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THE THEORY OF DIFFERENTIAL INVARIANTS AND KDV HAMILTONIAN EVOLUTIONS

BY GLORIA MARÍ BEFFA (*)

ABSTRACT. — In this paper I prove that the second KdV Hamiltonian evolution associated to $\mathrm{SL}(n,\mathbb{R})$ can be view as the most general evolution of projective curves, invariant under the $\mathrm{SL}(n,\mathbb{R})$-projective action on $\mathbb{RP}^{n-1}$, provided that certain integrability conditions are satisfied. This way, I establish a very close relationship between the theory of geometrical invariance, and KdV Hamiltonian evolutions. This relationship was conjectured in [4].

RÉSUMÉ. — LA THÉORIE DES INVARIANTS DIFFÉRENTIELS ET LES ÉVOLUTIONS HAMILTONIENNES DE KDV. — Dans cet article, je prouve que la seconde évolution hamiltonienne de KdV, associée au groupe $\mathrm{SL}(n,\mathbb{R})$, peut être considérée comme l'évolution la plus générale des courbes projectives qui sont invariantes par l'action projective de $\mathrm{SL}(n,\mathbb{R})$ sur $\mathbb{RP}^{n-1}$, si une certaine condition d'integrabilité est satisfaite. Je mets alors en évidence une connection très étroite entre la théorie d'invariance géométrique et les évolutions hamiltoniennes de KdV. Cette relation a été conjecturée en [4].

1. Introduction

Consider the following problem: Let $\phi(t,\theta) \in \mathbb{RP}^{n-1}$ be a family of projective curves. We ask the following question: is there a formula describing the most general evolution for ϕ of the form

$$\phi_t = F(\phi, \phi', \phi'', \dots)$$

invariant under the projective action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{RP}^{n-1}$? Here

$$\phi' = \phi_\theta = \frac{\mathrm{d}\phi}{\mathrm{d}\theta}, \quad \phi_t = \frac{\mathrm{d}\phi}{\mathrm{d}t}\cdot$$

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The projective action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{RP}^{n-1}$ is the one induced on $\mathbb{RP}^{n-1}$ by the usual action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ via the lift

$$\mathbb{RP}^{n-1} \longrightarrow \mathbb{R}^n, \quad \phi \longmapsto (1,\phi).$$

As we showed in [4], such a formula can be found using the theory of projective differential invariance. In fact, one can prove that any evolution of projectives curves which is invariant under $\mathrm{SL}(n,\mathbb{R})$ can always be written as

$$\phi_t = \mu \mathcal{I} \tag{1.1}$$

where $\mathcal{I}$ is a *vector of differential invariants* for the action and μ is a particular (fixed) matrix of *relative invariants*, whose explicit formula was found in [4]. Roughly speaking, if a group G acts on a manifold M, one can define an action of the group on a given jet bundle $J^{(k)}$ of order k, where $J^{(k)}$ is the set of equivalence classes of submanifolds modulo k-order contact. This action, in coordinates looks like

$$\begin{aligned} G \times J^{(k)} &\longrightarrow J^{(k)}, \\ (g, u_K) &\longmapsto (gu)_K, \end{aligned}$$

for any differential subindex K of order less or equal to k, and it is called *the prolonged action.* A differential invariant is a map

$$I\colon J^{(k)} \longrightarrow \mathbb{R}$$

which is invariant under the prolonged action. A relative differential invariant is a map

$$J\colon J^{(k)} \longrightarrow \mathbb{R}$$

whose value gets multiplied by a factor under the prolonged action. The factor is usually called the multiplier. In our particular case, their infinitesimal definitions are given in the second part of section 2. Differential invariants and relative invariants are the tools one uses to describe invariant evolutions.

These two concepts belong to the theory of Klein geometries and geometric invariants which had its high point last century before the appearance of Cartan's approach to differential geometry. It is also closely related to equivalence problems. Namely, one poses the question of equivalence of two geometrical objects under the action of a certain group, that is, when can one of those objects be taken to the other one using a transformation belonging to the given group? For example, given two curves on the plane, when are they equivalent under an Euclidean motion? or, when are they the same curve, up to parametrization? *etc.* One answer can be given in terms of *invariants*, that is, expressions depending on the objects under study and that do not change under the action of the group. If two objects are to be equivalent, they must have the same invariants. If these invariants are functions on some jet space (for example, if they depend on the curve and its derivatives with respect to the parameter), then they are called differential invariants. In the case of curves on the Euclidean plane under the action of the Euclidean group, the basic differential invariant is known to be the Euclidean curvature, and any other differential invariant will be a function of the curvature and its derivatives. In the case of immersions

$$\phi\colon \mathbb{R} \longrightarrow \mathbb{RP}^1,$$

with $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{RP}^1$, the basic differential invariant is classically known to be the *Schwarzian derivative* of ϕ,

$$S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\Big(\frac{\phi''}{\phi'}\Big)^2.$$

Within the natural scope of the study of equivalence problems and their invariants lies also the description of invariant differential equations, symmetries, relative invariants, *etc.* For example, recently Olver *et al.* [12] used these ideas to characterize all scalar evolution equations invariant under the action of a subgroup of the projective group in the plane, a problem of interest in the theory of image processing. See Olver's book [11] for an account of the state of the subject.

A subject apparently unrelated to the Theory of differential invariance is the subject of Hamiltonian structures of partial differential equations, integrability and, in general, of infinite dimensional Poisson structures. The so-called KdV Poisson brackets lie within this area. These brackets were defined by Adler [1] in an attempt to generalize the bi-Hamiltonian character of the Korteweg-deVries (KdV) equation and its integrability. He defined a family of second Hamiltonian structures with respect to which the generalized higher-dimensional KdV equations could also be written as Hamiltonian systems. Jacobi's identity for these brackets was proved by Gel'fand and Dikii in [3]. These Poisson structures are called *second Hamiltonian KdV structures* or *Adler-Gel'fand-Dikii brackets*, and they are defined on the manifold of smooth Lax operators. Since the original definition of Adler was quite complicated and not very intuitive, alternative definitions have been subsequently offered by several authors, most notably by Kupershmidt and Wilson in [7], and by Drinfel'd and Sokolov in [2]. Once the second Hamiltonian structure was found, the integrability of generalized KdV equations was established via the usual construction of a sequence of Hamiltonian structures with commuting Hamiltonian operators. In this paper I will restrict to the case of the $\mathrm{SL}(n,\mathbb{R})$ Adler-Gel'fand-Dikii bracket, although brackets have been given for other groups (Drinfel'd and Sokolov described their definition for any semisimple Lie algebra). The second Hamiltonian Structure in this hierarchy of KdV brackets coincides with the usual second Poisson bracket for the KdV equation, that is, the canonical Lie-Poisson bracket on the dual of the Virasoro algebra. This is the only instance in which the second KdV bracket is linear.

The relationship between Lax operators (scalar n-th order ODE's) and projective curves was established by the classics and clearly described by Wilczynski in [13]. More recently (see [12]) the topology of these curves was used to identify one of the invariants of the symplectic leaves of the Adler–Gel'fand–Dikii Poisson foliation. Some comments with respect to the role of projective curves in these brackets can be found in [14] and [7]. In [4] it was conjectured that the second KdV Hamiltonian evolution and the general evolution for projective curves (1.1) found in [4] were, essentially, the same evolution under a 1-to-1 (up to $\mathrm{SL}(n,\mathbb{R})$ action) correspondence between Lax operators and projective curves. The only condition that needed to be imposed was that certain invariant combination of the components of the invariant vector $\mathcal{I}$ in (1.1) should be integrable to define the gradient of certain Hamiltonian operator (one can even describe the evolution so that both I and Hamiltonian coincide after the identification).

In this paper I prove this conjecture. Namely, I prove that there exists an invariant matrix $\mathcal{M}$, invertible, such that, if H is the *pseudo-differential operator associated to an operator* $\mathcal{H}$, and if

$$H = \mathcal{M}\mathcal{I}$$

then, whenever ϕ evolves following (1.1) with general invariant vector $\mathcal{I}$, then their associated Lax operators (associated in the sense of [4] and described again in the next chapter) will evolve following an AGD-evolution with Hamiltonian operator $\mathcal{H}$. I also prove the conjectured shape of $\mathcal{M}$, namely, lower triangular along the transverse diagonal with ones down the transverse diagonal and zeroes on the diagonal inmediately below the transverse one. The proof is based on a manipulation of Wilson's antiplectic pair for the $\mathrm{GL}(n,\mathbb{R})$-AGD bracket and on the comparison of the resulting formulas with the invariant formulas (1.1).

In chapter 2, I will briefly describe the 1–1 (up to $\mathrm{SL}(n,\mathbb{R})$ action) correspondence between the $\mathrm{SL}(n,\mathbb{R})$ AGD manifold and the manifold of projective curves, and I will also describe the evolutions we wish to relate. In chapter 3, I will describe Wilson's theory of antiplectic pairs and, in particular, the antiplectic pair for the $\mathrm{GL}(n,\mathbb{R})$-AGD bracket. In chapter 4, I will adapt Wilson's formulas and finally give the proof of the Main Theorem. I will also prove that $\mathcal{M}$ has the shape conjectured in [4]. For more information or further references about either the evolutions under study or the correspondence between Lax operators and projective curves, please see [4].

2. Description of the manifolds and their evolutions

Before describing evolutions, I will state the known parallelism between the manifold of Lax operators and the manifold of projective curves.

Let $\mathcal{A}_n$ be the infinite-dimensional Fréchet manifold of scalar differential operators (or Lax operators) with T-periodic, smooth coefficients of the form

$$L = \frac{\mathrm{d}^n}{\mathrm{d}\theta^n} + u_{n-2}\frac{\mathrm{d}^{n-2}}{\mathrm{d}\theta^{n-2}} + \cdots + u_1\frac{\mathrm{d}}{\mathrm{d}\theta} + u_0. \tag{2.1}$$

The manifold $\mathcal{A}_n$ is called the $\mathrm{SL}(n,\mathbb{R})$ Adler-Gel'fand-Dikii manifold, or manifold of $\mathrm{SL}(n,\mathbb{R})$-Lax operators. (Notice that for much of the study below the periodicity condition can be omitted.) The case when u_{n-1} does *not* vanish is refered to as the $\mathrm{GL}(n,\mathbb{R})$ AGD manifold or manifold of $\mathrm{GL}(n,\mathbb{R})$-Lax operators.

Let $\xi_L = (\xi_1,\ldots,\xi_n)$ be a solution curve associated to L, that is

$$L\xi_k = 0, \quad k = 1,\ldots,n,$$

the Wronskian of whose components equals one. Notice that, since the Wronskian of any solution curve is constant ($u_{n-1} = 0$), up to multiplication by a matrix in $\mathrm{SL}(n,\mathbb{R})$, ξ_L will be uniquely determined by its Wronskian being equal to 1. Now, due to the periodicity of the coefficients of L, there exists a matrix $M_L \in \mathrm{SL}(n,\mathbb{R})$, called the *monodromy* of L, such that

$$\xi_L(\theta+T) = M_L\xi_L(\theta), \qquad \text{for all } \theta\in\mathbb{R}.$$

(M_L is conjugate to the transposed of the Floquet matrix.) This same property is shared by the projective coordinates of its projection on $\mathbb{RP}^{n-1}$, as far as we consider the action of $\mathrm{SL}(n,\mathbb{R})$ on the projective space. Observe that the monodromy is not completely determined by the operator L, but by its solution curves. Namely, if one chooses a different solution curve, its monodromy won't be equal to M_L in general, but it will be the conjugate of M_L by an element of $\mathrm{GL}(n,\mathbb{R})$. If we additionally ask the solution curve to have Wronskian equals 1, then M_L is determined by the equation up to conjugation by an element of $\mathrm{SL}(n,\mathbb{R})$. Thus, to each Lax operator we can associate a projective curve (the projectivisation of certain solution curve) whose monodromy is an element of $\mathrm{SL}(n,\mathbb{R})$. This curve is unique up to the projective action of $\mathrm{SL}(n,\mathbb{R})$.

Conversely, let $\mathcal{C}_n$ be the space of curves

$$\phi:\mathbb{R} \longrightarrow \mathbb{RP}^{n-1}$$

with the following nondegeneracy condition: the determinant

$$\begin{vmatrix} \phi_1^{(n-1)} & \phi_2^{(n-1)} & \dots & \phi_{n-1}^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1'' & \phi_2'' & \dots & \phi_{n-1}'' \\ \phi_1' & \phi_2' & \dots & \phi_{n-1}' \end{vmatrix}$$

must be positive. (This is equivalent to the Wronskian of the components of $(1,\phi)$ being positive; for example, the curve would be convex and right-hand oriented in the case $n = 3$.) Assume also that the elements of $\mathcal{C}_n$ satisfy a *monodromy property*:

$$\phi(\theta + T) = (M \cdot \phi)(\theta), \qquad \text{for all } \theta \in \mathbb{R}, \tag{2.2}$$

for some $M \in \mathrm{SL}(n,\mathbb{R})$. Here $M \cdot \phi$ represents the projective action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{RP}^{n-1}$.

To each ϕ one can associate a differential operator of the form (2.1) in the following manner: We lift ϕ to a unique curve on $\mathbb{R}^n$ so that the Wronskian of the components of the lifted curve equals 1. There is a unique choice $f(\theta)(1,\phi)$, namely when

$$f = W(1,\phi_1,\dots,\phi_{n-1})^{-\frac{1}{n}} = W(\phi_1',\dots,\phi_{n-1}')^{-\frac{1}{n}} = W_\phi^{-\frac{1}{n}},$$

where $\phi = (\phi_1,\dots,\phi_{n-1})$ and W represents the Wronskian. It is not very hard to see that the coordinate functions of the lifted curve are solutions of a unique differential equation. The equation for the unknown y is of the form

$$(2.3)\qquad \begin{vmatrix} y^{(n)} & (f\phi_0)^{(n)} & \dots & (f\phi_{n-1})^{(n)} \\ y^{(n-1)} & (f\phi_0)^{(n-1)} & \dots & (f\phi_{n-1})^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ y & f\phi_0 & \dots & f\phi_{n-1} \end{vmatrix}$$

$$= y^{(n)} + u_{n-2}y^{(n-2)} + \cdots + u_1 y' + u_0 y = 0,$$

where $\phi_0 = 1$. One can easily see (*cf.* [4]) that the monodromy property (2.2) results on the coefficients u_i, $i = 0, 1, \dots, n-2$ being T-periodic, so that this equation corresponds to a Lax operator of the form (2.1).

It is known (see [13]) that the coefficients of this Lax operator form the so-called *generating set of differential invariants* for projective curves under the projective action of $\mathrm{SL}(n, \mathbb{R})$. Namely, if a function I, depending on ϕ and its derivatives, is invariant under the *prolonged* projective action of $\mathrm{SL}(n, \mathbb{R})$, then, necessarily I is a function of the coefficients u's and their derivatives. The u's can thus be called the "projective curvatures" of the curve ϕ. I will go back to this point in the description of the evolution of curves. For more information see [13] and [4].

Example. — In the case of immersions $\phi\colon \mathbb{R} \to \mathbb{RP}^1$, with $\mathrm{SL}(2, \mathbb{R})$ acting on $\mathbb{RP}^1$, the Wronskian W_ϕ equals ϕ'. Thus, the lift ξ is given by $(\xi_1, \xi_2) = (\phi'^{-\frac{1}{2}}, \phi'^{-\frac{1}{2}}\phi)$. If we write down the associated Lax operator for this particular example, we obtain

$$\begin{vmatrix} y'' & (\phi'^{-\frac{1}{2}})'' & (\phi'^{-\frac{1}{2}}\phi)'' \\ y' & (\phi'^{-\frac{1}{2}})' & (\phi'^{-\frac{1}{2}}\phi)' \\ y & \phi'^{-\frac{1}{2}} & \phi'^{-\frac{1}{2}}\phi \end{vmatrix} = y'' + \frac{1}{2}S(\phi)y,$$

where, again, $S(\phi) = \phi'''/\phi' - \frac{3}{2}(\phi''/\phi')^2$ is the classical Schwarzian derivative of ϕ, the basic differential invariant or projective curvature.

The above description establishes a 1–1 (up to $\mathrm{SL}(n, \mathbb{R})$) correspondence between $\mathcal{A}_n$ and $\mathcal{C}_n$. Next I will describe two different evolutions, one in each manifold.

The Adler-Gel'fand-Dikii bracket, or the evolution on $\mathcal{A}_n$.

Given a linear functional $\mathcal{H}$ on $\mathcal{A}_n$, one can associate to it a pseudo-differential operator symbol of the form

$$(2.4)\qquad H = \sum_{i=1}^{n} h_i \partial^{-i}, \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}\theta},$$

such that

$$\mathcal{H}(L) = \int_{S^1} \operatorname{res}(HL)\,\mathrm{d}\theta,$$

where 'res' selects the coefficient of ∂^{-1} and is called the *residue* of the pseudo-differential operator (see [1] or [3]). The coefficients h_i, $i = 1, \dots, n-1$ are combinations of the coefficients of the gradient $\partial\mathcal{H}/\partial u$ and their derivatives; the coefficient h_n will be determined below.

To any $\mathcal{H}$ we can associate a (Hamiltonian) vector field V_H defined as

$$V_H(L) = (LH)_+L - L(HL)_+,$$

where by $(\ )_+$ we denote the non-negative (or differential) part of the operator. The vector field V_H defines a bracket, namely

$$\{\mathcal{H}, \mathcal{F}\}(L) = \int_{S^1} \operatorname{res}\big(V_H(L)F\big)\,\mathrm{d}\theta, \tag{2.5}$$

which turns out to be a Poisson bracket with associated Hamiltonian evolutions given by

$$L_t = V_H(L) \tag{2.6}$$

cf. [1], [3] or [9]. The coefficient h_n of the operator H is fixed, and its value easily found, so as to make the vector field V_H tangent to the manifold $\mathcal{A}_n$ (that is, so that both sides of (2.6) have the $n-1$ term equal zero). These are called the $\mathrm{SL}(n,\mathbb{R})$ Adler-Gel'fand-Dikii evolutions. As I explained in the introduction, the original definition of the bracket was given by Adler [1] in an attempt to make generalized KdV equations bi-Hamiltonian systems. Gel'fand and Dikii proved Jacobi's identity in [3]. In the case $n = 2$, this bracket coincides with the Lie-Poisson structure on the dual of the Virasoro algebra. Two other equivalent definitions of the original bracket were found in [6] and in [2]. I will describe briefly the definition in [6] in the chapter where I prove the Main Theorem.

Invariant evolutions of projective curves, or the evolution on $\mathcal{C}_n$.

On $\mathcal{C}_n$ we are interested in evolutions of the form

$$\phi_t = F(\phi, \phi', \phi'', \dots), \quad \phi\colon \mathbb{R}^2 \longrightarrow \mathbb{RP}^{n-1}, \tag{2.7}$$

such that whenever $\phi(\theta,t)$ is a solution of (2.7) so is $(M\cdot\phi)(\theta,t)$, for all $M \in \mathrm{SL}(n,\mathbb{R})$. That is, evolutions of curves on $\mathbb{RP}^{n-1}$, invariant under the projective action of $\mathrm{SL}(n,\mathbb{R})$.

One can see that, if the initial condition has the monodromy property, the entire flow does and in fact the monodromy is preserved, as far as the solution is unique. Indeed, (2.7) is invariant under translations of the independent variable θ. Hence, if the initial condition $\phi(\,\cdot\,,0)$ of (2.7) has a matrix $M \in \mathrm{SL}(n,\mathbb{R})$ as monodromy, and we consider a different curve in the flow $\phi(\,\cdot\,,t)$, we have that $\phi(\theta - T, t)$ is also a solution. If (2.7) is $\mathrm{SL}(n,\mathbb{R})$-invariant, $M\cdot\phi(\theta-T,t)$ will also be a solution of (2.7). Applying uniqueness of solutions of (2.7) (whenever possible),

$$M \cdot \phi(\theta - T, t) = \phi(\theta, t),$$

so that $\phi(\,\cdot\,,t)$ has the same monodromy as $\phi(\,\cdot\,,0)$. If there is no uniqueness of solutions, both Hamiltonian and invariant evolutions are obviously much more complicated; I won't deal with those cases in this paper.

In [4] we found the explicit formula for the most general form of evolution (2.7). It could be described as follows: First of all, the infinitesimal generators of the projective $\mathrm{SL}(n,\mathbb{R})$ action on $\mathbb{RP}^{n-1}$ are easily found to be the following vector fields on $\mathbb{R}\times\mathbb{R}\times\mathbb{RP}^{n-1}$:

$$(2.8)\quad \boldsymbol{v}_i = \frac{\partial}{\partial\phi_i}, \quad \boldsymbol{v}_{ij} = \phi_i\frac{\partial}{\partial\phi_j}, \quad \boldsymbol{w}_i = \phi_i\sum_{j=1}^{n}\phi_j\frac{\partial}{\partial\phi_j}; \quad 1\le i,j\le n-1.$$

If $\boldsymbol{v} = \sum_{i=1}^{n-1}\eta_i(\theta,t,\phi)\frac{\partial}{\partial\phi_i}$ is a vertical vector field, its *prolongation* $\mathrm{pr}\,\boldsymbol{v}$ (see [9]) is the vector field defined on the jet space by

$$(2.9)\qquad \mathrm{pr}\,\boldsymbol{v} = \boldsymbol{v} + \sum_{\substack{j\ge1\\k\ge0}}\sum_{i=1}^{n-1}(D_t^k\,D^j\eta_i)\frac{\partial}{\partial(\partial_t^k\phi_i^{(j)})},$$

where $\phi_i^{(j)} = \partial^j\phi_i$, D is the total derivative operator with respect to θ

$$D = \partial + \sum_{j\ge0}\sum_{i=1}^{n-1}\phi_i^{(j+1)}\frac{\partial}{\partial\phi_i^{(j)}},$$

and

$$(2.10)\qquad D_t = \partial_t + \sum_{j\ge0}\sum_{i=1}^{n-1}(\partial_t\phi_i^{(j)})\frac{\partial}{\partial\phi_i^{(j)}}.$$

The prolongation of an infinitesimal generator is indeed the infinitesimal generator of the prolonged action on the jet space. In our case, it reduces to the vector field

$$\mathrm{pr}\, \boldsymbol{v} = \boldsymbol{v} + \sum_{j\geq 1}\sum_{i=1}^{n-1}(D^j\eta_i)\frac{\partial}{\partial\phi_i^{(j)}}, \tag{2.11}$$

defined on the infinite-dimensional jet space

$$J^\infty \equiv J^\infty(\mathbb{R}, \mathbb{RP}^{n-1})$$

with local coordinates θ, $\phi_i^{(j)}$ where $1 \leq i \leq n-1$ and $j \geq 0$ (to be correct, we should in fact restrict ourselves to the jet space $J^{(k)}$, for some order k as large as necessary; but for simplicity I will work on $J^{(\infty)}$).

DEFINITION 2.1.
a) We will say that I is a *differential invariant* for the $\mathrm{SL}(n, \mathbb{R})$-projective action if

$$\mathrm{pr}\, \boldsymbol{v}(I) = 0 \quad \text{for all } \boldsymbol{v} \in \mathrm{sl}(n, \mathbb{R}).$$

This is indeed the infinitesimal version of the definition we gave in the introduction: I is invariant under the prolonged action.

b) (Infinitesimal definition also) Let

$$\boldsymbol{v} = \sum_{i=1}^{n-1} \eta_i(\theta,t,\phi)\frac{\partial}{\partial\phi_i} \in \mathrm{sl}(n, \mathbb{R})$$

be an infinitesimal generator. We will say that F is a *relative vector differential invariant* of the Lie algebra $\mathrm{sl}(n, \mathbb{R})$ given by (2.8), whose associated *weight* is the matrix $\partial\eta/\partial\phi$ whenever

$$\mathrm{pr}\, \boldsymbol{v}(F) = \frac{\partial\eta}{\partial\phi}F, \tag{2.12}$$

for all $\boldsymbol{v} \in \mathrm{sl}(n, \mathbb{R})$, where $\partial\eta/\partial\phi$ is the $(n-1)\times(n-1)$ matrix with (i,j) entry $\partial\eta_i/\partial\phi_j$.

It is not hard to see (*cf.* [4]) that (2.7) is invariant if, and only if F is a relative vector differential invariant of the Lie algebra $\mathrm{sl}(n, \mathbb{R})$ with weight $\partial\eta/\partial\phi$.

THEOREM 2.2 (see [4]). — *The most general solution F of* (2.12) *is of the form*

$$F = \mu \mathcal{I},$$

where the $(n-1)\times(n-1)$ *matrix*

$$\mu = (\mu^1 \ \mu^2 \ \dots \ \mu^{n-1})$$

is any matrix with non-vanishing determinant and whose columns μ^i *are particular solutions of* (2.12), *and where*

$$\mathcal{I} = (I_k)_{k=1}^{n-1}$$

is an arbitrary absolute (vector) differential invariant *of the algebra* (2.8), *i.e. a solution of*

$$\operatorname{pr} \boldsymbol{v}(I_i) = 0, \quad \forall \boldsymbol{v} \in \mathfrak{sl}(n),\ i = 1,\dots,n-1.$$

The problem is thus splitted into two parts: first find the invariants, second find the matrix μ.

The first part was already solved by Wilczynski in [13]. He proved that, in the case of projective curves, a set of generating basic differential invariants are the coefficients of the differential operator associated to ϕ as in (2.3). That is, any differential invariant I_j will have to be a function of the coefficients u_i, $i = 0,\dots,n-2$ and their derivatives with respect to θ.

In [4] we solved the second part, finding an explicit expression for a regular matrix of relative differential invariants, μ, which I describe below.

DEFINITION 2.3.

a) For $i_1,\dots,i_k \geq 0$ and $1 \leq k \leq n-1$, let us denote

$$w_{i_1 i_2 \dots i_k} = \begin{vmatrix} \phi_1^{(i_1)} & \phi_2^{(i_1)} & \dots & \phi_k^{(i_1)} \\ \phi_1^{(i_2)} & \phi_2^{(i_2)} & \dots & \phi_k^{(i_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(i_k)} & \phi_2^{(i_k)} & \dots & \phi_k^{(i_k)} \end{vmatrix}$$

and

$$w^k = w_{12\dots k}.$$

b) We define the homogeneous variables $q_{i_1 i_2 \dots i_k}$ by

$$q_{i_1 i_2 \dots i_k} = \frac{w_{i_1 i_2 \dots i_k}}{w^k}.$$

c) For $k = 1,2,\dots,n$ the variables q_n^k are defined as follows:

$$q_n^k = q_{12\dots\widehat{k}\dots n},$$

where the notation $\widehat{k}$ means that the index k is to be omitted.

THEOREM 2.4. — *An invertible matrix μ of relative invariants with weight $\partial\eta/\partial\phi$ is given by a matrix of the form*

$$\mathcal{W}_\phi^T\,(\mathrm{Id}+A)$$

where $\mathcal{W}_\phi^T$ is the transposed of the Wronskian matrix of ϕ defined as

$$\mathcal{W}_\phi = \begin{pmatrix} \phi_1' & \phi_2' & \dots & \phi_{n-1}' \\ \phi_1'' & \phi_2'' & \dots & \phi_{n-1}'' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \dots & \phi_{n-1}^{(n-1)} \end{pmatrix}, \tag{2.13}$$

and where $A = (a_i^j)$ is defined by

$$a_i^j = \begin{cases} \dfrac{(-1)^{j-i}\binom{j}{i}}{\binom{n}{j-i}}\, q_n^{n-j+i} & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases} \tag{2.14}$$

As an immediate consequence one obtains:

COROLLARY 2.5. — *The most general equation for the evolution of curves on $\mathbb{RP}^{n-1}$ which is invariant under the projective action of $\mathrm{SL}(n,\mathbb{R})$ is given by*

$$\phi_t = \mathcal{W}_\phi^T\,(\mathrm{Id}+A)\,\mathcal{I}, \tag{2.15}$$

where $\mathcal{W}_\phi$ and A are given by (2.13) and (2.14), and where $\mathcal{I}$ is any vector differential invariant for the action.

Before finishing the section I will like to include an immediate corollary to Theorem 2.2 which will be of use later on.

COROLLARY 2.6. — *Let μ and $\widehat{\mu}$ be two nondegenerate matrices whose columns are relative vector differential invariants, with associated weights given as in* (2.12). *Then, there exists an invariant and nondegenerate matrix $\mathcal{M}$ such that*

$$\mu = \widehat{\mu}\mathcal{M}.$$

After defining the two parallel evolutions, and describing some of the theory of differential invariance, I would like to show that these are essentially the same evolution. This implies that certain relationship will have to be found between $\mathcal{I}$ and $\mathcal{H}$. Notice that both of them are functions of the coefficients u_i, $i = 0, \ldots, n-2$, and their derivatives. In fact, it was conjecture in [4] that both equations (2.6) and (2.15) are equivalent whenever

$$\frac{\delta \mathcal{H}}{\delta u} = T\mathcal{M}\mathcal{I}, \tag{2.16}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \partial & 1 & 0 & 0 & 0 & \ldots & 0 \\ \partial^2 & \binom{2}{1}\partial & 1 & 0 & 0 & \ldots & 0 \\ \partial^3 & \binom{3}{2}\partial^2 & \binom{3}{1}\partial & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \partial^{n-3} & \ldots & \binom{n-3}{3}\partial^3 & \binom{n-3}{2}\partial^2 & \binom{n-3}{1}\partial & 1 & 0 \\ \partial^{n-2} & \binom{n-2}{n-3}\partial^{n-3} & \ldots & \binom{n-2}{3}\partial^3 & \binom{n-2}{2}\partial^2 & \binom{n-2}{1}\partial & 1 \end{pmatrix} \tag{2.17}$$

and $\mathcal{M}$ is a certain lower triangular matrix of the form

$$\mathcal{M} = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & \ldots & 0 & 1 & 0 & m_1^1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & m_{n-4}^{n-4} & \ldots & m_1^{n-4} \\ 1 & 0 & m_{n-3}^{n-3} & m_{n-3}^{n-4} & \ldots & m_1^{n-3} \end{pmatrix}, \tag{2.18}$$

whose matrix elements m_i^j are all functions of the coefficients u_i and their derivatives. On the other hand, if

$$H = \sum_{k=1}^{n} h_k \partial^{-k}$$

is the pseudo-differential operator associated to $\mathcal{H}$, $(h_1, \dots, h_{n-1})$ is easily seen to be related to the gradient of $\mathcal{H}$ through the matrix T, exactly the same way $\mathcal{M}\mathcal{I}$ is conjectured to be in (2.16), namely,

$$\frac{\delta\mathcal{H}}{\delta u} = T\begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}.$$

That is, the evolutions will be equivalent provided that

$$\mathcal{M}\mathcal{I} = \begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix},$$

that is, provided that *certain linear combination of $\mathcal{I}$ with differential invariant coefficients coincides with the coefficients $(h_1, \dots, h_{n-1})$ of the pseudo-differential operator H defining the evolution of u.* The necessary and sufficient condition for this to be true is, of course, for $T\mathcal{M}I$ to have self-adjoint Fréchet derivative with respect to u. Notice the slight change in the shape of T and $\mathcal{M}$ as opposed to the ones conjectured in [4]. The change is due to the fact that $\mathcal{I}$ was compared in [4] to $(h_{n-1}, \dots, h_1)^T$ and here I have turned it into the more natural order $(h_1, \dots, h_{n-1})^T$.

Before I specify all the details of the proof in Section 4, I need to give a brief description of what is called the theory of antiplectic pairs, which was introduced by Wilson in [14], [15] and [16], and, in particular, of the antiplectic pair for the $\mathrm{GL}(n, \mathbb{R})$-AGD bracket. In the next section I will keep much of the notation as in [14], but I will try to avoid any confusion with the notation I have used up to this point.

3. Antiplectic pairs

Let (A, ∂) be the differential field of differential rational functions on independent variables $\xi_0, \xi_1, \dots, \xi_{n-1}$. That is, A is the field of rational functions on the infinite-dimensional jet space $J^\infty(\mathbb{R}, \mathbb{R}^n)$ with coordinates ξ_0, ξ_1, …, ξ_{n-1}, and ∂ is the unique derivation such that $\partial\xi_i^{(j)} = \xi_i^{(j+1)}$. I will assume familiarity with concepts such as Fréchet derivatives of differential operators on A, tensors, *etc.*, such as we did in [4]. For more information see [14] or [9].

Consider the action of $\mathrm{GL}(n, \mathbb{R})$ on A induced the usual way by the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. That is, if $M\xi$ indicates the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ given by matrix multiplication, $\xi \in \mathbb{R}^n$, then $M\xi^{(k)} = (M\xi)^{(k)}$, what we have called the prolonged action. Let (B, ∂) be the differential field of invariants of the action. As it happened in the $\mathrm{SL}(n,\mathbb{R})$ case, B is well known to be generated by the coefficients of the $\mathrm{GL}(n,\mathbb{R})$-Lax operator (2.1), that is, the case where $u_{n-1} \neq 0$; B is a differential field on the independent variables $u_0, u_1, \dots, u_{n-1}$, related to ξ analogously as in (2.3), with associated derivation ∂.

Let Ω_A and Ω_B be the modules of differentials. That is, they are free $A[\partial]$-module with basis $\{d\xi_i\}$, and free $B[\partial]$-module with basis $\{du_i\}$, respectively. Let $\widehat{\Omega}_B$ be the module Ω_B with scalars extended to $A[\partial]$. If we impose that the action of $\mathrm{GL}(n,\mathbb{R})$ conmutes with d, the $\mathrm{GL}(n,\mathbb{R})$ prolonged action can be extended to Ω_A the obvious way. Consider also Ω^* to be the dual R-module to Ω, that is, $\Omega^* = \mathrm{Hom}(\Omega, R)$ where $R = A[\partial]$ is the algebra of differential operators with coefficients in A. Analogously define Ω_A^*, Ω_B^* and $\widehat{\Omega}_B^*$.

Next, let us describe the general Hamiltonian formalism for evolution equations.

Definition 3.1. — A 2-form $\lambda \in \Omega_A \otimes \Omega_A$, $\lambda = \sum_i \nu_i \otimes \mu_i$, is called *special* if:

1) λ is $\mathrm{GL}(n,\mathbb{R})$-invariant;

2) the homomorphism $\lambda^{\#}\colon \Omega_A^* \to \Omega_A$ defined as

$$\lambda^{\#}(\kappa) = \sum_i \kappa(\nu_i)\mu_i$$

is injective and its image is $\widehat{\Omega}_B$.

If λ is special, we obtain a factorization of $\lambda^{\#}$ through the inclusion $i\colon \widehat{\Omega}_B \to \Omega_A$, namely,

$$\lambda^{\#}\colon \Omega_A^* \xrightarrow{\alpha} \widehat{\Omega}_B \xrightarrow{i} \Omega_A \tag{3.1}$$

where α is invertible (an isomorphism). By the way, notice that, with respect to the basis $\{d\xi_i\}$ and $\{du_i\}$ for Ω_A and Ω_B and their dual basis $\{d\xi_i{}^*\}$ and $\{du_i{}^*\}$ for Ω_A^* and Ω_B^* respectively, the matrix associated to the inclusion map has as (i,j) entry $Du_i/D\xi_j$, the Fréchet derivative of u_i with respect to ξ_j. That is, if we denote by $\mathcal{D}$ the matrix of the inclusion map $i\colon \widehat{\Omega}_B \to \Omega_A$, then $\mathcal{D}$ is the Fréchet Jacobian of u with respect to ξ.

Now, consider the map

$$\hat{\lambda}\colon \widehat{\Omega}_B \xrightarrow{\alpha^{-1}} \Omega_A^* \xrightarrow{i^*} \widehat{\Omega}_B^*. \tag{3.2}$$

Since λ is $\mathrm{GL}(n,\mathbb{R})$-invariant, the map (3.2) must also be invariant. Therefore, it must come from a homomorphism

$$\ell^{\#}\colon \Omega_B \longrightarrow \Omega_B^*$$

by perhaps extensions of scalars. The map $\ell^{\#}$ corresponds to a skew tensor $\ell \in \Omega_B^* \otimes \Omega_B^*$.

Let $S = (s_{ij})$ be the matrix of α^{-1} in the basis $\{\mathrm{d}u_i\}$, $\{\mathrm{d}\xi_i\}$ and $\{\mathrm{d}\xi_i{}^*\}$ that were fixed before. In the definitions that follows, let $*$ denote the formal adjoint of an operator. Then the matrix of λ (or of $\lambda^{\#}$ if one prefers it), abusing the notation, is given by

$$\lambda = S^{-1}\mathcal{D} = -\mathcal{D}^*(S^*)^{-1} = (\lambda_{ij}).$$

Also, in the basis $\{\mathrm{d}u_i\}$, $\{\mathrm{d}u_i{}^*\}$ and $\{\mathrm{d}\xi_i{}^*\}$, the matrix of ℓ (or $\ell^{\#}$) is

$$\ell = S\mathcal{D}^* = -\mathcal{D}S^* = (\ell_{ij}).$$

In this situation, and under all these conditions, there exists a unique evolutionary derivation of B such that

$$\partial_H u_i = -\sum_j \ell_{ij}\frac{\delta H}{\delta u_j}. \tag{3.3}$$

Furthermore, there also exists a $\mathrm{GL}(n,\mathbb{R})$-equivariant derivation ∂_H of A given by

$$\partial_H \xi_i = \sum_j s_{ji}^* \frac{\delta H}{\delta u_j} \tag{3.4}$$

whose restriction to B is (3.3).

DEFINITION 3.2. — The pair (λ,ℓ) is called *antiplectic* whenever ℓ is Hamiltonian, that is, whenever the derivation given in (3.3) holds

$$\partial_{\{H,G\}} = [\partial_H,\partial_G]$$

if it makes sense, where $\{H,G\} = \partial_H G$.

It can be proved that if λ is a closed form, then ℓ is Hamiltonian.

In [14], Wilson constructs an antiplectic pair for the $\mathrm{GL}(n,\mathbb{R})$ KdV evolution, restricting himself to a particular symplectic leaf of the Poisson foliation. Even though at the beginning of the paper he imposes two extra conditions which are not included here, namely that the functions $\{\xi_i\}$ must be periodic, and the field $\mathbb{C}$ rather than $\mathbb{R}$, the construction above, and the one that follows, can be carried out without such assumptions. (In fact, he seems to drop the periodicity assumption until he places the antiplectic pair within the Drinfel'd and Sokolov formulation. In that case, for algebraic and invertibility reasons, the solutions of the Lax operators need to be periodic — he is restricting to one symplectic leaf of the Poisson manifold with identity monodromy.)

Construction of Wilson's GL$(n,\mathbb{R})$*-KdV antiplectic pair.*

Consider the factorization of L into first order factors

$$L = (\partial + y_{n-1}) \cdots (\partial + y_1)(\partial + y_0) \tag{3.5}$$

such that for each $i = 0, 1, \dots, n-1$ the set $\{\xi_0, \dots, \xi_i\}$ is a basis for the kernel of the operator $(\partial + y_i) \cdots (\partial + y_1)(\partial + y_0)$. Let ω_i be the Wronskian of $\{\xi_0, \dots, \xi_i\}$. Define

$$\rho_i = \frac{\omega_i}{\omega_{i-1}}$$

and fix $\omega_{-1} = 1$.

As before, let Ω_i be the free module of differentials with base $\mathrm{d}\rho_i$, and let $\widehat{\Omega}_i$ be Ω_i with an extension of scalars to the algebra $A[\partial]$, for any $i = 0, 1, \dots, n-1$. Define the 2-forms

$$\lambda^{(i)} = -{\rho_i}^{-1} \mathrm{d}\rho_i \otimes \partial {\rho_i}^{-1} \mathrm{d}\rho_i, \quad i = 0, 1, \dots, n-1,$$

and let $i_j \colon \widehat{\Omega}_j \to \Omega_A$ be the natural inclusions $j = 0, 1, \dots, n-1$.

THEOREM 3.3 (see [14]). — *The 2-form*

$$\lambda_n = i_0 \lambda^{(0)} + i_1 \lambda^{(1)} + \cdots + i_{n-1} \lambda^{(n-1)} \tag{3.6}$$

is special under the action of GL$(n,\mathbb{R})$ *and forms an antiplectic pair with the* GL$(n,\mathbb{R})$*-AGD Poisson tensor.*

For more information about this very inspiring construction, please see [14].

I am most interested in one of the explicit formulas for the matrix S^{-1} defining the map α in (3.1). More precisely, I am interested in formula (5.9) in [14], which can be described as follows: Let

$$L^* = (-\partial)^n + \sum_{i=1}^{n-1} \bar{u}_i \partial^i$$

be the dual operator to L. There exists a nondegenerate canonical pairing between the kernels of L and L^* given by the so-called *bilinear concomitant.* It is defined as

$$\langle \xi, \eta \rangle_L = \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j \xi^{(i-j-1)} (u_i \eta)^{(j)}$$

(see [5] for more information). Let $\{\eta_i\}$ be the dual to $\{\xi_i\}$ with respect to this pairing. Also, fix the basis $\{\mathrm{d}u_i\}$ in $\widehat{\Omega}_B$ and $\{\mathrm{d}{\xi_i}^*\}$ in Ω^*_A.

PROPOSITION 3.4 (see [14]). — *If we denote by $\bar{u}$ the coefficients of L^*, the matrix S^{-1} associated to λ_n as in* (3.6) *is given by*

$$S^{-1} = \mathcal{W}_\eta{}^* C^* \tag{3.7}$$

where $D\bar{u}/Du = C^*$, and where $\mathcal{W}_\eta$ is the Wronskian matrix of η, that is, $\mathcal{W}_\eta = (y_{ij})$ where $y_{ij} = \eta_j^{(i)}$ for $i,j = 0,1,\dots,n-1$.

We now have most of the machinery I need to attack our problem, so we finally go into out last section.

4. The equivalence of evolutions on $\mathcal{A}_n$ and $\mathcal{C}_n$

In this section, I will prove our Main Theorem, the equivalence of the two evolutions described in Section 2, and the description of $\mathcal{M}$. But let's first give a quick definition and let's set in place some preliminaries.

DEFINITION 4.1. — We will say that a vector funtion $H = (h_i)$ *comes from a gradient* whenever there exists an operator $\mathcal{H}$: $AGD \to \mathbb{R}$ such that its associated pseudodifferential operator, in the sense of (2.4) is

$$X = \sum_{i=1}^{n} h_i \partial^{-i}.$$

That is, a vector function will come from a gradient whenever certain combinations of its entries and their derivatives satisfy the corresponding integrability condition so that they can be the gradient of an operator defined on the AGD manifold.

The next thing to achieve is the rewriting of formula (3.4) in the $\mathrm{SL}(n,\mathbb{R})$ case, and to show that it induces an evolution on the projectivisation of ξ. This evolution will be invariant under the projective action of $\mathrm{SL}(n,\mathbb{R})$. The proof of the Theorem will finally come from the comparison of the formula we will obtain and (2.15).

First of all let us consider the $\mathrm{GL}(n,\mathbb{R})$-AGD evolutions (3.3) restricted to the submanifold $u_{n-1} = 0$, with the added condition $\delta\mathcal{H}/\delta u_{n-1} = 0$. I claim that, for Hamiltonians independent of u_{n-1}, the submanifold $u_{n-1} = 0$ is left invariant by the flow of evolution (3.3), and so, there exists an induced evolution on such submanifold. Furthermore, I claim that such an evolution is the $\mathrm{SL}(n,\mathbb{R})$-AGD evolution. This fact can be seen in many different ways, but the easiest might be to follow Kupershmidt and Wilson's definition of the AGD evolution given in [6]. Before stating the theorem that describes the bracket, I need some definitions.

Consider y defined as in (3.5) and let v be defined through the relation

$$y_k = v_0 + r^k v_1 + r^{2k} v_2 + \cdots + r^{(n-1)k} v_{n-1}, \tag{4.1}$$

$0 \leq k \leq n-1$, where $r = \mathrm{e}^{\frac{2\pi i}{n}}$. Kupershmidt and Wilson [6] called v_i the "modified" variables. In their paper, they proved the following theorem (which I have somehow rephrased).

THEOREM 4.2 (see [6]). — *Consider the following two evolutions for the modified variables v:*

$$\partial_t v_i = -\frac{1}{n} \partial x_{n-i}, \quad i = 1,2,\ldots,n-1 \tag{4.2}$$

$$\partial_t v_0 = -\frac{1}{n} \partial x_0, \quad \partial_t v_i = -\frac{1}{n} \partial x_{n-i}, \quad i = 1,2,\ldots,n-1 \tag{4.3}$$

where $(x_0,\ldots,x_{n-1})$ is the 1-form defined by the gradient of a Hamiltonian operator $\mathcal{H}$, and where x_0 corresponds exactly to $\delta\mathcal{H}/\delta v_0$. If u and v are related by (3.5) *and* (4.1), *then u will evolve following the* $\mathrm{SL}(n,\mathbb{R})$ *AGD evolution whenever v evolves following* (4.2), *and the* $\mathrm{GL}(n,\mathbb{R})$ *AGD evolution whenever v follows* (4.3). *The Hamiltonian will be the same, but evaluated either on u or v, depending on which evolution we consider.*

That is, in modified variables the $\mathrm{SL}(n,\mathbb{R})$ and the $\mathrm{GL}(n,\mathbb{R})$ AGD brackets are given by (4.2) and by (4.3), respectively. Clearly, the submanifold $v_0 = 0$ is invariant under the later flow (4.3), provided that $x_0 = 0$, and the evolution induced on this manifold is the former (4.2). Finally, noticed that $u_{n-1} = nv_0$ so that both submanifold $v_0 = 0$ and $u_{n-1} = 0$ coincide. Thus, the restriction of (3.3) to $u_{n-1} = 0$ is the $\mathrm{SL}(n,\mathbb{R})$-AGD Hamiltonian evolution.

In second place we inquire: how does the restriction $u_{n-1} = 0$ look when carried over to the ξ evolution (3.4)?

The condition $u_{n-1} = 0$ imposes a condition on ξ, namely, given L such that $u_{n-1} = 0$, any set of independent elements in the kernel $\xi_0,\ldots,\xi_{n-1}$, will have constant Wronskian, that is, independent of the parameter θ. Call the Wronskian $W_\xi = \det \mathcal{W}_\xi$. Furthermore, if they evolve using evolution (3.4), we get the following claim.

PROPOSITION 4.3. — *In the* $\mathrm{SL}(n,\mathbb{R})$ *case (that is, whenever $u_{n-1} = 0$), W_ξ is independent of t and θ along the solutions of* (3.4).

Proof. — Indeed, if ξ evolves following (3.4), due to the antiplectic relation between λ and ℓ, u will evolve following the $\mathrm{SL}(n,\mathbb{R})$-AGD evolution. It is known that the AGD evolution preserves the $\mathrm{SL}(n,\mathbb{R})$ conjugation class of any of the monodromies associated to L. (It preserves the $\mathrm{GL}(n,\mathbb{R})$ class in the $\mathrm{GL}(n,\mathbb{R})$ case.) In fact, the conjugation class of the monodromy is one of the invariants of the symplectic leaves of the AGD Poisson bracket, the so called continuous invariant. The second invariant is discrete (it does not change locally) and, as I remarked earlier, it depends on the topology of the projective curves (see [12] for the description of the invariants). That is, the determinant of the monodromy is unchanged along flows of the t-evolution for ξ. This determinant coincides with the Wronskian of ξ at $\theta = T$, the period. $\square$

Thus, equation (3.3) restricted to $u_{n-1} = 0$ is the restriction of the evolution (3.4) to the ring B generated by u_i, $i = 0, \dots, n-2$, whenever one adds the extra condition to (3.4) of the Wronskian of ξ being constant. Notice that this is a restriction on the set of initial conditions that are allowed in (3.4), rather than on the equation itself. The ξ evolution is, of course, invariant under the real action of $\mathrm{SL}(n,\mathbb{R})$ (action on $\mathbb{R}^n$), since it is invariant under the real action of $\mathrm{GL}(n,\mathbb{R})$.

I must say that a $\mathrm{SL}(n,\mathbb{R})$-invariant evolution on ξ induced by the AGD bracket was found in [7]; even though the approach used in that paper was quite nice, it was completely different from the one used by Wilson (and the one used in this paper) and it is not obvious that both formulas coincide. In that paper the author also claims that it suffices to take the restriction of the ξ-evolution to the field generated by $\phi_i = \xi_i/\xi_0$, $i = 1, \dots, n-1$ in order to obtain the so-called Ur-KdV equations, which will also be $\mathrm{SL}(n,\mathbb{R})$-invariant. In fact, if we write the evolution for the proportions ϕ_i induced by (3.4), these equations will *not* be homogeneous. One needs to futher substitute the relationship $\xi_i = W_\phi^{-\frac{1}{n}} \phi_i$, $i = 1, \dots, n-1$ and $\xi_0 = W_\phi^{-\frac{1}{n}}$ to obtain an evolution on ϕ. These relationships exist only under the condition $W_\xi = 1$, and not simply constant.

Finally, I will describe the evolution induced on ϕ, on the projectivisation of the solution curve, by the evolution (3.4) in the $\mathrm{SL}(n,\mathbb{R})$ case.

Consider (3.4) restricted to curves with the condition $W_\xi = 1$ (we are simply restricting further the possible initial conditions). Then, it is not hard to see that, if $\phi_i = \xi_i/\xi_0$ is the projectivisation of ξ

$$W_\phi = \begin{vmatrix} \phi_1' & \dots & \phi_{n-1}' \\ \vdots & \vdots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_{n-1}^{(n-1)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & 0 \\ \phi_1 & \phi_1' & \dots & \phi_{n-1}' \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n-1} & \phi_1^{(n-1)} & \dots & \phi_{n-1}^{(n-1)} \end{vmatrix}$$

$$= \xi_0^{-n} \begin{vmatrix} \xi_0 & \xi_0' & \cdots & \xi_0^{(n-1)} \\ \xi_1 & \xi_1' & \cdots & \xi_1^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{n-1} & \xi_{n-1}' & \cdots & \xi_{n-1}^{(n-1)} \end{vmatrix} = \xi_0^{-n}.$$

Therefore, the evolution (3.4) restricts to ϕ as

$$\phi_t = \frac{\xi_t}{\xi_0} - \phi \frac{(\xi_0)_t}{\xi_0} \tag{4.4}$$

where the right hand side depends on ϕ after the substitution

$$\xi_i = W_\phi^{-\frac{1}{n}} \phi_i, \quad i = 1, \dots, n-1 \quad \text{and} \quad \xi_0 = W_\phi^{-\frac{1}{n}}.$$

LEMMA 4.4. — *Let* (3.4) *be an evolution on* ξ, *with* $W_\xi = 1$, *invariant under the real action of* $\mathrm{SL}(n, \mathbb{R})$. *Let* (4.4) *be the induced evolution on the projectivisation of* ξ. *Then,* (4.4) *is invariant under the projective action of* $\mathrm{SL}(n, \mathbb{R})$ *on* $\mathbb{RP}^{n-1}$.

Proof. — The proof is very simple of course. We need to show that, if we denote by $M \cdot \phi$ the projective action of $\mathrm{SL}(n, \mathbb{R})$ on $\mathbb{RP}^{n-1}$ (that is, the projectivisation of $M\binom{1}{\phi}$), then $M \cdot \phi$ is a solution of (4.4) whenever ϕ is a solution itself. But notice that $M\binom{1}{\phi} = \xi_0^{-1} M$, and so, the projectivisation of $M\binom{1}{\phi}$ coincides with the projectivisation of $M\xi$. If ϕ is a solution of (4.4), it is clear that ξ is a solution of (3.4); because of the invariance of (3.4), $M\xi$ will also be a solution of (3.4) and its Wronskian will be 1 since $M \in \mathrm{SL}(n, \mathbb{R})$. Hence, $M \cdot \phi$ will be a solution of (4.4). ∎

After these three steps I proceed now to state and prove the Main Theorem, conjectured in [4].

THEOREM 4.5. — *Let* $\phi(t, \theta)$ *be a solution curve of the evolution*

$$\phi_t = \mathcal{W}_\phi^T (\mathrm{Id} + A) \mathcal{I} \tag{4.5}$$

defined in (2.15). *Let* $u(t, \theta)$ *be the vector of associated basic* $\mathrm{SL}(n, \mathbb{R})$-*differential invariants defined in* (2.3). *Then, there exists an invariant and invertible matrix* $\mathcal{M}$ *such that, if* $H = \mathcal{M}\mathcal{I}$ *and if* H *comes from a gradient, then* $u(t, \theta)$ *evolves following the* $\mathrm{SL}(n, \mathbb{R})$-*AGD Hamiltonian evolution*

$$u_t = \ell \frac{\delta \mathcal{H}}{\delta u}$$

where $\mathcal{H}$ *is the operator associated to* H.

Proof. — The proof is based on the analysis of (4.4) and its comparison to (2.15). Recall that the ξ-evolution can be written as

$$\xi_t = S^* \frac{\delta \mathcal{H}}{\delta u},$$

where $S^* = {\mathcal{W}_\eta}^{-1} C$ is defined in (3.7), that is, where $C^* = D\bar{u}/Du$ and where $\mathcal{W}_\eta$ is the Wronskian matrix of η. Let's call ${\mathcal{W}_\eta}^{-1} = (g_{ij})$ and $C = (c_{ij})$. Using this expression for ξ_t we can show that

1) In general, the ϕ-evolution (4.4) does not depend on $\delta\mathcal{H}/\delta u_{n-1}$, even if (3.3) was dependent of it.

2) In expression (4.4), the coefficients of c_{n-1j}, $j = 1, 2, \ldots, n-1$ are always zero.

Indeed this is true. The coefficient of $\delta\mathcal{H}/\delta u_{n-1}$ in (3.4) is $(-1)^{n-1} g_{in-1}$ since, clearly, C is lower triangular and $c_{n-1n-1} = (-1)^{n-1}$. Now, it is known (see [14]) that, if two basis are dual with respect to the bilinear concomitant, then any of the basis form the last column of the inverse of the Wronskian matrix of the other basis. Thus, the last column of ${\mathcal{W}_\eta}^{-1}$ is $(\xi_0, \xi_1, \ldots, \xi_{n-1})^T$. Therefore $g_{in-1} = \xi_i$. Finally, the $\delta\mathcal{H}/\delta u_{n-1}$ term in the ϕ-evolution has coefficient

$$\frac{1}{\xi_0}(-1)^{n-1}\xi_i - \phi_i(-1)^{n-1}\frac{\xi_0}{\xi_0} = 0.$$

Also, the coefficient of the entry c_{n-1j} in evolution (3.4) is given by $g_{in-1} = \xi_i$. Hence, as before, the coefficient of c_{n-1j} in evolution (4.4) is given by

$$\frac{g_{in-1}}{\xi_0} - \phi_i \frac{g_{0n-1}}{\xi_0} = \frac{\xi_i}{\xi_0} - \phi_i \frac{\xi_0}{\xi_0} = 0.$$

In view of 1) and 2), we can deduce that the projective evolution (4.4) can in turn be written as

$$\phi_t = B\overline{C}\,\overline{\frac{\delta \mathcal{H}}{\delta u}} \tag{4.6}$$

where

$$\overline{\frac{\delta \mathcal{H}}{\delta u}} = \begin{pmatrix} \delta\mathcal{H}/\delta u_0 \\ \vdots \\ \delta\mathcal{H}/\delta u_{n-2} \end{pmatrix},$$

$\overline{C}$ is the upper $(n-1)\times(n-1)$ block of C,

$$C = \begin{pmatrix} & & & 0 \\ & \overline{C} & & \vdots \\ & & & 0 \\ c_{n1} & \dots & c_{nn-1} & c_{nn} \end{pmatrix},$$

and where, if I denote by $\widetilde{G}$ the last $n-1$ rows of ${\mathcal{W}_\eta}^{-1}$ and by G_0 its first row, then B can be written as

$$B = \frac{1}{\xi_0}(\widetilde{G} - \phi G_0). \tag{4.7}$$

The final step is to realize what $\overline{C} = (\bar{c}_{ij})$ is. It is easy to see that, in our notation,

$$\overline{c_{ij}} = (-1)^{j-1}\binom{i-1}{j-1}\partial^{i-j}, \quad i,j = 1,2,\dots,n-1.$$

That is, $\overline{C}R = T$, where $R = (r_{ij})$ is given by $r_{ij} = (-1)^{j-1}\delta_i^j$, ($\delta$ represents the Delta of Kronecker), and where T is defined in (2.17). Up to R, a matrix changing the sign of every other row, the matrix $D\bar{u}/Du$ and T such that $\delta\mathcal{H}/\delta u = TH$ are the same. (Notice that $T = \overline{C}R$ and $\overline{C}^2 = \mathrm{Id}$ by definition. Hence $T^{-1} = RTR$.)

Finally, equation (4.4) can be written as

$$\phi_t = BTR\frac{\delta\mathcal{H}}{\delta u} = BRH. \tag{4.8}$$

This evolution is $\mathrm{SL}(n,\mathbb{R})$-projectively invariant for any gradient vector $\delta\mathcal{H}/\delta u$. The vector $TR\,\delta\mathcal{H}/\delta u = RH$ depends on the u's and their derivatives. It is, therefore, an invariant vector. Now, in [4] we showed that such an evolution is invariant if and only if BRH is a relative invariant for the action with weights as in (2.12), and this should be true for any invariant vector H coming from a gradient. But if H is invariant we have that, if $\boldsymbol{v} \in \mathrm{sl}(n,\mathbb{R})$ is an infinitesimal generator $\boldsymbol{v} = \sum_{i=1}^{n-1} \eta_i(\theta,t,\phi)\frac{\partial}{\partial\phi_i}$, then,

$$\mathrm{pr}\,\boldsymbol{v}(BRH) = \mathrm{pr}\,\boldsymbol{v}(BR)H = \frac{\partial\eta}{\partial\phi}BRH.$$

Running H through an appropiate independent family of vectors we obtain that BR is indeed a relative invariant with the necessary weights, that is

$$\mathrm{pr}\,\boldsymbol{v}(BR) = \frac{\partial\eta}{\partial\phi}BR,$$

for any infinitesimal generator $\boldsymbol{v} = \sum_i \eta_i\frac{\partial}{\partial\phi_i}$.

The end of the proof of our theorem is now clear. Using the result of Corollary 2.6 we can deduce the existence of an invertible matrix of differential invariants $\mathcal{M}$ such that

$$\mathcal{W}_\phi^T(\mathrm{Id}+A) = BR\mathcal{M}. \tag{4.9}$$

Therefore, if $T^{-1}\delta\mathcal{H}/\delta u = H = \mathcal{M}\mathcal{I}$, evolutions (2.15) and (4.4) are identical. Finally, the basic differential invariants u, which coincide with the coefficients of the Lax operator L, evolve following the $\mathrm{SL}(n,\mathbb{R})$-AGD Hamiltonian evolution (3.3). ▯

A short comment on the meaning of this theorem. From Theorem 2.2 we see that the matrix $\widehat{\mu} = \mathcal{W}_\phi^T(\mathrm{Id}+A)\mathcal{M}^{-1}$ is a relative invariant of the $\mathrm{SL}(n,\mathbb{R})$ action, since $\mathcal{M}$ is a nondegenerate matrix of differential invariants. Thus, from Theorem 2.2, the most general invariant evolution of projective curves can also be written as

$$\phi_t = \widehat{\mu}\mathcal{I} \tag{4.10}$$

where $\mathcal{I}$ is any general vector of differential invariants. If $\mathcal{I}$ comes from a gradient, then (4.10) can be view as an alternative definition for the AGD bracket, written in projective coordinates. They are the same evolution whenever one identifies $\mathcal{I}$, the invariant vector, with the pseudodifferential operator associated to a functional $\mathcal{H}$, the Hamiltonian.

The last part of this chapter is dedicated to show that $\mathcal{M}$ is indeed of the form (2.18). This is the result of the following two lemmas and their projective counterparts. As before, I will call W_ξ the Wronskian of ξ and $\mathcal{W}_\xi$ its Wronskian matrix. Analogous notation is used for η.

LEMMA 4.6. — *Let $\{\xi_i\}$ and $\{\eta_i\}$ be basis for* kernel(L) *and* kernel(L^*), *respectively, and assume they are dual with respect to the bilinear concomitant. Then*

$$\sum_{i=0}^{n-1} \xi_i^{(k)}\eta_i^{(r)} = \begin{cases} 0 & \textit{if } \; k < n-r-1, \\ (-1)^r & \textit{if } \; k = n-r-1, \; r = 0,1,\dots,n-1. \end{cases}$$

Proof. — This lemma is a simple consequence of the relationship

$$\sum_i \xi_i^{(k)}\eta_i = \begin{cases} 0 & \text{if } \; 0 < k < n-1, \\ 1 & \text{if } \; k = n-1 \end{cases}$$

which holds since η conforms the last column of the inverse of the Wronskian matrix of ξ. The result of the lemma is obtained by simply applying Leibniz's rule.

LEMMA 4.7. — *If* $W_\xi = 1$, *then* $W_\eta = (-1)^{m_1}$, *where* $m_1 = \frac{1}{2}n(n-1)$.

Proof. — The previous lemma implies that $\mathcal{W}_\xi^T \mathcal{W}_\eta$ is triangular along the transverse diagonal. Hence, W_η is the product of entries in the transverse diagonal, namely $(-1)^{m_1}$, m_1 as in the statement. □

LEMMA 4.8. — *Let* $\{\phi_i\}$ *be the projectivisation of* $\{\xi_i\}$, *and let* $\{\xi_i\}$ *be such that* $W_\xi = 1$. *Let* $\{\eta_i\}$ *be as in Lemma* 4.6. *Then*

$$\sum_{i=1}^{n-1} \phi_i^{(k)} \eta_i^{(r)} = \begin{cases} 0 & \text{if } 0 < k < n-r-1, \\ (-1)^r W_\phi^{\frac{1}{n}} & \text{if } k = n-1-r,\ r = 0,1,\dots,n-2. \end{cases}$$

Proof. — The proof is straightforward from Lemma 4.6. Namely, if $0 < k < n-r-1$, then

$$\begin{aligned} \sum_{i=1}^{n-1} \phi_i^{(k)} \eta_i^{(r)} &= \sum_{j=1}^{n-1} \Big(\frac{\xi_j}{\xi_0}\Big)^{(k)} \eta_j^{(r)} \\ &= \sum_{j=0}^{n-1} \sum_{s=0}^{k} \binom{k}{s} (\xi_0^{-1})^{(k-s)} \xi_j^{(s)} \eta_j^{(r)} = 0 \end{aligned}$$

since $0 \le s < k \le n-r-1$, while, if $k = n-r-1$ one gets

$$\begin{aligned} \sum_{j=1}^{n-1} \Big(\frac{\xi_j}{\xi_0}\Big)^{(k)} \eta_i^{(r)} &= \sum_{j=0}^{n-1} \sum_{s=0}^{k} \binom{k}{s} (\xi_0^{-1})^{(k-s)} \xi_j^{(s)} \eta_j^{(r)} \\ &= \xi_0^{-1}(-1)^r = (-1)^r W_\phi^{\frac{1}{n}}. \quad \square \end{aligned}$$

LEMMA 4.9. — *Let* $\widetilde{\mathcal{W}_\eta}$ *be the upper right* $(n-1)\times(n-1)$ *block of* $\mathcal{W}_\eta$, *that is,*

$$\widetilde{\mathcal{W}_\eta} = \begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \eta_1^{(n-2)} & \eta_2^{(n-2)} & \dots & \eta_{n-1}^{(n-2)} \end{pmatrix}.$$

If $W_\xi = 1$, *then* $\widetilde{W_\eta} = (-1)^{m_2} W_\phi^{-\frac{1}{n}}$, *where* $m_2 = \frac{1}{2}(n-2)(n-1)$.

Proof. — From Lemma 4.8 we have that $\mathcal{W}_\phi^T \widetilde{\mathcal{W}_\eta}$ is upper triangular along the tranverse diagonal. Hence, $\widetilde{W_\eta}$ equals W_ϕ^{-1} times the product of the elements on the transverse diagonal. That is

$$W_\eta = W_\phi^{-1} W_\phi^{\frac{n-1}{n}} (-1)^{m_2} = (-1)^{m_2} W_\phi^{-\frac{1}{n}},$$

m_2 as in the statement. □

Also, we have the following expected result

LEMMA 4.10. — *Let B be defined as in* (4.7). *Then* $B = (-1)^{n-1}\widetilde{\mathcal{W}}_\eta^{-1}$

Proof. — Following the notation in (4.7), the $(k, j+1)$ entry in B is given by

$$g_{k+1j+1} - \phi_k g_{1j+1} = (-1)^{m_1}\left[C^\eta_{j+1,k+1} - \phi_k C^\eta_{j+1,1}\right],$$

where, in general, I denote the cofactor (i,j) of the Wronskian of the vector function ψ by $C^\psi_{i,j}$. This entry can be rewritten as

$$(4.11)\quad (-1)^{m_1+k+j}\begin{vmatrix} \eta_0 + \phi_k\eta_k & \eta_1 & \dots & \widehat{\eta}_k & \dots & \eta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widehat{\eta}_0^{(j)} + \widehat{\phi}_k\widehat{\eta}_k^{(j)} & \widehat{\eta}_1^{(j)} & \dots & \widehat{\eta}_k^{(j)} & \dots & \widehat{\eta}_{n-1}^{(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_0^{(n-1)} + \phi_k\eta_k^{(n-1)} & \eta_1^{(n-1)} & \dots & \widehat{\eta}_k^{(n-1)} & \dots & \eta_{n-1}^{(n-1)} \end{vmatrix},$$

where, again, $\hat{}$ signs the deletion of the term. Using the fact that all η_j can be expressed in terms of cofactors of $\mathcal{W}_\xi$, one can easily prove that $\eta_0 = -\sum_{i=1}^{n-1}\phi_i\eta_i$. This fact, together with Lemma 4.8, implies that determinant (4.11) has, in fact, zeros along the first column, except for the last entry, which is equal to $\sum_{r=0}^{n-2}\binom{n-1}{r}(-1)^r W_\phi^{\frac{1}{n}}$. Since $\sum_{r=0}^{n-1}\binom{n-1}{r}(-1)^r = 0$, we get that (4.11) equals

$$(-1)^{2n+1}(-1)^{m_1}(-1)^{k+j}W_\phi^{\frac{1}{n}}\begin{vmatrix} \eta_1 & \dots & \widehat{\eta}_k & \dots & \eta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widehat{\eta}_1^{(j)} & \dots & \widehat{\eta}_k^{(j)} & \dots & \widehat{\eta}_{n-1}^{(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_1^{(n-2)} & \dots & \widehat{\eta}_k^{(n-2)} & \dots & \eta_{n-1}^{(n-1)} \end{vmatrix}.$$

The use of the result of Lemma 4.9 leads as to the conclusion of the proof. □

At last we can analyze the shape of $\mathcal{M}$. We can directly obtain it from the relationship (4.9), rewritten as

$$\mathcal{M} = (-1)^{n-1}R\widetilde{\mathcal{W}}_\eta\mathcal{W}_\phi^T(\mathrm{Id} + A).$$

We know from Lemma 4.8 that $\widetilde{\mathcal{W}_\eta}\mathcal{W}_\phi^T$ is lower triangular along the transverse diagonal. Since $\mathrm{Id}+A$ is upper triangular along the main diagonal, we have that $\mathcal{M}$ must be lower triangular along the transverse diagonal, as conjectured (notice that $R^{-1}=R$ is a diagonal matrix).

Furthermore, one can calculate the entries of the diagonal strictly below the transverse diagonal of $\mathcal{M}$. Let $(\, s_{2n-1}\; s_{3n-2} \dots s_{n-12}\,)$ be the diagonal strictly below the transverse diagonal in $\widetilde{\mathcal{W}_\eta}\mathcal{W}_\phi^T$. Using Lemma 4.8 one can find explicitly s_{kn-k+1}, for all k.

First of all, $\sum_{j=1}^{n-1}\phi_j^{(n-1)}\eta_j = W_\phi^{\frac{1}{n}}$, and so

$$\sum_{j=1}^{n-1}\phi_j^{(n-1)}\eta_j' = \frac{1}{n}\,W_\phi^{-\frac{n-1}{n}}\,W_\phi' - \sum_{j=1}^{n-1}\phi_j^{(n)}\eta_j.$$

But, since $\eta_j = C^\xi_{nj+1} = W_\phi^{-\frac{n-1}{n}}\,C^\phi_{n-1j}$ (again, C_{ij} is indicating the cofactor of the Wronskian matrix of the superindex), we have that

$$\sum_{j=1}^{n-1}\phi_j^{(n)}\eta_j = W_\phi^{-\frac{n-1}{n}}\,W_\phi'$$

and so

$$\sum_{j=1}^{n-1}\phi_j^{(n-1)}\eta_j' = -\frac{n-1}{n}\,W_\phi^{-\frac{n-1}{n}}\,W_\phi'.$$

Assume that

$$\sum_{j=1}^{n-1}\phi_j^{(n-r)}\eta_j^{(r)} = (-1)^r\frac{n-r}{n}\,W_\phi^{-\frac{n-1}{n}}\,W_\phi'.$$

Again, differentiating the relationship in Lemma 4.8

$$\sum_{j=1}^{n-1}\phi_j^{(n-r-1)}\eta_j^{(r)} = (-1)^r W_\phi^{\frac{1}{n}}$$

we can easily see that the same relationship holds for $r+1$. Therefore

$$s_{kn-k+1} = \sum_{j=1}^{n-1}\phi_j^{(n-k+1)}\eta_j^{(k-1)} = (-1)^{k-1}\frac{n-k+1}{n}\,W_\phi^{-\frac{n-1}{n}}\,W_\phi'.$$

In [4], we found that

$$r_{kk+1} = -\frac{\binom{k+1}{k}}{\binom{n}{1}} q_n^{n-1} = -\frac{k+1}{n}\frac{W_\phi'}{W_\phi}.$$

Finally, ignoring the action of the matrix R, the diagonal in $\mathcal{M}$ strictly below the transverse diagonal is given by

$$\begin{aligned}(-1)^{k+1}&W_\phi^{\frac{1}{n}}\, r_{n-kn-k+1} + s_{kn-k+1}\\ &= (-1)^k W_\phi^{\frac{1}{n}}\,\frac{n-k+1}{n}\,\frac{W_\phi'}{W_\phi} + (-1)^{k-1}\frac{n-k+1}{n}\, W_\phi^{-\frac{n-1}{n}}\, W_\phi' = 0,\end{aligned}$$

concluding the proof of the fact that the shape of the matrix $\mathcal{M}$ is the one conjectured in [4].

To finish the paper I will point out at some of the advantages of regarding AGD evolutions as evolutions of basic differential invariants. Apart from the obvious connection between two apparently unrelated subjects, there are some extra advantages about viewing the AGD evolution in the terrain of differential invariants. Namely, it opens a road to generalizations of Schwarz derivatives and KdV evolutions to the case of several independent variables. These generalizations might have a strong relevance in various subjects such as the theory of solitons, inverse scattering and other topological branches. In fact, in [8] we found and classified a complete set of basic differential invariants for projective surfaces. We also described the generalizations of the Schwarz derivative to two independent variables and the generalization of KdV to this same case. Of course, these generalizations assume that the $\mathrm{SL}(n,\mathbb{R})$ symmetry is also held in the generalized case. It is not clear yet what connection this might have with the Poisson Geometry of PDEs and the Theory of integrable systems.

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