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CHARACTERIZATIONS OF CLASSES OF LEFT EXT-REPRODUCED GROUPS

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1. Introduction.

In a previous paper [4], we considered classes \mathfrak{M} of abelian groups G , which are maximal with respect to the property : There exists a reduced group X such that

$$G \cong \text{Ext}(G, X) \quad \text{for all } G \in \mathfrak{M}.$$

Such classes are called *maximal classes of left Ext-reproduced groups*. We found two such classes, viz. \mathfrak{F} and \mathfrak{A} (see [4], Theorem 2.7 and Theorem 2.9).

In this paper, we give characterizations of two classes of left Ext-reproduced groups, viz. \mathfrak{F} and \mathfrak{G} , where

\mathfrak{F} denotes the class of all groups $Q^n \oplus T$ where n is a non-negative integer and T is a finite group, and

\mathfrak{G} denotes the class of all groups $\prod_{p \in P} T_p$ where, for all primes p , T_p is a finite p -group.

The latter class is a subclass of the class \mathfrak{A} , and it is therefore not a maximal class of left Ext-reproduced groups. We mention, in passing, the fact that all groups in \mathfrak{A} are reduced and adjusted cotorsion groups (see [5], p. 373), and that \mathfrak{G} contains all finite groups. The classes \mathfrak{F} and \mathfrak{G} are linked together in a very special way. Let us consider a number of properties of a class \mathfrak{M} of left Ext-reproduced groups, i. e. a class of groups for which there exists a reduced group X such that

$$G \cong \text{Ext}(G, X) \quad \text{for all } G \in \mathfrak{M}.$$

- (I) If $G \in \mathfrak{M}$, then every direct summand U of G belongs to \mathfrak{M} .
 (II) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, then $\text{Hom}(G, H) \in \mathfrak{M}$.
 (III) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, and if $\varphi : G \rightarrow H$ is a homomorphism, then $\text{Ker } \varphi \in \mathfrak{M}$ and $\text{Coker } \varphi \in \mathfrak{M}$.
 (IV) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, then $\text{Ext}(G, H) \in \mathfrak{M}$.
 (V) If $G \in \mathfrak{M}$, then $\text{Hom}(G, X) = 0$.

Now the following statement summarizes the main results :

\mathfrak{F} and \mathfrak{G} are classes of left Ext-reproduced groups which are maximal with respect to each of the properties (I) to (V). Conversely, if \mathfrak{M} is a class of left Ext-reproduced groups which is maximal with respect to any one of the properties (I) to (V), then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$, the latter being the case if \mathfrak{M} contains only reduced groups.

NOTATION.

$A \oplus B, \bigoplus_{i \in I} A_i, A^{(m)}$, direct sum;

$\prod_{i \in I} A_i, A^m$, direct product;

$A \otimes B$, tensor product of A and B ;

tG , maximal torsion subgroup of G ;

G_p , p -component of G ;

Z , additive group of integers;

Q , additive group of rational numbers;

$Z(p)$, additive group of p -adic integers;

$C(n)$, cyclic group of order n ;

$C(p^\infty)$, quasi-cyclic group;

\aleph , the power of the continuum;

P , the set of all prime numbers;

cotorsion group, a group X such that $\text{Ext}(Q, X) = 0$; adjusted reduced cotorsion group, a reduced cotorsion group G such that G/tG is divisible.

All groups under consideration are additively written abelian groups.

2. Characterizations of classes of left Ext-reproduced groups.

Let \mathfrak{M} be a class of left Ext-reproduced groups. Throughout this paper, X will denote a reduced group such that

$$\text{Ext}(G, X) \cong G \text{ for all } G \in \mathfrak{M}.$$

Recall that \mathfrak{F} is the class of all groups $Q^n \oplus T$ where n is a non-negative integer and T is a finite group. For this class of groups, we have that

$\prod_{p \in P} Z(p)/X \cong Q$ ([4], Example 2.3). Let \mathfrak{G} denote the class of all

groups $\prod_{p \in P} T_p$, where T_p is a finite p -group for all primes p . Note that \mathfrak{G} is a proper subclass of \mathfrak{A} , consequently \mathfrak{G} is also a class of left Ext-reproduced groups ([4], Example 2.8).

The classes \mathfrak{F} and \mathfrak{G} are quite remarkable, and we give characterizations of them in this paragraph. First, we prove several lemmas which we shall need subsequently.

LEMMA 1. — *Let \mathfrak{H} be a class of left Ext-reproduced groups such that $\mathfrak{H} \supseteq \mathfrak{G}$. Then $X \cong \prod_{p \in P} Z(p)$.*

Proof. — Since $C(p^k) \in \mathfrak{H}$ for all primes p and all natural numbers k , it follows from [4] (Lemma 2.4) that X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ and that $X/pX \cong C(p)$ for all primes p . Note that $\text{Ext}(Q, X) = 0$, this follows from the exact sequences

$$0 \rightarrow \bigoplus_{p \in P} C(p) \rightarrow \prod_{p \in P} C(p) \rightarrow Q^{(\mathbb{N})} \rightarrow 0$$

and

$$0 \rightarrow (\text{Ext}(Q, X))^{\mathbb{N}} \rightarrow \text{Ext}\left(\prod_{p \in P} C(p), X\right) \cong \prod_{p \in P} C(p)$$

since $\prod_{p \in P} C(p)$ is reduced and $\prod_{p \in P} C(p) \in \mathfrak{H}$. Hence ([5], p. 372)

$$\begin{aligned} X &\cong \prod_{p \in P} \text{Ext}(C(p^{\infty}), X) \cong \prod_{p \in P} \text{Hom}(C(p^{\infty}), X \otimes C(p^{\infty})) \\ &\cong \prod_{p \in P} \text{Hom}(C(p^{\infty}), C(p^{\infty})) \cong \prod_{p \in P} Z(p), \end{aligned}$$

which completes the proof.

LEMMA 2. — *Let \mathfrak{M} be a class of left Ext-reproduced groups and let $G \in \mathfrak{M}$.*

1° *If $G_p \neq 0$, then $X/pX \neq 0$.*

2° *If the reduced part of G contains an unbounded p -component for some prime p then*

(i) $X_p = 0$;

(ii) G has a direct summand $Z(p)^{\mathbb{N}} \oplus \prod_{i=1}^{\infty} ((C(p^i))^{\mathfrak{m}_i})$ where \mathfrak{m}_i is

finite and $\mathfrak{m}_{p_i} \neq 0$ for an infinite number of i 's;

- (iii) $\text{Hom}(G, G)$ is not left Ext-reproduced;
 (iv) $\text{Ext}(G, G)$ is not left Ext-reproduced.

Proof.

1° Let $G_p \neq 0$, and suppose that $X = pX$. Then, by [1] (p. 245), we have $pG = G$ since $G \cong \text{Ext}(G, X)$, consequently $G = (C(p^\infty))^{(n)} \oplus G'$, where $G'_p = 0$ and $pG' = G'$. Hence

$$(1) \quad G \cong \text{Ext}(G, X) \cong (\text{Ext}(C(p^\infty), X))^{(n)} \oplus \text{Ext}(G', X)$$

and since X is reduced and $pX = X$ it follows that $X_p = 0$. The exact sequences

$$0 \rightarrow tX \rightarrow X \rightarrow X/tX \rightarrow 0$$

and

$$\text{Ext}(C(p^\infty), tX) = 0 \rightarrow \text{Ext}(C(p^\infty), X) \rightarrow \text{Ext}(C(p^\infty), X/tX) \rightarrow 0$$

show that

$$(2) \quad \begin{aligned} \text{Ext}(C(p^\infty), X) &\cong \text{Ext}(C(p^\infty), X/tX) \\ &\cong \text{Hom}(C(p^\infty), (X/tX) \otimes (Q/Z)) = 0 \end{aligned}$$

since $(X/tX) \otimes C(p^\infty) = 0$ ([1], p. 251). In addition, $G'_p = 0$ and $pG' = G'$ imply (see [1], p. 245)

$$(3) \quad \text{Ext}(G', X)_p = 0$$

and hence it follows from (1), (2) and (3) that $G_p = 0$, contrary to the assumption $G_p \neq 0$. We conclude that $X/pX \neq 0$. This proves 1°.

2° Let $G \in \mathfrak{M}$ be such that $G = D \oplus G'$, where D is divisible and G' is reduced. Suppose that G'_p is unbounded for some prime p and let $B^{(p)}$ denote a basic subgroup of G'_p . Then $B^{(p)}$ is unbounded, and if we put $B^{(p)} = B_1 \oplus \dots \oplus B_i \oplus \dots$, where $B_i = (C(p^i))^{(m_{p_i})}$, then each m_{p_i} is finite ($i = 1, 2, \dots$) (see [3], p. 136), and $m_{p_i} \neq 0$ for an infinite number of i 's. Now $B_1 \oplus \dots \oplus B_i$ is a direct summand of G'_p and hence of G , that is ([1], p. 243)

$$\begin{aligned} \text{Ext}(B_1 \oplus \dots \oplus B_i, X) &\cong \text{Ext}(B_1, X) \oplus \dots \oplus \text{Ext}(B_i, X) \\ &\cong (X/pX)^{(m_{p_1})} \oplus \dots \oplus (X/p^i X)^{(m_{p_i})} \end{aligned}$$

is a direct summand of $\text{Ext}(G, X) \cong G$. Bearing in mind the fact that $B_1 \oplus \dots \oplus B_i$ is a maximal p^i -bounded direct summand of G for every i ([1], p. 99), and that $X/pX \neq 0$, we conclude that, for every i ,

$$\text{Ext}(B_1 \oplus \dots \oplus B_i, X) \cong B_1 \oplus \dots \oplus B_i.$$

(i) Suppose that $X_p \neq 0$, then $X = C(p^k) \oplus X'$ where k is a natural number ([1], p. 80), and

$$\text{Ext}(B_{k+j}, X) \cong (X/p^{k+j} X)^{(m_{p_{k+j}})} \cong (C(p^k) \oplus X'/p^{k+j} X')^{(m_{p_{k+j}})}$$

contradicts the fact that

$$\text{Ext}(B_1 \oplus \dots \oplus B_{k+j}, X) \cong \text{Ext}(B_1, X) \oplus \dots \oplus \text{Ext}(B_{k+j}, X)$$

is isomorphic to $B_1 \oplus \dots \oplus B_{k+j}$ for all $j \geq 1$. Hence we conclude that $X_p = 0$. This proves (i).

(ii) Consider the exact sequences

$$0 \rightarrow tG \xrightarrow{i} G \xrightarrow{\nu} G/tG \rightarrow 0$$

and

$$(4) \quad \text{Ext}(G/tG, X) \xrightarrow{\nu^*} \text{Ext}(G, X) \cong G \rightarrow \text{Ext}(tG, X) \rightarrow 0.$$

It follows from (4) that

$$(5) \quad G \cong \nu^*(\text{Ext}(G/tG, X)) \oplus \prod_{p \in P} \text{Ext}((tG)_p, X)$$

since $\text{Ext}(G/tG, X)$ is divisible. Recall that $X_p = 0$, and hence the exact sequences

$$0 \rightarrow tX \rightarrow X \rightarrow X/tX \rightarrow 0$$

and

$$\text{Ext}((tG)_p, tX) = 0 \rightarrow \text{Ext}((tG)_p, X) \rightarrow \text{Ext}((tG)_p, X/tX) \rightarrow 0$$

show that

$$(6) \quad \text{Ext}((tG)_p, X) \cong \text{Ext}((tG)_p, X/tX) \cong \text{Hom}((tG)_p, (X/tX) \otimes C(p^\infty)).$$

Now, $X_p = 0$ and $X/pX \neq 0$ imply $(X/tX)/p(X/tX) \neq 0$ and hence

$$(X/tX) \otimes C(p^\infty) \cong (C(p^\infty))^{(n_p)},$$

where

$$n_p = r((X/tX)/p(X/tX)) = r(X/pX)$$

([1], p. 255). Hence $\text{Ext}((tG)_p, X)$ has a direct summand $\text{Hom}((tG)_p, C(p^\infty))$ and, by [2] (p. 137),

$$\text{Hom}((tG)_p, C(p^\infty)) \cong Z(p)^{\aleph} \oplus \prod_{i=1}^{\infty} (C(p^i)^{m_{p^i}}).$$

However, the latter group is a direct summand of G by virtue of (5) and (6). This proves (ii).

(iii) By making use of the result in (ii), we see that G contains direct summands $Z(p)^{\aleph}$ and $C(p^i)$ for a suitable i . Hence $\text{Hom}(G, G)$ has a direct summand $\text{Hom}(Z(p)^{\aleph}, C(p^i))$. However,

$$V = \text{Hom}(Z(p)^{\aleph}, C(p^i)) \cong \text{Hom}(Z(p)^{\aleph}/p^i(Z(p)^{\aleph}), C(p^i))$$

and since $Z(p)\mathfrak{N}/p^i(Z(p)\mathfrak{N})$ is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$, it follows that V is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$. Hence, by [3] (p. 136), $\text{Hom}(G, G)$ is not left Ext-reproduced. This proves (iii).

(iv) By (ii), G contains direct summands $\prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}})$ and $Z(p)\mathfrak{N}$, and hence $\text{Ext}(G, G)$ has a direct summand

$$\text{Ext}\left(\prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}}), Z(p)\mathfrak{N}\right).$$

Now, by [2] (p. 137),

$$\begin{aligned} & \text{Ext}\left(\prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}}), Z(p)\mathfrak{N}\right) \\ & \cong \left(\text{Ext}\left(\prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}}), Z(p)\right)^{\mathfrak{N}}\right) \\ & \cong \left(\text{Hom}\left(t\prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}}), Z(p) \otimes C(p^z)\right)^{\mathfrak{N}}\right) \\ & \cong \left(Z(p)\mathfrak{N} \oplus \prod_{i=1}^{\infty} ((C(p^i))^{m_{p^i}})\right)^{\mathfrak{N}}. \end{aligned}$$

The latter group, and consequently $\text{Ext}(G, G)$ as well, contains a bounded direct summand of power $2^{\mathfrak{N}}$. By [3] (p. 136), $\text{Ext}(G, G)$ is not left Ext-reproduced. This proves (iv), and the proof of the lemma is complete.

LEMMA 3. — *Let*

$$G = K \oplus \prod_{p \in P} T_p,$$

where T_p is a finite p -group and $T_p \neq 0$ for an infinite number of primes p and where K has no reduced torsion direct summand. Then G does not belong to any class \mathfrak{M} of left Ext-reproduced groups for which $\text{Ext}(Q, X) \neq 0$.

Proof. — Suppose, to the contrary, that there exists a class \mathfrak{M} of left Ext-reproduced groups with $\text{Ext}(Q, X) \neq 0$, and which contains a group $G = K \oplus \prod_{p \in P} T_p$ satisfying the above-mentioned properties. The exact sequence

$$0 \rightarrow \bigoplus_{p \in P} T_p \rightarrow G \rightarrow Q^{(\mathfrak{N})} \oplus K \rightarrow 0$$

gives rise to the exact sequence

$$(7) \quad \text{Hom}\left(\bigoplus_{\rho \in P} T_\rho, X\right) \rightarrow \text{Ext}(Q^{(\aleph)} \oplus K, X) \rightarrow \text{Ext}(G, X) \cong G$$

and we have that

$$(8) \quad \left| \prod_{\rho \in P} \text{Hom}(T_\rho, X) \right| \leq \aleph.$$

This follows from the following observation : If $T_\rho \neq 0$ then by Lemma 2, $X/pX \neq 0$ and if we put $T_\rho = B_1 \oplus \dots \oplus B_n$ where $B_i = (C(p^i))^{m_i}$, then we deduce from

$$\text{Ext}(T_\rho, X) \cong (X/pX)^{m_1} \oplus \dots \oplus (X/p^n X)^{m_n}$$

that $\text{Ext}(T_\rho, X) \cong T_\rho$. This implies in particular that X_ρ is finite for all primes p for which $T_\rho \neq 0$. This proves (8).

However,

$$\text{Ext}(Q^{(\aleph)} \oplus K, X) \cong (\text{Ext}(Q, X))^\aleph \oplus \text{Ext}(K, X)$$

and our initial assumption implies that

$$|(\text{Ext}(Q, X))^\aleph| \geq 2^\aleph.$$

Hence we deduce from (7) that G contains a torsion-free divisible subgroup of infinite rank, and this contradiction completes the proof.

LEMMA 4. — Let \mathfrak{M} be a class of left Ext-reproduced groups for which $\text{Ext}(Q, X) \neq 0$. Suppose that $G \in \mathfrak{M}$ is not reduced, and let

$$G = D \oplus \left(\bigoplus_{\rho \in P} D_\rho \right) \oplus G'$$

where $D = Q^n$ (n a non-negative integer), $D_\rho = (C(p^z))^{(n_\rho)}$, and where G' is reduced.

1° If $\bigoplus_{\rho \in P} D_\rho \neq 0$, then

(i) $G' = L' \oplus tG'$, where $L' \neq 0$ is a reduced and torsion-free cotorsion group and where tG' is finite;

(ii) $X_\rho = 0$ for all primes p for which $D_\rho \neq 0$;

(iii) $\text{Ext}\left(\bigoplus_{\rho \in P} D_\rho, X\right) \cong L'$, $\text{Ext}(L', X) \cong \bigoplus_{\rho \in P} D_\rho$.

2° If $\text{Ext}(Z(p), X) \neq 0$ for some prime p , then $D_p = 0$.

Proof. — Let $G \in \mathfrak{M}$ and let

$$G = D \oplus \left(\bigoplus_{\rho \in P} D_\rho \right) \oplus G'$$

where D, D_ρ and G' are defined as above.

1° Let us assume that $\bigoplus_{p \in P} D_p \neq 0$, then it is clear that G' is not torsion. Moreover, $(tG')_p$ is bounded for all primes p . In fact, if $(tG')_p$ is unbounded for some prime p then by Lemma 2, $X_p = 0$, and G has a direct summand

$$U = \prod_{i=1}^{\infty} ((C(p^i))^{m_{p_i}}),$$

where m_{p_i} is finite for all i , and $m_{p_i} \neq 0$ for an infinite number of i 's. By [4] (Lemma 2.5), $U/tU \cong Q^{(m)} \oplus K$, where m is infinite and K is reduced. The exact sequences

$$0 \rightarrow tU \rightarrow U \rightarrow Q^{(m)} \oplus K \rightarrow 0$$

and

$$\text{Hom}(tU, X) = 0 \rightarrow \text{Ext}(Q^{(m)} \oplus K, X) \rightarrow \text{Ext}(U, X)$$

show that $\text{Ext}(U, X)$ contains a direct summand $(\text{Ext}(Q, X))^m \cong Q^{(m)}$. However, $\text{Ext}(U, X)$ is a direct summand of G , and hence G contains a torsion-free divisible subgroup of infinite rank. This contradiction proves that $(tG')_p$ is bounded and therefore finite for all primes p .

Now G' , being a direct summand of a cotorsion group, is also cotorsion and hence the exact sequences ([5], p. 372),

$$(9) \quad 0 \rightarrow tG' \rightarrow G' \rightarrow G'/tG' \rightarrow 0$$

and

$$(10) \quad 0 \rightarrow \text{Ext}(Q/Z, tG') \cong \prod_{p \in P} (tG')_p \rightarrow \text{Ext}(Q/Z, G') \cong G' \\ \rightarrow \text{Ext}(Q/Z, G'/tG') = L' \rightarrow 0$$

imply

$$G' \cong \prod_{p \in P} (tG')_p \oplus L'$$

since $\prod_{p \in P} (tG')_p$ is cotorsion and L' is torsion-free cotorsion. Moreover, by Lemma 3, $(tG')_p \neq 0$ for at most a finite number of primes p and hence

$$G' \cong tG' \oplus L',$$

where $L' \neq 0$ since, by assumption, $\bigoplus_{p \in P} D_p \neq 0$, and where tG' is finite. This proves (i).

In order to prove (ii), put $tG' = \bigoplus_{i=1}^n (tG')_{p_i}$ where

$$(tG')_{p_i} = B^{[p_i]} \oplus \dots \oplus B^{[p_i]} \text{ and } B^{[p_i]} = (C(p^i))^{m_{p_i}},$$

m_{ij} a non-negative integer. Then

$$\text{Ext}(tG', X) \cong \bigoplus_{i=1}^n ((X/p_i X)^{m_{i1}} \oplus \dots \oplus (X/p_i^i X)^{m_{ii}})$$

and hence it follows from Lemma 2 that $\text{Ext}(tG', X) \cong tG'$. Furthermore, it is clear that $\text{Ext}(D, X) \cong D$. Hence, if $X_p \neq 0$ for some prime p for which $D_p \neq 0$ then $\text{Ext}(C(p^\infty), X)$ will contain a finite p -group and therefore, if we consider $\text{Ext}(G, X) \cong G$ then tG' is no longer the maximal reduced torsion subgroup of G . Consequently $X_p = 0$ for all primes p for which $D_p \neq 0$ and this proves (ii).

We have $G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus L' \oplus tG'$ and, hence, if we consider $\text{Ext}(G, X) \cong G$, then $\text{Ext}(D, X) \cong D$ and $\text{Ext}(tG', X) \cong tG'$ imply

$$\text{Ext}\left(\left(\bigoplus_{p \in P} D_p\right) \oplus L', X\right) \cong \left(\bigoplus_{p \in P} D_p\right) \oplus L'.$$

However,

$$\text{Ext}\left(\left(\bigoplus_{p \in P} D_p\right) \oplus L', X\right) \cong \prod_{p \in P} \text{Ext}(D_p, X) \oplus \text{Ext}(L', X)$$

and by (ii)

$$(11) \quad \prod_{p \in P} \text{Ext}(D_p, X) \cong \prod_{p \in P} \text{Hom}(D_p, (X/tX) \otimes C(p^\infty))$$

and the latter group is a reduced torsion-free cotorsion group. Moreover, $\text{Ext}(L', X)$ is divisible since L' is torsion-free and hence assertion (iii) follows.

We turn our attention to 2° , and we suppose that $\text{Ext}(Z(p), X) \neq 0$. We shall prove that $D_p = 0$. Assume, to the contrary, that $D_p \neq 0$. Then it follows from (11) that G has a direct summand

$$\text{Hom}(D_p, (X/tX) \otimes C(p^\infty)).$$

Let $D_p = (C(p^\infty))^{(m)}$. We assert that m is finite. In fact, if $m \geq \aleph_0$, then $\text{Hom}(D_p, (X/tX) \otimes C(p^\infty))$ has a direct summand

$$\text{Hom}(C(p^\infty)^{(m)}, C(p^\infty)) \cong (\text{Hom}(C(p^\infty), C(p^\infty))) \cong Z(p)^m$$

and hence the exact sequences

$$0 \rightarrow Z(p)^{(m)} \rightarrow Z(p)^m \quad \text{and} \quad \text{Ext}(Z(p)^m, X) \rightarrow (\text{Ext}(Z(p), X))^m \rightarrow 0,$$

and $\text{Ext}(Z(p), X) \neq 0$, show that $\text{Ext}(Z(p)^m, X)$ has a torsion-free divisible subgroup of infinite rank. This is a contradiction since $\text{Ext}(Z(p)^m, X)$ is a direct summand of G . Hence m is finite, i. e.

$$D_p = (C(p^\infty))^{m_p}, \quad m_p, \text{ a non-negative integer.}$$

Next, we assert that $r((X/tX)/p(X/tX)) = r(X/pX)$ is finite. Indeed, if $r(X/pX) = n \geq \aleph_0$, then $\text{Hom}(D_p, (X/tX) \otimes C(p^\infty))$ has a direct summand

$$\text{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)) \cong \text{Hom}(C(p^\infty), C(p^\infty)^{(n)}) = V,$$

and the latter group contains a subgroup

$$(\text{Hom}(C(p^\infty), C(p^\infty)))^{(n)} \cong Z(p)^{(n)}.$$

Now the exact sequences $0 \rightarrow Z(p)^{(n)} \rightarrow V$ and

$$\text{Ext}(V, X) \rightarrow (\text{Ext}(Z(p), X))^n \rightarrow 0 \quad \text{and} \quad \text{Ext}(Z(p), X) \neq 0,$$

imply that $\text{Ext}(V, X)$ has a torsion-free divisible subgroup of infinite rank. This gives rise to a contradiction since $\text{Ext}(V, X)$ is a direct summand of G , and we conclude that n is finite. Consequently, G has a direct summand

$$\text{Hom}(D_p, (X/tX) \otimes C(p^\infty)) \cong Z(p)^{n_p}, \quad n_p \text{ a natural number.}$$

Moreover, it follows from (iii) and [1] (p. 245) that

$$(12) \quad \text{Ext}(Z(p)^{n_p}, X) \cong D_p \cong (C(p^\infty))^{n_p}.$$

We assert that (12) is impossible. Indeed, suppose that (12) holds, and consider the exact sequences (see [1], p. 250 and p. 252)

$$0 \rightarrow Z(p)^{n_p} \otimes Z \cong Z(p)^{n_p} \rightarrow (Z(p) \otimes Q)^{n_p} \rightarrow Z(p)^{n_p} \otimes (Q/Z) \cong (C(p^\infty))^{n_p} \rightarrow 0$$

and

$$(13) \quad \text{Ext}((C(p^\infty))^{n_p}, X) \rightarrow (\text{Ext}(Q, X))^{\aleph} \rightarrow \text{Ext}(Z(p)^{n_p}, X) \rightarrow 0.$$

We have that

$$(a) \quad |\text{Ext}((C(p^\infty))^{n_p}, X)| = |(\text{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)))^{n_p}| = \aleph$$

since $r((X/tX)/p(X/tX))$ is finite;

$$(b) \quad |(\text{Ext}(Q, X))^{\aleph}| = 2^{\aleph}$$

since, by assumption, $\text{Ext}(Q, X) \neq 0$;

$$(c) \quad |\text{Ext}(Z(p)^{n_p}, X)| = \aleph_0,$$

on account of (12), and the exact sequence (13) implies that this is clearly impossible. Hence (12) leads to a contradiction, and we conclude that $D_p = 0$. This proves 2°, and completes the proof of the lemma.

We now derive properties which are characteristic of the classes \mathfrak{F} and \mathfrak{G} .

THEOREM 5.

(i) If $G \in \mathfrak{F}$, then for every direct summand U of G we have $U \in \mathfrak{F}$.

(ii) \mathfrak{G} is a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{G}$, then for every direct summand U of G we have $U \in \mathfrak{G}$.

Proof.

(i) This statement is an immediate consequence of [4] (Theorem 2.14).

(ii) Let $G = G' \oplus G'' \in \mathfrak{G}$ where $G' \neq 0$, $G'' \neq 0$. Then $tG' \neq 0$, $tG'' \neq 0$ and recall that $G \cong \text{Ext}(Q/Z, tG)$ since G is adjusted ([5], p. 375). Now, both G' and G'' are adjusted and hence

$$\begin{aligned} G &= G' \oplus G'' \cong \text{Ext}(Q/Z, tG') \oplus \text{Ext}(Q/Z, tG'') \\ &\cong \prod_{p \in P} \text{Ext}(C(p^\infty), (tG')_p) \oplus \prod_{p \in P} \text{Ext}(C(p^\infty), (tG'')_p) \\ &\cong \prod_{p \in P} (tG')_p \oplus \prod_{p \in P} (tG'')_p \end{aligned}$$

whence it follows that

$$G' \cong \prod_{p \in P} (tG')_p \in \mathfrak{G}, \quad G'' \cong \prod_{p \in P} (tG'')_p \in \mathfrak{G}.$$

Let \mathfrak{H} be a class of left Ext-reproduced groups such that if $G \in \mathfrak{H}$ then for every direct summand U of G we have $U \in \mathfrak{H}$, and let $\mathfrak{H} \supseteq \mathfrak{G}$. Then it follows from Lemma 1 that $X \cong \prod_{p \in P} Z(p)$. Let $G \in \mathfrak{H}$, then G_p is

bounded for all primes p . Indeed, if G_p is unbounded for some prime p then by Lemma 2, G has a direct summand $Z(p)^{\aleph}$ and by assumption $Z(p)^{\aleph} \in \mathfrak{H}$, which is clearly impossible. Hence G_p is bounded and consequently finite for all primes p , and we have (see [4], Example 2.8)

$$G \cong \text{Ext}\left(tG, \prod_{p \in P} Z(p)\right) \cong \text{Hom}(tG, Q/Z) \cong \prod_{p \in P} (tG)_p.$$

Hence $G \in \mathfrak{H}$ implies $G \in \mathfrak{G}$ and consequently $\mathfrak{H} \subseteq \mathfrak{G}$. This completes the proof.

THEOREM 6. — Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{M}$ then for all direct summands U of G , we have $U \in \mathfrak{M}$. Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Proof. — Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the above mentioned property. Then there are two possibilities, viz. either

- (i) there exists a group $G \in \mathfrak{M}$ which is not reduced, or
- (ii) all groups G in \mathfrak{M} are reduced.

Let us first consider case (i). If $G \in \mathfrak{M}$ is not reduced, then $G = Q^n \oplus G'$ where n is a natural number, and G' is reduced. By assumption, $Q \in \mathfrak{M}$ and hence, by [4] (Theorem 2.2), $H/X \cong Q$ where H is a reduced cotorsion group. Moreover, we also have that $G' \in \mathfrak{M}$. Now \mathfrak{M} , being maximal, cannot contain only divisible groups and hence there exists a non-zero reduced group $G' \in \mathfrak{M}$. We contend that

(a) G' is finite.

In order to prove this, consider the exact sequences

$$0 \rightarrow tG' \rightarrow G' \rightarrow G'/tG' \rightarrow 0$$

and

$$\text{Ext}(G'/tG', X) \rightarrow \text{Ext}(G', X) \cong G' \rightarrow \text{Ext}(tG', X) \rightarrow 0.$$

We conclude that

$$G' \cong \text{Ext}(tG', X) \cong \prod_{p \in P} \text{Ext}((tG')_p, X)$$

since G' is reduced. In the first instance, $(tG')_p$ is bounded for all primes p . In fact, if $(tG')_p$ is unbounded for some prime p then it follows from Lemma 2 that G' has a direct summand $Z(p)^{\aleph}$ and hence our initial assumption implies that $Z(p)^{\aleph} \in \mathfrak{M}$. This contradiction shows that $(tG')_p$ is bounded and hence finite for all primes p . Moreover, by Lemma 3, $(tG')_p \neq 0$ for only a finite number of primes p whence it follows that tG' , and consequently $G' \cong \text{Ext}(tG', X)$ as well, is finite. This proves (a).

To recapitulate, if (i) holds and if $G \in \mathfrak{M}$ then $G \cong Q^n \oplus T$ where n is a non-negative integer and T is a finite group, that is to say $G \in \mathfrak{F}$. Hence $\mathfrak{M} \subseteq \mathfrak{F}$ and the maximality of \mathfrak{M} implies that $\mathfrak{M} = \mathfrak{F}$. This settles the first case.

Next we consider case (ii). In this case, $tG \neq 0$ for all $G \in \mathfrak{M}$. Again, if $G \in \mathfrak{M}$, then $(tG)_p$ is bounded for all primes p , for if $(tG)_p$ is unbounded for some prime p then by Lemma 2, $Z(p)^{\aleph} \in \mathfrak{M}$. This contradiction shows that $(tG)_p$ is bounded and hence finite for all primes p . Since $(tG)_p$ is finite and pure it follows from [1] (p. 80) that $(tG)_p \in \mathfrak{M}$ for all primes p whence

$$(14) \quad G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p.$$

We maintain that $\text{Ext}(Q, X) = 0$.

Assume, to the contrary, that $\text{Ext}(Q, X) \neq 0$. The maximality of \mathfrak{M} implies that tG cannot be finite for all $G \in \mathfrak{M}$ and hence there exists an $H \in \mathfrak{M}$ with tH an infinite torsion group. This implies that $(tH)_p \neq 0$ for an infinite number of primes p and hence

$$H \cong \text{Ext}\left(\bigoplus_{p \in P} (tH)_p, X\right) \cong \prod_{p \in P} (tH)_p.$$

By Lemma 3, $H \notin \mathfrak{M}$. This contradiction shows that $\text{Ext}(Q, X) = 0$.

The latter fact and (14) show that if $G \in \mathfrak{M}$ then $G \in \mathfrak{G}$, that is, $\mathfrak{M} \subseteq \mathfrak{G}$ and hence $\mathfrak{M} = \mathfrak{G}$ since \mathfrak{M} is maximal. This completes the proof of the theorem.

THEOREM 7.

(i) If $G \in \mathfrak{F}$ and $H \in \mathfrak{F}$, then $\text{Hom}(G, H) \in \mathfrak{F}$.

(ii) \mathfrak{G} is a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $\text{Hom}(G, H) \in \mathfrak{G}$.

Proof.

(i) If $G \in \mathfrak{F}$ and $H \in \mathfrak{F}$ then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers and S and T are finite groups. Consequently,

$$\text{Hom}(G, H) \cong Q^{mn} \oplus \text{Hom}(S, T) \in \mathfrak{F}.$$

This proves (i).

(ii) Let $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $G \cong \prod_{p \in P} G_p$, $H \cong \prod_{p \in P} H_p$ where G_p and H_p are finite p -groups for all primes p . Then we have

$$\text{Hom}(G, H) \cong \text{Hom}(tG, H) \cong \text{Hom}(tG, tH) \cong \prod_{p \in P} \text{Hom}((tG)_p, (tH)_p) \in \mathfrak{G}.$$

Let \mathfrak{H} be a class of left Ext-reproduced groups such that if $G \in \mathfrak{H}$, $H \in \mathfrak{H}$ then $\text{Hom}(G, H) \in \mathfrak{H}$, and suppose that $\mathfrak{H} \supseteq \mathfrak{G}$. Then, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$. All groups $G \in \mathfrak{H}$ are reduced since

$$G \cong \text{Ext}(G, X) \cong \text{Ext}(tG, X),$$

and hence if $0 \neq G \in \mathfrak{H}$, then $tG \neq 0$.

Let $G \in \mathfrak{H}$. Then G_p is bounded for all primes p , for if G_p is unbounded for some prime p then it follows from Lemma 2 that $\text{Hom}(G, G)$ is not left Ext-reproduced. Hence G_p is bounded and therefore finite for all

primes p . Consequently,

$$G \cong \text{Ext}(tG, X) \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} \text{Hom}((tG)_p, Q/Z) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{G}$. Therefore $\mathfrak{H} \subseteq \mathfrak{G}$, and the proof is complete.

THEOREM 8. — *Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the property :*

If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, then $\text{Hom}(G, H) \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Proof. — There are two possibilities, viz. either

- (i) \mathfrak{M} contains a group G which is not reduced, or
- (ii) all groups G in \mathfrak{M} are reduced.

Consider case (i), and let $G = D \oplus G'$ where $D \neq 0$ is divisible and G' is reduced. Then D contains no subgroup $C(p^\infty)$, for if $D = C(p^\infty) \oplus D'$, then $\text{Hom}(C(p^\infty), C(p^\infty)) \cong Z(p)$ is a direct summand of $\text{Hom}(G, G) \in \mathfrak{M}$ and, by assumption, $\text{Hom}(\text{Hom}(G, G), G) \in \mathfrak{M}$. However, the latter group has a direct summand $\text{Hom}(Z(p), C(p^\infty)) \cong Q^{(2^{\aleph})} \oplus C(p^\infty)$ ([2], p. 136), and this gives rise to a contradiction since any left Ext-reproduced group contains at most a finite number of copies of Q . This proves that $G = Q^n \oplus G'$ where n is a natural number and G' is reduced. It is also clear that $G' \in \mathfrak{M}$.

Now G'_p is bounded for all primes p , for if G'_p is unbounded for some prime p then it follows from Lemma 2 that $\text{Hom}(G, G)$ is not left Ext-reproduced. This shows that G'_p is bounded and consequently finite for all primes p . Furthermore, $(tG')_p \neq 0$ for at most a finite number of primes p , for if $(tG')_p \neq 0$ for an infinite number of primes p , then by Lemma 3,

$$G' \cong \prod_{p \in P} \text{Ext}((tG')_p, X) \cong \prod_{p \in P} (tG')_p \notin \mathfrak{M}.$$

[We mention in passing the fact that if $(tG')_p \neq 0$ for an infinite number of primes p , then it can also easily be shown that $\text{Hom}(G, G)$ contains a torsion-free divisible subgroup of infinite rank.] This contradiction shows that $(tG')_p \neq 0$ for at most a finite number of primes p whence we deduce that G' is finite. The proof thus far shows in fact that the reduced part of each group $G \in \mathfrak{M}$ is finite.

Hence, if condition (i) holds and $G \in \mathfrak{M}$ then $G = Q^n \oplus T$ where n is a non-negative integer and T is finite, that is to say, $G \in \mathfrak{F}$ and hence $\mathfrak{M} \subseteq \mathfrak{F}$. The maximality of \mathfrak{M} implies that $\mathfrak{M} = \mathfrak{F}$.

We turn our attention to case (ii). Since all $G \in \mathfrak{M}$ are reduced, it follows that if $0 \neq G \in \mathfrak{M}$, then $tG \neq 0$ for we have

$$G \cong \text{Ext}(tG, X) \text{ for all } G \in \mathfrak{M},$$

recall (4) and (5) in the proof of Lemma 2. Moreover, Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes p . Consequently, if $G \in \mathfrak{M}$ then

$$G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{G}$. This proves that $\mathfrak{M} \subseteq \mathfrak{G}$ and since \mathfrak{M} is maximal it follows that $\mathfrak{M} = \mathfrak{G}$. This completes the proof.

THEOREM 9.

(i) Let $G \in \mathfrak{F}$ and $H \in \mathfrak{F}$, and let $\varphi : G \rightarrow H$ be a homomorphism. Then $\text{Ker } \varphi \in \mathfrak{F}$ and $\text{Coker } \varphi \in \mathfrak{F}$.

(ii) \mathfrak{G} is a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, and if $\varphi : G \rightarrow H$ is a homomorphism, then $\text{Ker } \varphi \in \mathfrak{G}$ and $\text{Coker } \varphi \in \mathfrak{G}$.

Proof.

(i) Let $G \in \mathfrak{F}$, $H \in \mathfrak{F}$, and let $\varphi : G \rightarrow H$ be a homomorphism. Then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers, and S and T are finite groups. It is clear that $\varphi(Q^n) \subseteq Q^m$ and that $\varphi(S) \subseteq T$. Hence $\text{Ker } \varphi = (\text{Ker } \varphi \cap Q^n) \oplus (\text{Ker } \varphi \cap S)$ and since $\text{Ker } \varphi \cap Q^n$ is divisible and $\text{Ker } \varphi \cap S$ is finite, our assertion follows.

(ii) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, and let $\varphi : G \rightarrow H$ be a homomorphism. $\text{Ker } \varphi$ and $\text{Im } \varphi$ are reduced, and hence it follows from the exact sequences

$$0 \rightarrow \text{Ker } \varphi \rightarrow G \rightarrow \text{Im } \varphi \rightarrow 0$$

and

$$0 \rightarrow \text{Ext}(Q, \text{Ker } \varphi) \rightarrow \text{Ext}(Q, G) = 0 \rightarrow \text{Ext}(Q, \text{Im } \varphi) \rightarrow 0$$

that $\text{Ker } \varphi$ and $\text{Im } \varphi$ are reduced cotorsion groups.

Note that if $tG \subseteq \text{Ker } \varphi$, then $\varphi = 0$ for then we have

$$G/\text{Ker } \varphi \cong (G/tG)/(\text{Ker } \varphi/tG) \subseteq H$$

and G/tG is divisible, whence $G/\text{Ker } \varphi$ is divisible and hence 0 . Hence $\text{Ker } \varphi = G$, that is, $\varphi = 0$. Note further that if $\text{Ker } \varphi$ is a torsion group then it is necessarily finite and then our assertion is obvious. We may therefore assume that $\text{Ker } \varphi$ is infinite and that $\varphi \neq 0$. We then have

$$\text{Ker } \varphi \cap tG \neq tG, \quad (\text{Ker } \varphi)_p \subseteq (tG)_p \text{ for all primes } p.$$

We maintain that $\text{Ker } \varphi$ is adjusted. In order to prove this, assume to the contrary that $\text{Ker } \varphi$ is not adjusted. Then it follows from [5] (p. 373-374) that

$$\text{Ker } \varphi \cong \prod_{p \in P} (\text{Ker } \varphi)_p \oplus L,$$

where $L \neq 0$ is a reduced torsion-free cotorsion group, and hence L has a direct summand $Z(p)$ for some prime p ([5], p. 372). Hence $\text{Ker } \varphi$, and therefore G as well, contains a subgroup $Z(p)$. That is to say, G contains elements of infinite order which are divisible by arbitrarily high powers of primes $q (q \neq p)$ since $qZ(p) = Z(p)$ for all primes $q \neq p$. This is evidently impossible, and we conclude that $\text{Ker } \varphi$ is adjusted,

i. e. $\text{Ker } \varphi \cong \prod_{p \in P} (\text{Ker } \varphi)_p \in \mathfrak{G}.$

We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & t \text{Ker } \varphi & \longrightarrow & tG & \longrightarrow & \bigoplus_{p \in P} ((tG)_p / (t \text{Ker } \varphi)_p) \longrightarrow 0 \\ & & \downarrow \iota & & \downarrow \mu & & \downarrow \lambda \\ 0 & \longrightarrow & \text{Ker } \varphi & \longrightarrow & G & \longrightarrow & \text{Im } \varphi \longrightarrow 0 \end{array}$$

where ι, μ and λ are the obvious mappings. This gives rise to the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow \text{Ext}(Q/Z, t \text{Ker } \varphi) \longrightarrow \text{Ext}(Q/Z, tG) \longrightarrow \text{Ext}(Q/Z, \bigoplus_{p \in P} ((tG)_p / (t \text{Ker } \varphi)_p)) \longrightarrow 0 \\ \downarrow \iota_* \qquad \qquad \qquad \downarrow \mu_* \qquad \qquad \qquad \downarrow \lambda_* \\ 0 \longrightarrow \text{Ext}(Q/Z, \text{Ker } \varphi) \longrightarrow \text{Ext}(Q/Z, G) \longrightarrow \text{Ext}(Q/Z, \text{Im } \varphi) \longrightarrow 0 \end{array}$$

and since ι_* and μ_* are isomorphisms ([5], p. 375), it follows that λ_* is also an isomorphism. Hence

$$\text{Im } \varphi \cong \text{Ext}(Q/Z, \bigoplus_{p \in P} ((tG)_p / (t \text{Ker } \varphi)_p)) \cong \prod_{p \in P} ((tG)_p / (t \text{Ker } \varphi)_p) \in \mathfrak{G}.$$

The exact sequences

$$(15) \qquad 0 \rightarrow \text{Im } \varphi \rightarrow H \rightarrow \text{Coker } \varphi \rightarrow 0$$

and

$$\text{Hom}(Q, H) = 0 \rightarrow \text{Hom}(Q, \text{Coker } \varphi) \rightarrow \text{Ext}(Q, \text{Im } \varphi) = 0$$

show that $\text{Coker } \varphi$ is a reduced cotorsion group. If we consider the exact sequence (15), then we obtain a situation entirely similar to that in the above commutative diagrams, and we conclude that $\text{Coker } \varphi \in \mathfrak{G}$.

If \mathfrak{H} is a class of left Ext-reproduced groups such that if $K \in \mathfrak{H}, M \in \mathfrak{H}$, and if $\psi : K \rightarrow M$ is a homomorphism, then $\text{Ker } \psi \in \mathfrak{H}, \text{Coker } \psi \in \mathfrak{H}$, and if

$\mathfrak{H} \supseteq \mathfrak{G}$, then $\mathfrak{H} = \mathfrak{G}$. In fact, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$ whence it follows that all groups in \mathfrak{H} are reduced. Hence if $K \in \mathfrak{H}$ then, obviously, every direct summand of K belongs to \mathfrak{H} and hence, by Theorem 6, $K \in \mathfrak{G}$. We have therefore shown that $\mathfrak{H} \supseteq \mathfrak{G}$, and the proof is complete.

Remark. — We turn our attention to the converse of the above theorem.

Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{M}$, $H \in \mathfrak{M}$ and if $\varphi : G \rightarrow H$ is a homomorphism then $\text{Ker } \varphi \in \mathfrak{M}$. $\text{Coker } \varphi \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Indeed, if $G \in \mathfrak{M}$ then, manifestly, every direct summand of G belongs to \mathfrak{M} and hence Theorem 6 implies our assertion.

THEOREM 10.

(i) If $G \in \mathfrak{F}$ and $H \in \mathfrak{F}$, then $\text{Ext}(G, H) \in \mathfrak{F}$.

(ii) \mathfrak{G} is a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $\text{Ext}(G, H) \in \mathfrak{G}$.

Proof.

(i) Let $G \in \mathfrak{F}$, $H \in \mathfrak{F}$, then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers and S and T are finite groups. It is clear that $\text{Ext}(G, H) \cong \text{Ext}(S, T)$ and since the latter group is finite it follows that $\text{Ext}(G, H) \in \mathfrak{F}$.

(ii) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, then $G \cong \prod_{p \in P} (tG)_p$, $H \cong \prod_{p \in P} (tH)_p$ where $(tH)_p$ and $(tG)_p$ are finite p -groups for all primes p . The exact sequences

$$0 \rightarrow \bigoplus_{p \in P} (tG)_p \rightarrow \prod_{p \in P} (tG)_p \rightarrow Q^{(\mathbb{N})} \rightarrow 0$$

and

$$\text{Ext}(Q^{(\mathbb{N})}, H) = 0 \rightarrow \text{Ext}\left(\prod_{p \in P} (tG)_p, H\right) \rightarrow \text{Ext}\left(\bigoplus_{p \in P} (tG)_p, H\right) \rightarrow 0$$

imply

$$\text{Ext}(G, H) \cong \prod_{p \in P} \text{Ext}((tG)_p, H) \cong \prod_{p \in P} \text{Ext}((tG)_p, (tH)_p)$$

and $\text{Ext}((tG)_p, (tH)_p)$ is a finite p -group for all primes p so that $\text{Ext}(G, H) \in \mathfrak{G}$.

Let \mathfrak{S} be a class of left Ext-reproduced groups such that $\text{Ext}(G, H) \in \mathfrak{S}$ whenever $G \in \mathfrak{S}$ and $H \in \mathfrak{S}$, and suppose that $\mathfrak{S} \supseteq \mathfrak{G}$. Then, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$ and hence all groups in \mathfrak{S} are reduced, by virtue of the fact that $G \cong \text{Ext}(tG, X)$ for all $G \in \mathfrak{S}$.

Let $0 \neq G \in \mathfrak{S}$, then $tG \neq 0$. Then Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes p . Thus we have (see [4], Example 2.8)

$$G \cong \text{Ext}(tG, X) \cong \text{Hom}(tG, X \otimes (Q/Z)) \\ \cong \prod_{p \in P} \text{Hom}((tG)_p, Q/Z) \cong \prod_{p \in P} (tG)_p,$$

in other words, $G \in \mathfrak{G}$. Hence $\mathfrak{S} \subseteq \mathfrak{G}$, and the proof is complete.

THEOREM 11. — *Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the property :*

*If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$ then $\text{Ext}(G, H) \in \mathfrak{M}$.
Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.*

Proof. — The method of approach is basically the same as in Theorems 6 and 8. There are two possibilities, viz. either

- (i) all groups G in \mathfrak{M} are reduced, or
- (ii) \mathfrak{M} contains a group which is not reduced.

If (i) holds then for all $G \in \mathfrak{M}$, we have

$$G \cong \text{Ext}(tG, X) \cong \prod_{p \in P} \text{Ext}((tG)_p, X).$$

By Lemma 2, $(tG)_p$ is bounded and hence finite for all primes p . Consequently,

$$G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$. The maximality of \mathfrak{M} implies that $\mathfrak{M} = \mathfrak{G}$.

Let (ii) be valid, then it is clear that $\text{Ext}(Q, X) \neq 0$. Suppose that $G \in \mathfrak{M}$ is not reduced and let

$$G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus G'$$

where $D = Q^n$, n a non-negative integer, $D_p = (C(p^\infty))^{(n)}$ and where G' is reduced. We assert that $\bigoplus_{p \in P} D_p = 0$. In fact, if this is not the case then it follows from Lemma 4 that

$$G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus L' \oplus tG'$$

where $L' \neq 0$ is a reduced torsion-free cotorsion group and tG' is finite. By assumption, $\text{Ext}(G, G) \in \mathfrak{M}$. However,

$$\begin{aligned} \text{Ext}(G, G) \cong & \text{Ext}\left(\bigoplus_{p \in P} D_p, L'\right) \oplus \text{Ext}\left(\bigoplus_{p \in P} D_p, tG'\right) \\ & \oplus \text{Ext}(tG', L') \oplus \text{Ext}(tG', tG') \end{aligned}$$

and we have

- (a) $\text{Ext}\left(\bigoplus_{p \in P} D_p, L'\right) \neq 0$ is a torsion-free and reduced cotorsion group [this follows from Lemma 4, (ii)];
- (b) $\text{Ext}\left(\bigoplus_{p \in P} D_p, tG'\right)$ is a finite group or else $\text{Ext}(G, G) \notin \mathfrak{M}$;
- (c) $\text{Ext}(tG', L')$ is a finite group for the same reason as in (b);
- (d) $\text{Ext}(tG', tG')$ is a finite group — this is obvious.

In other words, $\text{Ext}(G, G)$ is the direct sum of a reduced and non-zero torsion-free cotorsion group, and a finite group, and hence it is not left Ext-reproduced, contrary to $\text{Ext}(G, G) \in \mathfrak{M}$.

This contradiction shows that $\bigoplus_{p \in P} D_p = 0$ and hence

$$G = D \oplus G' = Q^n \oplus G'$$

where n is a natural number and G' is a finite group. It follows from Lemma 2 and Lemma 3 that the reduced part of every group $G \in \mathfrak{M}$ is finite and hence if $G \in \mathfrak{M}$ then $G \in \mathfrak{F}$. Hence $\mathfrak{M} = \mathfrak{F}$ and the proof is complete.

Remark. — If we consider \mathfrak{F} then X satisfies $\prod_{p \in P} Z(p)/X \cong Q$ ([4, Example 2.3]) and it is clear that $\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{F}$. For the class \mathfrak{G} , we have $X \cong \prod_{p \in P} Z(p)$ and we also have that $\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{G}$. Moreover, \mathfrak{G} is a class of left Ext-reproduced groups which is maximal with respect to the property: $\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{G}$. In fact, if \mathfrak{H} is a class of left Ext-reproduced groups which contains \mathfrak{G} and which is such that $\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{H}$, then $\mathfrak{H} = \mathfrak{G}$. This follows from following: If $G \in \mathfrak{H}$ then G_p is bounded for all primes p , for if G_p is unbounded for some prime p then G contains a direct summand $Z(p)^{\aleph}$ (Lemma 2), and since $X \cong \prod_{p \in P} Z(p)$ (Lemma 1), we deduce that $\text{Hom}(G, X) \neq 0$. Hence G_p is bounded and therefore finite for all primes p , consequently

$$G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p \in \mathfrak{G}$$

whence $\mathfrak{H} \subseteq \mathfrak{G}$.

This property can also be used to characterize the classes \mathfrak{F} and \mathfrak{G} .

THEOREM 12. — *Let \mathfrak{M} be a class of left Ext-reproduced groups which is maximal with respect to the property :*

$\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{M}$.
 Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Proof. — There are two alternatives, viz. either

- (i) all groups G in \mathfrak{M} are reduced, or
- (ii) \mathfrak{M} contains a group G which is not reduced.

Let us consider case (i) and let $G \in \mathfrak{M}$. The exact sequence

$$0 \rightarrow tG \rightarrow G \rightarrow G/tG \rightarrow 0$$

yields the exact sequence

$$0 \rightarrow \text{Hom}(tG, X) \rightarrow \text{Ext}(G/tG, X) \rightarrow \text{Ext}(G, X) \cong G \rightarrow \text{Ext}(tG, X) \rightarrow 0$$

and since G and $\text{Hom}(tG, X)$ are reduced it follows that

$$(16) \quad \text{Hom}(tG, X) = 0 = \text{Ext}(G/tG, X)$$

and hence $G \cong \prod_{p \in P} \text{Ext}((tG)_p, X)$ shows that if $0 \neq G \in \mathfrak{M}$ then $tG \neq 0$.

if $G \in \mathfrak{M}$, then $(tG)_p$ is bounded for all primes p . Assume, to the contrary, that $(tG)_p$ is unbounded for some prime p , then by Lemma 2, G has a direct summand $Z(p)^{\aleph}$, and hence $\text{Hom}(G, X) = 0$ and (16) imply

$$\text{Hom}(Z(p), X) = 0 = \text{Ext}(Z(p), X).$$

The exact sequence (see [1], p. 252 and 255)

$$0 \rightarrow Z(p) \otimes Z \cong Z(p) \rightarrow Z(p) \otimes Q \rightarrow Z(p) \otimes (Q/Z) \cong C(p^\infty) \rightarrow 0$$

leads to the exact sequence

$$0 \rightarrow \text{Ext}(C(p^\infty), X) \rightarrow (\text{Ext}(Q, X))^{\aleph} \rightarrow \text{Ext}(Z(p), X) = 0$$

and hence

$$(17) \quad \text{Ext}(C(p^\infty), X) = 0 = \text{Ext}(Q, X).$$

However, $\text{Hom}(Z(p), X) = 0$ implies $X_p = 0$ and, by Lemma 2, $(tG)_p \neq 0$ implies $X/pX \neq 0$. Hence

$$\text{Ext}(C(p^\infty), X) \cong \text{Ext}(C(p^\infty), X/tX) \cong \text{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)) \neq 0$$

since $X/tX \neq p(X/tX)$ ([1], p. 255). This is however contrary to (17) and hence we conclude that $(tG)_p$ is bounded and consequently finite for all primes p . This implies that

$$G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

or alternatively, $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$ and hence $\mathfrak{M} \subseteq \mathfrak{G}$ so that $\mathfrak{M} = \mathfrak{G}$.

We now turn our attention to (ii), and we notice that $\text{Ext}(Q, X) \neq 0$. Let $G \in \mathfrak{M}$ be a group which is not reduced and let

$$G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus G'$$

where $D = Q^n$, n a non-negative integer, $D_p = (C(p^*))^{(n)}$ and where G' is reduced. We assert that G'_p is bounded for all primes p . In fact, if G'_p is unbounded for some prime p , then by Lemma 2, G contains a direct summand $Z(p)^{\aleph}$ and hence $\text{Hom}(Z(p), X) = 0$. The exact sequence

$$(18) \quad 0 \rightarrow Z(p) \rightarrow Z(p) \otimes Q \rightarrow C(p^*) \rightarrow 0$$

yields the exact sequence

$$(19) \quad 0 \rightarrow \text{Ext}(C(p^*), X) \rightarrow (\text{Ext}(Q, X))^{\aleph} \rightarrow \text{Ext}(Z(p), X) \rightarrow 0$$

and hence it follows from (19) that $\text{Ext}(Z(p), X) \neq 0$ since $\text{Ext}(C(p^*), X)$ is reduced and $\text{Ext}(Q, X) \neq 0$. Now, $\text{Ext}(Z(p)^{\aleph}, X)$ is a direct summand of $G \cong \text{Ext}(G, X)$ and hence the exact sequences

$$0 \rightarrow Z(p)^{(\aleph)} \rightarrow Z(p)^{\aleph} \text{ and } \text{Ext}(Z(p)^{\aleph}, X) \rightarrow (\text{Ext}(Z(p), X))^{\aleph} \rightarrow 0$$

show that $\text{Ext}(Z(p)^{\aleph}, X)$, and consequently G as well, contains a torsion-free divisible subgroup of infinite rank. This contradiction shows that G'_p is bounded and therefore finite for all primes p .

We contend that $\bigoplus_{p \in P} D_p = 0$. Indeed, if $\bigoplus_{p \in P} D_p \neq 0$ then by Lemma 4, we have $G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus L' \oplus tG'$, where $L' \neq 0$ is a reduced and torsion-free cotorsion group and where tG' is finite. Moreover, it follows from Lemma 4, (iii), that for some prime p for which $D_p \neq 0$, L' contains a direct summand $Z(p)$. Hence $\text{Hom}(Z(p), X) = 0$, and we deduce from the exact sequences (18) and (19) that $\text{Ext}(Z(p), X) \neq 0$. Now it follows from Lemma 4, 2°, that $D_p = 0$. This contradiction shows that $\bigoplus_{p \in P} D_p = 0$ and hence

$$G = Q^n \oplus G'$$

where n is a natural number and G' is reduced. Moreover, the finiteness of G'_p for all primes p implies that

$$G' \cong \text{Ext}(tG', X) \cong \prod_{p \in P} \text{Ext}((tG')_p, X) \cong \prod_{p \in P} (tG')_p$$

and by Lemma 3, $(tG')_p \neq 0$ for only a finite number of primes p , that is to say, G' is finite. It is also clear that the reduced part of each group $G \in \mathfrak{M}$ is finite. Hence if $G \in \mathfrak{M}$ then $G \in \mathfrak{F}$ and hence $\mathfrak{M} \subseteq \mathfrak{F}$. Consequently $\mathfrak{M} = \mathfrak{F}$, and the proof is complete.

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