = \mathcal{D}^b(\operatorname{mod} k\Delta)/\varphi$ where φ is an auto-equivalence of $\mathcal{D}^b(\operatorname{mod} k\Delta)$.

5. Construction of a covering functor

From now, we suppose that the AR-quiver Γ of $\mathcal{T}$ is connected. We know its structure. It is natural to ask: Is the category $\mathcal{T}$ *standard*, *i.e.* equivalent as a k-linear category to the mesh category $k(\Gamma)$? First, in this part we construct a covering functor $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$.

5.1. Construction. — We write $\pi : \mathbb{Z}\Delta \to \Gamma$ for the canonical projection. As G is a weakly admissible group, this projection verifies the following property: if x is a vertex of $\mathbb{Z}\Delta$, the number of arrows of $\mathbb{Z}\Delta$ with source x is equal to the number of arrows of $\mathbb{Z}\Delta/G$ with source πx. Let $\mathcal{S}$ be a system of representatives of the isoclasses of indecomposables of $\mathcal{T}$. We write $\operatorname{ind}\mathcal{T}$ for the full subcategory of $\mathcal{T}$ whose set of objects is $\mathcal{S}$. For a tree Δ, we write $k(\mathbb{Z}\Delta)$ for the mesh category (see [39]). Using the same proof as Riedtmann [39], one shows the following theorem.

THEOREM 5.1. — *There exists a k-linear functor $F : k(\mathbb{Z}\Delta) \to \operatorname{ind}\mathcal{T}$ which is surjective and induces bijections:*

$$\bigoplus_{Fz=Fy} \operatorname{Hom}_{k(\mathbb{Z}\Delta)}(x,z) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(Fx,Fy),$$

for all vertices x and y of $\mathbb{Z}\Delta$.

5.2. Infinite case. — If the category $\mathcal{T}$ is locally finite not finite *i.e.* if there is infinitely many indecomposables, the constructed functor F is immediately fully faithful. Thus we get the corollary.

COROLLARY 5.2. — *If $\operatorname{ind}\mathcal{T}$ is not finite, then we have a k-linear equivalence between $\mathcal{T}$ and the mesh category $k(\mathbb{Z}\Delta)$.*

5.3. Uniqueness criterion. — The covering functor F can be seen as a k-linear functor from the derived category $\mathcal{D}^b(\operatorname{mod} k\Delta)$ to the category $\mathcal{T}$. By construction, it satisfies the following property called the *AR-property*:

For each AR-triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} SX$ of $\mathcal{D}^b(\operatorname{mod} k\Delta)$, there exists a triangle of $\mathcal{T}$ of the form $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\epsilon} SFX$.

In fact, thanks to this property, F is determined by its restriction to the subcategory $\operatorname{proj} k\Delta = k(\Delta)$, *i.e.* we have the following lemma.

LEMMA 5.3. — *Let F and G be k-linear functors from $\mathcal{D}^b(\operatorname{mod} k\Delta)$ to $\mathcal{T}$. Suppose that F and G satisfy the AR-property and that the restrictions $F_{|k(\Delta)}$ and $G_{|k(\Delta)}$ are isomorphic. Then the functors F and G are isomorphic as k-linear functors.*

Proof. — It is easy to construct this isomorphism by induction using the (TR3) axiom of the triangulated categories (see [36]). □

6. Particular cases of k-linear equivalence

From now we suppose that the category $\mathcal{T}$ is finite, *i.e.* $\mathcal{T}$ has finitely many isoclasses of indecomposable objects.

6.1. Equivalence criterion. — Let Γ be the AR-quiver of $\mathcal{T}$ and suppose that it is isomorphic to $\mathbb{Z}\Delta/G$. Let φ be a generator of G. It induces an automorphism in the mesh category $k(\mathbb{Z}\Delta)$ that we still denote by φ. Then we have the following equivalence criterion.

PROPOSITION 6.1. — *The categories $k(\Gamma)$ and $\operatorname{ind}\mathcal{T}$ are equivalent as k-categories if and only if there exists a covering functor $F : k(\mathbb{Z}\Delta) \to \operatorname{ind}\mathcal{T}$ and an isomorphism of functors $\Phi : F \circ \varphi \to F$.*

The proof consists in constructing a k-linear equivalence between $\operatorname{ind}\mathcal{T}$ and the orbit category $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ using the universal property of the orbit category (see [30]), and then constructing an equivalence between $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ and $k(\Gamma)$.

6.2. Cylindric case for $\mathbb{A}_n$

THEOREM 6.2. — *If $\Delta = \mathbb{A}_n$ and $\varphi = \tau^r$ for some $r \geq 1$, then there exists a functor isomorphism $\Phi : F \circ \varphi \to F$, i.e. for each object x of $k(\mathbb{Z}\Delta)$ there exists an automorphism Φ_x of Fx such that for each arrow $\alpha : x \to y$ of $\mathbb{Z}\Delta$, the following diagram commutes:*

$$\begin{array}{ccc} Fx & \xrightarrow{\Phi_x} & Fx \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle F\varphi\alpha} \\ Fy & \xrightarrow{\Phi_y} & Fy. \end{array}$$

To prove this, we need the following lemma.

LEMMA 6.3. — *Let $\alpha : x \to y$ be an arrow of $\mathbb{Z}\mathbb{A}_n$ and let c be a path from x to $\tau^r y$, $r \in \mathbb{Z}$, which is not zero in the mesh category $k(\mathbb{Z}\mathbb{A}_n)$. Then c can be written $c'\alpha$ where c' is a path from y to $\tau^r y$ (up to sign).*

Proof of the lemma. — There is a path from x to $\tau^r y$, thus, we have $\operatorname{Hom}_{k(\mathbb{Z}\Delta)}(x, \tau^r y) \simeq k$, and x and $\tau^r y$ are opposite vertices of a 'rectangle' in $\mathbb{Z}\mathbb{A}_n$. This implies that there exists a path from x to $\tau^r y$ beginning by α. □

Proof of Theorem 6.2. — Combining Proposition 6.1 and Lemma 5.3, we have just to construct an isomorphism between the restriction of F and $F \circ \varphi$ to a subquiver $\mathbb{A}_n$.

Let us fix a full subquiver of $\mathbb{Z}\mathbb{A}_n$ of the following form

$$x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} x_n$$

such that $x_1, \ldots, x_n$ are representatives of the τ-orbits in $\mathbb{Z}\mathbb{A}_n$. We define the $(\Phi_{x_i})_{i=1\ldots n}$ by induction. We fix $\Phi_{x_1} = \operatorname{Id}_{Fx_1}$. Now suppose we have constructed some automorphisms $\Phi_{x_1},\dots,\Phi_{x_i}$ such that for each $j\leq i$ the following diagram is commutative:

$$\begin{array}{ccc} Fx_{j-1} & \xrightarrow{\Phi_{x_{j-1}}} & Fx_{j-1} \\ {\scriptstyle F\alpha_{j-1}}\downarrow & & \downarrow{\scriptstyle F\varphi\alpha_{j-1}} \\ Fx_j & \xrightarrow{\Phi_{x_j}} & Fx_j. \end{array}$$

The composition $(F\varphi\alpha_i)\circ\Phi_{x_i}$ is in the morphism space $\mathrm{Hom}_{\mathcal{T}}(Fx_i,Fx_{i+1})$, which is isomorphic, by Theorem 5.1, to the space

$$\bigoplus_{Fz=Fx_{i+1}}\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x_i,z).$$

Thus we can write

$$(F\varphi\alpha_i)\Phi_{x_i}=\lambda F\alpha_i+\sum_{z\neq x_{i+1}}F\beta_z$$

where β_z belongs to $\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x_i,z)$ and $Fz=Fx_{i+1}$. But Fz is equal to Fx_{i+1} if and only if z is of the form $\tau^{r\ell}x_{i+1}$ for an ℓ in $\mathbb{Z}$. By the lemma, we can write $\beta_z=\beta'_z\alpha_i$. Thus we have the equality

$$(F\varphi\alpha_i)\Phi_{x_i}=F(\lambda\mathrm{Id}_{x_{i+1}}+\sum_z\beta'_z)F\alpha_i.$$

The scalar λ is not zero. Indeed, Φ_{x_i} is an automorphism, thus the image of $(F\varphi\alpha_i)\Phi_{x_i}$ is not zero in the quotient

$$\mathcal{R}(Fx_i,Fx_{i+1})/\mathcal{R}^2(Fx_i,Fx_{i+1}).$$

Thus $\Phi_{x_{i+1}}=F(\lambda\mathrm{Id}_{x_{i+1}}+\sum_z\beta'_z)$ is an automorphism of Fx_{i+1} which verifies the commutation relation

$$(F\varphi\alpha_i)\circ\Phi_{x_i}=\Phi_{x_{i+1}}\circ F\alpha_i.\qquad\square$$

6.3. Other standard cases. — In the mesh category $k(\mathbb{Z}\Delta)$, where Δ is a Dynkin tree, the length of the non zero paths is bounded. Thus there exist automorphisms φ such that, for an arrow $\alpha:x\to y$ of Δ, the paths from x to $\varphi^r y$ vanish in the mesh category for all $r\neq 0$. In other words, for each arrow $\alpha:x\to y$ of $\mathbb{Z}\Delta$, we have

$$\mathrm{Hom}_{k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}}(x,y)=\bigoplus_{r\in\mathbb{Z}}\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x,\varphi^r y)=\mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x,y)\simeq k,$$

where $k(\mathbb{Z}\Delta)/\varphi^{\mathbb{Z}}$ is the orbit category (see Section 6.1).

LEMMA 6.4. — *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G and suppose that φ verifies for each arrow $x \to y$ of $\mathbb{Z}\Delta$*

$$\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) = \mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x, y) \simeq k.$$

Let $F : k(\mathbb{Z}\Delta) \to \mathcal{T}$ and $G : k(\mathbb{Z}\Delta) \to \mathcal{T}$ be covering functors satisfying the AR-property. Suppose that F and G agree up to isomorphism on the objects of $k(\mathbb{Z}\Delta)$. Then F and G are isomorphic as k-linear functors.

Proof. — Using Lemma 5.3, we have just to construct an isomorphism between the functors restricted to Δ. Let $\alpha : x \to y$ be an arrow of Δ. Using Theorem 5.1 and the hypothesis, we have the isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(Fx, Fy) \simeq \bigoplus_{Fz=Fy} \mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x, z) \simeq \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{k(\mathbb{Z}\Delta)}(x, \varphi^r y) \simeq k$$

and then

$$\mathrm{Hom}_{\mathcal{T}}(Gx, Gy) \simeq \mathrm{Hom}_{\mathcal{T}}(Fx, Fy) \simeq k.$$

Thus there exists a scalar λ such that $G\alpha = \lambda F\alpha$. This scalar does not vanish since F and G are covering functors. As Δ is a tree, we can find some λ_x for $x \in \Delta$ by induction such that

$$G\alpha = \lambda_x \lambda_y^{-1} F\alpha.$$

Now it is easy to check that $\Phi_x = \lambda_x \mathrm{Id}_{Fx}$ is the functor isomorphism. □

This lemma gives us an isomorphism between the functors F and $F \circ \varphi$, Moreover, using the same argument, one can show that the covering functor F is an S-functor and a τ-functor.

For each Dynkin tree Δ we can determine the automorphisms φ which satisfy this combinatorial property. Using the preceding lemma and the equivalence criterion we deduce the following theorem:

THEOREM 6.5. — *Let $\mathcal{T}$ be a finite triangulated category with AR-quiver $\Gamma = \mathbb{Z}\Delta/G$. Let φ be a generator of G. If one of these cases holds,*

- *$\Delta = \mathbb{A}_n$ with n odd and G is generated by τ^r or $\varphi = \tau^r\phi$ with $r \geq \frac{1}{2}(n-1)$ and $\phi = \tau^{\frac{1}{2}(n+1)}S$;*
- *$\Delta = \mathbb{A}_n$ with n even and G is generated by ρ^r with $r \geq n-1$ and $\rho = \tau^{\frac{1}{2}n}S$;*
- *$\Delta = \mathbb{D}_n$ with $n \geq 5$ and G is generated by τ^r or $\tau^r\phi$ with $r \geq n-2$ and ϕ as in Theorem 2.1;*
- *$\Delta = \mathbb{D}_4$ and G is generated by $\phi\tau^r$, where $r \geq 2$ and ϕ runs over σ_3;*

- *$\Delta = \mathbb{E}_6$ and G is generated by τ^r or $\tau^r\phi$ where $r \geq 5$ and ϕ is as in Theorem 2.1;*
- *$\Delta = \mathbb{E}_7$ and G is generated by τ^r, $r \geq 8$;*
- *$\Delta = \mathbb{E}_8$ and G is generated by τ^r, $r \geq 14$;*

then $\mathcal{T}$ is standard, i.e. the categories $\mathcal{T}$ and $k(\Gamma)$ are equivalent as k-linear categories.

Corollary 6.6. — *A finite maximal d-Calabi-Yau (see [30, 8]) triangulated category $\mathcal{T}$, with $d \geq 2$, is standard,* i.e. *there exists a k-linear equivalence between $\mathcal{T}$ and the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\tau^{-1}S^{d-1}$ where Δ is Dynkin of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$*

7. Algebraic case

For some automorphism groups G, we know the k-linear structure of $\mathcal{T}$. But what about the triangulated structure? We can only give an answer adding hypothesis on the triangulated structure. We distinguish two cases.

If $\mathcal{T}$ is locally finite, not finite, we have the following theorem which is proved in Section 7.2.

Theorem 7.1. — *Let $\mathcal{T}$ be a connected locally finite triangulated category with infinitely many indecomposables. If $\mathcal{T}$ is the base of a tower of triangulated categories [29], then $\mathcal{T}$ is triangle equivalent to $\mathcal{D}^b(\operatorname{mod} k\Delta)$ for some Dynkin diagram Δ.*

Now if $\mathcal{T}$ is a finite standard category which is algebraic, *i.e.* $\mathcal{T}$ is triangle equivalent to $\underline{\mathcal{E}}$ for some k-linear Frobenius category $\mathcal{E}$ (see [31, 3.6]), then we have the following result which is proved in Section 7.3.

Theorem 7.2. — *Let $\mathcal{T}$ be a finite triangulated category, which is connected, algebraic and standard. Then, there exists a Dynkin diagram Δ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ and an auto-equivalence Φ of $\mathcal{D}^b(\operatorname{mod} k\Delta)$ such that $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$.*

This theorem combined with Corollary 6.6 yields the following result (compare to [30, Cor. 8.4]).

Corollary 7.3. — *If $\mathcal{T}$ is a finite algebraic maximal d-Calabi-Yau category with $d \geq 2$, then $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/S^d\nu^{-1}$ for some Dynkin diagram Δ.*

7.1. ∂-functor. — We recall the following definition from [29] and [46].

Definition 7.1.1. — Let $\mathcal{H}$ be an exact category and $\mathcal{T}$ a triangulated category. A *∂-functor* $(I, \partial) : \mathcal{H} \to \mathcal{T}$ is given by:

— an additive k-linear functor $I : \mathcal{H} \to \mathcal{T}$;
— for each conflation $\epsilon : X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ of $\mathcal{H}$, a morphism $\partial\epsilon : IZ \to SIX$ functorial in ϵ such that $IX \xrightarrow{Ii} IY \xrightarrow{\ell Ip} IZ \xrightarrow{\partial\epsilon} SIX$ is a triangle of $\mathcal{T}$.

For each exact category $\mathcal{H}$, the inclusion $I : \mathcal{H} \to \mathcal{D}^b(\mathcal{H})$ can be completed to a ∂-functor (I, ∂) in a unique way. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. If $(F, \varphi) : \mathcal{T} \to \mathcal{T}'$ is an S-functor and $(I, \partial) : \mathcal{H} \to \mathcal{T}$ is a ∂-functor, we say that F *respects* ∂ if $(F \circ I, \varphi(F\partial)) : \mathcal{H} \to \mathcal{T}'$ is a ∂-functor. Obviously each triangle functor respects ∂.

Proposition 7.4. — *Let $\mathcal{H}$ be a k-linear hereditary abelian category and let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a ∂-functor. Then there exists a unique (up to isomorphism) k-linear S-functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects ∂.*

Proof. — On $\mathcal{H}$ (which can be seen as a full subcategory of $\mathcal{D}^b(\mathcal{H})$), the functor F is uniquely determined. We want F to be an S-functor, so F is uniquely determined on $S^n\mathcal{H}$ for $n \in \mathbb{Z}$ too. Since $\mathcal{H}$ is hereditary, each object of $\mathcal{D}^b(\mathcal{H})$ is isomorphic to a direct sum of stalk complexes, *i.e.* complexes concentrated in a single degree. Thus, the functor F is uniquely determined on the objects.

Now, let X and Y be stalk complexes of $\mathcal{D}^b(\mathcal{H})$ and $f : X \to Y$ a non-zero morphism. We can suppose that X is in $\mathcal{H}$ and Y is in $S^n\mathcal{H}$. If $n \neq 0, 1$, f is necessarily zero. If $n = 0$, then f is a morphism in $\mathcal{H}$ and Ff is uniquely determined. If $n = 1$, f is an element of $\mathrm{Ext}^1_{\mathcal{H}}(X, S^{-1}Y)$, so gives us a conflation $\epsilon : S^{-1}Y \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} X$ in $\mathcal{H}$. The functor F respects ∂, thus Ff has to be equal to $\varphi \circ \partial\epsilon$ where φ is the natural isomorphism between $SFS^{-1}Y$ and FY. Since ∂ is functorial, F is a functor. The result follows. □

A priori this functor is not a triangle functor. We recall a theorem proved by B. Keller [29, Cor. 2.7].

Theorem 7.5. — *Let $\mathcal{H}$ be a k-linear exact category, and $\mathcal{T}$ be the base of a tower of triangulated categories* [29]. *Let $(I, \partial) : \mathcal{H} \to \mathcal{T}$ be a ∂-functor such that for each $n < 0$, and all objects X and Y of $\mathcal{H}$, the space* $\mathrm{Hom}_{\mathcal{T}}(IX, S^nIY)$

vanishes. Then there exists a triangle functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ such that the following diagram commutes up to isomorphism of ∂-functors:

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{D}^b(\mathcal{H}) \\ & {\scriptstyle (I,\partial)} \searrow \quad \swarrow {\scriptstyle F} & \\ & \mathcal{T} & \end{array}$$

From Theorem 7.5, and the proposition above we deduce the following corollary.

Corollary 7.6 (compare [44]). — *Let $\mathcal{T}$, $\mathcal{H}$ and $(I,\partial) : \mathcal{H} \to \mathcal{T}$ be as in Theorem 7.5. If $\mathcal{H}$ is hereditary, then the unique functor $F : \mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$ which respects ∂ is a triangle functor.*

7.2. Proof of Theorem 7.1. — Let F be the k-linear equivalence constructed in Theorem 5.1 between an algebraic triangulated category $\mathcal{T}$ and $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H} = \operatorname{mod} k\Delta$ and Δ is a simply-laced Dynkin graph. As we saw in Section 6, the covering functor is an S-functor.

The category $\mathcal{H}$ is the heart of the standard t-structure on $\mathcal{D}^b(\mathcal{H})$. The image of this t-structure through F is a t-structure on $\mathcal{T}$. Indeed, F is an S-equivalence, so the conditions (i) and (ii) from [7, Def. 1.3.1] hold obviously. And since $\mathcal{H}$ is hereditary, for an object X of $\mathcal{D}^b(\mathcal{H})$, the morphism $\tau_{>0}X \to S\tau_{\leq 0}X$ of the triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{>0}X \longrightarrow S\tau_{\leq 0}X$$

vanishes. Thus the image of this triangle through F is a triangle of $\mathcal{T}$ and condition (iii) of [7, Def. 1.3.1] holds. Then we get a t-structure on $\mathcal{T}$ whose heart is $\mathcal{H}$.

It results from [7, Prop. 1.2.4] that the inclusion of the heart of a t-structure can be uniquely completed to a ∂-functor. Thus we obtain a ∂-functor $(F_0,\partial) : \mathcal{H} \to \mathcal{T}$ with $F_0 = F_{|\mathcal{H}}$.

The functor F is an S-equivalence. Thus for each $n < 0$, and all objects X and Y of $\mathcal{H}$, the space $\operatorname{Hom}_{\mathcal{T}}(FX, S^nFY)$ vanishes. Now we can apply Theorem 7.5 and we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{D}^b(\mathcal{H}), \\ & {\scriptstyle (F_0,\partial)} \searrow \quad \swarrow {\scriptstyle F} \swarrow {\scriptstyle G} & \\ & \mathcal{T} & \end{array}$$

where F is the S-equivalence and G is a triangle functor. Note that *a priori* F is an S-functor which does not respect ∂. The functors $F_{|\mathcal{H}}$ and $G_{|\mathcal{H}}$ are isomorphic. The functor F is an S-functor thus we have an isomorphism

$F_{|S^n\mathcal{H}} \simeq G_{|S^n\mathcal{H}}$ for each $n \in \mathbb{Z}$. Thus the functor G is essentially surjective. Since $\mathcal{H}$ is the category $\operatorname{mod} k\Delta$, to show that G is fully faithful, we have just to show that for each $p \in \mathbb{Z}$, there is an isomorphism induced by G

$$\operatorname{Hom}_{\mathcal{D}^b(\mathcal{H})}(A, S^pA) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(GA, S^pGA)$$

where A is the free module $k\Delta$. For $p = 0$, this is clear because A is in $\mathcal{H}$. And for $p \neq 0$ both sides vanish.

Thus G is a triangle equivalence between $\mathcal{D}^b(\mathcal{H})$ and $\mathcal{T}$.

7.3. Finite algebraic standard case. — For a small dg category $\mathcal{A}$, we denote by $\mathcal{C}\mathcal{A}$ the category of dg $\mathcal{A}$-modules, by $\mathcal{D}\mathcal{A}$ the derived category of $\mathcal{A}$ and by $\operatorname{per}\mathcal{A}$ the *perfect derived category* of $\mathcal{A}$, *i.e.* the smallest triangulated subcategory of $\mathcal{D}\mathcal{A}$ which is stable under passage to direct factors and contains the free $\mathcal{A}$-modules $\mathcal{A}(?, A)$, where A runs through the objects of $\mathcal{A}$. Recall that a small triangulated category is *algebraic* if it is triangle equivalent to $\operatorname{per}\mathcal{A}$ for a dg category $\mathcal{A}$. For two small dg categories $\mathcal{A}$ and $\mathcal{B}$, a triangle functor $\operatorname{per}\mathcal{A} \to \operatorname{per}\mathcal{B}$ is *algebraic* if it is isomorphic to the functor

$$F_X = ? \overset{L}{\otimes}_{\mathcal{A}} X$$

associated with a dg bimodule X, *i.e.* an object of the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$.

Let Φ be an algebraic autoequivalence of $\mathcal{D}^b(\operatorname{mod} k\Delta)$ such that the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ is triangulated. Let Y be a dg $k\Delta$-$k\Delta$-bimodule such that $\Phi = F_Y$. In Section 9.3 of [30], it was shown that there is a canonical triangle equivalence between this orbit category and the perfect derived category of a certain small dg category. Thus, the orbit category is algebraic, and endowed with a canonical triangle equivalence to the perfect derived category of a small dg category. Moreover, by the construction in [*loc. cit.*], the projection functor

$$\pi : \mathcal{D}^b(\operatorname{mod} k\Delta) \longrightarrow \mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$$

is algebraic.

The proof of Theorem 7.0.5 is based on the following universal property of the triangulated orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$. For the proof, we refer to Section 9.3 of [30].

Proposition 7.7. — *Let $\mathcal{B}$ be a small dg category and*

$$F_X = ? \overset{L}{\otimes}_{k\Delta} X : \mathcal{D}^b(\operatorname{mod} k\Delta) \longrightarrow \operatorname{per}\mathcal{B}$$

an algebraic triangle functor given by a dg $k\Delta$-$\mathcal{A}$-bimodule X. Suppose that there is an isomorphism between $Y \otimes^L_{k\Delta} X$ and X in the derived bimodule *category $\mathcal{D}(k\Delta^{\mathrm{op}} \otimes \mathcal{B})$. Then the functor F_X factors, up to isomorphism of triangle functors, through the projection*

$$\pi : \mathcal{D}^b(\operatorname{mod} k\Delta) \longrightarrow \mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi.$$

Moreover, the induced triangle functor is algebraic.

Let us recall a lemma of Van den Bergh [33].

Lemma 7.8. — *Let Q be a quiver without oriented cycles and $\mathcal{A}$ be a dg category. We denote by $k(Q)$ the category of paths of Q and by* Can : $\mathcal{CA} \to \mathcal{DA}$ *the canonical functor. Then we have the following properties:*

a) *Each functor $F : k(Q) \to \mathcal{DA}$ lifts, up to isomorphism, to a functor $\widetilde{F} : k(Q) \to \mathcal{CA}$ which verifies the following property: For each vertex j of Q, the induced morphism*

$$\bigoplus_i \widetilde{F}i \to \widetilde{F}j,$$

where i runs through the immediate predecessors of j, is a monomorphism which splits as a morphism of graded $\mathcal{A}$-modules.

b) *Let F and G be functors from $k(Q)$ to $\mathcal{CA}$, and suppose that F satisfies the property of a). Then any morphism of functors φ :* $\mathrm{Can} \circ F \to \mathrm{Can} \circ G$ *lifts to a morphism $\widetilde{\varphi} : F \to G$.*

Proof. — a) For each vertex i of Q, the object Fi is isomorphic in $\mathcal{DA}$ to its cofibrant resolution X_i. Thus for each arrow $\alpha : i \to j$, F induces a morphism $f_\alpha : X_i \to X_j$ which can be lifted to $\mathcal{CA}$ since the X_i are cofibrant. Since Q has no oriented cycle, it is easy to choose the f_α such that the property is satisfied.

b) For each vertex i of Q, we may assume that Fi is cofibrant. Then we can lift $\varphi_i : \mathrm{Can} \circ Fi \to \mathrm{Can} \circ Gi$ to $\psi_i : Fi \to Gi$. For each arrow α of Q, the square

$$\begin{array}{ccc} Fi & \xrightarrow{F_\alpha} & Fj \\ {\scriptstyle\psi_i}\downarrow & & \downarrow{\scriptstyle\psi_j} \\ Gi & \xrightarrow{G_\alpha} & Gj \end{array}$$

is commutative in $\mathcal{DA}$. Thus the square

$$\begin{array}{ccc} \bigoplus_i Fi & \xrightarrow{F_\alpha} & Fj \\ {\scriptstyle(\psi_i)}\downarrow & & \downarrow{\scriptstyle\psi_j} \\ \bigoplus_i Gi & \xrightarrow[G_\alpha]{} & Gj \end{array}$$

is commutative up to nullhomotopic morphism $h : \bigoplus_i Fi \to Gj$. Since the morphism $f : \bigoplus_i Fi \to Fj$ is split mono in the category of graded $\mathcal{A}$-modules, h

extends along f and we can modify Ψ_j so that the square becomes commutative in $\mathcal{CA}$. The quiver Q does not have oriented cycles, so we can construct $\widetilde{\varphi}$ by induction. □

Proof of Theorem 7.2. — The category $\mathcal{T}$ is small and algebraic, thus we may assume that $\mathcal{T} = \operatorname{per}\mathcal{A}$ for some small dg category $\mathcal{A}$. Let $F : \mathcal{D}^b(\operatorname{mod} k\Delta) \to \mathcal{T}$ be the covering functor of Theorem 5.1. Let Φ be an auto-equivalence of $\mathcal{D}^b(\operatorname{mod} k\Delta)$ such that the AR-quiver of the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ is isomorphic (as translation quiver) to the AR-quiver of $\mathcal{T}$. We may assume that $\Phi = -\otimes^L_{k\Delta} Y$ for an object Y of $\mathcal{D}(k\Delta^{\mathrm{op}} \otimes k\Delta)$. The orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ is algebraic, thus it is $\operatorname{per}\mathcal{B}$ for some dg category $\mathcal{B}$.

The functor $F_{|k(\Delta)}$ lifts by Lemma 7.8 to a functor $\widetilde{F}$ from $k(\Delta)$ to $\mathcal{CA}$. This means that the object $X = \widetilde{F}(k\Delta)$ has a structure of dg $k\Delta^{\mathrm{op}} \otimes \mathcal{A}$-module. We denote by X the image of this object in $\mathcal{D}(k\Delta^{\mathrm{op}} \otimes \mathcal{A})$.

The functors F and $-\otimes^L_{k\Delta} X$ become isomorphic when restricted to $k(\Delta)$. Moreover $-\otimes^L_{k\Delta} X$ satisfies the AR-property since it is a triangulated functor. Thus by Lemma 5.3, they are isomorphic as k-linear functors. So we have the following diagram:

$$\mathcal{D}^b(\operatorname{mod} k\Delta) \xrightarrow{-\overset{L}{\otimes}_{k\Delta} X} \operatorname{per}\mathcal{A} = \mathcal{T}. \qquad \circlearrowleft\, -\overset{L}{\otimes}_{k\Delta} Y$$

The category $\mathcal{T}$ is standard, thus there exists an isomorphism of k-linear functors

$$c : -\overset{L}{\otimes}_{k\Delta} X \longrightarrow -\overset{L}{\otimes}_{k\Delta} Y \overset{L}{\otimes}_{k\Delta} X.$$

The functor $-\otimes^L_{k\Delta} X$ restricted to the category $k(\Delta)$ satisfies the property of a) of Lemma 7.8. Thus we can apply b) and lift $c_{|k(\Delta)}$ to an isomorphism $\tilde{c}$ between X and $Y \otimes^L_{k\Delta} X$ as dg-$k\Delta^{\mathrm{op}} \otimes \mathcal{A}$-modules.

By the universal property of the orbit category, the bimodule X endowed with the isomorphism $\tilde{c}$ yields a triangle functor from $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ to $\mathcal{T}$ which comes from a bimodule Z in $\mathcal{D}(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A})$.

$$\begin{array}{ccc} -\overset{L}{\otimes}_{k\Delta} Y \circlearrowright \mathcal{D}^b(\operatorname{mod} k\Delta) & \xrightarrow{-\overset{L}{\otimes}_{k\Delta} X} & \operatorname{per}\mathcal{A} = \mathcal{T} \\ \downarrow \pi & \nearrow_{-\overset{L}{\otimes}_{k\Delta} Z} & \\ \mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi = \operatorname{per}\mathcal{B} & & \end{array}$$

The functor $-\otimes^{L}_{k\Delta} Z$ is essentially surjective. Let us show that it is fully faithful. For M and N objects of $\mathcal{D}^b(\operatorname{mod} k\Delta)$ we have the commutative diagram

$$\begin{array}{ccc} & \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(M,\Phi^n N) & \\ \swarrow^{\pi} & & \searrow^{-\overset{L}{\otimes}_{k\Delta} X = F} \\ \operatorname{Hom}_{\mathcal{D}/\Phi}(\pi M,\pi N) & \xrightarrow[-\overset{L}{\otimes}_{k\Delta} Z]{} & \operatorname{Hom}_{\mathcal{T}}(FM,FN), \end{array}$$

where $\mathcal{D}$ means $\mathcal{D}^b(\operatorname{mod} k\Delta)$. The two diagonal morphisms are isomorphisms, thus so is the horizontal morphism. This proves that $-\otimes^{L}_{k\Delta} Z$ is a triangle equivalence between the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ and $\mathcal{T}$. □

8. Triangulated structure on the category of projectives

Let k be an algebraically closed field and $\mathcal{P}$ a k-linear category with split idempotents. The category $\operatorname{mod}\mathcal{P}$ of contravariant finitely presented functors from $\mathcal{P}$ to $\operatorname{mod} k$ is exact. As the idempotents split, the projectives of $\operatorname{mod}\mathcal{P}$ coincide with the representables. Thus the Yoneda functor gives a natural equivalence between $\mathcal{P}$ and $\operatorname{proj}\mathcal{P}$. Assume besides that $\operatorname{mod}\mathcal{P}$ has a structure of Frobenius category. The stable category $\underline{\operatorname{mod}}\mathcal{P}$ is a triangulated category, we write Σ for the suspension functor.

Let S be an auto-equivalence of $\mathcal{P}$. It can be extended to an exact functor from $\operatorname{mod}\mathcal{P}$ to $\operatorname{mod}\mathcal{P}$ and thus to a triangle functor of $\underline{\operatorname{mod}}\mathcal{P}$. The aim of this part is to find a necessary condition on the functor S such that the category $(\mathcal{P},S)$ has a triangulated structure. Heller already showed [25, Thm. 16.4] that if there exists an isomorphism of triangle functors between S and Σ^3, then $\mathcal{P}$ has a pretriangulated structure. But he did not succeed in proving the octahedral axiom. We are going to impose a stronger condition on the functor S and prove the following theorem.

THEOREM 8.1. — *Assume there exists an exact sequence of exact functors from* $\operatorname{mod}\mathcal{P}$ *to* $\operatorname{mod}\mathcal{P}$*:*

$$0\to \operatorname{Id}\longrightarrow X^0\longrightarrow X^1\longrightarrow X^2\longrightarrow S\to 0,$$

where the X^i*,* $i=0,1,2$*, take values in* $\operatorname{proj}\mathcal{P}$*. Then the category* $\mathcal{P}$ *has a structure of triangulated category with suspension functor* S*.*

For an M in $\operatorname{mod}\mathcal{P}$, denote $T_M : X^0M\to X^1M\to X^2M\to SX^0M$ a *standard triangle*. A triangle of $\mathcal{P}$ will be a sequence $X : P\xrightarrow{u} Q\xrightarrow{v} R\xrightarrow{w} SP$ which is isomorphic to a standard triangle T_M for an M in $\operatorname{mod}\mathcal{P}$.

8.1. S-complexes, Φ-S-complexes and standard triangles. — Let $\mathcal{A}cp(\operatorname{mod}\mathcal{P})$ be the category of acyclic complexes with projective components. It is a Frobenius category whose projective-injectives are the contractible complexes, *i.e.* the complexes homotopic to zero. The functor $Z^0 : \mathcal{A}cp(\operatorname{mod}\mathcal{P}) \to \operatorname{mod}\mathcal{P}$ which sends a complex

$$\cdots \longrightarrow X^{-1} \xrightarrow{x^{-1}} X^0 \xrightarrow{x^0} X^1 \xrightarrow{x^1} \cdots$$

to the kernel of x^0 is an exact functor. It sends the projective-injectives to projective-injectives and induces a triangle equivalence between $\underline{\mathcal{A}cp(\operatorname{mod}\mathcal{P})}$ and $\underline{\operatorname{mod}}\mathcal{P}$.

Definition 8.1.1. — An object of $\mathcal{A}cp(\operatorname{mod}\mathcal{P})$ is called an *S-complex* if it is S-periodic, *i.e.* if it has the following form:

$$\cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots .$$

The category S-comp of S-complexes with S-periodic morphisms is a non full subcategory of $\mathcal{A}cp(\operatorname{mod}\mathcal{P})$. It is a Frobenius category. The projective-injectives are the S-contractibles, *i.e.* the complexes homotopic to zero with an S-periodic homotopy. Using the functor Z^0, we get an exact functor from S-comp to $\operatorname{mod}\mathcal{P}$ which induces a triangle functor

$$\underline{Z^0} : \underline{S\text{-comp}} \longrightarrow \underline{\operatorname{mod}}\mathcal{P}.$$

Fix a sequence as in Theorem 8.1. Clearly, it induces for each object M of $\underline{\operatorname{mod}}\mathcal{P}$, a functorial isomorphism in $\underline{\operatorname{mod}}\mathcal{P}$, $\Phi_M : \Sigma^3 M \to SM$.

Let Y be an S-complex:

$$Y: \ \cdots \longrightarrow P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP \xrightarrow{Su} SQ \longrightarrow \cdots$$

Let M be the kernel of u. Then Y induces an isomorphism θ (in $\underline{\operatorname{mod}}\mathcal{P}$) between $\Sigma^3 M$ and SM. If θ is equal to Φ_M, we will say that X is a *Φ-S-complex*.

Let M be an object of $\operatorname{mod}\mathcal{P}$. The standard triangle T_M can be see as a Φ-S-complex:

$$\cdots \longrightarrow X^0 M \longrightarrow X^1 M \longrightarrow X^2 M \longrightarrow SX^0 M \longrightarrow SX^1 M \longrightarrow \cdots$$

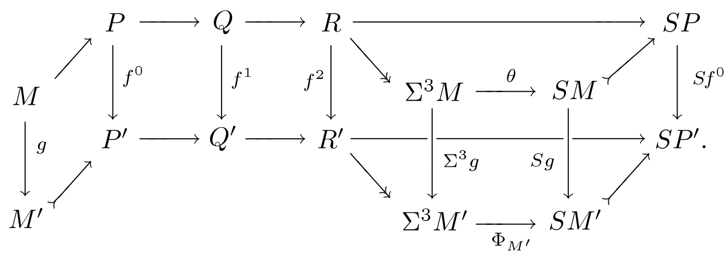
The functor T which sends an object M of $\operatorname{mod}\mathcal{P}$ to the S-complex T_M is exact since the X^i are exact. It satisfies the relation $Z^0 \circ T \simeq \mathrm{Id}_{\operatorname{mod}\mathcal{P}}$. Moreover, as it preserves the projective-injectives, it induces a triangle functor

$$T : \underline{\operatorname{mod}}\mathcal{P} \longrightarrow \underline{S\text{-comp}}$$

8.2. Properties of the functors Z^0 and T

LEMMA 8.2. — *An S-complex which is homotopy-equivalent to a Φ-S-complex is a Φ-S-complex.*

Proof. — Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be an S-complex homotopy-equivalent to the Φ-S-complex $X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$. Let M be the kernel of u and M' the kernel of u'. By assumption, there exists a S-periodic homotopy equivalence f from X to X', which induces a morphism $g = Z^0 f : M \to M'$. Thus, we get the following commutative diagram:



The morphism g is an isomorphism of $\underline{\mathrm{mod}}\,\mathcal{P}$ since f is an isomorphism of $\underline{S\text{-comp}}$. Thus the morphisms $\Sigma^3 g$ and Sg are isomorphisms of $\underline{\mathrm{mod}}\,\mathcal{P}$. The following equality in $\underline{\mathrm{mod}}\,\mathcal{P}$

$$\theta = (Sg)^{-1}\Phi_{M'}\Sigma^3 g = \Phi_M$$

shows that the complex X is a Φ-S-complex. □

LEMMA 8.3. — *Let*

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP,$$

$$X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ-S-complexes. Suppose that we have a commutative square

$$\begin{array}{ccc} P & \xrightarrow{u} & Q \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f^1} \\ P' & \xrightarrow{u} & Q'. \end{array}$$

Then, there exists a morphism $f^2 : R \to R'$ such that (f^0, f^1, f^2) extends to an S-periodic morphism from X to X'.

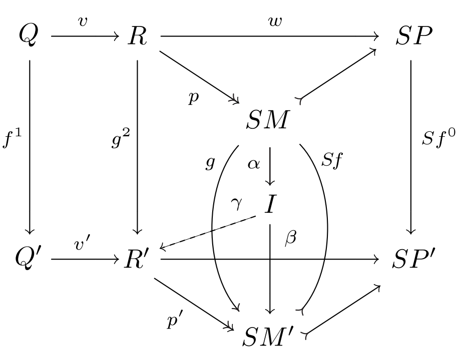
Proof. — Let M be the kernel of u, M' be the kernel of u' and $f : M \to M'$ be the morphism induced by the commutative square. As R and R' are projective-injective objects, we can find a morphism $g^2 : R \to R'$ such that the following square commutes:

$$\begin{array}{ccc} Q & \xrightarrow{v} & R \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle g^2} \\ Q' & \xrightarrow{v'} & R'. \end{array}$$

The morphism g^2 induces a morphism $g : SM \to SM'$ such that the following square is commutative in $\underline{\mathrm{mod}}\,\mathcal{P}$:

$$\begin{array}{ccc} \Sigma^3 M & \xrightarrow{\Phi_M} & SM \\ {\scriptstyle \Sigma^3 f}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma^3 M' & \xrightarrow{\Phi_{M'}} & SM'. \end{array}$$

Thus the morphisms Sf and g are equal in $\underline{\mathrm{mod}}\,\mathcal{P}$, *i.e.* there exists a projective-injective I of $\mathrm{mod}\,\mathcal{P}$ and morphisms $\alpha : SM \to I$ and $\beta : I \to SM'$ such that $g - Sf = \beta\alpha$. Let p (resp. p') be the epimorphism from R onto SM (resp. from R' onto SM'). Then, as I is projective, β factors through p'.



We put $f^2 = g^2 - \gamma\alpha p$. Then obviously, we have the equalities $f^2 v = v' f^1$ and $w' f^2 = Sf^0 w$. Thus the morphism (f^0, f^1, f^2) extends to a morphism of S-comp. □

PROPOSITION 8.4. — *The functor* $\underline{Z}^0 : \underline{\Phi\text{-}S\text{-comp}} \to \underline{\mathrm{mod}}\,\mathcal{P}$ *is full and essentially surjective. Its kernel is an ideal whose square vanishes.*

Proof. — The functor $\underline{Z^0}$ is essentially surjective since we have the relation $\underline{Z^0} \circ \underline{T} = \mathrm{Id}_{\underline{\mathrm{mod}}\,\mathcal{P}}$. Let us show that $\underline{Z^0}$ is full. Let

$$X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP,$$
$$X' : P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} SP'$$

be two Φ-S-complexes. Let M (resp. M') be the kernel of u (resp. u'). As P, Q, P' and Q' are projective-injective, there exist morphisms $f^0 : P \to P'$ and $f^1 : Q \to Q'$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \rightarrowtail & P & \xrightarrow{u} & Q \\ \downarrow f & & \downarrow f^0 & & \downarrow f^1 \\ M' & \rightarrowtail & P' & \xrightarrow{u'} & Q'. \end{array}$$

Now the result follows from Lemma 8.3.

Now let $\underline{f} : X \to X'$ be a morphism in the kernel of $\underline{Z^0}$. Up to homotopy, we can suppose that $\underline{f}$ has the following form:

$$\begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle f^2}\downarrow & & {\scriptstyle 0}\downarrow \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP'. \end{array}$$

As the composition $w'f^2$ vanishes and as Q' is projective-injective, f^2 factors through v'. For the same argument, f^2 factors through w. If $\underline{f}$ and $\underline{f'}$ are composable morphisms of the kernel of $\underline{Z^0}$, we get the following diagram:

$$\begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\ {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & \swarrow{\scriptstyle h^2} & {\scriptstyle f^2}\downarrow & & \downarrow{\scriptstyle 0} \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP' \\ {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle f'^2}\downarrow & \swarrow{\scriptstyle h'^3} & \downarrow{\scriptstyle 0} \\ P'' & \xrightarrow{u''} & Q'' & \xrightarrow{v''} & R'' & \xrightarrow{w''} & SP''. \end{array}$$

The composition $\underline{f'} \circ \underline{f}$ vanishes obviously. □

Corollary 8.5. — *A Φ-S-complex morphism f which induces an isomorphism $\underline{Z^0}(f)$ in $\underline{\mathrm{mod}}\,\mathcal{P}$ is an homotopy-equivalence.*

This corollary comes from the previous theorem and from the following lemma.

LEMMA 8.6. — *Let $F : \mathcal{C} \to \mathcal{C}'$ be a full functor between two additive categories. If the kernel of F is an ideal whose square vanishes, then F detects isomorphisms.*

Proof. — Let $u \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism in $\mathcal{C}$ such that Fu is an isomorphism. Since the functor F is full, there exists v in $\mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $Fv = (Fu)^{-1}$. The morphism $w = uv - \mathrm{Id}_B$ is in the kernel of F, thus w^2 vanishes. Then the morphism $v(\mathrm{Id}_B - w)$ is a right inverse of u. In the same way we show that u has a left inverse, so u is an isomorphism. □

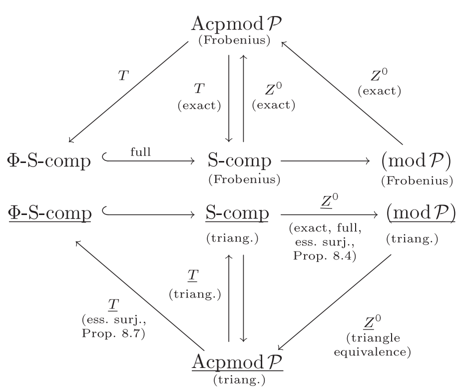
PROPOSITION 8.7. — *The category of Φ-S-complexes is equivalent to the category of S-complexes which are homotopy-equivalent to standard triangles.*

Proof. — Since standard triangles are ϕ-S-complexes, each S-complex that is homotopy equivalent to a standard triangle is a Φ-S-complex (Lemma 8.2). Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a Φ-S-complex. Let M be the kernel of u. Then there exist morphisms $f^1 : P \to X^0M$ and $f^1 : Q \to X^1M$ such that the following diagram is commutative:

$$\begin{array}{ccccc} M & \rightarrowtail & P & \xrightarrow{u} & Q \\ \| & & \downarrow f^0 & & \downarrow f^1 \\ M & \rightarrowtail & X^0M & \longrightarrow & X^1M. \end{array}$$

We can complete (Lemma 8.3) f into an S-periodic morphism from X in T_M. The morphism f satisfies $Z^0 f = \mathrm{Id}_M$, so $\underline{Z}^0(T_M)$ and $Z^0(X)$ are equal in $\underline{\mathrm{mod}}\,\mathcal{P}$. By the corollary, T_M and X are homotopy-equivalent. Thus the inclusion functor T is essentially surjective. □

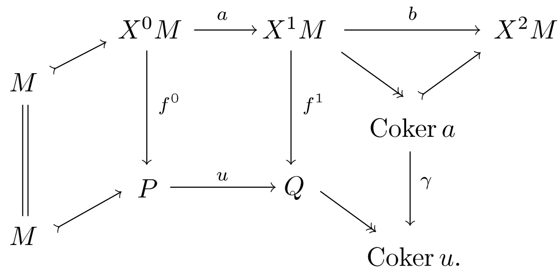
These two diagrams summarize the results of this section:



8.3. Proof of Theorem 8.1. — We are going to show that the Φ-S-complexes form a system of triangles of the category $\mathcal{P}$. We use triangle axioms as in [36].

TR0: For each object M of $\mathcal{P}$, the S-complex $M = M \to 0 \to SM$ is homotopy-equivalent to the zero complex, so is a Φ-S-complex.

TR1: Let $u : P \to Q$ be a morphism of $\mathcal{P}$, and let M be its kernel. We can find morphisms f^0 and f^1 so as to obtain a commutative square:



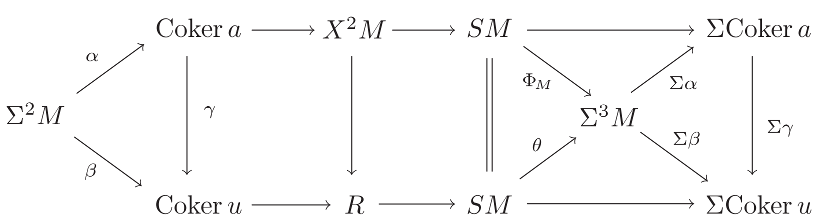
We form the push-out

$$\begin{array}{ccccccc} 0 \to & \operatorname{Coker} a & \longrightarrow & X^2M & \longrightarrow & SM & \to 0 \\ & {\scriptstyle\gamma}\downarrow & \text{PO} & \downarrow & & \| & \\ 0 \to & \operatorname{Coker} u & \longrightarrow & R & \longrightarrow & SM & \to 0. \end{array}$$

It induces a triangle morphism of the triangulated category $\underline{\operatorname{mod}}\,\mathcal{P}$:

$$\begin{array}{ccccccc} \operatorname{Coker} a & \longrightarrow & X^2M & \longrightarrow & SM & \longrightarrow & \Sigma\operatorname{Coker} a \\ {\scriptstyle\gamma}\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle\Sigma\gamma} \\ \operatorname{Coker} u & \longrightarrow & R & \longrightarrow & SM & \longrightarrow & \Sigma\operatorname{Coker} u. \end{array}$$

The morphism γ is an isomorphism in $\underline{\operatorname{mod}}\,\mathcal{P}$ since $\operatorname{Coker} a$ and $\operatorname{Coker} u$ are canonically isomorphic to $\Sigma^2 M$ in $\underline{\operatorname{mod}}\,\mathcal{P}$. By the five lemma, $X^2M \to R$ is an isomorphism in $\underline{\operatorname{mod}}\,\mathcal{P}$. Since X^2M is projective-injective, so is R. Thus the complex $P \xrightarrow{u} Q \to R \to SP$ is an S-complex. Then we have to see that it is a Φ-S-complex. Let θ be the isomorphism between SM and $\Sigma^3 M$ induced by this complex. We write α (resp. β) for the canonical isomorphism in $\underline{\operatorname{mod}}\,\mathcal{P}$ between $\Sigma^2 M$ and $\operatorname{Coker} a$ (resp. $\operatorname{Coker} u$). From the commutative diagram



we deduce the equality $\theta = (\Sigma\beta)^{-1}\Sigma\gamma\Sigma\alpha\Phi_M = \Phi_M$ in $\underline{\mathrm{mod}}\,\mathcal{P}$. The constructed S-complex is a Φ-S-complex.

TR2: Let $X : P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} SP$ be a Φ-S-complex. It is homotopy-equivalent to a standard triangle T_M. Thus the S-complex

$$X' : \; Q \xrightarrow{-v} R \xrightarrow{-w} SP \xrightarrow{-Su} SQ$$

is homotopy-equivalent to $T_M[1]$. Since $\underline{T}$ is a triangle functor, the objects $T_{\Sigma M}$ and $T_M[1]$ are isomorphic in the stable category $\underline{S - \mathrm{comp}}$, *i.e.* they are homotopy-equivalent. Thus, by Lemma 8.2, $T_M[1]$ is a Φ-S-complex and then so is X'.

TR3: This axiom is a direct consequence of Lemma 8.3.

TR4: Let X and X' be two Φ-S-complexes and suppose we have a commutative diagram:

$$\begin{array}{ccccccccc}
X: & P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & SP \\
 & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle Sf^0} \\
X': & P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & SP'.
\end{array}$$

Let M (resp. M') be the kernel of u (resp. u'), and $g : M \to M'$ the induced morphism. The morphism $Tg : T_M \to T_{M'}$ induces a S-complex morphism $\widetilde{g} = (g^0, g^1, g^2)$ between X and X'.

We are going to show that we can find a morphism $f^2 : R \to R'$ such that (f^0, f^1, f^2) can be extended in an S-complex morphism that is homotopic to $\widetilde{g}$. As (g^0, g^1) and (f^0, f^1) induce the same morphism g in the kernels, we have some morphisms $h^1 : Q \to P'$ and $h^2 : R \to Q'$ such that $f^0 - g^0 = h^1u$ and $f^1 - g^1 = u'h^1 + h^2v$. We put $f^2 = g^2 + v'h^2$. We have the equalities

$$f^2v = g^2v + v'h^2v = v'(g^1 + h^2v) = v'(f^1 - u'h^1) = v'f^1,$$
$$w'f^2 = w'g^2 = (Sg^0)w = (Sf^0 - Sh^1Su)w = (Sf^0)w.$$

Thus (f^0, f^1, f^2) can be extended to an S-periodic morphism $\widetilde{f}$ which is S-homotopic to $\widetilde{g}$. Their respective cones $C(\widetilde{f})$ and $C(\widetilde{g})$ are isomorphic as S-complexes. Moreover, since $\widetilde{g}$ is a composition of $Tg : T_M \to T_{M'}$ with homotopy-equivalences, the cones $C(\widetilde{g})$ and $C(Tg)$ are homotopy-equivalent.

In $\underline{\mathrm{mod}}\,\mathcal{P}$, we have a triangle

$$M \xrightarrow{g} M' \longrightarrow C(g) \longrightarrow \Sigma M.$$

Since $\underline{T}$ is a triangle functor, the sequence

$$T_M \xrightarrow{T_g} T_{M'} \longrightarrow T_{C(g)} \longrightarrow T_{\Sigma M}$$

is a triangle in $\underline{S\text{-comp}}$. But we know that

$$T_M \xrightarrow{Tg} T_{M'} \longrightarrow C(Tg) \longrightarrow T_M[1]$$

is a triangle in $\underline{S\text{-comp}}$ Thus the objects $C(Tg)$ and $T_{C(g)}$ are isomorphic in $\underline{S\text{-comp}}$, *i.e.* homotopy-equivalent. Thus, the cone $C(\widetilde{f}\,)$ of $\widetilde{f}$ is a Φ-S-complex by Lemma 8.2.

9. Application to the deformed preprojective algebras

In this section, we apply Theorem 8.1 to show that the category of finite dimensional projective modules over a deformed preprojective algebra of generalized Dynkin type (see [8]) is triangulated. This will give us some examples of non standard triangulated categories with finitely many indecomposables.

9.1. Preprojective algebra of generalized Dynkin type. — Recall the notations of [8]. Let Δ be a generalized Dynkin graph of type $\mathbb{A}_n$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(n = 6,7,8)$, or $\mathbb{L}_n$. Let Q_Δ be the following associated quiver:

$\Delta = \mathbb{A}_n \ (n \geq 1)$:

$$0 \underset{\bar a_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{1}{\overset{a_1}{\rightleftarrows}} 2 \rightleftarrows \cdots \rightleftarrows n-2 \underset{n-2}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

$\Delta = \mathbb{D}_n \ (n \geq 4)$:

$$\begin{array}{l} 0 \\ \ \ \ \searrow^{a_0} \nwarrow_{\bar a_0} \\ \qquad 2 \underset{2}{\overset{a_2}{\rightleftarrows}} 3 \rightleftarrows \cdots \rightleftarrows n-2 \underset{n-2}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \\ \ \ \ \nearrow^{a_1} \swarrow_{1} \\ 1 \end{array}$$

$\Delta = \mathbb{E}_n \ (n = 6,7,8)$:

$$\begin{array}{c} \qquad\qquad 0 \\ \qquad\qquad \bar a_0 \uparrow\downarrow a_0 \\ 1 \underset{1}{\overset{a_1}{\rightleftarrows}} 2 \underset{2}{\overset{a_2}{\rightleftarrows}} 3 \underset{3}{\overset{a_3}{\rightleftarrows}} 4 \rightleftarrows \cdots \rightleftarrows n-2 \underset{n-2}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \end{array}$$

$\Delta = \mathbb{L}_n \ (n \geq 1)$:

$$\epsilon=\bar\epsilon \circlearrowright 0 \underset{\bar a_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{1}{\overset{a_1}{\rightleftarrows}} 2 \rightleftarrows \cdots \rightleftarrows n-2 \underset{n-2}{\overset{a_{n-2}}{\rightleftarrows}} n-1.$$

The *preprojective algebra* $P(\Delta)$ associated to the graph Δ is the quotient of the path algebra kQ_Δ by the relations

$$\sum_{sa=i} a\bar a, \quad \text{for each vertex } i \text{ of } Q_\Delta.$$

The following proposition is classical [8, Prop 2.1].

PROPOSITION 9.1. — *The preprojective algebra $P(\Delta)$ is finite dimensional and selfinjective. Its Nakayama permutation ν is the identity for $\Delta = \mathbb{A}_1$, $\mathbb{D}_{2n}$, $\mathbb{E}_7$, $\mathbb{E}_8$ and $\mathbb{L}_n$, and is of order 2 in all other cases.*

9.2. Deformed preprojective algebras of generalized Dynkin type. — Let us recall the definition of deformed preprojective algebra introduced by [8]. Let Δ be a graph of generalized Dynkin type. We define an associated algebra $R(\Delta)$ as follows:

$$R(\mathbb{A}_n) = k, \quad R(\mathbb{D}_n) = k\langle x,y\rangle/\big(x^2, y^2, (x+y)^{n-2}\big),$$
$$R(\mathbb{E}_n) = k\langle x,y\rangle/\big(x^2, y^3, (x+y)^{n-3}\big), \quad R(\mathbb{L}_n) = k[x]/(x^{2n}).$$

Further, we fix an exceptional vertex in each graph as follows (with the notations of the previous section):

$$0 \text{ for } \Delta = \mathbb{A}_n \text{ or } \mathbb{L}_n, \quad 2 \text{ for } \Delta = \mathbb{D}_n, \quad 3 \text{ for } \Delta = \mathbb{E}_n.$$

Let f be an element of the square $\operatorname{rad}^2 R(\Delta)$ of the radical of $R(\Delta)$. The *deformed preprojective algebra* $P^f(\Delta)$ is the quotient of the path algebra kQ_Δ by the relations

$$\sum_{sa=i} a\bar{a}, \quad \text{for each non exceptional vertex } i \text{ of } Q,$$

and

$$\begin{aligned}
&a_0\bar{a}_0 && \text{for } \Delta = \mathbb{A}_n,\\
&\bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2 + f(\bar{a}_0a_0, \bar{a}_1a_1), \text{ and } (\bar{a}_0a_0 + \bar{a}_1a_1)^{n-2} && \text{for } \Delta = \mathbb{D}_n,\\
&\bar{a}_0a_0 + \bar{a}_2a_2 + a_3\bar{a}_3 + f(\bar{a}_0a_0, \bar{a}_2a_2), \text{ and } (\bar{a}_0a_0 + \bar{a}_2a_2)^{n-3} && \text{for } \Delta = \mathbb{E}_n,\\
&\epsilon^2 + a_0\bar{a}_0 + \epsilon f(\epsilon), \text{ and } \epsilon^{2n} && \text{for } \Delta = \mathbb{L}_n.
\end{aligned}$$

Note that if f is zero, we get the preprojective algebra $P(\Delta)$.

9.3. Corollaries of [8]. — The following proposition [8, Prop. 3.4] shows that the category $\operatorname{proj} P^f(\Delta)$ of finite-dimensional projective modules over a deformed preprojective algebra satisfies the hypothesis of Theorem 8.1.

PROPOSITION 9.2. — *Let $A = P^f(\Delta)$ be a deformed preprojective algebra. Then there exists an exact sequence of A-A-bimodules*

$$0 \to {}_1A_{\Phi^{-1}} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \to 0,$$

where Φ is an automorphism of A and where the P_i's are projective as bimodules. Moreover, for each idempotent e_i of A, we have $\Phi(e_i) = e_{\nu(i)}$.

So we can easily deduce the corollary:

Corollary 9.3. — *Let* $P^f(\Delta)$ *be a deformed preprojective algebra of generalized Dynkin type. Then the category* $\operatorname{proj} P^f(\Delta)$ *of finite dimensional projective modules is triangulated. The suspension is the Nakayama functor.*

Indeed, if $P_i = e_i A$ is a projective indecomposable, then $P_i \otimes_A A_\Phi$ is equal to $\Phi(e_i)A = e_{\nu(i)}A$ thus to $\nu(P_i)$.

Now we are able to answer to the question of the previous part and find a triangulated category with finitely many indecomposables which is not standard. The proof of the following theorem comes essentially from [8], Theorem 1.3.

Theorem 9.4. — *Let* k *be an algebraically closed field of characteristic* 2*. Then there exist* k*-linear triangulated categories with finitely many indecomposables which are not standard.*

Proof. — By [8, Thm. 1.3], we know that there exist basic deformed preprojective algebras of generalized Dynkin type $P^f(\Delta)$ which are not isomorphic to $P(\Delta)$. Thus the categories $\operatorname{proj} P^f(\Delta)$ and $\operatorname{proj} P(\Delta)$ can not be equivalent. But both are triangulated by Corollary 9.3 and have the same AR-quiver $\mathbb{Z}\Delta/\tau = Q_\Delta$. □

Conversely, we have the following theorem:

Theorem 9.5. — *Let* $\mathcal{T}$ *be a finite* 1*-Calabi-Yau triangulated category. Then* $\mathcal{T}$ *is equivalent to* $\operatorname{proj}\Lambda$ *as* k*-category, where* Λ *is a deformed preprojective algebra of generalized Dynkin type.*

Proof. — Let $M_1, \ldots, M_n$ be representatives of the isoclasses of indecomposable objects of $\mathcal{T}$. The k-algebra $\Lambda = \operatorname{End}(\bigoplus_{i=1}^n M_i)$ is basic, finite-dimensional and selfinjective since $\mathcal{T}$ has a Serre duality. It is easy to see that $\mathcal{T}$ and $\operatorname{proj}\Lambda$ are equivalent as k-categories.

Let $\operatorname{mod}\Lambda$ be the category of finitely presented Λ-modules. It is a Frobenius category. Denote by Σ the suspension functor of the triangulated category $\underline{\operatorname{mod}}\,\Lambda$. The category $\mathcal{T}$ is 1-Calabi-Yau, that is to say that the suspension functor S of the triangulated category $\mathcal{T}$ and the Serre functor ν are isomorphic. But in $\underline{\operatorname{mod}}\,\Lambda$, the functors S and Σ^3 are isomorphic. Thus, for each non projective simple Λ-module M we have an isomorphism $\Sigma^3 M \simeq \nu M$. We get immediately the result by [8, Thm. 1.2]. □

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