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HOLOMORPHIC FUNCTIONS ON FULLY NUCLEAR SPACES

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RÉSUMÉ. — Un espace localement convexe E est dit pleinement nucléaire si E et E'_β sont des espaces nucléaires complets et réflexifs. Soient E un espace pleinement nucléaire admettant une base de Schauder, et U un polydisque ouvert de E; alors $H(U)$ désigne l'espace des fonctions holomorphes sur U. On montre et applique que les monômes z^m forment une base absolue dans $H(U)$ pour la topologie τ_0 de la convergence compacte et pour la topologie τ_ω de L. NACHBIN.

ABSTRACT. — A locally convex space E is fully nuclear if both E and E'_β are complete reflexive nuclear spaces. If E is a fully nuclear space with a Schauder basis, and U is an open polydisc in E, then $H(U)$ is the space of holomorphic functions on U. We show and apply the result that the monomials z^m form an absolute basis in $H(U)$ for the topology τ_0 of uniform convergence on compact subsets and for the L. NACHBIN topology τ_ω.

If $U \subset C^n$ is a Reinhardt domain (i. e. a connected open set such that $z = (z_1, \ldots, z_n) \in U$ if and only if $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in U$ for all $(\theta_1, \ldots, \theta_n) \in R^n$) containing 0, and f is a holomorphic function on U, then $f(z) = \sum_{m \in N^n} a_m z^m$ where the coefficients are uniquely determined and the series converges normally in $H(U)$. Since $H(U)$, the set of all holomorphic functions on U, is a Fréchet nuclear space when endowed with the topology of compact convergence τ_0 this says that the monomials $(z^m)_{m \in N^n}$ form an absolute basis for $(H(U), \tau_0)$.

In this article, we introduce the class of fully nuclear locally convex spaces, and show that a result similar to the above holds for certain subsets of fully nuclear locally convex spaces with an equicontinuous basis. Using this result, we characterize $(H(U), \tau_0)'$ algebraically as a space of holomorphic germs and topologically as an inductive limit of Banach spaces. This characterization allows us to compare different topologies on $H(U)$, and in this way, we partially answer a question of BIERSTEDT and MEISE [4]. In the final section, we investigate entire functions on a fully nuclear space, and prove in particular that $\tau_0 = \tau_\omega$ on $H(E)$ for any Fréchet nuclear or dual of Fréchet nuclear space E.

We refer to [15] and [19] for the general theory of locally convex spaces, to [16] and [23] for the theory of nuclear spaces, to ([1], [10], [11], [20], [21], [22]) for the theory of holomorphic functions on locally convex spaces, and to ([6], [7], [9]) for the theory of holomorphic functions on nuclear spaces.

1. Fully nuclear spaces

In this section, we develop the linear properties of locally convex spaces that we shall use in the remaining sections. All locally convex spaces are over the field of complex numbers.

DEFINITION 1. — A sequence of elements, $(e_n)_{n=1}^{\infty}$, in a locally convex space E, is called a *basis* if, for each $x \in E$, there is a unique sequence of scalars, $(x_n)_{n=1}^{\infty}$, such that

$$x = \lim_{m \to \infty} \sum_{n=1}^{m} x_n e_n .$$

The correspondence $x \to x_n$ defines a linear functional on E, e'_n, and if each e'_n is continuous, we call $(e_n)_{n=1}^{\infty}$ a *Schauder* basis. The basis is said to be *equicontinuous* if the corresponding sequence of finite dimensional projections $(S_n)_{n=1}^{\infty}$ $(S_n(x) = \sum_{m=1}^{n} x_m e_m)$ belongs to $\mathscr{L}(E, E)$ and is equicontinuous. A basis, $(e_n)_{n=1}^{\infty}$, is called an *absolute* basis if, for each absolutely convex neighbourhood U of 0 in E, there exists an absolutely convex neighbourhood V of 0 such that

$$\sum_{n=1}^{\infty} |e'_n(x)| p_U(e_n) \leqslant p_V(x) \quad \text{for all } x \text{ in } E,$$

where p_U and p_V denote the usual Minkowski semi-norms associated with U and V.

Remark. — An absolute basis is an equicontinuous basis. Every Schauder basis in a barrelled locally convex space is an equicontinuous basis and every equicontinuous basis in a nuclear space is an absolute basis [23].

DEFINITION 2 [19]. — Let P be a collection of sequences, $(\alpha_n)_{n=1}^{\infty}$ of non negative real numbers such that, for each $r \in N$, there exists $\alpha = (\alpha_n)_{n=1}^{\infty} \in P$ such that $\alpha_r > 0$.

The sequence space $\Lambda(P)$ is the set of all sequences of complex numbers, $(x_n)_{n=1}^{\infty}$, such that

$$\sum_{n=1}^{\infty} |x_n| \alpha_n < \infty \quad \text{for all } \alpha = (\alpha_n)_{n=1}^{\infty} \in P.$$

We endow $\Lambda(P)$ with the topology defined by the semi-norms p_α, $\alpha = (\alpha_n)_{n=1}^\infty \in P$, where

$$(\star) \qquad p_\alpha((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty \alpha_n |x_n|.$$

We shall assume that P is complete in the following sense; if $\alpha = (\alpha_n)_{n=1}^\infty$ is any sequence of non-negative real numbers such that

$$p_\alpha : \ (x_n)_{n=1}^\infty \to \sum_{n=1}^\infty \alpha_n |x_n|$$

is continuous, then $\alpha \in P$. $\Lambda(P)$ is a complete locally convex space. We state the following basic result concerning nuclear sequence spaces ([19], [14]).

THEOREM 3. — *The locally convex space $\Lambda(P)$ is nuclear if, and only if, for each $(\alpha_n)_{n=1}^\infty \in P$ there exists $(u_n)_{n=1}^\infty \in l_1$ and $(\alpha'_n)_{n=1}^\infty \in P$ such that $\alpha_n \leqslant |u_n| . \alpha'_n$ for all n.*

Using this result, we obtain another representation of nuclear sequence spaces. If $\Lambda(P)$ is nuclear then

$$\begin{aligned}\Lambda(P) &= \{(x_n)_{n=1}^\infty ; \sup_n |x_n| \alpha_n < \infty, \text{ for all } (\alpha_n)_{n=1}^\infty \in P\} \\ &= \{(x_n)_{n=1}^\infty ; |x_n| \alpha_n \to 0 \text{ as } n \to \infty \text{ for all } \alpha = (\alpha_n)_{n=1}^\infty \in P\}.\end{aligned}$$

Furthermore the topology of $\Lambda(P)$ is also generated by all seminorms of the form

$$\| (x_n)_{n=1}^\infty \|_{(\alpha_n)_{n=1}^\infty} = \sup_n |x_n \alpha_n|,$$

where $\alpha = (\alpha_n)_{n=1}^\infty$ ranges over P.

Any complete nuclear space E, with an equicontinuous basis, can be identified with a complete nuclear sequence space $\Lambda(P)$. To see this, let $(e_n)_{n=1}^\infty$ be an equicontinuous and hence absolute basis for E. Then if $P = \{ (p_U(e_n))_{n=1}^\infty ; \ U$ an absolutely convex neighbourhood of zero $\}$, we have $E \simeq \Lambda(P)$.

We now define fully nuclear spaces.

DEFINITION 4. — A locally convex space E is a *fully nuclear space* if both E and E'_β (the strong dual of E) are complete reflexive nuclear spaces. A locally convex space E is a *fully nuclear space with a basis* if it is fully nuclear and has a Schauder (and hence equicontinuous and absolute) basis.

Remarks.

1° A quasi-complete nuclear space is always semi-reflexive and hence it is reflexive if, and only if, it is infrabarrelled.

2° If E is a complete nuclear space with a Schauder basis, then E'_β also has a Schauder basis. Hence the strong dual of a fully nuclear space with a basis is a fully nuclear space with a basis.

3° Any Fréchet nuclear or DFN (strong dual of a Fréchet nuclear space) space is fully nuclear. The space of distributions $\mathcal{D}$ of Schwartz is fully nuclear. If E is a Fréchet nuclear space or a DFN space with a Schauder basis, then E is a fully nuclear space with a basis.

4° If E is fully nuclear with a basis, we fix, once and for all, a representation of E and E'_β as sequence spaces $\Lambda(P)$ and $\Lambda(P')$ and denote the duality between E and E'_β as follows:

$$\omega(z) = \langle \omega, z \rangle = \langle (\omega_n)_{n=1}^\infty, (z_n)_{n=1}^\infty \rangle = \sum_{n=1}^\infty \omega_n z_n$$

where $z \in E$ and $\omega \in E'_\beta$.

5° We prove most of our results for holomorphic functions defined on open subsets of fully nuclear spaces with bases. This provides a framework for a reasonably clear presentation and avoids technical discussions. We could however, weaken our hypothesis on a number of occasions but the resulting gain in generality does not appear to lead to any significant new examples.

PROPOSITION 5. — *Let U be a neighbourhood of zero in the nuclear space $\Lambda(P)$. Then there exists an absolutely convex neighbourhood V of zero and a sequence $\delta = (\delta_n)_{n=1}^\infty$ where $\delta_n > 1$ for all n and $1/\delta = \sum_{n=1}^\infty 1/\delta_n < +\infty$, such that*

$$\delta V = \{(\delta_n x_n)_{n=1}^\infty; (x_n)_{n=1}^\infty \in V, (\delta_n x_n)_{n=1}^\infty \in \Lambda(P)\} \subset U.$$

Proof. — Without loss of generality, we may assume that

$$U = \{(x_n)_{n=1}^\infty; \sup_n |x_n \alpha_n| < \varepsilon\} \qquad \text{where} \quad (\alpha_n)_{n=1}^\infty \in P.$$

Now let $(\alpha'_n)_{n=1}^\infty \in P$ and $(u_n)_{n=1}^\infty \in l_1$ be such that $\alpha'_n > \alpha_n$ for all n and $\alpha_n \leqslant |u_n| \alpha'_n$ for all n. Let $V = \{(x_n)_{n=1}^\infty; \sup_n |x_n \alpha'_n| < \varepsilon\}$ and le $\delta_n = \alpha'_n/\alpha_n$ if $\alpha_n \neq 0$, and $\delta_n = 2^n$ otherwise. Clearly $\delta_n > 1$ for all n and

$$\sum_{n=1}^\infty \frac{1}{\delta_n} \leqslant \sum_{n=1}^\infty \frac{1}{2^n} + \sum_{n=1}^\infty |u_n| < \infty.$$

Furthermore, if $(x_n)_{n=1}^\infty \in V$, then

$$\sup_n |\delta_n x_n \alpha_n| \leqslant \sup_n |x_n \alpha'_n| < \varepsilon$$

and hence $\delta V \subset U$.

DEFINITION 6. — Let $E = \Lambda(P)$ denote a sequence space, and let A denote a subset of E.

(*a*) A is said to be *Reinhardt* if whenever $z = (z_n)_{n=1}^{\infty} \in A$ and

$$(\theta_n)_{n=1}^{\infty} \in R^N, \qquad \text{then} \quad (e^{i\theta_n} z_n)_{n=1}^{\infty} \in A.$$

(*b*) A is said to be *modularly decreasing* if $(z_n)_{n=1}^{\infty} \in A$ then

$$(y_n)_{n=1}^{\infty} \in A \quad \text{whenever} \quad |y_n| \leqslant |z_n| \quad \text{for all } n.$$

The Reinhardt hull and the modularly decreasing hull of arbitrary subsets of $\Lambda(P)$ are defined in an obvious way.

If E is a complete nuclear space with an equicontinuous basis then we say a subset A of E is Reinhardt (resp. modularly decreasing) if A is Reinhardt (resp. modularly decreasing) when identified with a subset of $\Lambda(P)$ as previously described. We refer to [17] for further information concerning Reinhardt domains in the theory of infinite dimensional holomorphy.

PROPOSITION 7. — *Let $\Lambda(P)$ be a reflexive nuclear space and let $U \subset (\Lambda(P))'_\beta$ be an open modularly decreasing set. Then if B is a compact subset of U there exists a $\delta = (\delta_n)_{n=1}^{\infty}$ where $\delta_n > 1$ for all n and $\sum_{n=1}^{\infty} 1/\delta_n < +\infty$ such that $\delta B = \{ (\delta_n z_n)_{n=1}^{\infty};\ z = (z_n)_{n=1}^{\infty} \in B \}$ is a relatively compact subset of U.*

Proof. — Without loss of generality, B is a modularly decreasing set.

Since $\Lambda(P)$ is infrabarrelled, every bounded subset of $(\Lambda(P))'_\beta$ is equicontinuous and hence we can find a sequence $(\alpha_n)_{n=1}^{\infty}$ in P such that

$$B \subset \{(z_n)_{n=1}^{\infty};\ \textstyle\sum_{n=1}^{\infty} |z_n \alpha_n| \leqslant 1\}^0 = \{(\omega_n)_{n=1}^{\infty};\ |\omega_n| \leqslant \alpha_n \text{ for all } n\}.$$

Now choose $(\alpha'_n)_{n=1}^{\infty} \in P$ and $(u_n)_{n=1}^{\infty} \in l_1$ such that $\alpha_n \leqslant |u_n|\, \alpha'_n$ for all n and

$$B + \varepsilon\{(z_n)_{n=1}^{\infty};\ \textstyle\sum_{n=1}^{\infty} |z_n \alpha'_n| \leqslant 1\}^0 \subset U.$$

Hence $B' = B + \varepsilon \{ (\omega_n)_{n=1}^{\infty};\ |\omega_n| \leqslant |\alpha'_n| \}$ is a relatively compact subset of U.

Let

$$\delta_n = \begin{cases} 2^n & \text{if } \alpha_n = 0, \\ 1 + \dfrac{\varepsilon \alpha'_n}{\alpha_n} & \text{if } \alpha_n \neq 0. \end{cases}$$

Then $\delta_n > 1$ for all n and

$$\sum_{n=1}^{\infty} \frac{1}{\delta_n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{\alpha_n \neq 0} \frac{1}{1+(\varepsilon\alpha_n'/\alpha_n)}$$

$$\leqslant 1+\sum_{\alpha_n \neq 0} \frac{\alpha_n}{\varepsilon\alpha_n'}$$

$$\leqslant 1+\sum_{\alpha_n \neq 0} \frac{|u_n|}{\varepsilon} < +\infty.$$

Moreover, if $\omega = (\omega_n)_{n=1}^{\infty} \in B$ then $|\omega_n \delta_n| \leqslant |\omega_n| + \varepsilon\alpha_n'$ and hence $\delta B \subset B'$. This completes the proof.

Subsets of $\Lambda(P)$, a nuclear sequence space, which have either of the following forms

$$A = \{(z_n)_{n=1}^{\infty} \in \Lambda(P);\ \sup_n |z_n \alpha_n| < 1\}$$

or

$$B = \{(z_n)_{n=1}^{\infty} \in \Lambda(P);\ \sup_n |z_n \alpha_n| \leqslant 1\},$$

where $\alpha_n \in [0, +\infty]$ for all n and $a.(+\infty) = +\infty$ if $a > 0$, are called polydiscs.

It is immediate that the polydisc A is open if and only if $(\alpha_n)_{n=1}^{\infty} \in P$, and the polydisc B is always closed. We note that $\Lambda(P)$ is an open polydisc, and 0 is a compact polydisc. Since every fully nuclear space with a basis is a nuclear sequence space, this defines polydiscs in fully nuclear spaces with a basis.

DEFINITION 8. — If E is a fully nuclear space with a basis and $A \subset E$, we define A^M (the multiplicative polar of A) as

$$A^M = \{(\omega_n)_{n=1}^{\infty} \in E'_\beta;\ \sup_n |\omega_n z_n| \leqslant 1 \text{ for all } z = (z_n)_{n=1}^{\infty} \in A\}.$$

It is immediate that A^M is a closed modularly decreasing subset of E'_β, and

$$A^{MM} = (A^M)^M = \{(z_n)_{n=1}^{\infty} \in E;\ \sup |\omega_n z_n| \leqslant 1 \text{ for all } (\omega_n)_{n=1}^{\infty} \in A^M\}$$

is a closed subset of E which contains A.

LEMMA 9. — *Let U be an open polydisc in a fully nuclear space with a basis, E. Then U^M is a compact polydisc in E'_β. Moreover U contains a undamental system of compact sets consisting of compact polydiscs, and the* *open polydiscs containing U^M form a fundamental neighbourhood system for U^M. If $K(U)$ denotes the set of compact polydiscs in U, then the mapping*

$$K \in K(U) \to \text{Interior}(K^M)$$

defines a $(1-1)$*-correspondence between the compact polydiscs in U and the open polydiscs containing U^M.*

Proof. — Let $U = \{ (z_n)_{n=1}^{\infty} \in E;\ \sup | z_n \alpha_n | < 1 \}$ for some $\alpha = (\alpha_n)_{n=1}^{\infty} \in P$, and let $V = \{ (z_n)_{n=1}^{\infty} \in E;\ \sum_{n=1}^{\infty} | z_n \alpha_n | < 1 \}$. Then

$$U^M = V^0 = \{(\omega_n)_{n=1}^{\infty} \in E'_\beta;\ |\omega_n| \leqslant \alpha_n \text{ for all } n\}.$$

Since E is complete and dual nuclear, it follows that U^M is a compact polydiscs in E'_β.

Now let W denote a neighbourhood of U^M in E'_β. Thus we can choose $(\alpha'_n)_{n=1}^{\infty} \in P'$ such that

$$\begin{aligned} W &\supset U^M + \{(\omega_n)_{n=1}^{\infty} \in E'_\beta : \sup | \omega_n \alpha'_n | < 1\} \\ &= \{(\omega_n)_{n=1}^{\infty} \in E'_\beta : \sup | \omega_n \beta_n | < 1\}, \end{aligned}$$

where

$$\beta_n = \begin{cases} 0 & \text{if } \alpha'_n = 0, \\ \dfrac{1}{\alpha_n + (1/\alpha'_n)} & \text{if } \alpha'_n \neq 0. \end{cases}$$

Since $(\beta_n)_{n=1}^{\infty} \in P'$, it follows that U^M has a fundamental neighbourhood system consisting of open polydiscs. With the above notation we see that $W^M \subset \{ (z_n)_{n=1}^{\infty} \in E, | z_n | \leqslant \beta_n \text{ for all } n \}$, a compact polydisc in E. Since U^M is a compact subset of W we can choose $\lambda > 1$ such that $\lambda\, U^M \subset W$. Hence $W^M \subset (1/\lambda)\,(U^M)^M = (1/\lambda)\,\overline{U}$, and therefore $W^M \subset \text{Interior}\,\overline{U} = U$. Thus W^M is a compact subset of U.

Now suppose K is a compact subset of U. We may assume without loss of generality that K is modularly decreasing. Since $K^0 \subset K^M$, it follows that K^M is a neighbourhood of 0 in E'_β. We choose $\lambda > 1$ such that $\lambda\, K \subset U$. Then $(\lambda\, K)^M = (1/\lambda)\, K^M \supset U^M$, and hence

$$K^M = \frac{1}{\lambda} K^M + \left(1 - \frac{1}{\lambda}\right) K^M \supset U^M + \left(1 - \frac{1}{\lambda}\right) K^M.$$

Since $(1-(1/\lambda))\, K^M$ is a neighbourhood of 0 in E'_β, K^M is a neighbourhood of U^M. From the above $K \subset (K^M)^M$ and since $(K^M)^M$ is a compact polydisc in U, U contains a fundamental system of compact polydiscs.

Now a compact $K \subset U$ is a polydisc if, and only if, $K = K^{MM}$, and a neighbourhood W of U^M is an open polydisc if, and only if, $W = \text{Interior}\,(W^M)^M$. From this it follows that the mapping

$$K \in K(U) \to \text{Interior}\,(K^M).$$

defines a one to one correspondence between the compact polydiscs of U and the open polydiscs which contain U^M.

2. Holomorphic functions on nuclear spaces

If U is an open subset of a locally convex space E, then $H(U)$ will denote the space of holomorphic functions from U to C, i. e. $H(U)=\{f; f : U \to C$, f continuous and f is G-holomorphic $\}$. $H_{HY}(U)$ will denote the space of hypoanalytic functions from U to C, i. e. $f \in H_{HY}(U)$ if f is G-holomorphic and continuous on the compact subsets of U. τ_0 will denote the topology of compact convergence on $H(U)$ and $H_{HY}(U)$, and τ_ω will denote the topology on $H(U)$ generated by all seminorms ported by the compact subsets of U. (p a semi-norm on $H(U)$ is ported by the compact subset K of U if for all open V, $K \subset V \subset U$, there exists $C_V > 0$ such that

$$p(f) \leqslant C_V \| f \|_V \quad \text{for all } f \in H(U)).$$

DEFINITION 10. — Let $N^{(N)} = \{ (m_1, m_2, \ldots,); m_i \geqslant 0$, and m_i is eventually zero $\}$. If $m \in N^{(N)}$ and $z = (z_n)_{n=1}^\infty \in \Lambda(P)$, we let $z^m = \prod_{n=1}^\infty z_n^{m_n}$ where $m = (m_1, m_2, \ldots)$. z^m is called a monomial for each $m \in N^{(N)}$.

THEOREM 11. — *Let $E \simeq \Lambda(P)$ be a reflexive nuclear space and let $U \subset (\Lambda(P))'_\beta$ be a modularly decreasing open set. Then the monomials form an absolute basis for the complete nuclear space $(H_{HY}(U), \tau_0)$.*

Proof. — Let $f \in H_{HY}(U)$. If $b \in U$, we let

$$[b]_r = \{(z_n)_{n=1}^\infty \in E' \mid |z_i| \leqslant |b_i| \text{ for } 1 \leqslant i \leqslant r, \text{ and } z_i = 0 \text{ for } i > r\}.$$

$[b]_r$ is a finite dimensional polydisc in E'.

Now let K be any modularly decreasing compact subset of U. By proposition 7, there exists $\delta = (\delta_n)_{n=1}^\infty$ where $\delta_n > 1$ for all n, $\sum_{n=1}^\infty 1/\delta_n < \infty$ and δK is a relatively compact subset of U.

Now if $\xi = (\xi_n)_{n=1}^{\infty} \in K$, we have, by using the finite dimensional Cauchy integral formula

$$f(z) = \sum_{m \in N^r} a_m z^m \quad \text{for all } z \in [\xi]_r,$$

where

$$a_m = \frac{1}{(2\pi i)^r} \int \dots \int_T \frac{f(\eta_1, \eta_2, \dots, \eta_r, 0, 0, \dots)}{\eta_1^{m_1+1} \dots \eta_r^{m_r+1}} d\eta_1 \dots d\eta_r,$$

$$T = \{(\eta_i)_{i=1}^r ; |\eta_i| = |\xi_i| \text{ for } i = 1, \dots, r\},$$

and $m = (m_1, \dots, m_r) \in N^r$.

Hence

$$|a_m| \leqslant \frac{\|f\|_{[\xi]_r}}{|\xi_1|^{m_1} \dots |\xi_r|^{m_r}} \leqslant \frac{\|f\|_K}{|\xi^m|} \quad \text{for all } m \in N^r.$$

Applying this result to δK, we get

$$|a_m| \leqslant \frac{\|f\|_{\delta K}}{|(\delta\xi)^m|} \quad \text{for all } m \in N^r.$$

Therefore:

$$\sum_{m \in N^{(N)}} |a_m \xi^m| \leqslant \sum_{m \in N^{(N)}} \frac{\|f\|_{\delta K}}{\delta^m} = \|f\|_{\delta K} \sum_{m \in N^{(N)}} \frac{1}{\delta^m}$$

$$= \|f\|_{\delta K} \prod_{n=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{\delta_n}\right)^j$$

$$= \|f\|_{\delta K} \prod_{n=1}^{\infty} \frac{1}{1-(1/\delta_n)} = \|f\|_{\delta K} \Big/ \left(\prod_{n=1}^{\infty} \left(1 - \frac{1}{\delta_n}\right)\right).$$

Since $\sum_{n=1}^{\infty} 1/\delta_n < \infty$, this means

$$\sum_{m \in N^{(N)}} \sup_{\xi \in K} |a_m \xi^m| \leqslant C \|f\|_{\delta K} \quad \text{for some constant } C.$$

Hence $\tilde{f}(z) = \sum_{m \in N^{(N)}} a_m z^m$ defines a hypoanalytic function on U. Since f and $\tilde{f}$ agree on a dense subset of U, it follows that $f = \tilde{f}$. The coefficients a_m, $m \in N^{(N)}$, are obviously uniquely determined by f.

Let $\varepsilon > 0$ be arbitrary. Choose J a finite subset of $N^{(N)}$ such that

$$\|f\|_{\delta K} \sum_{m \in N^{(N)}/J} \frac{1}{\delta^m} \leqslant \varepsilon.$$

Then if J' is any finite subset of $N^{(N)}$ which contains J, we have

$$\| f-\textstyle\sum_{J'} a_m z^m \|_K \leqslant \| f \|_{\delta K} \sum_{m \in N^{(N)}/J} \dfrac{1}{\delta^m} \leqslant \varepsilon.$$

Thus

$$\textstyle\sum_{m \in N^{(N)}} a_m z^m = f \quad \text{in } (H_{HY}(U), \tau_0).$$

Finally, since

$$\| f \|_K \leqslant \textstyle\sum_{m \in N^{(N)}} | a_m | . \| z^m \|_K \leqslant \left(\sum_{m \in N^{(N)}} \dfrac{1}{\delta^m} \right) . \| f \|_{\delta K},$$

it follows that the monomials form an absolute basis for $(H_{HY}(U), \tau_0)$.

Since the uniform limit of continuous functions on compact sets is continuous on compact sets, $(H_{HY}(U), \tau_0)$ is complete.

Thus $(H_{HY}(U), \tau_0)$ is isomorphic to the sequence space $\Lambda(Q)$ where:

$$Q = \{(\| z^m \|_K)_{m \in N^{(N)}} ; K \subset U\}.$$

Since

$$\| z^m \|_K \leqslant \frac{1}{\delta^m} \| z^m \|_{\delta K} \quad \text{and} \quad \textstyle\sum_{m \in N^{(N)}} \dfrac{1}{\delta^m} < \infty,$$

it follows that $(H_{HY}(U), \tau_0)$ is nuclear. This completes the proof.

COROLLARY 12. — *The monomials form an absolute basis for the nuclear space* $(H(U), \tau_0)$.

Proof. — Since the monomials are continuous and $(H(U), \tau_0)$ is a (topological) subspace of $(H_{HY}(U), \tau_0)$, this follows immediately from theorem 11.

COROLLARY 13. — *The completion of* $(H(U), \tau_0)$ *is* $(H_{HY}(U), \tau_0)$.

COROLLARY 14. — *A G-holomorphic function f on U is hypoanalytic if and only if it is bounded on compact sets and*

$$f(z) = \textstyle\sum_{m \in N^{(N)}} a_m z^m \quad \textit{for all } z \textit{ in } U.$$

Remark. — A more general result concerning nuclearity is to be found in [8] and [23]: if E is a quasi-complete locally convex space whose strong dual is nuclear, then $(H(U), \tau_0)$ is nuclear for any open subset U of E.

THEOREM 15. — *Let* $E \simeq \Lambda(P)$ *denote a fully nuclear space with a basis, and let* $U \subset (\Lambda(P))$ *be a modularly decreasing open set. Then the monomials form an absolute basis for* $(H(U), \tau_\omega)$.

Proof. – Let p denote a τ_ω continuous semi-norm on $H(U)$ ported by the compact set K. Without loss of generality, K is modularly decreasing. Let $f \in H(U)$, and let V be a modularly decreasing neighbourhood of K such that $K \subset V \subset U$ and $\|f\|_V < \infty$. By propositions 5 and 7, we can find a modularly decreasing neighbourhood of 0, W, and $\delta = (\delta_n)_{n=1}^\infty$ where $\delta_n > 1$ and $\sum_{n=1}^\infty 1/\delta_n < +\infty$, such that $\delta(K+W) \subset V$.

Now if $z \in K+W$ and $m \in N^{(N)}$, then:

$$|a_m z^m| \leqslant \frac{\|f\|_{\delta(K+W)}}{\delta^m},$$

where a_m is defined as in theorem 11.

Let C_{K+W} denote a constant such that

$$p(g) \leqslant C_{K+W} \|g\|_{K+W} \quad \text{for all } g \in H(U).$$

If J is any finite subset of $N^{(N)}$, then

$$\begin{aligned} p(f-\textstyle\sum_{m \in J} a_m z^m) &\leqslant C_{K+W} \|f-\textstyle\sum_{m \in J} a_m z^m\|_{K+W} \\ &\leqslant C_{K+W} \|\textstyle\sum_{m \in N^{(N)} \setminus J} a_m z^m\|_{K+W} \\ &\leqslant C_{K+W} \|f\|_{\delta(K+W)} \textstyle\sum_{m \in N^{(N)} \setminus J} \dfrac{1}{\delta^m}. \end{aligned}$$

Since $\sum_{m \in N^{(N)}} 1/\delta^m < \infty$, it follows that

$$\lim_J p(f-\textstyle\sum_{m \in J} a_m z^m) = 0.$$

Therefore, as p was arbitrary, the monomials form a basis for $(H(U), \tau_\omega)$.

Now let W' denote an arbitrary modularly decreasing neighbourhood of K. By propositions 5 and 7, we can find a modularly decreasing neighbourhood of 0, V', and $\delta' = (\delta'_n)_{n=1}^\infty$ such that $\delta_n > 1$ all n, $\sum_{n=1}^\infty 1/\delta'_n < \infty$ and $\delta'(K+V') \subset W'$.

Hence:

$$\begin{aligned} p'(f) \equiv \textstyle\sum_{m \in N^{(N)}} p(a_m z^m) &\leqslant \textstyle\sum_{m \in N^{(N)}} C_{K+V'} \|a_m z^m\|_{K+V'} \\ &\leqslant C_{K+V'} \textstyle\sum_{m \in N^{(N)}} \|f\|_{\delta'(K+V')} \dfrac{1}{(\delta')^m} \\ &\leqslant C_{K+V'} \textstyle\sum_{m \in N^{(N)}} \dfrac{1}{(\delta')^m} \|f\|_{W'} \\ &= C\|f\|_{W'}, \end{aligned}$$

where $C_{K+V'}$ was chosen so that

$$p(g) \leqslant C_{K+V'} \| g \|_{K+V'} \quad \text{for all } g \in H(U).$$

Thus the semi-norm p' is τ_ω-continuous and the monomials form an absolute basis for $(H(U), \tau_\omega)$. This completes the proof.

In [4], the authors show that $(H(U), \tau_\omega)$ is nuclear when U is a balanced open subset of a metrizable nuclear space. Using the preceding theorem we obtain a further criterion for the nuclearity of $(H(U), \tau_\omega)$.

PROPOSITION 16. — *Let E denote a fully nuclear space with a basis, $E \simeq \Lambda(P)$. If there exists a sequence $\delta = (\delta_n)_{n=1}^\infty$ where $\delta_n > 1$ for all n and $\sum_{n=1}^\infty 1/\delta_n < \infty$, such that $(\delta_n \alpha_n)_{n=1}^\infty \in P$ whenever $(\alpha_n)_{n=1}^\infty \in P$, then $(H(U), \tau_\omega)$ is nuclear for any modularly decreasing open subset U of E.*

Proof. — We have already seen that $(H(U), \tau_\omega)$ is isomorphic to a subspace of a sequence space with weights

$$Q = \{(p(z^m))_{m \in N^{(N)}};\ p \text{ a } \tau_\omega\text{-continuous semi-norm on } H(U)\}.$$

Now suppose p is ported by the compact subset K of U.

By proposition 7, we can choose $\delta' = (\delta'_n)_{n=1}^\infty$ such that $\delta_n \geqslant \delta'_n > 1$ for all n, $\sum_{n=1}^\infty 1/\delta'_n < \infty$, and $K' = \delta' \delta' K$ is a relatively compact subset of U. Our hypothesis implies that $(\delta' V)_{V \in \mathscr{V}}$ is a fundamental system of neighbourhoods of 0 in E whenever $\mathscr{V}$ is a fundamental system of neighbourhoods of 0 in E.

Let

$$p'(f) = \sum_{m \in N^{(N)}} \delta'^m \, | a_m | \, p(z^m) \quad \text{for all } f = \sum_{m \in N^{(N)}} a_m z^m \in H(U).$$

Then

$$\begin{aligned} p'(f) &\leqslant \sum_{m \in N^{(N)}} \delta'^m \, | a_m | . \| z^m \|_{K+V} C(K+V) \\ &\leqslant C(K+V) \sum_{m \in N^{(N)}} \frac{1}{\delta'^m} \| a_m z^m \|_{\delta' \delta' K + \delta' \delta' V} \\ &\leqslant C(K+V) \left(\sum_{m \in N^{(N)}} \frac{1}{(\delta')^m} \right) \| f \|_{\delta' \delta' K + \delta' \delta' V}. \end{aligned}$$

Hence p' is a τ_ω-continuous semi-norm on $H(U)$ ported by $\delta' \delta' K$, and $(\delta'^m p(z^m))_{m \in N^{(N)}} \in Q$.

Since

$$\frac{p(z^m)}{(\delta')^m p(z^m)} = \frac{1}{(\delta')^m} \quad \text{and} \quad \sum_{m \in N^{(N)}} \frac{1}{(\delta')^m} < \infty,$$

this shows that $(H(U), \tau_\omega)$ is nuclear.

Remark. – If $E = \sum_N C \times \prod_N C$, then E satisfies the conditions of proposition 16. Therefore $H(U)$, τ_ω ($\neq (H(U), \tau_0)$) is nuclear for any modularly decreasing open subset U of E.

If U is a connected Reinhardt domain containing 0 in E'_β, where $E \simeq \Lambda(P)$ is a reflexive nuclear space, we let

$$\tilde{U} = \{(z_n)_{n=1}^\infty \in E'_\beta;\ \text{there exists } \omega = (\omega_n)_{n=1}^\infty \in U,\ |z_n| \leqslant |\omega_n| \text{ for all } n\}.$$

$\tilde{U}$ is a modularly decreasing open subset of E'_β and is the Reinhardt hull of U. For $f \in H_{HY}(U)$, we can define the "Taylor series" coefficients of f, $(a_m)_{m \in N^{(N)}}$, as in theorem 11, and we obtain by the finite dimensional theory of Reinhardt domains, the following results:

1° If K is a compact subset of $\tilde{U}$ then $\sum_{m \in N^{(N)}} \| a_m \omega^m \|_K < \infty$;

2° If in addition $f \in H(U)$, then whenever K is a compact subset of $\tilde{U}$, there exists a neighbourhood V of K, $V \subset \tilde{U}$, such that $\sum_{m \in N^{(N)}} \| a_m \omega^m \|_V < \infty$.

Hence each $f \in H_{HY}(U)$ can be extended in a unique fashion to $\tilde{f} \in H_{HY}(\tilde{U})$ and each $f \in H(U)$ can be extended in a unique fashion to $\tilde{f} \in H(\tilde{U})$.

For this reason, we have worked with modularly decreasing domains and not with Reinhardt domains.

3. Duality for spaces of holomorphic functions

If K is a compact subset of a locally convex space, $H(K)$ will denote the space of holomorphic germs on K. We endow $H(K)$ with the inductive limit topology

$$H(K) = \operatorname{ind\,lim}_{V \supset K} (H_\infty(V), \| \ \|_V),$$

where V ranges over all open neighbourhoods of K and

$$H_\infty(V) = \{f;\ f \in H(V), \|f\|_V < \infty\}.$$

$(H_\infty(V), \| \ \|_V)$ is a Banach space. Similarly, $H_{HY}(K)$ is the space of hypoanalytic germs about K endowed with the inductive limit topology

$$H_{HY}(K) = \operatorname{ind}\lim_{V \supset K}(H_{HY}(V), \tau_0),$$

where V ranges over all neighbourhoods of K.

THEOREM 17. — *Let U denote an open polydisc in the fully nuclear space with a basis E. Then the strong dual of $(H(U), \tau_0)$ is algebraically isomorphic to the space $H(U^M)$ of holomorphic germs on the compact polydisc U^M.*

Proof. — We will define a mapping $\beta : (H(U), \tau_0)' \to H(U^M)$ which is an algebraic isomorphism.

Let $U = \{ (z_n)_{n=1}^\infty \in E;\ \sup | z_n \alpha_n | < 1$ for some $\alpha = (\alpha_n)_{n=1}^\infty \in P \}$.

Now suppose $T \in (H(U), \tau_0)'$. Then there exist $C > 0$ and $K = K^{MM}$, a compact subset of U, such that $| T(f) | \leqslant C \| f \|_K$ for all $f \in H(U)$. By proposition 7, we can choose $\delta_K = (\delta_n)_{n=1}^\infty$ such that $\delta_n > 1$ for all n, $\sum_{n=1}^\infty 1/\delta_n < \infty$ and $\delta_K K$ is relatively compact in U. For each $m \in N^{(N)}$, let $b_m = T(z^m)$. Now $(\delta_K K)^M$ is a neighbourhood of U^M in E'_β and

$$\| b_m \omega^m \|_{(\delta_K K)^M} \leqslant C \| z^m \omega^m \|_{K \times (\delta_K K)^M}$$

$$\leqslant \frac{C}{\delta_K^m} \cdot \| z^m \omega^m \|_{K \times K^M} \leqslant \frac{C}{\delta_K^m}.$$

Hence

$$\sum_{m \in N^{(N)}} \| b_m \omega^m \|_{(\delta_K K)^M} \leqslant C . \sum_{m \in N^{(N)}} \frac{1}{\delta_K^m} < +\infty.$$

Therefore $\sum_{m \in N^{(N)}} b_m \omega^m$ represents an element of H_∞ (Interior $(\delta_K K)^M$) which we call g_T.

We define βT to be the germ of this function on U^M. It is clear that β is well defined and linear. Since the monomials form a basis for $(H(U), \tau_0)$, β is injective.

We now show that β is surjective. Let $g \in H(U^M)$. There exists an open polydisc in E'_β, V, which contains U^M and a $\tilde{g} \in H_\infty(V)$ whose germ on U^M is g such that

$$\tilde{g}(\omega) = \sum_{m \in N^{(N)}} b_m \omega^m \quad \text{for all } \omega \in V$$

and

$$\textstyle\sum_{m \in N^{(N)}} \| b_m \omega^m \|_V = C < \infty.$$

Let $V = \{ (\omega_n)_{n=1}^{\infty} \in E'_\beta;\ \sup_n | \omega_n \alpha'_n | < 1 \}$. If $m = (m_i)_{i=1}^{\infty} \in N^{(N)}$, then $m_i = 0$ for all except a finite number of i.

If $m \in N^{(N)}$ and $m_i \neq 0 \Rightarrow \alpha'_i \neq 0$, then:

$$\| z^m \omega^m \|_{V^M \times V} = 1 \qquad \text{and} \qquad | b_m | \leqslant \frac{C}{\| \omega^m \|_V} = C \| z^m \|_{V^M}.$$

If $m_i \neq 0$ and $\alpha'_i = 0$ for some i, then $b_m = 0$ and $| b_m | \leqslant C \| z^m \|_{V^M}$ also in this case.

We now define T_g on $H(U)$ by

$$T_g(f) = T_g(\textstyle\sum_{m \in N^{(N)}} a_m z^m) = \sum_{m \in N^{(N)}} a_m b_m .$$

$V^M = K$ is a compact subset of U.

Now

$$\textstyle\sum_{m \in N^{(N)}} | a_m b_m | \leqslant C \sum_{m \in N^{(N)}} | a_m z^m |_K \leqslant C \| f \|_{\delta_K K} \sum_{m \in N^{(N)}} \frac{1}{\delta_K^m} .$$

Hence T_g is well defined and τ_0-continuous on $H(U)$. Since $\beta(T_g) = g$ this shows that β is surjective and completes the proof.

THEOREM 18. — *Let U denote an open polydisc in the fully nuclear space with a basis E. Then $(H(U), \tau_\omega)'$ is algebraically isomorphic to the space $H_{HY}(U^M)$.*

Proof. — We extend the mapping β of theorem 17 to prove this result. Let $U = \{ (z_n)_{n=1}^{\infty} \in E;\ \sup_n | z_n \alpha_n | < 1 \}$ where $\alpha = (\alpha_n)_{n=1}^{\infty} \in P$.

Let $T \in (H(U), \tau_\omega)'$. We can find K compact in U, $K = K^{MM}$, such that, if V is any neighbourhood of K, $K \subset V \subset U$, then $| T(f) | \leqslant C(V) \|f\|_V$ for all $f \in H(U)$ and the constant $C(V)$ is independent of f. Let $b_m = T(z^m)$ for all $m \in N^{(N)}$ and let $\tilde{g}_T(\omega) = \sum_{m \in N^{(N)}} b_m \omega^m$. To show that $\tilde{g}_T$ defines an element of H_{HY} (Interior K^M), it suffices to show that $\sum_{m \in N^{(N)}} \| b_m \omega^m \|_L < \infty$ for each compact subset L of Interior (K^M). By lemma 9, it suffices to show $\sum_{m \in N^{(N)}} \| b_m \omega^m \|_{V^M} < \infty$ for each neighbourhood V of K. Using propositions 5 and 7, we can find a sequence $\delta = (\delta_n)_{n=1}^{\infty}$ such that $\delta_n > 1$ for all n and $\sum_{n=1}^{\infty} 1/\delta_n < \infty$, and W a neighbourhood of 0 in E such that δK is a relatively compact subset of U and $\delta(K+W) = \delta K + \delta W \subset V$.

Hence

$$\sum_{m \in N^{(N)}} \| b_m \omega^m \|_{V^M} \leqslant \sum_{m \in N^{(N)}} C(K+W) \| z^m \|_{K+W} \| \omega^m \|_{V^M}$$

$$\leqslant C(K+W) \sum_{m \in N^{(N)}} \frac{1}{\delta^m} \| z^m \|_{\delta(K+W)} \| \omega^m \|_{V^M}$$

$$\leqslant C(K+W) \sum_{m \in N^{(N)}} \frac{1}{\delta^m} \| z^m \omega^m \|_{V \times V^M}$$

$$\leqslant C(K+W) \sum_{m \in N^{(N)}} \frac{1}{\delta^m} < \infty.$$

We define βT to be the germ of $\tilde{g}_T$ on U^M. Hence $\beta T \in H_{HY}(U^M)$. Clearly β is linear and since the monomials form a basis for $(H(U), \tau_\omega)$ it follows that β is an injective mapping.

We now show that β is surjective. Let $g(\omega) = \sum_{m \in N^{(N)}} b_m \omega^m \in H_{HY}(U^M)$. By lemma 9 and theorem 11, there exists a function

$$\tilde{g}(\omega) = \sum b_m \omega^m \in H_{HY}(\text{Interior } K^M)$$

where K is a compact subset of U, $K = K^{MM}$, and

$$\sum_{m \in N^{(N)}} | b_m | \; \| \omega^m \|_L < +\infty$$

for every compact subset L of Interior K^M, such that g is the germ of $\tilde{g}$ on U^M.

Now let V denote a neighbourhood of K which lies in U. By propositions 5 and 7, we can find a sequence $\delta = (\delta_n)_{n=1}^{\infty}$ where $\delta_n > 1$ for all n, $\sum_{n=1}^{\infty} 1/\delta_n < +\infty$, and W a neighbourhood of zero such that δK is a relatively compact subset of U, $K \subset \delta K + \delta W \subset V$ and $(K+W)^{MM} = K+W$. If $m \in N^{(N)}$, then:

$$\| z^m \|_{K+W} \text{ is finite if and only if } \| z^m \|_{K+W} \| \omega^m \|_{(K+W)^M} = 1$$

(and hence $1/\| \omega^m \|_{(K+W)^M} = \| z^m \|_{K+W}$), while $\| z^m \|_{K+W} = \infty$ if, and only if, $\| \omega^m \|_{(K+W)^M} = 0$.

Since $(K+W)^M$ is a compact subset of interior of K^M,

$$\sum_{m \in N^{(N)}} \| b_m \omega^m \|_{(K+W)^M} = C < \infty.$$

If

$$f \in H(U), \qquad f = \sum_{m \in N^{(N)}} a_m z^m,$$

then:

$$\begin{aligned}
\left|\sum_{m\in N^{(N)}} a_m b_m\right| &\leqslant \sum_{m\in N^{(N)}} |a_m b_m| \\
&\leqslant \sum_{m\in N^{(N)},\, \|\omega^m\|_{(K+W)^M}>0} |a_m| \frac{C}{\|\omega^m\|_{(K+W)^M}} \\
&\quad + \sum_{m\in N^{(N)},\, \|\omega^m\|_{(K+W)^M}=0} |a_m|\, C\, \|z^m\|_{K+W} \\
&\leqslant C \sum_{m\in N^{(N)}} \|a_m z^m\|_{K+W} \\
&\leqslant C\left(\sum_{m\in N^{(N)}} \frac{1}{\delta^m}\right) \cdot \|f\|_{\delta(K+W)} \leqslant C' \|f\|_V.
\end{aligned}$$

Now let $T_g(f) = \sum_{m\in N^{(N)}} a_m b_m$. The above shows that T_g is well defined and is τ_ω continuous (being ported by K).

Since $\beta(T_g) = g$ this shows that β is surjective and this completes the proof.

LEMMA 19. — $H(K) = \text{ind}\lim_{V\supset K} H_\infty(V) = \text{ind}\lim_{V\supset K}(H(V), \tau_\omega)$.

Proof. — It suffices to show that the two inductive limits define the same topology. Since the injection $H_\infty(V) \to (H(V), \tau_\omega)$ is continuous, it follows thet the identity mapping $\text{ind}\lim_{V\supset K} H_\infty(V) \to \text{ind}\lim_{V\supset K}(H(V), \tau_\omega)$ is continuous.

Let p denote a continuous semi-norm on $\text{ind}\lim_{V\supset K} H_\infty(V)$, and let W denote an open neighbourhood of K. For each U open, $K \subset U \subset W$ there exists $C(U) > 0$ such that

$$B = \{f \in H(K);\ p(f) \leqslant 1\} \supset \left\{f \in H_\infty(U);\ \|f\|_U \leqslant \frac{1}{C(U)}\right\}.$$

Hence $B \supset \{f \in H(W);\ \|f\|_U \leqslant 1/C(U)\}$ and thus $p(f) \leqslant C(U)\|f\|_U$ for all $f \in H(W)$. It thus follows that $p\,|_{H(W)}$ defines a τ_ω-continuous semi-norm which is ported by K.

Hence p is a continuous semi-norm on $\text{ind}\lim_{V\supset K}(H(V), \tau_\omega)$. This completes the proof.

THEOREM 20. — *Let E denote a fully nuclear space with a basis, and let U denote an open polydisc in E. Then $(H(U), \tau_0)'_\beta \cong H(U^M)$.*

Proof. — We have already seen that there exists a linear bijection from $(H(U), \tau_0)'_\beta$ onto $H(U^M)$. Now $(H(U), \tau_0)'_\beta = (H_{HY}(U), \tau_0)'_\beta$ and

$(H_{HY}(U), \tau_0)$ is a complete nuclear space, hence $(H(U), \tau_0)'_\beta$ has the Mackey topology. Since $H(U^M)$ is an inductive limit of Banach spaces it also has the Mackey topology. To complete the proof it thus suffices to show

$$(H(U), \tau_0)'' = (H(U^M))'.$$

Since $(H_{HY}(U), \tau_0)$ is a complete nuclear space $(H(U), \tau_0)'' \cong H_{HY}(U)$. Now $T \in H(U^M)'$ if and only if for every neighbourhood V of U^M, $T|_{H(V)} \in (H(V), \tau_\omega)'$ (lemma 19). Hence if $a_m = T(\omega^m)$, then

$$\sum_{m \in N^{(N)}} a_m \omega^m \in H_{HY}(\text{Interior } V^M)$$

for all open V containing U^M. As V ranges over all neighbourhoods of U^M, V^M ranges over all compact polydiscs of U. Thus

$$\sum_{m \in N^{(N)}} a_m \omega^m \in H_{HY}(U).$$

This completes the proof.

Remark. — In [9], the author shows that $(H(E), \tau_0)'_\beta \cong H(0)$ where E is the strong dual of a Fréchet nuclear space, and $H(0)$ is the space of germs at $0 \in E'_\beta$.

We now apply these results.

PROPOSITION 21. — *Let E denote a fully nuclear space with a basis. The following are equivalent:*

(*a*) τ_0 *and* τ_ω *are compatible topologies on* $H(U)$ (i. e. *they define the same dual*) *for any open polydisc U in E;*

(*b*) $(H(V), \tau_0)$ *is complete for any open set V in* E'_β*;*

(*c*) $H(V) = H_{HY}(V)$ *for any open set V in* E'_β*;*

(*d*) $(H(V), \tau_0)$ *is semi-reflexive for any open set V in* E'_β.

Proof. — We have already noted that (*b*), (*c*) and (*d*) are equivalent for open polydiscs V, and it easily follows that they are equivalent for arbitrary open V.

If (*c*) is true then $H(K) = H_{HY}(K)$ for any compact set in E'_β. Since $H(U^M) = (H(U), \tau_0)'$ and $H_{HY}(U^M) = (H(U), \tau_\omega)'$ we see that (*c*) $\Rightarrow$ (*a*).

If (*c*) is not true then there exists a compact polydisc $K = U^M$ in E'_β such that $H(K) \neq H_{HY}(K)$. It then follows that $(H(U), \tau_0)' \neq (H(U), \tau_\omega)'$ and hence (*a*) $\Rightarrow$ (*c*). This completes the proof.

PROPOSITION 22. — *Let E denote a fully nuclear space with a basis. The following are equivalent:*

(a) $(H(U), \tau_0) = (H(U), \tau_\omega)$ *for any open polydisc U in E;*

(b) If V is any open set in E'_β *and* $B \subset (H(V), \tau_0)$ *is bounded, then for each* $x \in V$ *there exists a neighbourhood* V_x *of x such that* $\sup_{f \in B} \| f \|_{V_x} < \infty$.

Proof. — We first suppose that (*b*) holds. By theorem 11 and proposition 21, this implies that τ_0 and τ_ω are compatible topologies on $H(U)$. Since $\tau_\omega \geqslant \tau_0$ in all cases we only need show that any τ_ω-neighbourhood of zero, W, contains a τ_0-neighbourhood of zero.

Without loss of generality we may assume $W = \{ f \in H(U); p(f) \leqslant 1 \}$ where p is a τ_ω-continuous semi-norm ported by $K = K^{MM}$, and $W = W^{00}$.

Hence for each neighbourhood V of $K \subset V \subset U$, there exists a positive real number $C(V)$ such that

$$\{ f : p(f) \leqslant 1 \} \supset \bigcup_{K \subset V} \left\{ f \in H(U); \| f \|_V \leqslant \frac{1}{C(V)} \right\}.$$

Thus:

$$W^0 \subset \bigcap_{K \subset V} \{ T \in (H(U), \tau_\omega)'; | T(f) | \leqslant C(V) \| f \|_V \text{ for all } f \in H(U) \}.$$

By the proof of theorem 18,

$$\bigcap_{K \subset V} \{ T \in (H(U), \tau_\omega)'; | T(f) | \leqslant C(V) \| f \|_V \text{ for all } f \in H(U) \}$$

may be identified with a set of functions in H_{HY} (Interior K^M) which are uniformly bounded on compact subsets of Interior K^M. By condition (*b*),

$$\bigcap_{K \subset V} \{ T \in (H(U), \tau_\omega)'; | T(f) | \leqslant C(V) \| f \|_V \text{ for all } f \in H(U) \}$$
$$\subset \{ g \in H(V'); \| g \|_{V'} \leqslant C \}$$

for some neighbourhood V' of U^M and some positive number C.

Hence:

$$W = W^{00} \supset \{ g \in H(V'); \| g \|_{V'} \leqslant C \}^0 \supset \{ f \in H(U); \| f \|_{\delta (V')^M} \leqslant C' \}$$

for $C' > 0$, and some $\delta = (\delta_n)_{n=1}^\infty$ where $\delta (V')^M$ is a compact subset of U by theorem 17. Hence $(H(U), \tau_0) = (H(U), \tau_\omega)$ when (*b*) is satisfied.

If (*b*) is not satisfied then there exists a compact polydisc U^M in E'_β, V an open polydisc neighbourhood of U^M, and B a bounded subset of $(H(V), \tau_0)$ which is not uniformly bounded in any neighbourhood of U^M. The set B, as above, may be identified with an equicontinuous subset of $(H(U), \tau_\omega)'$ but is not an equicontinuous subset of $(H(U), \tau_0)'$. Hence $\tau_0 \neq \tau_\omega$ if (b) is not satisfied. Thus $(a) \Leftrightarrow (b)$. This completes the proof.

Remarks. — In [1], J. A. BARROSO shows that

$$(H(\prod_{n=1}^{\infty} C), \tau_0) = (H(\prod_{n=1}^{\infty} C), \tau_\omega)$$

and this result has recently been extended to arbitarry open subsets of $\prod_{n=1}^{\infty} C$ by BARROSO and NACHBIN [3], and M. SCHOTTENLOHER [24].

Condition (b) of propositions 21 and 22 are frequently easy to verify. Every k-space, [18], satisfies (b) of proposition 21, (and indeed in [13] a space with property (b) of proposition 21 is called a $\tilde{k}$ space.) $\prod_A C$, A uncountable, is an example of a space which has property (b) of proposition 21, but which is not a k-space. By corollary 13, every fully nuclear space with a basis which satisfies (b) of proposition 22 also satisfies (b) of proposition 21. $\sum_{n=1}^{\infty} C \times \prod_{n=1}^{\infty} C$ is an example of a fully nuclear space with a basis which does not satisfy (b) of proposition 21 (the function

$$f : \sum_{n=1}^{\infty} C \times \prod_{n=1}^{\infty} C \to C,$$

$$(z_n)_{n=1}^{\infty} \times (\omega_n)_{n=1}^{\infty} \to \sum_{n=1}^{\infty} (z_n \omega_n)^n$$

is hypoanalytic but not holomorphic).

All our examples of fully nuclear spaces with a basis which satisfy (b) of proposition 21 also satisfy (b) of proposition 22.

$$C_0(A) = \operatorname{proj\,lim}_{A' \subset A} C_0(A'),$$

A uncountable and A' ranges over all countable subsets of A, is an example of a space which satisfies property (b) of proposition 22, but which does not satisfy the corresponding property (b) of proposition 21. (see [10] for further details).

Spaces which satisfy (b) of proposition 22 for Banach valued holomorphic functions are studied in [2], where they are called holomorphically infrabarrelled locally convex spaces. Results concerning such spaces are also given in [10] where it is shown that for any completely regular Hausdorff space X, $C(X)$ is holomorphically infrabarrelled (and hence satisfies property (b) of proposition 22) if and only if $C(X)$ is infrabarrelled.

It is also possible to generate spaces with properties (b) of propositions 21 and 22 by means of surjective limits [10].

THEOREM 23. — *Let E denote a fully nuclear space with a basis. The following are equivalent*

(*a*) $(H(U), \tau_\omega)'_\beta = H_{HY}(U^M) \equiv \text{ind}\lim_{V \supset U^M}(H_{HY}(V), \tau_0)$ *for any open polydisc U in E;*

(*b*) *If B is a bounded subset of* $(H(U), \tau_\omega)$, *U an open polydisc in E, then for each* $x \in U$ *there exists a neighbourhood* V_x *of x such that* $\sup_{f \in B} \|f\|_{V_x} < \infty$.

Proof. — Since $(H_{HY}(V), \tau_0)' = (H(V), \tau_0)' = H(V^M)$ for any polydisc V, it follows that $H_{HY}(U^M)' = H(\text{Interior}(\bigcup_{V \supset U^M} V^M)) = H(U)$. Hence the two topologies coincide if and only if the τ_ω bounded subsets of $H(U)$ coincide with the equi-continuous subsets of $(H_{HY}(U^M))'$.

Now W, a convex balanced set, is a neighbourhood of zero in $H_{HY}(U^M)$ if and only if for each V open, $V \supset U^M$, there exists K_V, a compact subset of V and $C(K_V) > 0$ such that

$$W \supset \{f\,;\, f \in H_{HY}(V),\ \|f\|_{K_V} \leqslant C(K_V)\}.$$

Hence W' is an equicontinuous subset of $(H_{HY}(U^M))'$ if and only if for each compact subset K of U there exists a neighbourhood V of K and $C(K, V) > 0$ such that

$$(\bigstar) \qquad W' \subset \bigcap_{K \subset U} \{f\,;\, f \in H(U),\ \|f\|_V \leqslant C(K, V)\}.$$

This implies that W' is bounded in $(H(U), \tau_\omega)$ and the strong topology on $(H(U), \tau_\omega)'$ is always finer than the inductive limit topology. The converse will be true if and only if every τ_ω bounded subset of $H(U)$ is contained in a set which has the form ($\bigstar$), i. e. if, and only if, condition (*b*) is satisfied. This completes the proof.

Remarks. — Since $\tau_\omega \geqslant \tau_0$ it follows that if E is a fully nuclear space with a basis which satisfies (*b*) of proposition 22, then E'_β satisfies (*b*) of theorem 23. $\sum_{n=1}^{\infty} C \times \prod_{n=1}^{\infty} C$ is an example of a fully nuclear space which satisfies (*b*) of theorem 23 ([11], proposition 16). Further examples may be constructed using results in [10] and [11]. We also note that any fully nuclear Fréchet space with a basis or its dual satisfy (*b*) of proposition 21, 22 and theorem 23. Therefore in particulier, $\tau_0 = \tau_\omega$ on $H(E)$ for any Fréchet nuclear or dual Fréchet nuclear space with a basis E.

To complete the duality between τ_0 and τ_ω, we conjecture that $(H(U), \tau_\omega)$ is semi-reflexive for any open polydisc U in the fully nuclear space with a basis E if and only if $(H(U), \tau_\omega)$ is quasi-complete.

We have the following partial answer to this question.

PROPOSITION 24. — *Let E denote a fully nuclear space with a basis, anp let U denote an open polydisc in E. If* $(H(U), \tau_\omega)$ *is quasi-complete then it is semi-reflexive if either of the following conditions hold;*

(i) $(H(U), \tau_\omega)$ *is nuclear;*

(ii) *E satisfies condition (b) of theorem* 23.

4. Entire functions on fully nuclear spaces

In this section, we discuss entire functions on a fully nuclear space. A fully nuclear space need not possess an equi-continuous basis and there exist Fréchet nuclear spaces without bases (*see* [16]). When a fully nuclear space does not possess a (equi-continuous) basis, we do not have a basis for spaces of entire functions, and hence we cannot use the techniques of the preceding sections. If E is a fully nuclear space without a (equi-continuous) basis, we have no analogue of open polydiscs as in the case of with a basis. For this reason, we confine our study to entire functions.

PROPOSITION 25. — *Let E denote a fully nuclear space. Then there exists an (algebraic) isomorphism* β *between* $(H(E), \tau_0)'$ *and the space of holomorphic germs at* 0 *in* E'_β. *Moreover, under this isomorphism* β, *the equicontinuous subsets of* $(H(E), \tau_0)'$ *correspond with sets of germs which are defined and uniformly bounded on a neighbourhood of* 0 *in* E'_β.

Proof. — We note that if E is a fully nuclear space and n is a positive integer, then the mapping $\beta_n : (P(^nE), \tau_0)' \to P(^nE')$ defined by

$$\beta_n(T)(\Phi) = T(\Phi^n) \qquad \text{for} \quad \Phi \in E',$$

is an (algebraic) isomorphism onto. (*see* for example [6]).

Now suppose $T \in (H(E), \tau_0)'$, and we will define $\beta\, T$. As $T \in (H(E), \tau_0)'$ there exist $C > 0$ and K a compact set in E such that $|T(f)| \leqslant C\,||f||_K$ for all $f \in H(E)$. For each $n \in N$, let $T_n = T|_{P(^nE)}$. Then $\beta_n T_n \in P(^nE')$, and

$$||\beta_n T_n||_{1/2(K^0)} = \sup_{\Phi \in E',\, ||\Phi||_K \leqq 1/2} T(\Phi^n) \leqslant \frac{C}{2^n}.$$

Now K^0 is a neighbourhood of 0 in E', and hence $g_T = \sum_{n=0}^{\infty} \beta_n T_n$ is holomorphic and bounded by C on $1/2(K^0)$. We define $\beta\, T$ to be the germ of g_T at 0.

It is clear that β is linear, and since β_n is an isomorphism for each n, it follows that β is $1-1$. From the above definition, it is clear that the image *via* β *of an* equi-continuous subset of $(H(E), \tau_0)'$ is uniformly bounded in some neighbourhood of $0 \in E'$.

Next suppose that $B \subset H_\infty(V)$ is a family of holomorphic functions on the neighbourhood V of 0 in E'_β which are uniformly bounded on V. Then there exists a compact set K in E and $C > 0$ such that $\| \hat{d}^n g(0)/n! \|_{K^0} \leqslant C$ for all $g \in B$ and all n. For each $g \in B$ and $n \geqslant 0$, let $T_{g,n} = \beta_n^{-1}(\hat{d}^n g(0)/n!)$ By proposition 1.3 of [6], $T_{g,n}$ is a well defined element of $(P(^nE), \tau_0)'$, and there exists a compact set K_1 in E (depending only on K and C) such that

$$| T_{g,n}(p) | \leqslant \| p \|_{K_1} \quad \text{for all } p \in P(^nE), \quad g \in B, \quad \text{and all } n = 0, 1, \ldots$$

For $g \in B$, let $T_g \in (H(E), \tau_0)'$ be defined by

$$T_g(f) = \sum_{n=0}^{\infty} T_{g,n}\left(\frac{\hat{d}^n f(0)}{n!}\right) \quad \text{for} \quad f \in H(E).$$

Then $T_g \in (H(E), \tau_0)'$ and $\beta(T_g)$ is the germ of g at 0 in E' (and hence β is surjective). Moreover:

$$| T_g(f) | \leqslant \sum_{n=0}^{\infty} \left| T_{g,n}\left(\frac{\hat{d}^n f(0)}{n!}\right) \right|$$
$$\leqslant C' \sum_{n=0}^{\infty} \left\| \frac{d^n f(0)}{n!} \right\|_{K_1} \leqslant C' \| f \|_{2K_1}$$

for all $f \in H(E)$ and $g \in B$. Hence the set $\{ \beta^{-1} g; g \in B \} = \{ T_g; g \in B \}$ is an equi-continuous subset of $(H(E), \tau_0)'$, and this completes the proof.

The following lemmas will be used in the proofs of proposition 28 and theorem 29.

LEMMA 26. — *Let E be a fully nuclear space. Then $P_f(^mE)$, the space of m homogeneous continuous polynomials of finite type, is dense in $(P(^mE), \tau_\omega)$.*

Proof. — Suppose $q \in P(^mE)$, p is a τ_ω-continuous semi-norm on $P(^mE)$, and $\varepsilon > 0$ is arbitrary. By [6], [7], there exists a sequence of continuous linear forms $(\Phi_i)_{i=1}^{\infty}$ on E and a neighbourhood V of 0 in E such that

$$q(z) = \sum_{i=1}^{\infty} \Phi_i^m(z) \quad \text{for all } z \in E \quad \text{and} \quad \sum_{i=1}^{\infty} \| \Phi_i^m \|_V < \infty.$$

As p is τ_ω-continuous, there exists a $C(V) > 0$ such that $p(r) \leqslant C(V) \| r \|_V$ for all polynomials $r \in P(^mE)$. Let N be such that $\sum_{i=N+1}^{\infty} \| \Phi_i^m \|_V < \varepsilon / C(V)$. Then

$$p(q - \sum_{i=1}^{N} \Phi_i^m) \leqslant C(V) \| q - \sum_{i=1}^{N} \Phi_i^m \|_V = C(V) \| \sum_{i=N+1}^{\infty} \Phi_i^m \|_V \leqslant \varepsilon.$$

This shows that $P_f(^mE)$ is dense in $P(^mE)$ for τ_ω.

LEMMA 27. — *Let E be a fully nuclear space, and U be open in E. If f is a G-holomorphic function on U which is bounded on the compact subsets of U, then $f \in H_{HY}(U)$.*

Proof. — Since E is fully nuclear and hence Montel, f must be bounded on the complete bounded subsets of U. Therefore f is s-holomorphic, i. e. $f \in H_s(U)$ (*see* [11], p. 459). As E'_β is a reflexive nuclear space (therefore an infrabarrelled Schwartz space), and $E \simeq (E'_\beta)'_\beta$, it follows by proposition 3.5 ([11], p. 459), that $f \in H_s(U) = H_{HY}(U)$.

PROPOSITION 28. — *Let E be a fully nuclear space. Then τ_0 and τ_ω are compatible topologies on $H(E)$ if $H(U) = H_{HY}(U)$ for all open subsets U of E'_β (in particular, if E'_β is a k-space).*

Proof. — Since $\tau_\omega \geqslant \tau_0$, it suffices to show every $T \in (H(E), \tau_\omega)'$ is continuous for the τ_0-topology. If $T \in (H(E), \tau_\omega)'$, then T is ported by some absolutely convex compact set K in E. Now let $T_n = T/P(^nE)$ for each n. Then $T = \sum_{n=0}^{\infty} T_n$. Moreover if V is open and $V \supset K$, there exists $C_V > 0$ such that $| T_n(\Phi^n) | \leqslant C_V \| \Phi^n \|_V$ for all $\Phi \in E'$ and all n. Hence if $\hat{T}_n : \Phi \to T_n(\Phi^n)$, then $\hat{T}_n \in P_{HY}(^nE') = P(^nE')$. Moreover if $g_T = \sum \hat{T}_n$, then

$$\| g_T \|_{1/2(V^0)} \leqslant \sum_{n=0}^{\infty} \| \hat{T}_n \|_{1/2(V^0)} \leqslant C_V.$$

As sets of the form $1/2(V^0)$ (where V is a neighbourhood of K in E) form a fundamental system of compact subsets of interior ((1/2) K^0), it follows by lemma 27 that $g_T \in H_{HY}$ (Interior (1/2) K^0) $= H$ (Interior (1/2) K^0). From the construction of the isomorphism β in proposition 25 and lemma 26, it follows that $T \in (H(E), \tau_0)'$, and this completes the proof.

THEOREM 29. — *If E is fully nuclear, then $(H(E), \tau_0) = (H(E), \tau_\omega)$ if bounded subsets of $(H_{HY}(U), \tau_0)$ are locally bounded for every open subset U of E'_β. In particular, $\tau_0 = \tau_\omega$ on $H(E)$ if E is a Fréchet nuclear or dual Fréchet nuclear space.*

Proof. — As in the proof of proposition 22, it follows that every equicontinuous subset of $(H(E), \tau_\omega)'$ may be identified with a set of germs which are defined and uniformly bounded in a neighbourhood of 0 in E'_β. It follows from proposition 25 that the equi-continuous subsets of $(H(E), \tau_0)'$ and $(H(E), \tau_\omega)'$ are the same, and hence $\tau_0 = \tau_\omega$.

The following has been pointed out to us by K. D. BIERSTEDT and R. MEISE.

If U is an open subset of a locally convex space then

$$(H(U), \tau_\pi) = \text{proj lim } H(K), \text{ with } K \subset U, \ K \text{ compact},$$

where each $H(K)$ has the inductive limit topology as given in section 3 ([4], [20]). τ_0 is a sheaf topology and thus we have shown that $(H(U), \tau_0) = (H(U), \tau_\pi)$ for any open subset U of a Fréchet nuclear space with a basis. In particular, we have $(H(U), \tau_0) = (H(U), \tau_\omega)$ for any balanced open subset of a Fréchet nuclear space with a basis

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