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POSITIVITY OF QUADRATIC BASE CHANGE L-FUNCTIONS

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ABSTRACT. — We show that certain quadratic base change L-functions for Gl(2) are non-negative at their center of symmetry.

RÉSUMÉ (*Positivité des fonctions L du changement de base quadratique*)
On montre que certaines des fonctions L de Gl(2) obtenues par changement de base quadratique sont positives en leur centre de symétrie.

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1. The main theorem

Let E/F be a quadratic extension of number fields and $\eta_{E/F}$ or simply η the quadratic character of F attached to E, $\{1,\tau\}$ the Galois group of E/F. We will often write

$$\tau(z)=\overline{z}.$$

We will denote by U_1 the unitary group in one variable, that is, the group of elements of norm 1 in $E^\times$. Suppose that π is an automorphic cuspidal representation of $\mathrm{Gl}(2,F_{\mathbb{A}})$ whose central character ω is trivial on the group of norms. In other words $\omega=1$ or $\omega=\eta$. We assume that π is not dihedral with respect to E so that the base change representation Π of π to $\mathrm{Gl}(2,E)$ is still automorphic and cuspidal. Let Ω be an idele class character of E whose restriction to $F_{\mathbb{A}}^\times$ is equal to ω. Our main result is the following theorem:

Theorem 1. — *With the previous notations:* $L\big(\frac{1}{2},\Pi\otimes\Omega^{-1}\big)\geq 0.$

If $\omega=1$ and $\Omega=1$ then $L(s,\Pi)=L(s,\pi)L(s,\pi\otimes\eta)$ and the result has been established by Guo (see [G1], under some restrictions on E/F). As a matter of fact, by using results on averages of L-functions (see [FH]), Guo is able to prove that $L(\frac{1}{2},\pi)\geq 0$, which then implies our result for $\omega=1$, $\Omega=1$, without restriction on E/F. At any rate, Baruch and Mao [BM] have independently established that $L(\frac{1}{2},\pi)\geq 0$ if $\omega=1$. However, the present result–where Ω needs not be trivial–is more general, even in the case $\omega=1$.

Results on the positivity of $\mathrm{Gl}(2)$ L-functions have been considered by many mathematicians (see, for instance, [BFH], [Gr], [K], [Kk], [KS], [KZ], [S], [R], [S], [W3], [Ya]). Specially, the positivity of the twisted L-function at hand has been investigated (for holomorphic forms) in [GZ].

We note that $\Omega^\tau=\Omega^{-1}$ and Π is self-contragredient: $\widetilde{\Pi}=\Pi$. Thus

$$L(s,\Pi\otimes\Omega^{-1})=L\big(s,\Pi^\tau\otimes(\Omega^{-1})^\tau\big)=L(s,\Pi\otimes\Omega)=L(s,\widetilde{\Pi}\otimes\Omega).$$

Likewise,

$$\epsilon(s,\Pi\otimes\Omega^{-1})\epsilon(1-s,\Pi\otimes\Omega^{-1})=\epsilon(s,\Pi\otimes\Omega^{-1})\epsilon(1-s,\widetilde{\Pi}\otimes\Omega)=1.$$

In particular $\epsilon(\frac{1}{2},\Pi\otimes\Omega^{-1})=\pm1$. Thus, despite the fact that $\Pi\otimes\Omega^{-1}$ is not necessarily self-contragredient, the L-function $L(s,\Pi\otimes\Omega^{-1})$ is symmetric:

$$L(s,\Pi\otimes\Omega^{-1})=\epsilon(s,\Pi\otimes\Omega^{-1})L(1-s,\Pi\otimes\Omega^{-1}).$$

The following lemma is easily verified:

Lemma 1. — *Let v_0 be a place of F. If v_0 is inert and v is the corresponding place of E then:*

$$L\big(\tfrac{1}{2},\Pi_v\otimes\Omega_v^{-1}\big)>0.$$

If v_0 splits into v_1 and v_2 then:

$$L\big(\tfrac{1}{2},\Pi_{v_1}\otimes\Omega_{v_1}^{-1}\big)L\big(\tfrac{1}{2},\Pi_{v_2}\otimes\Omega_{v_2}^{-1}\big)>0.$$

Let S_0 be a finite set of places of F and S the corresponding set of places of E. Set

$$L^S(s, \Pi \otimes \Omega^{-1}) = \prod_{v \notin S} L(s, \Pi_v \otimes \Omega_v^{-1}).$$

In view of the lemma, the statement of the theorem is equivalent to the positivity of $L^S(\frac{1}{2}, \Pi \otimes \Omega^{-1})$.

If Π is dihedral with respect to E, then Π is associated with an idele class character Ξ of E whose restriction to $F_{\mathbb{A}}^{\times}$ is $\omega\eta$. Thus $\Xi^{\tau} = \Xi^{-1}$ and

$$L(s, \Pi \otimes \Omega^{-1}) = L(s, \Xi\,\Omega^{-1}) L(s, \Xi^{\tau}\Omega^{-1}) = L(s, \Xi\,\Omega^{-1}) L(s, \Xi^{-1}\Omega^{-1}).$$

If Ω is trivial or even quadratic this is ≥ 0. At any rate, in general, $\Xi\,\Omega^{-1}$ and $\Xi^{-1}\Omega^{-1}$ have η for restriction to $F_{\mathbb{A}}^{\times}$. Thus there are cuspidal representations π_1 and π_2 of $\mathrm{Gl}(2, F_{\mathbb{A}}^{\times})$ with trivial central character such that:

$$L(s, \Xi\,\Omega^{-1}) = L(s, \pi_1), L(s, \Xi^{-1}\Omega^{-1}) = L(s, \pi_2)$$

and by the results already quoted each factor is ≥ 0 at $s = \frac{1}{2}$. We will not discuss this case but remark that, by considering the discrete but non-cuspidal terms in our trace formula, we could probably handle this case as well.

The proof of the theorem is based on a careful analysis of the relative trace formula of [J2] (In the case $\Omega = 1$ we could, like Guo, use the simpler trace formula of [J1].) Namely, we consider an inner form G of $\mathrm{Gl}(2, F)$ which contains a torus T isomorphic to $E^{\times}$. There is then an $\epsilon \in F^{\times}$, uniquely determined modulo $\mathrm{Norm}(E^{\times})$, such that the pair (G, T) is isomorphic to the pair (G_{ϵ}, T) defined as follows. We denote by $\mathbb{H}_{\epsilon}$ the semi-simple algebra of matrices $g \in M(2, E)$ of the form

$$g = \begin{pmatrix} a & \epsilon b \\ \overline{b} & \overline{a} \end{pmatrix} \tag{1}$$

and by G_{ϵ} its multiplicative group. Then

$$T = \left\{ t = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \right\}.$$

We let Z be the center of G_{ϵ}. We regard Ω as a character of $T(F_{\mathbb{A}})$: $t \mapsto \Omega(a)$. Suppose that f is a smooth function of compact support on $G_{\epsilon}(F_{\mathbb{A}})$. We form as usual a kernel

$$K_f(x, y) := \int_{Z(F_{\mathbb{A}})/F^{\times}} \sum_{\xi \in G_{\epsilon}(F)} f(x^{-1} z \xi y)\, \omega(z) \mathrm{d}z$$

and a distribution

$$J_{\epsilon}(f) := \int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}}))^2} K_f(t_1, t_2)\, \Omega(t_1)^{-1} \mathrm{d}t_1\, \Omega(t_2) \mathrm{d}t_2.$$

We have a spectral decomposition of the kernel:

$$K_f = \sum_\sigma K_{f,\sigma} + K_{f,\mathrm{cont}}\,;$$

the sum on the right is over all irreducible (cuspidal) automorphic representations σ of G_ϵ: if G_ϵ is not split, by cuspidal we mean an irreducible automorphic representation which is not one-dimensional. The term $K_{f,\mathrm{cont}}$ represents the contribution of the one dimensional representations and the continuous spectrum which is present only if G_ϵ is split, that is, ϵ is a norm. For every σ the kernel $K_{f,\sigma}$ is defined by

$$K_{f,\sigma}(x,y) = \sum_\phi \rho(f)\phi(x)\overline{\phi(y)},$$

the sum over an orthonormal basis of the space of σ. We define then:

$$J_\sigma(f) := \int_{(Z(F_\mathbb{A})T(F)\backslash T(F_\mathbb{A}))^2} K_{f,\sigma}(t_1,t_2)\,\Omega(t_1)^{-1}\mathrm{d}t_1\,\Omega(t_2)\mathrm{d}t_2.$$

This is a distribution of positive type: if $f = f_1 * f_1^*$ where $f_1^*(g) := \overline{f}_1(g^{-1})$ then

$$J_\sigma(f) = \sum \nu\big(\rho(f_1)\phi\big)\,\overline{\nu\big(\rho(f_1)\phi\big)},$$

where we have set

$$\nu(\phi) := \int_{Z(F_\mathbb{A})T(F)\backslash T(F_\mathbb{A})} \phi(t)\Omega(t)^{-1}\,\mathrm{d}t; \tag{2}$$

thus $J_\sigma(f) \geq 0$. Moreover, if ν is not identically zero on the space of σ, or as we shall say, if σ is *distinguished* by (T,Ω), then every local component σ_{v_0} is *distinguished* by (T_{v_0},Ω_{v_0}), that is, admits a non-zero continuous linear form ν_{v_0} such that $\nu_{v_0}(\pi_{v_0}(t)u) = \Omega_{v_0}(t)\nu_{v_0}(u)$ for all $t \in T_{v_0}$ and all smooth vectors u. The dimension of the space of such linear forms is one. One can then define a local distribution

$$J_{\sigma_{v_0}}(f_{v_0}) = \sum \nu_{v_0}\big(\rho(f_{v_0})u\big)\,\overline{\nu_{v_0}(u)},$$

the sum over an orthonormal basis. The distribution $J_{\sigma_{v_0}}$ is defined within a positive factor. It is of positive type. Normalizing in an appropriate way we get

$$J_\sigma(f) = C(\sigma)\prod_{v_0} J_{v_0}(f_{v_0}), \tag{3}$$

where the constant $C(\sigma)$ is positive. Assuming that $L(\frac{1}{2},\Pi\otimes\Omega^{-1}) \neq 0$ we can find an ϵ such that there is an automorphic representation σ of G_ϵ corresponding to π and distinguished by (T,Ω) (see [J2], [W4]). Another goal of the paper is to obtain an *explicit* decomposition of the above form, with a specific normalization (Theorem 2). The crux of the matter is then to show that $C(\sigma)$ is essentially equal to $L(\frac{1}{2},\Pi\otimes\Omega^{-1})$ which gives the positivity result. Possibly, this can be used to provide lower bounds for $L(\frac{1}{2},\Pi\otimes\Omega^{-1})$.

We note that if ϵ exists then it is unique. Indeed, this follows at once from the following local fact: if v_0 is inert, ϵ not a norm at v_0, π is a square integrable representation of $\mathrm{Gl}(2, F_{v_0})$, and σ the representation of $G_\epsilon(F_{v_0})$ corresponding to π then π and σ cannot be both distinguished by (T_{v_0}, Ω_{v_0}) (see [W4]).

We stress that there is no direct way to compute the constant $C(\sigma)$ because there is no direct relation between the global linear form ν and the local linear forms ν_{v_0}. The situation at hand (a globally defined distribution of positive type decomposed as a product over all places of F of local distributions of positive type, times the appropriate special values of L-functions) is, conjecturally, quite general. In this situation, the positivity of the special value of the L-function follows. One can view this question as a generalization of the problem of computing the Tamagawa number. This is our motivation for investigating in detail the present situation.

We proceed as follows. We introduce the matrices

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w_\epsilon = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}. \tag{4}$$

It will be more convenient to consider instead the distributions

$$\theta_\epsilon(f) := \int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}}))^2} K_f(t_1, t_2)\, \Omega(t_1 t_2)^{-1} \mathrm{d}t_1 \mathrm{d}t_2,$$

and, for σ an automorphic representation of G_ϵ,

$$\theta_\sigma(f) := \int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}})^2} K_{f,\sigma}(t_1, t_2)\, \Omega(t_1 t_2)^{-1} \mathrm{d}t_1 \mathrm{d}t_2.$$

Thus

$$J_\sigma(f) = \theta_\sigma\bigl(\rho(w_\epsilon) f\bigr)$$

and likewise for J_ϵ. We will decompose explicitly θ_σ into a product over all places v_0 of F of local distributions $\theta_{\sigma_{v_0}}$.

To that end, we compute the geometric expression for $\theta_\epsilon(f)$. A set of representatives for the double cosets of $T(F)$ in $G_\epsilon(F)$ is given by the matrices:

$$\mathbf{1}_2, \quad \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \epsilon\,\beta \\ \overline{\beta} & 1 \end{pmatrix}, \quad \beta \in E^\times / U_1(F).$$

We define orbital integrals. For $\xi \neq 1$ in $\mathrm{Norm}(E^\times)\epsilon$ we write $\xi = \beta\overline{\beta}\epsilon$ and set:

$$H(\xi; f) = \iint f\Bigl[t_1 \begin{pmatrix} \beta^{-1} & \epsilon \\ 1 & \overline{\beta}^{-1} \end{pmatrix} t_2\Bigr] \Omega(t_1 t_2^{-1}) \mathrm{d}t_1 \mathrm{d}t_2. \tag{5}$$

Note that the right hand side of the integral depends only on $\beta\overline{\beta}\epsilon$, which justifies the notations. In addition, we define

$$H(\infty; f) := \int f\Bigl[t_1 \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\Bigr] \Omega(t_1) \mathrm{d}t_1, \tag{6}$$

and, if Ω^2 is trivial,

$$H(0;f) := \int f(t_1)\Omega(t_1)\,\mathrm{d}t_1. \tag{7}$$

It will be useful to observe that the condition $\Omega^2 = 1$ is equivalent to $\Omega^\tau = \Omega$ or $\Omega_{|U_1}(F_{\mathbb{A}}) = 1$. The integrals are for $t_1 \in T(F_{\mathbb{A}})$ and $t_2 \in T(F_{\mathbb{A}})/Z(F_{\mathbb{A}})$. An idele class character χ of E is *normalized* if it is trivial on the subgroup E_∞^+ of ideles with all finite components trivial and all infinite components equal to some common positive number. For such a character we set $\delta(\chi) = 1$ if $\chi = 1$ and $\delta(\chi) = 0$ otherwise. All idele class characters will be assumed to be normalized. We have then

$$\begin{aligned}\theta_\epsilon(f) = &\sum_{\xi \in \mathrm{Norm}(E^\times)\epsilon - \{1\}} H(\xi; f)\\ &+\big(\delta(\Omega^2)H(0;f) + H(\infty;f)\big)\,\mathrm{vol}\big(T(F_{\mathbb{A}})/T(F)Z(F_{\mathbb{A}})\big).\end{aligned} \tag{8}$$

In what follows we denote by G the group $\mathrm{Gl}(2)$, by A the group of diagonal matrices, by Z the group of scalar matrices, by P the group of upper triangular matrices, by P_1 the subgroup of matrices with $(0,1)$ for second row and by N the subgroup of triangular matrices with unit diagonal. Depending on the context these groups are regarded as algebraic groups over F or E. We now view the group G as an algebraic group defined over E and we let H_0 be the unitary group for the matrix w. We denote by H the corresponding similitude group and by κ the similitude ratio. It is a result of [HLR] that Π is *distinguished* by $(H, \eta\omega)$, in the sense that there is a vector ϕ in the space of Π such that

$$\mathcal{P}(\phi) := \int_{H(F)Z(E_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} \phi(h)\eta\,\omega\big(\kappa(h)\big)\,\mathrm{d}h \neq 0.$$

This condition characterizes representations which are base change of representations of $\mathrm{Gl}(2, F_{\mathbb{A}})$ with central character ω (*loc. cit.*).

We may regard Ω as a character of $A(E_{\mathbb{A}})$ trivial on $Z(E_{\mathbb{A}})$:

$$\Omega\big[(\mathrm{diag}(a_1, a_2)\big] = \Omega(a_1 a_2^{-1}).$$

Then $L(\frac{1}{2}, \Pi \otimes \Omega^{-1}) \neq 0$ if and only if there is ϕ in the space of Π such that

$$\lambda(\phi) := \int_{Z(E_{\mathbb{A}})A(E)\backslash A(E_{\mathbb{A}})} \phi(a)\Omega^{-1}(a)\,\mathrm{d}a \neq 0. \tag{9}$$

These facts suggest the following construction. Let f be a smooth function of compact support on $G(E_{\mathbb{A}})$. Define a kernel

$$K_f(x,y) = \int_{Z(E_{\mathbb{A}})/Z(E)} \sum_{\xi \in G(E)} f(x^{-1} z \xi y)\,\mathrm{d}^\times z,$$

and a distribution:

$$\Theta(f) := \int_{Z(E_{\mathbb{A}})A(E)\backslash A(E_{\mathbb{A}})} \int_{Z(E_{\mathbb{A}})H(F)\backslash H(F_{\mathbb{A}})} K_f(a,h)\Omega^{-1}(a)\eta\omega\big(\kappa(h)\big)\,\mathrm{d}h\,\mathrm{d}^{\times}a. \tag{10}$$

The outer integral is not convergent and must be regularized. If Π is a cuspidal automorphic representation of $G(E_{\mathbb{A}})$ with trivial central character then we define similarly the distribution Θ_{Π}. This distribution is non-zero if and only if Π is distinguished by $(H,\omega\eta)$ and $L(\frac{1}{2},\Pi\otimes\Omega^{-1}) \neq 0$. It can decomposed *explicitly* into a product of local distributions.

The distribution Θ can be computed in terms of orbital integrals in the following way. We let $\mathcal{S}$ be the space of invertible Hermitian matrices and $\mathcal{S}_s$ the space of split Hermitian matrices. The group $\mathrm{Gl}(2,E)$ operates on $\mathcal{S}$ by $s \mapsto gs\,{}^t\overline{g}$. We let Φ_f or simply Φ be the function on the space $\mathcal{S}(F_{\mathbb{A}})$ such that

$$\Phi_f(gw\,{}^t\overline{g}) = \int_{H_0(F_{\mathbb{A}})} f(gh_0)\,\mathrm{d}h_0$$

and Φ_f vanishes outside $\mathcal{S}_s(F_{\mathbb{A}})$. We have

$$\int K_f\Big[\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}, h\Big]\eta\omega\big(\kappa(h)\big)\,\mathrm{d}h = \int \sum_{\xi\in\mathcal{S}(F)} \Phi\Big[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\xi\begin{pmatrix} \overline{a} & 0 \\ 0 & 1\end{pmatrix}z\Big]\omega\eta(z)\,\mathrm{d}^{\times}z.$$

This leads us to introduce the action of $E^{\times}\times F^{\times}$ defined by:

$$s \xmapsto{(\alpha,\zeta)} \begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix} s \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \zeta & 0 \\ 0 & \zeta\end{pmatrix}.$$

A system of representatives for the orbits of $E^{\times}\times F^{\times}$ on $\mathcal{S}(F)$ is constituted of the following matrices:

$$\begin{pmatrix} \xi^{-1} & 1 \\ 1 & 1\end{pmatrix},\ \xi\in F^{\times}-1, \quad \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1\end{pmatrix},\ \epsilon\in F^{\times}/\operatorname{Norm}(E^{\times}), \tag{11}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1\end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}. \tag{13}$$

For $\xi\in F^{\times}-\{1\}$, we set

$$U(\xi;\Phi) := \int_{E_{\mathbb{A}}^{\times}}\int_{F_{\mathbb{A}}^{\times}} \Phi\Big[\begin{pmatrix} z & 0 \\ 0 & z\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \xi^{-1} & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} \overline{y} & 0 \\ 0 & 1\end{pmatrix}\Big] \times \eta\,\omega(z)\,\mathrm{d}^{\times}z\,\Omega(y)\,\mathrm{d}^{\times}y.$$

If Ω is trivial on U_1 we define

$$U\Big[\begin{pmatrix}\epsilon^{-1} & 0\\ 0 & 1\end{pmatrix};\Phi\Big] := \int_{F_{\mathbb{A}}^{\times}}\int_{E_{\mathbb{A}}^{\times}/U_1(F_{\mathcal{A}})}\Phi\Big[\begin{pmatrix}\epsilon^{-1}y\overline{y}z & 0\\ 0 & z\end{pmatrix}\Big]\omega\eta(z)\Omega(y)\,\mathrm{d}^{\times}y\,\mathrm{d}^{\times}z.$$

The stabilizer of (12) in $E_{\mathbb{A}}^{\times}\times F_{\mathbb{A}}^{\times}$ is the set of pairs (y,z) with $yz=1$. However, because of our assumptions, the character $(y,z)\mapsto\Omega(y)\omega\eta(z)$ is non-trivial on that subgroup and so the element (12) does not contribute to the sum below. We also introduce two *unipotent* orbital integrals

$$U\Big[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix};\Phi\Big] := \iint\Phi\Big[\begin{pmatrix}0 & y\\ \overline{y} & z\end{pmatrix}\Big]\Omega(y)\,\mathrm{d}^{\times}y\,\eta(z)\,\mathrm{d}^{\times}z,$$

$$U\Big[\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix};\Phi\Big] := \iint\Phi\Big[\begin{pmatrix}z & y\\ \overline{y} & 0\end{pmatrix}\Big]\Omega(y)\,\mathrm{d}^{\times}y\,\eta(z)\,\mathrm{d}^{\times}z.$$

These integrals are improper integrals. For the first one, for instance, we remark that the function ϕ defined by:

$$\phi(z) := \int\Phi\Big[\begin{pmatrix}0 & y\\ \overline{y} & z\end{pmatrix}\Big]\Omega(y)\,\mathrm{d}^{\times}y$$

is a Schwartz-Bruhat function on $F_{\mathbb{A}}$. The unipotent integral is then the analytic continuation to the point $s=0$ of the Tate integral:

$$\int_{F_{\mathbb{A}}^{\times}}\phi(z)\eta(z)|z|^{s}\,\mathrm{d}^{\times}z.$$

We have then

$$\begin{aligned}(14)\quad \Theta(f) = &\sum_{\xi\in F^{\times}-\{1\}}U(\xi;\Phi)+\delta(\Omega^2)\operatorname{vol}\big(U_1(F_{\mathbb{A}})/U_1(F)\big)\\ &\times\sum_{\epsilon\in F^{\times}/\operatorname{Norm}(E^{\times})}U\Big[\begin{pmatrix}\epsilon^{-1} & 0\\ 0 & 1\end{pmatrix};\Phi\Big]+U\Big[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix};\Phi\Big]+U\Big[\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix};\Phi\Big].\end{aligned}$$

We say that f and a family of functions (f_ϵ), $\epsilon\in F^{\times}/\operatorname{Norm}(E^{\times})$, have *matching orbital integrals* if

$$U(\xi;\Phi)=H(\xi;f_\epsilon),$$

for $\xi\in\epsilon\operatorname{Norm}(E^{\times})-\{1\}$. Implicit in this definition is the fact that $f_\epsilon=0$ for almost all ϵ. We show that for any f there is a family (f_ϵ) with matching orbital integrals. Note that the converse is not true: to have a converse one would need to consider all unitary groups. When f and the family (f_ϵ) have matching orbital integrals, we have then

$$\Theta(f)=\sum_{\epsilon}\theta_\epsilon(f_\epsilon),$$

that is, the terms corresponding to all orbitals integrals match (see [J2]).

Now we apply standard arguments: by identifying the contribution of the continuous (and residual) spectrum on both sides we arrive at the identity

$$\Theta_\Pi(f) = \sum \theta_{\sigma_\epsilon}(f_\epsilon).$$

In a precise way, the representation Π is the base change of π and $\pi \otimes \eta$. The sum on the right is over all representations σ_ϵ which correspond to π or $\pi \otimes \eta$. As noted above, the sum on the right reduces to two terms, that is, there is a unique ϵ and a representation σ_ϵ of $G_\epsilon(F_\mathbb{A})$ which corresponds to π such that

$$\Theta_\Pi(f) = \theta_{\sigma_\epsilon}(f_\epsilon) + \theta_{\sigma_\epsilon \otimes \eta}(f_\epsilon).$$

The distribution Θ_Π can be decomposed explicitly into a product; the above identity allows us to decompose θ_{σ_ϵ} and, finally, prove the theorem.

The method is quite general and should apply to many situations. For instance, the work of [G2] suggests a possible direct generalization of the present set-up to $\mathrm{Gl}(2n)$, the group T being replaced by the group $\mathrm{Gl}(n, E)$ embedded in $\mathrm{Gl}(2n, F)$ and some inner forms of it. [G2] is concerned with the generalization of the simpler trace formula in [J1].

We remark that it would be interesting to compare our explicit result with the results of [W4].

The paper is arranged as follows. In sections 2 and 3 we review the results of [J2] on the matching of orbital integrals, reformulating the results in terms of symmetric spaces; we carefully normalize the various Haar measures. In section 4 we decompose the distribution Θ_Π explicitly into a product of local distributions. This is mainly a review of the material in [HLR]. The heart of the paper is section 5 where we compare the local distributions at hand.

The proof of the main theorem is given in section 6, with some additional comments in section 7. The paper concludes with sections 8 and 9, an appendix where we discuss the absolute convergence of the term coming from the continuous spectrum in the trace formulas at hand. Unfortunately, reference [J2] (also [J1]) is somewhat deficient on this point: the main point is that [J2] omits infinite sums on idele class characters unramified at all places. To make the argument rigorous we introduce, in the case at hand, a new form of truncation which may be more appropriate than the standard truncation operator for the investigation of period integrals (*cf.* [JLRo]) and a new device (special to $\mathrm{Gl}(2)$ or closely related groups) to estimate some period integrals. Note that [G1] which is based on [J1] is also deficient and so is [BM]. Thus we take care at once of a gap in several papers.

Finally, we would like to thank the referee for a careful and thorough reading of the manuscript.

2. Choice of measures and local matching

We review the results of [J2] on local matching. It will be essential to keep track of the choice of the various Haar measures.

2.1. Inert case. — We consider a local quadratic extension E/F. We choose an additive character ψ_F of F and set

$$\psi_E(z) = \psi_F(z + \overline{z}).$$

On the additive groups F and E we consider the self-dual Haar measures. In particular, if we write $E = F[\sqrt{\delta}\,]$ and $z = a + b\sqrt{\delta}$ then $\mathrm{d}z = \mathrm{d}a\mathrm{d}b\,|2|_F\,|\delta|_F^{1/2}$. The multiplicative Tamagawa measures on $F^\times$ and $E^\times$ are respectively:

$$\mathrm{d}^\times x = L(1, 1_F)\frac{\mathrm{d}x}{|x|_F}, \quad \mathrm{d}^\times z = L(1, 1_E)\frac{\mathrm{d}z}{|z|_E}\cdot$$

The following integration formula will be used below:

LEMMA 2. — *One has*

$$\int_{E^\times} \phi(z)\frac{\mathrm{d}z}{|z\overline{z}|_F} = \int_{\mathrm{Norm}(E^\times)} \widetilde{\phi}(x)\frac{\mathrm{d}x}{|x|_F},$$

where $\widetilde{\phi}(x)$ *is the function on* $\mathrm{Norm}(E^\times)$ *defined by*

$$\widetilde{\phi}(x) := \int_{E^\times/F^\times} \phi\Big(z\frac{u}{\overline{u}}\Big)\mathrm{d}^0u, \quad x = z\,\overline{z},$$

and the measure d^0u *is the quotient of* $\mathrm{d}z/|z|_E$ *by* $\mathrm{d}x/|x|_F$.

Since $T(F)$ is isomorphic to $E^\times$ via the map $\mathrm{diag}(a, \overline{a}) \mapsto a$, we obtain the Tamagawa measure on $T(F)$. Likewise the center Z of $\mathrm{Gl}(2)$ is isomorphic to $\mathrm{Gl}(1)$ and so we have the Tamagawa measure on $Z(F)$ and $Z(E)$.

On the group $\mathrm{Gl}(2, F)$ we have the Tamagawa measure

$$\mathrm{d}^\times g = L(1, 1_F)\frac{\mathrm{d}p\mathrm{d}q\mathrm{d}r\mathrm{d}s}{|\det g|_F^2}, \quad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Using the Iwasawa decomposition we can write

$$\mathrm{d}^\times g = \mathrm{d}^\times a\mathrm{d}^\times b\mathrm{d}x\mathrm{d}k,$$

if

$$g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}k,$$

where $\mathrm{d}k$ is a Haar measure on the standard maximal compact subgroup K_F of $\mathrm{Gl}(2, F)$. Let Φ be a Schwartz-Bruhat function on F^2. The following lemma is easily verified:

LEMMA 3. — *One has*

$$\iint \Phi(x,y)\mathrm{d}x\mathrm{d}y = \iint \Phi\big[(0,t)k\big]\cdot |t|^2\mathrm{d}^\times t\mathrm{d}k.$$

In particular, in the unramified situation, we can take for Φ the characteristic function of the set $\mathcal{O}_F^2$ and obtain $\mathrm{vol}(K_F) = L(2,1_F)^{-1}$. On the group $\mathrm{Gl}(2,E)$ we also have the Tamagawa measure

$$\mathrm{d}^\times g = L(1,1_E)\frac{\mathrm{d}g}{|\det g|_E^2}, \quad \mathrm{d}g = \mathrm{d}p\mathrm{d}q\mathrm{d}r\mathrm{d}s$$

if

$$g = \begin{pmatrix} p & q \\ r & s\end{pmatrix}.$$

On the group G_ϵ we take for the Tamagawa measure the measure:

$$\mathrm{d}^\times g = L(1,1_F)|\epsilon|_F\frac{\mathrm{d}a\mathrm{d}b}{|a\overline{a}-\epsilon b\overline{b}|_F^2},$$

if

$$g = \begin{pmatrix} a & \epsilon b \\ \overline{b} & \overline{a}\end{pmatrix}.$$

If ϵ is a norm the group $G_\epsilon(F)$ is isomorphic to $\mathrm{Gl}(2,F)$ and the isomorphism takes the above measure to the Tamagawa measure on $\mathrm{Gl}(2,F)$. From Lemma 2 we get the following integration formula:

$$\int_{G_\epsilon} F(g)\mathrm{d}^\times g \tag{15}$$
$$= \frac{1}{L(1,\eta)^2}\int_{x=\epsilon a\overline{a}}\Big\{\iint F\Big[t_1\begin{pmatrix} a^{-1} & \epsilon \\ 1 & \overline{a}^{-1}\end{pmatrix}t_2\Big]\mathrm{d}t_1\mathrm{d}t_2\Big\}\frac{\mathrm{d}x}{|1-x|_F^2}$$

with $t_1\in T(F)$, $t_2\in T(F)/Z(F)$.

In view of the isomorphism

$$\mathrm{Gl}(2,F)Z(E)/Z(E)\simeq \mathrm{Gl}(2,F)/Z(F)$$

we give to $\mathrm{Gl}(2,F)Z(E)$ the measure defined by

$$\mathrm{d}g = L(1,1_E)\frac{\mathrm{d}a}{|a|_F}\frac{\mathrm{d}b}{|b|_E}\mathrm{d}x\mathrm{d}y$$

if

$$g = \begin{pmatrix} ab & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}.$$

On the group H isomorphic to $\mathrm{Gl}(2,F)Z(E)$ the corresponding measure is given by

$$L(1,1_E)\frac{\mathrm{d}a}{|a|_F}\frac{\mathrm{d}b}{|b|_E}\mathrm{d}x_1\mathrm{d}y_1$$

for

$$g = \begin{pmatrix} ab & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

with $a \in F^\times$, $b \in E^\times$, $x = x_1\sqrt{\delta}$, $y = y_1\sqrt{\delta}^{-1}$, $x_1, y_1 \in F$. Of course, this measure does not depend on the choice of δ. Using the exact sequence

$$1 \to H_0 \longrightarrow H \xrightarrow{\kappa} F^\times \to 1,$$

we get the Tamagawa measure on H_0:

$$\mathrm{d}h_0 = L(1,\eta)\frac{\mathrm{d}b}{|b|_E}\mathrm{d}x_1\mathrm{d}y_1$$

if

$$h_0 = \begin{pmatrix} b & 0 \\ 0 & \overline{b}^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

We need the following integration formula:

LEMMA 4. — *The quotient of the Tamagawa measure on* $\mathrm{Gl}(2,E)$ *by the Tamagawa measure on* H_0 *is the following measure on the symmetric space* $\mathcal{S}_s$*:*

$$\mathrm{d}s = L(1,1_F)\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{|xy - z\overline{z}|_F^2}, \quad s = \begin{pmatrix} x & z \\ \overline{z} & y \end{pmatrix}.$$

For f a smooth function of compact support on $G_\epsilon(F)$ we define

$$H(x;f) := \int_{T(F)/Z(F)}\int_{T(F)} f\Big[t_1\begin{pmatrix} b^{-1} & \epsilon \\ 1 & \overline{b}^{-1} \end{pmatrix}t_2\Big]\Omega(t_1t_2^{-1})\mathrm{d}t_1\mathrm{d}t_2 \tag{16}$$

if $x = \epsilon b\overline{b}$, $x \neq 1$. The integrand is a smooth function of b depending only on $b\overline{b}\epsilon$ which justifies the notation. Next we define the orbital integrals for the space $\mathcal{S}(F)$ of invertible Hermitian matrices. As in the global case, the group $E^\times \times F^\times$ operates. Relative to this action we have the local orbital integrals of a function $\Phi \in \mathcal{C}_0^\infty(\mathcal{S}(F))$:

$$U(x;\Phi) := \int_{E^\times}\int_{F^\times} \Phi\Big[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \overline{y} & 0 \\ 0 & 1 \end{pmatrix}\Big] \times \eta\,\omega(z)\mathrm{d}^\times z\Omega(y)\mathrm{d}^\times y.$$

Now let $\{\epsilon_1, \epsilon_2\}$ be a system of representatives for the classes of $\mathrm{Norm}(E^\times)$ in $F^\times$, where ϵ_1 is a norm. Let $\Phi \in \mathcal{C}_0^\infty(\mathcal{S}(F))$ and f_{ϵ_i}, $i = 1,2$, be functions in $\mathcal{C}_0^\infty(G_{\epsilon_i}(F))$. We say that Φ and the pair $(f_{\epsilon_1}, f_{\epsilon_2})$ have *matching orbital integrals* if

$$U(x;\Phi) = H(x;f_{\epsilon_i}) \quad \text{for} \quad x \in \epsilon_i\,\mathrm{Norm}(E^\times).$$

PROPOSITION 1. — *Given* Φ *there is a pair* $(f_{\epsilon_1}, f_{\epsilon_2})$ *with matching orbital integrals.*

To prove the result one needs to consider the behavior at infinity of the orbital integrals defined above as well as the other orbital integrals. This is done in [J2]. We will use this result in the following way: f will be a smooth function of compact support on $\mathrm{Gl}(2,E)$ and we will set

$$\Phi_f(s) = \int_{H_0(F)} f(gh_0)\mathrm{d}h_0, \quad s = g\,w\,{}^t\overline{g},$$
$$\Phi_f(s) = 0, \quad s \notin \mathcal{S}_s(F).$$

If the pair $(f_{\epsilon_1}, f_{\epsilon_2})$ has matching orbital integrals with Φ_f we shall say that it has *matching orbital integrals* with f.

Consider now the unramified situation: the residual characteristic is odd, the extension E/F is unramified, the conductor of ψ_F is $\mathcal{O}_F$, ϵ_1 is a unit, Ω is unramified. Then $K_E := \mathrm{Gl}(2,\mathcal{O}_E)$ has volume $L(2,1_E)^{-1}$ and $K_F := \mathrm{Gl}(2,\mathcal{O}_F)$ has volume $L(2,1_F)^{-1}$. Likewise, the group

$$K_0 := \mathrm{Gl}(2,\mathcal{O}_E) \cap H_0(F)$$

is a maximal compact subgroup of $H_0(F)$ with volume $L(2,1_F)^{-1}$ and

$$K_\epsilon := \mathrm{Gl}(2,\mathcal{O}_E) \cap G_\epsilon(F)$$

is a maximal compact subgroup of G_ϵ with volume $L(2,1_F)^{-1}$. The *unramified Tamagawa measures* are obtained by multiplying the Tamagawa measures by the inverse of those volumes. Using the Cartan decomposition and the methods of [J1, pp. 199–204], one can prove the following proposition:

PROPOSITION 2. — *Suppose the situation is unramified. Let q be the cardinality of the residual field of F. For $n \geq 0$ let Φ_{2n} be the characteristic function of the set of matrices $s \in \mathcal{S}(F)$, with integral entries such that $|\det s|_F = q^{-2n}$. Let f_{2n} be the characteristic function of the set of matrices $g \in G_{\epsilon_1}(F)$ with integral entries such that $|\det g|_F = q^{-2n}$. Then Φ_{2n} and the pair $(f_{2n}, 0)$ have matching orbital integrals.*

Finally, we recall the fundamental lemma for the unramified situation. Since ϵ_1 is a norm, there is an isomorphism of $G_{\epsilon_1}(F)$ onto $G(F)$ taking K_{ϵ_1} to K_F and the Hecke algebra $\mathcal{H}(G_{\epsilon_1}(F), K_{\epsilon_1})$ to the Hecke algebra $\mathcal{H}(\mathrm{Gl}(2,F), K_F)$. We have thus a base change homomorphism b from $\mathcal{H}(\mathrm{Gl}(2,E), K_E)$ to $\mathcal{H}(G_{\epsilon_1}(F), K_{\epsilon_1})$.

PROPOSITION 3. — *For $f \in \mathcal{H}(K)$ the function Φ_f and the pair $(b(f), 0)$ have matching orbital integrals.*

Proof. — This is really a reformulation of the corresponding result of [J2]. It can derived more directly from the previous proposition by using the methods of [JLR, pp. 318–322]. $\square$

2.2. Split case. — *Mutatis mutandis*, the above discussion applies to the split case where the quadratic extension is replaced by the algebra $E = F \oplus F$ with F embedded diagonally in E and Galois action $\tau(x,y) = (y,x)$. If we write $z = (x,y)$ then $\psi_E(z) = \psi(x+y)$ and so the self dual Haar measure on E is $\mathrm{d}z = \mathrm{d}x\mathrm{d}y$. To obtain the Tamagawa measure on $\mathrm{Gl}(2,E)$ we must divide the self-dual additive measure $\mathrm{d}g = \mathrm{d}g_1\mathrm{d}g_2$ by $|\det g_1g_2|_F$ and then multiply by $L(1,1_F)^2$. So the isomorphism $\mathrm{Gl}(2,E) \simeq \mathrm{Gl}(2,F) \times \mathrm{Gl}(2,F)$ preserves the Tamagawa measures.

Likewise, the group H_0 becomes the group of pairs (h_1,h_2) with $h_2 = w\,{}^t h_1^{-1} w$. Choosing the first factor in the decomposition $E = F \oplus F$, we have an isomorphism $H_0(F) \simeq \mathrm{Gl}(2,F)$ which is also compatible with the Tamagawa measures.

The group $G_\epsilon(F)$ is the group of invertible matrices of the form

$$\begin{pmatrix} a & \epsilon b \\ \bar{b} & \bar{a} \end{pmatrix},$$

with a,b in E. We consider the measure

$$|\epsilon|_F\, L(1,1_F)\frac{\mathrm{d}a\mathrm{d}b}{|a\bar{a} - \epsilon b\bar{b}|_F^2}.$$

Thus $G_\epsilon(F)$ is the group of pairs

$$(g_1, g_2)$$

with $g_2 = w_\epsilon g w_\epsilon^{-1}$. Again $(g_1,g_2) \mapsto g_1$ defines an isomorphism $G_\epsilon(F) \simeq \mathrm{Gl}(2,F)$ which takes the above measure to the Tamagawa measure.

The manifold $S(F)$ is the submanifold of pairs

$$S(F) = \{(s_1,s_2) \mid s_2 = {}^t s_1\}$$

and is again isomorphic to $G(F)$ via $(s_1,s_2) \mapsto s_1$.

The torus $T(F)$ is the subgroup of pairs:

$$t = \left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ o & a_1 \end{pmatrix}\right).$$

A character Ω of $T(F)$ trivial on $Z(F)$ has the form

$$\Omega(t) = \Omega_1(a_1)\Omega_2(a_2)$$

with $\Omega_1\Omega_2 = 1$.

In particular, if $f_1 \otimes f_2$ is a product function on $G(E)$ then

$$\Phi_{f_1\otimes f_2}(s_1,s_2) = \int f_1(g_1h_1)f_2(g_2h_2)\mathrm{d}h_0$$

becomes

$$\Phi_{f_1\otimes f_2}(g) = \int f_1(gwh)f_2(w\,{}^t h^{-1}w)\mathrm{d}h = f_1 * \tilde{f}_2(gw)$$

where we have set

$$\tilde{f}_2(g) = f_2(w\,{}^t g w).$$

With this identification, the orbital integrals for $\Phi \in \mathcal{C}_0^\infty(\mathcal{S}(F))$ become:

$$U(x;\Phi) = \iiint \Phi\Big[\begin{pmatrix} z & 0 \\ 0 & z\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} x^{-1} & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix}\Big] \\ \times \Omega_1(a)\Omega_2(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}^\times z.$$

Similarly, for $f_\epsilon \in \mathcal{C}_0^\infty(G_\epsilon(F))$, the orbital integrals take the form:

$$H(x;f_\epsilon) = \iiint f_\epsilon\Big[\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} u_1^{-1} & \epsilon \\ 1 & u_2^{-1}\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 1\end{pmatrix}\Big] \\ \times \Omega_1(a)\Omega_2(b)\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\Omega_1(c)^{-1}\,\mathrm{d}^\times c,$$

with $x = u_1 u_2 \epsilon$. The condition of matching $U(x;\Phi) = H(x;f_\epsilon)$ is trivially verified with

$$f_\epsilon(g) = \Phi\Big[g\begin{pmatrix} \epsilon & 0 \\ 0 & 1\end{pmatrix}\Big].$$

If $\Phi = \Phi_{f_1\otimes f_2}$ this becomes the condition of matching for the split case:

$$f_\epsilon(g) = f_1 * \tilde{f}_2(g w_\epsilon). \tag{17}$$

If F is non-Archimedean, Ω_1 unramified, the functions f_i bi-invariant under $K_F = \mathrm{Gl}(2,\mathcal{O}_F)$, and the Haar measure of K_F is 1, then $\tilde{f}_2 = f_2$, $w_\epsilon \in K_F$ and

$$f = \Phi = f_1 * f_2.$$

3. Global matching

We now consider an extension E/F of number fields.

3.1. Global Haar measures. — On the group $F_\mathbb{A}^\times$ we consider the Tamagawa measure $d^\times x$ which is the (convergent) product of the local Tamagawa measures. We let F^1 be the group of ideles of norm 1 and use the exact sequence

$$1 \to F^1 \longrightarrow F_\mathbb{A}^\times \to \mathbb{R}_+^\times$$

to define a measure on F^1 for which

$$\mathrm{vol}(F^1/F^\times) = \mathrm{res}_{s=1} L(s, 1_F).$$

Likewise for E.

We also consider a finite set places S_0 of F and the let S be the corresponding set of places of E. We assume that S_0 contains all the places at infinity, the places of even residual characteristic, the finite places which ramify in E, and all the places where the character ψ is ramified (that is, the conductor is not the ring of integers). We also assume that Ω is unramified outside S. We enlarge S_0 as the need arises.

On the group $\mathrm{Gl}(2, F_{\mathbb{A}})$ we consider the Tamagawa measure $\mathrm{d}^\times g$ which is the convergent product of the local Tamagawa measures and multiply it by the factor $L^{S_0}(2, 1_F) := \prod_{v_0 \notin S_0} L(2, 1_{F_{v_0}})$. In other words, the measure we are considering is the product of the Tamagawa measures $\mathrm{d}^\times g_{v_0}$ for $v_0 \in S_0$ times the measures $\mathrm{d}g_{v_0}$, $v_0 \notin S_0$, for which the measure of $\mathrm{Gl}(2, \mathcal{O}_{v_0})$ is one. Using the Iwasawa decomposition

$$g = \mathrm{diag}(a, b) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k$$

we obtain a measure on the standard maximal compact K:

$$\mathrm{d}g = \mathrm{d}^\times a \mathrm{d}^\times b \mathrm{d}x \mathrm{d}k.$$

We have then the following identity for a Schwartz-Bruhat function Φ in two variables:

$$\iint \Phi(x, y) \mathrm{d}x \mathrm{d}y = \iint \Phi\big[(0, t)k\big] \cdot |t|^2 \mathrm{d}^\times t \, \mathrm{d}k \frac{1}{L^{S_0}(2, 1_F)} \cdot$$

If $\Phi = \prod \Phi_{v_0}$ with Φ_{v_0} the characteristic function of $\mathcal{O}_{v_0}^2$ outside S_0 we get then:

$$\iint \Phi(x, y) \mathrm{d}x \mathrm{d}y = \int_{K_{S_0}} \int_{F_S^\times} \Phi_{S_0}\big[(0, t)k\big] \cdot |t|^2 \mathrm{d}^\times t \, \mathrm{d}k.$$

We use the similar measure on $\mathrm{Gl}(2, E_{\mathbb{A}})$. Likewise for $H_0(F_{\mathbb{A}})$ and $H(F_{\mathbb{A}})$ we multiply the Tamagawa measure by $L^{S_0}(2, 1_F)$.

3.2. Matching. — As explained in the introduction, we consider a smooth function of compact support f on $G(E_{\mathbb{A}})$ and the corresponding function $\Phi = \Phi_f$ on $\mathcal{S}(F_{\mathbb{A}})$. The function f is a product of local functions f_v. For $v \notin S$ the function f_v is bi-invariant under $K_v := \mathrm{Gl}(2, \mathcal{O}_v)$. We choose a set of representative $\{\epsilon\}$ for $F^\times / \mathrm{Norm}(E^\times)$. For each ϵ we choose a smooth function of compact support f_ϵ on $G_\epsilon(F_{\mathbb{A}})$. It is a product of local functions that we choose as follows. For a place v_0 inert in F and the corresponding place v of E we demand that f_{ϵ, v_0} and Φ_{v_0} have matching orbital integrals, that is, $H(x; f_{\epsilon, v_0}) = U(x; \Phi_{v_0})$ for $x \in \epsilon \, \mathrm{Norm}(E_v^\times)$. If $v_0 \notin S_0$ and ϵ is not a norm at the place v_0, we have seen that $H(x; \Phi_{v_0}) = 0$ for $x \in \epsilon \, \mathrm{Norm}(E_v^\times)$ and so we take $f_{\epsilon, v_0} = 0$. Thus we have $f_\epsilon = 0$ unless ϵ is a norm at all places not in S_0. Thus there is only a finite set Ξ of ϵ such that $f_\epsilon \neq 0$. There is a finite set $S_0' \supseteq S_0$ of places of F such that the $\epsilon \in \Xi$ are unit at all places not in S_0'. We let S' be the set of places of E above a place in S_0'.

For v_0 inert not in S_0' we may and do assume that $f_{\epsilon, v_0} = b(f_v)$ where v is the corresponding place of E and b is the base change homomorphism. For v_0 split into v_1 and v_2 in E we assume that

$$f_{\epsilon, v_0}(g) = \Phi_{v_0}\Big[g \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\Big].$$

In particular, for $\epsilon \in \Xi$ and $v_0 \notin S_0'$ this means that (with the usual identifications) $f_{\epsilon,v_0} = f_1 * f_2$. This being so, we change our notations and now take the set S_0 to be the set S_0'.

We compare the geometric terms in the two relative trace formulas. As usual we set

$$\mathrm{e}^{H(g)} = |a_1 a_2^{-1}|$$

if $g = nak$, $a = \mathrm{diag}(a_1, a_2)$.

As in [J2], we consider a *compact* truncation along the diagonal. Namely, if ϕ is a function on $A(E_{\mathbb{A}})$ invariant under $A(E)$ and $Z(E_\infty^+)$, we set:

$$\Lambda_c^T \phi(a) = \phi(a) \qquad \text{if } -T < H(a) < T, \tag{18}$$

$$\Lambda_c^T \phi(a) = 0 \qquad \text{otherwise.} \tag{19}$$

We define then

$$\Theta_T(f) := \iint_{Z(E_{\mathbb{A}})H(F)\backslash H(F_{\mathbb{A}})} \Lambda_c^T K_f(a,h)\Omega^{-1}(a)\,\mathrm{d}a\omega\eta\bigl(\kappa(h)\bigr)\,\mathrm{d}h.$$

The outer integral is over $A(E_{\mathbb{A}})/A(E)Z(E_{\mathbb{A}})$. This integral is absolutely convergent and equal to:

$$\int_{\mathrm{e}^{-T}\le|a|\le e^T} \int_{F_{\mathbb{A}}^\times/F^\times} \sum_{\xi\in\mathcal{S}(F)} \Phi\Bigl[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\xi\begin{pmatrix} \overline{a} & 0\\ 0 & 1\end{pmatrix} z\Bigr]\Omega^{-1}(a)\omega\eta(z)\,\mathrm{d}^\times z.$$

In [J2] it is shown that as $T \to +\infty$ this tends to the right hand side of (14). Thus we define

$$\Theta(f) =: \lim_{T\longrightarrow+\infty} \Theta_T(f),$$

and we arrive at (14). Comparing with (8) we get:

$$\Theta(f) = \sum_{\epsilon\in\Xi} \theta(f_\epsilon).$$

Indeed given $\xi \in F^\times - \{1\}$ we have $\xi \in \epsilon\,\mathrm{Norm}(E^\times)$ for a unique ϵ and

$$U(\xi;\Phi) = H(\xi; f_\epsilon).$$

The matching of the other terms is explained in [J2].

4. Factorization of Θ_Π over E

Most of the material of this section is a review of [HLR] (see also [A], [F2], [FZ]).

We now consider the Epstein Eisenstein series associated to a Schwartz-Bruhat function Φ on $F_{\mathbb{A}}^2$; we assume that Φ is a product, the local component

Φ_{v_0} being the characteristic function of $\mathcal{O}_{v_0}^2$ for all $v_0 \notin S_0$:

$$E(g,\Phi,s) = \int_{F_{\mathbb{A}}^\times/F^\times} \sum_{\xi\in F^2-\{(0,0)\}} \Phi[\xi t g]\cdot |\det t|^{2s}\mathrm{d}^\times t\,|\det g|^s.$$

It is meromorphic, with simple pole at $s=1$ and residue:

$$\frac{1}{2}\operatorname{vol}\Big(\frac{F^1}{F^\times}\Big)\iint \Phi(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{2}\Big(\frac{F^1}{F^\times}\Big)\int_{K_{S_0}}\int_{F_{S_0}^\times} \Phi_{S_0}\big[(0,t)k\big]\cdot |t|^2\mathrm{d}^\times t\mathrm{d}k.$$

Let Π be the base change of a representation π with central character η and ϕ a cusp form in the space of Π which is invariant under translations by the compact subgroup $K^S := \prod_{v\notin S} K_{E_v}$. Let us compute the integral:

$$\Psi(s,\phi,\Phi) := \int_{G(F)Z(F_{\mathbb{A}})\backslash G(F_{\mathbb{A}})} E(g,\Phi,s)\phi(g)\mathrm{d}g.$$

It is a meromorphic function of s, with simple pole at $s=1$ and residue:

$$\frac{1}{2}\operatorname{vol}\Big(\frac{F^1}{F^\times}\Big)\int_{K_{S_0}}\int_{F_{S_0}^\times} \Phi_{S_0}\big[(0,t)k\big]\cdot |t|^2\mathrm{d}^\times t\mathrm{d}k\int\phi(g)\mathrm{d}g. \tag{20}$$

On the other hand, for $\operatorname{Re} s \gg 0$,

$$\Psi(s,\phi,\Phi) = \Psi(s,W,\Phi)$$

where we have set:

$$W(g) := \int_{E_{\mathbb{A}}} \phi\Big[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\Big]\psi_E(-x)\mathrm{d}x$$

and, writing $E = F[\sqrt{\delta}\,]$,

$$\Psi(s,W,\Phi) := \int_{N(F_{\mathbb{A}})\backslash \mathrm{Gl}(2,F_{\mathbb{A}})} W\Big[\begin{pmatrix}\sqrt{\delta}^{\,-1} & 0\\ 0 & 1\end{pmatrix} g\Big]\Phi\big[(0,1)g\big]\cdot|\det g|^s\mathrm{d}^\times g.$$

We assume that W is a product

$$W(g) = \prod_v W_v(g_v)$$

over all places of E, with $W_{v|K_{E_v}} = 1$ outside S. The local component is an element of the local Whittaker model $\mathcal{W}(\Pi_v,\psi_v)$. If v_0 is a place of F we write $E_{v_0} = E\otimes F_{v_0}$. If v_0 is inert in E we also write W_{v_0} for W_v. If v_0 splits into v_1, v_2 we write W_{v_0} for $W_{v_1}\otimes W_{v_2}$, a function on $\mathrm{Gl}(2,E_{v_0})$. We can then set in all cases:

$$\Psi(s,W_{v_0},\Phi_{v_0}) = \int_{N(F_{v_0})\backslash \mathrm{Gl}(2,F_{v_0})} W_{v_0}\Big[\begin{pmatrix}\sqrt{\delta}^{\,-1} & 0\\ 0 & 1\end{pmatrix} g\Big]\Phi_{v_0}\big[(0,1)g\big]\cdot|\det g|^s\mathrm{d}^\times g.$$

Explicitly, if v_0 splits, that is, δ is the square of two elements ϵ_1 and ϵ_2 of F_{v_0}, the above integral is in fact equal to:

$$\int W_{v_1}\Big[\begin{pmatrix}\epsilon_1^{-1} & 0\\ 0 & 1\end{pmatrix}g\Big]W_{v_2}\Big[\begin{pmatrix}\epsilon_2^{-1} & 0\\ 0 & 1\end{pmatrix}g\Big]\Phi_{v_0}\big[(0,1)g\big]\cdot|\det g|^s\mathrm{d}^\times g.$$

We recall that S_0 is so large that all the data is unramified outside S_0 (or S).

We introduce the partial Asai L-function attached to Π noted $L^{S_0}(s,\Pi,\mathrm{Asai})$ (see [HLR]). For every place $v_0\notin S_0$ the integral $\Psi(s,W_{v_0},\Phi_{v_0})$ is equal to the corresponding local L-factor of the Asai L-function. Thus

$$\Psi(s,\phi,\Phi)=L^{S_0}(s,\Pi,\mathrm{Asai})\prod_{v_0\in S_0}\Psi(s,W_{v_0},\Phi_{v_0}).$$

Taking the residue at $s=1$ we obtain that (20) is equal to:

$$\mathrm{res}_{s=1}\,L^{S_0}(s,\Pi,\mathrm{Asai})\prod_{v_0\in S_0}\Psi(1,W_{v_0},\Phi_{v_0}).$$

This equality implies that, as a linear form in W_{v_0}, $\Psi(1,W_{v_0},\Phi_{v_0})$ is invariant under right shifts by $\mathrm{Gl}(2,F_{v_0})$. As a linear form in Φ_{v_0}, it is proportional to

$$\int_{K_{v_0}}\int_{F_{v_0}^\times}\Phi_{v_0}[(0,t)k]\cdot|t|^2\mathrm{d}^\times t\,\mathrm{d}k.$$

Now let us set:

$$\mathcal{P}_{v_0}(W_{v_0}):=\int_{F_{v_0}^\times}W_{v_0}\begin{pmatrix}\sqrt{\delta}^{\,-1}a & 0\\ 0 & 1\end{pmatrix}\mathrm{d}^\times a.$$

This is a linear form invariant under $\mathrm{Gl}(2,F_{v_0})$. Moreover, in the Archimedean case, it is continuous for the topology of the smooth vectors. The invariance is well known for v_0 split (see also [B]) and is established in [H] for v_0 inert. At any rate, it follows from the above considerations. Then

$$\Psi(1,W_{v_0},\Phi_{v_0})=\mathcal{P}_{v_0}(W_{v_0})\int_{K_{v_0}}\int_{F_{v_0}^\times}\Phi_{v_0}\big[(0,t)k\big]\cdot|t|^2\mathrm{d}^\times t\,\mathrm{d}k.$$

The above residue becomes:

$$\mathrm{res}_{s=1}\,L^{S_0}(s,\Pi,\mathrm{Asai})\prod_{S_0}\mathcal{P}_{v_0}(W_{v_0})\int_{K_{S_0}}\int_{F_{S_0}^\times}\Phi_{S_0}\big[(0,t)k\big]\cdot|t|^2\mathrm{d}^\times t\,\mathrm{d}k.$$

Comparing with the previous residue computation, we get:

$$\int\phi(g)\mathrm{d}g=\prod_{v_0\in S_0}\mathcal{P}_{v_0}(W_{v_0})\frac{2\,\mathrm{res}_{s=1}\,L^{S_0}(s,\Pi,\mathrm{Asai})}{\mathrm{vol}(F^1/F^\times)}\cdot\tag{21}$$

The identity is to be interpreted as the statement that the period integral of ϕ is non-zero if and only the Asai L-function has a pole at $s=1$ and each of the local integral $\mathcal{P}_{v_0}$ does not vanish on W_{v_0}.

REMARK. — The convergence of the integral defining $\mathcal{P}_{v_0}$ follows from the local analog of the integral representation for Whittaker functions given below in (44). Formula (21) implies that the linear form $\mathcal{P}_{v_0}$ is continuous for the topology of smooth vectors, since the left hand side is.

Recall that we write $E = F[\sqrt{\delta}\,]$. Then

$$\begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1 \end{pmatrix} H(F) \begin{pmatrix} \sqrt{\delta}^{\,-1} & 0 \\ 0 & 1 \end{pmatrix} = Z(E)\,\mathrm{Gl}(2,F). \tag{22}$$

Accordingly:

$$\int_{H(F)Z(E_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} \phi(h)\,\mathrm{d}h = \int_{Z(F_{\mathbb{A}})\,\mathrm{Gl}(2,F)\backslash\,\mathrm{Gl}(2,F_{\mathbb{A}})} \phi\Big[g\begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1 \end{pmatrix}\Big]\,\mathrm{d}g$$

or

$$\int_{H(F)Z(E_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} \phi(h)\,\mathrm{d}h = \prod_{S_0} \mathcal{P}_{v_0}(W_{v_0})\,\frac{2\,\mathrm{res}_{s=1}\,L^{S_0}(s,\Pi,\mathrm{Asai})}{\mathrm{vol}(F^1/F^\times)} \tag{23}$$

where here we have set:

$$\mathcal{P}_{v_0}(W_{v_0}) = \int_{F^\times_{v_0}} W_{v_0}\Big[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big]\,\mathrm{d}^\times a.$$

We also denote by $L^{S_0}(s,\Pi,\mathrm{Asai};1)$ the previous Asai L-function and by $L(s,\Pi,\mathrm{Asai};\eta)$ the function $L(s,\Pi\otimes\mu,\mathrm{Asai};1)$ where μ is an idele class character of E whose restriction to F is η. Suppose that now Π is the base change of π with trivial central character. Then:

$$\begin{aligned}\int_{H(F)Z(E_{\mathbb{A}})\backslash H(F_{\mathbb{A}})} &\phi(h)\eta\big(\kappa(h)\big)\,\mathrm{d}h \\ &= \prod_{S_0} \mathcal{P}_{v_0}(W_{v_0})\,\frac{2\,\mathrm{res}_{s=1}\,L^{S_0}(s,\Pi,\mathrm{Asai};\eta)}{\mathrm{vol}(F^1/F^\times)}\end{aligned} \tag{24}$$

where here we have set:

$$\mathcal{P}_{v_0}(W_{v_0}) = \int_{F^\times_{v_0}} W_{v_0}\Big[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big]\eta(a)\,\mathrm{d}^\times a.$$

From now on we consider both cases at once, that is, Π is the base change of π with central character $\omega = 1$ or $\omega = \eta$.

A similar discussion applies to the computation of the scalar product of two forms ϕ_1, ϕ_2. As usual we assume that the corresponding functions W_i are product and the data is unramified outside S. We introduce a scalar product $\mathcal{B}_v$ on $\mathcal{W}(\Pi_v,\psi_v)$ for any place $v \in S$:

$$\mathcal{B}_v(W_1,W_2) := \int_{E^\times_v} W_1\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}\overline{W_2\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}}\,\mathrm{d}^\times a_1.$$

Then:

$$\int \phi_1(g)\overline{\phi_2}(g)\,\mathrm{d}g = \prod_{v\in S} \mathcal{B}_v\big((W_1)_v,(W_2)_v\big)\frac{2\,\mathrm{res}_{s=1} L^S(s,\Pi\times\widetilde{\Pi})}{\mathrm{vol}(E^1/E^\times)}.$$

We set:

$$\lambda(\phi) := \int_{E_{\mathbb{A}}^\times/E^\times} \phi\Big[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\Big]\Omega^{-1}(a)\,\mathrm{d}^\times a.$$

At a place $v\in S$, we define

$$\lambda_v(W_v) := \int_{E_v^\times} W_v\Big[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\Big]\Omega^{-1}(a)\,\mathrm{d}^\times a.$$

Then

$$\lambda(\phi) = \prod_{v\in S}\lambda_v(W_v)\, L^S\big(\tfrac12,\Pi\otimes\Omega^{-1}\big).$$

We consider the global distribution

$$\Theta_\Pi(f) := \sum_\phi \lambda\big(\Pi(f)\phi\big)\overline{\int \phi(h)\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h},$$

where the sum is over an orthonormal basis. Likewise, for a place v_0 inert in E with corresponding place v of E we define

$$\Theta_{\Pi,v_0}(f_v) := \sum_W \lambda_v\big(\Pi_v(f_v)W\big)\overline{\mathcal{P}_v(W)},$$

the sum over an orthonormal basis for the scalar product $\mathcal{B}_v$. Whenever convenient, we write f_{v_0} for f_v

Similarly, at a place v_0 which splits into v_1,v_2 we define

$$\Theta_{\Pi,v_0}(f_{v_1}\otimes f_{v_2}) := \sum_{W_1,W_2}\lambda_{v_1}\big(\Pi_{v_1}(f_{v_1})W_1)\big)\lambda_{v_2}\big(\Pi_{v_2}(f_{v_2})W_2)\big)\overline{\mathcal{P}_{v_0}(W_{v_1}\otimes W_{v_2})}.$$

We again write f_{v_0} for $f_{v_1}\otimes f_{v_2}$.

Now we assume that the function f is a product with f^S the characteristic function of K^S. Then:

$$\Theta_\Pi(f) = \prod_{v_0\in S_0}\Theta_{\Pi,v_0}(f_{v_0})\,\frac{\mathrm{vol}(E^1/E^\times)\,\mathrm{res}_{s=1} L^{S_0}(s,\Pi,\mathrm{Asai};\omega\eta)L^S(\frac12,\Pi\otimes\Omega^{-1})}{\mathrm{vol}(F^1/F^\times)\,\mathrm{res}_{s=1} L^S(s,\Pi\times\widetilde{\Pi})}.$$

Recall that here $\widetilde{\Pi}=\Pi=\Pi^\tau$ and Π is the base change of π with central character ω. We have the relation

$$L^S(s,\Pi\times\Pi^\tau) = L^{S_0}(s,\pi\times\pi)L(s,\pi\times\pi\otimes\eta).$$

If $\omega=1$ we have the relation

$$L^{S_0}(s,\Pi,\mathrm{Asai};\eta) = \frac{L^{S_0}(s,\pi\times\pi\otimes\eta)L^{S_0}(s,1_F)}{L^{S_0}(s,\eta)}.$$

Moreover since π is self contragredient then

$$L^{S_0}(s,\pi\times\pi)=L^{S_0}(s,1_F)L^{S_0}(s,\pi;\mathrm{Ad}).$$

where the last factor on the right is the partial adjoint L-function. Thus

$$\frac{L^{S_0}(s,\Pi,\mathrm{Asai};\eta)}{L^{S}(s,\Pi\times\widetilde{\Pi})}=\frac{1}{L^{S_0}(s,\eta)L^{S_0}(s,\pi,\mathrm{Ad})}\cdot$$

If $\omega=\eta$ then

$$L^{S_0}(s,\Pi,\mathrm{Asai};1)=\frac{L^{S_0}(s,\pi\times\pi)L^{S_0}(s,1_F)}{L^{S_0}(s,\eta)}\cdot$$

On the other hand

$$L^{S_0}(s,\pi\times\pi\otimes\eta)=L^{S_0}(s,\pi\times\widetilde{\pi})=L^{S_0}(s,1_F)L^{S_0}(s,\pi,\mathrm{Ad}).$$

Thus

$$\frac{L^{S_0}(s,\Pi,\mathrm{Asai};1)}{L^{S}(s,\Pi\times\widetilde{\Pi})}=\frac{1}{L^{S_0}(s,\eta)L^{S_0}(s,\pi,\mathrm{Ad})}\cdot$$

Also

$$\mathrm{vol}(E^1/E^\times)=L(1,\eta)\,\mathrm{vol}(F^1/F^\times).$$

We can thus simplify our ratio of L-functions to obtain the following result:

PROPOSITION 4. — *One has*

$$\Theta_\Pi(f)=\prod_{v_0\in S_0}\Theta_{\Pi,v_0}(f_{v_0})\frac{L_{S_0}(1,\eta)L^S(\frac12,\Pi\otimes\Omega^{-1})}{L^{S_0}(1,\pi,\mathrm{Ad})}\cdot$$

5. Local comparisons

As explained in the introduction, when f and the family (f_ϵ) match we have a spectral identity, with a single ϵ on the right, which is uniquely determined by Π:

$$\Theta_\Pi(f)=\theta_{\sigma_\epsilon}(f_\epsilon)+\theta_{\sigma_\epsilon\otimes\eta}(f_\epsilon).$$

We recall the expression for θ_{σ_ϵ}:

$$\theta_{\sigma_\epsilon}(f_\epsilon)=\sum_\phi\int\sigma_\epsilon(f_\epsilon)\phi(t)\Omega(t)^{-1}\,\mathrm{d}t\overline{\int\phi(t)\Omega(t)\,\mathrm{d}t}.$$

Let us write σ for σ_ϵ.

5.1. Local comparison: inert case. — Now consider a place v_0 of F, inert in E. Let v be the corresponding place of E. If the distribution θ_σ is not identically 0, then σ_{v_0} admits non-zero linear forms transforming under the characters Ω_v and Ω_v^{-1} of T_{v_0}, a group compact modulo the center. Thus there are smooth unit vectors e_T and e'_T such that

$$\sigma_{v_0}(t)e_T = \Omega(t)^{-1}e_T, \quad \sigma_{v_0}(t)e'_T = \Omega(t)e'_T.$$

We may assume that

$$e'_T = \sigma_{v_0}(w_\epsilon)(e_T).$$

We define a local distribution:

$$\theta_{\sigma_{v_0}}(f_{v_0}) = \sum_{u_i}\big\langle\sigma_{v_0}(f_{v_0})u_i, e'_T\big\rangle\overline{\langle u_i, e_T\rangle},$$

the sum over an orthonormal basis. This can also be written as

$$\int f(g)\omega_{\sigma_{v_0}}(g)\mathrm{d}g,$$

where we have set

$$\omega_{\sigma_{v_0}}(g) = \big\langle\sigma_{v_0}(g)e_T, e'_T\big\rangle. \tag{25}$$

Note that $\omega_{\sigma_{v_0}}(w_\epsilon) = 1$. There is a similar distribution $\theta_{\sigma_{v_0}\otimes\eta_{v_0}}$ defined by:

$$\theta_{\sigma_{v_0}\otimes\eta_{v_0}}(f_{v_0}) = \sum_{u_i}\big\langle\sigma_{v_0}\otimes\eta_{v_0}(f_{v_0})u_i, e''_T\big\rangle\overline{\langle u_i, e_T\rangle},$$

as well as a function

$$\omega_{\sigma_{v_0}\otimes\eta_{v_0}}(g) = \big\langle\sigma_{v_0}\otimes\eta_{v_0}(g)e_T, e''_T\big\rangle, \tag{26}$$

where we have set:

$$e''_T = \sigma_{v_0}\otimes\eta_{v_0}(w_\epsilon)(e_T) = \eta_{v_0}(-\epsilon)e'_T.$$

Thus

$$\omega_{\sigma_{v_0}\otimes\eta_{v_0}}(g) = \eta_{v_0}(\det g)\eta_{v_0}(-\epsilon)\omega_{v_0}(g). \tag{27}$$

In particular, if $\det g \in -\epsilon\,\mathrm{Norm}(E^\times)$ then $\omega_{\sigma_{v_0}}(g) = \omega_{\sigma_{v_0}\otimes\eta_{v_0}}(g)$.

The global distribution θ_σ decomposes into a tensor product

$$\theta_\sigma(f_\epsilon^{v_0}\otimes f_{\epsilon,v_0}) = \theta^{v_0}(f_\epsilon^{v_0})\theta_{\sigma_{v_0}}(f_{\epsilon,v_0}).$$

Likewise, we have obtained a decomposition of the global distribution Θ_Π into a tensor product $\Theta^v\otimes\Theta_{\Pi_{\sigma_v}}$. The global spectral identity gives then a linear relation of the form:

$$\Theta_{\Pi_{\sigma_v}}(f_v) = C_1\theta_{\sigma_{v_0}}(f_{\epsilon,v_0}) + C_2\theta_{\sigma_{v_0}\otimes\eta_{v_0}}(f_{\epsilon,v_0}).$$

To analyze the situation conveniently, let us go back to a local situation. That is, let E/F be a local quadratic extension and Π an irreducible unitary representation of $G(E)$ with trivial central character. We assume that Π is the base change of a representation π of $G(F)$ with central character ω, and $\omega = 1$ or $\omega = \eta_{E/F}$. We assume that ϵ is such that the representation σ of $G_\epsilon(F)$ corresponding to π has a vector transforming under Ω and Ω^{-1} with $\Omega_{|F^\times} = \omega$. We assume that we have a relation of the form:

$$\Theta_\Pi(f) = C_1\theta_\sigma(f_\epsilon) + C_2\theta_{\sigma\otimes\eta}(f_\epsilon)$$

for any pair of functions (f, f_ϵ) with matching orbital integrals; that is, f correspond to a function Φ on $\mathcal{S}_s$ and $U(x;\Phi) = H(x; f_\epsilon)$ for $x \in \epsilon\,\mathrm{Norm}(E^\times)$. In particular, if f_ϵ is supported on the set of $g \in G_\epsilon$ such that $\det(g) \in -\epsilon\,\mathrm{Norm}(E^\times)$ then this simplifies to

$$\Theta_\Pi(f) = C\int_{G_\epsilon(F)} \omega_\sigma(g) f_\epsilon(g)\,\mathrm{d}g, \quad C = C_1 + C_2. \tag{28}$$

Our goal in this section is to compute the constant C.

Proposition 5. — *Suppose that f and f_ϵ have matching orbital integrals and f_ϵ is supported on the set $\{g \in G_\epsilon(F) \mid \det(g) \in -\epsilon\,\mathrm{Norm}(E^\times)\}$. Then*

$$\Theta_\Pi(f) = \epsilon(1, \eta, \psi_F)2\eta(-\epsilon)L(0, \eta)\theta_\sigma(f_\epsilon).$$

Moreover, we can choose the pair in such a way that both sides are non-zero.

5.2. Computation of Θ_Π. — To prove the proposition, we first consider a sequence of functions (f_j) on $G(E)$ which tends to δ_e. That is, $f_j \geq 0$, $\int f_j\,\mathrm{d}g = 1$ and $\mathrm{supp} f_j \to e$. We let Φ_j be the associated function on $\mathcal{S}$. It tends to δ_w on $\mathcal{S}(F)$. Note that we can start with an approximation Φ_j of δ_w on $\mathcal{S}$ and then choose for the functions f_j an approximation of δ_e. We also choose a function $\phi \in C_0^\infty(E)$ of small support. We set

$$f^j(g) = \int_E \phi(x) f_j\Big[\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\Big]\,\mathrm{d}x \tag{29}$$

and we denote by Φ^j the corresponding function on $\mathcal{S}(F)$.

We first compute $\Theta_\Pi(f^j)$. Recall that the scalar product is given by:

$$\langle W_1, W_2\rangle = \int W_1\overline{W}_2\big[\mathrm{diag}(a, 1)\big]\,\mathrm{d}^\times a.$$

and

$$\Theta_\Pi(f) = \sum \lambda\big(\Pi(f)W_i\big)\overline{\mathcal{P}(W_i)},$$

where W_i is an orthonormal basis. We have:

$$\begin{aligned}\lambda\big(\Pi(f^j)W_i\big) &= \iint W_i\Big[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\Big]\Omega^{-1}(a) f^j(g)\,\mathrm{d}^\times a\,\mathrm{d}g \\ &= \iiint W_i\Big[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1\end{pmatrix} g\Big]\Omega^{-1}(a) f_j(g)\phi(x)\,\mathrm{d}^\times a\,\mathrm{d}x\,\mathrm{d}g \\ &= \iint \widehat{\phi}(a) W_i\Big[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\Big]\Omega^{-1}(a) f_j(g)\,\mathrm{d}^\times a\,\mathrm{d}g\end{aligned}$$

where $\widehat{\phi}$ denotes the Fourier transform of ϕ.

For now we assume that E is non-Archimedean. We choose ϕ in such a way that $\widehat{\phi}(0) = 0$. Then there is an element $W^0 \in \mathcal{W}(\Pi, \psi)$ such that

$$W^0\Big[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\Big] = \overline{\widehat{\phi}(a)\Omega^{-1}(a)}.$$

Then the above expression is

$$\big\langle \Pi(f_j)W_i, W^0\big\rangle = \big\langle W_i, \Pi(f_j^*)W^0\big\rangle.$$

Since

$$\Pi(f_j^*)W^0 = \sum_i \big\langle \Pi(f_j^*)W^0, W_i\big\rangle W_i$$

we see that:

$$\Theta_\Pi(f^j) = \overline{\mathcal{P}\big(\Pi(f_j^*)W^0\big)}.$$

Since f_j is an approximation of δ_e, for j large enough, $\Pi(f_j^*)W^0 = W^0$ and

$$\Theta_\Pi(f^j) = \overline{\mathcal{P}(W^0)} = \overline{\int W^0\Big[\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix}\Big]\omega\eta(b)\,\mathrm{d}^\times b} = \int_{F^\times} \widehat{\phi}(b)\eta(b)\,\mathrm{d}^\times b.$$

Clearly, we can choose ϕ with $\widehat{\phi}(0) = 0$ in such a way that this is non-zero.

We pass to the Archimedean case. We first explain how to choose the function ϕ. We assume, as we may, that $\psi_F(x) = \exp(2i\pi x)$. Let ϕ_0 be a smooth function of compact support on F with $\int \phi_0(x)\,\mathrm{d}x \neq 0$. Then the function $\phi_1 = \phi_0 * \phi_0^*$ is smooth of compact support. Its Fourier transform is $\widehat{\phi}_1 = |\widehat{\phi}_0|^2$ and is thus ≥ 0. Moreover, $\widehat{\phi}_1(0) = |\widehat{\phi}_0(0)|^2 > 0$. Now consider the function on E defined by:

$$\phi_2(x+iy) = \phi_1(x)\phi_1(y).$$

The Fourier transform on E of this function is given by

$$\widehat{\phi}_2(x+iy) = 2\widehat{\phi}_1(2x)\widehat{\phi}_1(-2y).$$

Let $Q > 0$ be an integer. There is a smooth function of compact support ϕ on E, whose Fourier transform is given by:

$$\widehat{\phi}(z) = 2^{2Q+1}(z\overline{z})^Q z\widehat{\phi}_1(2x)\widehat{\phi}_1(-2y), \quad z = x+iy.$$

Then

$$\int_{F^\times} \widehat{\phi}(x)\eta(x)\mathrm{d}^\times x = \widehat{\phi}_1(0)\int |x|_F^{2Q+1}\widehat{\phi}_1(x)\mathrm{d}^\times x > 0.$$

This function, which will be also denoted by ϕ_Q to stress the dependence on Q, will have the required properties, provided the integer Q is sufficiently large.

To continue we recall that the unitary representation Π can be realized in the Hilbert space $\mathcal{H}$ of square integrable functions on $E^\times$. Let P_1 be the group of matrices with last row $(0,1)$. The representation of $P_1(E)$ on $\mathcal{H}$ is the representation induced by the generic character of $N(E)$ determined by ψ_E. It is (topologically) irreducible. The subspace $\mathcal{V}$ of smooth vectors may be viewed as a certain space of smooth functions on $E^\times$ (Kirilov model). If W is in the Whittaker model of Π then the function Φ defined by

$$\Phi(a) = W\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]$$

is in $\mathcal{V}$ and determines W. Let $\mathcal{V}'$ be the topological dual of $\mathcal{V}$ (space of *generalized vectors*). We have the inclusions: $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. We extend the scalar product $\langle \bullet\,,\bullet\rangle$ on $\mathcal{H}$ to $\mathcal{V}\times\mathcal{V}'$ and $\mathcal{V}'\times\mathcal{V}$. The group and its enveloping algebra operate on $\mathcal{V}$ and $\mathcal{V}'$. For instance, the Whittaker linear from $\mathcal{W}$ is the element of $\mathcal{V}'$ such that, for $\Phi \in \mathcal{V}$:

$$\langle \Phi, \mathcal{W}\rangle = \Phi(e).$$

Likewise, λ and $\mathcal{P}$ are elements of $\mathcal{V}'$ such that

$$\langle \Pi, \lambda\rangle = \int_{E^\times} \Phi(a)\Omega^{-1}(a)\mathrm{d}^\times a, \quad \langle \Phi, \mathcal{P}\rangle = \int_{F^\times} \Phi(b)\omega\eta(b)\mathrm{d}^\times b,$$

and

$$\Theta_\Pi(f) = \big\langle \Pi(f)\mathcal{P}, \lambda\big\rangle = \big\langle \mathcal{P}, \Pi(f^*)\lambda\big\rangle.$$

We have already remarked that the convergence of these integrals follows at any rate from the local analog of the integral representation for Whittaker functions given below (see (44)). Moreover, the continuity follows for instance from the global theory (since we assume anyway that Π is a component of a cuspidal representation). The continuity of $\mathcal{P}$ means that there is M such that, if (X_α), $1 \le \alpha \le N$, is a basis of the space of elements of the enveloping algebra with (filtration) degree $\le M$, then, for suitable constants $c_\alpha > 0$,

$$\big|\langle \Phi, \mathcal{P}\rangle\big| \le \sum_\alpha c_\alpha \big\|\Pi(X_\alpha)\Phi\big\|.$$

In this sum $X_1 = 1$. Let $\mathcal{H}^N$ be the Hilbert sum of N copies of $\mathcal{H}$ and $\mathcal{V}_0$ the (non-closed) subspace of $N-$tuples of the form $v = (\Pi(X_\alpha)\phi)$, with $\phi \in \mathcal{V}$. Let $\mathcal{P}_0$ be the linear form on $\mathcal{V}_0$ defined by $\mathcal{P}_0(v) = \langle \phi, \mathcal{P}\rangle$. This linear form is continuous for the topology induced by the topology of $\mathcal{H}^N$, thus extends to the closure of $\mathcal{V}_0$ in $\mathcal{H}^N$ and is therefore given by the scalar product with a vector $(\Phi_\alpha) \in \mathcal{H}^N$. In other words,

$$\langle \Phi, \mathcal{P}\rangle = \sum_\alpha \big\langle \Pi(X_\alpha)\Phi, \Phi_\alpha\big\rangle \tag{30}$$

with $\Phi_\alpha \in \mathcal{H}$.

Now the scalar product $\langle \Phi, \mathcal{P}\rangle$ is defined if Φ is a vector of class C^M. We will prove the following assertion:

LEMMA 5. — *If Q is sufficiently large, then the function Φ^0 defined*

$$\Phi^0(z) = \overline{\widehat{\phi}_Q(z)\Omega(z)^{-1}}$$

is square integrable and is, in fact, a vector of class C^M in the representation. Moreover:

$$\mathcal{P}(\Phi^0) = \int_{F^\times} \Phi^0(x)\omega\eta(x)d^\times x = \overline{\int_{F^\times} \widehat{\phi}_Q(x)\eta(x)\mathrm{d}^\times x}.$$

We prove the lemma in the case where $\omega = \eta$. Then the restriction of Ω to $F^\times$ is η. Thus Ω has the form:

$$\Omega(z) = \frac{z^m}{(z\overline{z})^{m/2}},$$

where m is an odd integer. Taking a large enough Q, we see that the function Φ^0 of the above lemma has in turn the form

$$\Phi^0(z) = (z\overline{z})^{\frac{1}{2}}(z\overline{z})^P\Psi(z)$$

where P is another integer which tends to infinity with Q. The function Ψ (which depends on Q) is in $\mathcal{S}(E)$, the space of Schwartz functions on E. Let us denote by $\mathcal{V}(P)$ the space of functions of the above form, with $\Psi \in \mathcal{S}(E)$. It is contained in $\mathcal{H}$. It will suffice to prove the following general lemma:

LEMMA 6. — *If P is large enough the space $\mathcal{V}(P)$ is contained in the space of vectors of class C^M and, for $\Phi \in \mathcal{V}(P)$,*

$$\mathcal{P}(\Phi) = \int_{F^\times} \Phi(x)\omega\eta(x)\mathrm{d}^\times x.$$

Proof. — To prove the lemma we study the action of the enveloping algebra of $\mathrm{Gl}(2,E)$ (viewed as a real Lie group) on $\mathcal{V}$. We regard the space $M(2\times 2,\mathbb{C})$ of 2×2 matrices with complex entries as a real vector space and identify it to the Lie algebra of $\mathrm{Gl}(2,E)$ as a real Lie group. Entries of such a matrix are written in the form $a + b\sqrt{-1}$ with a, b real. We define the following elements of $M(2\times 2,\mathbb{C})\otimes\mathbb{C}$:

$$X_+ = \frac{1}{2}\Big\{\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} - i\begin{pmatrix} 0 & \sqrt{-1}\\ 0 & 0\end{pmatrix}\Big\},\quad X_- = \frac{1}{2}\Big\{\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} - i\begin{pmatrix} 0 & 0\\ \sqrt{-1} & 0\end{pmatrix}\Big\},$$

$$H=\frac{1}{2}\Big\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}-i\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix}\Big\},\quad T=\frac{1}{2}\Big\{\begin{pmatrix}1&0\\0&0\end{pmatrix}-i\begin{pmatrix}\sqrt{-1}&0\\0&0\end{pmatrix}\Big\}.$$

We define similarly conjugate elements $\overline{X}_+$ and so on, by replacing i by $-i$ in the above formulas. They commute to the previous ones. The Casimir element

$$\Omega = X_+X_- + X_-X_+\frac{1}{2}H^2$$

is in the center of the enveloping algebra and $T-\frac{1}{2}H$ in the center of the Lie algebra. Thus X_+X_- can be expressed in terms of T and elements of the center of the enveloping algebra. As a consequence,

$$\Pi(X_+)\Pi(X_-)=R\big(\Pi(T)\big),$$

where R is a polynomial (of degree 2). We have then, for $\Phi\in\mathcal{V}$,

$$\Pi(X_+)\Phi(z)=2i\pi z\Phi(z),\quad \Pi(T)\Phi(z)=z\frac{\partial\Phi(z)}{\partial z},$$

$$2i\pi z\Pi(X_-)\Phi(z)=\Pi(X_+)\Pi(X_-)\Phi(z)=R\big(\Pi(T)\big)\Phi(z).$$

In other words (after a change of notations) we see that there is a polynomial R such that

$$\Pi(X_-)\Phi(z)=z^{-1}R\Big(z\frac{\partial}{\partial z}\Big)\Phi(z).$$

Similarly, there is a polynomial R' such that

$$\Pi(\overline{X}_-)\Phi(z)=\overline{z}^{-1}R'\Big(\overline{z}\frac{\partial}{\partial \overline{z}}\Big)\Phi(z).$$

Now the space $\mathcal{V}(P)$ is invariant under the operators of multiplication by $z,\overline{z}$ and the differential operators $z\partial/\partial z,\overline{z}\partial/\partial\overline{z}$. This implies that in fact $\mathcal{V}(P)$ is contained in the space of smooth vectors for the group $P_1(E)$. Now multiplication by z^{-1} sends $\mathcal{V}(P)$ to $\mathcal{V}(P-1)$. It follows that if P is sufficiently large and $\Phi\in\mathcal{V}(P)$ the vector $\Pi(X_-)\Phi$ which, *a priori*, is only a vector in $\mathcal{V}'$ is in fact in $\mathcal{H}$, more precisely, in $\mathcal{V}(P-1)$. Moreover, it is given by the same formula as above. Likewise for $\Pi(\overline{X}_-)\Phi$. The first assertion of the lemma follows. To prove the second assertion, we let $\mathcal{V}_0$ be the subspace of Φ in $\mathcal{V}(P)$ for which the function Ψ is flat at $0\in E$, that is, all its derivatives vanish at 0. From the above analysis, it follows that such a vector is in $\mathcal{V}$. Then $\mathcal{P}(\Phi)$ is given by the integral over $F^\times$. Now recall the decomposition (30). From the above computation, there exist (non commutative) polynomials P_α in $(z^{-1},\overline{z}^{-1},z\partial/\partial z,\overline{z}\partial/\partial\overline{z})$ such that

$$\mathcal{P}(\Phi)=\sum_\alpha\int(P_\alpha\Phi)(z)\,\overline{\Phi_\alpha(z)}\,\mathrm{d}^\times z.$$

This applies in particular to a $\Phi\in\mathcal{V}(P)$ for P large enough. Thus we have to show that the difference

$$\int_{F^\times}|x|^{2P+1}\Psi(x)\,\mathrm{d}^\times x-\sum_\alpha\int_{E^\times}\big(P_\alpha\big((z\overline{z})^{\frac12+P}\Psi(z)\big)\big)\overline{\Phi}_\alpha(z)\,\mathrm{d}^\times z$$

vanishes for all $\Psi\in\mathcal{S}(E)$, if P is large enough. At any rate, this is true for Ψ flat at 0. Since the difference is a distribution (linear form in Ψ), the difference is at most a linear combination of derivatives of Ψ at 0. Thus it will vanish if Ψ vanishes at sufficiently high order at 0. Thus it might not vanish on $\mathcal{V}(P)$ but it will vanish on the smaller space $\mathcal{V}(P_1)$ for $P_1\gg P$. We are done. □

In summary then, we have found a smooth function of small support ϕ on E and an element $\Phi^0\in\mathcal{H}$ of class C^M such that

$$\lambda\Big(\int_{N(E)}\phi(z)\Pi(n^{-1})\,\mathrm{d}n\Phi\Big)=\langle\Phi,\Phi_0\rangle,\quad n=\begin{pmatrix}1&z\\0&1\end{pmatrix},\ \Phi\in\mathcal{V},$$

$$\mathcal{P}(\Phi^0)=\overline{\int_{F^\times}\widehat{\phi}(b)\eta(b)\,\mathrm{d}^\times b}\neq 0.$$

Suppose that f_j is a smooth approximation of δ_e. From (30) we get:

$$\mathcal{P}\big(\Pi(f_j)\Phi^0\big)=\sum_\alpha\int\big\langle\Pi(X_\alpha)\Pi(g)\Phi^0,\Phi_\alpha\big\rangle f_j(g)\,\mathrm{d}g$$

If we write

$$\operatorname{Ad}g^{-1}X_\alpha=\sum_\beta\lambda_{\alpha,\beta}(g)X_\beta$$

this becomes

$$\sum_{\alpha,\beta}\int\lambda_{\alpha,\beta}(g)\big\langle\Pi(X_\beta)\Phi^0,\Pi(g)^{-1}\Phi_\alpha\big\rangle f_j(g)\,\mathrm{d}g.$$

As $j\to+\infty$ this tends to

$$\sum_{\alpha,\beta}\lambda_{\alpha,\beta}(e)\big\langle\Pi(X_\beta)\Phi^0,\Phi_\alpha\big\rangle=\sum_\alpha\big\langle\Pi(X_\alpha)\Phi_0,\Phi_\alpha\big\rangle=\mathcal{P}(\Phi^0).$$

We thus consider an approximation Φ_j of δ_w on $\mathcal{S}$. We can choose an approximation f_j of δ_e on $G(E)$ such that Φ_j is the function associated to f_j. Just as in the $p-$adic case, we define f^j in terms of ϕ and Φ^j. Then

$$\lambda\big(\Pi(f^j)\Phi\big)=\lambda\Big(\int_{N(E)}\phi(z)\Pi(n^{-1})\,\mathrm{d}n\Pi(f_j)\Phi\Big)=\big\langle\Phi(f_j)\Phi,\Phi_0\big\rangle,$$

or

$$\Pi(f^{j,*})\lambda=\Pi(f_j^*)\Phi_0.$$

Then

$$\Theta_\Pi(f)=\big\langle\mathcal{P},\Pi(f^{j,*})\lambda\big\rangle=\big\langle\mathcal{P},\Pi(f_j^*)\Phi_0\big\rangle$$

which tends to $\langle \mathcal{P}, \Phi_0\rangle = \overline{\mathcal{P}(\Phi_0)}$. Thus we find again

$$\lim_{j\to+\infty} \Theta_\Pi(f^j) = \int_{F^\times} \widehat{\phi}(b)\eta(b)\,\mathrm{d}^\times b.$$

We introduce the following function on F:

$$\phi_1(x) = \int \phi\big(x + y\sqrt{\delta}\,\big)|2|_F \cdot |\delta|_F^{\frac{1}{2}}\,\mathrm{d}y. \tag{31}$$

Then, denoting by $\mathcal{F}$ the Fourier transform for a function on F, we get for $x \in F$:

$$\widehat{\phi}(x) = \mathcal{F}(\phi_1)(2x).$$

Thus, in all cases,

$$\lim_{j\to+\infty} \Theta_\Pi(f^j) = \eta(2)\int_{F^\times} \mathcal{F}(\phi_1)(x)\eta(x)\,\mathrm{d}^\times x. \tag{32}$$

5.3. Computation on G_ϵ. — We now compute the orbital integral of Φ^j. After a change of variables, we get

$$U(x;\Phi^j) = \iint \Phi^j\Big[\begin{pmatrix} y\overline{y}z^{-1}x^{-1} & y \\ \overline{y} & z\end{pmatrix}\Big]\Omega(y)\eta(z)\,\mathrm{d}^\times y\,\mathrm{d}^\times z.$$

From (29) we get

$$\Phi^j(s) = \int \Phi_j\Big[\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix} s \begin{pmatrix} 1 & 0 \\ \overline{t} & 1\end{pmatrix}\Big]\phi(t)\,\mathrm{d}t.$$

After changing t to $-t$ we find that the orbital integral of Φ^j is equal to

$$\iiint \Phi_j\Big[\begin{pmatrix} y\overline{y}z^{-1}x^{-1} - t\overline{y} - \overline{t}y + t\overline{t}z & y - tz \\ \overline{y} - \overline{t}z & z\end{pmatrix}\Big]\phi(-t)\Omega(y)\eta(z)\,\mathrm{d}t\,\mathrm{d}^\times y\,\mathrm{d}^\times z.$$

Recall ϕ has compact support and the support of Φ_j tends to w. Thus there is a sequence $\{c_j\}$ of positive real numbers such that $\lim_{j\to+\infty} c_j = +\infty$ and the non-vanishing of $U(x;\Phi^j)$ implies $|x| > c_j$. Now we choose a function f_ϵ^j which matches f^j (or Φ^j). For $x \in \epsilon\,\mathrm{Norm}(E^\times)$ we write $x = \epsilon a\overline{a}$ and then

$$U(x;\Phi^j) = H(x; f_\epsilon^j) = \iint f_\epsilon^j\Big[t_1\begin{pmatrix} a^{-1} & \epsilon \\ 1 & \overline{a}^{-1}\end{pmatrix}t_2\Big]\Omega(t_1t_2^{-1})\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

Suppose the integral is non-zero. Then $|x| > c_j$. If j is large enough, then in the integrand, the determinant of the matrix $g = t_1(*)t_2$ belongs to $-\epsilon\,\mathrm{Norm}(E^\times)$. Thus we may and will assume that f_ϵ^j is supported on the set $\{g \mid \det g \in -\epsilon\,\mathrm{Norm}(E^\times)\}$. From (15) we get

$$\int \omega_\sigma(g)f_\epsilon^j(g)d^\times g = \frac{1}{L(1,\eta)^2}\int_{x=\epsilon a\overline{a}} \omega_\sigma\Big[\begin{pmatrix} a^{-1} & \epsilon \\ 1 & \overline{a}^{-1}\end{pmatrix}\Big]U(x;\Phi^j)\frac{\mathrm{d}x}{|1-x|^2}.$$

Now we define a function ω on $\epsilon\,\mathrm{Norm}(E^\times)-\{1\}$ by

$$\omega(x) := \omega_\sigma\Big[\begin{pmatrix} a^{-1} & \epsilon \\ 1 & \overline{a}^{-1}\end{pmatrix}\Big], \quad \text{for } x=\epsilon a\overline{a}.$$

One can check that indeed the right hand side depends only on x. The function $\omega(x^{-1})$ extends to a smooth function near 0 on $\epsilon\,\mathrm{Norm}(E)$, whose value at 0 is

$$\omega_\sigma\Big[\begin{pmatrix} 0 & \epsilon \\ 1 & 0\end{pmatrix}\Big]=1.$$

We denote by $\widetilde{\omega}$ a smooth function on F, supported on a neighborhood of 0, and equal to $\omega(x^{-1})|1-x|^{-2}$ for x near 0 in $\epsilon\,\mathrm{Norm}(E)$. In particular $\widetilde{\omega}(0)=1$. Thus, for large enough j,

$$\begin{aligned}\int \omega_\sigma(g)f_\epsilon^j(g)\mathrm{d}g &= \frac{1}{L(1,\eta)^2}\int_{\epsilon\,\mathrm{Norm}(E^\times)} U(x;\Phi^j)\frac{\omega(x)\mathrm{d}x}{|1-x|^2}\\ &= \frac{1}{L(1,\eta)^2}\int_{\epsilon\,\mathrm{Norm}(E^\times)} U(x^{-1};\Phi^j)\widetilde{\omega}(x)\mathrm{d}x.\end{aligned}$$

Equivalently,

$$\int \omega_\sigma(g)f_\epsilon^j(g)\mathrm{d}g = \frac{1}{2L(1,\eta)^2}\big[I_1+\eta(\epsilon)I_\eta\big] \tag{33}$$

where we have set

$$\begin{aligned}I_{\left\{\substack{1\\ \eta}\right\}} &:= \int_E U(x^{-1};\Phi^j)\begin{Bmatrix}1\\ \eta(x)\end{Bmatrix}\widetilde{\omega}(x)\mathrm{d}x\\ &= \int \Phi_j\Big[\begin{pmatrix} y\overline{y}z^{-1}x-t\overline{y}-\overline{t}y+t\,\overline{t}z & y-tz \\ \overline{y}-\overline{t}z & z\end{pmatrix}\Big]\\ &\qquad\qquad \times \phi(-t)\widetilde{\omega}(x)\Omega(y)\eta(z)\begin{Bmatrix}1\\ \eta(x)\end{Bmatrix}\mathrm{d}t\mathrm{d}^\times y\mathrm{d}^\times z\mathrm{d}x.\end{aligned}$$

To compute this, we first change x to $xz/y\overline{y}$:

$$\begin{aligned}&= \int \Phi_j\Big[\begin{pmatrix} x-t\overline{y}-\overline{t}y+t\overline{t}z & y-tz \\ \overline{y}-\overline{t}z & z\end{pmatrix}\Big]\phi(-t)\widetilde{\omega}\Big(\frac{xz}{y\overline{y}}\Big)\\ &\qquad\qquad \times \Omega(y)\begin{Bmatrix}\eta(z)\\ \eta(x)\end{Bmatrix}\mathrm{d}t|y|_E^{-1}\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x.\end{aligned}$$

Next we change y to $y+tz$ to get:

$$\begin{aligned}&\int \Phi_j\Big[\begin{pmatrix} x-t\overline{y}-\overline{t}y-t\overline{t}z & y \\ \overline{y} & z\end{pmatrix}\Big]\phi(-t)\widetilde{\omega}\Big(\frac{xz}{(y+tz)(\overline{y}+\overline{t}z)}\Big)\\ &\qquad\qquad \times \Omega(y+tz)\begin{Bmatrix}\eta(z)\\ \eta(x)\end{Bmatrix}\mathrm{d}t|y+tz|_E^{-2}|y|_E\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x.\end{aligned}$$

Now we change t to $t/\overline{y}$ to get:

$$\int \Phi_j\Big[\begin{pmatrix} x-(t+\overline{t}+t\overline{t}z/y\overline{y}) & y \\ \overline{y} & z \end{pmatrix}\Big]\phi\Big(\frac{-t}{\overline{y}}\Big)\widetilde{\omega}\Big(\frac{xz}{(y+tz/\overline{y})(\overline{y}+\overline{t}z/y)}\Big) \times \Omega\Big(y+\frac{tz}{\overline{y}}\Big)\left\{\begin{matrix}\eta(z)\\ \eta(x)\end{matrix}\right\}\mathrm{d}t\big|y+tz/\overline{y}\big|_E^{-2}\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x.$$

We finally change z to $zy\overline{y}$:

$$(34)\quad I_{\{\substack{1\\ \eta}\}} = \int \Phi_j\Big[\begin{pmatrix} x-(t+\overline{t}+t\overline{t}z) & y \\ \overline{y} & zy\overline{y} \end{pmatrix}\Big]\phi\Big(\frac{-t}{\overline{y}}\Big)\widetilde{\omega}\Big(\frac{xz}{(1+tz)(1+\overline{t}z)}\Big) \times\Omega(y)\frac{\Omega(1+tz)}{|1+tz|_E^2}\left\{\begin{matrix}\eta(z)\\ \eta(x)\end{matrix}\right\}\mathrm{d}t|y|_E^{-1}\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x.$$

To evaluate the limit of the integrals we remind ourselves that Φ_j is an approximation of δ_w on $\mathcal{S}_s$. In particular (see Lemma 4)

$$L(1,1_F)\int \Phi_j\Big[\begin{pmatrix} x & y \\ \overline{y} & z\end{pmatrix}\Big]\frac{\mathrm{d}x\mathrm{d}z\mathrm{d}y}{|xz-y\overline{y}|_F^2} = 1,$$

with $x,z\in F$ and $y\in E$. If we change z to $zy\overline{y}$ and take into account the relations $L(1,1_E)=L(1,1_F)L(1,\eta)$ and $\mathrm{d}^\times y = L(1,1_E)\mathrm{d}y\,|y|_E^{-1}$ we get:

$$(35)\qquad \int \Phi_j\Big[\begin{pmatrix} x & y \\ \overline{y} & zy\overline{y}\end{pmatrix}\Big]\frac{\mathrm{d}x\mathrm{d}z\mathrm{d}^\times y}{|1-xz|_F^2} = L(1,\eta).$$

We are going to see that the limit of the integral I_1 corresponding to the factor $\eta(z)$ in (34) is 0. We prove this in the Archimedean case, leaving the easier non-Archimedean case to the reader. We change x to $x+(t+\overline{t}+t\overline{t}z)$ to arrive (up to a multiplicative constant) at

$$\int \Psi_j(x,z,y)A(x,z,y)\eta(z)\mathrm{d}x\mathrm{d}z\mathrm{d}^\times y$$

where we have set

$$A(x,z,y) = \big|1-xz\big|_F^2\cdot\Omega(y)|y|_E^{-1}\int\phi\Big(\frac{-t}{\overline{y}}\Big) \times\widetilde{\omega}\Big(\frac{xz+(1+tz)(1+\overline{t}z)-1}{(1+tz)(1+\overline{t}z)}\Big)\frac{\Omega(1+tz)}{|1+tz|_F^2}\mathrm{d}t,$$

and

$$\Psi_j(x,z,y) = \Phi_j\Big[\begin{pmatrix} x & y \\ \overline{y} & zy\overline{y}\end{pmatrix}\Big]\frac{1}{|1-xz|_F^2}\cdot$$

Recall that Φ_j is an arbitrary smooth approximation of δ_w. Equivalently, Ψ_j is an arbitrary approximation of $L(1,\eta)\delta_{(0,0,1)}$ on the space $F\times F\times E$ for the measure $\mathrm{d}x\mathrm{d}z\mathrm{d}^\times y$. Thus, there is no harm in assuming that our approximation of δ satisfies:

$$\Psi_j(x,z,y) = \Psi_j(x,-z,y).$$

For ϕ with small support, A is continuous near $(0,0,1)$ on $F\times F\times E$. We can write $A=A_{+}+A_{-}$ where $A_{\pm}(x,-z,y)=\pm A_{\pm}(x,z,y)$. In view of our assumption on Ψ_j the integral of $A_{+}(x,z,y)\eta(z)$ against Ψ_j is 0. Now $A_{-}(x,z,y)\eta(z)$ is continuous and vanishes at $(0,0,1)$. We are integrating on $F\times F\times E$ against an approximation of a multiple of $\delta_{(0,0,1)}$. The limit of the integral is thus 0, as claimed.

To compute the integral I_η corresponding to the factor $\eta(x)$ in (34), we first replace t by a new variable s such that:

$$s+\overline{s}=t+\overline{t}+t\overline{t}z,\quad s-\overline{s}=t-\overline{t}.$$

Explicitly, for ϕ of small support:

$$\begin{aligned}
&t=p+q\sqrt{\delta},\quad s=p_1+q_1\sqrt{\delta},\quad q=q_1,\\
&p=\frac{-1+\sqrt{1+2p_1z+q_1^2\delta z^2}}{z}=\frac{2p_1+q_1^2\delta z}{1+\sqrt{1+2p_1z+q_1^2\delta z^2}}\mathrm{d}t\\
&\quad=\mathrm{d}s|1+2p_1z+q_1^2\delta z^2|_F^{-\frac12}.
\end{aligned}$$

Then the integral becomes:

$$\begin{aligned}
I_\eta=\int\Phi_j\Big[\begin{pmatrix} x-2p_1 & y\\ \overline{y} & zy\overline{y}\end{pmatrix}\Big]\phi\Big(\frac{-t}{\overline{y}}\Big)&\\
\times\widetilde{\omega}\Big(\frac{xz}{(1+tz)(1+\overline{t}z)}\Big)\Omega(y)|y|_E^{-1}\frac{\Omega(1+tz)}{|1+tz|_E^2}&\\
\times\eta(x)|1+2p_1z+q_1^2\delta z^2|_F^{-\frac12}\,\mathrm{d}s\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x&,
\end{aligned}$$

where now

$$t=\frac{-1+\sqrt{1+2p_1z+q_1^2\delta z^2}}{z}+q_1\sqrt{\delta}.$$

We change p_1 to $p_1+\frac12x$ and then p_1 to $-\frac12p_1$:

$$\begin{aligned}
\int\Phi_j\Big[\begin{pmatrix} p_1 & y\\ \overline{y} & zy\overline{y}\end{pmatrix}\Big]\phi\Big(\frac{-t}{\overline{y}}\Big)\widetilde{\omega}\Big(\frac{xz}{(1+tz)(1+\overline{t}z)}\Big)&\\
\times\Omega(y)|y|_E^{-1}\frac{\Omega(1+tz)}{|1+tz|_E^2}\eta(x)\cdot|1-p_1z+xz+q_1^2\delta z^2|_F^{-\frac12}&\\
\times\mathrm{d}p_1\mathrm{d}q_1|\delta|_F^{\frac12}\mathrm{d}^\times y|z|_F\mathrm{d}^\times z\mathrm{d}x&,
\end{aligned}$$

where

$$\begin{aligned}
t&=\frac{-1+\sqrt{1-p_1z+zx+q_1^2\delta z^2}}{z}+q_1\sqrt{\delta}\\
&=\frac{-p_1+x+q_1^2\delta z}{1+\sqrt{1-p_1z+zx+q_1^2\delta z^2}}+q_1\sqrt{\delta}.
\end{aligned}$$

Since $|z|_F \mathrm{d}^\times z \mathrm{d}x = \mathrm{d}z |x|_F \mathrm{d}^\times x$, this can be written as

$$\int \Phi_j\Big[\begin{pmatrix} p_1 & y \\ \overline{y} & zy\overline{y}\end{pmatrix}\Big] A(p_1,z,y) \frac{\mathrm{d}p_1 \mathrm{d}z \mathrm{d}^\times y}{|1-p_1 z|_F^2}$$

with

$$\begin{aligned} A(p_1,z,y) := |1-p_1z|_F^2 \int \phi\Big(\frac{-t}{\overline{y}}\Big)\widetilde{\omega}\Big(\frac{xz}{(1+tz)(1+\overline{t}z)}\Big) \\ \times |1-p_1 z + xz + q_1^2\delta z^2|_F^{-\frac{1}{2}} \Omega(y)|y|_E^{-1} \\ \times \frac{\Omega(1+tz)}{|1+tz|_E^2}\eta(x)|x|_F \mathrm{d}^\times x |\delta|_F^{\frac{1}{2}} \mathrm{d}q_1. \end{aligned}$$

For ϕ of small support, A is continuous on $F\times F\times E$ near $(0,0,1)$ with

$$\begin{aligned} A(0,0,1) &= \iint \phi\big(-\tfrac{1}{2}x - q_1\sqrt{\delta}\,\big)\eta(x)|x|\mathrm{d}^\times x |\delta|_F^{\frac{1}{2}} \mathrm{d}q_1 \\ &= \eta(-2)\int \phi_1(x)\eta(x)|x|_F \mathrm{d}^\times x, \end{aligned}$$

the last equation by definition of ϕ_1 (see (31)). Thus (see (35))

$$\lim_{j\to+\infty} I_\eta = L(1,\eta)\eta(-2)\int \phi_1(x)\eta(x)|x|_F \mathrm{d}^\times x$$

and we find from (33)

$$\lim_{j\to+\infty}\int \omega_\sigma(g) f_\epsilon^j(g)\mathrm{d}g = \frac{\eta(2\epsilon)}{2L(1,\eta)}\int \phi_1(x)\eta(x)|x|_F \mathrm{d}^\times x. \tag{36}$$

We compare this with (32):

$$\lim_{j\longrightarrow+\infty}\Theta_\Pi(f^j) = \eta(2)\int \mathcal{F}(\phi_1)(x)\eta(x)\mathrm{d}^\times x,$$

which, by the Tate functional equation, is equal to:

$$\begin{aligned} &= \eta(2)\epsilon(1,\eta,\psi_F)\frac{L(0,\eta)}{L(1,\eta)}\int \phi_1(x)|x|_F\eta(x)\mathrm{d}^\times x \\ &= \epsilon(1,\eta,\psi_F) 2\eta(-\epsilon)L(0,\eta)\lim_{j\to+\infty}\int \omega_\sigma(g) f_\epsilon^j(g)\mathrm{d}g. \end{aligned}$$

Proposition 5 follows.

5.4. Local comparison: split case. — We now consider the similar decomposition at a split place v_0. The isomorphism of (2.2) takes $G_\epsilon(F_{v_0})$ to $\mathrm{Gl}(2,F_{v_0})$, T_{v_0} to the group of A_{v_0} diagonal matrices and the character Ω to the character $\mathrm{diag}(a_1,a_2)\mapsto \Omega_{v_1}(a_1a_2^{-1})$. We may realize the representation σ_{v_0} in its Whittaker model. Then we have again a decomposition

$$\theta_\sigma(f^{v_0}\otimes f_{v_0}) = \theta^{v_0}(f^{v_0})\theta_{\sigma_{v_0}}(f_{v_0}),$$

where the local distribution at the place v_0 is now given by

$$\theta_{\sigma_{v_0}}(f_{v_0}) = \sum_W \int \sigma_{v_0}(f_{\epsilon,v_0})W(a)\Omega_{v_1}(a)^{-1}\,\mathrm{d}^\times a \overline{\int W(aw_\epsilon^{-1})\Omega_{v_1}(a)^{-1}\,\mathrm{d}^\times a}.$$

On the other hand, recall the distribution

$$\Theta_{\Pi,v_0}(f_{v_1}\otimes f_{v_2}) = \sum_{W_1,W_2} \lambda_{v_1}\big(\Pi_{v_1}(f_{v_1})W_1)\big)\lambda_{v_2}\big(\Pi_{v_2}(f_{v_2})W_2)\big)\overline{\mathcal{P}_{v_0}(W_1\otimes W_2)}.$$

Proposition 6. — *If $f_{v_1}\otimes f_{v_2}$ and f_{ϵ,v_0} match then*

$$\Theta_{\Pi_{v_0}}(f_{v_1}\otimes f_{v_2}) = \theta_{\sigma_{v_0}}(f_{\epsilon v_0}).$$

The distribution $\Theta_{\Pi_{v_0}}$ can be simplified as follows. Let us identify E_{v_1}, E_{v_2} and F_{v_0}. The representations Π_{v_1} and Π_{v_2} are in fact identical. The Hermitian form $\mathcal{P}_{v_0}(W_1\otimes W_2)$ is invariant under the unitary group at v_0. It follows that

$$\Theta_{\Pi,v_0}(f_{v_1}\otimes f_{v_2}) = \sum_{W_1,W_2} \lambda_{v_1}\big(\Pi_{v_1}(f_{v_1}*\tilde f_{v_2})W_1)\big)\lambda_{v_2}(W_2)\big)\overline{\mathcal{P}_{v_0}(W_1\otimes W_2)}$$

where we have set

$$\tilde f(g) = f(w\,{}^t g w).$$

Since Π_{v_1} is unitary and self contragredient, for $W\in\mathcal{W}(\Pi_{v_1},\psi_{v_0})$ the function $g\mapsto \overline{W}(\mathrm{diag}(-1,1)g)$ is still in the same space. Thus we can choose the functions W_1 in some orthonormal basis and choose then for the functions W_2 the functions of the form

$$g\longmapsto \overline{W}_1\big(\mathrm{diag}(-1,1)g\,\mathrm{diag}(-1,1)\big).$$

Since $\Omega_{v_2}\Omega_{v_1}=1$ our distribution reads then

$$\sum_{W_1,W_2}\int \Pi_{v_1}(f_{v_1}*\tilde f_{v_2})W_1(a_1)\Omega_{v_1}^{-1}(a_1)\,\mathrm{d}^\times a_1 \overline{\int W_2(a_2)\Omega_{v_1}^{-1}(a_2)\,\mathrm{d}^\times a_2}$$
$$\times\int_{F_{v_0}^\times} W_1\overline{W}_2\begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}\mathrm{d}^\times b,$$

where W_1, W_2 vary independently in the *same* orthonormal basis. The orthogonality relations give:

$$\sum_W \int \Pi_{v_1}(f_{v_1}*\tilde f_{v_2})W(a_1)\Omega_{v_1}^{-1}(a_1)\,\mathrm{d}^\times a_1 \overline{\int W(a_2)\Omega_{v_1}^{-1}(a_2)\,\mathrm{d}^\times a_2}.$$

Recall that the matching condition (17) amounts to:

$$f_{v_1}*\tilde f_{v_2}(g) = f_{\epsilon,v_0}(gw_\epsilon^{-1}).$$

Using the fact that we can change the orthonormal basis, we see that the above expression becomes, after a change of variables,

$$\sum_W \int \Pi_{v_1}(f_{\epsilon,v_0}) W(a_1)\Omega_{v_1}^{-1}(a_1)\,\mathrm{d}^\times a_1 \overline{\int W(a_2 w_\epsilon^{-1})\Omega_{v_1}^{-1}(a_2)\,\mathrm{d}^\times a_2}$$

which is equal to $\theta_{\sigma_\epsilon, v_0}(f_{v_0})$.

6. Global comparison

We go back to the global relation:

$$\Theta_\Pi(f) = \theta_\sigma(f_\epsilon) + \theta_{\sigma\otimes\eta}(f_\epsilon).$$

We assume f is a product with f^S the characteristic function of K^S. For v_0 inert not in S_0 the function f_{ϵ,v_0} is the characteristic function of $\mathrm{Gl}(2,\mathcal{O}_E)\cap G_\epsilon(F)$ and in particular supported on the set of g with $\det g \in -\epsilon\,\mathrm{Norm}(E_v^\times)$. Let us make the assumption that for any v_0 inert in S_0 the function f_{ϵ,v_0} is also supported on the set of g with $\det g \in -\epsilon\,\mathrm{Norm}(E_v^\times)$. Then the function $\rho(w_\epsilon)^{-1} f_\epsilon$ is supported on the set of g with $\det g \in \mathrm{Norm}(E_{\mathbb{A}}^\times)$ so that the distributions J_σ and $J_{\sigma\otimes\eta}$ take the same value on that function. Equivalently, $\theta_\sigma(f_\epsilon) = \theta_{\sigma\otimes\eta}(f_\epsilon)$ so that

$$\theta_\epsilon(f_\epsilon) = \frac{1}{2}\Theta_\Pi(f).$$

At a split place $v_0 \in S_0$, we have

$$\Theta_{\Pi_{v_0}}(f_{v_0}) = \theta_{\sigma_{v_0}}(f_{v_0}).$$

At a place $v_0 \in S_0$ inert in E, we have (Proposition 5)

$$\Theta_{\Pi_v}(f_v) = \epsilon(1,\eta_{v_0},\psi_{v_0})\, 2\eta_{v_0}(-\epsilon) L(0,\eta_{v_0})\, \theta_{\sigma_{v_0}}(f_{\epsilon,v_0}).$$

Then

$$\theta_\sigma(f_\epsilon) = \frac{1}{2}\Theta_\Pi(f) = \frac{1}{2}\prod_{\substack{v_0\in S_0,\\ \text{inert}}} \epsilon(1,\eta_{v_0},\psi_{v_0}) 2\eta_{v_0}(-\epsilon) L(0,\eta_{v_0})$$
$$\times \frac{L_{S_0}(1,\eta) L^S(\frac{1}{2},\Pi\otimes\Omega^{-1})}{L^{S_0}(1,\pi,\mathrm{Ad})} \prod_{v_0\in S_0} \theta_{\sigma_{v_0}}(f_{v_0}).$$

At this point we go back to distributions of positive type:

$$J_\sigma(f) := \theta_\sigma\big(\rho(w_\epsilon)f\big), \quad \widetilde{J}_{\sigma_{v_0}}(f) := \theta_{\sigma_{v_0}}\big(\rho(w_\epsilon)f\big).$$

Explicitly:

$$J_\sigma(f) = \sum \int \sigma(f)\phi(t)\Omega(t)^{-1}\,\mathrm{d}t \overline{\int \phi(t)\Omega^{-1}(t)\,\mathrm{d}t}.$$

For v_0 inert:

$$\widetilde{J}_{\sigma_{v_0}}(f_{v_0}) = \int_{G_{\epsilon\, v_0}} f_{v_0}(g)\langle \sigma_{v_0}(g)e'_T, e'_T\rangle \mathrm{d}g.$$

For v_0 split:

$$\widetilde{J}_{\sigma_{v_0}}(f) = \sum_W \int \rho(f)W(a)\Omega_{v_1}^{-1}(a)\mathrm{d}a \overline{\int W(a)\Omega_{v_1}^{-1}(a)\mathrm{d}a}.$$

Because S_0 is large enough the product of the factors $\eta_{v_0}(-\epsilon)$ is one.

THEOREM 2. — *For any smooth function $f = \prod_{v_0\in S_0} f_{v_0} f^{S_0}$ of compact support:*

$$J_\sigma(f) = \prod_{v_0\in S_0} \widetilde{J}_{\sigma_{v_0}}(f_{v_0}) \times \frac{1}{2}\Big(\prod_{\substack{v_0\in S_0,\\ \text{inert}}} \epsilon(1,\eta_{v_0},\psi_{v_0})2L(0,\eta_{v_0})\Big) \times \frac{L_{S_0}(1,\eta)L^S(\frac{1}{2},\Pi\otimes\Omega^{-1})}{L^{S_0}(1,\pi,\mathrm{Ad})}.$$

Indeed, we know in advance that the left-hand side decomposes into a product $C\prod_{v_0\in S_0}\widetilde{J}_{\sigma_{v_0}}(f_{v_0})$ for a suitable constant C. To evaluate C we choose the data as above. The result follows. Moreover, since the factors of C other than the one we are interested in are > 0, we conclude that $L^S(\frac{1}{2},\Pi\otimes\Omega^{-1}) > 0$, as was claimed.

7. Concluding remarks

The distributions $\widetilde{J}_{\sigma_{v_0}}$ have the following property. We let f_j be an approximation of δ on G_{ϵ,v_0}.

If v_0 is split, we choose a Schwartz-Bruhat function ϕ on F_{v_0} such that $\widehat{\phi}(0) = 0$. We also choose also an isomorphism of $G_\epsilon(F_{v_0})$ with $\mathrm{Gl}(2,F_{v_0})$ and set:

$$f^j(g) = \int f_j\Big[\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}g\Big]\phi(x)\mathrm{d}x$$

We have then

$$\lim_{j\to+\infty}\widetilde{J}_{\sigma_{v_0}}(f^j) = \int \widehat{\phi}(x)\mathrm{d}^\times x.$$

Now the Tate functional equation gives:

$$\int \widehat{\phi}(x)\mathrm{d}^\times x = \epsilon(1,1_{v_0},\psi_{v_0})\,L(1-s,1_{v_0})\int \phi(x)|x|^{s-1}\mathrm{d}x\Big|_{s=1}.$$

To have our local distribution independent of the choice of the additive character it is reasonable to set

$$\widetilde{J}_{\sigma_{v_0}}(f) = \epsilon(1,1_{v_0},\psi_{v_0})J_{\sigma_{v_0}}(f).$$

This new distribution is still of positive type and now:

$$\lim_{j\to+\infty} J_{\sigma_{v_0}}(f^j) = L(s,1_{v_0}) \int \phi(x)|x|^s\,\mathrm{d}x\Big|_{s=0}.$$

Note that the last expression is a multiple of $\int \phi(x)\log(|x|)\mathrm{d}x$. For instance in the non-Archimedean case, the expression is $-\int \phi(x)\nu(x)\mathrm{d}x$ where ν is the valuation. In the p-adic case there is an asymptotic theory of spherical characters (see [RR]). This is worked in [F4] for the case of the space $\mathrm{Gl}(2,F)/A(F)$. Let us assume for simplicity that $\Omega = 1$. The result is then that a spherical character like $J_{\sigma_{v_0}}$ can be represented, near the origin, as a unique linear combination

$$c_1 \int f\Big[\begin{pmatrix} 1 & u \\ v & 1\end{pmatrix} a\Big]\mathrm{d}a\mathrm{d}u\mathrm{d}v - c_2 \int f\Big[\begin{pmatrix} 1 & u \\ v & 1\end{pmatrix} a\Big]\mathrm{d}a\nu(uv)\mathrm{d}u\mathrm{d}v.$$

For the distribution $J_{\sigma_{v_0}}$ we have just defined, the above assertion amounts to saying that the constant c_2 is 1.

On the other hand, if v_0 is inert, then $\widetilde{J}_{\sigma_{v_0}}$ is a smooth function and we have simply: $\lim_{j\to+\infty} \widetilde{J}_{\sigma_{v_0}}(f_j) = 1$.

Finally, the reader can check that the left hand side of the formula in the theorem does not depend on the choice of the Haar measure on the group $G_\epsilon(F_{\mathbb{A}})$, and the formulas for a finite set of places S_0 and a larger set S_0' are compatible.

8. Appendix: Continuous spectrum over E

8.1. A diagonal truncation operator. — To prepare for the computation of the integral of the spectral kernel, we introduce another *diagonal truncation* operator for functions ϕ on $G(E_{\mathbb{A}})$ invariant under $Z(E_{\mathbb{A}})G(E)$:

$$\Lambda_d^T\phi(a) := \phi(a) - \sum_{\gamma\in A(E)\backslash \mathrm{Norm}(A)} \phi_N(\gamma a)\widehat{\tau}_P\big(H(\gamma a) - T\big).$$

Here $\widehat{\tau}_P(x) = 1$ if $x > 0$ and equal 0 otherwise. The sum is over the normalizer of $A(E)$ in $G(E)$ modulo $A(E)$, thus, has only two elements, that we can take to be e and w. The result of the truncation is a function on $A(E_{\mathbb{A}})$ invariant under $A(E)Z(E_{\mathbb{A}})$. It is useful to remember the following facts:

(i) $\widehat{\tau}_P(H(a) - T) = 1 \iff |a_1/a_2| > \mathrm{e}^T$;

(ii) for any function f invariant under $Z(E_{\mathbb{A}})$, $f(a) = f(wa^{-1}w)$;

(iii) $H(wa) = H(a^{-1}) = -H(a)$. Thus we can write more explicitly

$$\Lambda_d^T\phi(a) = \phi(a) - \phi_N(a)\widehat{\tau}_P(H(a)-T) - \phi_N(a^{-1}w)\widehat{\tau}_P(-H(a)-T),$$

$$\begin{aligned}
\Lambda_d^T\phi(a) &= \phi(a) && \text{if } -T < H(a) < T,\\
\Lambda_d^T\phi(a) &= \phi(a) - \phi_N(a) && \text{if } H(a) \geq T,\\
\Lambda_d^T\phi(a) &= \phi(a^{-1}w) - \phi_N(a^{-1}w) && \text{if } H(a) \leq -T.
\end{aligned}$$

If ϕ is of slow increase (as well as all its derivatives) then the function $a \mapsto \Lambda_d^T\phi(\mathrm{diag}(a,1))$ is rapidly decreasing for $|a|$ large and $|a|$ small. Furthermore, if Λ_c^T denotes the compact truncation operator introduced in (18), then

$$\lim_{T\to+\infty}\int(\Lambda_d^T - \Lambda_c^T)\phi(a)\Omega^{-1}(a)\,\mathrm{d}^\times a = 0.$$

Consider again a smooth function of compact support f on $\mathrm{Gl}(2,E_{\mathbb{A}})$. We have a standard majorization [Ar, Lemma 4.3]:

$$\bigl|\rho_1(X_1)\rho_2(X_2)K_f(x,y)\bigr| \leq C(f)\|x\|. \tag{37}$$

Here X_1 and X_2 are elements of the enveloping algebra of G_∞ acting as left invariant differential operators on the first and second variables respectively. For given X_1 and X_2 the scalar $C(f)$ is bounded for f in a bounded set. Thus the function

$$g \longmapsto \int K_f(g,h)\omega\eta\bigl(\kappa(h)\bigr)\,\mathrm{d}h$$

is of slow increase as well as all its derivatives. Thus we arrive at the following lemma:

Lemma 7. — *One has*

$$\Theta(f) = \lim_{T\to+\infty}\iint\Lambda_{1,d}^T K_f(a,h)\Omega^{-1}(a)\,\mathrm{d}a\,\omega\kappa(h)\,\mathrm{d}h.$$

In [JL] and [JLRo] we have introduced a *mixed truncation operator* Λ_m^T for the group $\mathrm{Gl}(2,F)$. It satisfies the following easy lemma:

Lemma 8. — *Suppose that ϕ is a smooth function on $\mathrm{Gl}(2,E_{\mathbb{A}})$ invariant under the center, uniformly bounded with derivatives of slow increase. Then*

$$\int_{Z(F_{\mathbb{A}})\,\mathrm{Gl}(2,F)\backslash\,\mathrm{Gl}(2,F_{\mathbb{A}})}\phi(h)\,\mathrm{d}h = \lim_{T\to+\infty}\int\Lambda_m^T\phi(h)\,\mathrm{d}h.$$

Proof. — Since ϕ is integrable over the quotient we may write

$$\int\Lambda_m^T\phi(h)\,\mathrm{d}h = \int\phi(h)\,\mathrm{d}h - \int\sum_{\gamma\in P(F)\backslash\,\mathrm{Gl}(2,F)}\phi_N(\gamma h)\widehat{\tau}_P(\gamma h - T)\,\mathrm{d}h.$$

In this expression the second term can be computed as

$$\int_{P(F)Z(F_{\mathbb{A}})\backslash \mathrm{Gl}(2,F_{\mathbb{A}})} \phi_N(h)\widehat{\tau}_P(h-T)\,\mathrm{d}h.$$

Since ϕ_N is also uniformly bounded this is majorized by a constant multiple of

$$\int_{P(F)Z(F_{\mathbb{A}})\backslash \mathrm{Gl}(2,F_{\mathbb{A}})} \widehat{\tau}_P(h-T)\,\mathrm{d}h$$

itself a multiple of e^{-T}. The lemma follows. $\square$

Using (22) we may define a mixed truncation operator Λ_m^T for the group $H(F)$ which also satisfies the above lemma. It is easy to conclude from (37) that

$$K_f(g,h) = \lim_{T_2\to+\infty} \Lambda_{2,m}^{T_2} K_f(g,h),$$

uniformly for g and h in compact sets. Since the computation of $\Lambda_d^{T_2}\phi(a)$ depends only on the values of ϕ on a compact set (depending on a) we get also

$$\Lambda_{1,d}^{T_1} K_f(a,h) = \lim_{T_2\to+\infty} \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_f(a,h),$$

uniformly for a and h in compact sets. Our next task is the following lemma:

Lemma 9. — *Given T_1:*

$$\iint \Lambda_{1,d}^{T_1} K_f(a,h)\Omega^{-1}(a)\,\mathrm{d}a\,\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h = \lim_{T_2\to+\infty} \iint \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_f(a,h)\Omega^{-1}(a)\,\mathrm{d}a\,\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h.$$

For the proof of the lemma we use again (37) to show that $|\Lambda_{1,d}^{T_1} K_f(a,g)|$ is bounded by $C|a|^{-N}$ for $H(a) \ge T_1$ and $H(a) \le -T_1$ and by a constant for $-T_1 < H(a) < T_1$. The derivatives with respect to the second variable satisfy similar majorizations. It follows that the function defined by

$$m(g) = \int \Lambda_{1,d}^{T_1} K_f(a,g)\Omega^{-1}(a)\,\mathrm{d}a$$

satisfy the conditions of the previous lemma. Moreover:

$$\Lambda_m^{T_2} m(g) = \int \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_f(a,g)\Omega^{-1}(a)\,\mathrm{d}a.$$

Thus the lemma follows from the previous one. In conclusion:

Lemma 10. — *Set*

$$\Theta_{T_1,T_2}(f) = \iint \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_f(a,h)\Omega^{-1}(a)\,\mathrm{d}a\,\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h.$$

Then:

$$\Theta(f) = \lim_{T_1\to+\infty}\lim_{T_2\to+\infty} \Theta_{T_1,T_2}(f).$$

8.2. The continuous kernel over E. — We let f be a smooth function of compact support on $G(E_{\mathbb{A}})$ which, for now, is K-finite, where K is the standard maximal compact subgroup. Then K_f has a spectral expression

$$K_f = \sum_{\Pi} K_{f,\Pi},$$

where the sum is over all classes Π of cuspidal data for the parabolic subgroups P of G. We set

$$\Theta_{\Pi,T_1,T_2}(f) = \iint \Lambda_{1,d}^{T_1}\Lambda_{2,m}^{T_2} K_{f,\Pi}(a,h)\Omega^{-1}(a)\,\mathrm{d}a\,\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h.$$

Standard estimates [Ar, Lemma 4.4] allow us to write

$$\Theta_{T_1,T_2}(f) = \sum_{\Pi} \Theta_{\Pi,T_1,T_2}(f).$$

If $P = G$, then Π is a cuspidal automorphic representation with trivial central character and we have simply $\Theta_{\Pi,T_1,T_2}(f) = \Theta_{\Pi}(f)$.

If P is the minimal parabolic subgroup, then Π is an equivalence class of pairs of (normalized) characters (Π_1,Π_2) of $E_{\mathbb{A}}^{\times}/E^{\times}$; the equivalence relation is $(\Pi_1,\Pi_2) \simeq (\Pi_2,\Pi_1)$. Here we need only consider pairs such that $\Pi_1\Pi_2 = 1$. The expression for $K_{f,\Pi}$ is [Ar, p. 935]

$$K_{f,\Pi}(x,y) = \frac{1}{2}\sum_{(\Pi_1,\Pi_2)\in\Pi} \int \sum_{\phi} E(x, I_{\lambda,\Pi_1,\Pi_2}(f)\phi;\lambda,\Pi_1,\Pi_2) \overline{E(y,\phi;\lambda,\Pi_1,\Pi_2)}\,\mathrm{d}\lambda + \cdots$$

where the dots represent residual terms. Thus the outer sum has either two terms for $(\Pi_1,\Pi_2),(\Pi_2,\Pi_1)$ (the terms being in fact equal) or just one term (Π_1,Π_1) (where $\Pi_1^2 = 1$).

It is a little simpler to sum over all pairs $\Pi = (\Pi_1,\Pi_2)$ with $\Pi_2 = \Pi_1^{-1}$ and for such a pair define $K_{f,\Pi}$ by

$$K_{f,\Pi}(x,y) = \frac{1}{2}\int \sum_{\phi} E(x, I_{\lambda,\Pi}(f)\phi;\lambda,\Pi)\overline{E(y,\phi;\lambda,\Pi)}\,\mathrm{d}\lambda + \frac{\delta(\Pi_1\Pi_2^{-1})}{\mathrm{vol}(G(E)\backslash G^1)}\int f(g)\Pi_1(\det g)\,\mathrm{d}g\,\Pi_1(xy^{-1}).$$

We are using standard notations. We use the same notation Π for the representation induced by the pair Π. In particular

$$E(g,\phi;\lambda,\Pi) = \sum_{P(E)\backslash G(E)} \phi(\gamma g)\mathrm{e}^{\langle\lambda+\rho,H(\gamma g)\rangle}.$$

The variable λ is identified to a complex number:

$$\mathrm{e}^{\langle\lambda+\rho,H(\gamma g)\rangle} = |a_1a_2^{-1}|_E^{\lambda+\frac{1}{2}}.$$

We often drop Π from the notation.

Then, apart from a residual term, $\Theta_{\Pi,T_1,T_2}(f)$ is given by:

$$\frac{1}{2}\sum_{\Pi}\int\sum_{\phi}\Big(\int \Lambda_d^{T_1}E\big(a,I_\lambda(f)\phi;\lambda\big)\Omega^{-1}(a)\,\mathrm{d}a\Big) \times \overline{\Big(\int \Lambda_m^{T_2}E(h,\phi;\lambda)\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h\Big)}\,\mathrm{d}\lambda.$$

From now on, we will assume that f is a convolution product of three K-finite functions: $f = f_1 * f_2^*$, $f_1 = f_1^1 * f_1^2$. Later, we will assume that f_2 is itself a convolution product of sufficiently many K-finite functions. Then this is:

$$\frac{1}{2}\int\sum_{\phi,\phi'}\langle I_\lambda(f_1^2)\phi,\phi'\rangle\Big(\int \Lambda_d^{T_1}E\big(a,I_\lambda(f_1^1)\phi';\lambda\big)\Omega^{-1}(a)\,\mathrm{d}^\times a\Big) \times \overline{\Big(\int \Lambda_m^{T_2}E\big(h,I_\lambda(f_2)\phi;\lambda\big)\omega\eta\big(\kappa(h)\big)\,\mathrm{d}h\Big)}\,\mathrm{d}\lambda.$$

Our goal is now to obtain an *absolutely convergent* expression for the spectral terms.

We will establish convergence in the case $\omega = \eta$. Then, by hypothesis, the restriction of Ω to $F_{\mathbb{A}}^\times$ is η. The case $\omega = 1$ is similar. Also, to establish the convergence, it will be more convenient to replace the unitary similitude group by the group $\mathrm{Gl}(2,F)$. Then the above expression is (with $h \in Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$):

$$\frac{1}{2}\sum_{\Pi}\int \mathrm{d}\lambda\sum_{\phi,\phi'}\langle I_\lambda(f_1^2)\phi,\phi'\rangle \tag{38}$$

$$\times\Big(\int \Lambda_d^{T_1}E\big(a,I_\lambda(f_1^1)\phi';\lambda\big)\Omega^{-1}(a)\,\mathrm{d}^\times a\Big) \tag{39}$$

$$\times\overline{\Big(\int \Lambda_m^{T_2}E\big(h,I_\lambda(f_2)\phi;\lambda\big)\,\mathrm{d}h\Big)}\,\mathrm{d}\lambda. \tag{40}$$

In this part of the paper, we do not pay much attention to the normalization of the Haar measures.

8.3. Period integral over A. — Let (Π_1,Π_2) be a pairs of idele class characters with $\Pi_2 = \Pi_1^{-1}$. We set

$$\nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then, in the domain of convergence we get:

$$E(g,\phi;\lambda) = \sum_{\alpha\in Z(E)\backslash A(E)}\phi(w\nu\alpha g)\mathrm{e}^{\langle\lambda+\rho,H(w\nu\alpha g)\rangle} + \phi(g)\mathrm{e}^{\langle\lambda+\rho,H(g)\rangle} + \phi(wg)\mathrm{e}^{\langle\lambda+\rho,H(wg)\rangle}.$$

On the other hand:

$$E_N(g,\phi;\lambda) = \phi(g)\mathrm{e}^{\langle\lambda+\rho,H(g)\rangle} + M(w,\lambda)\phi(g)\mathrm{e}^{\langle-\lambda+\rho,H(g)\rangle}.$$

Then:

$$\begin{aligned}\Lambda_d^T E(a,\phi;\lambda) = \sum_\alpha &\phi(w\nu\alpha a)\mathrm{e}^{\langle\lambda+\rho,H(w\nu\alpha a)\rangle}\\
&+\phi(a)\mathrm{e}^{\langle\lambda+\rho,H(a)\rangle}\big(1-\widehat{\tau}_P(H(a)-T)\big)\\
&+\phi(wa)\mathrm{e}^{\langle\lambda+\rho,\mathrm{H(wa)}\rangle}\big(1-\widehat{\tau}_{\mathrm{P}}(\mathrm{H(wa)-T})\big)\\
&-M(w,\lambda)\phi(a)\mathrm{e}^{\langle\lambda+\rho,H(a)\rangle}\big((\widehat{\tau}_P(H(a)-T)\big)\\
&-M(w,\lambda)\phi(wa)\mathrm{e}^{\langle-\lambda+\rho,H(wa)\rangle}\big(\widehat{\tau}_P(H(wa)-T)\big).\end{aligned}$$

A simple computation gives

$$\begin{aligned}(41)\quad \int \Lambda_d^T E(a,\phi,\lambda,\Pi)\Omega^{-1}(a)\mathrm{d}^\times a &= \mu(\phi;\lambda,\Pi_1,\Pi_2)\\
&+\big(\delta(\Pi_1\Omega^{-1})\phi(\mathrm{e})+\delta(\Pi_2\Omega^{-1})\phi(w)\big)\frac{\mathrm{e}^{T(\frac12+\lambda)}}{\frac12+\lambda}\\
&+\big(\delta(\Pi_2\Omega^{-1})M(w,\lambda)\phi(\mathrm{e})+\delta(\Pi_1\Omega^{-1})M(w,\lambda)\phi(w)\big)\frac{\mathrm{e}^{T(\frac12-\lambda)}}{\frac12-\lambda}\end{aligned}$$

where we have set $a=\mathrm{diag}(a_1,1)$ and

$$\mu(\phi;\lambda,\Pi_1,\Pi_2)=\int_{E_{\mathbb{A}}^\times}\phi(w\nu a)\Omega^{-1}(a_1)\mathrm{e}^{\langle\lambda+\rho,H(w\nu a)\rangle}\mathrm{d}^\times a_1.$$

This formula gives the analytic continuation of μ. However, we will need to have more information on $\mu(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi)$ where f is a smooth function of compact support on $G(E_{\mathbb{A}})$. We write $f=f_Sf^S$ where f_S is a smooth function of compact support on $G(E_S)$ and f^S is the characteristic function of $K^S:=\prod_{v\notin S}K_v$ and $K_v=\mathrm{Gl}(2,\mathcal{O}_v)$. We may then assume that ϕ is invariant under K^S. We have, for $g\in G_S$,

$$I_{\lambda,\Pi}(f)\phi(g)\mathrm{e}^{\langle\lambda+\rho,H(g)\rangle}=\int_{G_S}\phi(gx)\mathrm{e}^{\langle\lambda+\rho,H(gx)\rangle}f_0(x^{-1})\mathrm{d}x$$

where we have set $f_0=\check{f}$. This can be written as

$$\int_{G_S}\phi(x)\mathrm{e}^{\langle\lambda+\rho,H(x)\rangle}f_0(x^{-1}g)\mathrm{d}x$$

or

$$\int f_1\Big[n\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix}g\Big]\mathrm{d}n\,|a_2|^{\lambda+\frac12}\Pi_2(a_2)^{-1}|a_1|^{-\lambda-\frac12}\Pi_1(a_1)^{-1}\mathrm{d}^\times a_1,$$

where we have set

$$f_1(g)=\int_K\phi(k)f_0(k^{-1}g)\mathrm{d}k.$$

At this point we introduce new functions on G_S:

$$f_1(g;\lambda,\Pi_1):=f_1(g)\Pi_1(\det g)^{-1}|\det g|^{-\lambda-\frac12},$$

$$\phi_S[g;\lambda,\Pi_1] := \int_{E_S^\times}\int_{N_S} f_1\Big[n\begin{pmatrix} a_1 & 0\\ 0 & 1\end{pmatrix}g;\lambda,\Pi_1\Big]\,\mathrm{d}n\,\mathrm{d}^\times a_1.$$

We let P_1 be the group of matrices with second row $(0,1)$. Clearly ϕ_S is invariant on the left under $P_1(E_S)$. Thus there is a Schwartz-Bruhat function $\Phi_S[(x,y);\lambda,\Pi_1]$ on $E_S\times E_S$ such that, for $g\in G_S$,

$$\Phi_S[(0,1)g;\lambda,\Pi_1] = \phi_S[g;\lambda,\Pi_1].$$

The support of Φ_S is contained in a compact set of $E_S^2-\{(0,0)\}$ which is independent of λ. We have then for $g\in G_S$:

$$I_{\lambda,\Pi}(f)\phi(g)\mathrm{e}^{\langle\lambda+\rho,H(g)\rangle} = \int_{E_S^\times}\Phi_S[(0,t)g;\lambda,\Pi_1]|t|^{2\lambda+1}\Pi_1\Pi_2^{-1}(t)\,\mathrm{d}^\times t\,\Pi_1(g)\,|\det g|^{\lambda+\frac12}.$$

To obtain a formula valid for all g we introduce

$$\Phi\big[(x,y);\lambda,\Pi_1\big] := \Phi_S\big[(x_S,y_S);\lambda,\Pi_1\big]\prod_{v\notin S}\Phi_v(x_v,y_v),$$

where Φ_v, $v\notin S$, is the characteristic function of $\mathcal{O}_v^2$. Then, for $g\in G(E_{\mathbb{A}})$,

$$I_{\lambda,\Pi}(f)\phi(g)\mathrm{e}^{\langle\lambda+\rho,H(g)\rangle} = \frac{\Pi_1(g)\,|\det g|^{\lambda+\frac12}}{L^S(2\lambda+1,\Pi_1\Pi_2^{-1})}\int_{E_{\mathbb{A}}^\times}\Phi\big[(0,t)g;\lambda,\Pi_1\big]\cdot|t|^{2\lambda+1}\Pi_1\Pi_2^{-1}(t)\,\mathrm{d}^\times t. \tag{42}$$

After a change of variables we find:

$$\mu(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi) = \frac{1}{L^S(2\lambda+1,\Pi_1\Pi_2^{-1})} \times\iint\Phi\big[(a_1,a_2);\lambda,\Pi_1\big]\Omega^{-1}\Pi_1(a_1)|a_1|^{\lambda+\frac12} \times\mathrm{d}^\times a_1\,\Omega\Pi_2^{-1}(a_2)|a_2|^{\lambda+\frac12}\,\mathrm{d}^\times a_2. \tag{43}$$

The computations are justified for $\operatorname{Re}\lambda\gg 0$. For a given Π, a given ϕ and a given f, this expression gives the analytic continuation of μ as a meromorphic function of λ, which has no singularity on the line $\operatorname{Re}\lambda=0$. It has a zero at $\lambda=0$ if $\Pi_1=\Pi_2$, that is, if $\Pi_1^2=1$. Moreover, this function of λ is at most of polynomial increase on the line $\operatorname{Re}\lambda=0$. All its derivatives have the same property.

However, for our purposes, we will need to have uniform estimates. We proceed as follows. First we take f in a fixed bounded set $\mathcal{B}$ of the space of smooth functions of compact support. This means that $f=f_Sf^S$ as before, with S fixed; the function f_S has support in a fixed set Ξ, open and relatively compact; we may as well assume $K_S\Xi=\Xi K_S=\Xi$; the function is invariant on the right and the left under a compact open normal subgroup K' of $\prod_{v\in S,\mathrm{finite}}K_v$ (and thus under the subgroup $K'' = K'K^S$ of K^∞). For each element X of the enveloping algebra of G_∞ there is a constant C_X such that

$$\sup\bigl|\rho(X)f\bigr| \le C_X.$$

Consider the function:

$$f_1(g) = \int \phi(k) f_0(k^{-1}g)\phi(k)\,\mathrm{d}k.$$

The integral does not change if we replace ϕ by ϕ_1 defined by

$$\phi_1(k) = \mathrm{vol}(K'')^{-1}\int \phi(kk'')\,\mathrm{d}k''.$$

Then ϕ_1 is invariant on the right and therefore on the left under K''. Thus the same is true of f_1. Moreover $f_1 = f_{1,S}f^S$. For any X,

$$\sup\bigl|\rho(X)f_1\bigr| \le \|\phi\|C_X.$$

Finally, the support of $f_{1,S}$ is still contained in Ξ. If $I_{\lambda,\Pi}(f) \neq 0$ for a $f \in \mathcal{B}$, then Π_1 is unramified outside S and, for each finite $v \in S$, $\Pi_{1,v|\mathcal{O}_v^\times}$ belongs to a finite set. We now consider the function $f_1(g;\lambda,\Pi_1)$. It has a support contained in Ξ. It is invariant under a compact subgroup K_0' of $\prod_{v\in S,\mathrm{finite}} K_v$ (depending only on K'). Suppose λ is in a strip $a \le \mathrm{Re}\,\lambda \le b$. Then, for any X, there is a polynomial $P_X(\Pi_1,\lambda)$ such that

$$\sup\bigl|\rho(X)f_1(\bullet;\lambda,\Pi_1)\bigr| \le \|\phi\|P_X(\Pi_1,\lambda).$$

By a *polynomial* we means a function $P(\Pi_1,\lambda)$ of the following form. We may identify the enveloping algebra of $E_\infty^\times$ to the space of distributions with support $\{e\}$. Thus if f is a smooth function on $E_\infty^\times$ or $E_\mathbb{A}^\times$ and X in the enveloping algebra we can define $\langle X,f\rangle$. Then a polynomial is a function of the form $P(\Pi_1,\lambda) = \langle X,f\rangle$ where $f(a) = \Pi_1(a)|a|^\lambda$ and X is in the enveloping algebra. Note that such a function depends only on λ and the infinite components of Π_1. In fact, if we write, for v real,

$$\Pi_{1,v}(x) = |x|_v^{iu_v}\Bigl(\frac{x}{|x|}\Bigr)^{\epsilon_v}, \quad \epsilon_v = 0,1, \quad d_v = 1,$$

and, for v complex,

$$\Pi_{1,v}(z) = (z\,\overline{z})^{iu_v}\frac{z^{n_v}}{(z\,\overline{z})^{n_v/2}}, \quad n_v \in \mathbb{Z}, \quad d_v = 2,$$

with $\sum_{v\in\infty} d_v u_v = 0$, then a polynomial $P(\Pi_1,\lambda)$ is just a polynomial in the variables (n_v,u_v) and λ. To allow for the case where λ is complex, we will say that any linear combination of products $P(\Pi_1,\lambda)\,\overline{Q(\Pi_1,\lambda)}$, where P and Q are polynomials, is also a polynomial. For instance, for any integer N, $(1+\lambda\,\overline{\lambda})^N$ is a polynomial in that sense. It will be understood that all polynomials considered are positive.

This being so, all derivatives of $f_1(\bullet\,; \lambda, \Pi_1)$ have the same properties (with different polynomials). A similar assertion is valid for the function $\phi_S(\bullet\,; \lambda, \Pi_1)$ except that its support is contained in a fixed set, compact modulo $P_1(E_S)$.

The Schwartz-Bruhat function $\Phi_S(\bullet\,; \lambda, \Pi_1)$ is compactly supported: its support is contained in a fixed open set of $E_S^2 - 0$, relatively compact in $E_S^2 - 0$. It is invariant on the right under K_0'. Thus its support is a compact set of E_S^2 and it is invariant under translation by a compact open subgroup of $E_{S_{\mathrm{finite}}}$. Take λ in a strip. Then $\Phi_S[\bullet\,; \lambda, \Pi_1]$ is bounded by a bound for $\phi_S(\bullet\,; \lambda, \Pi_1)$. Since G_∞ operates on E_∞^2 every element X of the enveloping algebra defines a differential operator X_a on E_∞^2 (with non constant coefficients). We have

$$(X_a \Phi_S)\big[(0,1)g; \lambda, \Pi_1\big] = \big(\rho(X)\phi_S\big)(g; \lambda, \Pi_1).$$

Consider a differential operator ξ on E_∞^2 with constant coefficients. On the complement of $(0,0)$ we can write $\xi = \sum_i c_i X_{i,a}$ where the X_i are elements of the enveloping algebra and the functions c_i are smooth functions on $E_\infty^2 - (0,0)$. Using the majorization for the functions $\rho(X_i)\phi_S$ we conclude that

$$\big|\,\xi\,\Phi_S[(x,y); \lambda, \Pi_1]\big| \le \|\phi\| P_\xi(\Pi_1, \lambda).$$

Thus the functions $\Phi_S[(x,y); \lambda, \Pi_1]$ remain in a bounded set of $\mathcal{C}_c^\infty(E_S^2)$ and *a fortiori* of the Schwartz-Bruhat space $\mathcal{S}(E_S^2)$. Note that the functions

$$\xi\,\Phi_S\big[(x,y); \lambda, \Pi_1\big]$$

are continuous in (x,y,λ) and holomorphic in λ. In (43) the analytic continuation of the Tate integral is obtained by using the Poisson summation formula. Thus in a strip, say $-\frac{1}{4} \le \operatorname{Re}\lambda \le \frac{1}{4}$, we have then a uniform estimate:

$$\Big|\iint \Phi\big[(a_1,a_2); \lambda, \Pi_1\big]\Omega^{-1}\Pi_1(a_1)|a_1|^{\lambda+\frac{1}{2}}\,\mathrm{d}^\times a_1 \Omega\Pi_2^{-1}(a_1)|a_1|^{\lambda+\frac{1}{2}}\,\mathrm{d}^\times a_1\Big| \le \|\phi\| P(\Pi_1, \lambda).$$

We have similar estimates for the derivatives by Cauchy integral formula.

We stress that we need only consider characters Π_1 such that $I_{\lambda,\Pi}(f) \ne 0$ for $f \in \mathcal{B}$. Thus for $v \notin S$ the character $\Pi_{1,v}$ is unramified and for v finite in S the restriction of $\Pi_{1,v}$ to $\mathcal{O}_v^\times$ belongs to a finite set. If furthermore $\Pi_1^2 = 1$ this forces Π_1 to belong to a finite set. There is then a polynomial $P(\Pi_1, \lambda)$ such that, on the line $\operatorname{Re}\lambda = 0$, for characters Π_1 of the above type,

$$\Big|\frac{1}{L^S(2\lambda+1, \Pi_1^2)}\Big|$$

is bounded by $P(\Pi_1, \lambda)$. Combining with the previous inequality, we get:

Lemma 11. — *Suppose f is in a bounded set $\mathcal{B}$. Then there is a polynomial $P(\Pi_1, \lambda)$ such that, for $\operatorname{Re}\lambda = 0$, and all characters Π_1:*

$$\big|\mu(I_{\lambda,(\Pi_1,\Pi_1^{-1})}(f)\phi; \lambda,\, \Pi_1, \Pi_1^{-1})\big| \le P(\Pi_1, \lambda)\|\phi\|.$$

On the other hand, we can obtain similar estimates for the functions

$$L^S(2\lambda+1,\Pi_1^2)$$

in a strip (with modification for a pole at $\lambda=0$ if $\Pi_1^2=1$), hence also for their derivatives. Thus we have also similar estimates for the derivative of the expression in the proposition on the line $\operatorname{Re}\lambda=0$. We can use the notations of generalized vectors (dual to the smooth vectors). There is a generalized vector $\mu_{\lambda,\Pi}$ such that

$$\mu\big(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi\big)=\big\langle\phi,I_{\lambda,\Pi}(f^*)\mu_{\lambda,\Pi}\big\rangle$$

and our estimate amounts to

$$\big\|I_{\lambda,\Pi}(f^*)\mu_{\lambda,\Pi}\big\|\le P(\Pi_1,\lambda).$$

Remark. — We define the Whittaker linear form $\mathcal{W}$ on the space of $I_{\lambda,\Pi}$ by analytic continuation of the integral

$$\mathcal{W}(\phi;\lambda,\Pi)=\int_{E_{\mathbb{A}}}\phi(wn)\mathrm{e}^{\langle\lambda+\rho,H(wn)\rangle}\psi_E(-x)\,\mathrm{d}x,\quad n=\begin{pmatrix}1&x\\0&1\end{pmatrix}.$$

After a change of variables, we easily find

$$\begin{aligned}(44)\quad &\mathcal{W}\Big[I_{\lambda,\Pi}\begin{pmatrix}a&0\\0&1\end{pmatrix}I_{\lambda,\Pi}(f)\phi\Big]\\ &\qquad=\frac{\Pi_1(a)\,|a|^{\lambda+\frac12}}{L^S(2\lambda+1,\Pi_1\Pi_2^{-1})}\int_{E_{\mathbb{A}}^\times}\widetilde{\Phi}\big[(at,t^{-1};\lambda,\Pi_1\big]\cdot|t|^{2\lambda}\Pi_1\Pi_2^{-1}(t)\,\mathrm{d}^\times t\end{aligned}$$

where $\widetilde{\Phi}$ denotes the Fourier transform with respect to the second variable. This integral is absolutely convergent and gives the analytic continuation of $\mathcal{W}$.

8.4. Intertwining period and intertwining operator. — We need to consider only pairs $\Pi=(\Pi_1,\Pi_2)$ (with $\Pi_2=\Pi_1^{-1}$) such that the integral of a truncated Eisenstein series over $\mathrm{Gl}(2,F_{\mathbb{A}})$ is non-zero. Then (see [JL], [HLRo]) either Π_1 (and Π_2) have a trivial restriction to $F_{\mathbb{A}}^\times$ or $\Pi_2^\tau=\Pi_1^{-1}$ (or, what amounts to the same, $\Pi_1=\Pi_1^\tau$). Of course the two conditions can be both satisfied.

In this section, we assume that $\Pi_1^\tau=\Pi_1$. Then we can define an *intertwining period* as follows (see [JL] and [JLRo]):

$$\mathcal{P}(\phi;\lambda,\Pi):=\int_{T(F_{\mathbb{A}})\backslash G(F_{\mathbb{A}})}\phi(\eta h)\mathrm{e}^{\langle\lambda+\rho,H(\eta h)\rangle}\,\mathrm{d}h.$$

Here $\eta\in G(E)$ is an element such that $\eta\,\overline{\eta}^{-1}=w$ and T is the torus such that

$$T(F)=\eta^{-1}A(E)\eta\cap\mathrm{Gl}(2,F).$$

The integral converges for $\operatorname{Re}\lambda \gg 0$ and can be analytically continued. Recall the formula (42). We let $\Phi_\eta[m;\lambda,\Pi_1]$ be the Schwartz-Bruhat function on $M(2\times 2, F_{\mathbb{A}})$ defined by

$$\Phi_\eta[m;\lambda,\Pi_1] := \Phi[(0,1)\eta m;\lambda,\Pi_1].$$

Then we get:

$$\begin{aligned}&\mathcal{P}(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi)\\&\quad=\frac{1}{L^S(2\lambda+1,\Pi_1^2)}\times\int_{\mathrm{Gl}(2,F_{\mathbb{A}})}\Phi_\eta[m;\lambda,\Pi_1]\,\Pi_1(\det m)|\det m|_F^{2\lambda+1}\,\mathrm{d}m.\end{aligned}$$

The integral is a Tate integral for the group $\mathrm{Gl}(2,F)$. Using the Iwasawa decomposition we see that the Tate integral is an entire multiple of

$$L\big(2\lambda+1,\Pi_{1|F_{\mathbb{A}}^\times}\big)L\big(2\lambda,\Pi_{1|F_{\mathbb{A}}^\times}\big).$$

In particular, it has no pole on the line $\operatorname{Re}\lambda=0$ except at $\lambda=0$ when $\Pi_{1|F_{\mathbb{A}}^\times}=1$. Because we assume that $\Pi_1^\tau=\Pi_2^{-1}=\Pi_1$ this implies that $\Pi_1^2=1$. Then the above ratio is an entire multiple of

$$\frac{L_{S_0}(2\lambda+1,1_F)L(2\lambda,1_F)}{L^{S_0}(2\lambda+1,\eta)}\cdot$$

These observations combined with standard estimates for L-functions can be used to prove the following lemma.

Lemma 12. — *Let $\mathcal{B}$ be a bounded set. Then there is a polynomial $P(\Pi_1,\lambda)$ such that, for all Π_1 with $\Pi_1^\tau=\Pi_1$ and $\Pi_{1|F_{\mathbb{A}}^\times}\neq 1$, all λ with $\operatorname{Re}\lambda=0$, and all $f\in\mathcal{B}$:*

$$\big|\mathcal{P}(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi)\big|\le P(\Pi_1,\lambda)\|\phi\|.$$

This can expressed again in terms of generalized vectors. There is a generalized vector $\mathcal{P}_{\lambda,\Pi}$ such that

$$\mathcal{P}\big(I_{\lambda,\Pi}(f)\phi;\lambda,\Pi\big)=\big\langle\phi,I_{\lambda,\Pi}(f^*)\mathcal{P}_{\lambda,\Pi}\big\rangle.$$

Then:

$$\big\|I_{\lambda,\Pi}(f^*)\mathcal{P}_{\lambda,\Pi}\big\|\le P(\Pi_1,\lambda).$$

Combining with the results of the previous subsection, we arrive at the following result.

Proposition 7. — *Let $\mathcal{B}$ be a bounded set of the space of smooth functions of compact support. Then there is a polynomial $P(\Pi_1,\lambda)>0$ such that, for any function f of the form $f=f_1*f_2*f_3$ with $f_1,f_3\in\mathcal{B}$, any Π_1 with $\Pi_1^\tau=\Pi_1$, any λ with $\operatorname{Re}\lambda=0$:*

$$\big|\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi},\mu_{\lambda,\Pi}\rangle\big|\le P(\Pi_1,\lambda)\big\|I_{\lambda,\Pi}(f_2)\big\|.$$

For the proof we remark that the expression in the proposition has the form

$$\langle I_{\lambda,\Pi}(f_2)I_{\lambda,\Pi}(f_3)\mathcal{P}_{\lambda,\Pi}, I_{\lambda,\Pi}(f_1^*)\mu_{\lambda,\Pi}\rangle$$

so that our assertion follows from the previous lemmas, for the characters Π_1 whose restriction to $F_{\mathbb{A}}^\times$ is non-trivial. On the other hand, if this restriction is trivial then $\Pi_1^2 = 1$. We need only consider those Π_1 such that $I_{\lambda,\Pi_1}(f_1) \neq 0$ for $f_1 \in \mathcal{B}$. There are only finitely many such Π_1. So we do not need to have estimates uniform in this case. We have written

$$\mu_{\lambda,\Pi}(I_{\lambda,\Pi}(f)\phi) = L^S(2\lambda+1, 1_E)^{-1}\widetilde{\mu}_{\lambda,\Pi}\big(I_{\lambda,\Pi}(f)\phi\big)$$

where $\widetilde{\mu}(\bullet)$ has no singularity on $\operatorname{Re}\lambda = 0$. Since $\mathcal{P}_{\Pi_1,\lambda}(I_{\lambda,\Pi}(f)\phi))$ has a pole at $\lambda = 0$ in this case, we need to modify the previous lemma and replace $\overline{\mathcal{P}_{\Pi_1,\lambda}(I_{\lambda,\Pi}(f)\phi)}$ by its product with $L^S(2\lambda+1, 1_E)^{-1}$ which has a zero at $\lambda = 0$. The product has then no singularity at $\lambda = 0$. It is then easy to obtain the required estimates in this case.

Using similar arguments, we can obtain estimates for the intertwining operator, more precisely, for expressions of the form

$$\langle M(w,\lambda)I_{\lambda,\Pi}(f)(\phi), \phi'\rangle.$$

8.5. Continuous contribution. — We study the terms corresponding to pairs Π with $\Pi_2^\tau = \Pi_1^{-1}$, that is, $\Pi_1^\tau = \Pi_1$. Each such term will ultimately contribute an integral. We first consider the terms with $\Pi_1 \neq \Omega$, (a condition equivalent to $\Pi_2 \neq \Omega$ since $\Omega^\tau = \Omega^{-1}$) and $\Pi_{1|F_{\mathbb{A}}^\times} \neq 1$. Then in fact

$$\int \Lambda_d^{T_1} E(a,\phi,\lambda)\Omega^{-1}(a)\mathrm{d}a, \qquad \int \Lambda_m^{T_2} E(h_1,\phi,\lambda)\mathrm{d}h_1$$

are independent of T_1 and T_2, being equal respectively, to $\mu(\phi;\lambda,\Pi)$ and $\mathcal{P}(\phi;\lambda,\Pi)$. Thus:

$$\Theta_{\Pi,T_1,T_2}(f) = \frac{1}{2}\int \sum_{\phi,\phi'} \langle I_\lambda(f_2^2)\phi, \phi'\rangle\, \mu(I_\lambda(f_1^1)\phi', \lambda)\, \overline{\mathcal{P}(I_\lambda(f_2)\phi, \lambda)}\mathrm{d}\lambda.$$

Note that if $\Pi_1 = \Pi_2$ then the expression for $K_{f,\Pi}(x,y)$ contains a residual term: $\int \Pi_1(\det g)f(g)\mathrm{d}g\,\Pi_1(\det x)\overline{\Pi_1(\det y)}$. However, since $\Omega \neq \Pi_1$, the integral of the truncation of this term over A against Ω^{-1} contains as a factor the integral of $\Pi_1\Omega^{-1}$ over the quotient of the ideles of absolute value 1 and is thus 0. With the notation of generalized vectors, the above integral becomes:

$$\int \langle I_\lambda(f)\mathcal{P}_{\lambda,\Pi}, \mu_{\lambda,\Pi}\rangle \mathrm{d}\lambda.$$

For $f = f_1^1 * f_1^2 * f_2^*$:

$$\langle I_\lambda(f)P_{\lambda,\Pi}, \mu_{\lambda,\Pi}\rangle = \langle I_\lambda(f_1^2)I_\lambda(f_2^*)P_{\lambda,\Pi}, I_\lambda(f_1^{1*})\mu_{\lambda,\Pi}\rangle.$$

By Proposition 7, there is then a polynomial $P(\Pi_1,\lambda)$ such that

$$\big|\langle I_\lambda(f)P_{\lambda,\Pi},\mu_{\lambda,\Pi}\rangle\big| \le P(\Pi_1,\lambda)\big\|I_{\lambda,\Pi}(f_1^2)\big\|.$$

In fact, this is true for f_1^1, f_2^* smooth in a bounded set and any smooth function f_1^2.

We will make repeated use of the following lemma:

LEMMA 13. — *Suppose that $\mathcal{B}$ is a bounded set of the space of smooth functions of compact support and $P(\Pi_1,\lambda)$ a polynomial. Then there is a constant $C>0$ such that, for any $f\in\mathcal{B}$,*

$$\sum_{\Pi_1}\int P(\Pi_1,\lambda)\big\|I_{\lambda,\Pi_1,\Pi_1^{-1}}(f)\big\|\mathrm{d}\lambda < C,$$

the sum over all *(normalized) idele class characters of E.*

Proof of the Lemma. — The operator $I_{\lambda,\Pi_1,\Pi_1^{-1}}(f)$ is represented by the following kernel function on $K\times K$:

$$H_{f,\lambda,\Pi}(k_1,k_2) := \int f\Big[k_1^{-1}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0\\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}k_2\Big]\Pi_1(a)|a|_E^{\lambda+\frac12}\,\mathrm{d}^\times a\,\mathrm{d}^\times b\,\mathrm{d}x.$$

Thus its operator norm is bounded by the supremum over $K\times K$ of

$$\Big|\int\phi(a:k_1:k_2)\Pi_1(a)|a|_E^{\lambda}\mathrm{d}^\times a\Big|$$

where

$$\phi(a:k_1:k_2) := \int f\Big[k_1^{-1}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0\\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}k_2\Big]|a|_E^{\frac12}\mathrm{d}^\times b\,\mathrm{d}x.$$

For $f\in\mathcal{B}$, $k_i\in K$, the function $a\mapsto\phi(a:k_1:k_2)$ remains in a bounded set $\mathcal{B}_0$ of the space of smooth functions of compact support on $E_{\mathbb{A}}^\times$. Thus it will suffice to show there is C such that, for $\phi\in\mathcal{B}_0$,

$$\sum_{\Pi_1}\int P(\Pi_1,\lambda)\Big|\int\phi(a)\Pi_1(a)|a|_E^{\lambda}\mathrm{d}^\times a\Big|\mathrm{d}\lambda < C.$$

This is clear if $P=1$. In general

$$\begin{aligned}P(\Pi_1,\lambda)\Big|\int\phi(a)\Pi_1(a)|a|_E^{\lambda}\mathrm{d}^\times a\Big| &= \Big|P(\Pi_1,\lambda)\int\phi(a)\Pi_1(a)|a|_E^{\lambda}\mathrm{d}^\times a\Big|\\ &= \Big|\int\phi_1(a)\Pi_1(a)|a|_E^{\lambda}\mathrm{d}^\times a\Big|\end{aligned}$$

where $\phi_1=\rho(X)\phi$ for a suitable element X of the enveloping algebra of $E_\infty^\times$. Since ϕ_1 remains in another bounded set, our assertion follows. □

From our observations on the independence of the integrals on the truncation parameters, we see that we can sum over all Π_1 of the above type and get

$$\lim_{T_1\to+\infty}\lim_{T_2\to+\infty}\sum_{\Pi}\Theta_{\Pi,T_1,T_2}(f)=\sum_{\Pi_1}\frac{1}{2}\int\big\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi},\mu_{\lambda,\Pi}\big\rangle\,\mathrm{d}\lambda,$$

with

$$\sum_{\Pi_1}\int\big|\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi},\mu_{\lambda,\Pi}\rangle\big|\,\mathrm{d}\lambda<+\infty.$$

Moreover, if $\mathcal{B}$ is a bounded set, there is $C>0$ such that, for f of the form $f=f_1*f_2*f_3$, with each f_i in $\mathcal{B}$, then

$$\sum_{\Pi_1}\int\big|\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi},\mu_{\lambda,\Pi}\rangle\big|\,\mathrm{d}\lambda<C.$$

Next, we discuss the contribution of the term $\Pi_1=\Omega$, if any. Since we are assuming that $\Omega^\tau=\Omega^{-1}$ in any case and for now we are assuming that $\Pi_1^\tau=\Pi_2^{-1}$ this implies that $\Pi_2=\Pi_1=\Omega$ so that Ω is a quadratic character satisfying $\Omega_{|F_{\mathbb{A}}^\times}=\eta$. Moreover $\Pi_{1|F_{\mathbb{A}}^\times}$ is then non-trivial. The expression for K_Π contains again a residual term; however its truncated integral over $\mathrm{Gl}(2,F_{\mathbb{A}})$ is zero. Indeed, it contains as a factor:

$$\int_{Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})}\eta(\det h)\,\mathrm{d}h-\int_{Z(F_{\mathbb{A}})A(F)N(F_{\mathbb{A}})\backslash G(F_{\mathbb{A}})}\eta(\det h)\widehat{\tau}_P\big(H(h)-T\big)\,\mathrm{d}h$$

which is zero. Then $\int\Lambda_m^{T_2}E(h_1,\phi,\lambda)\,\mathrm{d}h_1$ is again independent of T_2 but

$$\int\Lambda_d^{T_1}E(a,\phi,\lambda)\Omega^{-1}(a)\,\mathrm{d}a$$

does depend on T_1. We have then for $f=f_1^1*f_1^2*f_2^*$, all K-finite functions:

$$\begin{aligned}
\Theta_{\Pi,T_1,T_2}(f)&=\frac{1}{2}\int\langle I_\lambda(f)\mathcal{P}_{\lambda,\Pi},\mu_{\lambda,\Pi}\rangle\,\mathrm{d}\lambda\\
&+\tfrac12\int\sum_{\phi}\big(I_\lambda(f_1^1*f_1^2)\phi(\mathrm{e})+I_\lambda(f_1^1*f_1^2)\phi(w)\big)\\
&\qquad\qquad\times\frac{\mathrm{e}^{T_1(\frac12+\lambda)}}{\frac12+\lambda}\overline{\mathcal{P}(I_\lambda(f_2)\phi,\lambda)}\,\mathrm{d}\lambda\\
&+\tfrac12\int\sum_{\phi}\big((I_{-\lambda}(f_1^1*f_1^2)M(w,\lambda)\phi(\mathrm{e})+I_{-\lambda}(f_1^1*f_1^2)M(w,\lambda)\phi(w)\big)\\
&\qquad\qquad\times\frac{\mathrm{e}^{T_1(\frac12-\lambda)}}{\frac12-\lambda}\overline{\mathcal{P}(I_{-\lambda}(f_2)M(w,\lambda)\phi,-\lambda)}\,\mathrm{d}\lambda
\end{aligned}$$

where for the last expression we have used the functional equation of the period $\mathcal{P}$ (inherited from the functional equation of the Eisenstein series, see [JLRo]) to insert an intertwining operator. In the last expression, the sum over

ϕ does not depend on the choice of the basis. Thus we can ignore the intertwining operator. In the second expression, we can also change λ to $-\lambda$; we see that the second expression is equal to the third. That is:

$$\Theta_{\Pi,T_1,T_2}(f) = \frac{1}{2}\int \langle I_\lambda(f)P_{\lambda,\Pi}, \mu_{\lambda,\Pi}\rangle \mathrm{d}\lambda$$
$$+\int\sum_\phi \Big\{ \big(I_{-\lambda}(f_1^1 * f_1^2)\phi(\mathrm{e}) + I_{-\lambda}(f_1^1 * f_1^2)\phi(w)\big)\frac{\mathrm{e}^{\mathrm{T}_1(\frac{1}{2}-\lambda)}}{\frac{1}{2}-\lambda}\overline{\mathcal{P}(I_{\bar\lambda}(f_2)\phi, \overline{\lambda})}\Big\}\mathrm{d}\lambda.$$

We will move the integral to the line $\mathrm{Re}\,\lambda = 1$ rather than to the line $\mathrm{Re}\,\lambda = \frac{1}{2}$ as we did in [J2] (a similar contour integration for Gl(2) is done in [F3]). The term $\overline{\mathcal{P}(*,\overline{\lambda})}$ is the product of a Tate integral which is entire, hence bounded in vertical strips and the factor $(L^S(2\lambda+1, 1_E))^{-1}$. Thus this moving of the contour of integration is legitimate. In doing so we pick up a residue at $\lambda = \frac{1}{2}$. The integral on the line $\mathrm{Re}\,\lambda = 1$ tends to 0 as T_1 tends to infinity. So we are left with

$$\lim_{T_1\to+\infty}\lim_{T_2\to+\infty}\Theta_{\Pi,T_1,T_2}(f) = \frac{1}{2}\int\langle I_\lambda(f)P_{\lambda,\Pi}, \mu_{\lambda,\Pi}\rangle\mathrm{d}\lambda$$
$$+\sum_\phi\big(I_{-\frac{1}{2}}(f_1^1 * f_1^2)\phi(\mathrm{e}) + I_{-\frac{1}{2}}(f_1^1 * f_1^2)\phi(w)\big)\overline{\mathcal{P}(I_{\frac{1}{2}}(f_2)\phi, \tfrac{1}{2})}.$$

We view again $\mathcal{P}$ as a generalized vector in the (non-unitary) representation $I_{-\frac{1}{2},\Omega,\Omega}$. Then $I_{-\frac{1}{2},\Omega,\Omega}(f)\mathcal{P}$ is a smooth vector in this representation and the second term is the sum $I_{-\frac{1}{2},\Omega,\Omega}(f)\mathcal{P}(e) + I_{-\frac{1}{2},\Omega,\Omega}(f)\mathcal{P}(w)$.

Finally, we study the contribution of the terms where $\Pi_1^\tau = \Pi_1$ and $\Pi_{1|F_{\mathbb{A}}^\times}=1$. Then $\Pi_1 \neq \Omega$. Also $\Pi_1^2 = 1$. Thus, for a given f, or even f in a given bounded set, there are only *finitely* many such Π_1 with $I_{\lambda,\Pi_1,\Pi_2}(f) \neq 0$. Thus the sum $\sum \Theta_{\Pi,T_1,T_2}(f)$ is finite, and in the sum, each term is itself written as a finite sum of terms. The kernel $K_{f,\Pi}$ has once more a residual component whose truncated integral over the diagonal is 0. This time $\int \Lambda_d^{T_1}E(a,\phi,\lambda)\Omega^{-1}(a)\mathrm{d}a$ is independent of T_1 but

$$(45)\qquad \int \Lambda_m^{T_2}E(h_1,\phi,\lambda)\mathrm{d}h_1 = \frac{\mathrm{e}^{\lambda T_2}}{\lambda}\int_{K\cap H_1}\phi(k)\mathrm{d}k$$
$$+\frac{\mathrm{e}^{-\lambda T_2}}{-\lambda}\int_{K\cap H_1}M(w,\lambda)\phi(k)\mathrm{d}k + \mathcal{P}(\phi,\lambda,\Pi).$$

Since $\Pi_1 = \Pi_2$ we have $M(w,0) = -1$ and the two first terms have the same residue at $\lambda = 0$. The last term has also a simple pole at $\lambda = 0$. However the factor (39) reduces to $\mu(I_\lambda(f_1^1)\phi,\lambda,\Pi)$ which has a zero at $\lambda = 0$ because in (43) the reciprocal of the L-factor does. Thus the poles of the terms in (40) are compensated by the zero of (39). As T_2 tends to infinity the terms containing

$e^{\pm\lambda T_2}$ tend to 0 and we find the contribution of those finitely many terms to be the same as before (see Proposition 7).

8.6. Discrete terms. — Finally, we discuss the terms whose contributions will be purely discrete. We consider those Π such that $\Pi_1|_{F_{\mathbb{A}}^\times} = 1$ but $\Pi_1^\tau \neq \Pi_1$. This condition amounts to $\Pi_1^2 \neq 1$. We also have then $\Pi_1 \neq \Omega$ and the integral $\int \Lambda_d^{T_1} E(a,\phi,\lambda)\Omega^{-1}(a)\mathrm{d}a$ does not depend on T_1. However, we have then, in the sense of analytic continuation,

$$(46)\qquad \begin{cases} \displaystyle\int \Lambda_m^{T_2} E(h_1,\phi,\lambda)\mathrm{d}h_1 = \frac{e^{\lambda T_2}}{\lambda}\mathcal{P}_c(\phi) + \frac{e^{-\lambda T_2}}{-\lambda}\mathcal{P}_c\big(M(\lambda,w)\phi\big), \\ \displaystyle\mathcal{P}_c(\phi) = \int_{K_F} \phi(k)\mathrm{d}k, \end{cases}$$

where K_F denotes the standard maximal compact subgroup of $\mathrm{Gl}(2,F_{\mathbb{A}})$. Since the left hand side is holomorphic at $\lambda = 0$ we get (See also [JL, Lemma 7], for a computational proof.)

$$(47)\qquad \mathcal{P}_c(\phi) = \mathcal{P}_c\big(M(0,w)\phi\big).$$

It follows that

$$\Theta_{\Pi,T_1,T_2}(f) = \frac{1}{2}\sum_\phi \int \mu\big(I_\lambda(f)\phi\big)\overline{\Big(\frac{e^{\lambda T_2}}{\lambda}\mathcal{P}_c(\phi) + \frac{e^{-\lambda T_2}}{-\lambda}\mathcal{P}_c\big(M(\lambda,w)\phi\big)\Big)}\mathrm{d}\lambda.$$

Again, in terms of generalized vectors, this can be written in the form:

$$\frac{1}{2}\int \Big(\big\langle I_{\lambda,\Pi}(f)\mathcal{P}_c, \mu_{\lambda,\Pi}\big\rangle \frac{e^{-\lambda T_2}}{-\lambda} + \big\langle I_{\lambda,\Pi}(f)M(w,\lambda)^*\mathcal{P}_c, \mu_{\lambda,\Pi}\big\rangle \frac{e^{\lambda T_2}}{\lambda}\Big)\mathrm{d}\lambda.$$

Consider then the functions:

$$\Phi_1(\lambda) := \sum_\Pi \big\langle I_{\lambda,\Pi}(f)\mathcal{P}_c, \mu_{\lambda,\Pi}\big\rangle, \quad \Phi_2(\lambda) := \sum_\Pi \big\langle I_{\lambda,\Pi}(f)M(w,\lambda)^*\mathcal{P}_c, \mu_{\lambda,\Pi}\big\rangle,$$

the sums over all the Π of the above type.

Lemma 14. — *The above series converge absolutely, uniformly on compact sets, and define continuous functions on the line* $\operatorname{Re}\lambda = 0$. *Moreover* $\Phi_1(0) = \Phi_2(0)$. *If f_2 is the convolution of sufficiently many K-finite functions, the functions are differentiable on the line* $\operatorname{Re}\lambda = 0$.

Indeed, for the first assertion, we write as before $f_1^1 * f_1^2 * f_2^*$. Then

$$\big\langle I_{\lambda,\Pi}(f)\mathcal{P}_c, \mu_{\lambda,\Pi}\big\rangle = \big\langle I_{\lambda,\Pi}(f_1^2)I_{\lambda,\Pi}(f_2^*)\mathcal{P}_c, I_{\lambda,\Pi}(f_1^{1*})\mu_{\lambda,\Pi}\big\rangle.$$

As before, there is a polynomial $P(\Pi_1,\lambda)$ such that

$$\big\|I_{\lambda,\Pi}(f_1^{1*})\mu_{\lambda,\Pi}\big\| \le P(\Pi_1,\lambda).$$

It is easy to see that

$$\big\|I_{\lambda,\Pi}(f_2^*)\mathcal{P}_c\big\| \le C$$

where C is a constant. Hence

$$\big\|\langle I_{\lambda,\Pi}(f)\mathcal{P}_c, \mu_{\lambda,\Pi}\rangle\big\| \le CP(\Pi_1,\lambda)\big\|I_{\lambda,\Pi}(f_1^2)\big\|.$$

Since

$$\sum_{\Pi_1} P(\Pi_1,\lambda)\big\|I_{\lambda,\Pi}(f_1^2)\big\| < +\infty$$

the first assertion follows for Φ_1 (with uniform convergence on compact sets, even for f_1^2, f_1^2, f_2 smooth in a bounded set). Since the intertwining operator is unitary, the argument for Φ_2 is similar. For the differentiability, we proceed as follows. We do assume that the functions are K-finite. Then we need only consider those Π such that the corresponding operators are non-zero. This means that the restriction of Π_1 to the maximal compact subgroup of $E_{\mathbb{A}}^{\times}$ takes only finitely values. For the purpose of convergence, we may as well fix this restriction. Then all the representations $I_{\lambda,\Pi}$ operate on the same Hilbert of functions on the maximal compact subgroup K. Thus it suffices to show that

$$\sum_{\phi,\phi'}\sum_{\Pi}\big\langle I_{\lambda,\Pi}(f_1^2)\phi,\phi'\big\rangle\mu\big(I_{\lambda,\Pi}(f_1^1)\phi',\lambda,\Pi\big)\,\overline{\mathcal{P}_c(I_{\lambda,\Pi}(f_2)\phi)}$$

is differentiable, the sum over ϕ,ϕ' being *finite*. We may as well fix ϕ and ϕ'. At this point, we do assume that $f_2 = f_2^1 * f_2^2$ where the functions f_2^i, $i=1,2$, are K-finite. Then the terms $\langle I_{\lambda,\Pi}(f_1^2)\phi,\phi'\rangle$ and $\overline{\mathcal{P}_c(I_{\lambda,\Pi}(f_2))\phi)}$ are bounded by constant multiples of $\|I_{\lambda,\Pi}(f_1^2)\|$ and $\|I_{\lambda,\Pi}(f_2^1)\|$, respectively. Their respective derivatives are bounded, uniformly in (Π_1,λ). As for $\mu(I_{\lambda,\Pi}(f_1^1)\phi',\lambda,\Pi)$ it is bounded as well as its derivatives by a polynomial in (Π_1,λ). Our assertion follows for Φ_1. For Φ_2, we observe that

$$\begin{aligned}\sum_{\Pi_1}&\langle I_{\lambda,\Pi}(f_1^2)\phi,\phi'\rangle\mu(I_{\lambda,\Pi}(f_1^1)\phi',\lambda,\Pi)\,\overline{\mathcal{P}_c(I_{\lambda,\Pi}(f_2)M(w,\lambda)\phi)}\\ &= \sum_{\phi''}\sum_{\Pi_1}\big\langle I_{\lambda,\Pi}(f_1^1)\phi,\phi'\big\rangle\mu\big(I_{\lambda,\Pi}(f_1^2)\phi',\lambda,\Pi\big)\\ &\qquad\times\overline{\mathcal{P}_c(I_{\lambda,\Pi}(f_2^1))\phi'')}\cdot\overline{\big\langle I_{\lambda,\Pi}(f_2^2)M(w,\lambda)\phi,\phi''\big\rangle}\end{aligned}$$

with a finite sum over ϕ''. We proceed as before, this time assuming that, in turn, f_2^1 is the convolution of two K-finite functions. □

We have then, at least if f_2 is the convolution of sufficiently many K-finite functions,

$$\sum_{\Pi}\Theta_{\Pi,T_1,T_2}(f) = \int\Big(\Phi_1(\lambda)\frac{\mathrm{e}^{-\lambda T_2}}{-\lambda} + \Phi_2(\lambda)\frac{\mathrm{e}^{\lambda T_2}}{\lambda}\Big)\mathrm{d}\lambda.$$

This expression is independent of T_1. So we only need to take the limit as $T_2 \to +\infty$. It is of the form $c\Phi_1(0)$.

8.7. Conclusion. — We are now ready to state the main result of this section:

THEOREM 3. — *For any smooth function of compact support f:*

$$\begin{aligned}\Theta(f) = \sum_{\Pi\,\text{cuspidal}} \Theta_\Pi(f) &+ \sum_{\Pi_1^\tau=\Pi_1} \frac{1}{2}\sum_{\Pi}\int \big\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi}, \mu_{\lambda,\Pi}\big\rangle d\lambda \\ &+ c \sum_{\substack{\Pi_1|F_\mathbb{A}^\times=1\\ \Pi_1^\tau\neq\Pi_1}} \big\langle I_{0,\Pi}(f)\mathcal{P}_c, \mu_{0,\Pi}\big\rangle \\ &+ c_1\big\{\big(I_{-\frac{1}{2},\Omega,\Omega}(f)\mathcal{P}\big)(\mathrm{e}) + \big(I_{-\frac{1}{2},\Omega,\Omega}(f)\big)\mathcal{P}(w)\big\}.\end{aligned}$$

The expression is absolutely convergent:

$$\begin{aligned}&\sum_{\Pi_1^\tau=\Pi_1}\int \Big|\big\langle I_{\lambda,\Pi}(f)\mathcal{P}_{\lambda,\Pi}, \mu_{\lambda,\Pi}\big\rangle\Big| \mathrm{d}\lambda < +\infty,\\ &\sum_{\substack{\Pi_1|F_\mathbb{A}^\times=1\\ \Pi_1^\tau\neq\Pi_1}} \big|\langle I_{0,\Pi}(f)P_c, \mu_{0,\Pi}\rangle\big| < +\infty.\end{aligned}$$

By [DM], every smooth function of compact support can be written as a finite sum of triple convolution products. Thus it suffices to prove our assertion when $f = f_1 * f_2 * f_3$. If each f_i remains in a bounded set both sides are well defined and bounded. Since the space of smooth functions of compact support is a bornological space, each side may be regarded as a distribution on the product of three copies of the group. We have shown both sides agree when each f_i is K-finite and f_3 is itself the convolution of sufficiently many K-finite functions. Thus they agree for arbitrary f_i and our assertion follows. Note that the right hand side is in fact a distribution, as follows for instance from the kernel theorem.

9. Appendix: Continuous spectrum over F

We pass to the much easier discussion of the convergence of the spectral expression for G_ϵ. For simplicity we only consider the case $\omega = 1$, $\Omega = 1$. The only difficult case is the case where ϵ is a norm. Of course, no truncation is needed. We then identify G_ϵ with $\mathrm{Gl}(2, F)$. Consider a pair of characters $\pi = (\pi_1, \pi_2)$ with $\pi_1\pi_2 = 1$. Since $\mathrm{Gl}(2, F) = P(F)T(F)$ we have:

$$\int E(t, \phi; \pi, \lambda)\,\mathrm{d}t = \nu(\phi; \lambda, \pi)$$

where

$$\nu(\phi;\pi,\lambda) := \int_{T(F)Z(F_{\mathbb{A}}))\backslash T(F_{\mathbb{A}})} \phi(t)\mathrm{e}^{\langle\lambda+\rho,H(t)\rangle}\,\mathrm{d}t.$$

Using again the notation of generalized vectors, what we have to show is that

$$\sum\int \big\langle I_{\lambda,\pi}(f)\nu_{\lambda,\pi},\nu_{\lambda,\pi}\big\rangle\mathrm{d}\lambda$$

is absolutely convergent. This can be seen directly when f is K-finite. For our purposes this is not quite enough since we want to use smooth functions of compact support. We use again an integral representation of the form (42):

$$(48)\quad I_{\lambda,\Pi}(f)\phi(g) = \frac{\pi_1(g)\,|\det g|^{\lambda+\frac12}}{L^{S_0}(2\lambda+1,\pi_1\pi_2^{-1})} \times \int_{E_{\mathbb{A}}^\times} \Phi\big[(0,t)g;\lambda,\pi_1\big]\cdot|t|^{2\lambda+1}\pi_1\pi_2^{-1}(t)\,\mathrm{d}^\times t.$$

Let us introduce the Schwartz-Bruhat function

$$\phi[t;\lambda,\pi_1] := \Phi\big[(0,1)t;\lambda,\pi_1\big]$$

on the simple algebra of which T is the multiplicative group. Then

$$\nu(I_{\lambda,\pi}(f)\phi) = \frac{1}{L^{S_0}(2\lambda+1,\pi_1\pi_2^{-1})}\int \phi(t)\pi_1(\det t)\,|\det t|_F\,\mathrm{d}t.$$

The integral is then a Tate integral for E. In particular, it is a holomorphic multiple of $L(\lambda+\frac12,\pi_1\circ\mathrm{Norm})$. We can then finish the proof as before.

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