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ON N-HIGH SUBGROUPS OF ABELIAN GROUPS

BY

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1. Introduction.

This paper is based on a curious property of N-high subgroups when N is a subgroup of G^1 the subgroup of elements of infinite height of a group G. Let G be a group, N a subgroup of G, we say that a subgroup H of G is N-high if H is maximal with respect to the property $H \cap N = 0$. Our first result (theorem 2.4) is that given a group G and N a subgroup of G^1, then $G = \langle H, K \rangle$ whenever H is an N-high subgroup of G and K is a pure subgroup of G containing N. A close look at the proof of this result shows that the assumption that K is pure can be replaced by the weaker one that $N \subset K^1$. An immediate consequence is the classical theorem that divisible subgroups of a group are absolute summands of the group.

N-high subgroups where $N \subset G^1$ were first introduced and studied by IRWIN and WALKER in [3]. These authors proved that N-high subgroups are pure and that the factor groups they induce are divisible. It turns out (theorem 2.5) that H is an N-high subgroup of a group G, where $N \subset G^1$ if and only if H is pure, $H \cap N = 0$, $G = \langle H, K \rangle$ for all K pure containing N and G/H is divisible. We use this property of N-high subgroups where $N \subset G^1$, to generalize and simplify many results in [4]. In particular, we obtain a criterion for a pure subgroup of a group G containing $N \subset G^1$ to be a summand of G (theorem 3.1).

In the fourth part, we define the concept of quasi-essential and strongly quasi-essential subsocles of a p-group (definition 4.1) and proceed to characterize those quasi-essential subsocles which are also centers of purity (theorem 4.4) and those which are strongly quasi-essential (theorem 4.8).

We use standard notation from [1]. The symbol Z^+ denotes the set of positive integers. If G is a p-group, R a subgroup of G and $g \in R$, the symbol $h_R(g)$ denotes the height of the element g in the subgroup R. All groups considered are Abelian.

2. A characterization of N-high subgroups of a group G, with $N \subset G^1$.

We need the following lemmas.

LEMMA 2.1. — *Let G be a p-group, N subgroup of G^1 and K a pure subgroup of G containing N. Then for any N-high subgroup H of G, $G = \langle H, K \rangle$.*

Proof. — Clearly $\langle H, K \rangle \supset H[p] \oplus N[p] = G[p]$. By induction suppose $\langle H, K \rangle \supset G[p^n]$. Let $g \in G$, $o(g) = p^{n+1}$, if $g \notin H$, $\langle g, H \rangle \cap N \neq 0$ thus there exists $h \in H$, $g' \in N$, and $m < n+1$, such that

$$p^m g + h = g' \neq 0$$

since K is pure, $g' \in K^1$, thus there exists $k \in K$, such that $g' = p^m k$, or $h = p^m(g - k)$. If $h \neq 0$, by purity of H (*see* [3], theorem 5) there exists $h' \in H$, such that $p^m h' = h$, therefore $p^m(g - k - h') = 0$. This implies $g - k - h' \in \langle H, K \rangle$, and $g \in \langle H, K \rangle$, thus $\langle H, K \rangle \supset G[p^{n+1}]$. By induction

$$G = \langle H, K \rangle.$$

LEMMA 2.2. — *Let G be a torsion group, N a subgroup of G^1 and K a pure subgroup of G containing N. Then for any N-high subgroup H of G, $G = \langle H, K \rangle$.*

Proof. — Let $G = \sum Gp$, $H = \sum Hp$, $K = \sum Kp$ and $N = \sum Np$ then for each prime p, Hp is Np-high in Gp (*see* [2], lemma 11) and since Kp is pure containing Np, lemma 2.1 holds and $Gp = \langle Hp, Kp \rangle$. Therefore

$$G = \sum \langle Hp, Kp \rangle = \langle H, K \rangle.$$

LEMMA 2.3. — *Let G be a group, N a subgroup of G^1 and H an N-high subgroup of G. Then H_t is N_t-high in G_t.*

Proof. — Clearly $H_t \cap N_t = 0$, let $g \in G_t$, $o(g) = b$, $g \notin H$ then $\langle g, H \rangle \cap N \neq 0$, thus there exists $h \in H$, $n \in N$, and a positive integer a such that $ag + h = n \neq 0$. Clearly $a \neq b$. Now $bag + bh = bn$, thus $bh = bn = 0$, and $h \in H_t$, $n \in N_t$, therefore $\langle g, H_t \rangle \cap N_t \neq 0$. This implies that H_t is N_t-high in G_t.

THEOREM 2.4. — *Let G be a group, N a subgroup of G^1 and K a pure subgroup of G containing N. Then for any N-high subgroup H of G,*

$$G = \langle H, K \rangle.$$

Proof. — Suppose $g \in G$, $g \notin H$, then $\langle g, H \rangle \cap N \notin 0$, thus there exists $h \in H$, $n \in N$ and a positive integer a, such that

$$ag + h = n \neq 0.$$

By an argument similar to the one used in lemma 2.1, there exists $h' \in H$ and $k \in K$ such that $a(g + h' - k) = 0$. Thus $g + h' - k \in G_t$. But, by lemmas 2.3 and 2.2, we know that $G_t = \langle H_t, K_t \rangle$, therefore $g \in \langle H, K \rangle$ and $G = \langle H, K \rangle$.

A classical theorem follows immediately from theorem 2.4.

COROLLARY. — *If D is a divisible subgroup of a group G, then D is an absolute summand of G.*

Proof. — Let $D = N$ in theorem 2.4, since D is divisible it is pure in G. Thus $G = D \oplus H$, for any D-high subgroup H of G.

THEOREM 2.5. — *Let G be a group, N a subgroup of G^1 and H a subgroup of G disjoint from N. Then H is N-high in G if and only if H is pure, G/H is divisible and $G = \langle H, K \rangle$ for any pure subgroup K of G containing N.*

Proof. — The necessity follows from theorem 2.4. Suppose then that H satisfies the conditions of the theorem. Since $H \cap N = 0$ there exists an N-high subgroup H' of G containing H. Since H' is pure in G, H'/H is pure in G/H which is divisible, therefore H'/H is divisible and $G/H = (H'/H) \oplus (R/H)$ where R can be chosen to contain N. Since H is pure in G and R/H is pure in G/H, R is pure in G, and since $R \supset N$,

$$R = \langle R, H \rangle = G.$$

Therefore

$$H = R \cap H' = G \cap H' = H'$$

and H is N-high in G.

3. Some applications.

We first obtain a criterion for pure subgroups of a group G to be summands of G.

THEOREM 3.1. — *Let G be a group, K a pure subgroup of G containing a subgroup N of G^1. Then K is a direct summand of G if and only if there exists an N-high subgroup H of G such that $H \cap K$ is a direct summand of H.*

Proof. — Suppose $G = K \oplus L$, let M be any N-high subgroup of K, then it is easy to see that $H = L \oplus M$ is N-high in G and $H \cap K = M$ is a summand of H.

Suppose now that there exists an N-high subgroup H of G such that $H = (H \cap K) \oplus R$, by theorem 2.4 :

$$G = \langle H, K \rangle = \langle (H \cap K) \oplus R, K \rangle = \langle R, K \rangle$$

and since $R \cap K = 0$, $G = R \oplus K$.

The following corollary contains theorem 2 in [4].

COROLLARY. — *A reduced group G splits over its maximal torsion subgroup G_t if and only if some N-high subgroup of G splits, where $N \subset G^1 \cap G_t$.*

Proof. — If G is reduced and $G = G_t \oplus L$ then $G^1 \subset G_t$ and since G_t is pure theorem 3.1 implies there exists an N-high subgroup such that $H \cap G_t = H_t$ is a summand of H. Now if H is N-high and $H = H_t \oplus L$ since $N \subset G_t \cap G^1$ by theorem 3.1, G_t is a summand of G.

For what follows we need the following lemmas.

LEMMA 3.2. — *Let G be a group, H a subgroup of G then if K/H_t is an (H/H_t)-high subgroup of G/H_t, then K is pure in G and $K \supset G_t$.*

Proof. — Suppose $ng \in K$ where $g \in G$. Let $0 \neq h = ag + k \in \langle K, g \rangle \cap H$ then $nag + nk = nh \in K \cap H = H_t$, therefore $h \in H_t$, thus $\langle K, g \rangle \cap H = H_t$ which implies $\langle K, g \rangle = K$, therefore $g \in K$ and thus K is pure in G. Now if $g \in G_t$ then letting $n = o(g)$ in the above argument we see that $K \supset G_t$.

LEMMA 3.3. — *Let G be a group, N a subgroup of G^1, H an N-high subgroup of G and K a pure subgroup of G containing $\langle N, G_t \rangle$ and such that $K \cap H = H_t$. Then for any N-high subgroup H' we have $K \cap H' = H'_t$.*

Proof. — Such K do exist (lemma 3.2). Clearly $K \cap H' \supset H'_t$. Let $h' \in K \cap H'$ and suppose $h' \notin H$ then there exists $h \in H$, $g \in N$ and a positive integer a such that $ah' + h = g \neq 0$, thus $h \in K \cap H = H_t$ let $b = o(h)$, then

$$bah' = bah' + bh = bg \in H' \cap N = 0$$

thus $bah' = 0$ and consequently $h' \in H'_t$. Therefore $K \cap H' = H'_t$.

COROLLARY 1 ([4], lemma). — *If G is a group, N a subgroup of G^1 and H is an N-high subgroup of G, then H/H_t is a summand of G/H_t.*

Proof. — Let K/H_t be H/H_t-high in G/H_t. Choose $K \supset N$. Then, since K is pure in G (lemma 3.2), it follows from theorem 2.4 that $G = \langle K, H \rangle$. Therefore $G/H_t = (H/H_t) \oplus (K/H_t)$.

COROLLARY 2 ([4], theorem 4). — *Let H and H' be two N-high subgroups of a reduced group G where N is a subgroup of G^1. Then $H/H_t \simeq H'/H'_t$ and $G/H_t \simeq G/H'_t$.*

Proof. — From corollary 1, $G/H_t = (H/H_t) \oplus (K/H_t)$. From lemma 3.3, $K \cap H' = H'_t$, therefore $G/H'_t = (H'/H'_t) \oplus (K/H'_t)$. The result follows from this and the fact that $G/H \simeq G/H'$ (*see* [3]).

COROLLARY 3 ([4], theorem 1). — *Let G be a reduced group, N a subgroup of G^1 and H an N-high subgroup of G then if $H = H_t \oplus L$, we have $G = K \oplus L$ where K/G_t is the divisible part of G/G_t.*

Proof. — $G/H_t = H/H_t \oplus K/H_t$ from corollary 1.

Now K/H_t is divisible since $K/H_t \simeq G/H$, and $H/H_t \simeq L$ is reduced. Thus K/G_t is the divisible part of G/H_t. Now $K \cap H = H_t$ implies $K \cap L = 0$ and $\langle K, H \rangle = G$ implies $\langle K, L \rangle = G$, therefore

$$G = K \oplus L.$$

4. Generalizations. Quasi-Essential subsocles of p-groups.

It is natural to ask, what kind of subgroups of a group G have properties similar to subgroups of G^1. We consider first p-groups. It is trivial to verify that two subgroups of a p-group are disjoint if and only if their socles are. Thus it suffices to consider subgroups of the socle of a p-group which we will call subsocles.

DEFINITION 4.1. — Let G be a p-group, a subsocle S of G is said to be quasi-essential (q. e.) if $G = \langle H, K \rangle$ whenever H is an S-high subgroup of G and K a pure subgroup of G containing S. S is said to be strongly quasi-essential (s. q. e.) if every subgroup of S is q. e.

We now proceed to characterize those quasi-essential subsocles of G a p-group G which are also centers of purity (*see* [7] and [6]).

THEOREM 4.2. — *Let G be a p-group, S a center of purity, $S \subset G[p]$. If S is not quasi-essential in G then there exists $n \in Z$, $g \in G[p]$, $g \notin S$ and $s \in S$ such that*

$$h(s) = h(g) = n \qquad \text{and} \qquad h(s+g) = n+1.$$

Proof. — Set $P_n = (p^n G)[p]$, $P_\infty = G^1[p]$ and $P_{\infty+1} = 0$ then it is known (*see* [6]) that S is a center of purity if and only if

$$P_n \supset S \supset P_{n+2} \qquad \text{for some} \quad n \in \{1, 2, \ldots, \infty, \infty+1\}.$$

From lemma 2.1, we see that if $n = \infty$, i. e. $S \subset G^1$, S is q. e. Thus if S is not q. e. there exists $n \in Z^+$, such that

$$P_n \supset S \supset P_{n+2}.$$

Also S is not q. e. implies that there exist a pure subgroup K of G containing S and an S-high subgroup H, of G such that $\langle H, K \rangle \neq G$. Let

$\langle H, K \rangle = R$. Since $R \supset G[p]$ and $R \neq G$, R is not pure in G (*see* [5], lemma 12). Therefore there exists an element $x \in R[p]$ such that $h(x) > h_R(x)$. H and K being both pure in G implies that $x \notin H$ and $x \notin K$ Therefore there exists $g \in H[p]$ and $s \in S$ such that $x = g + s$, $g \neq 0 \neq s$. It is easy to verify that

$$h_R(g) = h_H(g) = h(g) \qquad \text{and} \qquad h_R(s) = h_K(s) = h(s),$$

therefore

$$h(g) = h(s) \leq h_R(g + s) < h(g + s).$$

Now $s \in S$ implies $h(s) \geq p^n$, $g \notin S$ implies $h(g) \leq n + 1$ and since $S \supset P_{n+2}$ we conclude that $h(s) = h(g) = n$ and $h(g + s) = n + 1$ as stated.

COROLLARY 1. — *Let G be a p-group, S a subsocle of G such that*

$$P_n \supset S \supset P_{n+1} \quad \textit{then } S \textit{ is quasi-essential.}$$

Proof. — S is a center of purity, thus theorem 4.2 applies and clearly there exists no pair $g \in G[p]$, $g \notin S$ and $s \in S$ that satisfy the conditions of the theorem. Thus S is q. e.

COROLLARY 2. — *Let G be a p-group, S subsocle of G such that S supports an absolute summand A of G then S is quasi-essential.*

Proof. — S is a center of purity, thus theorem 4.2 holds and again if $g \notin S$ and $s \in S$ and $h(g) = h(s)$ then, since g can be embedded in a complementary summand of A in G, $h(g + s) = h(g) = h(s)$. Therefore the condition of the theorem cannot be satisfied and S must be q. e.

COROLLARY 3. — *Let G be a p-group, K a pure subgroup of G containing P_n for some $n \in Z^+$, then K is a direct summand containing $p^n G$.*

Proof. — Since P_n is q. e., $G = \langle K, H \rangle$, where H is a P_n-high subgroup of G. Now H is bounded, in fact $p^n H = 0$, and $G/K \simeq H/H \cap K$, therefore K is a direct summand of G and $p^n G \subset K$.

In fact, it turns out that the conditions on S in corollary 1 and 2 as well as the condition that S be quasi-essential and a center of purity, are equivalent provided $S \not\subset G^1$. To prove this, we need the following lemma.

LEMMA 4.3. — Let G be a p-group, H a pure subgroup of G such that G/H is pure-complete. Let S be a subsocle of G such that $H[p] \subset S$. Then S supports a pure subgroup K of G containing H.

Proof. — Since G/H is a pure-complete group, by definition, every subsocle of G/H supports a pure subgroup of G/H. Now $\langle S, H \rangle / H$ is clearly a subsocle of G/H, therefore there exists K/H a pure subgroup of G/H such that

$$(K/H)[p] = \langle S, H \rangle / H.$$

Since H is pure in G, K is pure in G (*see* [5], lemma 2). Clearly $K[p] \supset S$, let $k \in K[p]$, then $k + H \in (K/H)[p] = \langle S, H \rangle / H$, thus there exists $s \in S$ and $h \in H$ such that $k - s = h$, but $ph = p(k - s) = 0$, and since $S \supset H[p]$, we conclude that $k \in S$. Therefore $K[p] = S$.

COROLLARY. — *Let G be a p-group, S a subsocle containing P^u* (*see* theorem 4.2) *for some $n \in Z^+$, then S supports a pure subgroup of G containing $p^n G$.*

Proof. — Let G_n be as in [1], p. 98. Then G_n is pure in G, $G_n[p] = P_n$ and G/G_n is bounded and therefore pure complete. Thus lemma 4.3 holds, and S supports a pure subgroup of G containing G_n.

THEOREM 4.4. — *Let G be a p-group, S subsocle of G not contained in G^1 then the following are equivalent:*

(i) *S is both a center of purity and a quais-essential subsocle of G;*
(ii) *S supports an absolute direct summand of G;*
(iii) *There exists $n \in Z^+$ such that $P_n \supset S \supset P_{n+1}$.*

Proof. — (i) implies (ii). Suppose S satisfies (i), then since S is a center of purity $S \supset P_m$ for some $m \in Z^+$ and by the corollary to lemma 4.3, S supports a pure subgroup K of G. Since S is also quasi-essential K is an absolute summand of G.

(ii) implies (i). Suppose S supports an absolute summand K. Then S is clearly a center of purity and by corollary 2 to theorem 4.2, S is q. e.

(i) implies (iii). Suppose S satisfies (i), then S supports an absolute summand K of G. Since S is a center of purity, we know there exists $m \in Z^+ \ni P_m \supset S \supset P_{m+2}$. Suppose $P_{m+1} \not\supset S$, we will show, by contradiction, that $S \supset P_{m+1}$. Indeed, suppose not, i. e. there is $x \in G[p]$ such that $x \notin S$ and $h(x) = m + 1$. Now $P_{m+1} \not\subset S$ implies there exists $s \in S$, $h(s) = m$, otherwise $P_m \subset S \subset P_{m+1}$, and we would be done. Let

$$y = x - s \quad \text{then } h(y) = m, \qquad y \notin S \qquad \text{and} \qquad h(y+s) = m + 1.$$

Since $y \notin S$ there is an S-high subgroup H of G such that $y \in H$. But, $G = K \oplus H$ and $h(y) = h(s)$ imply that $h(y+s) = h(y) = h(s)$ which is a contradiction. Therefore $S \supset P_{m+1}$.

(iii) implies (i). If S satisfies (iii) it is a center of purity (*see* theorem 4.2, proof) and by corollary 1 to theorem 4.2 it is also q. e.

At this point we have completely characterized those quasi-essential subsocles of a p-group which are also centers of purity. An immediate consequence is the following.

COROLLARY. — *Let G be a p-group, A a pure subgroup of G, then A is an absolute direct summand of G if and only if A is divisible or $P_n \supset A[p] \supset P_{n+1}$ for some $n \in Z^+$.*

The strongly quasi-essential subsocles have also a simple characterization which can be obtained from the previous result. We need the following lemmas.

LEMMA 4.5. — *Let G be a group; A, B, C three subgroups of G then*

$$\langle A \cap B, C \cap B \rangle = \langle A \cap B, C \rangle \cap B = \langle A, C \cap B \rangle \subset B.$$

LEMMA 4.6. — *Let G be a group, N a subgroup of M a subgroup of G, if a subgroup H is N-high in G then $H \cap M$ is N-high in M. Conversely if H' is an N-high subgroup of M then $H' = H \cap M$ for any N-high subgroup H of G containing H'.*

Proof. — Let H be N-high in G then for all $x \notin H$, we have

$$\langle H, x \rangle \cap N \neq \mathrm{o}.$$

Suppose $m \in M$, $m \notin H$, then

$$\langle H \cap M, m \rangle \cap N = \langle H, m \rangle \cap M \cap N = \langle H, m \rangle \cap N \neq \mathrm{o}.$$

and since $(H \cap M) \cap N = \mathrm{o}$, $H \cap M$ is N-high in M.

Let H' be an N-high subgroup of M, and let H be any N-high subgroup of G then $H \cap M \supset H'$ and $(H \cap M) \cap N = \mathrm{o}$. The maximality of H' implies $H \cap M = H'$.

LEMMA 4.7. — *Let G be a p-group, S a quasi-essential subsocle of G. Let K be a pure subgroup of G containing S. Then S is quasi-essential in K.*

Proof. — Let M be a pure subgroup of K containing S and let H be an S-high subgroup of K. Let H' be an S-high subgroup of G containing H then, since S is q. e. and M is also pure in G we have $\langle M, H' \rangle = G$, thus by lemma 4.5 and 4.6,

$$K = \langle M, H' \rangle \cap K = \langle M \cap K, H' \rangle \cap K = \langle M \cap K, H' \cap K \rangle = \langle M, H \rangle,$$

and S is q. e. in K as stated.

THEOREM 4.8. — *Let G be a p-group, S a subsocle of G then S is strongly quasi-essential if and only if either $S \subset G^1$ or there exists $n \in Z^+$, such that $p^n G = \mathrm{o}$ and $(p^{n-1} G)[p] \supset S$.*

Proof. — If $S \subset G^1$, S is s. q. e. follows from lemma 2.1. If there exists $n \in Z^+$ such that $p^{n-1} G[p] \supset S \supset p^n G = \mathrm{o}$ then S is s. q. e. as a consequence of corollary 1 to theorem 4.2. Suppose now that S is s. q. e. and $S \not\subset G^1$ then there exists $s \in S$ such that $h(s) < \infty$. By corollary 24.2 in [1], s can be embedded in a finite pure subgroup K such that $K[p] = \langle s \rangle$. Since S is s. q. e., K is an absolute summand of G. Thus by theorem 4.4, there exists $m \in Z^+$, such that $(p^m G)[p] \subset \langle s \rangle \subset (p^{m-1} G)[p]$ but $\langle s \rangle$ is a cyclic group of order p, therefore G is a bounded group. This implies that S supports a pure subgroup M of G, and since S is q. e., M is an absolute summand of G. From lemma 4.7, we see that every subsocle of M is q. e. in M, and thus every summand of M is an absolute summand. By problem 11 (*b*), p. 93 in [1], $M = \sum C(p^n)$ for some $n \in Z^+$ and

$$S = M[p] \subset (p^{n-1} G)[p].$$

Clearly $M[p] \not\subset (p^n G)[p]$, therefore $(p^n G)[p] \subset S \subset (p^{n-1} G)[p]$, and since M is pure $P^n G \subset M$. Thus

$$p^n G = (p^n G) \cap M = p^n M = 0,$$

and the proof is complete.

The following characterization follows immediately from theorem 4.8.

THEOREM 4.9. — *Let G be a p-group, every subsocle of G is quasi-essential if and only if G is divisible or G is a direct sum of cyclic groups of same order.*

We have not been able to decide whether a quasi-essential subsocle is necessarily a center of purity or not. But in the next theorem, we have a case where quasi-essential subsocles are centers of purity.

THEOREM 4.10. — *Let G be a p-group, if G is pure-complete then every quasi-essential subsocle of G is a center of purity.*

Proof. — Let S be q. e. Since G is pure-complete, S supports a pure subgroup K of G. This K is an absolute summand and therefore the result follows from corollary 2 to theorem 4.2

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