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**LINEARIZATION OF GROUP STACK ACTIONS AND  
THE PICARD GROUP OF THE MODULI OF  
 $SL_r/\mu_s$ -BUNDLES ON A CURVE**

PAR YVES LASZLO (\*)

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ABSTRACT. — In this paper, we consider morphisms of algebraic stacks  $\mathcal{X} \rightarrow \mathcal{Y}$  which are torsors under a *group stack*  $\mathcal{G}$ . We show that line bundles on  $\mathcal{Y}$  correspond exactly with  $\mathcal{G}$ -linearized line bundles on  $\mathcal{X}$  (with a suitable definition of a  $\mathcal{G}$ -linearization). We use this fact to determine the precise structure of the Picard group of the moduli stack of  $G$ -bundles on an algebraic curve when  $G$  is any group of type  $A_n$ .

RÉSUMÉ. — Dans cet article, on considère les morphismes de champs algébriques  $\mathcal{X} \rightarrow \mathcal{Y}$  qui sont des *torseurs* sous un champ en groupes  $\mathcal{G}$ . Nous prouvons que les fibrés en droites sur  $\mathcal{Y}$  correspondent exactement aux fibrés en droites sur  $\mathcal{X}$  munis d'une  $\mathcal{G}$ -linéarisation (avec une définition convenable d'une  $\mathcal{G}$ -linéarisation). Nous utilisons ceci pour déterminer la structure exacte du groupe de Picard du champ des  $G$ -fibrés sur une courbe algébrique lorsque  $G$  est un groupe algébrique (non nécessairement simplement connexe) de type  $A_n$ .

### 1. Introduction

Let  $G$  be a complex simple group and  $\tilde{G} \twoheadrightarrow G$  the universal covering. For simplicity, let us consider the moduli stack  $\mathcal{M}_G$  (resp.  $\mathcal{M}_{\tilde{G}}$ ) of degree  $1 \in \pi_1(G)$  principal  $G$ -bundles (resp.  $\tilde{G}$ -bundles) over a curve  $X$ . In [B-L-S], we have studied the natural morphism

$$\iota : \text{Pic}(\mathcal{M}_G) \longrightarrow \text{Pic}(\mathcal{M}_{\tilde{G}}),$$

the group  $\text{Pic}(\mathcal{M}_{\tilde{G}})$  being infinite cyclic by [L-S]. It is proved in [B-L-S] that the kernel of  $\iota$  is naturally identified with the finite group  $H_{\text{ét}}^1(X, \pi_1(G)^\vee)$  reducing the study of  $\text{Pic}(\mathcal{M}_G)$  to the computation of the cardinality of  $\text{Coker}(\iota)$ . Among other things, it has been possible to

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perform this computation in the case where  $G = \mathbf{PSL}_r$  but not in the case where  $G = \mathbf{SL}_r/\mu_s$ , where  $s \mid r$ , although we were able to give partial results. The reason was that the technical background to study the descent of modules through the morphism  $p : \mathcal{M}_{\tilde{G}} \rightarrow \mathcal{M}_G$  wasn't at our disposal.

The aim of this paper is to compute  $\text{card Coker}(\iota)$  when  $G = \mathbf{SL}_r/\mu_s$ .

It turns out to be that  $p$  is a torsor under some group stack, not far from a Galois étale cover in the usual schematic picture. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a torsor under a group scheme  $\mathcal{G}$ . We know that a line bundle on  $\mathcal{X}$  descends if and only if it has a  $\mathcal{G}$ -linearization (easy consequence of descent theory). Now, the descent theory of Grothendieck has been adapted to the set-up of fpqc morphisms of stacks in [L-M]. If  $\mathcal{G}$  is now only assumed to be a group stack, there is a notion of  $\mathcal{G}$ -linearization of line bundles on  $\mathcal{X}$  (see section 2). One obtains (theorem 4.1) that a line bundle on  $\mathcal{X}$  descends if and only if it admits a linearization.

We then use this technical result to compute  $\text{card Coker}(\iota)$  when  $G = \mathbf{SL}_r/\mu_s$  (theorem 5.7 and its corollary).

I would like to thank L. BREEN for having taught me both the notions of torsor and of linearization of a vector bundle in the set-up of group-stack actions and for his comments on a preliminary version of this paper.

**Notations.**

Throughout this paper, all the stacks will be implicitly assumed to be algebraic over an algebraically closed field  $k$  and the morphisms locally of finite type. We fix once and for all a projective, smooth, connected genus  $g$  curve  $X$  and a closed point  $x$  of  $X$ . For simplicity, we assume  $g > 0$  (see remarks 5.6 and 5.10 for the case of  $\mathbf{P}^1$ ). The Picard stack parametrizing families of line bundles of degree 0 on  $X$  will be denoted by  $\mathcal{J}(X)$  and the jacobian variety of  $X$  by  $JX$ . If  $G$  is an algebraic group over  $k$ , the quotient stack  $\text{Spec}(k)/G$  (where  $G$  acts trivially on  $\text{Spec}(k)$ ) whose category over a  $k$ -scheme  $S$  is the category of  $G$ -torsors (or  $G$ -bundles) over  $S$  will be denoted by  $BG$ . If  $n$  is an integer and  $A = \mathcal{J}(X), JX$  or  $BG_m$  we denote by  $n_A$  the  $n^{\text{th}}$ -power morphism  $a \mapsto a^n$ . We denote by  $\mathcal{J}_n$  (resp.  $J_n$ ) the 0-fiber  $A \times_A \text{Spec}(k)$  of  $n_A$  when  $A = \mathcal{J}(X)$  (resp.  $A = JX$ ).

**1. Generalities.** — Following [Br], for any diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{g} & & \\
 A & \xrightarrow{h} & B & \begin{array}{c} \uparrow \lambda \\ \uparrow \lambda \end{array} & C & \xrightarrow{\ell} & D \\
 & & & \xrightarrow{f} & & & 
 \end{array}$$

of 2-categories, we will denote by

$$\ell * \lambda : \ell \circ f \Rightarrow \ell \circ g \quad (\text{resp. } \lambda * h : f \circ h \Rightarrow g \circ h)$$

the 2-morphism deduced from  $\lambda$ .

1.1. — For the convenience of the reader, let us prove a simple formal lemma which will be useful in section 4. Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be three 2-categories. Let diagram

$$(1.1.1) \quad \begin{array}{ccc} & & \mathcal{C} \\ & \nearrow \delta_0 & \uparrow d_0 \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ & \searrow \delta_1 & \downarrow d_1 \\ & & \mathcal{C} \end{array}$$

be a 2-commutative diagram and let  $\mu : \delta_0 \Rightarrow \delta_1$  be a 2-morphism.

LEMMA 1.2. — *Assume that  $f$  is an equivalence. There exists a unique 2-morphism*

$$\mu * f^{-1} : d_0 \Rightarrow d_1$$

such that  $(\mu * f^{-1}) * f = \mu$ .

*Proof.* — Let  $\epsilon_k$ , for  $k = 0, 1$  be the 2-morphism  $d_k \circ f \Rightarrow \delta_k$ . Let  $b$  be an object of  $\mathcal{B}$ . Pick an object  $a$  of  $\mathcal{A}$  and an isomorphism  $\alpha : f(a) \xrightarrow{\sim} b$ . Let  $\varphi_\alpha : d_0(b) \xrightarrow{\sim} d_1(b)$  be the unique isomorphism making the diagram

$$\begin{array}{ccccc} \delta_0(a) & \xrightarrow{\epsilon_0(a)} & d_0 \circ f(a) & \xrightarrow{d_0(\alpha)} & d_0(b) \\ \mu_a \downarrow & & & & \downarrow \varphi_\alpha \\ \delta_1(a) & \xrightarrow{\epsilon_1(a)} & d_1 \circ f(a) & \xrightarrow{d_1(\alpha)} & d_1(b) \end{array}$$

commutative. We have to show that  $\varphi_\alpha$  does not depend on  $\alpha$  but only on  $b$ . Let  $\alpha' : f(a') \xrightarrow{\sim} b$  be another isomorphism. There exists a unique isomorphism  $\iota : a' \xrightarrow{\sim} a$  such that  $\alpha \circ f(\iota) = \alpha'$ . Then one has the equality  $\varphi_{\alpha'} = d_1(\alpha) \circ \Phi \circ d_0(\alpha)^{-1}$  where

$$\Phi = [d_1 \circ f(\iota)] \circ \epsilon_1(a') \circ \mu_{a'} \circ \epsilon_0(a')^{-1} \circ [d_0 \circ f(\iota)]^{-1}.$$

The functoriality of  $\epsilon_i$  and  $\mu$  ensures that one has the equalities

$$d_k \circ f(\iota) \circ \epsilon_k(a') = \epsilon_k(a) \circ \delta_k(\iota)$$

and

$$\mu_a = \delta_1(\iota) \circ \mu_{a'} \circ \delta_0(\iota)^{-1}.$$

This shows the equality

$$\Phi = \epsilon_1(a) \circ \mu_a \circ \epsilon_0(a)^{-1}$$

which proves the equality  $\varphi_\alpha = \varphi_{\alpha'}$ . We can therefore define  $\mu_b$  to be the isomorphism  $\varphi_\alpha$  for one isomorphism  $\alpha : f(a) \xrightarrow{\sim} b$ . We check that the construction is functorial in  $b$  and the lemma follows.  $\square$

**2. Linearizations of line bundles on stacks.**

Let us first recall the notion of torsor in the stack context according to [Br].

2.1. — Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a faithfully flat morphism of stacks. Let us assume that an algebraic  $gr$ -stack  $\mathcal{G}$  acts on  $f$  (the product of  $\mathcal{G}$  is denoted by  $m_{\mathcal{G}}$  and the unit object by 1). Following [Br], this means that there exists a 1-morphism of  $\mathcal{Y}$ -stacks  $m : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$  and a 2-morphism  $\mu : m \circ (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) \Rightarrow m \circ (\text{Id}_{\mathcal{G}} \times m)$  such that the obvious associativity condition (see diagram (6.1.3) in [Br]) is satisfied and such that there exists a 2-morphism  $\epsilon : m \circ (1 \times \text{Id}_{\mathcal{X}}) \Rightarrow \text{Id}_{\mathcal{X}}$  which is compatible to  $\mu$  in the obvious sense (see (6.1.4) of [Br]).

REMARK 2.2. — To say that  $m$  is a morphism of  $\mathcal{Y}$ -stacks means that the diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{X} & \xrightarrow{m} & \mathcal{X} \\ & \searrow & \swarrow \\ & \mathcal{Y} & \end{array}$$

is 2-commutative. In other words, if we denote for simplicity the image of a pair of objects  $m(g, x)$  by  $g \cdot x$ . This means that there exists a functorial isomorphism  $\iota_{g,x} : f(g \cdot x) \rightarrow f(x)$ .

2.3. — Suppose that  $\mathcal{G}$  acts on such another morphism  $f' : \mathcal{X}' \rightarrow \mathcal{Y}$ . A morphism  $p : \mathcal{X}' \rightarrow \mathcal{X}$  will be said *equivariant* if there exists a 2-morphism

$$q : m \circ (\text{Id} \times p) \Rightarrow p \circ m'$$

which is compatible to  $\mu$  (as in [Br, (6.1.6)]) and  $\epsilon$  (which is implicit in [Br]) in the obvious sense.

DEFINITION 2.4. — With the above notations, we say that  $f$  (or  $\mathcal{X}$ ) is a  $\mathcal{G}$ -torsor over  $\mathcal{Y}$  if the morphism  $\text{pr}_2 \times m : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$  is an isomorphism (of stacks) and the geometrical fibers of  $f$  are not empty.

REMARK 2.5. — In down to earth terms, this means that if

$$\iota : f(x) \longrightarrow f(x')$$

is an isomorphism in  $\mathcal{Y}$  ( $x, x'$  being objects of  $\mathcal{X}$ ), there exists an object  $g$  of  $\mathcal{G}$  and a unique isomorphism  $(x, g \cdot x) \xrightarrow{\sim} (x, x')$  which induces  $\iota$  by way of  $\iota_{g,x}$  (cf. 2.2).

EXAMPLE 2.6. — Let  $\mathcal{M}_X(G_m)$  be the Picard stack of  $X$ . Then, the morphism

$$\mathcal{M}_X(G_m) \longrightarrow \mathcal{M}_X(G_m)$$

of multiplication by  $n \in \mathbb{Z}$  is a torsor under  $B\mu_n \times J_n(X)$  (cf. (3.1)).

2.7. — Let  $\mathcal{L}$  be a line bundle on  $\mathcal{X}$ . By definition, the datum  $\mathcal{L}$  is equivalent to the datum of a morphism  $\ell : \mathcal{X} \rightarrow BG_m$  (see [L-M, prop. 6.15]). If  $\mathcal{L}, \mathcal{L}'$  are two line bundles on  $\mathcal{X}$  defined by  $\ell, \ell'$ , we will view an isomorphism  $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$  as a 2-morphism  $\ell \Rightarrow \ell'$ .

DEFINITION 2.8. — A  $\mathcal{G}$ -linearization of  $\mathcal{L}$  is a 2-morphism

$$\lambda : \ell \circ m \Rightarrow \ell \circ \text{pr}_2$$

such that the two diagrams of 2-morphisms

$$(2.8.1) \quad \begin{array}{ccc} \ell \circ m \circ (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) & \xrightarrow{\ell * \mu} & \ell \circ m \circ (\text{Id}_{\mathcal{G}} \times m) \\ \lambda * (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) \Downarrow & & \Downarrow \lambda * (\text{Id}_{\mathcal{G}} \times m) \\ \ell \circ \text{pr}_2 \circ (m_{\mathcal{G}} \times \text{Id}_{\mathcal{X}}) & \xleftarrow{\lambda * \text{pr}_{23}} & \ell \circ \text{pr}_2 \circ (\text{Id}_{\mathcal{G}} \times m) \\ \parallel & & \parallel \\ \ell \circ \text{pr}_2 \circ \text{pr}_{23} & & \ell \circ m \circ \text{pr}_{23} \end{array}$$

and

$$(2.8.2) \quad \begin{array}{ccc} \ell \circ m \circ (1 \times \text{Id}_{\mathcal{X}}) & \xrightarrow{\ell * \epsilon} & \ell \\ \lambda * (1 \times \text{Id}_{\mathcal{X}}) \Downarrow & & \Downarrow \\ \ell & \xlongequal{\quad} & \ell \end{array}$$

(strictly) commute.

REMARK 2.9. — If  $g_1, g_2$  are objects of  $\mathcal{G}$  and  $d$  is an object of  $\mathcal{X}$ , the commutativity of diagram (2.8.1) means that the diagram

$$\begin{CD} \mathcal{L}_{(g_1 \cdot g_2)x} @>\sim>> \mathcal{L}_{g_1(g_2 \cdot x)} \\ @V\wr VV @VV\wr V \\ \mathcal{L}_x @<\sim<< \mathcal{L}_{g_2 \cdot x} \end{CD}$$

is commutative and the commutativity of (2.8.2) that the two isomorphisms  $\mathcal{L}_{1 \cdot x} \simeq \mathcal{L}_x$  defined by the linearization  $\lambda$  and  $\epsilon$  respectively are the same.

**3. An example.**

Let me recall that a closed point  $x$  of  $X$  has been fixed. Let  $S$  be a  $\mathbf{k}$ -scheme. The  $S$ -points of the jacobian variety of  $X$  are by definition isomorphism classes of line bundles on  $X_S$  together with a trivialization along  $\{x\} \times S$  (such a pair will be called a *rigidified line bundle*). For the convenience of the reader, let me state this well known lemma which can be found in SGA4, exp. XVIII, (1.5.4).

LEMMA 3.1. — *The Picard stack  $\mathcal{J}(X)$  is canonically isomorphic (as a  $\mathbf{k}$ -group stack) to  $JX \times BG_m$ .*

*Proof.* — Let  $f : \mathcal{J}(X) \rightarrow JX \times BG_m$  be the morphism which associates

- to the line bundle  $L$  on  $X_S$  the pair  $(L \boxtimes L_{|\{x\} \times S}^{-1}, L_{|\{x\} \times S})$  where  $\boxtimes$  denotes the external tensor product (this pair is thought of as an object of  $JX \times BG_m$  over  $S$ );

- to an isomorphism  $L \xrightarrow{\sim} L'$  on  $X_S$  its restriction to  $\{x\} \times S$ .

Let  $f' : JX \times BG_m \rightarrow \mathcal{J}(X)$  be the morphism which associates

- to the pair  $(L, V)$  where  $L$  is a rigidified bundle on  $X_S$  and  $V$  a line bundle on  $S$  (thought of as an object of  $JX \times BG_m$  over  $S$ ), the line bundle  $L \boxtimes_{X_S} V$ ;

- to an isomorphism  $(\ell, v) : (L, V) \xrightarrow{\sim} (L', V')$  the tensor product  $\ell \boxtimes_{X_S} v$ .

The morphisms  $f$  and  $f'$  are (quasi)-inverse to each other and are morphisms of  $\mathbf{k}$ -stacks.  $\square$

We will identify from now  $\mathcal{J}(X)$  and  $JX \times BG_m$ . Let  $\mathcal{L}$  (resp.  $\mathcal{P}$  and  $\mathcal{T}$ ) be the universal bundle on  $X \times \mathcal{J}(X)$  (resp. on  $X \times JX$  and  $BG_m$ ) and let  $\Theta = (\det R\Gamma\mathcal{P})^{-1}$  be the theta line bundle on  $JX$ . The isomorphism  $\mathcal{L} \xrightarrow{\sim} \mathcal{P} \boxtimes \mathcal{T}$  yields an isomorphism

$$(3.1.1) \quad \det R\Gamma\mathcal{L}^n(m \cdot x) \xrightarrow{\sim} \Theta^{-n^2} \boxtimes \mathcal{T}^{(m+1-g)}.$$

**4. Descent of  $\mathcal{G}$ -line bundles.**

The object of this section is to prove the following statement.

**THEOREM 4.1.** — *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a  $\mathcal{G}$ -torsor as above. Let  $\text{Pic}^{\mathcal{G}}(\mathcal{X})$  be the group of isomorphism classes of  $\mathcal{G}$ -linearized line bundles on  $\mathcal{X}$ . Then, the pull-back morphism  $f^* : \text{Pic}(\mathcal{Y}) \xrightarrow{\sim} \text{Pic}^{\mathcal{G}}(\mathcal{X})$  is an isomorphism.*

The descent theory of Grothendieck has been adapted in the case of algebraic 1-stacks in [L-M], essentially in proposition (6.23).

Let  $\mathcal{X}_\bullet \rightarrow \mathcal{Y}$  be the (augmented) simplicial complex of stacks coskeleton of  $f$  (as defined in [De, (5.1.4)] for instance). By proposition (6.23) of [L-M], one just has to construct a cartesian  $\mathcal{O}_{D_\bullet}$ -module  $\mathcal{L}_\bullet$  such that  $\mathcal{L}_0$  is the  $\mathcal{O}_{\mathcal{X}}$ -module  $\mathcal{L}$  to prove the theorem. The  $n$ -th piece  $\mathcal{X}_n$  is inductively defined by

$$\mathcal{X}_0 = \mathcal{X}, \quad \mathcal{X}_n = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}_{n-1} \quad \text{for } n > 0.$$

Let  $p_n : \mathcal{X}_n \rightarrow \mathcal{X}$  be the projection on the first factor. It is the simplicial morphism associated to the map

$$\tilde{p}_n : \begin{cases} \Delta_0 \rightarrow \Delta_n, \\ 0 \mapsto 0. \end{cases}$$

Let  $\mathcal{L}_n$  be the line bundle defined by the morphism (see (2.7))

$$(4.1.1) \quad \ell_n : \mathcal{X}_n \xrightarrow{p_n} \mathcal{X} \xrightarrow{\ell} BG_m.$$

4.2. — Let  $\delta_i$  (resp.  $s_j$ ) be the face (resp. degeneracy) operators (see for instance [De, 5.1.1]). By abuse of notation, we use the same notation for  $\delta_j, s_j$  and their image by  $\mathcal{X}$ ). The category  $(\Delta_\bullet)$  is generated by the face and degeneracy operators with the following relations (see for instance the proposition VII.5.2, p. 174 of [McL])

$$(4.2.1) \quad \delta_i \circ \delta_j = \delta_{j+1} \circ \delta_i \quad \text{if } i \leq j,$$

$$(4.2.2) \quad s_j \circ s_i = s_i \circ s_{j+1} \quad \text{if } i \leq j,$$

$$(4.2.3) \quad s_j \circ \delta_i = \begin{cases} \delta_i \circ s_{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, i = j + 1, \\ \delta_{i-1} \circ s_j & \text{if } i > j + 1. \end{cases}$$

Therefore, the datum of a cartesian  $\mathcal{O}_{\mathcal{X}_\bullet}$ -module  $\mathcal{L}_\bullet$  is equivalent to the data of isomorphisms

$$\alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1}, \quad j = 0, \dots, n + 1,$$

$$\beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n, \quad j = 0, \dots, n,$$



(where  $n$  is a non negative integer) which are compatible with relations (4.2.1), (4.2.2) and (4.2.32).

Let  $n$  be a non negative integer.

4.3. — We have first to define for  $j = 0, \dots, n + 1$  an isomorphism

$$\alpha_j : \delta_j^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_{n+1}.$$

The line bundle  $\delta_j^* \mathcal{L}_n$  is defined by the morphism  $\ell \circ p_n \circ \delta_j : \mathcal{X}_{n+1} \rightarrow BG_m$  and  $\tilde{p}_n \circ \delta_j$  is associated to the map

$$\begin{cases} \Delta_0 \longrightarrow \Delta_{n+1}, \\ 0 \longmapsto \delta_j(0). \end{cases}$$

• If  $j \neq 0$ , one has therefore  $\tilde{p}_n \circ \delta_j = \tilde{p}_{n+1}$  and  $\delta_j^* \mathcal{L}_n = \mathcal{L}_{n+1}$ . We define  $\alpha_j$  by the identity in this case.

• Suppose now that  $j = 0$ . Let  $\pi_n : \mathcal{X}_n \rightarrow \mathcal{X}_1$  be the projection on the two first factors (associated to the canonical inclusion  $\Delta_1 \hookrightarrow \Delta_n$ ). The commutativity of the two diagrams

$$\begin{array}{ccc} \mathcal{X}_{n+1} & \xrightarrow{\delta_0} & \mathcal{X}_n \\ \pi_{n+1} \downarrow & & p_n \downarrow \\ \mathcal{X}_1 & \xrightarrow{\delta_0} & \mathcal{X} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}_{n+1} & \xrightarrow{p_{n+1}} & \mathcal{X} \\ \downarrow \pi_{n+1} & & \uparrow \delta_1 \\ \mathcal{X}_1 & \xlongequal{\quad} & \mathcal{X}_1 \end{array}$$

allows to reduce the problem to the construction of an isomorphism

$$\delta_0^* \mathcal{L} \xrightarrow{\sim} \delta_1^* \mathcal{L}$$

where  $\delta_i : \mathcal{X}_1 \rightarrow \mathcal{X}$  for  $i = 0, 1$  are the face morphisms or, what amounts to the same, to the construction of a 2-morphism  $\nu : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1$  (the morphism  $\alpha_j$  will be  $\alpha_j = \nu * \pi_{n+1}$ ). Now the diagram

$$(4.3.1) \quad \begin{array}{ccc} & & BG_m \\ & \nearrow \ell \circ m & \uparrow \ell \circ \delta_0 \\ \mathcal{G} \times \mathcal{X} & \xrightarrow{\text{pr}_2 \times m} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \\ & \searrow \ell \circ \text{pr}_2 & \downarrow \ell \circ \delta_1 \\ & & BG_m \end{array}$$

is strictly commutative and  $\text{pr}_2 \times m$  is an equivalence by definition of a torsor. According to lemma 1.2, the 2-morphism  $\lambda$  induces a canonical 2-morphism

$$\lambda * (\text{pr}_2 \times m)^{-1} : \ell \circ \delta_0 \Rightarrow \ell \circ \delta_1$$

which is the required 2-morphism  $\nu$ .

4.4. — We then have to define for  $j = 0, \dots, n$  an isomorphism

$$\beta_j : s_j^* \mathcal{L}_{n+1} \xrightarrow{\sim} \mathcal{L}_n.$$

The line bundle  $s_j^* \mathcal{L}$  is defined by the morphism  $\ell \circ p_{n+1} \circ s_j$  and  $p_{n+1} \circ s_j$  is associated to the canonical inclusion  $\Delta_0 \hookrightarrow \Delta_n$  which means  $p_{n+1} \circ s_j = p_n$ . Therefore, one has a canonical isomorphism  $s_j^* \mathcal{L}_{n+1} = \mathcal{L}_n$  and we define  $\beta_j$  to be the identity.

4.5. — We have to show that the data  $\mathcal{L}_\bullet$  and  $\alpha_j, \beta_j$ , for  $j \geq 0$  define a line bundle on the simplicial stack  $\mathcal{X}_\bullet$  as explained in 4.2. Notice that the fact that the definition of the  $\beta_j$  is compatible with relations (4.2.2) is tautological ( $\beta_j$  is the identity on the relevant  $\mathcal{L}_n$ ).

4.6. — In terms of  $\ell$ , relation (4.2.1) means the following. We have the two strictly 2-commutative diagrams

$$\begin{array}{ccccc} \mathcal{X}_{n+2} & \xrightarrow{\delta_i} & \mathcal{X}_{n+1} & \xrightarrow{\delta_j} & \mathcal{X}_n \\ & \searrow p_{n+2} & \downarrow p_{n+1} & \swarrow p_n & \\ & & \mathcal{X} & \xrightarrow{\ell} & BG_m \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{X}_{n+2} & \xrightarrow{\delta_{j+1}} & \mathcal{X}_{n+1} & \xrightarrow{\delta_i} & \mathcal{X}_n \\ & \searrow p_{n+2} & \downarrow p_{n+1} & \swarrow p_n & \\ & & \mathcal{X} & \xrightarrow{\ell} & BG_m \end{array}$$

inducing the two 2-morphisms

$$\alpha_i \circ (\alpha_j * \delta_i) : \ell \circ p_n \circ \delta_j \circ \delta_i \xrightarrow{\alpha_j * \delta_i} \ell \circ p_{n+1} \circ \delta_i \xrightarrow{\alpha_i} \ell \circ p_{n+2}$$

and

$$\alpha_{j+1} \circ (\alpha_i * \delta_{j+1}) : \ell \circ p_n \circ \delta_i \circ \delta_{j+1} \xrightarrow{\alpha_i * \delta_{j+1}} \ell \circ p_{n+1} \circ \delta_{j+1} \xrightarrow{\alpha_{j+1}} \ell \circ p_{n+2}.$$

The relation (4.2.1) means exactly the equality

$$(4.2.1') \quad \alpha_i \circ (\alpha_j * \delta_i) = \alpha_{j+1} \circ (\alpha_i * \delta_{j+1}), \quad \text{for } i \leq j.$$

- If  $j = 0$ , the relation (4.2.1') is just by definition of  $\alpha_j$  the condition (2.8.1) (see remark 2.9).

- If  $j > 0$ , both isomorphisms  $\alpha_j$  and  $\alpha_{j+1}$  are the relevant identities and the relation (4.2.1') is tautological.

4.7. — The only non tautological relation in (4.2.3) corresponds to the equality  $s_0 \circ \delta_0 = 1$  in  $(\Delta_\bullet)$  which means as before that  $\alpha_0 * \delta_0$  is the identity functor of  $\ell \circ p_n = \ell \circ p_n \circ \delta_0 \circ s_0$ . But, this is exactly the meaning of the relation (2.8.2) (see remark 2.9).

**5. Application to the Picard groups of some moduli spaces.**

Let us choose three integers  $r, s, d$  such that

$$r \geq 2 \quad \text{and} \quad s \mid r \mid ds.$$

If  $G$  is the group  $\mathbf{SL}_r/\mu_s$  we denote as in [B-L-S] by  $\mathcal{M}_G(d)$  the (connected) moduli stack of  $G$ -bundles on  $X$  of degree  $\exp(2i\pi d/r) \in H_{\text{ét}}^2(X, \mu_s) = \mu_s$  and by  $\mathcal{M}_{\mathbf{SL}_r}(d)$  the moduli stack of rank  $r$  vector bundles and determinant  $\mathcal{O}(d \cdot x)$ . If  $r = s$  (i.e.  $G = \mathbf{PSL}_r$ ), the natural morphism of algebraic stacks

$$\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)$$

is a  $\mathcal{J}_r$ -torsor (see the corollary of proposition 2 of [Gr] for instance). Let me explain how to deal with the general case.

5.1. — Let  $E$  be a rank  $r$  vector bundle on  $X_S$  endowed with an isomorphism

$$\tau : D^{r/s} \xrightarrow{\sim} \det(E)$$

where  $D$  is some line bundle. Let me define the  $\mathbf{SL}_r/\mu_s$ -bundle  $\pi(E)$  associated to  $E$  (more precisely to the pair  $(E, \tau)$ ).

DEFINITION 5.2.

• An  $s$ -trivialization of  $E$  on an étale neighborhood  $T \rightarrow X_S$  is a triple  $(M, \alpha, \sigma)$  where

- $\alpha : D \xrightarrow{\sim} M^s$  is an isomorphism ( $M$  is a line bundle on  $T$ );
- $\sigma : M^{\oplus r} \xrightarrow{\sim} E_T$  is an isomorphism;
- $\det(\sigma) \circ \alpha^{r/s} : D^{r/s} \xrightarrow{\sim} \det(E)$  is equal to  $\tau$ .

• Two  $s$ -trivializations  $(M, \alpha, \sigma)$  and  $(M', \alpha', \sigma')$  of  $E$  will be said *equivalent* if there exists an isomorphism  $\iota : M \xrightarrow{\sim} M'$  such that  $\iota^s \circ \alpha = \alpha'$ .

The principal homogeneous space

$$T \longmapsto \{ \text{equivalence classes of } s\text{-trivializations of } E_T \}$$

defines the  $\mathbf{SL}_r/\mu_s$ -bundle  $\pi(E)^\dagger$ . Now, the construction is obviously functorial and therefore defines the morphism  $\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \rightarrow \mathcal{M}_G(d)$  (observe that an object  $E$  of  $\mathcal{M}_{\mathbf{SL}_r}(d)$  has determinant  $\mathcal{O}(ds/r \cdot x)^{r/s}$ ). Let  $L$  be a line bundle and  $(M, \alpha, \tau)$  an  $s$ -trivialization of  $E_T$ . Then,  $(M \otimes L, \alpha \otimes \text{Id}_{L^s}, \sigma \otimes \text{Id}_L)$  is an  $s$ -trivialization of  $E \otimes L$  (which has determinant  $(D \otimes L^s)^{r/s}$ ). This shows that there exists a canonical functorial isomorphism

$$(5.2.1) \quad \pi(E) \xrightarrow{\sim} \pi(E \otimes L)$$

In particular,  $\pi$  is  $\mathcal{J}_s$ -equivariant.

LEMMA 5.3. — *The natural morphism of algebraic stacks*

$$\pi : \mathcal{M}_{\mathbf{SL}_r}(d) \longrightarrow \mathcal{M}_G(d)$$

is a  $\mathcal{J}_s$ -torsor.

*Proof.* — Let  $E, E'$  be two rank  $r$  vector bundles on  $X_S$  (with determinant equal to  $\mathcal{O}(d \cdot x)$ ) and let  $\iota : \pi(E) \xrightarrow{\sim} \pi(E)'$  be an isomorphism. As in the proof of the lemma 13.4 of [B-L-S], we have the exact sequence of sets

$$1 \rightarrow \mu_s \longrightarrow \text{Isom}(E, E') \longrightarrow \text{Isom}(\pi(E), \pi(E)') \xrightarrow{\pi_{E, E'}} H_{\text{ét}}^1(X_S, \mu_s).$$

Let  $L$  be a  $\mu_s$ -torsor such that  $\pi_{E, E'}(\iota) = [L]$ . Then,  $\pi(E \otimes L)$  is equal to  $\pi(E)$ ,  $\pi_{E \otimes L, E'} = 0$  and  $\iota$  is induced by an isomorphism  $E \otimes L \xrightarrow{\sim} E'$  well defined up to multiplication by  $\mu_s$ . The lemma follows.  $\square$

5.4. — Let  $\mathcal{U}$  be the universal bundle on  $X \times \mathcal{M}_{\mathbf{SL}_r}(d)$ . We would like to know which power of the determinant bundle  $\mathcal{D} = (\det R\Gamma\mathcal{U})^{-1}$  on  $\mathcal{M}_{\mathbf{SL}_r}(d)$  descends to  $\mathcal{M}_G(d)$ . As in I.3 of [B-L-S], the rank  $r$  bundle

$$\mathcal{F} = \mathcal{L}^{\oplus(r-1)} \oplus \mathcal{L}^{1-r}(d \cdot x)$$

on  $X \times \mathcal{J}(X)$  has determinant  $\mathcal{O}(d \cdot x)$  and therefore defines a morphism

$$f : \mathcal{J}(X) = JX \times BG_m \longrightarrow \mathcal{M}_{\mathbf{SL}_r}(d)$$

which is  $\mathcal{J}_s$ -equivariant.

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<sup>†</sup> We see here a  $G$ -bundle as a formal homogeneous space under  $G$ .

The vector bundle

$$\mathcal{F}' = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{L}^{-r/s}(d \cdot x)$$

on  $X \times \mathcal{J}(X)$  has determinant  $[\mathcal{L}^{-1}(ds/r \cdot x)]^{r/s}$ . The  $G$ -bundle  $\pi(\mathcal{F}')$  on  $X \times \mathcal{J}(X)$  defines a morphism  $f' : \mathcal{J} \rightarrow \mathcal{M}_G(d)$ . The relation

$$\mathcal{L} \otimes (\text{Id}_X \times s_{\mathcal{J}})^* * (\mathcal{F}') = \mathcal{F}$$

and (5.2.1) gives an isomorphism

$$\pi(\mathcal{F}) = (\text{Id}_X \times s_{\mathcal{J}})^* \pi(\mathcal{F}')$$

which means that the diagram

$$(5.4.1) \quad \begin{array}{ccc} \mathcal{J}(X) & \xrightarrow{f} & \mathcal{M}_{\mathbf{SL}_r}(d) \\ s_{\mathcal{J}} \downarrow & & \downarrow \pi \\ \mathcal{J}(X) & \xrightarrow{f'} & \mathcal{M}_G(d) \end{array}$$

is 2-commutative. Exactly as in I.3 of [B-L-S], let me prove the

LEMMA 5.5. — *The line bundle  $f^* \mathcal{D}^k$  on  $\mathcal{J}(X)$  descends through  $s_{\mathcal{J}}$  if and only if  $k$  multiples of  $s/(s, r/s)$ .*

*Proof.* — Let  $\chi = r(g-1) - d$  be the opposite of the Euler characteristic of  $(\mathbf{k})$ -points of  $\mathcal{M}_{\mathbf{SL}_r}(d)$ . By (3.1.1), one has an isomorphism

$$f^* \mathcal{D}^k \xrightarrow{\sim} \Theta^{kr(r-1)} \boxtimes \mathcal{T}^{k\chi}.$$

The theory of Mumford groups says that  $\Theta^{kr(r-1)}$  descends through  $s_{\mathcal{J}}$  if and only if  $k$  is a multiple of  $s/(s, r/s)$ . The line bundle  $\mathcal{T}^{k\chi}$  on  $BG_m$  descends through  $s_{BG_m}$  if and only if  $k\chi$  is a multiple of  $s$ . The lemma follows from the above isomorphism and from the observation that the condition  $s \mid r \mid ds$  forces  $s\chi$  to be a multiple of  $s$ .  $\square$

REMARK 5.6. — If  $g = 0$ , the jacobian  $J$  is a point and the condition on  $\Theta$  is empty. The only condition in this case is that  $k\chi$  is a multiple of  $s$ .

Let me recall that  $\mathcal{D}$  is the determinant bundle on  $\mathcal{M}_{\mathbf{SL}_r}(d)$  and  $G = \mathbf{SL}_r/\mu_s$ .

THEOREM 5.7. — *Assume that the characteristic of  $\mathbf{k}$  is 0. The integers  $k$  such that  $\mathcal{D}^k$  descends to  $\mathcal{M}_G(d)$  are the multiple of  $s/(s, r/s)$ .*

By the proposition 1.5 of [BLS], one gets the

COROLLARY 5.8. — *The natural morphism*

$$\text{Pic}(\mathcal{M}_G(d)) \longrightarrow \text{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d)) = \mathbb{Z} \cdot \mathcal{D}$$

makes the Picard group of  $\mathcal{M}_G(d)$  an extension of  $\mathbb{Z} = \mathbb{Z} \cdot \mathcal{D}^{s/(s,r/s)}$  by  $H_{\text{ét}}^1(X, \mathbb{Z}/d\mathbb{Z}) \xrightarrow{\sim} (\mathbb{Z}/d\mathbb{Z})^{2g}$ .

*Proof of the theorem.* — By lemma 5.5 and diagram (5.4.1), we just have to prove that  $\mathcal{D}^k$  effectively descends when  $k = s/(s, r/s)$ . By theorem 4.1 and lemma 5.3, this means exactly that  $\mathcal{D}^k$  has a  $\mathcal{J}_s$ -linearization. We know by lemma 5.5 that the pull-back  $f^*\mathcal{D}^k$  has such a linearization.

LEMMA. — *The pull-back morphism*

$$\text{Pic}(\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d)) \longrightarrow \text{Pic}(\mathcal{J}_s \times \mathcal{J}(X))$$

is injective.

*Proof.* — By lemma 3.1, one is reduced to prove that the natural morphism

$$\text{Pic}(B\mu_s \times \mathcal{M}_{\mathbf{SL}_r}(d)) \longrightarrow \text{Pic}(B\mu_s \times \mathcal{J}(X))$$

is injective. Let  $\mathcal{X}$  be any stack. The canonical morphism  $\mathcal{X} \rightarrow \mathcal{X} \times B\mu_s$  is a  $\mu_s$ -torsor (with the trivial action of  $\mu_s$  on  $\mathcal{X}$ ). By theorem 4.1, one has the equality

$$\text{Pic}(\mathcal{X} \times B\mu_s) = \text{Pic}^{\mu_s}(\mathcal{X}).$$

Assume further that  $H^0(\mathcal{X}, \mathcal{O}) = \mathbf{k}$ . The later group is then canonically isomorphic to

$$\text{Pic}(\mathcal{X}) \times \text{Hom}(\mu_s, G_m) = \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s).$$

Eventually, we get a functorial isomorphism

$$(5.9.1) \quad \iota_{\mathcal{X}} : \text{Pic}(\mathcal{X} \times B\mu_s) \xrightarrow{\sim} \text{Pic}(\mathcal{X}) \times \text{Pic}(B\mu_s).$$

By [L-S], the Picard group of  $\mathcal{M}_{\mathbf{SL}_r}(d)$  is the free abelian group  $\mathbb{Z} \cdot \mathcal{D}$  and the formula (3.1.1) proves that

$$f^* : \text{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d)) \longrightarrow \text{Pic}(\mathcal{J}(X))$$

is an injection. The diagram

$$\begin{array}{ccc}
 \text{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d)) \times \text{Pic}(B\mu_s) & \hookrightarrow & \text{Pic}(\mathcal{J}(X)) \times \text{Pic}(B\mu_s) \\
 \downarrow \iota & & \downarrow \iota \\
 \text{Pic}(\mathcal{M}_{\mathbf{SL}_r}(d) \times B\mu_s) & \longrightarrow & \text{Pic}(\mathcal{J}(X) \times B\mu_s)
 \end{array}$$

is commutative and the lemma follows from this commutative diagram.  $\square$

Let  $\mathcal{H}$  (resp.  $\mathcal{H}_{\mathcal{J}}$ ) be the line bundle on  $\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d)$  (resp.  $\mathcal{J}_s \times \mathcal{J}(X)$ )

$$\mathcal{H} = \text{Hom}(m_{\mathcal{M}}^* \mathcal{D}^k, \text{pr}_2^* \mathcal{D}^k)$$

$$(\text{resp. } \mathcal{H}_{\mathcal{J}} = \text{Hom}(m_{\mathcal{M}}^* f^* \mathcal{D}^k, \text{pr}_2^* f^* \mathcal{D}^k)).$$

Let us choose a  $\mathcal{J}_s$ -linearization  $\lambda_{\mathcal{J}}$  of  $f^* \mathcal{D}^k$ . It defines a trivialization of the line bundle  $\mathcal{H}_{\mathcal{J}}$ . The equivariance of  $f$  implies (cf. 2.3) that there exists a (compatible) 2-morphism

$$q : m_{\mathcal{M}} \circ (\text{Id} \times f) \implies f \circ m_{\mathcal{J}}$$

making the diagram

$$\begin{array}{ccc}
 \mathcal{J}_s \times \mathcal{J}(X) & \xrightarrow{m_{\mathcal{J}}} & \mathcal{J}(X) \\
 \text{Id} \times f \downarrow & & \downarrow f \\
 \mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d) & \xrightarrow{m_{\mathcal{M}}} & \mathcal{M}_{\mathbf{SL}_r}(d)
 \end{array}$$

2-commutative. The 2-morphism  $q$  defines an isomorphism from the pull-back  $m_{\mathcal{M}}^* \mathcal{D}^k$  on  $\mathcal{J}_s \times \mathcal{J}(X)$  to  $m_{\mathcal{J}}^*(f^* \mathcal{D}^k)$ . The pull-back of  $\text{pr}_2^* \mathcal{D}^k$  on  $\mathcal{J}_s \times \mathcal{J}(X)$  is tautologically isomorphic to  $\text{pr}_2^*(f^* \mathcal{D}^k)$ . The preceding isomorphisms induce an isomorphism

$$(\text{Id} \times f)^* \mathcal{H} \xrightarrow{\sim} \mathcal{H}_{\mathcal{J}}.$$

The later line bundle being trivial, so is  $(\text{Id} \times f)^* \mathcal{H}$ . The lemma above proves therefore that  $\mathcal{H}$  itself is *trivial*. Each ( $\mathbf{k}$ -)point  $j$  of  $\mathcal{J}_s$  defines a morphism

$$\mathcal{M}_{\mathbf{SL}_r}(d) \rightarrow \mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d) \quad (\text{resp. } \mathcal{J}(X) \rightarrow \mathcal{J}_s \times \mathcal{J}(X));$$

let me denote by  $\mathcal{H}_j$  (resp.  $f^* \mathcal{H}_j$ ) the pull-back of  $\mathcal{H}$  (resp.  $(\text{Id} \times f)^* \mathcal{H}$ ) by this morphism. The pull-back morphism

$$H^0(\mathcal{J}_s \times \mathcal{M}_{\mathbf{SL}_r}(d), \mathcal{H}) \longrightarrow H^0(\mathcal{J}_s \times \mathcal{J}(X), (\text{Id} \times f)^* \mathcal{H})$$

can be identified to the direct sum

$$\bigoplus_{j \in \mathcal{J}_s(\mathbf{k})} H^0(\mathcal{M}_{\mathbf{SL}_r}(d), \mathcal{H}_j) \longrightarrow H^0(\mathcal{J}(X), f^*\mathcal{H}_j).$$

As

$$(5.9.2) \quad H^0(\mathcal{M}_{\mathbf{SL}_r}(d), \mathcal{O}) = H^0(\mathcal{J}(X), \mathcal{O}) = \mathbf{k},$$

this morphism is a direct sum of non-zero homomorphisms of vector spaces of dimension 1 and therefore an isomorphism. In particular, a linearization  $\lambda_{\mathcal{J}}$  of  $f^*\mathcal{D}^k$  defines canonically an isomorphism

$$\lambda_{\mathcal{M}} : m_{\mathcal{M}}^* \mathcal{D}^k \xrightarrow{\sim} \text{pr}_2^* \mathcal{D}^k$$

such that  $(\text{Id} \times f)^* \lambda_{\mathcal{M}} = \lambda_{\mathcal{J}}$ .

Explicitly,  $\lambda_{\mathcal{M}}$  is characterized as follows. Let  $x$  be an object of  $\mathcal{M}_{\mathbf{SL}_r}(d)$  over a connected scheme  $S$  and  $g$  an object of  $\mathcal{J}_s(S) = \mathcal{J}_s(\mathbf{k})$ . The preceding discussion means that the functorial isomorphisms

$$\lambda_{\mathcal{M}}(g, x) : \mathcal{D}_{g \cdot x}^k \xrightarrow{\sim} \mathcal{D}_x^k$$

are determined when  $x$  lies in the essential image of  $f$ . In this case, let us choose an isomorphism  $f(x') \xrightarrow{\sim} x$  (inducing an isomorphism  $g \cdot f(x') \xrightarrow{\sim} g \cdot x$ ). Then, the diagram of isomorphisms of line bundles on  $S$

$$\begin{array}{ccccc} L'_{x'} = L_{f(x')} & \longrightarrow & L_x & & \\ \lambda_{\mathcal{J}}(g, x') \downarrow & & & \swarrow \lambda_{\mathcal{M}}(g, x) & \\ L'_{g \cdot x'} = L_{f(g \cdot x')} & \xrightarrow{q_{g, x'}} & L_{g \cdot f(x')} & \longrightarrow & L_{g \cdot x} \end{array}$$

is commutative (where  $L = \mathcal{D}^k$  and  $L' = f^*\mathcal{D}^k$ ).

Now, the pull-back of  $\lambda_{\mathcal{M}}$  on  $\mathcal{J}_s \times \mathcal{J}(X)$  satisfies conditions (2.8.1) and (2.8.2). Using (5.9.2) and the equivariance of  $f$  as above, this shows that  $\lambda_{\mathcal{M}}$  is a linearization. For instance, keeping the notation above, let us verify condition (2.8.2). We have to check that the isomorphism  $\iota$  of  $L$  induced by  $\epsilon$  is the identity. As above, it is enough to check that on  $\mathcal{J}(X)$ . With a slight abuse of notations, the two diagrams

$$\begin{array}{ccccc} L'_{x'} = L_{f(x')} & \longrightarrow & L_x & & \\ \uparrow \lambda_{\mathcal{J}}(1, x') & & & \swarrow \lambda_{\mathcal{M}}(1, x) & \\ L'_{1 \cdot x'} = L_{f(1 \cdot x')} & \xrightarrow{q_{1, x'}} & L_{1 \cdot f(x')} & \longrightarrow & L_{1 \cdot x} \end{array} \quad \text{and} \quad \begin{array}{ccc} L_x & \xrightarrow{\iota} & L_x \\ \swarrow \lambda_{\mathcal{M}}(1, x) & & \downarrow \epsilon(x) \\ L_{1 \cdot x} & & L_{1 \cdot x} \end{array}$$



are commutative (the commutativity of the first diagram follows from the equivariance of  $f$  and the commutativity of the second diagram follows exactly from the definition of  $\iota$ ). Because  $\lambda_{\mathcal{J}}$  is a linearization, condition (2.8.2) shows that the diagram

$$\begin{array}{ccc}
 L'_{x'} & \xlongequal{\quad} & L'_{x'} \\
 \swarrow \lambda_{\mathcal{J}}(1, x') & & \uparrow \epsilon'(x') \\
 & & L_{1 \cdot x'}
 \end{array}$$

is commutative. It follows that the equality  $\iota = \text{Id}$  will follow from the commutativity of the diagram

$$(5.9.3) \quad \begin{array}{ccc}
 L_{f(1 \cdot x')} & \xrightarrow{\epsilon'} & L_{f(x')} \\
 q_{1, x'} \downarrow & & \parallel \\
 L_{1 \cdot f(x')} & \xrightarrow{\epsilon} & L_{f(x')}
 \end{array}$$

But  $q$  is compatible with  $\epsilon$  as required in 2.3. The diagram

$$\begin{array}{ccc}
 f(1 \cdot x') & \xrightarrow{\epsilon'} & f(x') \\
 q_{1, x'} \downarrow & & \parallel \\
 1 \cdot f(x') & \xrightarrow{\epsilon} & f(x')
 \end{array}$$

is therefore commutative from which the commutativity of (5.9.3) follows. One would check condition (2.8.1) in an analogous way.  $\square$

REMARK 5.10. — In the case  $g = 0$ , the condition is an in remark 5.6.

REMARK 5.11. — This linearization can be certainly also deduced from a careful analysis of the first section of [Fa], but the method above seems simpler.

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