ON THE CONDUCTOR FORMULA OF BLOCH

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ABSTRACT

In [6], S. Bloch conjectures a formula for the Artin conductor of the ℓ-adic etale cohomology of a regular model of a variety over a local field and proves it for a curve. The formula, which we call the conductor formula of Bloch, enables us to compute the conductor that measures the wild ramification by using the sheaf of differential 1-forms. In this paper, we prove the formula in arbitrary dimension under the assumption that the reduced closed fiber has normal crossings.

0. Introduction

Let K be a discrete valuation field with perfect residue field F and let X_K be a proper smooth scheme over K of dimension d. We briefly recall the definition of the conductor. We give a detailed account in Section 6.1. The Swan conductor $\mathrm{Sw}(X_K/K)$ of X_K is defined to be the alternating sum $\mathrm{Sw}(X_K/K) = \sum_{q=0}^{2d}(-1)^q \mathrm{Sw} H^q(X_{\bar{K}}, \mathbf{Q}_\ell)$ of the Swan conductor of the ℓ-adic etale cohomology for a prime ℓ different from the characteristic p of F. The Swan conductor of an ℓ-adic representation V is defined to be the intertwining number

$$\mathrm{Sw}(V) = \frac{1}{[L:K]} \sum_{\sigma \in P_{L/K}} \mathrm{sw}_{L/K}(\sigma)\mathrm{Tr}(\sigma : V)$$

by taking a sufficiently large finite Galois extension L of K, where $\mathrm{sw}_{L/K}(\sigma)$ denotes the Swan character and $P_{L/K}$ denotes the wild inertia subgroup of Gal(L/K). For a proper flat and regular scheme X over $S = \mathrm{Spec}\ \mathscr{O}_K$ such that $X \otimes_{\mathscr{O}_K} K = X_K$, the Artin conductor $\mathrm{Art}(X/\mathscr{O}_K)$ is defined by

$$\mathrm{Art}(X/\mathscr{O}_K) = \chi(X_{\bar{K}}) - \chi(X_{\bar{F}}) + \mathrm{Sw}(X_K/K).$$

In the right hand side, χ denotes the ℓ-adic Euler number.

To state the conductor formula, Bloch introduces in [6] the localized self-intersection class

$$(\Delta_X, \Delta_X)_S = (-1)^{d+1} c_{d+1}{}^X_{X_F}\big(\Omega^1_{X/\mathscr{O}_K}\big) \cap [X] \in \mathrm{CH}_0(X_F)$$

where $c_{d+1}{}^X_{X_F}(\Omega^1_{X/\mathscr{O}_K}) \cap [X]$ denotes the localized Chern class of the coherent $\mathscr{O}_X$-module $\Omega^1_{X/\mathscr{O}_K}$ and $\dim X = d+1$. We give an explicit computation in Proposition 5.1.6. Let $\deg : \mathrm{CH}_0(X_F) \to \mathrm{CH}_0(F) = \mathbf{Z}$ be the degree map. Bloch formulates the following in [6].

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Conjecture **6.2.1.** — *Let* K *be a discrete valuation field with perfect residue field* F *and let* X *be a proper flat and regular scheme over* $\mathcal{O}_K$ *with smooth generic fiber. Then we have*

$$\mathrm{Art}(X/\mathcal{O}_K) = -\deg(\Delta_X, \Delta_X)_S.$$

If $\dim X_K = 1$, it is proved by him in the same paper [6]. If $\dim X_K = 0$, it is nothing but the classical conductor-discriminant formula in algebraic number theory. For an elliptic curve, the formula is known in [38] Corollary 2 of Theorem 1 to be equivalent to the Tate-Ogg formula [31] for the relation between the conductor and the discriminant. The Milnor formula ([10] Exp. XVI Conjecture 1.9) for isolated singularities is shown to follow from the conductor formula in [33].

The main result of this paper is the following.

Theorem **6.2.3.** — *Let* K *and* X *be as in Conjecture 6.2.1. Assume that the reduced closed fiber* $(X_F)_{\mathrm{red}}$ *is a divisor of* X *with normal crossings. Then Conjecture 6.2.1 is true.*

Under the stronger assumption that the multiplicities l_i in $X_F = \sum_i l_i D_i$ are prime to the residue characteristic, Theorem 6.2.3 is proved in [4] and [7] independently. In a geometric equi-characteristic situation, the conductor formula is studied in [22] (cf. [13] Example 14.1.5).

If we could assume an embedded resolution in a strong sense for the reduced closed fiber, Conjecture 6.2.1 would be a consequence of Theorem 6.2.3. Let X be as in Conjecture 6.2.1 and assume that there exists a sequence of blowing-ups $X' = X_m \to \cdots \to X_0 = X$ at regular closed subschemes supported in the closed fibers such that the reduced closed fiber $(X'_F)_{\mathrm{red}}$ has normal crossings. Then Theorem 6.2.3 applied to X′ together with Proposition 6.2.2 implies Conjecture 6.2.1 for X.

We also prove a generalization involving an algebraic correspondence. Let X_K be a proper smooth scheme of dimension d over K and ℓ be a prime number different from the characteristic of the residue field F as above. For an algebraic correspondence $\Gamma \in \mathrm{CH}_d(X_K \times_K X_K)$, its cycle class defines an endomorphism Γ^* of $H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$. We put $\mathrm{Sw}(\Gamma, X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw}(\Gamma^* : H^q(X_{\bar{K}}, \mathbf{Q}_\ell))$. For an endomorphism f of an ℓ-adic representation V, its Swan conductor is defined by

$$\mathrm{Sw}(f : V) = \frac{1}{[L : K]} \sum_{\sigma \in P_{L/K}} \mathrm{sw}_{L/K}(\sigma) \mathrm{Tr}(f \circ \sigma : V)$$

by taking a sufficiently large finite Galois extension L of K.

Let X be a proper and flat regular scheme over $S = \mathrm{Spec}\, \mathcal{O}_K$ such that $X \otimes_{\mathcal{O}_K} K = X_K$ and that the reduced closed fiber $(X_F)_{\mathrm{red}}$ has simple normal crossings. In Section 5.4, we define the logarithmic localized intersection product $[[X, \ ]] : \mathrm{Gr}^F_\bullet G(X_K \times_K X_K) \to \mathrm{Gr}^F_{\bullet - d} G(X_F)$ (5.4.2.4) on the graded quotients of the Grothendieck groups of coherent sheaves with respect to the topological filtration $F_\bullet$.

Theorem **6.3.1.** — *Let* K *be as above and* ℓ *be a prime number different from the characteristic of the residue field. Let* X_K *be a proper smooth scheme of dimension* $d = \dim X_K$ *and* $\Gamma \in CH^d(X_K \times_K X_K)$ *be an algebraic correspondence on* X_K.

1. The Swan conductor $\mathrm{Sw}(\Gamma, X_K/K)$ *is a rational number independent of* ℓ.

2. Let X *be a proper and flat regular scheme over* $\mathcal{O}_K$ *such that* $X \otimes_{\mathcal{O}_K} K = X_K$ *and that the reduced closed fiber* $(X_F)_{\mathrm{red}}$ *is a divisor with simple normal crossings. Let* $[[X, \Gamma]] \in \mathrm{Gr}^F_0 G(X_F)$ *be the image of* Γ *by the composition map* $CH_d(X_K \times_K X_K) \to \mathrm{Gr}^F_d G(X_K \times_K X_K) \xrightarrow{[[X, \ ]]} \mathrm{Gr}^F_0 G(X_F)$. *Then we have an equality of integers*

$$\mathrm{Sw}(\Gamma, X_K/K) = -\deg[[X, \Gamma]].$$

Theorem 6.3.1.1 is a consequence of Theorem 1 of [41]. We will give an independent proof. Theorem 6.3.1.2 is a generalization to higher dimension of a logarithmic version of the formulas in [26] and [1]. The localized product in the right hand side is studied in an unpublished preprint [24] when Γ is the graph of an "admissible" automorphism (cf. Corollary 6.3.3).

The main ingredients of the proof of the two theorems are the following.

1. Equivalence of the conductor formula with its log version.
2. K-theoretic localized intersection theory.
3. Log Lefschetz trace formula.

An outline of the proof, completed in Sections 6.4 and 6.5, of the conductor formula is summarized as follows. We show that Theorem 6.2.3 is equivalent to its log version

$$\mathrm{Sw}(X_K/K) = -\deg(\Delta_X, \Delta_X)^{\log}_S$$

Theorem 6.2.5. The logarithmic self-intersection class $(\Delta_X, \Delta_X)^{\log}_S \in CH_0(X_F)$ is defined by replacing $\Omega^1_{X/\mathcal{O}_K}$ in the definition of $(\Delta_X, \Delta_X)_S$ by the sheaf $\Omega^1_{X/\mathcal{O}_K}(\log / \log)$ of differential 1-forms with log poles. We define the logarithmic K-theoretic localized intersection product $[[X, \ ]] : G(X_K \times_K X_K) \to G(X_F)$ with the log diagonal map $X \to (X \times_S X)^\sim$ in Definition 5.4.2. It is defined as the difference of the classes of higher $\mathcal{T}or$-sheaves of even degree and odd degree. We show the equality

$$(\Delta_X, \Delta_X)^{\log}_S = [[X, \Delta_X]]$$

in Lemma 5.4.5.1. The log version, Theorem 6.2.5, is the special case of Theorem 6.3.1 where Γ is the diagonal Δ_X.

To prove Theorem 6.3.1, we take an alteration $W \to X$ where W is a projective and strictly semi-stable scheme over the integer ring $T = \text{Spec}\ \mathcal{O}_L$ of a finite normal extension L of K. Using this alteration, we compute the Swan conductor as

$$[W:X]\text{Sw}(\Gamma, X_K/K) = q \sum_{\sigma \in P_{L/K}} \text{sw}(\sigma)\text{Tr}\ \left(\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)\right)$$

in Corollary 6.4.5 where Γ_σ denotes the pull-back of Γ by $W_L \times_L W_L^\sigma \to X_K \times_K X_K$, W^σ denotes the conjugate of W by σ, and q is the inseparable degree of L over K. On the other hand, we compute the localized intersection product as

$$\textbf{(6.4.6.1)} \qquad [W:X]\deg_{X_F}[[X,\Gamma]] = -q \cdot \sum_{\sigma \in P_{L/K}} \text{sw}(\sigma) \cdot \deg_{W_t} \Delta_{W_t}^*(\Gamma_{\sigma,t})$$

in Proposition 6.4.6. In the right hand side, t denotes the closed point of T, $\Gamma_{\sigma,t} \in G((W \times_T W)_t^\sim)$ denotes the reduction of Γ_σ and $\Delta_{W_t}^* : G((W \times_T W)_t^\sim) \to G(W_t)$ denotes the pull-back by the log diagonal map. For the proof of the equality (6.4.6.1), we use associativity, Propositions 3.3.2 and 3.3.3 of the localized intersection product and an interpretation, Lemma 6.1.1.2, of the Swan character as the localized intersection product. Finally, we complete the proof of Theorem 6.3.1 by showing a log Lefschetz trace formula

$$\text{Tr}\ \left(\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)\right) = \deg_{W_t} \Delta_{W_t}^*(\Gamma_{\sigma,t})$$

in Theorem 6.5.1.

The proof outlined above is compared to Bloch's original proof in [6] as follows. In the original proof, the main steps are the following.

1′. Computation of the Euler characteristic of the closed fiber.
2′. Projection formula for localized intersection product.
3′. Computation of the trace on etale cohomology.

Each of items 1′–3′ corresponds to each of items 1–3 above, respectively. In the original proof, the step 1′ is carried out by a detailed combinatorial analysis peculiar to the intersection product on surfaces. In this paper, by introducing the log version, we avoid the difficulty in this step. The idea is that putting the log structure defined by the boundary has an effect similar to cutting off the boundary, the closed fiber in our case. A prototype of this idea is the Lefschetz trace formula for an open variety, Lemma 6.2.6. In this paper, it is realized as Theorem 5.4.3 which asserts that the logarithmic localized intersection product in fact depends only on the generic fiber. Non-logarithmic localized intersection product does not share this property in general. The step 2′ is generalized to the theory of localized intersection product using K-theory. An advantage of the use of K-theory lies in that the crucial associativity formulas, Propositions 3.3.2 and 3.3.3, are derived from the associativity of derived tensor product. The log Lefschetz trace formula, Theorem 6.5.1, replaces the computation in the step 3′ in higher dimension.

The idea behind the definition of the localized intersection product is as follows. If X is a smooth scheme over a field F, the intersection product of cycles V and W on X is defined to be the pull-back of $V \times W$ in $X \times_F X$ by the diagonal embedding $X \to X \times_F X$. Our aim is to generalize it to a regular flat scheme X over a discrete valuation ring $\mathcal{O}_K$. The difficulty here is that, contrary to the case over a field, the immersion $X \to X \times_{\mathcal{O}_K} X$ is not a regular immersion unless X is smooth over $\mathcal{O}_K$. If we had a base field F of $\mathcal{O}_K$, the fiber product $X \times_{\mathcal{O}_K} X$ should be a divisor of a regular scheme $X \times_F X$. If D is a divisor of a regular scheme P, one can almost recover the intersection product of cycles on D with respect to P using $\mathcal{T}or$-sheaves on D, as in Proposition 3.2.3. Although the product $X \times_{\mathcal{O}_K} X$ may not be globally a divisor of a regular scheme, we can make a suitable definition of product using $\mathcal{T}or$-sheaves, based on the fact that it is locally a divisor of a smooth scheme over X with respect to a projection. The product thus defined is in fact supported in the nonsmooth locus of X and is called the localized intersection product. A relation with the localized intersection product in the setting of Chow groups defined by Abbes in [1] is given in Theorem 3.4.3.

In the classical case, the Lefschetz trace formula is rather a formal consequence of the Poincaré duality, the Künneth formula, the cycle map and the compatibility of trace map with degree map. For log etale cohomology, the Poincaré duality and the Künneth formula are already established in [28]. We consider the Chern character map to log etale cohomology in place of the cycle map. The required compatibility is reduced to that for the usual etale cohomology.

The content of each section is as follows. In Section 1, we recall basic facts on derived exterior powers, cotangent complexes and on the Atiyah class map following [19]. We also introduce in 1.6 a spectral sequence computing $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)$ under a certain hypothesis and study its relation with the Atiyah class map in 1.7. We recall some basic facts on K-theoretic intersection product and localized Chern classes and relate the derived exterior power to the localized Chern class in a certain case in Section 2. In Section 3, we develop generality on localized K-theoretic intersection product. In Section 4, we develop generality on logarithmic product and its applications. In Section 5, we study localized intersection product on schemes over a discrete valuation ring using the results in Sections 3 and 4. In the final Section 6, we state the main result, Theorem 6.2.3, and its log version, Theorem 6.2.5, and prove their equivalence. We formulate Theorem 6.3.1, which contains Theorem 6.2.5 as a special case, in terms of logarithmic intersection product. In the final Subsection 6.5, we also state and prove logarithmic Lefschetz trace formula, Theorem 6.5.1 and prove Theorem 6.3.1 and thus complete the proof of Theorems 6.2.3 and 6.2.5.

The results in Subsections 1.3, 1.6, 1.7, 2.3, 2.4 and 3.4 are used to prove the equivalence of Theorems 6.2.3 and 6.2.5 and to show that Theorem 6.2.5 is a special case of Theorem 6.3.1. A reader only interested in the proof of Theorem 6.3.1 may skip them.

Some results in this paper are closely related to those in the paper [39]. In [39], there are mistakes in Definition (1.1), proof of Proposition (3.1), and Proposition (4.1). Definition (1.1) is corrected as Definition 1.2.1 and Lemma 1.2.6. Proposition (3.1) is reproved as Lemma 5.1.3. A corrected statement of Proposition (4.1) is given in Proposition 5.1.4. The author of [39] apologizes for the mistakes and inconvenience.

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CONTENTS

1. Derived exterior powers and cotangent complexes

We recall generalities on derived exterior powers and cotangent complexes. A basic reference is [19].

After recalling standard notations on simplicial modules in 1.1, we recall the definitions and some basic properties of derived exterior powers and symmetric powers in 1.2. We introduce Koszul simplicial algebras in 1.3. We recall briefly the definition of cotangent complexes and the Atiyah classes in 1.4 and the associativity and the projection formula for $\mathcal{T}or$ in 1.5. We define the excess conormal complex and a spectral sequence computing $\mathcal{T}or$ in 1.6. We study its relation with the Atiyah class in 1.7.

1.1. *Simplicial modules and chain complexes.* — As a preliminary, we recall the standard notations on simplicial objects. Basic references are [19] Chap. I 1 and [42] Chap. 8.

For an integer $n \geq 0$, let $[0, n]$ denote the finite ordered set $\{0, 1, \ldots, n\}$. Let Δ denote the category whose objects are $[0, n], n = 0, 1, 2, \ldots$ and morphisms are increasing maps. For $0 \leq i \leq n$, let $\delta_i : [0, n-1] \to [0, n]$ be the increasing injection skipping i and let $\sigma_i : [0, n+1] \to [0, n]$ be the increasing surjection repeating i. For a category $\mathcal{C}$, a simplicial object of $\mathcal{C}$ is a contravariant functor $\Delta^o \to \mathcal{C}$. A simplicial object $X : \Delta^o \to \mathcal{C}$ is determined by the objects $X_n = X([0, n])$ for $n = 0, 1, 2, \ldots$, the maps $d_{i,n} = \delta_i^* : X_n \to X_{n-1}$ and $s_{i,n} = \sigma_i^* : X_n \to X_{n+1}$ for $0 \leq i \leq n$. For an object X of $\mathcal{C}$, the constant simplicial object defined by $X_n = X$ for all $n \geq 0$ and $d_{i,n} = s_{i,n} = \mathrm{id}_X$ is denoted by KX. If there is no fear of confusion, we drop K and write simply X. Let Simpl($\mathcal{C}$) denote the category of simplicial objects of $\mathcal{C}$.

For a category $\mathcal{C}$, a bisimplicial object of $\mathcal{C}$ is a contravariant functor $(\Delta \times \Delta)^o \to \mathcal{C}$. Let Bisimpl($\mathcal{C}$) denote the category of bisimplicial objects of $\mathcal{C}$. The diagonal functor Δ : Bisimpl($\mathcal{C}$) $\to$ Simpl($\mathcal{C}$) is defined as the pull-back by the diagonal functor $\Delta \to \Delta \times \Delta$. For a bisimplicial object X, we let $X^\Delta = \Delta X$ denote the associated simplicial object. We identify a bisimplicial object $(\mathrm{X}_{m,n})_{m,n}$ with a simplicial simplicial object $((\mathrm{X}_n)_m)_{m,n}$. The functor $\mathrm{Bisimpl}(\mathscr{C}) \to \mathrm{Simpl}(\mathrm{Simpl}(\mathscr{C}))$ is an isomorphism of categories.

Let $\mathscr{A}$ be an abelian category. A chain complex is a complex $\mathrm{C} = (\mathrm{C}_n, d_n : \mathrm{C}_n \to \mathrm{C}_{n-1})_n$ satisfying $\mathrm{C}_n = 0$ for $n < 0$. Let $\mathrm{C}_\bullet(\mathscr{A})$ denote the full-subcategory of the category $\mathrm{C}(\mathscr{A})$ of complexes of $\mathscr{A}$ consisting of chain complexes. For an simplicial object $\mathrm{C} = (\mathrm{C}_n, d_{i,n}, s_{i,n})_{i,n}$ of $\mathscr{A}$, the normal complex $\mathrm{NC} = (\mathrm{NC}_n, d_n)_n$ is the chain complex defined by $\mathrm{NC}_n = \bigcap_{1\le i\le n} \mathrm{Ker}(d_{i,n} : \mathrm{C}_n \to \mathrm{C}_{n-1})$ and $d_n = d_{0,n}|_{\mathrm{NC}_n}$. We say a map $\mathrm{C} \to \mathrm{C}'$ of simplicial object is a quasi-isomorphism if the map $\mathrm{NC} \to \mathrm{NC}'$ of normal complexes is a quasi-isomorphism. We define a functor $\mathrm{N} : \mathrm{Simpl}(\mathscr{A}) \to \mathrm{C}_\bullet(\mathscr{A})$ by sending a simplicial object to its normal complex.

The Dold-Kan transform $\mathrm{K} : \mathrm{C}_\bullet(\mathscr{A}) \to \mathrm{Simpl}(\mathscr{A})$ gives a quasi-inverse of the functor $\mathrm{N} : \mathrm{Simpl}(\mathscr{A}) \to \mathrm{C}_\bullet(\mathscr{A})$ ([19] Chap. I 1.3.1, [42] 8.4). Further, the functors N and K are compatible with homotopies and induce quasi-inverse functors between the corresponding categories up to homotopy [42] Theorem 8.4.1.

A double chain complex is a naive double complex $\mathrm{C} = (\mathrm{C}_{m,n}, d'_{m,n} : \mathrm{C}_{m,n} \to \mathrm{C}_{m-1,n}, d''_{m,n} : \mathrm{C}_{m,n} \to \mathrm{C}_{m,n-1})_{m,n}$ satisfying $\mathrm{C}_{m,n} = 0$ for $n < 0$ or $m < 0$ and $d'_{m,n-1}d''_{m,n} = d''_{m-1,n}d'_{m,n}$. Let $\mathrm{C}_{\bullet,\bullet}(\mathscr{A})$ denote the category of double chain complexes. For a double chain complex $\mathrm{C} = (\mathrm{C}_{m,n}, d'_{m,n}, d''_{m,n})$, the associated simple chain complex $\int \mathrm{C}$ is defined by $(\bigoplus_{n=p+q} \mathrm{C}_{p,q}, \sum_{n=p+q}(d'_{p,q} + (-1)^p d''_{p,q}))_n$. We have a functor $\int : \mathrm{C}_{\bullet,\bullet}(\mathscr{A}) \to \mathrm{C}_\bullet(\mathscr{A})$. We identify a double chain complex $(\mathrm{C}_{m,n})_{m,n}$ with a chain complex of chain complexes $((\mathrm{C}_n)_m)_{m,n}$. The functor $\mathrm{C}_{\bullet,\bullet}(\mathscr{A}) \to \mathrm{C}_\bullet(\mathrm{C}_\bullet(\mathscr{A}))$ is an isomorphism of categories.

For a bisimplicial object C of $\mathscr{A}$, the normal complex NC is the double chain complex consisting of $\mathrm{NC}_{m,n} = \bigcap_{1\le i\le m} \mathrm{Ker}\ d'_{i,(m,n)} \cap \bigcap_{1\le j\le n} \mathrm{Ker}\ d''_{j,(m,n)}$ and $d'_{m,n} = d'_{0,(m,n)}|_{\mathrm{NC}_{m,n}}$, $d''_{m,n} = d''_{0,(m,n)}|_{\mathrm{NC}_{m,n}}$. The normal complexes define a functor $\mathrm{N} : \mathrm{Bisimpl}(\mathscr{A}) \to \mathrm{C}_{\bullet,\bullet}(\mathscr{A})$.

The diagram

$$\textbf{(1.1.0.1)} \qquad \begin{array}{ccc} \mathrm{Bisimpl}(\mathscr{A}) & \xrightarrow{\ \mathrm{N}\ } & \mathrm{C}_{\bullet,\bullet}(\mathscr{A}) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\int} \\ \mathrm{Simpl}(\mathscr{A}) & \xrightarrow[\ \mathrm{N}\ ]{} & \mathrm{C}_\bullet(\mathscr{A}) \end{array}$$

is commutative up to homotopy. Namely there exist a morphism $\mathrm{N} \circ \Delta \to \int \circ \mathrm{N}$ of functors called the Alexander-Whitney map and its inverse up to homotopy ([42] 8.5.4 and [19] I 1.2.2, 1.3.5). It induces an isomorphism of functors to the derived category.

The functor N for the abelian category $\mathrm{Simpl}(\mathscr{A})$ defines a functor $\mathrm{N}' : \mathrm{Bisimpl}(\mathscr{A}) = \mathrm{Simpl}(\mathrm{Simpl}(\mathscr{A})) \to \mathrm{C}_\bullet(\mathrm{Simpl}(\mathscr{A}))$. The functor $\mathrm{N} : \mathrm{Simpl}(\mathscr{A}) \to \mathrm{C}_\bullet\mathscr{A}$ induces a functor $\mathrm{N}'' : \mathrm{C}_\bullet(\mathrm{Simpl}(\mathscr{A})) \to \mathrm{C}_\bullet(\mathrm{C}_\bullet\mathscr{A}) = \mathrm{C}_{\bullet,\bullet}\mathscr{A}$. We have $\mathrm{N} =$

$N'' \circ N'$. Similarly, the partial Dold-Kan transforms $K' : C_\bullet(\mathrm{Simpl}(\mathscr{A})) \to \mathrm{Simpl}(\mathrm{Simpl}(\mathscr{A})) = \mathrm{Bisimpl}(\mathscr{A})$ and $K'' : C_{\bullet,\bullet}\mathscr{A} = C_\bullet(C_\bullet\mathscr{A}) \to C_\bullet(\mathrm{Simpl}(\mathscr{A}))$ are defined and the composition $K = K' \circ K''$ gives a quasi-inverse of N.

1.2. *Derived exterior powers and derived symmetric powers.* — We recall generalities on derived exterior power complexes and derived symmetric power complexes. For a chain complex of the form $[\mathscr{L} \to \mathscr{M}]$ where $\mathscr{M}$ is put on degree 0, we give an explicit description of the exterior powers and the symmetric powers in Corollary 1.2.7. A basic reference is [19] Chapitre I 1.3 and 4.2.

In this section, (X, A_X) denotes a ringed topos. In practice, we consider the following two cases. Let (T, A_T) be a ringed space. Besides (T, A_T) itself, we also consider the topos $X = \mathrm{Simpl}(T)$ of simplicial sheaves of sets on T with the constant simplicial ring $A_X = KA_T$. In the second case, the category (A_X-modules) is naturally identified with the category Simpl(A_T-modules) of simplicial A_T-modules.

We say a simplicial A_X-module $\mathscr{M}$ is flat if each component $\mathscr{M}_n$ is flat. We also say a chain complex of A_X-modules $\mathscr{K}$ is flat if each component $\mathscr{K}_n$ is flat. For simplicial A_X-modules $\mathscr{M}$ and $\mathscr{N}$, let $\mathscr{M} \otimes_{A_X} \mathscr{N}$ denote the simplicial module defined by $(\mathscr{M} \otimes_{A_X} \mathscr{N})_n = \mathscr{M}_n \otimes_{A_X} \mathscr{N}_n$ and let $\mathscr{M} \otimes^b_{A_X} \mathscr{N}$ denote the bisimplicial module defined by $(\mathscr{M} \otimes_{A_X} \mathscr{N})_{m,n} = \mathscr{M}_m \otimes_{A_X} \mathscr{N}_n$. For chain complexes of A_X-modules $\mathscr{K}$ and $\mathscr{K}'$, let $\mathscr{K} \otimes^d_{A_X} \mathscr{K}'$ denote the double chain complex defined by $(\mathscr{K} \otimes^d_{A_X} \mathscr{K}')_{m,n} = \mathscr{K}_m \otimes_{A_X} \mathscr{K}'_n$ and let $\mathscr{K} \otimes_{A_X} \mathscr{K}'$ be the associated simple complex $\int(\mathscr{K} \otimes^d_{A_X} \mathscr{K}')$. Since $\mathscr{M} \otimes_{A_X} \mathscr{N} = \Delta(\mathscr{M} \otimes^b_{A_X} \mathscr{N})$ and $N(\mathscr{M} \otimes^b_{A_X} \mathscr{N}) = N\mathscr{M} \otimes^d_{A_X} N\mathscr{N}$, the Alexander-Whitney map $N \circ \Delta \to \int \circ N$ induces a quasi-isomorphism $N(\mathscr{M} \otimes_{A_X} \mathscr{N}) \to \int(N\mathscr{M} \otimes_{A_X} N\mathscr{N})$. Hence, we have quasi-isomorphisms

$$\begin{aligned} \mathscr{K} \otimes_{A_X} \mathscr{K}' &= \textstyle\int \left(\mathscr{K} \otimes^d_{A_X} \mathscr{K}'\right) \\ \longrightarrow \textstyle\int \left(NK\mathscr{K} \otimes^d_{A_X} NK\mathscr{K}'\right) &= \textstyle\int N\left(K\mathscr{K} \otimes^b_{A_X} K\mathscr{K}'\right) \\ \longrightarrow N\Delta\left(K\mathscr{K} \otimes^b_{A_X} K\mathscr{K}'\right) &= N\left(K\mathscr{K} \otimes_{A_X} K\mathscr{K}'\right). \end{aligned} \tag{1.1.0.1}$$

We briefly describe the idea of the definition of derived exterior powers and derived symmetric powers for chain complexes on a ringed topos (X, A_X) ([19] Chap. I 4.2.2.2, Definition 1.2.1 below) before recalling it precisely. In 1.1, we have recalled an equivalence

$$C_\bullet(A_X\text{-modules}) \underset{N}{\overset{K}{\rightleftarrows}} \mathrm{Simpl}(A_X\text{-modules})$$

of the categories of chain complexes of A_X-modules and of simplicial A_X-modules. For simplicial A_X-modules, the exterior power and symmetric power are defined by simply taking the exterior powers and the symmetric powers componentwise. For chain complexes, the definitions are given by transferring the definitions for simplicial modules by using the functors N and K.

Let (X, A_X) be a ringed topos and $\mathscr{M}$ be a simplicial A_X-module. For an integer $p \geq 0$, the p-th symmetric power $S^p\mathscr{M}$ is defined as the composition $\Delta^o \xrightarrow{\mathscr{M}}$ (A_X-modules) $\xrightarrow{S^p}$ (A_X-modules) with the functor S^p : (A_X-modules) $\to$ (A_X-modules) sending an A_X-module to its p-th symmetric power. Similarly, for an integer $q \geq 0$, the q-th exterior power $\Lambda^q\mathscr{M}$ is defined as the composition $\Delta^o \xrightarrow{\mathscr{M}}$ (A_X-modules) $\xrightarrow{\Lambda^q}$ (A_X-modules) with the functor Λ^q : (A_X-modules) $\to$ (A_X-modules) sending an A_X-module to its q-th exterior power. The simplicial module $F^{\Delta}_{A_X}\mathscr{M}$ associated to the standard free resolution $F_{A_X}\mathscr{M}$ ([19] Chap. I (1.5.5.2)) has a canonical quasi-isomorphism $F^{\Delta}_{A_X}\mathscr{M} \to \mathscr{M}$ of simplicial modules.

Definition **1.2.1** ([19] Chap. I 4.2.2.2). — *Let* (X, A_X) *be a ringed topos and* $\mathscr{K}$ *be a chain complex of* A_X*-modules.*

1. For an integer $p \geq 0$*, the* p*-th* derived symmetric power $LS^p\mathscr{K}$ *is defined to be* $NS^pF^{\Delta}_{A_X}K\mathscr{K}$.

2. For an integer $q \geq 0$*, the* q*-th* derived exterior power $L\Lambda^q\mathscr{K}$ *is defined to be* $N\Lambda^qF^{\Delta}_{A_X}K\mathscr{K}$.

For an integer $q \geq 0$, we put $L^qS^p\mathscr{K} = \mathscr{H}_qLS^p\mathscr{K}$. For an integer $r \geq 0$, we also put $L^r\Lambda^q\mathscr{K} = \mathscr{H}_rL\Lambda^q\mathscr{K}$. If $\mathscr{K}' \to \mathscr{K}$ is a homotopy equivalence of chain complexes, the induced maps $LS^p\mathscr{K}' \to LS^p\mathscr{K}$ and $L\Lambda^q\mathscr{K}' \to L\Lambda^q\mathscr{K}$ are also homotopy equivalences. If each component of $\mathscr{K}$ is flat, the canonical maps $LS^p\mathscr{K} \to NS^pK\mathscr{K}$ and $L\Lambda^q\mathscr{K} \to N\Lambda^qK\mathscr{K}$ are quasi-isomorphisms. For an A_X-module $\mathscr{F}$, we have canonical isomorphisms $L^0S^p\mathscr{F} \to S^p\mathscr{F}$ and $L^0\Lambda^q\mathscr{F} \to \Lambda^q\mathscr{F}$. If $\mathscr{F}$ is flat, the canonical maps $LS^p\mathscr{F} \to S^p\mathscr{F}$ and $L\Lambda^q\mathscr{F} \to \Lambda^q\mathscr{F}$ are quasi-isomorphisms.

For a simplicial A_X-module $\mathscr{M}$ and an integer $p \geq 0$, the diagonal map $\mathscr{M} \to \mathscr{M} \oplus \mathscr{M}$ induces a map

(1.2.1.1) $$S^p\mathscr{M} \to S^p(\mathscr{M} \oplus \mathscr{M}) \longrightarrow \bigoplus_{p=p'+p''} S^{p'}\mathscr{M} \otimes_{A_X} S^{p''}\mathscr{M}.$$

For a chain complex $\mathscr{K}$ and integers $p = p' + p''$, it induces a canonical map

(1.2.1.2) $$LS^p\mathscr{K} \longrightarrow LS^{p'}\mathscr{K} \otimes^{L}_{A_X} LS^{p''}\mathscr{K}.$$

Similarly, canonical maps

(1.2.1.3) $$\Lambda^q\mathscr{M} \longrightarrow \bigoplus_{q=q'+q''} \Lambda^{q'}\mathscr{M} \otimes_{A_X} \Lambda^{q''}\mathscr{M}$$

and

(1.2.1.4) $$L\Lambda^q\mathscr{K} \longrightarrow L\Lambda^{q'}\mathscr{K} \otimes^{L}_{A_X} L\Lambda^{q''}\mathscr{K}$$

for $q = q' + q''$ are defined. The following elementary lemma is useful in the sequel.

Lemma **1.2.2.** — *Let* $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ *be an exact sequence of flat* A_X*-modules. Then, the canonical maps* (1.2.1.1) *and* (1.2.1.3) *define commutative diagrams of exact sequences*

(1.2.2.1)
$$\begin{array}{ccccccccc} 0 \to \mathscr{L} \otimes S^{p-1}\mathscr{N} & \to & S^p\mathscr{M}/(S^2\mathscr{L} \cdot S^{p-2}\mathscr{M}) & \to & S^p\mathscr{N} & \to 0 \\ \| & & \downarrow & & \downarrow & \\ 0 \to \mathscr{L} \otimes S^{p-1}\mathscr{N} & \to & \mathscr{M} \otimes S^{p-1}\mathscr{N} & \to & \mathscr{N} \otimes S^{p-1}\mathscr{N} & \to 0, \end{array}$$

(1.2.2.2)
$$\begin{array}{ccccccccc} 0 \to \mathscr{L} \otimes \Lambda^{p-1}\mathscr{N} & \to & \Lambda^p\mathscr{M}/(\Lambda^2\mathscr{L} \cdot \Lambda^{p-2}\mathscr{M}) & \to & \Lambda^p\mathscr{N} & \to 0 \\ \| & & \downarrow & & \downarrow & \\ 0 \to \mathscr{L} \otimes \Lambda^{p-1}\mathscr{N} & \to & \mathscr{M} \otimes \Lambda^{p-1}\mathscr{N} & \to & \mathscr{N} \otimes \Lambda^{p-1}\mathscr{N} & \to 0. \end{array}$$

Proof. — It suffices to show the exactness. By localization and a limit argument (cf. [19] I 4.2.1), it is reduced to the case where $\mathscr{L}, \mathscr{M}$ and $\mathscr{N}$ are free of finite rank and the sequence $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ splits. Then the assertion is clear. □

For chain complexes $\mathscr{M}$ and $\mathscr{N}$, we naturally identify the complexes $\mathscr{M}[1]\otimes\mathscr{N}$ and $(\mathscr{M} \otimes \mathscr{N})[1]$.

Corollary **1.2.3.** — *1. Let* $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ *be an exact sequence of flat simplicial* A_X*-modules. Then, for* $p \geq 0$*, the upper exact sequence in* (1.2.2.1) *defines a distinguished triangle*

(1.2.3.1)
$$\to N\mathscr{L} \otimes^L_{A_X} NS^{p-1}\mathscr{N} \to N(S^p\mathscr{M}/(S^2\mathscr{L} \cdot S^{p-2}\mathscr{M})) \to NS^p\mathscr{N} \to .$$

The boundary map $NS^p\mathscr{N} \to N\mathscr{L} \otimes^L_{A_X} NS^{p-1}\mathscr{N}[1]$ *is the composition*

$$NS^p\mathscr{N} \xrightarrow{(1.2.1.1)} N\mathscr{N} \otimes^L_{A_X} NS^{p-1}\mathscr{N} \longrightarrow N\mathscr{L}[1] \otimes^L_{A_X} NS^{p-1}\mathscr{N}.$$

2. Let $\mathscr{L}$ *be an invertible* A_X*-module,* $\mathscr{E}$ *be a flat* A_X*-module and* $\to \mathscr{L} \to \mathscr{E} \to \mathscr{K} \to$ *be a distinguished triangle of chain complexes of* A_X*-modules. For* $q \geq 0$*, the upper exact sequence in* (1.2.2.2) *defines a distinguished triangle*

(1.2.3.2)
$$\longrightarrow \mathscr{L} \otimes L\Lambda^q\mathscr{K} \longrightarrow \Lambda^{q+1}\mathscr{E} \longrightarrow L\Lambda^{q+1}\mathscr{K} \longrightarrow .$$

The boundary map $L\Lambda^{q+1}\mathscr{K} \to \mathscr{L} \otimes L\Lambda^q\mathscr{K}[1]$ *is the composition*

$$L\Lambda^{q+1}\mathscr{K} \xrightarrow{(1.2.1.4)} \mathscr{K} \otimes L\Lambda^q\mathscr{K} \longrightarrow \mathscr{L}[1] \otimes L\Lambda^q\mathscr{K}.$$

It induces an isomorphism $L^{p+1}\Lambda^{q+1}\mathscr{K} \to \mathscr{L} \otimes L^p\Lambda^q\mathscr{K}$ *either if* $p > 0$ *or if* $\mathscr{E}$ *is locally free of rank* $n \leq q$*.*

Proof. — 1. It is sufficient to apply Lemma 1.2.2.

2. We may assume $\mathscr{K}$ is the mapping cone of $\mathscr{L} \to \mathscr{E}$. Let $\mathscr{C}$ be the mapping cylinder of $\mathscr{L} \to \mathscr{E}$. Then, for the distinguished triangle (1.2.3.2) and the description of the boundary map, it is sufficient to apply Lemma 1.2.2 to the exact sequence $0 \to \mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{C} \to \mathrm{K}\mathscr{K} \to 0$ of simplicial modules. The last assertion is clear from the distinguished triangle (1.2.3.2). □

To study explicitly the derived exterior power complex, we recall the divided power modules $\Gamma^r\mathrm{M}$, see e.g. [16] Exp. XVII 5.5.2. Let A be a commutative ring and M be an A-module. We regard M as a functor attaching to a commutative A-algebra A′ the set $\mathrm{A}' \otimes_{\mathrm{A}} \mathrm{M}$. For an integer $r \geq 0$ and for A-modules M and N, a morphism $f : \mathrm{M} \to \mathrm{N}$ of functors is called r-ic if $f(ax) = a^r f(x)$ for an A-algebra A′, $a \in \mathrm{A}'$ and $x \in \mathrm{A}' \otimes_{\mathrm{A}} \mathrm{M}$. For an A-module M, the r-th divided power $\Gamma^r\mathrm{M}$ represents the functor attaching to an A-module N the set of r-ic morphisms $\mathrm{M} \to \mathrm{N}$. The universal r-ic morphism is denoted by $\gamma^r : \mathrm{M} \to \Gamma^r\mathrm{M}$. We have $\Gamma^0\mathrm{M} = \mathrm{A}$ and the map $\mathrm{M} \to \Gamma^1\mathrm{M} : x \mapsto \gamma^1 x$ is an isomorphism. If $r = r_1 + r_2$, the r-ic map $\mathrm{M} \to \Gamma^{r_1}\mathrm{M} \otimes \Gamma^{r_2}\mathrm{M}$ sending x to $\gamma^{r_1}(x) \otimes \gamma^{r_2}(x)$ induces a map $\Gamma^r\mathrm{M} \to \Gamma^{r_1}\mathrm{M} \otimes \Gamma^{r_2}\mathrm{M}$. If $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_2$, the r-ic map $\mathrm{M} \to \bigoplus_{r_1+r_2=r} \Gamma^{r_1}\mathrm{M}_1 \otimes \Gamma^{r_2}\mathrm{M}_2$ sending (x_1, x_2) to $(\gamma^{r_1}(x_1) \otimes \gamma^{r_2}(x_2))$ defines an isomorphism $\Gamma^r\mathrm{M} \to \bigoplus_{r_1+r_2=r} \Gamma^{r_1}\mathrm{M}_1 \otimes \Gamma^{r_2}\mathrm{M}_2$ ([16] Exp. XVII 5.5.2.6). If M is a free (resp. flat) A-module, its r-th power $\Gamma^r\mathrm{M}$ is also a free (resp. flat) A-module. More precisely, if M is a free A-module and $e_1, \ldots, e_n$ is a basis of M, $\Gamma^r\mathrm{M}$ is also a free A-module and $\gamma^{r_1}e_1 \otimes \cdots \otimes \gamma^{r_n}e_n$, $(r_1 + \cdots + r_n = r, r_1, \ldots, r_n \geq 0)$ is a basis of $\Gamma^r\mathrm{M}$. Similarly as (1.2.1.1) and (1.2.1.3), we have a canonical map

(1.2.4.1) $$\Gamma^r\mathrm{M} \longrightarrow \bigoplus_{r=r'+r''} \Gamma^{r'}\mathrm{M} \otimes_{\mathrm{A}} \Gamma^{r''}\mathrm{M}.$$

The definition of Γ^r and the properties as above are generalized to modules on a ringed topos.

Definition **1.2.4.** — *Let* $(\mathrm{X}, \mathrm{A}_{\mathrm{X}})$ *be a ringed topos and* $v : \mathscr{L} \to \mathscr{M}$ *be a morphism of* A_{X}*-modules.*

1. For an integer $p \geq 0$*, we define a chain complex*

$$\mathrm{S}^p(\mathscr{L} \xrightarrow{v} \mathscr{M}) = \left(\mathrm{S}^{p-q}\mathscr{M} \otimes \Lambda^q\mathscr{L}, d_q\right)$$

by putting d_q *to be the composition*

(1.2.4.2) $$\begin{array}{ccc} \mathrm{S}^{p-q-1}\mathscr{M} \otimes \Lambda^{q+1}\mathscr{L} & \xrightarrow{1\otimes(1.2.1.3)} & \mathrm{S}^{p-q-1}\mathscr{M} \otimes \mathscr{L} \otimes \Lambda^q\mathscr{L} \\ & & \big\downarrow 1\otimes v\otimes 1 \\ \mathrm{S}^{p-q}\mathscr{M} \otimes \Lambda^q\mathscr{L} & \xleftarrow{\cdot\otimes 1} & \mathrm{S}^{p-q-1}\mathscr{M} \otimes \mathscr{M} \otimes \Lambda^q\mathscr{L}. \end{array}$$

2. For an integer $q \geq 0$, we define a chain complex

$$\Lambda^q(\mathscr{L} \xrightarrow{v} \mathscr{M}) = \left(\Lambda^{q-r}\mathscr{M} \otimes \Gamma^r\mathscr{L}, d_r\right)$$

by putting d_r to be the composition

$$\text{(1.2.4.3)}\qquad \begin{array}{ccc} \Lambda^{q-r-1}\mathscr{M} \otimes \Gamma^{r+1}\mathscr{L} & \xrightarrow{1\otimes(1.2.4.1)} & \Lambda^{q-r-1}\mathscr{M} \otimes \mathscr{L} \otimes \Gamma^r\mathscr{L} \\ & & \Big\downarrow {\scriptstyle 1\otimes v\otimes 1} \\ \Lambda^{q-r}\mathscr{M} \otimes \Gamma^r\mathscr{L} & \xleftarrow{\wedge\otimes 1} & \Lambda^{q-r-1}\mathscr{M} \otimes \mathscr{M} \otimes \Gamma^r\mathscr{L}. \end{array}$$

The complex $S^p(\mathscr{L} \xrightarrow{v} \mathscr{M})$ is the same as the total degree p-part of the Koszul complex $\mathrm{Kos}_\bullet(v)$ and the complex $\Lambda^q(\mathscr{L} \xrightarrow{v} \mathscr{M})$ is the total degree q-part of the Koszul complex $\mathrm{Kos}^\bullet(v)$ defined in [19] I 4.3.1.3.

Lemma **1.2.5.** — *Let $\mathscr{L}$ and $\mathscr{E}$ be locally free A_X-modules of rank 1 and n. Let $u : \mathscr{L} \to \mathscr{E}$ be an A_X-linear map and $u^* : \mathscr{E}^* \to \mathscr{L}^*$ be its dual. Let*

$$\begin{array}{ccc} \Lambda^{n-p}\mathscr{E} \otimes \mathscr{L}^{\otimes p} & \longrightarrow & \mathscr{H}om_{A_X}(\Lambda^p\mathscr{E} \otimes \mathscr{L}^{\otimes n-p}, \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n}) \\ & & \Big\uparrow \\ & & \mathscr{L}^{*\otimes n-p} \otimes \Lambda^p\mathscr{E}^* \otimes \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n} \end{array}$$

be the isomorphism sending $x\otimes y$ to the map $x'\otimes y' \mapsto x\wedge x'\otimes y\otimes y'$ and the canonical isomorphism. Then they induce an isomorphism

$$\text{(1.2.5.1)}\qquad \Lambda^n(\mathscr{L} \to \mathscr{E}) \longrightarrow S^n(\mathscr{E}^* \to \mathscr{L}^*) \otimes \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n}$$

of chain complexes.

Proof. — The squares

$$\begin{array}{ccc} \Lambda^{n-p-1}\mathscr{E} \otimes \mathscr{L}^{\otimes p+1} & \longrightarrow & \Lambda^{n-p}\mathscr{E} \otimes \mathscr{L}^{\otimes p} \\ \Big\downarrow & & \Big\downarrow \\ \mathscr{H}om(\Lambda^{p+1}\mathscr{E} \otimes \mathscr{L}^{\otimes n-p-1}, \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n}) & \longrightarrow & \mathscr{H}om(\Lambda^p\mathscr{E} \otimes \mathscr{L}^{\otimes n-p}, \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n}) \\ \Big\uparrow & & \Big\uparrow \\ \mathscr{L}^{*\otimes n-p-1} \otimes \Lambda^{p+1}\mathscr{E}^* \otimes \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n} & \longrightarrow & \mathscr{L}^{*\otimes n-p} \otimes \Lambda^p\mathscr{E}^* \otimes \Lambda^n\mathscr{E} \otimes \mathscr{L}^{\otimes n} \end{array}$$

are commutative up to $(-1)^p$ and the assertion follows. □

Lemma **1.2.6** (cf. [34] 7.34, [19] I 4.3.2). — *Let* $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ *be an exact sequence of flat* A_X*-modules. Then, the natural maps*

$$\mathrm{S}^p(\mathscr{L} \xrightarrow{v} \mathscr{M}) \longrightarrow \mathrm{S}^p\mathscr{N}, \tag{1.2.6.1}$$

$$\Lambda^q(\mathscr{L} \xrightarrow{v} \mathscr{M}) \longrightarrow \Lambda^q\mathscr{N} \tag{1.2.6.2}$$

are quasi-isomorphisms.

Proof. — It is proved for the symmetric power in [19] I 4.3.2. The proof for the exterior power is similar. We briefly sketch it. For the direct sum, we have a canonical isomorphism

$$\begin{aligned}\Lambda^q(\mathscr{L} \oplus \mathscr{L}' &\xrightarrow{(v,v')} \mathscr{M} \oplus \mathscr{M}') \\ &\longrightarrow \bigoplus_{q=q'+q''} \textstyle\int \Lambda^{q'}(\mathscr{L} \xrightarrow{v} \mathscr{M}) \otimes \Lambda^{q''}(\mathscr{L}' \xrightarrow{v'} \mathscr{M}').\end{aligned}$$

Similarly as loc. cit., it is reduced to the case where $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ are free of finite rank and the sequence $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ splits. Hence, we may identify $\mathscr{L} \to \mathscr{M}$ with $\mathscr{L} \oplus 0 \xrightarrow{(1,0)} \mathscr{L} \oplus \mathscr{N}$. By induction on rank of $\mathscr{L}$, we see that $\Lambda^{q'}(\mathscr{L} \xrightarrow{\mathrm{id}} \mathscr{L})$ is acyclic except for $q' = 0$. Thus we obtain a quasi-isomorphism $\Lambda^q(\mathscr{L} \xrightarrow{v} \mathscr{M}) \to \Lambda^q\mathscr{N}$ and the assertion follows. □

Corollary **1.2.7.** — *Let* $u : \mathscr{L} \to \mathscr{M}$ *be a map of flat* A_X*-modules and let* $\mathscr{K} = [\mathscr{L} \xrightarrow{u} \mathscr{M}]$ *be the mapping cone. Then, the maps* (1.2.6.1) *and* (1.2.6.2) *induce isomorphisms*

$$\mathrm{S}^p(\mathscr{L} \xrightarrow{v} \mathscr{M}) \longrightarrow \mathrm{LS}^p\mathscr{K}, \tag{1.2.7.1}$$

$$\Lambda^q(\mathscr{L} \xrightarrow{v} \mathscr{M}) \longrightarrow \mathrm{L}\Lambda^q\mathscr{K} \tag{1.2.7.2}$$

in the derived category.

Proof. — Let $\mathscr{C} = [\mathscr{L} \xrightarrow{(u,-1)} \mathscr{M} \oplus \mathscr{L}]$ be the mapping cylinder. The exact sequence of chain complexes $0 \to \mathscr{L} \to \mathscr{C} \to \mathscr{K} \to 0$ induces an exact sequence of simplicial modules $0 \to \mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{C} \to \mathrm{K}\mathscr{K} \to 0$. By Lemma 1.2.6, we obtain a quasi-isomorphism $\mathrm{S}^p(\mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{C}) \to \mathrm{S}^p\mathrm{K}\mathscr{K}$ of complexes of simplicial modules. It induces a quasi-isomorphism $\int \mathrm{N}''\mathrm{S}^p(\mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{C}) \to \mathrm{NS}^p\mathrm{K}\mathscr{K}$ of chain complexes. Since the canonical map $\mathscr{M} \to \mathscr{C}$ is a quasi-isomorphism, it induces a quasi-isomorphism $\mathrm{K}\mathscr{M} \to \mathrm{K}\mathscr{C}$ of simplicial modules. It further induces a quasi-isomorphism $\mathrm{S}^p(\mathscr{L} \to \mathscr{M}) = \int \mathrm{N}''\mathrm{S}^p(\mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{M}) \to \int \mathrm{N}''\mathrm{S}^p(\mathrm{K}\mathscr{L} \to \mathrm{K}\mathscr{C})$. Thus we obtain an isomorphism (1.2.7.1). It is similar for the exterior power. □

Proposition **1.2.8** ([34] 7.21, [19] Chap. I Proposition 4.3.2.1). — *Let* $\mathscr{K}$ *be a chain complex of* A_X*-modules and* $p \geq 0$ *be an integer. Then, the map* (1.2.6.1) *induces an isomorphism*

$$(L\Lambda^p \mathscr{K})[p] \longrightarrow LS^p(\mathscr{K}[1]) \tag{1.2.8.1}$$

in the derived category.

Proof. — We briefly recall the proof of loc. cit. Replacing $\mathscr{K}$ by a flat resolution, we may assume $\mathscr{K}$ is flat. Let $\mathscr{C}$ be the mapping cone of the identity $\mathscr{K} \to \mathscr{K}$. Then, we have an exact sequence $0 \to \mathscr{K} \to \mathscr{C} \to \mathscr{K}[1] \to 0$. Applying Lemma 1.2.6 to the exact sequence $0 \to K\mathscr{K} \to K\mathscr{C} \to K\mathscr{K}[1] \to 0$ of simplicial modules, we obtain a quasi-isomorphism of complexes of simplicial modules $S^p(K\mathscr{K} \to K\mathscr{C}) \to S^p(K\mathscr{K}[1])$. Since $\mathscr{C}$ is acyclic, the map of associated simple complexes $\int N'' S^p(K\mathscr{K} \to K\mathscr{C}) \to N\Lambda^p(K\mathscr{K})[p]$ is a quasi-isomorphism. Thus the assertion follows. □

Lemma **1.2.9.** — *The isomorphism* (1.2.8.1) *and the maps* (1.2.1.2) *and* (1.2.1.4) *form a commutative diagram*

$$\begin{array}{ccc} L\Lambda^p \mathscr{K}[p] & \longrightarrow & L\Lambda^{p'} \mathscr{K}[p'] \otimes^L_{A_X} L\Lambda^{p''} \mathscr{K}[p''] \\ \downarrow & & \downarrow \\ LS^p(\mathscr{K}[1]) & \longrightarrow & LS^{p'}(\mathscr{K}[1]) \otimes^L_{A_X} LS^{p''}(\mathscr{K}[1]). \end{array} \tag{1.2.9.1}$$

Proof. — We use the notation in the proof of Proposition 1.2.8. As in the proof of Lemma 1.2.6, we obtain maps

$$\begin{aligned} S^p(K\mathscr{K} \to K\mathscr{C}) &\to S^p(K\mathscr{K}^{\oplus 2} \to K\mathscr{C}^{\oplus 2}) \\ &\to S^{p'}(K\mathscr{K} \to K\mathscr{C}) \otimes S^{p''}(K\mathscr{K} \to K\mathscr{C}) \end{aligned}$$

of complexes of simplicial modules. The composition is compatible with the map $\Lambda^p K\mathscr{K}[p] \to \Lambda^{p'} K\mathscr{K}[p'] \otimes \Lambda^{p''} K\mathscr{K}[p'']$. Hence the assertion follows. □

1.3. *Koszul algebras.* — We introduce Koszul simplicial algebras. We will use them in the proof of the degeneration, Proposition 1.6.7, of a spectral sequence computing $\mathscr{T}or$.

Let (X, A_X) be a ringed topos and $u : \mathscr{M} \to A_X$ be a morphism of A_X-modules. Let $[\mathscr{M} \xrightarrow{u} A_X]$ denote the chain complex where A_X is put on degree 0. The Dold-Kan transform $K[\mathscr{M} \xrightarrow{u} A_X]$ of the chain complex $[\mathscr{M} \xrightarrow{u} A_X]$ is a simplicial A_X-module. Let $S(K[\mathscr{M} \xrightarrow{u} A_X])$ denote the symmetric algebra of the simplicial A_X-module $K[\mathscr{M} \xrightarrow{u} A_X]$. The n-th component of $S(K[\mathscr{M} \xrightarrow{u} A_X])$ is the symmetric algebra over A_X of the n-th component $K_n[\mathscr{M} \xrightarrow{u} A_X]$. The simplicial algebra $S(K[\mathscr{M} \xrightarrow{u} A_X])$ is naturally an algebra over the constant simplicial A_X-algebra $S(K[0 \to A_X]) = S(KA_X) = KS(A_X)$.

Definition **1.3.1.** — *Let* (X, A_X) *be a ringed topos and* $u : \mathscr{M} \to A_X$ *be a morphism of* A_X*-modules.*

1. We define a simplicial A_X*-algebra* $\mathbf{A}(\mathscr{M} \xrightarrow{u} A_X)$ *by*

(1.3.1.1) $$\mathbf{A}(\mathscr{M} \xrightarrow{u} A_X) = S(K[\mathscr{M} \xrightarrow{u} A_X]) \otimes_{S(KA_X)} KA_X$$

with respect to the map $S(KA_X) \to KA_X$ *induced by* $\mathrm{id} : KA_X \to KA_X$. *We call the simplicial* A_X*-algebra* $\mathbf{A}(\mathscr{M} \xrightarrow{u} A_X)$ *the* Koszul simplicial algebra *of* $u : \mathscr{M} \to A_X$.

2. The chain complex $\mathbf{K}(\mathscr{M} \xrightarrow{u} A_X) = (\Lambda^n \mathscr{M}, u_n)$ *defined by putting* u_n *to be the composition*

$$\Lambda^n \mathscr{M} \longrightarrow \mathscr{M} \otimes \Lambda^{n-1} \mathscr{M} \xrightarrow{u \otimes 1} \Lambda^{n-1} \mathscr{M}$$

is called the Koszul complex *of* $u : \mathscr{M} \to A_X$.

If $\Lambda^{n+1} \mathscr{M} = 0$, we have

(1.3.1.2) $$\mathbf{K}(\mathscr{M} \xrightarrow{u} A_X) = S^n(\mathscr{M} \xrightarrow{u} A_X).$$

In general, we have $\mathbf{K}(\mathscr{M} \xrightarrow{u} A_X) = \varinjlim_n S^n(\mathscr{M} \xrightarrow{u} A_X)$ with respect to the natural maps.

Lemma **1.3.2.** — *Let* (X, A_X) *be a ringed topos and* $u : \mathscr{M} \to A_X$ *be a morphism of* A_X*-modules. We define an increasing filtration* $F_\bullet$ *on* $\mathbf{A}(\mathscr{M} \xrightarrow{u} A_X)$ *by putting* F_p *to be the image of* $\bigoplus_{p' \leq p} S^{p'}(K[\mathscr{M} \xrightarrow{u} A_X])$. *Then, we have a canonical isomorphism* $S^p K(\mathscr{M}[1]) \to \mathrm{Gr}^F_p \mathbf{A}(\mathscr{M} \xrightarrow{u} A_X)$ *of simplicial modules.*

Assume $\mathscr{M}$ *is flat. Then, the spectral sequence*

$$E^1_{p,q} = H_{p+q} N \mathrm{Gr}^F_p \mathbf{A}(\mathscr{M} \xrightarrow{u} A_X) \Rightarrow H_{p+q} N \mathbf{A}(\mathscr{M} \xrightarrow{u} A_X)$$

satisfies $E^1_{p,q} = 0$ *except for* $q = 0$. *The complex* $E^1_{\bullet,0}$ *is naturally identified with the Koszul complex* $\mathbf{K}(\mathscr{M} \xrightarrow{u} A_X)$.

Proof. — The exact sequence $0 \to A \to [\mathscr{M} \to A] \to \mathscr{M}[1] \to 0$ of chain complexes induces an exact sequence $0 \to KA \to K[\mathscr{M} \to A] \to K(\mathscr{M}[1]) \to 0$ of simplicial A-modules. By definition, we have a commutative diagram of exact sequences

(1.3.2.1) $$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ KA \otimes_{KA} S^{p-1}K(\mathscr{M}[1]) & \to & \mathrm{Gr}_F^{p-1}\mathbf{A}(\mathscr{M} \to A) \\ \downarrow & & \downarrow \\ S^pK[\mathscr{M} \to A]/S^2KA \cdot S^{p-2}K[\mathscr{M} \to A] & \to & F^{[p,p-1]}\mathbf{A}(\mathscr{M} \to A) \\ \downarrow & & \downarrow \\ S^pK(\mathscr{M}[1]) & \to & \mathrm{Gr}_F^{p}\mathbf{A}(\mathscr{M} \to A) \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

Since the map $S(KA_X) \to KA_X$ in Definition 1.3.1.1 is induced by the identity $KA_X \to KA_X$, the upper horizontal arrow maps $1 \otimes a$ to the class of a. Thus, it is easy to see that the horizontal arrows are isomorphisms.

Assume $\mathscr{M}$ is flat. Then, the bottom horizontal map in (1.3.2.1) induces an isomorphism $\Lambda^p \mathscr{M}[p] \to LS^p(\mathscr{M}[1]) \to NGr_F^p \mathbf{A}(\mathscr{M} \to A)$ by Proposition 1.2.8. By the diagram (1.3.2.1) and Corollary 1.2.3.1, we have a commutative diagram

$$\begin{array}{ccc} LS^p(\mathscr{M}[1]) & \longrightarrow & NGr_F^p\mathbf{A}(\mathscr{M} \to A) \\ \downarrow & & \\ \mathscr{M}[1] \otimes^L_A LS^{p-1}(\mathscr{M}[1]) & & \downarrow \\ {\scriptstyle u\otimes 1}\downarrow & & \\ A[1] \otimes_A LS^{p-1}(\mathscr{M}[1]) & \longrightarrow & NGr_F^{p-1}\mathbf{A}(\mathscr{M} \to A)[1]. \end{array} \quad \textbf{(1.3.2.2)}$$

Thus the assertion follows from Lemma 1.2.9. □

Lemma **1.3.3.** — *Let* $0 \to \mathscr{L} \xrightarrow{f} \mathscr{M} \xrightarrow{g} \mathscr{N} \to 0$ *be an exact sequence of flat* A_X*-modules and* $u : \mathscr{N} \to A_X$ *be a map of* A_X*-modules. We put* $\mathbf{A}_{\mathscr{L}} = \mathbf{A}(\mathscr{L} \xrightarrow{u\circ g\circ f} A_X)$, $\mathbf{A}_{\mathscr{M}} = \mathbf{A}(\mathscr{M} \xrightarrow{u\circ g} A_X)$ *and* $\mathbf{A}_{\mathscr{N}} = \mathbf{A}(\mathscr{N} \xrightarrow{u} A_X)$ *and we identify* $A_X = \mathbf{A}(0 \to A_X)$. *The commutative diagram*

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & \mathscr{N} \\ \uparrow & & \uparrow \\ \mathscr{L} & \longrightarrow & 0 \end{array}$$

induces an isomorphism

$$\textbf{(1.3.3.1)} \qquad \mathbf{A}_{\mathscr{M}} \otimes_{\mathbf{A}_{\mathscr{L}}} A_X \longrightarrow \mathbf{A}_{\mathscr{N}}.$$

Proof. — We have an exact sequence $0 \to K[\mathscr{L} \to 0] \to K[\mathscr{M} \to A_X] \to K[\mathscr{N} \to A_X] \to 0$ of flat simplicial modules. Hence, we obtain an isomorphism $S(K[\mathscr{M} \to A_X]) \otimes_{S(K[\mathscr{L}\to 0])} A_X \to S(K[\mathscr{N} \to A_X])$ of simplicial algebras. It induces the isomorphism (1.3.3.1). □

We define a generalization for chain complexes.

Definition **1.3.4.** — *Let* (X, A_X) *be a ringed topos and* $u : \mathscr{K} \to A_X$ *be a map of chain complexes. We regard* $K\mathscr{K} \to KA_X$ *be a map of* KA_X*-modules on the topos* Simpl(X) *and define the Koszul bisimplicial algebra* $\mathbf{A}(\mathscr{K} \to A_X)$ *to be the simplicial simplicial algebra* $\mathbf{A}(K\mathscr{K} \to KA_X)$ *regarded as a bisimplicial* A_X*-algebra. Let* $\mathbf{A}^{\Delta}(K\mathscr{K} \to KA_X) = \Delta\mathbf{A}(K\mathscr{K} \to KA_X)$ *denote the diagonal simplicial* A_X*-algebra.*

Lemma **1.3.5.** — *Let* $v : \mathscr{K}' \to \mathscr{K}$ *be a quasi-isomorphism of flat chain complexes of* A_X*-modules and* $u : \mathscr{K} \to A_X$ *be a map of chain complexes of* A_X*-modules. We put* $u' = u \circ v : \mathscr{K}' \to A_X$. *Then the natural map* $\mathbf{A}^\Delta(\mathscr{K}' \xrightarrow{u'} A_X) \to \mathbf{A}^\Delta(\mathscr{K} \xrightarrow{u} A_X)$ *is a quasi-isomorphism of simplicial* A_X*-algebras.*

Proof. — Let $K\mathscr{K}' \to K\mathscr{K}$ be the map of the Dold-Kan transforms. Let $F^\bullet$ denote the filtrations on $\mathbf{A}(\mathscr{K} \xrightarrow{u} A_X)$ and $\mathbf{A}(\mathscr{K}' \xrightarrow{u'} A_X)$ in Lemma 1.3.2. It is sufficient to show that $\int N\mathrm{Gr}^n_F\mathbf{A}(\mathscr{K}' \xrightarrow{u'} A_X) \to \int N\mathrm{Gr}^n_F\mathbf{A}(\mathscr{K} \xrightarrow{u} A_X)$ is a quasi-isomorphism for each $n \geq 0$. By Lemma 1.3.2, $\mathrm{Gr}^n_F\mathbf{A}(\mathscr{K} \xrightarrow{u} A_X)$ and $\mathrm{Gr}^n_F\mathbf{A}(\mathscr{K}' \xrightarrow{u'} A_X)$ are isomorphic to $S^nK'((K\mathscr{K})[1])$ and $S^nK'((K\mathscr{K}')[1])$ respectively. By Proposition 1.2.8, $N'S^nK'((K\mathscr{K})[1])$ and $N'S^nK'((K\mathscr{K}')[1])$ are quasi-isomorphic to $\Lambda^nK\mathscr{K}[n]$ and $\Lambda^nK\mathscr{K}'[n]$ respectively, as complexes of simplicial modules. Hence $\int NS^nK'((K\mathscr{K})[1])$ and $\int NS^nK'((K\mathscr{K}')[1])$ are isomorphic to $L\Lambda^n\mathscr{K}[n]$ and $L\Lambda^n\mathscr{K}'[n]$ respectively. Thus the assertion follows. □

Corollary **1.3.6.** — *Let* $\mathscr{M}$ *and* $\mathscr{L}$ *be flat* A_X*-modules and let* $u : \mathscr{M} \to A_X$ *and* $v : \mathscr{L} \to \mathscr{M}$ *be* A_X*-linear maps. Let* $\mathscr{K} = [\mathscr{L} \xrightarrow{v} \mathscr{M}]$ *be the mapping cone and* $\mathscr{C} = [\mathscr{L} \xrightarrow{(v,-1)} \mathscr{M} \oplus \mathscr{L}]$ *be the mapping cylinder. We define a map* $c : \mathscr{C} \to A_X$ *by* $(u, u \circ v)$.

1. The natural map $\mathscr{M} \to \mathscr{C}$ *induces a quasi-isomorphism*

$$\text{(1.3.6.1)} \qquad \mathbf{A}_{\mathscr{M}} = \mathbf{A}(\mathscr{M} \xrightarrow{u} A_X) \longrightarrow \mathbf{A}_{\mathscr{C}} = \mathbf{A}^\Delta(\mathscr{C} \xrightarrow{c} A_X)$$

of simplicial A_X*-algebras.*

2. Assume the composition $u \circ v : \mathscr{L} \to A_X$ *is the 0-map and let* $w : \mathscr{K} \to A_X$ *be the map of chain complexes defined by u. We put* $\mathbf{A}_{\mathscr{L}} = \mathbf{A}(\mathscr{L} \xrightarrow{0} A_X)$ *and* $\mathbf{A}_{\mathscr{K}} = \mathbf{A}^\Delta(\mathscr{K} \xrightarrow{w} A_X)$. *Then the commutative diagram*

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathscr{K} \\ \uparrow & & \uparrow \\ \mathscr{L} & \longrightarrow & 0 \end{array}$$

induces an isomorphism

$$\text{(1.3.6.2)} \qquad \mathbf{A}_{\mathscr{C}} \otimes_{\mathbf{A}_{\mathscr{L}}} A_X \longrightarrow \mathbf{A}_{\mathscr{K}}$$

of simplicial A_X*-algebras.*

Proof. — 1. Since the map $\mathscr{M} \to \mathscr{C}$ is a quasi-isomorphism, the assertion follows from Lemma 1.3.5.

2. We have an exact sequence $0 \to K\mathscr{L} \to K\mathscr{C} \to K\mathscr{K} \to 0$ of simplicial A_X-modules. By applying Lemma 1.3.3, we obtain an isomorphism

$$\mathbf{A}(\mathscr{C} \xrightarrow{c} \mathrm{A_X}) \otimes_{\mathrm{K}\mathbf{A}(\mathscr{L} \xrightarrow{0} \mathrm{A_X})} \mathrm{KA_X} \longrightarrow \mathbf{A}(\mathscr{K} \xrightarrow{w} \mathrm{A_X})$$

of bisimplicial $\mathrm{A_X}$-algebras. Taking the diagonals, we obtain the isomorphism (1.3.6.2). □

1.4. *Cotangent complexes and the Atiyah classes.* — We recall some definitions and facts on cotangent complexes and the Atiyah classes. A basic reference is [19] Chapitres II and IV.

Let (X, A) be a ringed topos. For an A-algebra B, a standard resolution $P_A(B) \to B$ by a free simplicial A-algebra $P_A(B)$ is constructed in [19] I 1.5.5.6. The cotangent complex $L_{B/A}$ is defined as the normal complex $N(\Omega^1_{P_A(B)/A} \otimes_{P_A(B)} KB)$ ([19] Chapitre II 1.2). There is a canonical isomorphism $\mathscr{H}_0 L_{B/A} \to \Omega^1_{B/A}$ (loc.cit. Proposition 1.2.4.2). If $A \to B$ is surjective and $I = \mathrm{Ker}(A \to B)$, we have $\mathscr{H}_0 L_{B/A} = 0$ and a canonical isomorphism $\mathscr{H}_1 L_{B/A} \to I/I^2$ (loc.cit. Corollaire 1.2.8.1).

Let (X, A) be a ringed topos. We say a simplicial A-algebra P is *weakly free* if, for each $n \geq 0$, there exist a flat A-module L_n such that the n-th component P_n of P is isomorphic to the symmetric algebra $S_A L_n$. For an A-algebra B, we say a morphism of simplicial A-algebra $P \to KB$ is a resolution $P \to B$ by a weakly free simplicial A-algebra if P is weakly free and $P \to B$ is a quasi-isomorphism in the sense that the map $NP \to NKB = B$ of normal complexes is a quasi-isomorphism. A resolution $P \to B$ by a weakly free simplicial A-algebra induces an isomorphism $L_{B/A} \to \Omega^1_{P/A}$ in the derived category as follows. Let $P^\Delta_A(P) \to P$ be the diagonal of the standard resolution by free bisimplicial A-algebras as in loc.cit. (1.2.2.1). Then the quasi-isomorphisms $P_A(B) \leftarrow P^\Delta_A(P) \to P$ induce quasi-isomorphisms $\Omega^1_{P_A(B)/A} \leftarrow \Omega^1_{P^\Delta_A(P)/A} \to \Omega^1_{P/A}$. Composing them with the quasi-isomorphism $\Omega^1_{P_A(B)/A} \to \Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$, we obtain an isomorphism $L_{B/A} \to \Omega^1_{P/A}$.

For a map $f : X \to S$ of ringed toposes, the cotangent complex $L_{X/S}$ is defined as $L_{A_X/f^{-1}A_S} = \Omega^1_{P_{f^{-1}A_S}(A_X)/f^{-1}A_S} \otimes_{P_{f^{-1}A_S}(A_X)} A_X$. We will recall an explicit computation of the cotangent complex in Lemma 1.6.2 for some morphisms of schemes. For maps $X \xrightarrow{f} Y \xrightarrow{g} S$ of ringed toposes, a distinguished triangle

(1.4.0.1) $$\longrightarrow Lf^* L_{Y/S} \longrightarrow L_{X/S} \longrightarrow L_{X/Y} \longrightarrow$$

is constructed as follows (loc.cit. Proposition 2.1.2). Let $P_S(A_Y) \to A_Y$ be the standard resolution by a free simplicial $g^{-1}A_S$-algebra and $P^\Delta_{P_S(A_Y)}(A_X) \to A_X$ be the diagonal of the standard resolution by a free bisimplicial $f^{-1}P_S(A_Y)$-algebra as in loc.cit. (1.2.2.1). Then, we have quasi-isomorphisms

$$\Omega^1_{P^\Delta_{P_S(A_Y)}(A_X)/(g\circ f)^{-1}A_S} \to \Omega^1_{P_S(A_X)/(g\circ f)^{-1}A_S},$$
$$\Omega^1_{P^\Delta_{P_S(A_Y)}(A_X)/f^{-1}P_S(A_Y)} \to \Omega^1_{P_Y(A_X)/f^{-1}A_Y}$$

and the distinguished triangle (1.4.0.1) is defined by the exact sequence

$$0 \longrightarrow f^{-1}\Omega^1_{P_S(A_Y)/f^{-1}A_S} \otimes_{f^{-1}P_S(A_Y)} P^{\Delta}_{P_S(A_Y)}(A_X) \longrightarrow \Omega^1_{P^{\Delta}_{P_S(A_Y)}(A_X)/(g\circ f)^{-1}A_S} \longrightarrow \Omega^1_{P^{\Delta}_{P_S(A_Y)}(A_X)/f^{-1}P_S(A_Y)} \longrightarrow 0.$$

Let $f : X \to S$ be a map of ringed toposes and $\mathscr{F}$ be an A_X-module. The Atiyah class map is a map

(1.4.0.2) $$\mathrm{at}_{X/S}(\mathscr{F}) : \mathscr{F} \longrightarrow L_{X/S} \otimes^L_{A_X} \mathscr{F}[1]$$

in the derived category defined in [19] Chapitre IV 2.3.6. We briefly recall the definition. We consider the graded A_X-algebra $A_X \oplus \mathscr{F}$ such that A_X is put on degree 0 and $\mathscr{F}$ is put on degree 1. Then, for the maps $(X, A_X \oplus \mathscr{F}) \to (X, A_X) \to (S, A_S)$ of ringed toposes, the distinguished triangle (1.4.0.1) gives

(1.4.0.3) $$\to L_{X/S} \otimes^L_{A_X} (A_X \oplus \mathscr{F}) \to L_{(X,A_X\oplus\mathscr{F})/S} \to L_{(X,A_X\oplus\mathscr{F})/X} \to .$$

The degree 1-part of the map $L_{(X,A_X\oplus\mathscr{F})/X} \to L_{X/S} \otimes^L_{A_X} (A_X \oplus \mathscr{F})[1]$ gives the *Atiyah class map* $\mathrm{at}_{X/S}(\mathscr{F}) : \mathscr{F} \to L_{X/S} \otimes^L_{A_X} \mathscr{F}[1]$.

We recall another description of the Atiyah class map. Let $P_S(A_X) = P_{f^{-1}A_S}(A_X) \to A_X$ be the standard resolution of A_X by free $f^{-1}A_S$-algebra and I be the kernel of the surjection $P_S(A_X) \otimes_{f^{-1}A_S} P_S(A_X) \to P_S(A_X)$. We have $\Omega^1_{P_S(A_X)/f^{-1}A_S} = I/I^2$. We put $P^1_{P_S(A_X)/f^{-1}A_S} = (P_S(A_X) \otimes_{f^{-1}A_S} P_S(A_X))/I^2$. The exact sequence

(1.4.0.4) $$0 \to \Omega^1_{P_S(A_X)/f^{-1}A_S} \longrightarrow P^1_{P_S(A_X)/f^{-1}A_S} \longrightarrow P_S(A_X) \to 0$$

of $P^1_{P_S(A_X)/f^{-1}A_S}$-modules splits as an exact sequence of $P_S(A_X)$-module with respect to the ring homomorphism $P_S(A_X) \to P^1_{P_S(A_X)/f^{-1}A_S}$ sending a to $1\otimes a$. We regard the A_X-module $\mathscr{F}$ as a $P_S(A_X)$-module by the quasi-isomorphism $P_S(A_X) \to A_X$. By applying $\otimes_{P_S(A_X)}\mathscr{F}$, we obtain an exact sequence

(1.4.0.5) $$0 \to \Omega^1_{P_S(A_X)/f^{-1}A_S} \otimes_{P_S(A_X)} \mathscr{F} \to P^1_{P_S(A_X)/f^{-1}A_S} \otimes_{P_S(A_X)} \mathscr{F} \to \mathscr{F} \to 0.$$

We regard it as an exact sequence of $P_S(A_X)$-modules by the ring homomorphism $P_S(A_X) \to P^1_{P_S(A_X)/f^{-1}A_S}$ sending a to $a \otimes 1$ (cf. [19] III (1.2.6.3)). Since $L_{X/S} = N(\Omega^1_{P_S(A_X)/f^{-1}A_S} \otimes_{P_S(A_X)} A_X)$, we have $N(\Omega^1_{P_S(A_X)/f^{-1}A_S} \otimes_{P_S(A_X)} \mathscr{F}) = L_{X/S} \otimes^L_{A_X} \mathscr{F}$. Thus the exact sequence (1.4.0.5) gives a distinguished triangle

(1.4.0.6) $$\to L_{X/S} \otimes^L_{A_X} \mathscr{F} \to N\big(P^1_{P_S(A_X)/f^{-1}A_S} \otimes_{P_S(A_X)} \mathscr{F}\big) \to \mathscr{F} \to$$

of complexes of $A_X = N(P_S(A_X))$-modules (cf. [19] I Corollaire 3.3.4.6). By [19] IV Proposition 2.3.7.3, the Atiyah class map $\mathscr{F} \to L_{X/S} \otimes^L_{A_X} \mathscr{F}[1]$ is defined by the distinguished triangle (1.4.0.6).

Let $i : X \to P$ be a map of ringed toposes such that $i^{-1}A_P \to A_X$ is a surjection. We put $I_X = \mathrm{Ker}(i^{-1}A_P \to A_X)$. By the long exact sequence defined by the short exact sequence $0 \to I_X \to i^{-1}A_P \to A_X \to 0$, $\mathscr{T}\!or_1^{i^{-1}A_P}(A_X, A_X)$ is canonically identified with the conormal sheaf $N_{X/P} = I_X/I_X^2 = I_X \otimes_{i^{-1}A_P} A_X$. More generally, for an A_X-module $\mathscr{F}$, the A_X-module $\mathscr{T}\!or_1^{i^{-1}A_P}(A_X, \mathscr{F})$ is canonically identified with $I_X \otimes_{i^{-1}A_P} \mathscr{F} = N_{X/P} \otimes_{A_X} \mathscr{F}$. We consider the distinguished triangle

(1.4.1.1) $$\to N_{X/P} \otimes_{A_X} \mathscr{F}[1] \to \tau_{[-1,0]}\big(A_X \otimes^{L}_{i^{-1}A_P} \mathscr{F}\big) \to \mathscr{F} \to$$

of A_X-modules. Here and in the following, $\tau_{[a,b]}\mathscr{K} = \tau_{\geq a}\tau_{\leq b}\mathscr{K} = \tau_{\leq b}\tau_{\geq a}\mathscr{K}$ denotes the canonical truncation for a complex $\mathscr{K}$. In the middle, $A_X \otimes^{L}_{i^{-1}A_P} \mathscr{F}$ is regarded as a complex of A_X-modules with respect to the A_X-module structure of A_X and is computed by taking a resolution of $\mathscr{F}$ by flat $i^{-1}A_P$-modules. Note that it can be different from that with respect to the A_X-module structure of $\mathscr{F}$ computed by taking a resolution of A_X by flat $i^{-1}A_P$-modules. The distinguished triangle (1.4.1.1) defines a canonical map $\mathscr{F} \to N_{X/P} \otimes_{A_X} \mathscr{F}[2]$.

Lemma **1.4.1** ([19] IV Corollary 3.1.9). — *Let $i : X \to P$ be a map of ringed toposes over a ringed topos S and $\mathscr{F}$ be an A_X-module. Assume $i^{-1}A_P \to A_X$ and $i^{-1}i_*\mathscr{F} \to \mathscr{F}$ are surjective. Let $L_{X/S} \to L_{X/P} \to N_{X/P}[1]$ be the canonical map. Then the composition*

(1.4.1.2) $$\mathscr{F} \xrightarrow{\mathrm{at}_{X/S}} L_{X/S} \otimes^{L}_{A_X} \mathscr{F}[1] \xrightarrow{\mathrm{can}\otimes 1} N_{X/P} \otimes_{A_X} \mathscr{F}[2]$$

is the same as the map defined by the distinguished triangle (1.4.1.1).

Proof. — We reproduce the proof of loc.cit. Replacing S by P, we may assume $S = P$. Let $\mathscr{L}$ be the free A_P-module $A_P^{(i_*\mathscr{F})}$. The natural map $i^{-1}\mathscr{L} \to A_X^{(i^{-1}i_*\mathscr{F})} \to \mathscr{F}$ is surjective. Let $S_{A_P}(\mathscr{L}) = A_P[i_*\mathscr{F}]$ be the free A_P-algebra generated by $i_*\mathscr{F}$. Let $X_{\mathscr{F}}$ denote the graded ringed topos $(X, A_X \oplus \mathscr{F})$ and $P_{\mathscr{L}}$ denote $(P, S_{A_P}(\mathscr{L}))$. We put $J = \mathrm{Ker}(i^{-1}S_{A_P}(\mathscr{L}) \to A_{X_{\mathscr{F}}})$ and $\mathscr{G} = \mathrm{Ker}(i^{-1}\mathscr{L} \to \mathscr{F})$. Since the canonical map $L_{P_{\mathscr{L}}/P} \to \Omega^1_{P_{\mathscr{L}}/P} = \mathscr{L} \otimes_{A_P} S_{A_P}(\mathscr{L})$ is an isomorphism ([19] II Proposition 1.2.4.4), we obtain an isomorphism $\tau_{[-1,0]}L_{X_{\mathscr{F}}/P} \to [J/J^2 \to i^{-1}\mathscr{L} \otimes_{i^{-1}A_P} A_{X_{\mathscr{F}}}]$. Since $J/J^2 = N_{X/P} \oplus (\mathscr{G} \otimes_{i^{-1}A_P} A_X) \oplus (\deg \geq 2)$, by taking the degree 1-part, we see that the Atiyah class map $\mathrm{at}_{X/P} : \mathscr{F} \to N_{X/P} \otimes_{A_X} \mathscr{F}[2]$ is induced by the distinguished triangle

$$\to N_{X/P} \otimes_{A_X} \mathscr{F}[1] \to \big[A_X \otimes_{i^{-1}A_P} \mathscr{G} \to A_X \otimes_{i^{-1}A_P} i^{-1}\mathscr{L}\big] \to \mathscr{F} \to.$$

Since the isomorphism $[\mathscr{G} \to i^{-1}\mathscr{L}] \to \mathscr{F}$ induces an isomorphism $[A_X \otimes_{i^{-1}A_P} \mathscr{G} \to A_X \otimes_{i^{-1}A_P} i^{-1}\mathscr{L}] \to \tau_{[-1,0]}(A_X \otimes^{L}_{i^{-1}A_P} \mathscr{F})$ in the derived category of A_X-modules, the assertion follows. $\square$

1.5. *Associativity, projection formula and the Atiyah class.* — We recall spectral sequences for $\mathcal{T}or$ arising from the associativity and the projection formula. We show that a map induced by the Atiyah class map is the same as the boundary map of a spectral sequence in Lemma 1.5.4. First, we introduce notations on tensor products.

For a scheme X, let $D^-(X)$ (resp. $D^b(X)$) denote the derived category of complexes of $\mathcal{O}_X$-modules bounded above (resp. bounded above and below). Let $D^-(X)_{qcoh}$ denote the full subcategory consisting of complexes whose cohomology sheaves are quasi-coherent $\mathcal{O}_X$-modules. If X is locally noetherian, let $D^-(X)_{coh}$ and $D^b(X)_{coh}$ denote the full subcategories consisting of complexes whose cohomology sheaves are coherent $\mathcal{O}_X$-modules. Let $f : W \to X$ be a morphism of schemes. For $\mathcal{F} \in D^-(X)$ and $\mathcal{G} \in D^-(W)$, we put $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} = Lf^*\mathcal{F} \otimes^L_{\mathcal{O}_W} \mathcal{G} \in D^-(W)$ (cf. [17] Exp. III Notation 1.6). For an integer q, let $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denote the homology sheaf $\mathcal{H}_q(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G})$. If $\mathcal{F}_1 \to \mathcal{F}$ is a flat resolution, we obtain an isomorphism $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} \to f^*\mathcal{F}_1 \otimes_{\mathcal{O}_W} \mathcal{G}$. Locally, the sheaf $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is computed as follows. If $X = \operatorname{Spec} A$ and $W = \operatorname{Spec} B$ are affine and if $\mathcal{F} = M^\sim, \mathcal{G} = N^\sim$ are quasi-coherent sheaves associated to an A-module M and to a B-module N respectively, then $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the quasi-coherent sheaf associated to the B-module $\operatorname{Tor}_q^A(M, N)$.

Let $i : V \to X$ be a closed immersion and $\mathcal{F}$ be an $\mathcal{O}_V$-module. By abuse of notation, we identify $i_*\mathcal{F} = \mathcal{F}$ and regard $\mathcal{F}$ as an $\mathcal{O}_X$-module. We put $T = V \times_X W$. Then, $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_T$-module for each q. If X and W are locally noetherian, if $\mathcal{F}$ is a coherent $\mathcal{O}_V$-module and if $\mathcal{G} \in D^-(W)_{coh}$, then the $\mathcal{O}_T$-modules $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent for all q.

Lemma **1.5.1.** — *Let* $X \xleftarrow{f} W \xleftarrow{g} W'$ *be morphisms of schemes and* $\mathcal{F} \in D^-(X)$, $\mathcal{G} \in D^-(W)$ *and* $\mathcal{H} \in D^-(W')$ *respectively. Then,*

1. The associativity isomorphism

$$(\mathbf{1.5.1.1}) \qquad \left(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}\right) \otimes^L_{\mathcal{O}_W} \mathcal{H} \to \mathcal{F} \otimes^L_{\mathcal{O}_X} \left(\mathcal{G} \otimes^L_{\mathcal{O}_W} \mathcal{H}\right)$$

in $D^-(W')$ *induces an isomorphism*

$$(\mathbf{1.5.1.2}) \qquad \mathcal{T}or_q^{\mathcal{O}_W}\left(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}\right) \to \mathcal{T}or_q^{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{G} \otimes^L_{\mathcal{O}_W} \mathcal{H}\right)$$

of $\mathcal{O}_{W'}$*-modules.*

2. The canonical filtrations on $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}$ *and* $\mathcal{G} \otimes^L_{\mathcal{O}_W} \mathcal{H}$ *define spectral sequences*

$$(\mathbf{1.5.1.3}) \qquad E^2_{p,q} = \mathcal{T}or_p^{\mathcal{O}_W}\left(\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \mathcal{H}\right) \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_W}\left(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}\right),$$

$$(\mathbf{1.5.1.4}) \qquad E^2_{p,q} = \mathcal{T}or_p^{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{T}or_q^{\mathcal{O}_W}(\mathcal{G}, \mathcal{H})\right) \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{G} \otimes^L_{\mathcal{O}_W} \mathcal{H}\right)$$

of $\mathcal{O}_{W'}$*-modules, respectively. If* V *is a closed subscheme of* X *and if* $\mathcal{F}$ *is an* $\mathcal{O}_V$*-module, then they are spectral sequences of* $\mathcal{O}_{T'}$*-modules where* $T' = V \times_X W'$.

Proof. — 1. We recall the definition of the isomorphism (1.5.1.1). It suffices to consider the case where each component of $\mathcal{F}$ and $\mathcal{G}$ are flat over $\mathcal{O}_X$ and over $\mathcal{O}_W$ respectively. Then, we have isomorphisms $g^*(f^*\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{G}) \otimes_{\mathcal{O}_{W'}} \mathcal{H} \to (\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}) \otimes^L_{\mathcal{O}_W} \mathcal{H}$ and $(f \circ g)^*\mathcal{F} \otimes_{\mathcal{O}_{W'}} (g^*\mathcal{G} \otimes_{\mathcal{O}_{W'}} \mathcal{H}) \to \mathcal{F} \otimes^L_{\mathcal{O}_X} (\mathcal{G} \otimes^L_{\mathcal{O}_W} \mathcal{H})$. Hence the canonical isomorphism $g^*(f^*\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{G}) \otimes_{\mathcal{O}_{W'}} \mathcal{H} \to (g^*f^*\mathcal{F} \otimes_{\mathcal{O}_{W'}} g^*\mathcal{G}) \otimes_{\mathcal{O}_{W'}} \mathcal{H} \to (f \circ g)^*\mathcal{F} \otimes_{\mathcal{O}_{W'}} (g^*\mathcal{G} \otimes_{\mathcal{O}_{W'}} \mathcal{H})$ defines an isomorphism (1.5.1.1).

Clearly, the isomorphism (1.5.1.1) induces an isomorphism (1.5.1.2).

2. The canonical filtration on $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}$ defines a spectral sequence $E^1_{p,q} = \mathcal{T}or^{\mathcal{O}_W}_{2p+q}(\mathcal{T}or^{\mathcal{O}_X}_{-p}(\mathcal{F}, \mathcal{G}), \mathcal{H}) \Rightarrow \mathcal{T}or^{\mathcal{O}_W}_{p+q}(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})$. We obtain the spectral sequence (1.5.1.3) by decalage. The spectral sequence (1.5.1.3) is defined similarly. □

Lemma **1.5.2.** — *Let* $X \leftarrow W \rightarrow X'$ *be morphisms of schemes and* $\mathcal{F} \in D^-(X)$, $\mathcal{G} \in D^-(W)$ *and* $\mathcal{F}' \in D^-(X')$ *respectively. Then,*

1. The composition

(1.5.2.1)
$$\mathcal{F} \otimes^L_{\mathcal{O}_X} \left(\mathcal{F}' \otimes^L_{\mathcal{O}_{X'}} \mathcal{G}\right) \to \mathcal{F} \otimes^L_{\mathcal{O}_X} \left(\mathcal{G} \otimes^L_{\mathcal{O}_{X'}} \mathcal{F}'\right) \to \left(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}\right) \otimes^L_{\mathcal{O}_{X'}} \mathcal{F}' \to \mathcal{F}' \otimes^L_{\mathcal{O}_{X'}} \left(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}\right)$$

of the commutativity and the associativity isomorphisms in $D^-(W)$ *induces an isomorphism*

(1.5.2.2)
$$\mathcal{T}or^{\mathcal{O}_X}_q\left(\mathcal{F}, \mathcal{F}' \otimes^L_{\mathcal{O}_{X'}} \mathcal{G}\right) \to \mathcal{T}or^{\mathcal{O}_{X'}}_q\left(\mathcal{F}', \mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}\right)$$

of $\mathcal{O}_W$*-modules.*

2. The canonical filtrations define spectral sequences

(1.5.2.3)
$$E^2_{p,q} = \mathcal{T}or^{\mathcal{O}_X}_p\left(\mathcal{F}, \mathcal{T}or^{\mathcal{O}_{X'}}_q(\mathcal{F}', \mathcal{G})\right) \Rightarrow \mathcal{T}or^{\mathcal{O}_X}_{p+q}\left(\mathcal{F}, \mathcal{F}' \otimes^L_{\mathcal{O}_{X'}} \mathcal{G}\right),$$

(1.5.2.4)
$$E^2_{p,q} = \mathcal{T}or^{\mathcal{O}_{X'}}_p\left(\mathcal{F}', \mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{F}, \mathcal{G})\right) \Rightarrow \mathcal{T}or^{\mathcal{O}_{X'}}_{p+q}\left(\mathcal{F}', \mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G}\right)$$

of $\mathcal{O}_W$*-modules. If* V *and* V′ *are closed subschemes of* X *and* X′ *and if* $\mathcal{F}$ *is an* $\mathcal{O}_V$*-module and* $\mathcal{F}'$ *is an* $\mathcal{O}_{V'}$*-module respectively, then they are spectral sequences of* $\mathcal{O}_T$*-modules where* $T = V \times_X W \times_{X'} V'$.

Proof. — The proof is similar to Lemma 1.5.1 and left to the reader. □

We also recall the projection formula.

Lemma **1.5.3.** — *Let* X *be a quasi-compact scheme and* $f : W' \to W$ *be a quasi-compact and quasi-separated morphism of quasi-compact schemes over* X. *Let* $\mathcal{F} \in D^-(X)_{\mathrm{qcoh}}$ *and* $\mathcal{G} \in D^-(W')$. *We assume that either of the following condition is satisfied.*

(i) ([17] Exp. III Proposition 3.7) *The complex* $\mathscr{F}$ *is a perfect complex of* $\mathscr{O}_X$*-modules and* $\mathscr{G} \in D^b(W')_{\mathrm{qcoh}}$.
(ii) ([18] II Proposition 5.6) *The schemes* W *and* W′ *are noetherian schemes of finite dimensions.*

1. There exists a canonical and functorial isomorphism

$$\mathscr{F} \otimes^{L}_{\mathscr{O}_X} Rf_*\mathscr{G} \to Rf_*\big(\mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G}\big) \tag{1.5.3.1}$$

in $D^-(W)$. *The isomorphism* (1.5.3.1) *induces an isomorphism*

$$\mathscr{T}\!or_q^{\mathscr{O}_X}(\mathscr{F}, Rf_*\mathscr{G}) \to R^{-q}f_*\big(\mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G}\big) \tag{1.5.3.2}$$

of $\mathscr{O}_W$*-modules.*

2. The canonical filtrations define spectral sequences

$$E^2_{p,q} = \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, R^{-q}f_*\mathscr{G}) \Rightarrow \mathscr{T}\!or_{p+q}^{\mathscr{O}_X}(\mathscr{F}, Rf_*\mathscr{G}), \tag{1.5.3.3}$$

$$E^2_{p,q} = R^{-p}f_*\mathscr{T}\!or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \Rightarrow R^{-p-q}f_*\big(\mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G}\big) \tag{1.5.3.4}$$

of $\mathscr{O}_W$*-modules. If* V *is a closed subschemes of* X *and if* $\mathscr{F}$ *is an* $\mathscr{O}_V$*-module, then they are spectral sequences of* $\mathscr{O}_V$*-modules.*

Let $X \to P$ be an immersion of schemes and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Let $W \to X$ be a morphism of schemes and $\mathscr{G} \in D^-(W)$. Then the composition of $\mathscr{F} \to L_{X/P} \otimes^{L}_{\mathscr{O}_X} \mathscr{F}[1] \to N_{X/P} \otimes^{L}_{\mathscr{O}_X} \mathscr{F}[2]$ (1.4.1.2) induces a map

$$\alpha_{\mathscr{F},\mathscr{G},X/P} : \mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G} \longrightarrow N_{X/P} \otimes^{L}_{\mathscr{O}_X} \mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G}[2] \tag{1.5.4.1}$$

in $D^-(W)$. It further induces a map

$$\alpha_{\mathscr{F},\mathscr{G},X/P} : \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{T}\!or_{p-2}^{\mathscr{O}_X}(N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{G}) \tag{1.5.4.2}$$

of $\mathscr{O}_W$-modules for $p \geq 0$.

Lemma **1.5.4.** — *Let* $X \to P$ *be an immersion of schemes and* $\mathscr{F}$ *be an* $\mathscr{O}_X$*-module. Let* $W \to X$ *be a morphism of schemes and* $\mathscr{G} \in D^-(W)$. *Let*

$$E^2_{p,q} = \mathscr{T}\!or_p^{\mathscr{O}_X}\big(\mathscr{T}\!or_q^{\mathscr{O}_P}(\mathscr{F}, \mathscr{O}_X), \mathscr{G}\big) \Longrightarrow E_{p+q} = \mathscr{T}\!or_{p+q}^{\mathscr{O}_P}(\mathscr{F}, \mathscr{G}) \tag{1.5.4.3}$$

be the spectral sequence (1.5.1.3) *combined with the isomorphism* (1.5.1.2). *We identify* $N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{F}$ *with* $\mathscr{F} \otimes_{\mathscr{O}_X} N_{X/P} = \mathscr{T}\!or_1^{\mathscr{O}_P}(\mathscr{F}, \mathscr{O}_X)$ *by the multiplication by* -1. *Then, the map* $\alpha_{\mathscr{F},\mathscr{G},X/P} : \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \to \mathscr{T}\!or_{p-2}^{\mathscr{O}_X}(N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{G})$ (1.5.4.2) *is equal to the boundary map* $E^2_{p,0} = \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \to E^2_{p-2,1} = \mathscr{T}\!or_{p-2}^{\mathscr{O}_X}(\mathscr{T}\!or_1^{\mathscr{O}_P}(\mathscr{F}, \mathscr{O}_X), \mathscr{G})$ *of* (1.5.4.3).

Proof. — The boundary map $\mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{p-2}^{\mathcal{O}_X}(\mathcal{T}or_1^{\mathcal{O}_P}(\mathcal{F},\mathcal{O}_X),\mathcal{G})$ is the boundary map defined by the distinguished triangle

$$\longrightarrow \mathcal{T}or_1^{\mathcal{O}_P}(\mathcal{F},\mathcal{O}_X)[1] \longrightarrow \tau_{[-1,0]}\big(\mathcal{F}\otimes^{L}_{\mathcal{O}_P}\mathcal{O}_X\big) \longrightarrow \mathcal{F} \longrightarrow$$

of complexes of $\mathcal{O}_X$-modules where $\mathcal{F}\otimes^{L}_{\mathcal{O}_P}\mathcal{O}_X$ in the middle is regarded as a complex of $\mathcal{O}_X$-modules by the $\mathcal{O}_X$-module structure of $\mathcal{O}_X$. Under the identification $\mathcal{T}or_1^{\mathcal{O}_P}(\mathcal{F},\mathcal{O}_X) = \mathcal{F}\otimes_{\mathcal{O}_X} N_{X/P}$ and the commutativity isomorphism $\mathcal{F}\otimes^{L}_{\mathcal{O}_P}\mathcal{O}_X \to \mathcal{O}_X\otimes^{L}_{\mathcal{O}_P}\mathcal{F}$, it is identified with (1.4.1.1). Thus it follows from Lemma 1.4.1. □

If the $\mathcal{O}_X$-module $N_{X/P}$ is flat, we identify $\mathcal{T}or_{p-2}^{\mathcal{O}_X}(N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{G}) = N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{p-2}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ and the map (1.5.4.2) defines a map

$$\textbf{(1.5.4.4)} \qquad \alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \longrightarrow N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{p-2}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}).$$

For a spectral sequence $E = (E^2_{p,q} \Rightarrow E_{p+q})$, let $E[0,2]$ denote the spectral sequence $E^2_{p,q-2} \Rightarrow E_{p+q-2}$.

Lemma **1.5.5.** — *Let* $X \to P$ *be an immersion of schemes and* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. We assume that the conormal sheaf* $N_{X/P}$ *is flat over* $\mathcal{O}_X$*. Let* $f : W' \to W$ *be a map of schemes over* X.

1. Let $\mathcal{G} \in D^-(W)$ *and* $\mathcal{H} \in D^-(W')$ *respectively. Let*

$$\textbf{(1.5.5.1)} \qquad E = \Big(E^2_{p,q} = \mathcal{T}or_p^{\mathcal{O}_W}\big(\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}),\mathcal{H}\big) \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_X}\big(\mathcal{F},\big(\mathcal{G}\otimes^{L}_{\mathcal{O}_W}\mathcal{H}\big)\big)\Big)$$

be the spectral sequence (1.5.1.3) *combined with the isomorphism* (1.5.1.2).

Then the map $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{F}\otimes^{L}_{\mathcal{O}_X}\mathcal{G} \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}\otimes^{L}_{\mathcal{O}_X}\mathcal{G}[2]$ (1.5.4.1) *induces a map*

$$\textbf{(1.5.5.2)} \qquad E \longrightarrow N_{X/P}\otimes_{\mathcal{O}_X} E[0,2]$$

of spectral sequences. The maps on E_2*-terms are induced by*

$$\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{q-2}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

and the maps on the abutments are

$$\alpha_{\mathcal{F},\mathcal{G}\otimes^{L}_{\mathcal{O}_W}\mathcal{H},X/P} : \mathcal{T}or_n^{\mathcal{O}_X}\big(\mathcal{F},\big(\mathcal{G}\otimes^{L}_{\mathcal{O}_W}\mathcal{H}\big)\big) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{n-2}^{\mathcal{O}_X}\big(\mathcal{F},\big(\mathcal{G}\otimes^{L}_{\mathcal{O}_W}\mathcal{H}\big)\big).$$

2. Let $f : W' \to W$ *be a morphism of schemes over* X *and* $\mathcal{G} \in D^-(W')$*. Assume either of the condition (i) or (ii) in Lemma 1.5.3 is satisfied. Let*

(1.5.5.3) $$\mathrm{E} = \big(\mathrm{E}^2_{p,q} = \mathrm{R}^{-p}f_*\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_X}(\mathcal{F},\mathrm{R}f_*\mathcal{G})\big)$$

be the spectral sequence (1.5.3.4) *combined with the isomorphism* (1.5.3.2).

Then the map $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G} \to \mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}[2]$ (1.5.4.1) *induces a map*

(1.5.5.4) $$\mathrm{E} \longrightarrow \mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathrm{E}[0,2]$$

of spectral sequences. The maps on E_2*-terms are induced by*

$$\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{q-2}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

and the maps on the abutments are

$$\alpha_{\mathcal{F},\mathrm{R}f_*\mathcal{G},X/P} : \mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F},\mathrm{R}f_*\mathcal{G}) \to \mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{n-2}^{\mathcal{O}_X}(\mathcal{F},\mathrm{R}f_*\mathcal{G}).$$

Proof. — 1. We consider the map $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G} \to (\mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}[2])\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}$ as a map of filtered complexes with respect to the canonical filtrations on $\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}$ and on $\mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}[2]$. It induces a map of filtered complexes $\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}\otimes^{\mathrm{L}}_{\mathcal{O}_W}\mathcal{H} \to \mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}\otimes^{\mathrm{L}}_{\mathcal{O}_W}\mathcal{H}[2]$. By identifying $\mathcal{T}or_{p+q}^{\mathcal{O}_W}(\mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}[2],\mathcal{H})$ with $\mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or_{p+q-2}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}\otimes^{\mathrm{L}}_{\mathcal{O}_W}\mathcal{H})$ by using the isomorphism (1.5.1.2), we obtain a map $\mathrm{E}\to\mathrm{N}_{X/P}\otimes_{\mathcal{O}_X}\mathrm{E}[0,2]$ of spectral sequences. It is clear from the construction that the maps on the E_2-terms are induced by $\alpha_{\mathcal{F},\mathcal{G},X/P}$ and the maps on the abutments are $\alpha_{\mathcal{F},\mathcal{G}\otimes^{\mathrm{L}}_{\mathcal{O}_W}\mathcal{H},X/P}$.

2. Proof is similar to 1 and left to the reader. □

Lemma **1.5.6.** — *Let* $X\to S$ *be a flat morphism of schemes and* $\mathcal{F}$ *and* $\mathcal{G}$ *be complexes of* $\mathcal{O}_X$*-modules bounded above. We define* $\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_S}\mathcal{G}$ *to be* $\mathrm{L}pr_1^*\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_{X\times_S X}}\mathrm{L}pr_2^*\mathcal{G}$*. Then the adjunction induces an isomorphism* $(\mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_S}\mathcal{G})\otimes^{\mathrm{L}}_{\mathcal{O}_{X\times_S X}}\mathcal{O}_X \to \mathcal{F}\otimes^{\mathrm{L}}_{\mathcal{O}_X}\mathcal{G}$*. It induces a spectral sequence*

(1.5.6.1) $$\mathrm{E}^2_{p,q} = \mathcal{T}or_p^{\mathcal{O}_{X\times_S X}}\big(\mathcal{T}or_q^{\mathcal{O}_S}(\mathcal{F},\mathcal{G}),\mathcal{O}_X\big) \Rightarrow \mathcal{T}or_{p+q}^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}).$$

Proof. — The proof is similar to Lemma 1.5.1 and left to the reader. □

Corollary **1.5.7.** — *Let* $X\to S$ *be a smooth morphism of relative dimension* n *and* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. Assume* $\mathcal{F}$ *is of tor-dimension* $\le m$ *as an* $\mathcal{O}_S$*-module. Then* $\mathcal{F}$ *is of tor-dimension* $\le m+n$ *as an* $\mathcal{O}_X$*-module.*

Proof. — The diagonal map $X\to X\times_S X$ is a section of the smooth map $X\times_S X\to X$ of relative dimension n and hence is of tor-dimension n. We consider the spectral sequence (1.5.6.1). Then, we have $\mathrm{E}^2_{p,q}=0$ if $p>n$ or $q>m$. Hence the assertion follows. □

1.6. *Excess conormal complex and $\mathcal{T}or$.* — We construct a spectral sequence computing $\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)$ for certain morphisms $V \to X \leftarrow W$ of schemes in Proposition 1.6.4.

Definition **1.6.1.** — *1.* ([17] Exp. VII Definition 1.4) *We say an immersion* $X \to P$ *of schemes is a regular immersion if the following condition is satisfied.*

For $x \in X$, *there exist an open neighborhood* U *of* x *in* P, *a locally free* $\mathcal{O}_U$*-module* $\mathcal{E}_U$ *of finite rank and an* $\mathcal{O}_U$*-linear map* $\mathcal{E}_U \to \mathcal{O}_U$ *such that the Koszul complex* $\mathbf{K}(\mathcal{E}_U \to \mathcal{O}_U)$ *is a resolution of* $\mathcal{O}_{X\cap U}$.

2. ([17] Exp. VIII Definition 1.1) *Let* $X \to S$ *be a morphism locally of finite presentation of schemes. We say* X *is locally of complete intersection over* S *if, for each* $x \in X$, *there exist an open neighborhood* U *of* x *in* X, *a smooth scheme* P *over* S *and a regular immersion* $U \to P$ *over* S.

We do not require flatness in the definition of locally of complete intersection as in [15] (19.3.6). By Lemma 1.3.2, the condition that the Koszul complex $\mathbf{K}(\mathcal{E}_U \to \mathcal{O}_U)$ is a resolution of $\mathcal{O}_{X\cap U}$ is equivalent to that the canonical surjection $\mathbf{A}(\mathcal{E}_U \to \mathcal{O}_U) \to \mathcal{O}_{X\cap U}$ is a resolution by a weakly free simplicial $\mathcal{O}_U$-algebra. The quasi-isomorphism $\mathbf{K}(\mathcal{E}_U \to \mathcal{O}_U) \to \mathcal{O}_{X\cap U}$ induces an isomorphism $\mathcal{E}_U \otimes_{\mathcal{O}_U} \mathcal{O}_{X\cap U} \to N_{X\cap U/U}$ to the conormal sheaf. If P is a noetherian scheme, the condition that $\mathbf{K}(\mathcal{E}_U \to \mathcal{O}_U)$ is a resolution of $\mathcal{O}_{X\cap U}$ is equivalent to that the image of a local basis of $\mathcal{E}_U$ is a regular sequence of $\mathcal{O}_U$. A map of finite type of regular noetherian schemes is locally of complete intersection. If $X \to S$ is locally of complete intersection and if $P \to S$ is smooth, then an immersion $X \to P$ over S is a regular immersion.

Lemma **1.6.2.** — *1.* ([19] III Proposition 3.1.2) *Let* $X \to S$ *be a smooth morphism of schemes. Then, the canonical map* $L_{X/S} \to \Omega^1_{X/S}$ *is an isomorphism.*

2. (loc.cit. Proposition 3.2.4) *Let* $X \to P$ *be a regular immersion. Then, the canonical map* $L_{X/P} \to N_{X/P}[1]$ *is an isomorphism.*

3. (loc.cit. Proposition 3.2.6) *Let* $X \to P$ *be a regular immersion and* $P \to S$ *be a smooth morphism. Then, we have a distinguished triangle* $\to N_{X/P} \to \Omega^1_{P/S} \otimes_{\mathcal{O}_P} \mathcal{O}_X \to L_{X/S} \to$.

Let $i : V \to X$ be an immersion of schemes and let

$$\begin{array}{ccc} T & \xrightarrow{i'} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[i]{} & X \end{array}$$

be a cartesian diagram of schemes. Assume that the immersion $i' : T \to W$ is a regular immersion. We define the conormal complex $M_{V/X}$, the excess conormal complex

$M'_{V/X,W}$ and the excess conormal sheaf $N'_{V/X,W}$. Recall that the standard resolution $P_X(\mathscr{O}_V) = P_{i^{-1}\mathscr{O}_X}(\mathscr{O}_V) \to \mathscr{O}_V$ is a resolution of $\mathscr{O}_V$ by a free simplicial $i^{-1}\mathscr{O}_X$-algebra and that the cotangent complex $L_{V/X}$ is defined as the normal complex $N(\Omega^1_{P_X(\mathscr{O}_V)/i^{-1}\mathscr{O}_X} \otimes_{P_X(\mathscr{O}_V)} \mathscr{O}_V)$.

Definition **1.6.3.** — *Let* $i : V \to X$ *be an immersion of schemes.*
1. We call

$$\textbf{(1.6.3.1)} \qquad M_{V/X} = L_{V/X}[-1] = N\big(\Omega^1_{P_X(\mathscr{O}_V)/i^{-1}\mathscr{O}_X} \otimes_{P_X(\mathscr{O}_V)} \mathscr{O}_V\big)[-1]$$

the conormal complex *of the immersion* $i : V \to X$.
2. Let

$$\begin{array}{ccc} T & \xrightarrow{i'} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[i]{} & X \end{array}$$

be a cartesian diagram of schemes and assume $i' : T \to W$ *is a regular immersion. We put*

$$A_{V/X,W} = g^{-1}P_X(\mathscr{O}_V) \otimes_{(i\circ g)^{-1}\mathscr{O}_X} i'^{-1}\mathscr{O}_W$$

and define an ideal $I_{V/X,W} \subset A_{V/X,W}$ *by the exact sequence*

$$0 \longrightarrow I_{V/X,W} \longrightarrow A_{V/X,W} \longrightarrow \mathscr{O}_T \longrightarrow 0.$$

We call the chain complex

$$\textbf{(1.6.3.2)} \qquad M'_{V/X,W} = N\big(I_{V/X,W}/I^2_{V/X,W}\big)[-1]$$

the excess conormal complex. *We call the map*

$$M'_{V/X,W} \longrightarrow Lg^*M_{V/X}$$

induced by $d : I_{V/X,W}/I^2_{V/X,W} \to \Omega^1_{A_{V/X,W}/i'^{-1}\mathscr{O}_W} \otimes_{A_{V/X,W}} \mathscr{O}_T$ *the canonical map.*
We define the excess conormal sheaf $N'_{V/X,W}$ *by the exact sequence*

$$\textbf{(1.6.3.3)} \qquad 0 \longrightarrow N'_{V/X,W} \longrightarrow g^*N_{V/X} \longrightarrow N_{T/W} \longrightarrow 0$$

where $g^*N_{V/X} \to N_{T/W}$ *is the canonical surjection of conormal sheaves.*

The cohomology sheaf $\mathscr{H}_0(M_{V/X}) = \mathscr{H}_1(L_{V/X})$ is canonically isomorphic to the conormal sheaf $N_{V/X}$. If the immersion $V \to X$ is a regular immersion, the canonical map $M_{V/X} \to N_{V/X}$ is an isomorphism.

Proposition **1.6.4** (cf. [19] III Proposition 3.3.6, [35] Theorem 6.3). — *Let* $i : \mathrm{V} \to \mathrm{X}$ *be an immersion of schemes and let*

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{i'} & \mathrm{W} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{V} & \xrightarrow[i]{} & \mathrm{X} \end{array}$$

be a cartesian diagram of schemes. Assume that the immersion $i' : \mathrm{T} \to \mathrm{W}$ *is a regular immersion. We put* $\mathrm{A}_{\mathrm{V/X,W}} = g^{-1}\mathrm{P}_{\mathrm{X}}(\mathcal{O}_{\mathrm{V}}) \otimes_{(i\circ g)^{-1}\mathcal{O}_{\mathrm{X}}} i'^{-1}\mathcal{O}_{\mathrm{W}}$ *and* $\mathrm{I}_{\mathrm{V/X,W}} = \mathrm{Ker}(\mathrm{A}_{\mathrm{V/X,W}} \to \mathcal{O}_{\mathrm{T}})$ *as in Definition 1.6.3.2. We define a decreasing filtration* $\mathrm{F}^{\bullet}$ *on* $\mathrm{A}_{\mathrm{V/X,W}}$ *by* $\mathrm{F}^p\mathrm{A}_{\mathrm{V/X,W}} = \mathrm{I}^p_{\mathrm{V/X,W}}$.

1. *For* $p \geq 0$, *the canonical map* $\mathrm{S}^p(\mathrm{I}_{\mathrm{V/X,W}}/\mathrm{I}^2_{\mathrm{V/X,W}}) \to \mathrm{Gr}^p_{\mathrm{F}}(\mathrm{A}_{\mathrm{V/X,W}}) = \mathrm{I}^p_{\mathrm{V/X,W}}/\mathrm{I}^{p+1}_{\mathrm{V/X,W}}$ *is an isomorphism and induces an isomorphism*

(1.6.4.1) $$\mathrm{L}\Lambda^p\mathrm{M}'_{\mathrm{V/X,W}}[p] \longrightarrow \mathrm{N}\mathrm{Gr}^p_{\mathrm{F}}(\mathrm{A}_{\mathrm{V/X,W}})$$

in $\mathrm{D}^-(\mathrm{T})$.

2. *We have a distinguished triangle*

(1.6.4.2) $$\longrightarrow \mathrm{M}'_{\mathrm{V/X,W}} \longrightarrow \mathrm{L}g^*\mathrm{M}_{\mathrm{V/X}} \longrightarrow \mathrm{N}_{\mathrm{T/W}} \longrightarrow .$$

In particular, if $\mathrm{W} = \mathrm{T}$ *is a scheme over* V, *the canonical map* $\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{L}g^*\mathrm{M}_{\mathrm{V/X}}$ *is an isomorphism. If* $\mathrm{V} \to \mathrm{X}$ *is a regular immersion, the canonical map* $\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{N}'_{\mathrm{V/X,W}}$ *is an isomorphism.*

3. *The filtration* $\mathrm{F}^{\bullet}$ *defines a spectral sequence*

(1.6.4.3) $$\mathrm{E}^1_{p,q} = \mathrm{L}^{2p+q}\Lambda^{-p}\mathrm{M}'_{\mathrm{V/X,W}} \Longrightarrow \mathcal{T}or^{\mathcal{O}_{\mathrm{X}}}_{p+q}(\mathcal{O}_{\mathrm{V}}, \mathcal{O}_{\mathrm{W}})$$

of $\mathcal{O}_{\mathrm{T}}$*-modules.*

Proof. — 1. Since $i' : \mathrm{T} \to \mathrm{W}$ is a regular immersion, the ideal $\mathrm{I}_{\mathrm{T}} = \mathrm{Ker}(i'^{-1}\mathcal{O}_{\mathrm{W}} \to \mathcal{O}_{\mathrm{T}})$ of $i'^{-1}\mathcal{O}_{\mathrm{W}}$ is weakly regular in the sense of [19] III 3.3.1. Hence by loc.cit. Proposition 3.3.6, the $i'^{-1}\mathcal{O}_{\mathrm{W}}$-algebra $\mathcal{O}_{\mathrm{T}}$ is weakly of complete intersection in the sense of loc.cit. 3.3.4. Further, the ideal $\mathrm{I}_{\mathrm{V/X,W}}$ of $\mathrm{A}_{\mathrm{V/X,W}}$ is weakly regular and the map $\mathrm{S}^p(\mathrm{I}_{\mathrm{V/X,W}}/\mathrm{I}^2_{\mathrm{V/X,W}}) \to \mathrm{Gr}^p_{\mathrm{F}}\mathrm{A}_{\mathrm{V/X,W}}$ is an isomorphism by loc.cit. Proposition 3.3.6. It induces an isomorphism $\mathrm{NS}^p(\mathrm{I}_{\mathrm{V/X,W}}/\mathrm{I}^2_{\mathrm{V/X,W}}) = \mathrm{NS}^p(\mathrm{M}'_{\mathrm{V/X,W}}[1]) \to \mathrm{N}(\mathrm{I}^p_{\mathrm{V/X,W}}/\mathrm{I}^{p+1}_{\mathrm{V/X,W}})$. Hence we obtain an isomorphism (1.6.4.1) by Proposition 1.2.8.

2. By the canonical isomorphism $\mathrm{L}_{\mathrm{T/W}}[-1] \to \mathrm{N}_{\mathrm{T/W}}$, it suffices to apply further loc.cit. Proposition 3.3.6 to the surjection $\mathrm{A}_{\mathrm{V/X,W}} \to \mathcal{O}_{\mathrm{T}}$. If $\mathrm{W} = \mathrm{T}$, we have $\mathrm{N}_{\mathrm{T/W}} = 0$. If $\mathrm{V} \to \mathrm{X}$ is a regular immersion, the canonical map $\mathrm{M}_{\mathrm{V/X}} \to \mathrm{N}_{\mathrm{V/X}}$ is an isomorphism.

3. We consider the spectral sequence $\mathrm{E}^1_{p,q} = \mathcal{H}_{p+q}\mathrm{N}(\mathrm{Gr}^{-p}_{\mathrm{F}}\mathrm{A}_{\mathrm{V/X,W}}) \Rightarrow \mathcal{H}_{p+q}\mathrm{N}(\mathrm{A}_{\mathrm{V/X,W}})$ defined by $\mathrm{F}^{\bullet}$. The quasi-isomorphism $\mathrm{P}_{\mathrm{X}}(\mathcal{O}_{\mathrm{V}}) \to \mathcal{O}_{\mathrm{V}}$ induces an isomorphism $\mathcal{H}_r\mathrm{N}(\mathrm{A}_{\mathrm{V/X,W}}) \to \mathcal{T}or^{\mathcal{O}_{\mathrm{X}}}_r(\mathcal{O}_{\mathrm{V}}, \mathcal{O}_{\mathrm{W}})$. The isomorphism (1.6.4.1) induces an isomorphism $\mathrm{L}^{2p+q}\Lambda^{-p}\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{E}^1_{p,q}$. Thus the assertion follows. □

Corollary **1.6.5.** — *Assume further that the immersion* $i : \mathrm{V} \to \mathrm{X}$ *is a regular immersion. Then, the spectral sequence* (1.6.4.3) *degenerates at* E^1*-terms and gives an isomorphism*

$$\textbf{(1.6.5.1)}\qquad \Lambda^r \mathrm{N}'_{\mathrm{V/X,W}} \longrightarrow \mathcal{T}or_r^{\mathcal{O}_\mathrm{X}}(\mathcal{O}_\mathrm{V}, \mathcal{O}_\mathrm{W})$$

of locally free $\mathcal{O}_\mathrm{T}$*-modules. In particular, if* $\mathrm{W} = \mathrm{T}$, *we have an isomorphism*

$$\textbf{(1.6.5.2)}\qquad \Lambda^r g^*\mathrm{N}_{\mathrm{V/X}} \longrightarrow \mathcal{T}or_r^{\mathcal{O}_\mathrm{X}}(\mathcal{O}_\mathrm{V}, \mathcal{O}_\mathrm{W}).$$

Proof. — By Proposition 1.6.4.2, the canonical map $\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{N}'_{\mathrm{V/X,W}}$ is an isomorphism. Since the conormal sheaf $\mathrm{N}'_{\mathrm{V/X,W}}$ is a locally free $\mathcal{O}_\mathrm{T}$-module under the assumption, the assertion follows. □

In Proposition 1.6.4, we may replace the resolution $\mathrm{P}_\mathrm{X}(\mathcal{O}_\mathrm{V}) \to \mathcal{O}_\mathrm{V}$ by any resolution by a weakly free simplicial $i^{-1}\mathcal{O}_\mathrm{X}$-algebra.

Lemma **1.6.6.** — *Let the notation be as in Proposition 1.6.4. Let* $\mathrm{A} \to \mathcal{O}_\mathrm{V}$ *be a resolution by a weakly free simplicial* $i^{-1}\mathcal{O}_\mathrm{X}$*-algebra. We put* $\mathrm{A}_\mathrm{W} = g^{-1}\mathrm{A} \otimes_{(i\circ g)^{-1}\mathcal{O}_\mathrm{X}} i'^{-1}\mathcal{O}_\mathrm{W}$ *and* $\mathrm{I} = \mathrm{Ker}(\mathrm{A}_\mathrm{W} \to \mathcal{O}_\mathrm{T})$. *We define filtrations* $\mathrm{F}^\bullet$ *on* A_W *by* $\mathrm{F}^p\mathrm{A}_\mathrm{W} = \mathrm{I}^p$. *Let* $\mathrm{E}_\mathcal{T}$ *be the spectral sequence* (1.6.4.3) *and* E_A *be the spectral sequence* $\mathrm{E}^1_{p,q} = \mathrm{H}_{p+q}\mathrm{N}\mathrm{Gr}_\mathrm{F}^{-p}\mathrm{A}_\mathrm{W} \Rightarrow \mathrm{H}_{p+q}\mathrm{N}\mathrm{A}_\mathrm{W}$ *defined by the filtered complex* $(\mathrm{N}\mathrm{A}_\mathrm{W}, \mathrm{F}^\bullet)$.

Then, the canonical map

$$\textbf{(1.6.6.1)}\qquad \mathrm{S}^p(\mathrm{I}/\mathrm{I}^2) \longrightarrow \mathrm{I}^p/\mathrm{I}^{p+1} = \mathrm{Gr}^p_\mathrm{F}\mathrm{A}_\mathrm{W}$$

is an isomorphism. The quasi-isomorphism $\mathrm{A} \to \mathcal{O}_\mathrm{V}$ *induces an isomorphism of distinguished triangles*

$$\textbf{(1.6.6.2)}\qquad \begin{array}{ccccccc} \to & \mathrm{M}'_{\mathrm{V/X,W}} & \longrightarrow & g^*\mathrm{M}_{\mathrm{V/X}} & \longrightarrow & \mathrm{N}_{\mathrm{T/W}} & \to \\ & \downarrow & & \downarrow & & \| & \\ \to & \mathrm{N}(\mathrm{I}/\mathrm{I}^2)[-1] & \longrightarrow & \mathrm{N}\big(\Omega^1_{\mathrm{A}_\mathrm{W}/\mathcal{O}_\mathrm{W}}\big)[-1] & \longrightarrow & \mathrm{N}_{\mathrm{T/W}} & \to \end{array}$$

and an isomorphism of spectral sequences

$$\textbf{(1.6.6.3)}\qquad \mathrm{E}_\mathrm{A} \longrightarrow \mathrm{E}_\mathcal{T}.$$

Proof. — Recall $\mathrm{I}_{\mathrm{V/X,W}} = \mathrm{Ker}(\mathrm{A}_{\mathrm{V/X,W}} = \mathrm{P}_\mathrm{X}(\mathcal{O}_\mathrm{V}) \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{W} \to \mathcal{O}_\mathrm{T})$ and $\mathrm{N}(\mathrm{I}_{\mathrm{V/X,W}}/\mathrm{I}^2_{\mathrm{V/X,W}}) = \mathrm{M}'_{\mathrm{V/X,W}}[1]$ in the notation of Definition 1.6.3.2. In the notation of [19] II 1.2.2, we have quasi-isomorphisms $\mathrm{P}_\mathrm{X}(\mathcal{O}_\mathrm{V}) \leftarrow \mathrm{P}^\Delta_\mathrm{X}(\mathrm{A}) \to \mathrm{A}$. They induce a map

$$\begin{array}{ccccccc} \to & \mathrm{N}\big(\mathrm{I}_{\mathrm{V/X,W}}/\mathrm{I}^2_{\mathrm{V/X,W}}\big)[-1] & \longrightarrow & \mathrm{N}\big(\Omega^1_{\mathrm{A}_{\mathrm{V/X,W}}/\mathcal{O}_\mathrm{W}}\big)[-1] & \longrightarrow & \mathrm{N}_{\mathrm{T/W}} & \to \\ & \downarrow & & \downarrow & & \| & \\ \to & \mathrm{N}(\mathrm{I}/\mathrm{I}^2)[-1] & \longrightarrow & \mathrm{N}\big(\Omega^1_{\mathrm{A}_\mathrm{W}/\mathcal{O}_\mathrm{W}}\big)[-1] & \longrightarrow & \mathrm{N}_{\mathrm{T/W}} & \to \end{array}$$

of distinguished triangles. Since $A \to \mathcal{O}_V$ is a resolution by weakly free simpicial $\mathcal{O}_X$-algebra, the middle vertical arrow is an isomorphism. Thus, the left vertical map $N(I_{V/X,W}/I^2_{V/X,W}) = M'_{V/X,W}[1] \to N(I/I^2)$ is also an isomorphism and we obtain an isomorphism (1.6.6.2).

By [19] Proposition 3.3.6, the ideals $I \subset A_W$ and $I_{V/X,W} \subset A_{V/X,W}$ are weakly regular. Thus by loc.cit. 3.3.1, the maps $S^p(I/I^2) \to I^p/I^{p+1}$ (1.6.6.1) and $S^p(I_{V/X,W}/I^2_{V/X,W}) \to I^p_{V/X,W}/I^{p+1}_{V/X,W}$ are isomorphism.

We consider the maps $P_X(\mathcal{O}_V) \leftarrow P^\Delta_X(A) \to A$. For $p \geq 0$, they induce an isomorphism $\mathrm{Gr}^p_F A_{V/X,W} = N(I^p_{V/X,W}/I^{p+1}_{V/X,W}) \to \mathrm{Gr}^p_F A_W = N(I^p/I^{p+1})$ by the isomorphisms $N(I_{V/X,W}/I^2_{V/X,W}) \to N(I/I^2)$, $S^p(I_{V/X,W}/I^2_{V/X,W}) \to I^p_{V/X,W}/I^{p+1}_{V/X,W}$ and $S^p(I/I^2) \to I^p/I^{p+1}$. Hence they define an isomorphism $N(A_{V/X,W}, F^\bullet) \to N(A_W, F^\bullet)$ in the derived category of filtered complexes. It defines an isomorphism $E_A \to E_{\mathcal{F}}$ (1.6.6.3) of the spectral sequences. □

The following result will be used only in the proof of Proposition 5.1.4 and will not be used in the proof of the main result, Theorem 6.3.1.

Proposition **1.6.7** (cf. [5] Theorem 8). — *Let* $i : V \to X$ *be an immersion. Assume that, for each* $x \in X$*, there is an open neighborhood* U *and a regular immersion* $U \to P$ *such that the composition* $V \cap U \to U \to P$ *is also a regular immersion. Then for a scheme* W *over* V*, the spectral sequence* (1.6.4.3) *degenerates at* E^1*-terms.*

Proof. — We give a proof using the Koszul simplicial algebra defined in Section 1.3. Since the question is local, we may assume that there exist locally free $\mathcal{O}_P$-modules $\mathcal{M}_P$ and $\mathcal{L}_P$ of finite rank and $\mathcal{O}_P$-linear maps $v_P : \mathcal{L}_P \to \mathcal{M}_P$ and $u_P : \mathcal{M}_P \to \mathcal{O}_P$ such that the Koszul complexes $\mathbf{K}(\mathcal{M}_P \xrightarrow{u_P} \mathcal{O}_P)$ and $\mathbf{K}(\mathcal{L}_P \xrightarrow{u_P \circ v_P} \mathcal{O}_P)$ are resolutions of the $\mathcal{O}_P$-modules $\mathcal{O}_V$ and $\mathcal{O}_X$ respectively. By Lemma 1.3.2, $\mathbf{A}_{\mathcal{M}_P} = \mathbf{A}(\mathcal{M}_P \xrightarrow{u_P} \mathcal{O}_P) \to \mathcal{O}_V$ and $\mathbf{A}_{\mathcal{L}_P} = \mathbf{A}(\mathcal{L}_P \xrightarrow{u_P \circ v_P} \mathcal{O}_P) \to \mathcal{O}_X$ are quasi-isomorphisms.

Let $\mathcal{C}_P = [\mathcal{L}_P \xrightarrow{(v_P, -1)} \mathcal{M}_P \oplus \mathcal{L}_P]$ be the mapping cylinder and define a map $c_P : \mathcal{C}_P \to \mathcal{O}_P$ by $(u_P, u_P \circ v_P)$. By Corollary 1.3.6.1, the natural map $\mathcal{M}_P \to \mathcal{C}_P$ induces a quasi-isomorphism $\mathbf{A}_{\mathcal{M}_P} \to \mathbf{A}_{\mathcal{C}_P} = \mathbf{A}^\Delta(\mathcal{C}_P \xrightarrow{c_P} \mathcal{O}_P)$. Thus, in the commutative diagram

$$\begin{array}{ccc} \mathbf{A}_{\mathcal{C}_P} & \longrightarrow & \mathcal{O}_V \\ \uparrow & & \uparrow \\ \mathbf{A}_{\mathcal{L}_P} & \longrightarrow & \mathcal{O}_X, \end{array}$$

the horizontal arrows are quasi-isomorphisms. Since $\mathbf{A}_{\mathcal{C}_P}$ is flat over $\mathbf{A}_{\mathcal{L}_P}$, the map $\mathbf{A}_{\mathcal{C}_P} = \mathbf{A}_{\mathcal{C}_P} \otimes_{\mathbf{A}_{\mathcal{L}_P}} \mathbf{A}_{\mathcal{L}_P} \to \mathbf{A}_{\mathcal{C}_P} \otimes_{\mathbf{A}_{\mathcal{L}_P}} \mathcal{O}_X$ is a quasi-isomorphism by [19] I Lemme 3.3.2.1. Thus we obtain a quasi-isomorphism $\mathbf{A}_{\mathcal{C}_P} \otimes_{\mathbf{A}_{\mathcal{L}_P}} \mathcal{O}_X \to \mathcal{O}_V$.

We put $\mathscr{L} = \mathscr{L}_{\mathrm{P}} \otimes_{\mathscr{O}_{\mathrm{P}}} \mathscr{O}_{\mathrm{X}}$, $\mathscr{M} = \mathscr{M}_{\mathrm{P}} \otimes_{\mathscr{O}_{\mathrm{P}}} \mathscr{O}_{\mathrm{X}}$ and $\mathscr{C} = \mathscr{C}_{\mathrm{P}} \otimes_{\mathscr{O}_{\mathrm{P}}} \mathscr{O}_{\mathrm{X}}$. Let $\mathscr{K} = [\mathscr{L} \to \mathscr{M}]$ be the mapping cone and $w : \mathscr{K} \to \mathscr{O}_{\mathrm{X}}$ be the map defined by $u = u_{\mathrm{P}} \otimes 1$. We put $\mathbf{A}_{\mathscr{L}} = \mathbf{A}(\mathscr{L} \xrightarrow{u \circ v} \mathscr{O}_{\mathrm{X}})$, $\mathbf{A}_{\mathscr{C}} = \mathbf{A}(\mathscr{C} \xrightarrow{c} \mathscr{O}_{\mathrm{X}})$ and $\mathbf{A}_{\mathscr{K}} = \mathbf{A}(\mathscr{K} \xrightarrow{w} \mathscr{O}_{\mathrm{X}})$. Then, we have $\mathbf{A}_{\mathscr{C}_{\mathrm{P}}} \otimes_{\mathbf{A}_{\mathscr{L}_{\mathrm{P}}}} \mathscr{O}_{\mathrm{X}} = \mathbf{A}_{\mathscr{C}} \otimes_{\mathbf{A}_{\mathscr{L}}} \mathscr{O}_{\mathrm{X}}$. Since the composition $\mathscr{L} \to \mathscr{O}_{\mathrm{X}}$ is the 0-map, we have an isomorphism $\mathbf{A}_{\mathscr{C}} \otimes_{\mathbf{A}_{\mathscr{L}}} \mathscr{O}_{\mathrm{X}} \to \mathbf{A}_{\mathscr{K}}$ by Corollary 1.3.6.2. Thus we obtain a resolution $\mathrm{A} = \mathbf{A}_{\mathscr{K}} \to \mathscr{O}_{\mathrm{V}}$ by weakly free simplicial $\mathscr{O}_{\mathrm{X}}$-algebra.

We consider the filtration $\mathrm{F}^{\bullet}$ on $\mathrm{A}_{\mathrm{W}} = \mathrm{A} \otimes_{\mathscr{O}_{\mathrm{X}}} \mathscr{O}_{\mathrm{W}}$ defined by the powers of the kernel of the surjection $\mathrm{A}_{\mathrm{W}} \to \mathscr{O}_{\mathrm{T}} = \mathscr{O}_{\mathrm{W}}$. By the assumption that W is a scheme over V, the map $w_{\mathrm{W}} : \mathscr{K}_{\mathrm{W}} \to \mathscr{O}_{\mathrm{W}}$ defining $\mathrm{A}_{\mathrm{W}} = \mathbf{A}(\mathscr{K}_{\mathrm{W}} \xrightarrow{w_{\mathrm{W}}} \mathscr{O}_{\mathrm{W}})$ is the 0-map. Hence the filtration $\mathrm{F}^{\bullet}$ on A_{W} splits. Thus the assertion follows by Lemma 1.6.6. □

The relation of Proposition 1.6.7 with [5] Theorem 8 is as follows. We keep the notation in the proof of Proposition 1.6.7. Since the Koszul complex $\mathbf{K}(\mathscr{M}_{\mathrm{P}} \xrightarrow{u_{\mathrm{P}}} \mathscr{O}_{\mathrm{P}})$ is a resolution of the $\mathscr{O}_{\mathrm{P}}$-modules $\mathscr{O}_{\mathrm{V}}$, the Koszul complex $\mathrm{E} = \mathbf{K}(\mathscr{M} \xrightarrow{u} \mathscr{O}_{\mathrm{X}})$ is isomorphic to $\mathscr{O}_{\mathrm{X}} \otimes^{\mathrm{L}}_{\mathscr{O}_{\mathrm{P}}} \mathscr{O}_{\mathrm{V}}$. Hence, by Corollary 1.6.5, the $\mathscr{O}_{\mathrm{V}}$-module $\mathrm{H}_1(\mathrm{E})$ is isomorphic to $\mathscr{T}\!or_1^{\mathscr{O}_{\mathrm{P}}}(\mathscr{O}_{\mathrm{X}}, \mathscr{O}_{\mathrm{V}})$ and is locally free. Further, the canonical map $\Lambda^p \mathrm{H}_1(\mathrm{E}) \to \mathrm{H}_p(\mathrm{E})$ is an isomorphism for $p \geq 0$. Thus the ideal of $\mathscr{O}_{\mathrm{X}}$ defining $\mathscr{O}_{\mathrm{V}}$ has locally Free Exterior Koszul Homology property in the sense of [5]. Therefore loc.cit. Theorem 8 together with the remark following its proof implies Proposition 1.6.7.

1.7. *Spectral sequence for $\mathscr{T}\!or$ and the Atiyah class.* — We give a relation between the spectral sequence (1.6.4.3) and the Atiyah class map in Proposition 1.7.2.

In this subsection, we consider a commutative diagram

(1.7.0.1)
$$\begin{array}{ccccc} \mathrm{T} & \xrightarrow{i'} & \mathrm{W} & & \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ \mathrm{V} & \xrightarrow{i} & \mathrm{X} & \longrightarrow & \mathrm{P} \end{array}$$

of schemes. We assume that the square is cartesian, the horizontal arrows are immersions and that the immersions $i' : \mathrm{T} \to \mathrm{W}$ and $\mathrm{X} \to \mathrm{P}$ are regular immersions. Shifting the distinguished triangle (1.4.0.1) for the lower line in the diagram (1.7.0.1), we obtain a distinguished triangle

(1.7.0.2)
$$\longrightarrow (i \circ g)^* \mathrm{N}_{\mathrm{X}/\mathrm{P}} \longrightarrow \mathrm{M}_{\mathrm{V}/\mathrm{P}} \longrightarrow \mathrm{M}_{\mathrm{V}/\mathrm{X}} \longrightarrow .$$

Throughout this subsection, we use the following notation. We consider the standard resolution $\mathscr{P} = \mathrm{P}_{\mathrm{P}}(\mathscr{O}_{\mathrm{X}}) = \mathrm{P}_{j^{-1}\mathscr{O}_{\mathrm{P}}}(\mathscr{O}_{\mathrm{X}}) \to \mathscr{O}_{\mathrm{X}}$ by free simplicial $j^{-1}\mathscr{O}_{\mathrm{P}}$-algebra and the diagonal of the standard resolution $\mathcal{Q} = P^{\Delta}_{\mathcal{P}}(\mathcal{O}_V) \to \mathcal{O}_V$ by free bisimplicial $i^{-1}\mathcal{P}$-algebra. We put $J = \mathrm{Ker}(\mathcal{P} \otimes_{j^{-1}\mathcal{O}_P} \mathcal{P} \to \mathcal{P})$. Further, we put

$$\begin{aligned} B &= i^{-1}\mathcal{O}_X \otimes_{(j\circ i)^{-1}\mathcal{O}_P} \mathcal{Q}, \\ A &= i^{-1}\mathcal{O}_X \otimes_{i^{-1}\mathcal{P}} \mathcal{Q} = B \otimes_{i^{-1}(\mathcal{P}\otimes_{j^{-1}\mathcal{O}_P}\mathcal{P})} i^{-1}\mathcal{P}, \\ J_B &= \mathrm{Ker}(B \to A) = \mathcal{Q} \otimes_{i^{-1}(\mathcal{P}\otimes_{j^{-1}\mathcal{O}_P}\mathcal{P})} i^{-1}J. \end{aligned}$$

We put $A_W = A \otimes_{\mathcal{O}_X} \mathcal{O}_W$, $B_W = B \otimes_{\mathcal{O}_X} \mathcal{O}_W$ and $J_{B_W} = \mathrm{Ker}(B_W \to A_W)$. Further we put $I = \mathrm{Ker}(A_W \to \mathcal{O}_T)$, $\tilde{I} = \mathrm{Ker}(B_W \to \mathcal{O}_T)$.

For each n, there exist flat $(j \circ i)^{-1}\mathcal{O}_P$-modules L_n and M_n and isomorphisms $S_{(j\circ i)^{-1}\mathcal{O}_P} L_n \to \mathcal{P}_n$ and $S_{(j\circ i)^{-1}\mathcal{O}_P}(L_n \oplus M_n) \to \mathcal{Q}_n$. We put $L_{n,X} = L_n \otimes_{(j\circ i)^{-1}\mathcal{O}_P} i^{-1}\mathcal{O}_X$ and $M_{n,X} = M_n \otimes_{(j\circ i)^{-1}\mathcal{O}_P} i^{-1}\mathcal{O}_X$. Then we obtain a commutative diagram

$$\begin{array}{ccc} S_{i^{-1}\mathcal{O}_X}(L_{n,X} \oplus M_{n,X}) & \longrightarrow & B_n \\ \downarrow & & \downarrow \\ S_{i^{-1}\mathcal{O}_X} M_{n,X} & \longrightarrow & A_n \end{array}$$

where the horizontal arrows are isomorphisms. The left vertical arrow is induced by an $i^{-1}\mathcal{O}_X$-linear form $L_{n,X} \to i^{-1}\mathcal{O}_X$. Thus, by modifying the isomorphism $S_{i^{-1}\mathcal{O}_X}(L_{n,X} \oplus M_{n,X}) \to B_n$ by the linear form $L_{n,X} \to i^{-1}\mathcal{O}_X$, we may assume that the left vertical arrow is induced by the 0-map $L_{n,X} \to i^{-1}\mathcal{O}_X$. Thus, we obtain an isomorphism

$$\textbf{(1.7.0.3)} \qquad S_{A_n}(A_n \otimes_{i^{-1}\mathcal{O}_X} L_{n,X}) \longrightarrow B_n$$

of A_n-algebras.

Lemma **1.7.1.** — *We keep the notation above. Then, the canonical maps defines a map*

$$\textbf{(1.7.1.1)} \qquad \begin{array}{ccccccccc} \longrightarrow & (i\circ g)^* N_{X/P} & \longrightarrow & M'_{V/P,W} & \longrightarrow & M'_{V/X,W} & \longrightarrow \\ & \| & & \downarrow & & \downarrow & \\ \longrightarrow & (i\circ g)^* N_{X/P} & \longrightarrow & M_{V/P} & \longrightarrow & M_{V/X} & \longrightarrow \end{array}$$

of distinguished triangles, where the lower line is the distinguished triangle (1.7.0.2). *In particular, if the composition* V → P *is a regular immersion, the upper line gives a distinguished triangle*

$$\textbf{(1.7.1.2)} \qquad \longrightarrow (i\circ g)^* N_{X/P} \longrightarrow N'_{V/P,W} \longrightarrow M'_{V/X,W} \longrightarrow .$$

Proof. — We consider the commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
J_{B_W}/J^2_{B_W} \otimes_{A_W} \mathcal{O}_T & \longrightarrow & \Omega^1_{P_P(\mathcal{O}_X)/j^{-1}\mathcal{O}_P} \otimes_{P_P(\mathcal{O}_X)} \mathcal{O}_T \\
\downarrow & & \downarrow \\
\tilde{I}/\tilde{I}^2 & \longrightarrow & \Omega^1_{B_W/\mathcal{O}_W} \otimes_{B_W} \mathcal{O}_T \\
\downarrow & & \downarrow \\
I/I^2 & \longrightarrow & \Omega^1_{A_W/\mathcal{O}_W} \otimes_{A_W} \mathcal{O}_T \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
\tag{1.7.1.3}
$$

Using the isomorphism (1.7.0.3), it is easy to see that the left column of (1.7.1.3) is exact. It follows from the construction of the distinguished triangle (1.4.0.1) recalled in Section 1.4 that the right exact sequence gives the lower distinguished triangle in (1.7.1.1). By Lemma 1.6.6, the horizontal arrows in (1.7.1.3) induce the vertical arrows in (1.7.1.1). Thus, we obtain a map of distinguished triangles (1.7.1.1).

If $V \to P$ is a regular immersion, the canonical map $M'_{V/P,W} \to N_{V/P,W}$ is an isomorphism by Proposition 1.6.4.2. Thus the upper line of (1.7.1.1) implies (1.7.1.2). □

Let $M'_{V/X,W} \to N_{X/P} \otimes_{\mathcal{O}_X} \mathcal{O}_T[1]$ be the map defining the distinguished triangle $\to N_{X/P} \otimes_{\mathcal{O}_X} \mathcal{O}_T \to N'_{V/P,W} \to M'_{V/X,W} \to$. We define a map

$$\lambda_{V/X/P,W} : L^p\Lambda^q M'_{V/X,W} \longrightarrow N_{X/P} \otimes_{\mathcal{O}_X} L^{p-1}\Lambda^{q-1} M'_{V/X,W} \tag{1.7.2.1}$$

to be that induced by the composition

$$L\Lambda^q M'_{V/X,W} \to M'_{V/X,W} \otimes_{\mathcal{O}_X} L\Lambda^{q-1} M'_{V/X,W} \to N_{X/P} \otimes_{\mathcal{O}_X} L\Lambda^{q-1} M'_{V/X,W}[1].$$

For a spectral sequence $E = (E^1_{p,q} \Rightarrow E_{p+q})$ and integers a and b, let $E[a, b]$ denote the spectral sequence $(E^1_{p-a,q-b} \Rightarrow E_{p+q-(a+b)})$.

The following result will be used in the proof of the excess intersection formula, Proposition 3.4.2.

Proposition **1.7.2.** — *Let*

$$
\begin{array}{ccccc}
T & \xrightarrow{i'} & W & & \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \\
V & \xrightarrow{i} & X & \xrightarrow{j} & P
\end{array}
$$

be a diagram of schemes. We assume that the square is cartesian, the horizontal arrows are immersions and that the immersions $X \to P$ *and* $i' : T \to W$ *are regular immersions. Let* $E_{\mathcal{T}}$ *denote the spectral sequence* (1.6.4.3).

Then, there exists a map

$$\textbf{(1.7.2.2)} \qquad \alpha : E_{\mathcal{T}} \longrightarrow N_{X/P} \otimes_{\mathcal{O}_X} E_{\mathcal{T}}[-1, 3]$$

of spectral sequences such that the maps on the abutments are $\alpha_{\mathcal{O}_V, \mathcal{O}_W, X/P} : \mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W) \to N_{X/P} \otimes \mathcal{T}or_{r-2}^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)$ (1.5.4.4) *and the maps on the* E^1*-terms are the maps* $\lambda_{V/X/P,W} : L^p\Lambda^q M'_{V/X,W} \to N_{X/P} \otimes_{\mathcal{O}_X} L^{p-1}\Lambda^{q-1}M'_{V/X,W}$.

Proof. — Proof is divided into the following three steps.

1. Define a map $E_{\mathcal{T}} \to N_{X/P} \otimes_{\mathcal{O}_X} E_{\mathcal{T}}[-1, 3]$ of spectral sequences.
2. Compute the map on abutments.
3. Compute the map on E^1-terms.

1. We keep the notation

$$B = i^{-1}\mathcal{O}_X \otimes_{(j\circ i)^{-1}\mathcal{O}_P} \mathcal{Q},$$
$$A = i^{-1}\mathcal{O}_X \otimes_{i^{-1}\mathcal{P}} \mathcal{Q} = B \otimes_{i^{-1}(\mathcal{P}\otimes_{j^{-1}\mathcal{O}_P}\mathcal{P})} i^{-1}\mathcal{P},$$
$$J_B = \mathrm{Ker}(B \to A) = \mathcal{Q} \otimes_{i^{-1}(\mathcal{P}\otimes_{j^{-1}\mathcal{O}_P}\mathcal{P})} i^{-1}J,$$

$A_W = A \otimes_{\mathcal{O}_X} \mathcal{O}_W$, $B_W = B \otimes_{\mathcal{O}_X} \mathcal{O}_W$, $J_{B_W} = \mathrm{Ker}(B_W \to A_W)$, $I = \mathrm{Ker}(A_W \to \mathcal{O}_T)$, $\tilde{I} = \mathrm{Ker}(B_W \to \mathcal{O}_T)$ above. We define filtrations $F^\bullet$ on A_W, $J_{B_W}/J_{B_W}^2$ and on $B_W/J_{B_W}^2$ by $F^pA_W = I^pA_W$, $F^p(J_{B_W}/J_{B_W}^2) = I^p(J_{B_W}/J_{B_W}^2)$ and by $F^p(B_W/J_{B_W}^2) = \tilde{I}^p(B_W/J_{B_W}^2)$. Let E_A and E_J be the spectral sequences $E_{p,q}^1 = H_{p+q}N\mathrm{Gr}_F^{-p}(A_W) \Rightarrow H_{p+q}N(A_W)$ and $E_{p,q}^1 = H_{p+q}N\mathrm{Gr}_F^{-p}(J_{B_W}/J_{B_W}^2) \Rightarrow H_{p+q}N(J_{B_W}/J_{B_W}^2)$ defined by the filtered complexes $(N(A_W/A_W^2), F^\bullet)$ and $(N(J_{B_W}/J_{B_W}^2), F^\bullet)$ respectively

The construction of $E_{\mathcal{T}} \to N_{X/P} \otimes_{\mathcal{O}_X} E_{\mathcal{T}}[-1, 3]$ is divided into the following three substeps.

i. Define an isomorphism $\beta : E_{\mathcal{T}} \to E_A$ of spectral sequences.
ii. Define a map $\gamma : E_A \to E_J[-1, 2]$ of spectral sequences.
iii. Define an isomorphism $\delta : N_{X/P} \otimes_{\mathcal{O}_X} E_A[0, 1] \to E_J$.

Transporting the composition $\delta^{-1}\circ\gamma$ by the isomorphism $\beta : E_{\mathcal{T}} \to E_A$, we will define a map $\alpha : E_{\mathcal{T}} \to N_{X/P} \otimes_{\mathcal{O}_X} E_{\mathcal{T}}[-1, 3]$.

i. We define an isomorphism $E_{\mathcal{T}} \to E_A$ of spectral sequences. In the commutative diagram

$$\textbf{(1.7.2.3)} \qquad \begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{O}_V \\ \uparrow & & \uparrow \\ i^{-1}\mathcal{P} & \longrightarrow & i^{-1}\mathcal{O}_X, \end{array}$$

the horizontal arrows are quasi-isomorphisms. We show that the induced map $A \to \mathcal{O}_V$ is a resolution by weakly free simplicial $i^{-1}\mathcal{O}_X$-algebra. Since $\mathcal{Q}$ is a free simplicial $i^{-1}\mathcal{P}$-algebra, the tensor product $A = \mathcal{Q} \otimes_{i^{-1}\mathcal{P}} i^{-1}\mathcal{O}_X$ is a free simplicial $i^{-1}\mathcal{O}_X$-algebra. Further, the quasi-isomorphism $\mathcal{P} \to \mathcal{O}_X$ induces a quasi-isomorphism $\mathcal{Q} = \mathcal{Q} \otimes_{i^{-1}\mathcal{P}} i^{-1}\mathcal{P} \to A = \mathcal{Q} \otimes_{i^{-1}\mathcal{P}} i^{-1}\mathcal{O}_X$ by [19] I Lemme 3.3.2.1. Thus the quasi-isomorphism $\mathcal{Q} \to \mathcal{O}_V$ induces a quasi-isomorphism $A \to \mathcal{O}_V$. By applying Lemma 1.6.6, we obtain an isomorphism $\beta : E_{\mathcal{T}} \to E_A$ of spectral sequences.

ii. We define $\gamma : E_A \to E_J[-1, 2]$. Using the isomorphism (1.7.0.3), it is easy to see that the sequence

$$\textbf{(1.7.2.4)} \qquad 0 \to \mathrm{Gr}_F^{p-1}\big(J_{B_W}/J_{B_W}^2\big) \longrightarrow \mathrm{Gr}_F^{p}\big(B_W/J_{B_W}^2\big) \longrightarrow \mathrm{Gr}_F^{p} A_W \to 0$$

is exact for each $p \geq 0$. Namely, the exact sequence $0 \to J_{B_W}/J_{B_W}^2 \to B_W/J_{B_W}^2 \to A_W \to 0$ defines an exact sequence

$$\textbf{(1.7.2.5)} \qquad 0 \to \big(J_{B_W}/J_{B_W}^2, F^{\bullet-1}\big) \longrightarrow \big(B_W/J_{B_W}^2, F^{\bullet}\big) \longrightarrow (A_W, F^{\bullet}) \to 0$$

of filtered simplicial modules. The exact sequence (1.7.2.5) defines a map $(NA_W, F^{\bullet}) \to (N(J_{B_W}/J_{B_W}^2), F^{\bullet-1})[1]$ of filtered complexes in the derived category and hence a map $E_A \to E_J[1, -2]$ of spectral sequences.

iii. We define an isomorphism $\delta : N_{X/P} \otimes_{\mathcal{O}_X} E_A[0, 1] \to E_J$. The natural map $A_W \otimes_{i^{-1}\mathcal{P}} i^{-1}(J/J^2) \to J_{B_W}/J_{B_W}^2$ is an isomorphism. Since J/J^2 is flat over $\mathcal{P}$, it defines an isomorphism

$$\textbf{(1.7.2.6)} \qquad i^{-1}(J/J^2) \otimes_{i^{-1}\mathcal{P}} (A_W, F^{\bullet}) \longrightarrow \big(J_{B_W}/J_{B_W}^2, F^{\bullet}\big)$$

of filtered modules. By the assumption that $X \to P$ is a regular immersion, we have a canonical isomorphism $L_{X/P} \to N_{X/P}[1]$. Since $L_{X/P} = N(J/J^2 \otimes_{\mathcal{P}} \mathcal{O}_X)$, we have an isomorphism

$$\textbf{(1.7.2.7)} \qquad N\big(i^{-1}(J/J^2) \otimes_{i^{-1}\mathcal{P}} (A_W, F^{\bullet})\big) \longrightarrow N_{X/P} \otimes_{\mathcal{O}_X} N(A_W, F^{\bullet})[1]$$

of filtered complexes in the derived category. The isomorphisms (1.7.2.6) and (1.7.2.7) induce an isomorphism $\delta : N_{X/P} \otimes_{\mathcal{O}_X} E_A[0, 1] \to E_J$.

2. We show that the maps on the abutments are induced by the map $\alpha_{\mathcal{O}_V, \mathcal{O}_W, X/P} : \mathcal{O}_V \otimes^L_{\mathcal{O}_X} \mathcal{O}_W \to N_{X/P} \otimes \mathcal{O}_V \otimes^L_{\mathcal{O}_X} \mathcal{O}_W[2]$ (1.5.4.1). Applying the functors $i^{-1}(\ ) \otimes_{i^{-1}\mathcal{P}} \mathcal{O}_V$, $i^{-1}(\ ) \otimes_{i^{-1}\mathcal{P}} \mathcal{Q}$ and $\mathcal{O}_X \otimes_{i^{-1}\mathcal{P}} i^{-1}(\ ) \otimes_{i^{-1}\mathcal{P}} \mathcal{Q}$ to the exact sequence $0 \to J/J^2 \to (\mathcal{P} \otimes_{j^{-1}\mathcal{O}_P} \mathcal{P})/J^2 \to \mathcal{P} \to 0$, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & i^{-1}(J/J^2) \otimes_{i^{-1}\mathcal{P}} \mathcal{O}_V & \to & i^{-1}(\mathcal{P} \otimes_{j^{-1}\mathcal{O}_P} \mathcal{P})/J^2 \otimes_{i^{-1}\mathcal{P}} \mathcal{O}_V & \to & \mathcal{O}_V & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \to & i^{-1}(J/J^2) \otimes_{i^{-1}\mathcal{P}} \mathcal{Q} & \to & i^{-1}(\mathcal{P} \otimes_{j^{-1}\mathcal{O}_P} \mathcal{P})/J^2 \otimes_{i^{-1}\mathcal{P}} \mathcal{Q} & \to & \mathcal{Q} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & J_B/J_B^2 & \to & B/J_B^2 & \to & A & \to 0.
\end{array}$$

of exact sequences. We regard the upper two lines as exact sequences of $i^{-1}\mathscr{P}$-modules with respect to the map $\mathscr{P} \to \mathscr{P} \otimes_{j^{-1}\mathscr{O}_P} \mathscr{P}$ sending a to $a \otimes 1$. The lower vertical arrows are compatible with the surjection $i^{-1}\mathscr{P} \to \mathscr{O}_X$.

Since $L_{X/P} = N(\Omega^1_{\mathscr{P}/j^{-1}\mathscr{O}_P} \otimes_{\mathscr{P}} \mathscr{O}_X) = N(J/J^2 \otimes_{\mathscr{P}} \mathscr{O}_X)$, we may identify $L_{X/P} \otimes_{\mathscr{O}_X} \mathscr{O}_V = N(i^{-1}(J/J^2) \otimes_{i^{-1}\mathscr{P}} \mathscr{O}_V)$ for the upper left term. Then, by the second description of the Atiyah class map recalled in Section 1.4, the Atiyah class map $\mathscr{O}_V \to L_{X/P} \otimes_{\mathscr{O}_X} \mathscr{O}_V[1]$ is defined by the boundary map of the top sequence. Since the vertical arrows are quasi-isomorphisms, we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_V & \xrightarrow{\mathrm{at}_{X/P,\mathscr{O}_V}} & L_{X/P} \otimes_{\mathscr{O}_X} \mathscr{O}_V[1] \\ \uparrow & & \uparrow \\ NA & \longrightarrow & N\big(J_B/J_B^2\big)[1] \end{array}$$

in the derived category of $\mathscr{O}_X$-modules. Thus, applying $\otimes_{\mathscr{O}_X} \mathscr{O}_W$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_V \otimes^L_{\mathscr{O}_X} \mathscr{O}_W & \xrightarrow{\alpha_{\mathscr{O}_V,\mathscr{O}_W,X/P}} & N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{O}_V \otimes^L_{\mathscr{O}_X} \mathscr{O}_W[2] \\ \uparrow & & \uparrow \\ NA_W & \longrightarrow & N\big(J_{B_W}/J_{B_W}^2\big)[1] \end{array}$$

in the derived category of $\mathscr{O}_W$-modules. Thus the assertion follows from the definition of the identifications $\beta : E_{\mathscr{T}} \to E_A$ and $\delta : N_{X/P} \otimes_{\mathscr{O}_X} E_A[0, 1] \to E_J$ in i and iii above.

3. We show that the maps on the E^1-terms are given by $\lambda_{V/X/P,W} : L^p \Lambda^q M'_{V/X,W} \to N_{X/P} \otimes_{\mathscr{O}_X} L^{p-1} \Lambda^{q-1} M'_{V/X,W}$. By the assumption that $T \to W$ is a regular immersion, the kernel of the surjection $B_W \to \mathscr{O}_T$ is weakly regular. By the isomorphism (1.7.0.3), it is easy to see that the isomorphism $S^p(\tilde{I}/\tilde{I}^2) \to \tilde{I}^p/\tilde{I}^{p+1}$ induces an isomorphism

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S^{p-1}(I/I^2) \otimes J_{B_W}/J_{B_W}^2 & \longrightarrow & \mathrm{Gr}_F^{p-1}\big(J_{B_W}/J_{B_W}^2\big) \\ \downarrow & & \downarrow \\ S^p(\tilde{I}/\tilde{I}^2)/\big(S^2\big(J_{B_W}/J_{B_W}^2\big) \cdot S^{p-2}(\tilde{I}/\tilde{I}^2)\big) & \longrightarrow & \mathrm{Gr}_F^p\big(B_W/J_{B_W}^2\big) \\ \downarrow & & \downarrow \\ S^p(I/I^2) & \longrightarrow & \mathrm{Gr}_F^p A_W \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

of exact sequences. The right column is the exact sequence (1.7.2.4). Hence by Corollary 1.2.3.1, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{NS}^p(\mathrm{I}/\mathrm{I}^2) & \longrightarrow & \mathrm{NGr}^p_{\mathrm{F}}\mathrm{A}_{\mathrm{W}} \\
\downarrow & & \downarrow \\
\mathrm{N}(\mathrm{I}/\mathrm{I}^2)\otimes \mathrm{NS}^{p-1}(\mathrm{I}/\mathrm{I}^2) & & \mathrm{NGr}^{p-1}_{\mathrm{F}}\big(\mathrm{J}_{\mathrm{B_W}}/\mathrm{J}^2_{\mathrm{B_W}}\big)[1] \\
\downarrow & & \uparrow \\
\mathrm{N}\big(\mathrm{J}_{\mathrm{B_W}}/\mathrm{J}^2_{\mathrm{B_W}}\otimes_{\mathrm{A_W}}\mathscr{O}_{\mathrm{T}}\big)[1]\otimes \mathrm{NS}^{p-1}(\mathrm{I}/\mathrm{I}^2) & \longrightarrow & \mathrm{N}\big(\mathrm{J}/\mathrm{J}^2\otimes_{\mathscr{O}_{\mathrm{X}}}\mathrm{Gr}^{p-1}_{\mathrm{F}}\mathrm{A}_{\mathrm{W}}\big)[1].
\end{array}$$

The upper left vertical arrow is induced by the map (1.2.1.1) and the lower left and the upper right vertical arrows are defined by the exact sequence (1.7.2.4). The rest are the natural maps. Recall that the distinguished triangle $\to (i\circ g)^*\mathrm{N}_{\mathrm{X/P}} \to \mathrm{N}'_{\mathrm{V/P,W}} \to \mathrm{M}'_{\mathrm{V/X,W}} \to$ (1.7.1.2) is defined by the exact sequence $0\to \mathrm{J}_{\mathrm{B_W}}/\mathrm{J}^2_{\mathrm{B_W}}\otimes_{\mathrm{A_W}}\mathscr{O}_{\mathrm{T}} \to \tilde{\mathrm{I}}/\tilde{\mathrm{I}}^2 \to \mathrm{I}/\mathrm{I}^2 \to 0$ in the proof of Lemma 1.7.1. Thus, by Lemmas 1.2.9 and 1.6.6, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{L}\Lambda^p\mathrm{M}'_{\mathrm{V/X,W}}[p] & \longrightarrow & \mathrm{NGr}^p_{\mathrm{F}}\mathrm{A}_{\mathrm{W}} \\
\downarrow & & \downarrow \\
\mathrm{N}_{\mathrm{X/P}}\otimes_{\mathscr{O}_{\mathrm{X}}}\mathrm{L}\Lambda^{p-1}\mathrm{M}'_{\mathrm{V/X,W}}[p] & \longrightarrow & \mathrm{NGr}^{p-1}_{\mathrm{F}}\big(\mathrm{J}_{\mathrm{B_W}}/\mathrm{J}^2_{\mathrm{B_W}}\big)[1]
\end{array}$$

and the assertion follows. □

2. K-theory and localized Chern classes

We briefly recall generalities on K-groups, Chow groups and Chern classes in 2.1. We interpret intersection theory à la Fulton-MacPherson in terms of K-theory in 2.2. We briefly recall generalities on localized Chern classes in 2.3. We compare the localized Chern class and the class of the derived exterior power complex in 2.4 for a complex satisfying a certain condition.

2.1. K-*theory and Chow groups.* — We recall generalities on K-theoretic intersection theory. Basic references are [17] and [14].

For a scheme X, let K(X) be the Grothendieck group of the category of locally free $\mathscr{O}_{\mathrm{X}}$-modules of finite rank. It is the quotient of the free abelian group generated by the isomorphism classes $[\mathscr{E}]$ of locally free $\mathscr{O}_{\mathrm{X}}$-modules of finite rank divided by the relations $[\mathscr{E}] = [\mathscr{E}'] + [\mathscr{E}'']$ for exact sequences $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$. For a noetherian scheme X, let G(X) be the Grothendieck group of the category of coherent $\mathscr{O}_{\mathrm{X}}$-modules. It is the quotient of the free abelian group generated by the isomorphism classes $[\mathscr{F}]$ of coherent $\mathscr{O}_X$-modules divided by the relations $[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$ for exact sequences $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$. For $\mathscr{G} \in D^b(X)_{\mathrm{coh}}$, its class $[\mathscr{G}] \in G(X)$ is defined as the alternating sum $\sum_q (-1)^q [\mathscr{H}_q(\mathscr{G})]$. For a distinguished triangle $\to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to$ in $D^b(X)_{\mathrm{coh}}$, we have $[\mathscr{G}] = [\mathscr{G}'] + [\mathscr{G}'']$.

We have a canonical map $K(X) \to G(X)$ sending the class $[\mathscr{E}]$ of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ to $[\mathscr{E}]$. If X is regular, noetherian and separated, then the canonical map $K(X) \to G(X)$ is an isomorphism by the following lemma.

Lemma **2.1.1** ([17] Exp. II Corollary 2.2.7.1). — *Let* X *be a separated regular noetherian scheme of dimension n and* $\mathscr{F}$ *be a coherent* $\mathscr{O}_X$*-module. Then there exists a resolution* $0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_0 \to \mathscr{F} \to 0$ *of* $\mathscr{F}$ *by locally free* $\mathscr{O}_X$*-modules of finite rank.*

In this case, we identify $G(X) = K(X)$. For a coherent $\mathscr{O}_X$-module $\mathscr{F}$, the inverse image of $[\mathscr{F}]$ in $K(X)$ is $\sum_{q=0}^n (-1)^q [\mathscr{E}_q]$ for a resolution $(\mathscr{E}_\bullet)$ as in Lemma 2.1.1.

The multiplication on $K(X)$ is defined by the tensor product $[\mathscr{E}] \cdot [\mathscr{E}'] = [\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}']$. If X is noetherian, $G(X)$ is a $K(X)$-module by the multiplication $[\mathscr{E}] \cdot [\mathscr{F}] = [\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}]$. More generally, if $f : W \to X$ is a map of schemes and W is noetherian, a bilinear map $(\ ,\ )_X : K(X) \times G(W) \to G(W)$ is defined by $([\mathscr{F}], [\mathscr{G}])_X = [\mathscr{F} \otimes^{L}_{\mathscr{O}_X} \mathscr{G}]$. If X is separated, regular and noetherian of dimension n, the multiplication on $G(X) = K(X)$ is given by $[\mathscr{F}] \cdot [\mathscr{F}'] = \sum_{q=0}^n (-1)^q [\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}')]$.

The γ-filtration $F^n K(X)$ on $K(X)$ is defined as follows. There is a canonical map $\lambda_t : K(X) \to 1 + tK(X)[[t]] \subset K(X)[[t]]^\times$ sending the class $[\mathscr{E}]$ of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ to $\sum_q [\Lambda^q \mathscr{E}] t^q$. For $x \in K(X)$, we put $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \sum_{n>0} \gamma_n(x) t^n$. For a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank n, we have

$$\textbf{(2.1.1.1)} \qquad \gamma_t([\mathscr{E}] - n) = \sum_{q=0}^{n} [\Lambda^q \mathscr{E}] t^q (1-t)^{n-q} = \sum_{r=0}^{n} \Big(\sum_{q=0}^{r} (-1)^{r-q} \binom{n-q}{r-q} [\Lambda^q \mathscr{E}] \Big) t^r.$$

For $r = n$, we have

$$\textbf{(2.1.1.2)} \qquad \gamma_n([\mathscr{E}] - n) = \sum_{q=0}^{n} (-1)^{n-q} [\Lambda^q \mathscr{E}].$$

If $\mathscr{L}$ is invertible, we have $\gamma_t([\mathscr{L}] - 1) = 1 + ([\mathscr{L}] - 1)t$. For $n = 1$, $F^1 K(X)$ is defined to be the kernel of the map $K(X) \to \mathbf{Z}^{\pi_0(X)}$ sending $\mathscr{E}$ to rank $\mathscr{E}$. For $n \geq 1$, $F^n K(X)$ is defined as the subgroup generated by the elements of the form $\gamma_{n_1}(x_1) \cdots \gamma_{n_r}(x_r)$ where $x_i \in F^1 K(X)$ and $\sum_i n_i \geq n$. We put $F^0 K(X) = K(X)$. We have $F^n K(X) \cdot F^m K(X) \subset F^{m+n} K(X)$.

In the rest of this section, S denotes an equidimensional regular noetherian scheme of finite dimension. For a scheme X of finite type over S, the topological filtration $F_n G(X)$ on $G(X)$ is defined as follows. It is called the lower filtration in [14]

Chapter VI §5. We recall that the dimension $\dim S$ is defined as the supremum of the dimensions of the local rings $\dim \mathcal{O}_{S,s}$. For a point s of S, we put $\dim_S s = \dim S - \dim \mathcal{O}_{S,s}$. Let X be a scheme of finite type over S and $f : X \to S$ denote the structural map. We put $\dim_S x = \mathrm{tr.deg}_{\kappa(f(x))}\kappa(x) + \dim_S f(x)$ for $x \in X$ as in [16] Exp. XIV 2. If S is the spectrum of a regular local noetherian ring and X is proper over S, we have an equality $\dim_S x = \dim \overline{\{x\}}$ for $x \in X$ by loc.cit. Proposition 2.3. For a closed subset $V \subset X$, we put $\dim_S V = \sup_{x\in V} \dim_S x$. Note that the function $\dim_S$ depends on the base scheme S. For an integer $n \geq 0$, let $F_nG(X)$ be the subgroup of $G(X)$ generated by the classes $[\mathscr{F}]$ of coherent $\mathcal{O}_X$-modules $\mathscr{F}$ such that the dimension of the support of $\mathscr{F}$ is at most n.

The γ-filtration and the topological filtration are related as follows.

Lemma **2.1.2.** — *Let* S *be an equidimensional regular noetherian scheme of finite dimension and* X *be a scheme of finite type over* S.

1. ([14] Chapter V Theorem 3.9, Chapter VI Proposition 5.2) *We have* $F^nK(X) \cdot F_mG(X) \subset F_{m-n}G(X)$. *In particular, if* X *is of dimension* d, *the canonical map* $K(X) \to G(X)$ *sends* $F^nK(X)$ *into* $F_{d-n}G(X)$.

2. ([14] Chapter VI Proposition 5.5) *If* X *is regular and equidimensional of dimension* d *and if there exists an ample invertible* $\mathcal{O}_X$*-module, the induced map* $\mathrm{Gr}^n_F K(X)_{\mathbf{Q}} \to \mathrm{Gr}^F_{d-n} G(X)_{\mathbf{Q}}$ *is an isomorphism.*

Let $f : X \to Y$ be a morphism of schemes. The pull-back of locally free sheaves defines a ring homomorphism $f^* : K(Y) \to K(X)$. We have $f^*F^nK(Y) \subset F^nK(X)$. Assume X and Y are noetherian. If f is proper, there is a map $f_* : G(X) \to G(Y)$ sending the class of a coherent $\mathcal{O}_X$-module $\mathscr{F}$ to the class of the complex $Rf_*\mathscr{F}$. If f is flat, there is a map $f^* : G(Y) \to G(X)$ sending the class of a coherent $\mathcal{O}_Y$-module $\mathscr{F}$ to the class of $f^*\mathscr{F}$.

Lemma **2.1.3.** — *Let* $f : X \to Y$ *be a morphism of schemes of finite type over a regular noetherian scheme* S *of finite dimension.*

1. ([14] Chapter VI Proposition 5.6) *If* f *is proper, we have* $f_*F_nG(X) \subset F_nG(Y)$.

2. ([14] Chapter VI Proposition 6.3) *If* f *is flat of relative dimension* m, *we have* $f^*F_nG(Y) \subset F_{n+m}G(X)$.

We recall the definition of Chow groups and bivariant Chow groups. Let S be an equidimensional regular noetherian scheme of finite dimension, X be a scheme of finite type over S and $i \geq 0$ be an integer. Let X_i denote the set $\{x \in X | \dim_S x = i\}$. The Chow group $CH_i(X)$ is defined as the cokernel $\mathrm{Coker}(\bigoplus_{y\in X_{i+1}} \kappa(y)^\times \xrightarrow{d} \bigoplus_{x\in X_i} \mathbf{Z})$. The (x, y)-component $d_{x,y} : \kappa(y)^\times \to \mathbf{Z}$ of d is characterized as follows. Let Y be the closure of $\{y\}$ with the reduced subscheme structure. If $x \in Y$, the map $d_{x,y}$ satisfies $d_{x,y}f = \mathrm{length}\,\mathcal{O}_{Y,x}/(f)$ for $f \in \mathcal{O}_{Y,x}, \neq 0$ and, if $x \notin Y$, it is the 0-map.

Let S be an equidimensional regular noetherian scheme of finite dimension. Let X be a scheme of finite type over S and Z be a closed subscheme of X. An element of the bivariant Chow cohomology group $\mathrm{CH}^i(\mathrm{Z} \to \mathrm{X})$ is a collection of maps $\mathrm{CH}_j(\mathrm{W}) \to \mathrm{CH}_{j-i}(\mathrm{Z} \times_{\mathrm{X}} \mathrm{W})$ defined for schemes W of finite type over X and for integers $j \geq i$, satisfying certain natural functorial properties ([13] Chapters 17 and 20). If $\mathrm{Z} = \mathrm{X}$, let $\mathrm{CH}^*(\mathrm{X})$ denote the bivariant Chow ring $\mathrm{CH}^*(\mathrm{X} \to \mathrm{X})$. If X is equidimensional of dimension d, a canonical map $\cap[\mathrm{X}] : \mathrm{CH}^q(\mathrm{X}) \to \mathrm{CH}_{d-q}(\mathrm{X})$ is defined. It is an isomorphism if X is smooth and $\mathrm{S} = \mathrm{Spec}\, k$ for a field k [13] Corollary 17.4.

The filtrations on K-groups and Chow groups are related as follows. The map $ch : \mathrm{K}(\mathrm{X}) \to \mathrm{CH}^*(\mathrm{X})_{\mathbf{Q}}$ sending the class $[\mathscr{E}]$ of a locally free $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{E}$ to its Chern character $(ch_i(\mathscr{E}))_i \in \mathrm{CH}^*(\mathrm{X})_{\mathbf{Q}}$ is a ring homomorphism.

Lemma **2.1.4.** — *Let* S *be an equidimensional regular noetherian scheme of finite dimension and* X *be a scheme of finite type over* S.

1. The Chern character map $ch : \mathrm{K}(\mathrm{X}) \to \mathrm{CH}^*(\mathrm{X})_{\mathbf{Q}}$ *is compatible with the* γ*-filtration and induces a homomorphism* $ch : \mathrm{Gr}^*_{\mathrm{F}}\mathrm{K}(\mathrm{X}) \to \mathrm{CH}^*(\mathrm{X})_{\mathbf{Q}}$ *of graded rings.*

2. (cf. [13] Example 15.1.5) *The map* $\mathrm{CH}_*(\mathrm{X}) \to \mathrm{Gr}^{\mathrm{F}}_*\mathrm{G}(\mathrm{X})$ *sending the class* [V] *of an integral subscheme* V *to* $[\mathscr{O}_{\mathrm{V}}]$ *is well-defined and is a surjection.*

3. Assume X *is equidimensional of dimension n. Let* $\mathscr{E}$ *be a locally free* $\mathscr{O}_{\mathrm{X}}$*-module of rank r. Then for an integer* $i \geq 0$, *the class in* $\mathrm{Gr}^{\mathrm{F}}_{n-i}\mathrm{G}(\mathrm{X})$ *of the image of* $\gamma_i([\mathscr{E}] - r) \in \mathrm{F}^i\mathrm{K}(\mathrm{X})$ *is equal to the image of* $c_i(\mathscr{E}) \in \mathrm{CH}^i(\mathrm{X})$. *In particular, for* $i = r$, *the image of* $\gamma_r([\mathscr{E}] - r) = (-1)^r \sum_q (-1)^q [\Lambda^q \mathscr{E}] \in \mathrm{F}^r\mathrm{K}(\mathrm{X})$ *is equal to the image of* $c_r(\mathscr{E}) \in \mathrm{CH}^r(\mathrm{X})$.

4. Assume X *is equidimensional of dimension n. Then the composition*

$$\mathrm{Gr}^*_{\mathrm{F}}\mathrm{K}(\mathrm{X})_{\mathbf{Q}} \xrightarrow{ch} \mathrm{CH}^*(\mathrm{X})_{\mathbf{Q}} \xrightarrow{\cap[\mathrm{X}]} \mathrm{CH}_{n-*}(\mathrm{X})_{\mathbf{Q}} \longrightarrow \mathrm{Gr}^{\mathrm{F}}_{n-*}\mathrm{G}(\mathrm{X})_{\mathbf{Q}}$$

is equal to the map induced by the canonical map $\mathrm{K}(\mathrm{X}) \to \mathrm{G}(\mathrm{X})$.

5. Assume X *is quasi-projective and smooth of dimension n over a field. Then the three maps in 4 are isomorphisms.*

By Lemma 2.1.4, the intersection product on $\mathrm{CH}_*(\mathrm{X})_{\mathbf{Q}}$ for a smooth quasi-projective scheme X over a field may be computed by the product on $\mathrm{K}(\mathrm{X})_{\mathbf{Q}}$.

Proof. — 1. It follows from the splitting principle and the equality $\gamma_t([\mathscr{L}] - 1) = 1 + ([\mathscr{L}] - 1)t$ for an invertible sheaf $\mathscr{L}$.

2. Let W be a closed subscheme of $\mathbf{P}^1_{\mathrm{X}}$ and let $\pi : \mathbf{P}^1_{\mathrm{X}} \to \mathrm{X}$ be the projection. Then we have $[\mathscr{O}_{\mathrm{W}_0}] - [\mathscr{O}_{\mathrm{W}_\infty}] = \pi_*(([\mathscr{O}(1) - \mathscr{O}] - [\mathscr{O}(1) - \mathscr{O}]) \cdot [\mathscr{O}_{\mathrm{W}}]) = 0$ in G(X).

3. It follows from the splitting principle and the equality (2.1.1.1).

4. It follows from the splitting principle and the equality $ch_1([\mathscr{O}(\mathrm{D})] - 1) \cap [\mathrm{X}] = [\mathrm{D}]$ for a Cartier divisor D.

5. The second arrow is an isomorphism by [13] Corollary 17.4. The composition is an isomorphism by 4 and by [14] Chapter VI Proposition 5.5. By Riemann-Roch for the immersion $V \to X$, we have $ch_i[\mathcal{O}_V] = [V]$ for a closed subscheme V of codimension i. Hence the composition map $CH_{n-i}(X)_{\mathbf{Q}} \to Gr^F_{n-i}G(X)_{\mathbf{Q}} \to Gr^i_F K(X)_{\mathbf{Q}} \to CH_{n-i}(X)_{\mathbf{Q}}$ is the identity. Thus the assertion follows. □

2.2. K-*theory and intersection theory.* — The intersection theory à la Fulton-MacPherson is translated in terms of K-theory as follows. We introduce some notation. Let $i : V \to X$ be a regular closed immersion of codimension c. Then the $\mathcal{O}_X$-module $\mathcal{O}_V$ is of finite tor-dimension. Let W be a noetherian scheme and

$$\begin{array}{ccc} T & \longrightarrow & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \underset{i}{\longrightarrow} & X \end{array}$$

be a cartesian diagram of schemes. For a coherent $\mathcal{O}_W$-module $\mathcal{G}$, the $\mathcal{T}or$-sheaves $\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{G})$ are coherent $\mathcal{O}_T$-modules and are 0 except for $0 \le q \le c$ since $\mathcal{O}_V$ is of tor-dimension c. We define a map $(V, \ )_X : G(W) \to G(T)$ by

$$(V, [\mathcal{G}]) = \sum_{q=0}^{c} (-1)^q \big[\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{G})\big]$$

for a coherent $\mathcal{O}_W$-module $\mathcal{G}$.

Lemma **2.2.1.** — *Let* $i : V \to X$ *be a regular closed immersion of codimension* c *and* $f : W \to X$ *be a map of schemes. We put* $T = V \times_X W$ *and assume the closed immersion* $i' : T \to W$ *is a regular immersion of codimension* c'. *Assume* W *is noetherian. Then, for the intersection product* $(V, W)_X \in G(T)$ *defined as* $\sum_q (-1)^q [\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)]$, *we have an equality*

$$\textbf{(2.2.1.1)} \qquad (V, W)_X = \sum_{q=0}^{c-c'} (-1)^q \big[\Lambda^q N'_{V/X,W}\big] = (-1)^{c-c'} \gamma_{c-c'}\big(\big[N'_{V/X,W}\big] - (c - c')\big).$$

If $W = T$ *and* $g : W \to V$ *is the induced map, we have*

$$\textbf{(2.2.1.2)} \qquad (V, W)_X = \sum_{q=0}^{c} (-1)^q \big[\Lambda^q g^* N_{V/X}\big] = (-1)^c \gamma_c\big(\big[g^* N_{V/X}\big] - c\big).$$

Proof. — It follows from Corollary 1.6.5 and the equality (2.1.1.2). □

We study the relation of the K-theoretic intersection product with the intersection product using Chow groups. We recall the definition of the Segre class. Let S be an equidimensional regular noetherian scheme of finite dimension. Let W be an integral scheme of finite type over S and $T \subset W$ be a closed subscheme. If $T = W$, we put $s(T, W) = [W] \in CH_*(W) = \bigoplus_i CH_i(W)$. Assume $T \neq W$. Let $\pi : W' \to W$ be the blow-up at T and $T' = W' \times_W T$ be the inverse image of T. The subscheme T′ is a Cartier divisor of W′. Then, the total Segre class is defined by

$$s(T, W) = \sum_{i>0} s_i(T, W) = \sum_{i>0} (-1)^{i-1} \pi_*(T'^{i-1} \cap [T'])$$

$\in CH_*(T) = \bigoplus_i CH_i(T)$ (cf. [13] Corollary 4.2.2).

Let S be a regular scheme of finite equidimension as above. Let X be a scheme of finite type over S and $V \to X$ be a regular immersion of codimension c. The intersection product $(V, \ )_X$ is defined as an element of the bivariant Chow cohomology group $CH^c(V \to X)$ as follows. Let W be an integral scheme of finite type over S and $W \to X$ be a morphism over S. We put $T = V \times_X W$ and let $g : T \to V$ be the projection. Then the intersection product $(V, W)_X \in CH_{\dim W - c}(T)$ is defined by

(2.2.2.1) $$(V, W)_X = \{c(g^* N_{V/X})^* \cap s(T, W)\}_{\dim W - c}.$$

Here $c(g^* N_{V/X})^*$ denotes $\sum_i (-1)^i c_i(g^* N_{V/X})$ and the subscript $\dim W - c$ means taking the dimension $\dim W - c$-part. If the closed immersion $T \to W$ is a regular immersion of codimension c' and $N'_{V/X,W}$ denotes the excess conormal sheaf, we have

(2.2.2.2) $$(V, W)_X = (-1)^{c-c'} c_{c-c'}(N'_{V/X,W}) \cap [T].$$

The equality (2.2.2.2) is called the *excess intersection formula*. Thus we obtain a collection of maps $(V, \ )_X : CH_i(W) \to CH_{i-c}(T)$ sending the class of a closed integral subscheme W′ to $(V, W')_X$ for a morphism $W \to X$ of schemes of finite type over S. They define an element $[V] \in CH^c(V \to X)$ of the bivariant Chow group. The bivariant class $[V] \in CH^c(V \to X)$ is characterized by the excess intersection formula (2.2.2.2) and the projection formula $(V, \pi_* W)_X = \pi_*(V, W)_X$.

Proposition **2.2.2.** — *Let* S *be an equidimensional regular noetherian scheme of finite dimension and* $f : W \to X$ *be a morphism of schemes of finite type over* S. *Let* $i : V \to X$ *be a regular closed immersion of codimension* c *and we put* $T = V \times_X W$.

Then the map $(V, \ )_X : G(W) \to G(T)$ *sends the topological filtration* $F_p G(W)$ *to* $F_{p-c} G(T)$. *For the induced map, the diagram*

(2.2.2.3) $$\begin{array}{ccc} CH_p(W) & \xrightarrow{(V,\ )_X} & CH_{p-c}(T) \\ \downarrow & & \downarrow \\ Gr_p G(W) & \xrightarrow[(V,\ )_X]{} & Gr_{p-c} G(T) \end{array}$$

is commutative. In particular, if W *is equidimensional of dimension* p *and if the immersion* $T \to W$ *is a regular immersion of codimension* c', *we have an equality*

(2.2.2.4) $$([V], [W]) = (-1)^{c-c'} c_{c-c'}(N'_{V,W/X}) \cap [T]$$

in $\mathrm{Gr}_{p-c}G(T)$.

The equality (2.2.2.4) is also called the excess intersection formula. We will later show a localized version, Theorem 3.4.3.

Proof. — The topological filtration $F_pG(W)$ is generated by the classes $[\mathscr{O}_Y]$ for integral closed subschemes $Y \subset W$ of dimension $\leq p$. We put $Z = V \times_X Y$. If $Y = Z$, we put $Y' = Y$ and $Z' = Z$. If otherwise, let $\pi : Y' \to Y$ be the blow-up of Y at Z and put $Z' = Z \times_Y Y'$. In the latter case $Z \subsetneq Y$, the exceptional divisor Z' is a Cartier divisor of Y'. Let $\pi_Z : Z' \to Z$ denote the induced map.

We show an equality $(V, [R\pi_*\mathscr{O}_{Y'}])_X = \pi_{Z*}(V, [\mathscr{O}_{Y'}])_X$ in G(Z). Since $\mathscr{O}_V$ is of finite tor-dimension and π is quasi-compact, we have a projection formula $\mathscr{O}_V \otimes^L_{\mathscr{O}_X} R\pi_*\mathscr{O}_{Y'} = R\pi_*(\mathscr{O}_V \otimes^L_{\mathscr{O}_X} \mathscr{O}_{Y'})$ (1.5.3.1) in $D^b(X)_{\mathrm{coh}}$. Thus, by the spectral sequences (1.5.3.3) and by the isomorphism (1.5.3.2), we have

$$\begin{aligned}(V, [R\pi_*\mathscr{O}_{Y'}])_X &= \sum_{p+q}(-1)^{p+q}\left[\mathscr{T}\!or_q^{\mathscr{O}_X}\left(\mathscr{O}_V, R^p\pi_*\mathscr{O}_{Y'}\right)\right]\\ &= \sum_q(-1)^q\left[\mathscr{T}\!or_q^{\mathscr{O}_X}(\mathscr{O}_V, R\pi_*\mathscr{O}_{Y'})\right] = \sum_q(-1)^q\left[R^q\pi_*\left(\mathscr{O}_V \otimes^L_{\mathscr{O}_X} \mathscr{O}_{Y'}\right)\right]\\ &= \sum_{p,q}(-1)^{p+q}\left[R^q\pi_*\mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{O}_{Y'})\right] = \pi_{Z*}(V, [\mathscr{O}_{Y'}]).\end{aligned}$$

The topological filtration $F_pG(W)$ is generated by the classes $\pi_*[\mathscr{O}_{Y'}] = [R\pi_*\mathscr{O}_{Y'}]$ for integral closed subschemes $Y \subset W$ of dimension $\leq p$. Hence it is reduced to showing that $(V, [\mathscr{O}_{Y'}])_X$ is in $F_{p-n}G(Z')$ and that its class in $\mathrm{Gr}^F_{p-n}G(Z')$ is equal to the image of $(V, Y') \in CH_{p-c}(Z')$ assuming $\dim_S Y = p$. Replacing W by Y and further by Y', we may assume $W = Y = Y'$ and $T = Z = Z'$. Thus we may assume $Y = W$ is of dimension p and either T is equal to W or T is a Cartier divisor of W. Let $g : T \to V$ be the canonical map.

If $W = T$, we have $(V, [\mathscr{O}_W])_X = (-1)^c\gamma_c([g^*N_{V/X}] - c)$ by Lemma 2.2.1. Hence $(V, [\mathscr{O}_W])_X$ is in $F_{p-c}G(T)$ and its class is equal to the image of $(V, W)_X = (-1)^c c_c(g^*N_{V/X})$ by Lemma 2.1.4.3. If T is a Cartier divisor of W, we have $(V, [\mathscr{O}_W])_X = (-1)^{c-1}\gamma_{c-1}([N'_{V/X,W}] - (c-1))$ by Lemma 2.2.1. Hence $(V, [\mathscr{O}_W])_X$ is in $F_{(p-1)-(c-1)}G(T)$ and its class is equal to the image of $(-1)^{c-1}c_{c-1}(N'_{V/X,W})$ by Lemma 2.1.4.3.

The excess intersection formula (2.2.2.4) follows from (2.2.2.2) and the commutative diagram (2.2.2.3). □

Let $f : X \to Y$ be a morphism locally of complete intersection of noetherian schemes. For a subscheme Z of Y, the pull-back map $f^* : G(Z) \to G(Z \times_Y X)$ is defined by sending the class of a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ to $\sum_q (-1)^q [\mathcal{T}or_q^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{G})]$ since the map $f : X \to Y$ is of finite tor-dimension.

Corollary **2.2.3.** — *Let* S *be an affine, equidimensional regular noetherian scheme of finite dimension and* X *and* Y *be regular schemes of finite type over* S. *Let* $f : X \to Y$ *be a morphism over* S. *Let* $Z \subset Y$ *be a subscheme and put* $Z' = Z \times_Y X$. *Assume* X *is quasi-projective over* S.

1. Assume that X *is equidimensional of dimension* n *and* Y *is equidimensional of dimension* m. *Then the map* $f^* : G(Z) \to G(Z')$ *sends* $F_pG(Z)$ *into* $F_{p+n-m}G(Z')$.

2. Assume further that $f : X \to Y$ *is proper, surjective, generically finite of constant rank* $[X : Y]$. *Then, we have* $n = m$ *and the composition* $f_*f^* : \mathrm{Gr}_p^F G(Z) \to \mathrm{Gr}_p^F G(Z)$ *is the multiplication by* $[X : Y]$.

Proof. — 1. Take an immersion $X \to \mathbf{P}_S^N$. The map $X \to Y$ is factorized as $X \to \mathbf{P}_S^N \times_S Y \to Y$. Since X and Y are regular, the immersion $X \to \mathbf{P}_S^N \times_S Y$ is regular of codimension $m+N-n$. Hence it follows from Lemma 2.1.3.2 and Proposition 2.2.2.

2. The direct image $Rf_*\mathcal{O}_X$ is a perfect complex of $\mathcal{O}_Y$-modules of rank $[X : Y]$. Hence we have $[Rf_*\mathcal{O}_X] \equiv [X : Y] \bmod F^1K(Y)$. Thus, for a coherent $\mathcal{O}_Z$-module $\mathcal{F}$ such that $\dim_S \mathrm{supp}\mathcal{F} = p$, we have $[Rf_*Lf^*\mathcal{F}] = [\mathcal{F} \otimes_{\mathcal{O}_Y} Rf_*\mathcal{O}_X] \equiv [X : Y] \cdot [\mathcal{F}] \bmod F_{p-1}G(Z)$. □

For a scheme over a discrete valuation ring, we have a reduction map. Let $S = \mathrm{Spec}\, \mathcal{O}_K$ be the spectrum of a discrete valuation ring and X be a scheme of finite type over S. Then, since the immersion $s \to S$ of the closed point is a regular immersion, the intersection product $(s, \ )_S : G(X) \to G(X_s)$ is defined.

Corollary **2.2.4.** — *Let* X *be a scheme of finite type over a discrete valuation ring* $S = \mathrm{Spec}\, \mathcal{O}_K$. *Then*

1. The map $(s, \ )_S : G(X) \to G(X_s)$ *induces a map* $(s, \ )_S : G(X_K) \to G(X_s)$.

2. The induced map $(s, \ )_S : G(X_K) \to G(X_s)$ *sends the topological filtration* $F_pG(X_K)$ *into* $F_pG(X_s)$.

Proof. — 1. We have an exact sequence $G(X_s) \to G(X) \to G(X_K) \to 0$. It is sufficient to show that the composition $G(X_s) \to G(X) \to G(X_s)$ is the 0-map. By (2.2.1.2), for a closed subscheme $W \subset X_s$, we have $(s, W)_S = -([N_{s/S} \otimes \mathcal{O}_W] - 1) = 0$ and the assertion follows.

2. The map $F_{p+1}G(X) \to F_pG(X_K)$ is surjective. By Proposition 2.2.2, the map $(s, \ )_S : G(X) \to G(X_s)$ sends $F_{p+1}G(X)$ to $F_pG(X_s)$. Thus the assertion follows. □

2.3. *Localized Chern classes.* — We recall the definition and basic properties of localized Chern classes. Basic references are [13] Chapters 18 and 20 and [6] Section 1.

Let S be an equidimensional regular noetherian scheme of finite dimension, X be a scheme of finite type over S and Z be a closed subscheme of X. Let $\mathcal{K} = (\mathcal{K}_q, d_q)_q$ be a bounded complex of locally free $\mathcal{O}_X$-modules of finite ranks. Assume that on the complement $U = X - Z$, the restriction $\mathcal{K}|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K})|_U$ is locally free of rank $n-1$. Then for $i \geq n$, the localized Chern class $c_{i\,Z}^{\,X}(\mathcal{K}) \in \mathrm{CH}^i(Z \to X)$ is defined in [6] Section 1. We define a ring $\mathrm{CH}^*(Z \to X)^{(n)}$ to be $\prod_{i<n} \mathrm{CH}^i(X \to X) \times \prod_{i\geq n} \mathrm{CH}^i(Z \to X)$ and regard the total localized Chern class $c_Z^X(\mathcal{K}) = ((c_i(\mathcal{K}))_{i<n}, (c_{i\,Z}^{\,X}(\mathcal{K}))_{i\geq n})$ as an invertible element of the ring $\mathrm{CH}^*(Z \to X)^{(n)}$.

The localized Chern classes satisfy the following properties.

Proposition **2.3.1** ([6] Proposition (1.1)). — *Let* Z *be a closed subscheme of* X *and* $\mathcal{K}$ *be a bounded complex of locally free* $\mathcal{O}_X$*-modules of finite ranks. Assume that on the complement* $U = X - Z$, *the restriction* $\mathcal{K}|_U$ *is acyclic except at degree 0 and the cohomology sheaf* $\mathcal{H}_0(\mathcal{K})|_U$ *is locally free of rank* $n-1$.

1. The image of $c_Z^X(\mathcal{K})$ *in* $\mathrm{CH}^*(X)$ *is* $\prod_q c(\mathcal{K}_q)^{(-1)^q}$.

2. For a quasi-isomorphism $\mathcal{K} \to \mathcal{K}'$, *we have* $c_Z^X(\mathcal{K}) = c_Z^X(\mathcal{K}')$.

3. Let $\mathcal{E}$ *be a locally free* $\mathcal{O}_X$*-module of finite rank. Then for* $i \geq n$ *and for an integer* i', *we have* $c_{i\,Z}^{\,X}(\mathcal{K}) c_{i'}(\mathcal{E}) = c_{i'}(\mathcal{E}|_Z) c_{i\,Z}^{\,X}(\mathcal{K})$. *Let* $\mathcal{K}'$ *be another bounded complex of locally free* $\mathcal{O}_X$*-modules of finite ranks such that the restriction* $\mathcal{K}'|_U$ *is acyclic except at degree 0 and the cohomology sheaf* $\mathcal{H}_0(\mathcal{K}')|_U$ *is locally free of rank* $n'-1$. *Then for* $i \geq n$ *and* $i' \geq n'$, *we have* $c_{i\,Z}^{\,X}(\mathcal{K}) c_{i'}(\mathcal{K}') = c_{i'\,Z}^{\,X}(\mathcal{K}') c_i(\mathcal{K})$.

4. ([2]) *Let* $\mathcal{K}'$ *and* $\mathcal{K}''$ *be bounded complexes of locally free* $\mathcal{O}_X$*-modules of finite ranks such that the restriction* $\mathcal{K}'|_U$ *and* $\mathcal{K}''|_U$ *are acyclic except at degree 0 and the cohomology sheaves* $\mathcal{H}_0(\mathcal{K}')|_U$ *and* $\mathcal{H}_0(\mathcal{K}'')|_U$ *are locally free of rank* $n'-1$ *and* $n''-1$ *respectively and let* $\mathcal{K}' \to \mathcal{K} \to \mathcal{K}'' \to$ *be a distinguished triangle. Then we have* $c_Z^X(\mathcal{K}) = c_Z^X(\mathcal{K}') c_Z^X(\mathcal{K}'')$ *in* $\mathrm{CH}^*(Z \to X)^{(n)}$.

5. Let $Z \overset{i}{\subset} Z' \subset X$ *be closed immersions. Let* i_* *denote the collection of the induced maps* $i_* : \mathrm{CH}_*(Z \times_X X') \to \mathrm{CH}_*(Z' \times_X X')$ *for schemes* X′ *of finite type over* X. *Then we have* $i_* \circ c_Z^X(\mathcal{K}) = c_{Z'}^X(\mathcal{K})$.

Let $f : X'' \to X'$ *be a morphism of finite type over* X *and let* $g : Z'' \to Z'$ *be the base change by* $Z \to X$.

6. Assume f *is proper and let* $f_* : \mathrm{CH}_*(X'') \to \mathrm{CH}_*(X')$ *and* $g_* : \mathrm{CH}_*(Z'') \to \mathrm{CH}_*(Z')$ *be the induced maps. Then we have* $c_Z^X(\mathcal{K}) \circ f_* = g_* \circ c_Z^X(\mathcal{K})$.

7. Assume f *is flat of relative dimension* n *and let* $f^* : \mathrm{CH}_*(X') \to \mathrm{CH}_{*+n}(X'')$ *and* $g^* : \mathrm{CH}_*(Z') \to \mathrm{CH}_{*+n}(Z'')$ *be the induced maps. Then we have* $c_Z^X(\mathcal{K}) \circ f^* = g^* \circ c_Z^X(\mathcal{K})$.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module such that the restriction $\mathscr{F}|_U$ is locally free of rank n. If $\mathscr{F}$ has a finite resolution $\mathscr{E}_\bullet \to \mathscr{F}$ by locally free $\mathscr{O}_X$-modules $\mathscr{E}_q$ of finite rank, the localized Chern class $c_{i\,Z}^{X}(\mathscr{F})$ for $i > n$ is defined as $c_{i\,Z}^{X}(\mathscr{E}_\bullet)$. By Proposition 2.3.1.2, it is independent of the choice of a resolution.

For a locally free sheaf on a divisor, its localized Chern class is computed as a special case of Riemann-Roch without denominator as follows.

Lemma **2.3.2** (cf. [13] Theorem 15.3). — *Let* D *be a Cartier divisor of a scheme* X *and* $i : D \to X$ *be the immersion. Let* $\mathscr{E}$ *be a locally free* $\mathscr{O}_D$*-module of rank* n*. Assume there exist a locally free* $\mathscr{O}_X$*-module* $\tilde{\mathscr{E}}$ *of finite rank and a surjection* $\tilde{\mathscr{E}} \to i_*\mathscr{E}$ *so that the localized Chern class* $c_D^X(i_*\mathscr{E}(D)) \in \mathrm{CH}^*(D \to X)^{(1)}$ *is defined. We put* $a_j(\mathscr{E}) = \sum_{k=j}^n \binom{k}{j} c_{n-k}(\mathscr{E}) \in \mathrm{CH}^*(D \to D)$.

Then we have $\sum_{k=0}^n c_k(\mathscr{E} \otimes \mathscr{L}) = \sum_{j=0}^n a_j(\mathscr{E}) c_1(\mathscr{L})^j$ *for an invertible* $\mathscr{O}_D$*-module* $\mathscr{L}$ *and we have equalities*

$$\left(c_D^X(i_*\mathscr{E}(D)) - 1\right) \cap [X] = c(\mathscr{E})^{-1} \sum_{j=1}^n a_j(\mathscr{E}) D^{j-1} \cap [D]$$

in $\mathrm{CH}_*(D)$.

Proof. — We have

$$\sum_{k=0}^n c_k(\mathscr{E} \otimes \mathscr{L}) = \sum_{k=0}^n (1 + c_1(\mathscr{L}))^{n-k} c_k(\mathscr{E}) = \sum_{k=0}^n \sum_{j=0}^k \binom{k}{j} c_1(\mathscr{L})^j c_{n-k}(\mathscr{E})$$
$$= \sum_{j=0}^n a_j(\mathscr{E}) c_1(\mathscr{L})^j.$$

By deformation to the normal bundle, we may assume X is a $\mathbf{P}^1$-bundle over D and the immersion $i : D \to X$ is a section. Then $\mathscr{E}$ is the restriction to D of the pull-back $\mathscr{E}_X$ of $\mathscr{E}$ to X. Since the map $i_* : \mathrm{CH}_*(D) \to \mathrm{CH}_*(X)$ is injective, it is reduced to the equality for the usual Chern class $c(i_*\mathscr{E}(D))$. By the locally free resolution $0 \to \mathscr{E}_X \to \mathscr{E}_X(D) \to i_*\mathscr{E}(D) \to 0$, we have $c(i_*\mathscr{E}(D)) - 1 = c(\mathscr{E}_X)^{-1}(c(\mathscr{E}_X(D)) - c(\mathscr{E}_X)) = c(\mathscr{E}_X)^{-1}(\sum_{j=0}^n a_j(\mathscr{E}) D^j - a_0(\mathscr{E}))$. □

Similarly as Lemma 2.3.2, the following formula is proved.

Corollary **2.3.3.** — *Let* D *be a Cartier divisor of* X. *Then we have*

$$\left(c_D^X(\mathscr{O}_D)^{-1} - 1\right) \cap [X] = -[D].$$

We compute the localized Chern class of a blowing-up.

Lemma **2.3.4.** — *Let* X *be a regular noetherian scheme of finite equidimension,* C *be a regular closed subscheme of codimension* c *and* $i : \mathrm{C} \to \mathrm{X}$ *be the immersion. Let* $\pi : \mathrm{X}' \to \mathrm{X}$ *be the blowing-up at* C *and* $\pi_{\mathrm{E}} : \mathrm{E} = \mathrm{C} \times_{\mathrm{X}} \mathrm{X}' \to \mathrm{C}$ *be the induced map. Then, we have an equality*

$$\pi_{\mathrm{E}*}\big(\big(c_{\mathrm{E}}^{\mathrm{X}'}\big(\Omega^1_{\mathrm{X}'/\mathrm{X}}\big) - 1\big) \cap [\mathrm{X}']\big) = (-1)^c(c-1)c(\mathrm{N}_{\mathrm{C}/\mathrm{X}})^{-1} \cap [\mathrm{C}]$$

in $\mathrm{CH}_*(\mathrm{C})$.

Proof. — The canonical map $\Omega^1_{\mathrm{X}'/\mathrm{X}} \to \Omega^1_{\mathrm{E}/\mathrm{C}}$ is an isomorphism. Since E is a $\mathbf{P}^{c-1}$-bundle $\mathbf{P}(\mathrm{N}_{\mathrm{C}/\mathrm{X}})$ associated to the conormal sheaf $\mathrm{N}_{\mathrm{C}/\mathrm{X}}$, we have an exact sequence $0 \to \Omega^1_{\mathrm{E}/\mathrm{C}} \to \pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}(-1) \to \mathscr{O}_{\mathrm{E}} \to 0$. Hence, we have $c_{\mathrm{E}}^{\mathrm{X}'}(\Omega^1_{\mathrm{X}'/\mathrm{X}}) = c_{\mathrm{E}}^{\mathrm{X}'}(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}(-1))c_{\mathrm{E}}^{\mathrm{X}'}(\mathscr{O}_{\mathrm{E}})^{-1}$. By Corollary 2.3.3 and Lemma 2.3.2, we have

$$\begin{aligned}
&\big(c_{\mathrm{E}}^{\mathrm{X}'}\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}(-1)\big)c_{\mathrm{E}}^{\mathrm{X}'}(\mathscr{O}_{\mathrm{E}})^{-1} - 1\big) \cap [\mathrm{X}'] \\
&= \big(c_{\mathrm{E}}^{\mathrm{X}'}\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}(\mathrm{E})\big) - 1\big) \cap [\mathrm{X}'] - c_{\mathrm{E}}\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}\big)^{-1} c_{\mathrm{E}}\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}(\mathrm{E})\big) \cap [\mathrm{E}] \\
&= c_{\mathrm{E}}\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}\big)^{-1}\left(\sum_{j=1}^{c} a_j\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}\big)\mathrm{E}^{j-1} \cap [\mathrm{E}] - \sum_{j=0}^{c} a_j\big(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}}\big)\mathrm{E}^{j} \cap [\mathrm{E}]\right).
\end{aligned}$$

We have $\mathrm{E}^c = -\sum_{j=1}^{c} \pi_{\mathrm{E}}^* c_j(\mathrm{N}_{\mathrm{C}/\mathrm{X}})\mathrm{E}^{c-j}$ since $c_c(\mathrm{Ker}(\pi_{\mathrm{E}}^*\mathrm{N}_{\mathrm{C}/\mathrm{X}} \to \mathscr{O}(1))) = 0$. Substituting this and using $\pi_{\mathrm{E}*}(\mathrm{E}^j \cap [\mathrm{E}]) = (-1)^{c-1}[\mathrm{C}]$ if $j = c-1$ and is 0 for $j < c-1$, we have

$$\begin{aligned}
&\pi_{\mathrm{E}*}\big(\big(c_{\mathrm{E}}^{\mathrm{X}'}\big(\Omega^1_{\mathrm{X}'/\mathrm{X}}\big) - 1\big) \cap [\mathrm{X}']\big) \\
&= (-1)^{c-1}c(\mathrm{N}_{\mathrm{C}/\mathrm{X}})^{-1}(a_c(\mathrm{N}_{\mathrm{C}/\mathrm{X}}) - a_{c-1}(\mathrm{N}_{\mathrm{C}/\mathrm{X}}) + a_c(\mathrm{N}_{\mathrm{C}/\mathrm{X}})c_1(\mathrm{N}_{\mathrm{C}/\mathrm{X}})) \cap [\mathrm{C}].
\end{aligned}$$

Since $a_c(\mathrm{N}_{\mathrm{C}/\mathrm{X}}) = 1$ and $a_{c-1}(\mathrm{N}_{\mathrm{C}/\mathrm{X}}) = c + c_1(\mathrm{N}_{\mathrm{C}/\mathrm{X}})$, the assertion follows. □

2.4. *Localized Chern class and derived exterior power.* — Let $\mathscr{K}$ be a complex of $\mathscr{O}_{\mathrm{X}}$-modules and $n \geq 0$ be an integer. In this subsection, we compute the class of the derived exterior power $\mathrm{L}\Lambda^n\mathscr{K}$ assuming that $\mathscr{K}$ satisfies the following condition:

(L(n)) For each $x \in \mathrm{X}$, there exist an open neighborhood U of x, a locally free $\mathscr{O}_{\mathrm{U}}$-module $\mathscr{E}_{\mathrm{U}}$ of rank n, an invertible $\mathscr{O}_{\mathrm{U}}$-module $\mathscr{L}_{\mathrm{U}}$, and a distinguished triangle $\to \mathscr{L}_{\mathrm{U}} \to \mathscr{E}_{\mathrm{U}} \to \mathscr{K}|_{\mathrm{U}} \to$ in $\mathrm{D}^b(\mathrm{U})$.

We put $\mathscr{F} = \mathscr{H}_0\mathscr{K}$ and let $i : \mathrm{Z} \to \mathrm{X}$ be the closed immersion defined by the annihilater ideal Ann $\Lambda^n\mathscr{F}$. We also relate the class $[\mathrm{L}\Lambda^n\mathscr{K}]$ to the localized Chern class $c_{n\mathrm{Z}}^{\mathrm{X}}(\mathscr{K}) \in \mathrm{CH}^n(\mathrm{Z} \to \mathrm{X})$ in Proposition 2.4.4 assuming $\mathscr{K}$ further satisfies the condition:

(G) There exist a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank and a map $\mathcal{E} \to \mathcal{K}$ in $D^b(X)$ such that the induced map $\mathcal{E} \to \mathcal{F} = \mathcal{H}_0\mathcal{K}$ is a surjection.

Lemma **2.4.1.** — *Let* X *be a scheme,* $n \geq 1$ *be an integer and* $\mathcal{K}$ *be a complex of* $\mathcal{O}_X$-*modules satisfying the condition* (L(n)) *above. We put* $\mathcal{F} = \mathcal{H}_0\mathcal{K}$ *and let* $i : Z \to X$ *be the closed immersion defined by the annihilater ideal* Ann $\Lambda^n\mathcal{F}$. *Then,*

1. The restriction $\mathcal{F}|_{X-Z}$ *is locally free of rank* $n-1$. *The* $\mathcal{O}_Z$-*module* $\mathcal{L}_Z = L^1 i^*\mathcal{K}$ *is invertible.*

2. For an $\mathcal{O}_X$-*module* $\mathcal{G}$, *the* $\mathcal{T}or$-*sheaves* $\mathcal{T}or_q^{\mathcal{O}_X}(L\Lambda^n\mathcal{K}, \mathcal{G})$ *are* $\mathcal{O}_Z$-*modules for all* q *and are 0 except for* $0 \leq q \leq n$. *In particular,* $L^q\Lambda^n\mathcal{K}$ *are* $\mathcal{O}_Z$-*modules for all* q *and are 0 except for* $0 \leq q \leq n$.

3. Let $\mathcal{T}$ *be an* $\mathcal{O}_Z$-*module. Then the canonical map* $\mathcal{L}_Z[1] \to Li^*\mathcal{K}$ *induces an ismorphism*

$$\textbf{(2.4.1.1)} \qquad \mathcal{L}_Z \otimes_{\mathcal{O}_Z} \mathcal{T} \longrightarrow \mathcal{T}or_1^{\mathcal{O}_Z}(Li^*\mathcal{K}, \mathcal{T}) = \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{K}, \mathcal{T}).$$

For locally free $\mathcal{O}_X$-*modules* $\mathcal{L}$ *and* $\mathcal{E}$ *of finite rank and a distinguished triangle* $\to \mathcal{L} \to \mathcal{E} \to \mathcal{K} \to$, *we have a commutative diagram*

$$\textbf{(2.4.1.2)} \qquad \begin{array}{ccc} \mathcal{L}_Z \otimes_{\mathcal{O}_Z} \mathcal{T} & \longrightarrow & \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{K}, \mathcal{T}) \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{T} & = & \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{L}[1], \mathcal{T}). \end{array}$$

The vertical maps are induced by the map $\mathcal{K} \to \mathcal{L}[1]$. *If* $\mathcal{L}$ *is invertible, the vertical arrows are isomorphisms.*

4. If $\mathcal{K}$ *further satisfies the condition* (G) *above, then there exist a locally free* $\mathcal{O}_X$-*module* $\mathcal{L}$ *of finite rank and a distingushed triangle* $\to \mathcal{L} \to \mathcal{E} \to \mathcal{K} \to$ *in* $D^b(X)$.

Proof. — 1. Since the question is local on X, we may assume that there is an distinguished triangle $\to \mathcal{L} \to \mathcal{E} \to \mathcal{K} \to$ where $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{E} = \mathcal{O}_X^n$. Let $(a_1, \ldots, a_n) \in \mathcal{E} = \mathcal{O}_X^n$ be the image of $1 \in \mathcal{L} = \mathcal{O}_X$. Then the closed subscheme $Z \subset X$ is defined by the ideal $(a_1, \ldots, a_n)$. Hence, on the complement $X \setminus Z$, the map $\mathcal{L} \to \mathcal{E}$ is a locally splitting injection. The natural map $L^1 i^*\mathcal{K} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ is an isomorphism.

2. The question is local on X and we keep the notation in the proof of 1. By Lemma 1.2.5 and by the isomorphism (1.3.1.2), we have an isomorphism $\Lambda^n(\mathcal{L} \to \mathcal{E}) \to S^n(\mathcal{E}^* \to \mathcal{L}^*) \to \mathbf{K}(\mathcal{E}^* \otimes \mathcal{L} \to \mathcal{O}_X)$. It induces an isomorphism $\mathcal{T}or_q^{\mathcal{O}_X}(L\Lambda^n\mathcal{K}, \mathcal{G}) \to \mathcal{H}_q(\mathbf{K}(\mathcal{E}^* \otimes \mathcal{L} \to \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G})$. Since $\mathcal{H}_q(\mathbf{K}(\mathcal{E}^* \otimes \mathcal{L} \to \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G})$ is an $\mathcal{H}_0\mathbf{K}(\mathcal{E}^* \otimes \mathcal{L} \to \mathcal{O}_X) = \mathcal{O}_Z$-module, the assertion follows.

3. It is clear that the diagram (2.4.1.2) is commutative. It is clear from the definition of Z that the vertical arrows are isomorphisms if $\mathscr{L}$ is invertible. For the isomorphism (2.4.1.1), the question is local on X and hence the assertion follows from (2.4.1.2).

4. There exists a distinguished triangle $\to \mathscr{K}' \to \mathscr{E} \to \mathscr{K} \to$ of complexes of $\mathscr{O}_X$-modules. By the condition (L(n)), $\mathscr{K}'$ is acyclic except at degree 0 and hence is identified with an $\mathscr{O}_X$-module $\mathscr{L}$. In the notation of (L(n)), the restriction $\mathscr{L}|_U$ is isomorphic to the kernel of a surjection $\mathscr{E}|_U \oplus \mathscr{L}_U \to \mathscr{E}_U$ of locally free $\mathscr{O}_U$-modules of finite rank and the assertion follows. □

Lemma **2.4.2.** — *Let the notation be as in Lemma 2.4.1.*

1. The homology sheaf $L^p\Lambda^q\mathscr{K} = \mathscr{H}_p(L\Lambda^q\mathscr{K})$ *is an* $\mathscr{O}_Z$*-module except for* $p = 0$ *and* $0 \leq q < n$ *and is 0 except for* $\max(0, q-n) \leq p \leq q$.

2. Assume either $q \geq n$, $p > 0$ *or* $Z = X$. *Then the composition*

(2.4.2.1)
$$\begin{aligned}\lambda_{\mathscr{K}} : L^{p+1}\Lambda^{q+1}\mathscr{K} \longrightarrow \mathscr{T}or_{p+1}^{\mathscr{O}_X}(\mathscr{K}, L\Lambda^q\mathscr{K}) &\longrightarrow \mathscr{T}or_1^{\mathscr{O}_X}(\mathscr{K}, L^p\Lambda^q\mathscr{K}) \\ &\longrightarrow \mathscr{L}_Z \otimes_{\mathscr{O}_Z} L^p\Lambda^q\mathscr{K}\end{aligned}$$

is an isomorphism. The first map is induced by the map $L\Lambda^{q+1}\mathscr{K} \to \mathscr{K} \otimes^L_{\mathscr{O}_X} L\Lambda^q\mathscr{K}$, *the second map is the boundary map of the spectral sequence* $E^2_{s,t} = \mathscr{T}or_s^{\mathscr{O}_X}(\mathscr{K}, L^t\Lambda^q\mathscr{K}) \Rightarrow \mathscr{T}or_{s+t}^{\mathscr{O}_X}(\mathscr{K}, L\Lambda^q\mathscr{K})$ *and the last map is the inverse of the isomorphism* (2.4.1.1).

3. Assume $Z = X$. *Then the* $\mathscr{O}_X$*-module* $\mathscr{E} = \mathscr{H}_0\mathscr{K}$ *is locally free of rank n and* $\mathscr{L} = \mathscr{H}_1\mathscr{K}$ *is invertible. An iteration of the isomorphism* $\lambda_{\mathscr{K}}$ (2.4.2.1) *defines an isomorphism* $L^p\Lambda^q\mathscr{K} \to \mathscr{L}^{\otimes p} \otimes \Lambda^{q-p}\mathscr{E}$.

4. Assume Z is a Cartier divisor of X. Then $\mathscr{F} = \mathscr{H}_0\mathscr{K}$ *is an extension of a locally free* $\mathscr{O}_X$*-module* $\mathscr{E}'$ *of rank* $n-1$ *by an invertible* $\mathscr{O}_Z$*-module* $\mathscr{L}'_Z = \mathscr{L}_Z(Z) = \mathscr{L}_Z \otimes_{\mathscr{O}_X} \mathscr{O}_X(Z)$. *The canonical map* $\mathscr{K} \to \mathscr{F}$ *is an isomorphism in the derived category.*

For $q \geq 0$, *the composition*

(2.4.2.2)
$$\begin{aligned}L^1\Lambda^{q+1}\mathscr{K} \longrightarrow \mathscr{T}or_1^{\mathscr{O}_X}(\mathscr{K}, L\Lambda^q\mathscr{K}) &\longrightarrow \mathscr{T}or_1^{\mathscr{O}_X}(\mathscr{K}, \Lambda^q\mathscr{F}) \\ \longrightarrow \mathscr{T}or_1^{\mathscr{O}_X}\big(\mathscr{K}, \mathscr{L}_Z(Z) \otimes \Lambda^{q-1}\mathscr{E}'\big) &\longrightarrow \mathscr{L}_Z^{\otimes 2}(Z) \otimes \Lambda^{q-1}\mathscr{E}'\end{aligned}$$

is an isomorphism of $\mathscr{O}_Z$*-modules. The first map is induced by the map* $L\Lambda^{q+1}\mathscr{K} \to \mathscr{K} \otimes^L_{\mathscr{O}_X} L\Lambda^q\mathscr{K}$, *the second map is induced by the canonical map* $L\Lambda^q\mathscr{K} \to \Lambda^q\mathscr{F}$, *the third map is the inverse of the isomorphism induced by the map* $\mathscr{L}_Z(Z) \otimes \Lambda^{q-1}\mathscr{E}' \to \Lambda^q\mathscr{F}$ *and the last map is the inverse of the isomorphism* (2.4.1.1).

Proof. — Since the questions are local on X, we may assume that there is an distinguished triangle $\to \mathscr{L} \to \mathscr{E} \to \mathscr{K} \to$ where $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{E} = \mathscr{O}_X^n$ as in the proof of Lemma 2.4.1. We put $\mathcal{F} = \mathcal{H}_0\mathcal{K}$. By Corollary 1.2.3.2, we have an exact sequence

$$\textbf{(2.4.2.3)} \qquad 0 \to L^1\Lambda^{q+1}\mathcal{K} \longrightarrow \mathcal{L} \otimes \Lambda^q\mathcal{F} \longrightarrow \Lambda^{q+1}\mathcal{E} \longrightarrow \Lambda^{q+1}\mathcal{F} \to 0$$

and isomorphism

$$\textbf{(2.4.2.4)} \qquad L^{p+1}\Lambda^{q+1}\mathcal{K} \to \mathcal{L} \otimes L^p\Lambda^q\mathcal{K}$$

for $q \geq 0$ and $p > 0$. If $X = Z$, we have an isomorphism (2.4.2.4) also for $p = 0$.

1. By the isomorphisms (2.4.2.4), it is reduced to the case $q = n$. Hence it follows from Lemma 2.4.1.2.

2. The composition of $\lambda_{\mathcal{K}} : L^{p+1}\Lambda^{q+1}\mathcal{K} \to \mathcal{L}_Z \otimes_{\mathcal{O}_Z} L^p\Lambda^q\mathcal{K}$ with the isomorphism $\mathcal{L}_Z \otimes_{\mathcal{O}_Z} L^p\Lambda^q\mathcal{K} \to \mathcal{L} \otimes_{\mathcal{O}_X} L^p\Lambda^q\mathcal{K}$ is the isomorphism (2.4.2.4) either if $q \geq n$, $p > 1$ or $X = Z$. Hence the assertion follows.

3. If $X = Z$, we have an isomorphism $\mathcal{K} \to \mathcal{E} \oplus \mathcal{L}[1]$ and the assertion follows.

4. We show that $\mathcal{F}$ is an extension of a locally free $\mathcal{O}_X$-module $\mathcal{E}'$ of rank $n-1$ by an invertible $\mathcal{O}_Z$-module $\mathcal{L}'_Z$ and $\mathcal{K} \to \mathcal{F}$ is an isomorphism. Let $(a_1, \ldots, a_n) \in \mathcal{E} = \mathcal{O}_X^n$ be the image of $1 \in \mathcal{L} = \mathcal{O}_X$. Shrinking further X and changing the isomorphism $\mathcal{O}_X^n \to \mathcal{E}$, we may assume a_1 is a non-zero divisor and $a_2 = \ldots = a_n = 0$. The assertion is clear from this.

We have a canonical isomorphism $\mathcal{L}_Z = \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{F}) \to \mathrm{Ker}(\mathcal{F} \otimes \mathcal{O}_X(-Z) \to \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-Z) = \mathcal{L}'_Z \otimes_{\mathcal{O}_X} \mathcal{O}_X(-Z)$. Thus we obtain an isomorphism $\mathcal{L}_Z \otimes \mathcal{O}_X(Z) \to \mathcal{L}'_Z$.

We show that the map (2.4.2.2) is an isomorphism. By the exact sequence $0 \to \mathcal{L}_Z(Z) \to \mathcal{F} \to \mathcal{E}' \to 0$, we obtain an exact sequence $0 \to \mathcal{L}_Z(Z) \otimes \Lambda^{q-1}\mathcal{E}' \to \Lambda^q\mathcal{F} \to \Lambda^q\mathcal{E}' \to 0$. From this, we see that the kernel of the map $\mathcal{L} \otimes \Lambda^q\mathcal{F} \to \Lambda^{q+1}\mathcal{E}$ in (2.4.2.3) is $\mathcal{L}_Z^{\otimes 2}(Z) \otimes \Lambda^{q-1}\mathcal{E}'$ and obtain an isomorphism $L^1\Lambda^{q+1}\mathcal{K} \to \mathcal{L}_Z^{\otimes 2}(Z) \otimes \Lambda^{q-1}\mathcal{E}'$. It is easy to see that this isomorphism is the same as the map (2.4.2.2). □

We compute the class of the exterior derived power $L\Lambda^n\mathcal{K}$ in the K-group.

Corollary **2.4.3.** — *Let the notation be as in Lemma 2.4.1.*

1. Assume $Z = X$. *Let* $\mathcal{E} = \mathcal{H}_0\mathcal{K}$ *be the locally free* $\mathcal{O}_X$*-module of rank* n *and* $\mathcal{L} = \mathcal{H}_1\mathcal{K}$ *be the invertible* $\mathcal{O}_X$*-module in Lemma 2.4.2.3. Then, we have an equality*

$$\textbf{(2.4.3.1)} \qquad \begin{aligned}[L\Lambda^n\mathcal{K}] &= (-1)^n \sum_{p=0}^{n} (-1)^p [\Lambda^p(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) \otimes \mathcal{L}^{\otimes n}] \\ &= \gamma_n([\mathcal{E} \otimes \mathcal{L}^{\otimes -1}] - n)[\mathcal{L}]^n\end{aligned}$$

in K(X).

2. Assume X *is a noetherian scheme and* Z *is a Cartier divisor of* X. *Let* $\mathscr{E}'$ *be the locally free* $\mathscr{O}_X$*-module of rank* $n-1$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$*-module as in Lemma 2.4.2.4. Then, we have an equality*

$$\textbf{(2.4.3.2)}\qquad \begin{aligned}\sum_{p=0}^{n-1}(-1)^p[L^p\Lambda^n\mathscr{K}] &= (-1)^{n-1}\sum_{p=0}^{n-1}(-1)^p\big[\Lambda^p\big(\mathscr{E}'\otimes_{\mathscr{O}_X}\mathscr{L}_Z^{\otimes-1}\big)\otimes\mathscr{L}_Z^{\otimes n}(Z)\big]\\ &= \gamma_{n-1}\big(\big[\mathscr{E}'\otimes_{\mathscr{O}_X}\mathscr{L}_Z^{\otimes-1}\big]-(n-1)\big)[\mathscr{L}_Z]^n[\mathscr{O}_Z(Z)]\end{aligned}$$

in G(Z).

Proof. — 1. We have an isomorphism $L^p\Lambda^n\mathscr{K} \to \Lambda^{n-p}\mathscr{E}\otimes\mathscr{L}^{\otimes p}$ by Lemma 2.4.2.3. Thus the first equality of (2.4.3.1) follows. The second equality in (2.4.3.1) follows from (2.1.1.2).

2. By the composition of an iteration of the isomorphisms (2.4.2.1) and the isomorphism (2.4.2.2), we obtain an isomorphism $L^p\Lambda^n\mathscr{K} \to \Lambda^{n-1-p}\mathscr{E}'\otimes\mathscr{L}^{\otimes p+1}(Z)$. Thus the first equality in (2.4.3.2) follows. The second equality in (2.4.3.2) follows from (2.1.1.2). □

We compare the localized Chern class and the class of the exterior derived power. We introduce some notations. Let S be an equidimensional regular noetherian scheme of finite dimension and X be a scheme of finite type over S. Let $\mathscr{K}$ be a complex of $\mathscr{O}_X$-modules satisfying the condition (L(n)). Let Z be the closed subscheme of X as in Lemma 2.4.1. For a coherent $\mathscr{O}_X$-module $\mathscr{G}$, the $\mathscr{T}or$-sheaves $\mathscr{T}or_q^{\mathscr{O}_X}(L\Lambda^n\mathscr{K},\mathscr{G})$ are coherent $\mathscr{O}_Z$-modules and are 0 except for $0\le q\le n$ by Lemma 2.4.1.2. Hence the map $([L\Lambda^n\mathscr{K}],\ )_X : G(X)\to G(Z)$ sending the class $[\mathscr{G}]$ of a coherent $\mathscr{O}_X$-module $\mathscr{G}$ to $\sum_{q=0}^n(-1)^q[\mathscr{T}or_q^{\mathscr{O}_X}(L\Lambda^n\mathscr{K},\mathscr{G})]$ is defined. If $\mathscr{K}$ further satisfies the condition (G) above, the localized Chern class $c_{nZ}^{X}(\mathscr{K})\in CH^n(Z\to X)$ is defined by Lemma 2.4.1.4.

Proposition **2.4.4.** — *Let* S *be an equidimensional regular noetherian scheme of finite dimension and* X *be a scheme of finite type over* S. *Let* $n\ge 1$ *be an integer and* $\mathscr{K}$ *be a complex of* $\mathscr{O}_X$*-modules satisfying the condition* (L(n)) *above. Let* $i: Z\to X$ *be the closed immersion defined by the annihilater ideal* Ann $\Lambda^n\mathscr{H}_0\mathscr{K}$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$*-module* $L^1i^*\mathscr{K}$.

Then the map $([L\Lambda^n\mathscr{K}],\ )_X : G(X)\to G(Z)$ *sends the topological filtration* $F_pG(X)$ *to the topological filtration* $F_{p-n}G(Z)$. *If* $\mathscr{K}$ *further satisfies the condition* (G), *the induced map makes a commutative diagram*

$$\begin{array}{ccc} CH_p(X) & \xrightarrow{c_{nZ}^{X}(\mathscr{K})\cap} & CH_{p-n}(Z)\\ \downarrow & & \downarrow\\ Gr_p^F G(X) & \xrightarrow[([L\Lambda^n\mathscr{K}],\ )_X]{} & Gr_{p-n}^F G(Z). \end{array}$$

Proof. — The proof is similar to that of Proposition 2.2.2. The topological filtration $F_pG(X)$ is generated by the classes $[\mathcal{O}_W]$ for integral closed subschemes $W \subset X$ of dimension $\leq p$. We put $T = W \times_X Z$. If $W = T \subset Z$, we put $W' = W$. If otherwise, let $\pi : W' \to W$ be the blow-up of W at T and put $T' = W' \times_W T$. Then, the topological filtration $F_pG(X)$ is generated by the classes $\pi_*[\mathcal{O}_{W'}] = [R\pi_*\mathcal{O}_{W'}]$ for integral closed subschemes $W \subset X$ of dimension $\leq p$.

Let $\mathcal{K}_W$ and $\mathcal{K}_{W'}$ denote $\mathcal{K} \otimes^L_{\mathcal{O}_X} \mathcal{O}_W$ and $\mathcal{K} \otimes^L_{\mathcal{O}_X} \mathcal{O}_{W'}$ respectively. We show the equality $([L\Lambda^n\mathcal{K}], \pi_*[\mathcal{O}_{W'}])_X = \pi_*[L\Lambda^n\mathcal{K}_{W'}]$ in G(T). Since $L\Lambda^n\mathcal{K}_W$ is a perfect complex of $\mathcal{O}_W$-modules and π is quasi-compact, we have a projection formula $L\Lambda^n\mathcal{K}_W \otimes^L R\pi_*\mathcal{O}_{W'} \simeq R\pi_*L\Lambda^n\mathcal{K}_{W'}$ (1.5.3.1) in $D^b(X)_{\mathrm{coh}}$. Thus, by the spectral sequences (1.5.3.3) and (1.5.3.4) and by the isomorphism (1.5.3.2), we have

$$
\begin{aligned}
([L\Lambda^n\mathcal{K}], [R\pi_*\mathcal{O}_{W'}])_X &= \sum_{p,q}(-1)^{p+q}\big[\mathcal{T}or_p^{\mathcal{O}_X}(L\Lambda^n\mathcal{K}, R^q\pi_*\mathcal{O}_{W'})\big] \\
= \sum_p(-1)^p\big[\mathcal{T}or_p^{\mathcal{O}_X}(L\Lambda^n\mathcal{K}, R\pi_*\mathcal{O}_{W'})\big] &= \sum_p(-1)^p\big[R^p\pi_*L\Lambda^n\mathcal{K}_{W'}\big] \\
= \sum_{p,q}(-1)^{p+q}\big[R^p\pi_*L^q\Lambda^n\mathcal{K}_{W'}\big] &= \pi_*\big[L\Lambda^n\mathcal{K}_{W'}\big].
\end{aligned}
$$

Hence it is reduced to showing that $[L\Lambda^n\mathcal{K}_{W'}]$ is in $F_{p-n}G(T')$ and its class in $\mathrm{Gr}^F_{p-n}G(T')$ is equal to the image of $c_n{}^X_Z(\mathcal{K}) \cap [W']$ assuming $\dim_S W = p$. Replacing X by W and further by W′ and $\mathcal{K}$ by $\mathcal{K}_{W'}$, we may assume $X = W = W'$ and $Z = T = T'$. Thus we may assume $X = W$ is of dimension p and $Z = T$ is either equal to X or is a Cartier divisor of X.

First, we assume $Z = X$. In the notation of Corollary 2.4.3.1, we have

$$
\begin{aligned}
([L\Lambda^n\mathcal{K}], [\mathcal{O}_X])_X &= \gamma_n([\mathcal{E} \otimes \mathcal{L}^{\otimes -1}] - n)[\mathcal{L}]^n \\
&\equiv \gamma_n([\mathcal{E} \otimes \mathcal{L}^{\otimes -1}] - n) \bmod F_{p-n-1}G(X).
\end{aligned}
$$

Hence it is contained in $F_{p-n}G(X)$ and its class in $\mathrm{Gr}^F_{p-n}G(X)$ is equal to the image of $c_n(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) \cap [X]$ by Lemma 2.1.4.3. Further, we have

(2.4.4.1)
$$
\begin{aligned}
&c_n{}^X_Z(\mathcal{K}) \cap [X] = (c(\mathcal{E})c(\mathcal{L})^{-1} \cap [X])_{\deg\ n} \\
&= \textstyle\sum_{i+j=n}(-1)^j c_i(\mathcal{E})c_1(\mathcal{L})^j \cap [X] = c_n(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) \cap [X]
\end{aligned}
$$

in $CH_{d-n}(W)$. Thus the assertion is proved in the case $Z = X$.

Next, we assume Z is a Cartier divisor of X. In the notation of Corollary 2.4.3.2, we have

$$
\begin{aligned}
\big([L\Lambda^n\mathcal{K}], [\mathcal{O}_X]\big)_X &= \gamma_{n-1}\big(\big[\mathcal{E}' \otimes \mathcal{L}_Z^{\otimes -1}\big] - (n-1)\big)[\mathcal{L}_Z]^n[\mathcal{O}_Z(Z)] \\
&\equiv \gamma_{n-1}\big(\big[\mathcal{E}' \otimes \mathcal{L}_Z^{\otimes -1}\big] - (n-1)\big) \bmod F_{p-n-1}G(Z).
\end{aligned}
$$

Hence by Lemma 2.1.4.3, it is contained in $F_{p-n}G(Z)$ and its image in $Gr^F_{p-n}G(Z)$ is equal to the image of $c_{n-1}(\mathcal{E}'|_Z \otimes \mathcal{L}_Z^{\otimes -1}) \cap [Z]$. We show the equality

$$\textbf{(2.4.4.2)} \qquad c_{n_Z}^X(\mathcal{K}) \cap [X] = c_{n-1}\big(\mathcal{E}'|_Z \otimes \mathcal{L}_Z^{\otimes -1}\big) \cap [Z]$$

in $CH_{d-n}(Z)$. By the exact sequence $0 \to \mathcal{L}_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(Z) \to \mathcal{F} \to \mathcal{E}' \to 0$, we have $c_Z^X(\mathcal{K}) \cap [X] = c(\mathcal{E}') c_Z^X(\mathcal{L}_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(Z)) \cap [X]$. By Lemma 2.3.2, we have $c_Z^X(\mathcal{K}) \cap [X] = c(\mathcal{E}')([X] + c(\mathcal{L}_Z)^{-1} \cap [Z])$. Its degree n-part is equal to $\sum_{p+q=n-1} (-1)^q c_p(\mathcal{E}'|_Z) c_1(\mathcal{L}_Z)^q \cap [Z]$ and further to the right hand side of (2.4.4.2). Thus the assertion is also proved in the case Z is a Cartier divisor of X. □

Corollary **2.4.5.** — *Let* X *be a separated regular noetherian scheme of finite dimension and* $\mathcal{F}$ *be a coherent* $\mathcal{O}_X$*-modules such that* $\mathcal{K} = \mathcal{F}$ *satisfies the condition* (L(n)) *for an integer* $n \geq 0$. *Let* $i : Z \to X$ *be the closed immersion defined by the annihilator ideal of* $\Lambda^n \mathcal{F}$. *Assume* $\mathcal{F}$ *is locally free of rank* $n-1$ *on a dense open subscheme of* X. *Let* $\pi : X' \to X$ *be the blow-up at* Z, $D = Z \times_X X'$ *be the exceptional divisor and* $\pi_D : D \to Z$ *be the restriction of* π. *Let* $\mathcal{E}'_{X'}$ *be the locally free quotient of rank* $n-1$ *of the* $\mathcal{O}_{X'}$*-module* $\pi^*\mathcal{F}$ *by the invertible* $\mathcal{O}_D$*-module* $\pi_D^*\mathcal{L}_Z \otimes_{\mathcal{O}_D} \mathcal{O}_D(D)$. *Then, we have*

$$c_{n_Z}^X(\mathcal{F}) \cap [X] = \pi_{D*}\big(c_{n-1}\big(\mathcal{E}'_{X'}|_D \otimes \pi_D^*\mathcal{L}_Z^{\otimes -1}\big) \cap [D]\big)$$

in $CH_{d-n}(Z)$.

Proof. — The complex $\mathcal{F}$ satisfies the condition (G) by Lemma 2.1.1. Since the cohomology sheaves $L^q i^* \mathcal{K}$ are locally free $\mathcal{O}_Z$-modules for all q, the $\mathcal{O}_D$-module $L^1(\pi_D \circ i)^* \mathcal{K}$ is the pull-back $\pi_D^* \mathcal{L}_Z$. Thus it follows from the equality (2.4.4.2) for $L\pi^*\mathcal{F}$. □

3. K-theoretic localized intersection product

In this section, we define and study K-theoretic localized intersection product, which plays an essential role in the proof of the conductor formula. To define the localized intersection product in Section 3.2, we prove a periodicity of $\mathcal{T}or$-sheaves in Theorem 3.1.3 using the Atiyah class map recalled in Section 1.4. We establish basic properties of the localized intersection product including the associativity formulas, Proposition 3.3.2 and 3.3.3, the projection formula, Proposition 3.3.5 and the excess intersection formula, Theorem 3.4.3. The excess intersection formula gives a relation with the localized Chern class introduced in Section 2.3 and also with the localized intersection theory defined by Abbes [1]. We prove the formula by using the map (1.7.2.2) of the spectral sequence (1.6.4.3).

3.1. *Periodicity.*

Definition **3.1.1.** — *Let* S *be a scheme. We say a scheme* X *locally of finite presentation over* S *is locally a* hypersurface *of virtual relative dimension* $n-1$ *if, for each* $x \in \mathrm{X}$*, there exist an open neighborhood* U *of* x *in* X *and a regular immersion* $\mathrm{U} \to \mathrm{P}$ *of codimension 1 over* S *into a smooth scheme* P *over* S *of relative dimension* n.

Clearly, if a scheme is locally a hypersurface, it is locally of complete intersection. In this section, for a scheme X over S that is locally a hypersurface of virtual relative dimension $n-1$, let $i : \mathrm{Z} \to \mathrm{X}$ denote the closed immersion defined by the annihilator ideal $\mathrm{Ann}\, \Omega^n_{\mathrm{X/S}}$ and let $\mathscr{L}_\mathrm{Z}$ denote the $\mathscr{O}_\mathrm{Z}$-module $\mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$. Locally on X, the closed subscheme Z is described as follows. Let the notation be as in Definition 3.1.1. Further let $\mathrm{P} \to \mathbf{A}^n_\mathrm{S}$ be an etale map defined by a coordinate $t_1, \ldots, t_n$ and assume U is defined by $g \in \Gamma(\mathrm{P}, \mathscr{O}_\mathrm{P})$. Then we have a distinguished triangle $\to \mathrm{N}_{\mathrm{U/P}} \to \Omega^1_{\mathrm{P/S}} \otimes_{\mathscr{O}_\mathrm{P}} \mathscr{O}_\mathrm{U} \to \mathrm{L}_{\mathrm{X/S}}|_\mathrm{U} \to$ and the map $\mathrm{N}_{\mathrm{U/P}} \to \Omega^1_{\mathrm{P/S}} \otimes_{\mathscr{O}_\mathrm{P}} \mathscr{O}_\mathrm{U}$ sends the basis g to $dg = \frac{\partial g}{\partial t_1} dt_1 + \cdots + \frac{\partial g}{\partial t_n} dt_n$. Thus the closed subscheme $\mathrm{Z} \cap \mathrm{U} \subset \mathrm{U}$ is defined by the ideal $(\frac{\partial g}{\partial t_1}, \ldots, \frac{\partial g}{\partial t_n})$.

Lemma **3.1.2.** — *Let* X *be a scheme over* S *that is locally a hypersurface of virtual relative dimension* $n-1$*. Let* $i : \mathrm{Z} \to \mathrm{X}$ *be the closed immersion defined by the annihilator ideal* $\mathrm{Ann}\, \Omega^n_{\mathrm{X/S}}$*. We put* $\mathscr{L}_\mathrm{Z} = \mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$.

1. The underlying set of Z *is equal to the closed subset* $\{x \in \mathrm{X} : \mathrm{X}$ *is not smooth at* x *over* $\mathrm{S}\}$.

2. The cotangent complex $\mathrm{L}_{\mathrm{X/S}}$ *satisfies the condition* (L(n)) *in Section 2.4. For* $\Omega^1_{\mathrm{X/S}} = \mathscr{H}_0 \mathrm{L}_{\mathrm{X/S}}$*, the restriction* $\Omega^1_{\mathrm{X/S}}|_{\mathrm{X}\setminus\mathrm{Z}}$ *to the complement of* Z *is locally free of rank* $n-1$*. The* $\mathscr{O}_\mathrm{Z}$*-module* $\mathscr{L}_\mathrm{Z} = \mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$ *is invertible.*

3. Let P *be a smooth scheme over* S *and* $\mathrm{X} \to \mathrm{P}$ *be a regular immersion over* S*. Then the canonical map* $\mathrm{L}_{\mathrm{X/S}} \to \mathrm{L}_{\mathrm{X/P}} \to \mathrm{N}_{\mathrm{X/P}}[1]$ *induces a locally splitting injection*

$$\textbf{(3.1.2.1)} \qquad \nu_{\mathrm{X/P/S}} : \mathscr{L}_\mathrm{Z} \to \mathrm{N}_{\mathrm{X/P}} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{O}_\mathrm{Z}.$$

If $\mathrm{P} \to \mathrm{S}$ *is smooth of relative dimension* n *and* $\mathrm{X} \to \mathrm{P}$ *is a regular immersion of codimension 1, the map* $\nu_{\mathrm{X/P/S}} : \mathscr{L}_\mathrm{Z} \to \mathrm{N}_{\mathrm{X/P}} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{O}_\mathrm{Z}$ *is an isomorphism.*

Proof. — 1. Clear from the local description above.

2. The condition (L(n)) is also clear from the local description above. The rest follows from this and Lemma 2.4.1.1.

3. By the distinguished triangle $\to \mathrm{N}_{\mathrm{X/P}} \to \Omega^1_{\mathrm{P/S}} \otimes_{\mathscr{O}_\mathrm{P}} \mathscr{O}_\mathrm{X} \to \mathrm{L}_{\mathrm{X/S}} \to$, we have an exact sequence $0 \to \mathscr{L}_\mathrm{Z} \to \mathrm{N}_{\mathrm{X/P}} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{O}_\mathrm{Z} \to \Omega^1_{\mathrm{P/S}} \otimes_{\mathscr{O}_\mathrm{P}} \mathscr{O}_\mathrm{Z} \to \Omega^1_{\mathrm{X/S}} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{O}_\mathrm{Z} \to 0$. Since $\Omega^1_{\mathrm{X/S}} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{O}_\mathrm{Z}$ is locally free of rank n, the assertion follows. □

In the following, for a scheme W over X, we put $\mathrm{Z}_\mathrm{W} = \mathrm{Z} \times_\mathrm{X} \mathrm{W}$. By Lemma 3.1.2.2, for an $\mathscr{O}_{\mathrm{Z}_\mathrm{W}}$-module $\mathscr{T}$, the isomorphism (2.4.1.1) defines an isomorphism

$$\textbf{(3.1.2.2)} \qquad \tau_{\mathscr{T},\mathrm{X/S}} : \mathscr{T}\!or_1^{\mathscr{O}_\mathrm{X}}(\mathrm{L}_{\mathrm{X/S}}, \mathscr{T}) \to \mathscr{L}_\mathrm{Z} \otimes_{\mathscr{O}_\mathrm{Z}} \mathscr{T}$$

of $\mathscr{O}_{\mathrm{Z}_\mathrm{W}}$-modules.

The following periodicity result is crucial in the definition of the localized intersection product.

Theorem **3.1.3.** — *Let* S *be a scheme and* X *be a scheme over* S *that is locally a hypersurface over* S *of virtual relative dimension* $n-1$. *Let* W *be a scheme over* X, $\mathscr{F}$ *be an* $\mathscr{O}_X$*-module and* $\mathscr{G}$ *be a complex of* $\mathscr{O}_W$*-modules. Assume that* $\mathscr{F}$ *is of tor-dimension* $\leq m$ *as an* $\mathscr{O}_S$*-module and that* $\mathscr{H}_q(\mathscr{G}) = 0$ *except for* $a \leq q \leq b$. *We put* $q_0 = m + n + b$.

Then we have the following.

1. The $\mathscr{O}_W$*-module* $\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ *is an* $\mathscr{O}_{Z_W}$*-module for* $q \geq q_0$.

2. For $q - 2 \geq q_0$, *the composition*

(3.1.3.1)
$$\alpha_{\mathscr{F},\mathscr{G},X/S} : \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

of the maps

(3.1.3.2)
$$\begin{array}{ccc} \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) & & \\ \downarrow & & \mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \\ \mathscr{T}or_q^{\mathscr{O}_X}\big(L_{X/S} \otimes^L_{\mathscr{O}_X} \mathscr{F}[1], \mathscr{G}\big) & & \uparrow \\ = \mathscr{T}or_{q-1}^{\mathscr{O}_X}\big(L_{X/S}, \mathscr{F} \otimes^L_{\mathscr{O}_X} \mathscr{G}\big) & \longrightarrow & \mathscr{T}or_1^{\mathscr{O}_X}\big(L_{X/S}, \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})\big) \end{array}$$

is an isomorphism of $\mathscr{O}_{Z_W}$*-modules. The first map is induced by the Atiyah class map* $\mathrm{at}_{X/S,\mathscr{F}} : \mathscr{F} \to L_{X/S} \otimes^L_{\mathscr{O}_X} \mathscr{F}[1]$ (1.4.0.2), *the second map is the boundary map of the spectral sequence* $E^2_{p,q} = \mathscr{T}or_p^{\mathscr{O}_X}(L_{X/S}, \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) \Rightarrow \mathscr{T}or_{p+q}^{\mathscr{O}_X}(L_{X/S}, \mathscr{F} \otimes^L_{\mathscr{O}_X} \mathscr{G})$ (1.5.1.4) *and the last upward map is the isomorphism* $\tau_{\mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}),X/S}$ (3.1.2.2).

3. Let P *be a smooth scheme over* S *and* $X \to P$ *be a regular immersion over* S. *Let* $\alpha_{\mathscr{F},\mathscr{G},X/P} : \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \to N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ *be the map* (1.5.4.4). *Then the diagram*

(3.1.3.3)
$$\begin{array}{ccc} \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) & \xrightarrow{\alpha_{\mathscr{F},\mathscr{G},X/S}} & \mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \\ \| & & \downarrow {\scriptstyle \nu_{X/P/S} \otimes 1} \\ \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) & \xrightarrow[\alpha_{\mathscr{F},\mathscr{G},X/P}]{} & N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \end{array}$$

is commutative.

Proof. — 1 and 2. The assertions are local on X. Shrinking X, we take a smooth scheme P over S and a regular immersion $X \to P$ over S. We consider the diagram

$$
\textbf{(3.1.3.4)}\qquad
\begin{array}{ccc}
 & & \Omega^1_{P/S}\otimes_{\mathcal{O}_P}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G}) \\
 & & \uparrow \\
\mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{F},\mathcal{G}) & \xrightarrow{\alpha_{\mathcal{F},\mathcal{G},X/P}} & N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G}) \\
\downarrow & & \uparrow \\
\mathcal{T}or^{\mathcal{O}_X}_q\big(L_{X/S}\otimes^{L}_{\mathcal{O}_X}\mathcal{F}[1],\mathcal{G}\big) & & \\
=\mathcal{T}or^{\mathcal{O}_X}_{q-1}\big(L_{X/S},\mathcal{F}\otimes^{L}_{\mathcal{O}_X}\mathcal{G}\big) & \longrightarrow & \mathcal{T}or^{\mathcal{O}_X}_1\big(L_{X/S},\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G})\big) \\
 & & \uparrow \\
 & & 0.
\end{array}
$$

The right column is the exact sequence defined by the distinguished triangle $\to N_{X/P} \to \Omega_{P/S}\otimes_{\mathcal{O}_P}\mathcal{O}_X \to L_{X/S} \to$. The lower left part is the same as in (3.1.3.2). Since the map $\alpha_{\mathcal{F},\mathcal{G},X/P}$ is induced by the composition of the Atiyah class map $\mathcal{F} \to L_{X/S}\otimes\mathcal{F}[1]$ and the map $L_{X/S}\to N_{X/P}[1]$, the square is commutative.

Now we assume $X \to P$ is a regular immersion of codimension 1. We show that the map $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{F},\mathcal{G}) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G})$ is an isomorphism for $q-2\geq q_0$. By Lemma 1.5.4, the map is the same as the boundary map $d^2_{p,0} : \mathcal{T}or^{\mathcal{O}_X}_p(\mathcal{F},\mathcal{G}) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{p-2}(\mathcal{F},\mathcal{G})$ of the spectral sequence $E^2_{p,q} = \mathcal{T}or^{\mathcal{O}_X}_p(\mathcal{T}or^{\mathcal{O}_P}_q(\mathcal{F},\mathcal{O}_X),\mathcal{G}) \Rightarrow E_{p+q} = \mathcal{T}or^{\mathcal{O}_P}_{p+q}(\mathcal{F},\mathcal{G})$ (1.5.4.3). Since $X\to P$ is a regular immersion of codimension 1, the E^2-term vanishes for $q>1$. By Corollary 1.5.7, the $\mathcal{O}_P$-module $\mathcal{F}$ is of tor-dimension $\leq m+n$. Hence we have $\mathcal{T}or^{\mathcal{O}_P}_r(\mathcal{F},\mathcal{G})=0$ for $r>q_0 = b+n+m$. Therefore the map $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{F},\mathcal{G}) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G})$ is an isomorphism if $q-2\geq q_0$.

Since $\alpha_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{F},\mathcal{G}) \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G})$ is an isomorphism, the top vertical map $N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G}) \to \Omega^1_{P/S}\otimes_{\mathcal{O}_P}\mathcal{T}or^{\mathcal{O}_X}_{q-2}(\mathcal{F},\mathcal{G})$ in (3.1.3.4) is the 0-map. Hence the assertion 1 follows by the definition of Z. Further, since $\nu_{X/P/S} : \mathcal{L}_Z \to N_{X/P}\otimes_{\mathcal{O}_X}\mathcal{O}_Z$ is an isomorphism, the assertion 2 follows.

3. Clear from the commutative diagram (3.1.3.4). □

3.2. K-*theoretic localized intersection product.* — In this subsection, we keep the notation in Theorem 3.1.3. Namely, X is locally a hypersurface of virtual relative dimension $n-1$ over a scheme S, Z is the closed subscheme defined by Ann $\Omega^n_{X/S}$ and $\mathcal{L}_Z$ is the invertible $\mathcal{O}_Z$-module $L^1 i^* L_{X/S}$. For a noetherian scheme Y over Z, let $G(Y)_{/\mathcal{L}_Z}$ denote the cokernel of the endomorphism $1-\mathcal{L}_Z\cdot : G(Y)\to G(Y)$ sending $[\mathcal{G}]$ to $[\mathcal{G}]-[\mathcal{L}_Z\otimes_{\mathcal{O}_Z}\mathcal{G}]$.

Theorem 3.1.3.2 has the following consequence.

Theorem **3.2.1.** — *Let* S *be a noetherian scheme and* X *be a scheme over* S *that is locally a hypersurface of virtual relative dimension* $n-1$ *over* S. *Let* Z *be the closed subscheme defined by* $\mathrm{Ann}\,\Omega^n_{X/S}$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$*-module* $L^1 i^* L_{X/S}$. *Let* V *be a closed subscheme of* X *and* $\mathscr{F}$ *be a coherent* $\mathscr{O}_V$*-module. Let* W *be a noetherian scheme over* X *and* $\mathscr{G} \in D^b(W)_{\mathrm{coh}}$. *Assume that* $\mathscr{F}$ *is of tor-dimension* $\leq m$ *as an* $\mathscr{O}_S$*-module and that* $\mathscr{H}_q(\mathscr{G}) = 0$ *except for* $a \leq q \leq b$. *We put* $q_0 = m + n + b$ *and* $T = V \times_X W$. *Then,*

1. For $q \geq q_0$, $\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ *is a coherent* $\mathscr{O}_{Z_T}$*-module and the class* $[\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G})] \in G(Z_T)_{/\mathscr{L}_Z}$ *depends only on the parity of* q *modulo* 2. *The class*

$$[[\mathscr{F},\mathscr{G}]]_X = (-1)^q\big[\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G})\big] + (-1)^{q+1}\big[\mathscr{T}or_{q+1}^{\mathscr{O}_X}(\mathscr{F},\mathscr{G})\big] \tag{3.2.1.1}$$

$\in G(Z_T)_{/\mathscr{L}_Z}$ *is independent of* q.

2. For an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *of coherent* $\mathscr{O}_V$*-modules, we have*

$$[[\mathscr{F},\mathscr{G}]] = [[\mathscr{F}',\mathscr{G}]] + [[\mathscr{F}'',\mathscr{G}]].$$

3. Let F *be an increasing filtration on* $\mathscr{G}$. *Assume that* $F_q\mathscr{G}$ *is acyclic for sufficiently small* q, $\mathscr{G}/F_q\mathscr{G}$ *is acyclic for sufficiently large* q *and that* $\mathrm{Gr}^F_q\mathscr{G} \in D^b(W)_{\mathrm{coh}}$ *for all* q. *Then we have*

$$[[\mathscr{F},\mathscr{G}]]_X = \sum_q \big[\big[\mathscr{F}, \mathrm{Gr}^F_q\mathscr{G}\big]\big]_X.$$

In particular, for an exact sequence $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ *of coherent* $\mathscr{O}_W$*-modules, we have*

$$[[\mathscr{F},\mathscr{G}]] = [[\mathscr{F},\mathscr{G}']] + [[\mathscr{F},\mathscr{G}'']].$$

4. If W *is also a closed subscheme of* X, $\mathscr{G}$ *is a coherent* $\mathscr{O}_W$*-module and if* $\mathscr{G}$ *is of finite tor-dimension as an* $\mathscr{O}_S$*-module, we have* $[[\mathscr{F},\mathscr{G}]]_X = [[\mathscr{G},\mathscr{F}]]_X$.

Proof. — 1. Clear from Theorem 3.1.3.

2. We have a long exact sequence

$$\begin{array}{llllll}
\longrightarrow & \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}',\mathscr{G}) & \xrightarrow{a} & \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) & \longrightarrow & \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}'',\mathscr{G}) \\
\longrightarrow & \mathscr{T}or_{q-1}^{\mathscr{O}_X}(\mathscr{F}',\mathscr{G}) & \longrightarrow & \mathscr{T}or_{q-1}^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) & \longrightarrow & \mathscr{T}or_{q-1}^{\mathscr{O}_X}(\mathscr{F}'',\mathscr{G}) \\
\longrightarrow & \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F}',\mathscr{G}) & \xrightarrow{b} & \mathscr{T}or_{q-2}^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) & \longrightarrow & \quad .
\end{array}$$

Since the canonical map $\alpha_{\mathscr{F},\mathscr{G},X/S}$ is functorial, it induces an isomorphism $\mathrm{Im}\, a \to \mathrm{Im}\, b \otimes \mathscr{L}_Z$. Hence the equality follows.

3. Assume $F^q\mathscr{G}$ is acyclic for $q \leq a$ and $\mathscr{G}/F^q\mathscr{G}$ is acyclic for $q \geq b$. By induction on $b - a$, it is reduced to the case where $a = -1$ and $b = 1$. In other words, it is sufficient to show an equality $[[\mathscr{F},\mathscr{G}]] = [[\mathscr{F},\mathscr{G}']] + [[\mathscr{F},\mathscr{G}'']]$ for an exact sequence $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ of complexes. It is proved similarly as in 2.

4. Clear from the definition. □

Definition **3.2.2.** — *Let* S *be a regular noetherian scheme of finite dimension and* X *be a scheme over* S *that is locally a hypersurface of virtual relative dimension* $n-1$ *over* S. *Let* Z *be the closed subscheme defined by* Ann $\Omega^n_{X/S}$ *and* $\mathscr{L}_Z$ *is the invertible* $\mathscr{O}_Z$*-module* $L^1 i^* L_{X/S}$. *Let* V *be a closed subscheme of* X *and* W *be a noetherian scheme over* X *and put* $T = V \times_X W$.

We call the bilinear map

$$\textbf{(3.2.2.1)} \qquad [[\ ,\ ]]_X : G(V) \times G(W) \longrightarrow G(Z_T)_{/\mathscr{L}_Z}$$

sending $([\mathscr{F}], [\mathscr{G}])$ *to* $[[\mathscr{F}, \mathscr{G}]]_X$ (3.2.1.1) *the* localized intersection product *on* X. *We put* $[[V, W]]_X = [[\mathscr{O}_V, \mathscr{O}_W]]_X$.

The localized product is related to the usual intersection product in the following way.

Proposition **3.2.3.** — *Let the notation be the same as in Definition 3.2.2. Let* P *be a smooth scheme over* S *and* $X \to P$ *be a regular immersion of codimension 1. Let* $G(T)_{/N_{X/P}}$ *denote the cokernel* $\mathrm{Coker}(1 - [N_{X/P}] : G(T) \to G(T))$.

Then, the canonical map $G(Z_T) \to G(T)$ *induces a map* $G(Z_T)_{/\mathscr{L}_Z} \to G(T)_{/N_{X/P}}$. *Further we have a commutative diagram*

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{[[\ ,\ ]]_X} & G(Z_T)_{/\mathscr{L}_Z} \\ \Big\| & & \downarrow \\ G(V) \times G(W) & \xrightarrow{(\ ,\ )_P} & G(T)_{/N_{X/P}}. \end{array}$$

Proof. — By the isomorphism $\nu_{X/P/S} : \mathscr{L}_Z \to N_{X/P} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ (3.1.2.1), the canonical map $G(Z_T) \to G(T)$ induces a map $G(Z_T)_{/\mathscr{L}_Z} \to G(T)_{/N_{X/P}}$.

We show the equality $(\mathscr{F}, \mathscr{G})_P = [[\mathscr{F}, \mathscr{G}]]_X$ in $G(T)_{/N_{X/P}}$ for a coherent $\mathscr{O}_V$-module $\mathscr{F}$ and a coherent $\mathscr{O}_W$-module $\mathscr{G}$. We consider the spectral sequence $E^2_{p,q} = \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{T}\!or_q^{\mathscr{O}_P}(\mathscr{F}, \mathscr{O}_X), \mathscr{G}) \Rightarrow \mathscr{T}\!or_{p+q}^{\mathscr{O}_P}(\mathscr{F}, \mathscr{G})$ (1.5.4.3). Since $E^2_{p,q} = 0$ for $q \neq 0, 1$, we have a long exact sequence

$$\to \mathscr{T}\!or_p^{\mathscr{O}_P}(\mathscr{F}, \mathscr{G}) \to \mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \to \mathscr{T}\!or_{p-2}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \otimes N_{X/P} \to .$$

For $p > m = n + \dim S$, we have $\mathscr{T}\!or_p^{\mathscr{O}_P}(\mathscr{F}, \mathscr{G}) = 0$. Hence we have $(\mathscr{F}, \mathscr{G})_P = \sum_{p=0}^m (-1)^p [\mathscr{T}\!or_p^{\mathscr{O}_P}(\mathscr{F}, \mathscr{G})]$ is equal to

$$\begin{aligned} &\sum_{p=0}^{m+1} (-1)^p \big[\mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})\big] - \sum_{p=0}^{m-1} (-1)^p \big[\mathscr{T}\!or_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \otimes N_{X/P}\big] \\ &= (-1)^m \big[\mathscr{T}\!or_m^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})\big] + (-1)^{m+1} \big[\mathscr{T}\!or_{m+1}^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \otimes N_{X/P}\big] \\ &= [[\mathscr{F}, \mathscr{G}]]_X \end{aligned}$$

in $G(T)_{/N_{X/P}}$. □

For a flat hypersurface, the localized intersection product commutes with base change in the following sense.

Lemma **3.2.4.** — *Let* X *be locally a flat hypersurface of virtual relative dimension* $n-1$ *over a scheme* S *and* V *be a closed subscheme of* X. *Let* $i : \mathrm{Z} \to \mathrm{X}$ *be the closed immersion defined by the ideal* $\mathrm{Ann}\Omega^n_{\mathrm{X/S}}$ *and* $\mathscr{L}_\mathrm{Z}$ *be the invertible* $\mathscr{O}_\mathrm{Z}$*-module* $\mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$ *as in Theorem* 3.1.3. *Let* $\mathrm{S}' \to \mathrm{S}$ *be a map of schemes.*

1. The base change $\mathrm{X}' = \mathrm{X} \times_\mathrm{S} \mathrm{S}'$ *is a flat hypersurface over* S'. *The closed immersion* $i' : \mathrm{Z}' \to \mathrm{X}'$ *defined by the ideal* $\mathrm{Ann}\Omega^n_{\mathrm{X}'/\mathrm{S}'}$ *is the base change of* $i : \mathrm{Z} \to \mathrm{X}$ *and the invertible* $\mathscr{O}_{\mathrm{Z}'}$*-module* $\mathrm{L}^1 i'^* \Omega^1_{\mathrm{X}'/\mathrm{S}'}$ *is the pull-back of* $\mathscr{L}_\mathrm{Z}$.

2. Assume S *and* S′ *are regular noetherian of finite dimension. Let* V *be a closed subscheme of* X *and* $\mathscr{F}$ *be a coherent* $\mathscr{O}_\mathrm{V}$*-module and assume* $\mathscr{F}$ *is flat as an* $\mathscr{O}_\mathrm{S}$*-module, We put* $\mathrm{V}' = \mathrm{V} \times_\mathrm{X} \mathrm{X}'$ *and let* $\mathscr{F}'$ *be the* $\mathscr{O}_{\mathrm{V}'}$*-module* $\mathscr{F} \otimes_{\mathscr{O}_\mathrm{S}} \mathscr{O}_{\mathrm{S}'}$. *Let* W *be a noetherian scheme over* X′ *and put* $\mathrm{T} = \mathrm{V} \times_\mathrm{X} \mathrm{W}$. *Then the two maps*

$$[[\mathscr{F},\ ]]_\mathrm{X}, [[\mathscr{F}',\ ]]_{\mathrm{X}'} : \mathrm{G(W)} \longrightarrow \mathrm{G}(\mathrm{Z}_\mathrm{T})_{/\mathscr{L}_\mathrm{Z}}$$

are equal.

Proof. — 1. Clear.

2. Since $\mathscr{F}' = \mathscr{F} \otimes^\mathrm{L}_{\mathscr{O}_\mathrm{X}} \mathscr{O}_{\mathrm{X}'}$, we have $\mathscr{F} \otimes^\mathrm{L}_{\mathscr{O}_\mathrm{X}} \mathscr{G} = \mathscr{F}' \otimes^\mathrm{L}_{\mathscr{O}_{\mathrm{X}'}} \mathscr{G}$ and the assertion follows. □

Corollary **3.2.5.** — *Let the notation be as in Lemma 3.2.4.2. Assume further that* $\mathrm{W} = \mathrm{X}'$, *the map* $\mathrm{S}' \to \mathrm{S}$ *is a closed immersion and that* $\mathscr{F} = \mathscr{O}_\mathrm{V}$ *is flat as an* $\mathscr{O}_\mathrm{S}$*-module. Then, we have* $\mathrm{T} = \mathrm{V}'$ *and the diagram*

$$\begin{array}{ccc} \mathrm{G(X)} & \xrightarrow{[[\mathrm{V},\ ]]_\mathrm{X}} & \mathrm{G}(\mathrm{Z}_\mathrm{V})_{/\mathscr{L}_\mathrm{Z}} \\ \uparrow & & \uparrow \\ \mathrm{G(X')} & \xrightarrow[{[[\mathrm{V}',\ ]]_{\mathrm{X}'}}]{} & \mathrm{G}(\mathrm{Z}_{\mathrm{V}'})_{/\mathscr{L}_\mathrm{Z}} \end{array}$$

is commutative.

Proof. — Clear from Lemma 3.2.4.2. □

Lemma **3.2.6.** — *Let* S *be a regular noetherian scheme and* $\mathrm{N} \geq 1$ *be an integer. Then* $\mathrm{X} = \mu_{\mathrm{N,S}}$ *is a flat hypersurface over* S *of virtual relative dimension* 0. *The invertible* $\mathscr{O}_\mathrm{Z}$*-module* $\mathscr{L}_\mathrm{Z} = \mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$ *on the closed subscheme* $i : \mathrm{Z} \to \mathrm{X}$ *defined by* $\mathrm{Ann}\Omega^1_{\mathrm{X/S}}$ *is trivial. We regard* S *as a closed subscheme of* $\mathrm{X} = \mu_{\mathrm{N,S}}$ *by the unit section* $i_1 : \mathrm{S} \to \mathrm{X}$. *Then, the composition*

$$\mathrm{G(X)} \xrightarrow{[[\mathrm{S},\ ]]_\mathrm{X}} \mathrm{G}(\mathrm{Z}_\mathrm{S})_{/\mathscr{L}_\mathrm{Z}} = \mathrm{G}(\mathrm{Z}_\mathrm{S}) \xrightarrow{i_*} \mathrm{G(S)}$$

is the 0-map.

Proof. — The closed subscheme Z is defined by the ideal (N). To show $\mathscr{L}_Z$ is trivial, we may assume $S = \mathrm{Spec}\,\mathbf{Z}$ by Lemma 3.2.4.1. The assertion is clear in this case.

We show that the composition $[[S, \ ]]_X : G(X) \to G(S)$ is equal to the composition $G(X) \to G(\mathbf{G}_{m,S}) \xrightarrow{i_1^*} G(S)$ where $i_1 : S \to \mathbf{G}_{m,S}$ is the unit section. It is sufficient to apply Proposition 3.2.3 by taking $S \to X \to \mathbf{G}_{m,S} \to S$ as $V \to X = W \to P \to S$.

We show that the composition $G(X) \to G(\mathbf{G}_{m,S}) \to G(S)$ is the 0-map. Let t be the coordinate of $\mathbf{G}_{m,S}$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Since $0 \to \mathscr{O}_{\mathbf{G}_{m,S}} \xrightarrow{(t-1)\times} \mathscr{O}_{\mathbf{G}_{m,S}} \to \mathscr{O}_S \to 0$ is a resolution of $\mathscr{O}_S$ by free $\mathscr{O}_{\mathbf{G}_{m,S}}$-modules, we have a quasi-isomorphism $[\mathscr{F} \xrightarrow{(t-1)\times} \mathscr{F}] \to \mathscr{F} \otimes^{L}_{\mathscr{O}_{\mathbf{G}_{m,S}}} \mathscr{O}_S$. Hence the class $i^*[\mathscr{F}] \in G(S)$ is equal to the image of $0 = [\mathscr{F}] - [\mathscr{F}] \in G(X)$ by the push-forward map $G(X) \to G(S)$. Thus the assertion follows. □

Example. — Let G be a finite cyclic group of order N and let $\mathbf{Z}[G]$ be the group algebra. We put $S = \mathrm{Spec}\,\mathbf{Z}$ and $X = \mathrm{Spec}\,\mathbf{Z}[G]$. Then we have $X = \mu_{N,S} = \mathrm{Spec}\,\mathbf{Z}[T]/(T^N - 1)$. The unit section $S \to X$ is defined by the augmentation $\mathbf{Z}[G] \to \mathbf{Z}$. By Theorem 3.1.3, for a G-module M, there is an isomorphism $\mathrm{Tor}_q^{\mathbf{Z}[G]}(\mathbf{Z}, M) \to \mathrm{Tor}_{q-2}^{\mathbf{Z}[G]}(\mathbf{Z}, M)$ for $q - 2 > 0$. Since $\mathrm{Tor}_q^{\mathbf{Z}[G]}(\mathbf{Z}, M)$ is equal to the homology group $H_q(G, M)$, the isomorphism is equivalent to the periodicity of the homology of cyclic group [36] Chapitre VIII Section 4.

The Grothendieck group $G(Z_S) = G(\mathbf{Z}/N\mathbf{Z}) \simeq \bigoplus_{p|N} \mathbf{Z}$ is naturally identified with the subgroup of $\mathbf{Q}^\times$ generated by the prime divisors of N. Then the localized intersection product $[[\mathbf{Z}, M]]_{\mathrm{Spec}\mathbf{Z}[G]} \in \mathbf{Q}^\times$ is identified with the Herbrand quotient $\#\hat{H}_0(G, M)/\#H_1(G, M)$.

3.3. *Associativity and projection formula.* — We prepare a technical lemma for the proof of the associativity formula and the projection formula. For a spectral sequence $E = (E^l_{p,q} \Rightarrow E_{p+q})$, let $E[s, t]$ denote the spectral sequence $E^l_{p-s,q-t} \Rightarrow E_{p+q-s-t}$.

Lemma **3.3.1.** — *Let* W *be a noetherian scheme,* T *be a closed subscheme of* W *and* $\mathscr{L}_T$ *be an invertible* $\mathscr{O}_T$*-module. Let* $E = (E^l_{p,q} \Rightarrow E_{p+q})$ *be a spectral sequence of coherent* $\mathscr{O}_W$*-modules. Let* r_0 *and* t *be integers. We assume that* $E^l_{p,q}$ *are* $\mathscr{O}_T$*-modules for* $p + q \geq r_0$ *and* E_r *are* $\mathscr{O}_T$*-modules for* $r \geq r_0$*. We also assume that there exist integers* $a \leq b$ *such that* $E^l_{p,q} = 0$ *unless* $a \leq (t+2)p + tq \leq b$*.*

Let $\alpha^l_{p,q} : E^l_{p,q} \to \mathscr{L}_T \otimes_{\mathscr{O}_T} E^l_{p+t,q-t-2}$ *and* $\alpha_r : E_r \to \mathscr{L}_T \otimes_{\mathscr{O}_T} E_{r-2}$ *be isomorphisms of* $\mathscr{O}_T$*-modules defined for* $p + q - 2 \geq r_0$ *and* $r - 2 \geq r_0$ *respectively. Assume that, for each* $x \in W$*, there exist an open neighborhood* $U \subset W$ *of* x*, an invertible* $\mathscr{O}_U$*-module* $\mathscr{L}_U$*, an isomorphism* $\mathscr{L}_U \otimes_{\mathscr{O}_U} \mathscr{O}_{T\cap U} \to \mathscr{L}_T|_{T\cap U}$ *and a map* $\alpha_U : E|_U \to \mathscr{L}_U \otimes_{\mathscr{O}_U} E|_U[-t, t+2]$ *of spectral sequences compatible with the restrictions of the maps* $\alpha^l_{p,q}|_{T\cap U}$ *and* $\alpha_r|_{T\cap U}$ *for* $p + q - 2 \geq r_0$ *and* $r - 2 \geq r_0$*.*

Then, we have

$$\textbf{(3.3.1.1)} \qquad \sum_{p+q=r,r+1} (-1)^{p+q}\big[\mathrm{E}^{l}_{p,q}\big] = (-1)^r[\mathrm{E}_r] + (-1)^{r+1}[\mathrm{E}_{r+1}]$$

for $r \geq r_0$ *in the cokernel* $\mathrm{G(T)}_{/\mathscr{L}_{\mathrm{T}}} = \mathrm{Coker}(1 - \mathscr{L}_{\mathrm{T}} : \mathrm{G(T)} \to \mathrm{G(T)})$ *of the map sending* $[\mathscr{F}]$ *to* $[\mathscr{F}] - [\mathscr{L}_{\mathrm{T}} \otimes_{\mathscr{O}_{\mathrm{T}}} \mathscr{F}]$.

Proof. — By the isomorphisms $\alpha^{l}_{p,q}$ and α_r, the both sides of (3.3.1.1) are independent of $r \geq r_0$ and we may replace r by a larger integer if necessary. The difference of the both sides is the sum for $m \geq l$ of

$$\begin{aligned} &\sum_{p+q=r,r+1} (-1)^{p+q}\big(\big[\mathrm{E}^{m}_{p,q}\big] - \big[\mathrm{E}^{m+1}_{p,q}\big]\big) \\ = &\sum_{p+q=r,r+1} (-1)^{p+q}\big(\big[\mathrm{Im}\ d^{m}_{p,q}\big] + \big[\mathrm{Im}\ d^{m}_{p+m,q-m+1}\big]\big) \\ = &(-1)^r \left(\sum_{p+q=r} \big[\mathrm{Im}\ d^{m}_{p,q}\big] - \sum_{p+q=r+2} \big[\mathrm{Im}\ d^{m}_{p,q}\big] \right). \end{aligned}$$

Hence it suffices to show that the isomorphisms $\alpha^{l}_{p,q}$ induces isomorphisms $\mathrm{Im}\ d^{m}_{p,q} \to \mathscr{L}_{\mathrm{T}} \otimes \mathrm{Im}\ d^{m}_{p+t,q-t-2}$ for $p + q - 2 > m - l + r_0$.

The assertion is local on W. Hence, replacing W by U, we may drop the subscript U and identify $\mathscr{L}_{\mathrm{T}} = \mathscr{L} \otimes_{\mathscr{O}_{\mathrm{W}}} \mathscr{O}_{\mathrm{T}}$. By induction on $m \geq l$, the map $\alpha^{m}_{p,q} : \mathrm{E}^{m}_{p,q} \to \mathscr{L} \otimes \mathrm{E}^{m}_{p+t,q-t-2}$ is an isomorphism for $p + q - 2 \geq (m - l) + r_0$. Hence the map $\mathrm{Im}\ d^{m}_{p,q} \to \mathscr{L}_{\mathrm{T}} \otimes \mathrm{Im}\ d^{m}_{p+t,q-t-2}$ is an isomorphism if $p + q > m - l + r_0$ as required. □

Proposition **3.3.2.** — *Let* X *be locally a hypersurface of virtual relative dimension* $n - 1$ *over a noetherian scheme* S *and* $i : \mathrm{Z} \to \mathrm{X}$ *be the closed immersion defined by* $\mathrm{Ann}\ \Omega^{n}_{\mathrm{X/S}}$ *and let* $\mathscr{L}_{\mathrm{Z}}$ *be the invertible* $\mathscr{O}_{\mathrm{Z}}$*-module* $\mathrm{L}^1 i^* \Omega^{1}_{\mathrm{X/S}}$. *Let* V *be a closed subscheme of* X *and* $\mathscr{F}$ *be a coherent* $\mathscr{O}_{\mathrm{V}}$*-module. Assume* $\mathscr{F}$ *is of finite tor-dimension as an* $\mathscr{O}_{\mathrm{S}}$*-module.*

Let

$$\begin{array}{ccccc} \mathrm{V} & \longleftarrow & \mathrm{T} & \longleftarrow & \mathrm{T}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{X} & \longleftarrow & \mathrm{W} & \longleftarrow & \mathrm{W}' \end{array}$$

be a cartesian diagram of noetherian schemes over S *and* $\mathscr{G} \in \mathrm{D}^b(\mathrm{W})_{\mathrm{coh}}$ *and* $\mathscr{H} \in \mathrm{D}^b(\mathrm{W}')_{\mathrm{coh}}$. *Assume* $\mathscr{H}$ *is of finite tor-dimension as a complex of* $\mathscr{O}_{\mathrm{W}}$*-modules. Then the map* $(\ ,\mathscr{H})_{\mathrm{W}}$:

$G(Z_T) \to G(Z_{T'})$ *induces a map* $(\ ,\mathscr{H})_W : G(Z_T)_{/\mathscr{L}_Z} \to G(Z_{T'})_{/\mathscr{L}_Z}$ *and we have an equality*

(3.3.2.1) $$([[\mathscr{F},\mathscr{G}]]_X,\mathscr{H})_W = [[\mathscr{F},(\mathscr{G},\mathscr{H})_W]]_X$$

in $G(Z_{T'})_{/\mathscr{L}_Z}$.

Proof. — For an $\mathscr{O}_{Z_T}$-module $\mathscr{T}$, we have a canonical isomorphism $\mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T}or_q^{\mathscr{O}_W}(\mathscr{T},\mathscr{H}) \to \mathscr{T}or_q^{\mathscr{O}_W}(\mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T},\mathscr{H})$ of $\mathscr{O}_{Z_{T'}}$-modules. Hence the map $(\ ,\mathscr{H})_W : G(Z_T)_{/\mathscr{L}_Z} \to G(Z_{T'})_{/\mathscr{L}_Z}$ is well-defined.

We show the equality (3.3.2.1). We consider the spectral sequence $E = (E^2_{p,q} = \mathscr{T}or_p^{\mathscr{O}_W}(\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}),\mathscr{H}) \Rightarrow E_{p+q} = \mathscr{T}or_{p+q}^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}\otimes^{L}_{\mathscr{O}_W}\mathscr{H}))$ (1.5.5.1). We have $([[\mathscr{F},\mathscr{G}]]_X,\mathscr{H})_W = \sum_p(-1)^{p+q}[E^2_{p,q}] + \sum_p(-1)^{p+q+1}[E^2_{p,q+1}]$ for a sufficiently large integer q. Since $[E^2_{p,q}] = [E^2_{p-2,q}]$ for a sufficiently large p, it is further equal to $\sum_{p+q=r,r+1}(-1)^{p+q}[E^2_{p,q}]$ for sufficiently large r. For the left hand side, we have $[[\mathscr{F},\mathscr{G}\otimes^{L}_{\mathscr{O}_W}\mathscr{H}]]_X = (-1)^r[E_r] + (-1)^{r+1}[E_{r+1}]$ for a sufficiently large integer r. Hence it is sufficient to verify that the assumption of Lemma 3.3.1 is satisfied with $t = 0$.

By the assumption that $\mathscr{H}$ is of finite tor-dimension, there exists an integer b such that $E^2_{p,q} = 0$ except for $0 \le p \le b$. By Theorem 3.1.3.1, $E^2_{p,q}$ are $\mathscr{O}_{Z_{T'}}$-modules for sufficiently large q and E_r are $\mathscr{O}_{Z_{T'}}$-modules for sufficiently large r. We consider the maps $\alpha_{\mathscr{F},\mathscr{G},X/S,*} : E^2_{p,q} = \mathscr{T}or_p^{\mathscr{O}_W}(\mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}),\mathscr{H})) \to \mathscr{L}_Z \otimes_{\mathscr{O}_Z} E^2_{p,q-2}$ induced by the Atiyah class maps and the Atiyah class maps $\alpha_{\mathscr{F},\mathscr{G}\otimes^{L}_{\mathscr{O}_W}\mathscr{H},X/S} : E_r = \mathscr{T}or_r^{\mathscr{O}_X}(\mathscr{F},\mathscr{G}\otimes^{L}_{\mathscr{O}_W}\mathscr{H}) \to \mathscr{L}_Z \otimes_{\mathscr{O}_Z} E_{r-2}$ themselves. Let $U \subset X$ be an open subscheme, P be a smooth scheme of relative dimension n over S and $U \to P$ be a regular immersion of codimension 1. Then, by Lemma 1.5.5.1, the Atiyah class map defines a map $\alpha_{U/P} : E|_U \to N_{U/P} \otimes E|_U[0,2]$ (1.5.5.2) of spectral sequences. By the commutative diagram (3.1.3.3), the map $\alpha_{U/P}$ is compatible with the maps $\alpha_{\mathscr{F},\mathscr{G},X/S,*} : E^2_{p,q} \to \mathscr{L}_Z \otimes_{\mathscr{O}_Z} E^2_{p,q-2}$ and $\alpha_{\mathscr{F},\mathscr{G}\otimes^{L}_{\mathscr{O}_W}\mathscr{H},X/S} : E_r \to \mathscr{L}_Z \otimes_{\mathscr{O}_Z} E_{r-2}$. Thus, it suffices to apply Lemma 3.3.1 to show the equality (3.3.2.1). □

Proposition **3.3.3.** — *Let*

$$\begin{array}{ccccccccc} V & \longrightarrow & X & \longleftarrow & W & \longrightarrow & X' & \longleftarrow & V' \\ & & \downarrow & & & & \downarrow & & \\ & & S & & & & S' & & \end{array}$$

be a diagram of noetherian schemes. Assume that $V \to X$ *and* $V' \to X'$ *are closed immersions. Assume further that* X *is locally a hypersurface of virtual relative dimension* $n-1$ *over* S *and* X' *is locally a hypersurface of virtual relative dimension* $n'-1$ *over* S'. *Let* $i : Z \to X$ *be the closed subscheme of* X *defined by* $\mathrm{Ann}\,\Omega^n_{X/S}$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$*-module* $L^1 i^*\Omega^1_{X/S}$. *Let*

$i' : Z' \to X'$ *be the closed subscheme of* X' *defined by* $\mathrm{Ann}\ \Omega^{n'}_{X'/S'}$ *and* $\mathscr{L}'_{Z'}$ *be the invertible* $\mathscr{O}_{Z'}$*-module* $L^1 i'^* \Omega^1_{X'/S'}$.

Let Z_1 *be a closed subset of* W. *Assume that the underlying sets of* $Z_T = Z \times_X V \times_X W$ *and* $Z'_{T'} = Z' \times_{X'} V' \times_{X'} W$ *are subsets of* Z_1 *and let* $G(Z_1)_{/\mathscr{L}_Z,\mathscr{L}'_{Z'}}$ *be the cokernel of the map* $(\mathrm{can} \circ ([\mathscr{L}_Z] - 1), \mathrm{can} \circ ([\mathscr{L}'_{Z'}] - 1)) : G(Z_T) \oplus G(Z'_{T'}) \to G(Z_1)$ *so that the canonical maps induce* $G(Z_T)_{/\mathscr{L}_Z} \to G(Z_1)_{/\mathscr{L}_Z,\mathscr{L}'_{Z'}}$ *and* $G(Z_{T'})_{/\mathscr{L}'_{Z'}} \to G(Z_1)_{/\mathscr{L}_Z,\mathscr{L}'_{Z'}}$.

Let $\mathscr{F}$ *be a coherent* $\mathscr{O}_V$*-module and* $\mathscr{F}'$ *be a coherent* $\mathscr{O}_{V'}$*-module. Assume* $\mathscr{F}$ *is of finite tor-dimension as an* $\mathscr{O}_S$*-module and* $\mathscr{F}'$ *is of finite tor-dimension as an* $\mathscr{O}_{S'}$*-module. Let* $\mathscr{G} \in D^b(W)_{\mathrm{coh}}$. *Assume that the complex* $\mathscr{G}$ *is of finite tor-dimension as a complex of* $\mathscr{O}_X$*-modules and as a complex of* $\mathscr{O}_{X'}$*-modules so that the maps* $(\ ,\mathscr{G})_{X'} : G(X') \to G(W)$ *and* $(\ ,\mathscr{G})_X : G(X) \to G(W)$ *are defined. Then we have an equality*

$$[[\mathscr{F}, (\mathscr{F}', \mathscr{G})_{X'}]]_X = [[\mathscr{F}', (\mathscr{F}, \mathscr{G})_X]]_{X'}$$

in $G(Z_1)_{/\mathscr{L}_Z,\mathscr{L}'_{Z'}}$.

Proof. — By Theorem 3.2.1.3, we have $[[\mathscr{F}, (\mathscr{F}', \mathscr{G})_{X'}]]_X = [[\mathscr{F}, \mathscr{G} \otimes^L_{\mathscr{O}_{X'}} \mathscr{F}']]$ and $[[\mathscr{F}', (\mathscr{F}, \mathscr{G})_X]]_{X'} = [[\mathscr{F}', \mathscr{F} \otimes^L_{\mathscr{O}_X} \mathscr{G}]]$. Hence it follows from the isomorphism $\mathscr{T}or_r^{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}' \otimes^L_{\mathscr{O}_{X'}} \mathscr{G}) \to \mathscr{T}or_r^{\mathscr{O}_{X'}}(\mathscr{F}', \mathscr{F} \otimes^L_{\mathscr{O}_X} \mathscr{G})$ (1.5.2.2). □

In the proof of conductor formula, we will use the following special cases of Propositions 3.3.2 and 3.3.3.

Corollary **3.3.4.** — *Let* S *be a regular noetherian scheme of finite dimension and* X *be a scheme of finite type over* S *that is locally a hypersurface over* S. *Let* $f : W \to X$ *be a morphism of noetherian schemes.*

1. Let $g : W' \to W$ *be a morphism of finite tor-dimension of noetherian schemes over* X. *Then, for* $\Gamma \in G(X)$, *we have an equality*

$$g^*[[\Gamma, W]]_X = [[\Gamma, W']]_X.$$

Here $[[\ , W]]_X : G(X) \to G(Z_W)_{/\mathscr{L}_Z}$ *and* $[[\ , W']]_X : G(X) \to G(Z_{W'})_{/\mathscr{L}_Z}$ *denotes the localized intersection product respectively and* $g^* : G(Z_W)_{/\mathscr{L}_Z} \to G(Z_{W'})_{/\mathscr{L}_Z}$ *in the left hand side denotes the pull-back defined by* Lg^*.

2. Let $g : W' \to W$ *be a morphism of noetherian scheme and* V *be a closed subscheme of* X. *Assume* W *is regular of finite dimension so that the functor* $\otimes^L_{\mathscr{O}_W}$ *induces an intersection product* $(\ ,\ )_W : G(Z_T)_{/\mathscr{L}_Z} \times G(W') \to G(Z_{T'})_{/\mathscr{L}_Z}$. *Then, for* $\Gamma \in G(W')$, *we have*

$$[[V, \Gamma]]_X = ([[V, W]]_X, \Gamma)_W.$$

In each side, $[[V,\ ]]_X : G(W') \to G(Z_{T'})_{/\mathscr{L}_Z}$ *and* $[[V,\ ]]_X : G(W) \to G(Z_T)_{/\mathscr{L}_Z}$ *denotes the localized intersection product respectively and* $([[V, W]]_X,\ )_W : G(W') \to G(Z_{T'})_{/\mathscr{L}_Z}$ *in the right hand side denotes the intersection product above.*

3. Let S' *be another regular noetherian scheme and* X' *be locally a hypersurface over* S'. *Let* $g : W \to X'$ *be a flat morphism,* V' *be a closed subscheme of* X' *and put* $W' = W \times_{X'} V'$. *Assume that* $f : W \to X$ *is a morphism of finite tor-dimension, that the closed subset* $Z_{W'} = Z \times_X W'$ *of* W' *is set-theoretically a subset of* $Z'_{W'} = Z' \times_{X'} W'$ *and that we have* $G(Z_{W'})_{/\mathscr{L}_Z} = G(Z_{W'})$ *and* $G(Z'_{W'})_{/\mathscr{L}'_{Z'}} = G(Z'_{W'})$. *Then, for* $\Gamma \in G(X)$, *we have*

$$[[\Gamma, W']]_X = [[V', f^*\Gamma]]_{X'}.$$

In each side, $[[\ , W']]_X : G(X) \to G(Z_{W'})_{/\mathscr{L}_Z}$ *and* $[[V',\ ]]_{X'} : G(W) \to G(Z'_{W'})_{/\mathscr{L}'_{Z'}}$ *denotes the localized intersection product respectively and* $f^* : G(X) \to G(W)$ *in the right hand side denotes the pull-back.*

Proof. — 1. It is sufficient to show the equality $g^*[[\mathscr{F}, W]]_X = [[\mathscr{F}, W']]_X$ for a coherent $\mathscr{O}_X$-module $\mathscr{F}$. This is the special case of Proposition 3.3.2 where $\mathscr{G} = \mathscr{O}_W$ and $\mathscr{H} = \mathscr{O}_{W'}$.

2. It is sufficient to show the equality $[[V, \mathscr{H}]]_X = ([[V, W]]_X, \mathscr{H})_W$ for a coherent $\mathscr{O}_{W'}$-module $\mathscr{H}$. This is the special case of Proposition 3.3.2 where $\mathscr{F} = \mathscr{O}_V$ and $\mathscr{G} = \mathscr{O}_W$.

3. It is sufficient to show the equality $[[\mathscr{F}, W']]_X = [[V', Lf^*\mathscr{F}]]_{X'}$ for a coherent $\mathscr{O}_X$-module $\mathscr{F}$. By the flatness of $W \to X'$, we have $Lg^*\mathscr{O}_{V'} = \mathscr{O}_{W'}$. By the assumption, $G(Z'_{W'})_{/\mathscr{L}_Z, \mathscr{L}'_{Z'}}$ in the notation Proposition 3.3.3 is equal to $G(Z'_{W'})$. Hence this is the special case of Proposition 3.3.3 where $\mathscr{G} = \mathscr{O}_W$ and $\mathscr{F}' = \mathscr{O}_{V'}$. □

Proposition **3.3.5.** — *Let* X *be locally a hypersurface of virtual relative dimension* $n-1$ *over a noetherian scheme* S. *Let* $\mathscr{F}$ *be a coherent* $\mathscr{O}_V$*-module on a closed subscheme* V *of* X. *Assume* $\mathscr{F}$ *is of finite tor-dimension as an* $\mathscr{O}_S$*-module. Let* $i : Z \to X$ *be the closed subscheme of* X *defined by* $\mathrm{Ann}\ \Omega^n_{X/S}$ *and put* $\mathscr{L}_Z = L^1 i^*\Omega^1_{X/S}$.

Let $\pi : W' \to W$ *be a proper morphism of noetherian schemes of finite dimension over* X *and* $\mathscr{G} \in D^b(W')_{coh}$. *We put* $T = V \times_X W$ *and* $T' = V \times_X W'$. *Then the map* $\pi_* : G(Z_{T'}) \to G(Z_T)$ *induces a map* $\pi_* : G(Z_{T'})_{/\mathscr{L}_Z} \to G(Z_T)_{/\mathscr{L}_Z}$ *and we have an equality*

(3.3.5.1) $$[[\mathscr{F}, R\pi_*\mathscr{G}]]_X = \pi_*[[\mathscr{F}, \mathscr{G}]]_X$$

in $G(Z_T)_{/\mathscr{L}_Z}$.

Proof. — For an $\mathscr{O}_{Z_{T'}}$-module $\mathscr{T}$, we have a canonical isomorphism $\mathscr{L}_Z \otimes_{\mathscr{O}_Z} R^q\pi_*\mathscr{T} \to R^q\pi_*(\mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{T})$ of $\mathscr{O}_{Z_T}$-modules. Hence the map $\pi_* : G(Z_{T'})_{/\mathscr{L}_Z} \to G(Z_T)_{/\mathscr{L}_Z}$ is well-defined.

We show the equality (3.3.5.1). The proof is similar to that of (3.3.2.1). By the assumption that W is a noetherian scheme of finite dimension, the condition (ii) in Lemma 1.5.3 is satisfied. Applying Lemma 1.5.5.2, we obtain a spectral sequence $E^2_{p,q} = R^{-p}\pi_* \mathscr{T}or_q^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \Rightarrow \mathscr{T}or_{p+q}^{\mathscr{O}_X}(\mathscr{F}, R\pi_*\mathscr{G})$ (1.5.5.3). We have $[[\mathscr{F}, R\pi_*\mathscr{G}]]_X =$

$(-1)^r[\mathrm{E}_r]+(-1)^{r+1}[\mathrm{E}_{r+1}]$ for a sufficiently large integer r. We also have $\pi_*[[\mathcal{F},\mathcal{G}]]_{\mathrm{X}} = \sum_p(-1)^{p+q}[\mathrm{E}^2_{p,q}]+\sum_p(-1)^{p+q+1}[\mathrm{E}^2_{p,q+1}]$ for a sufficiently large integer q. Similarly as in the proof of (3.3.2.1) it is sufficient to verify the assumption of Lemma 3.3.1.

We consider $\alpha_{\mathcal{F},\mathcal{G},\mathrm{X/S},*} : \mathrm{E}^2_{p,q} = \mathrm{R}^{-p}\pi_*\mathcal{T}or_q^{\mathcal{O}_\mathrm{X}}(\mathcal{F},\mathcal{G}) \to \mathcal{L}_\mathrm{Z}\otimes_{\mathcal{O}_\mathrm{Z}}\mathrm{E}^2_{p,q-2}$ and $\alpha_{\mathcal{F},\mathrm{R}\pi_*\mathcal{G},\mathrm{X/S}} : \mathrm{E}_r = \mathcal{T}or_r^{\mathcal{O}_\mathrm{X}}(\mathcal{F},\mathrm{R}\pi_*\mathcal{G}) \to \mathcal{L}_\mathrm{Z}\otimes_{\mathcal{O}_\mathrm{Z}}\mathrm{E}_{r-2}$. Let $\mathrm{U}\subset\mathrm{X}$ be an open subscheme, P be a smooth scheme of relative dimension n over S and $\mathrm{U}\to\mathrm{P}$ be a regular immersion of codimension 1. Then, by Lemma 1.5.5.2, the Atiyah class map defines a map $\alpha_{\mathrm{U/P}} : \mathrm{E}|_\mathrm{U} \to \mathrm{N}_{\mathrm{U/P}}\otimes\mathrm{E}|_\mathrm{U}[0,2]$ (1.5.5.4) of spectral sequence. By the commutative diagram (3.1.3.3), the map $\alpha_{\mathrm{U/P}}$ is compatible with the maps $\alpha_{\mathcal{F},\mathcal{G},\mathrm{X/S},*} : \mathrm{E}^2_{p,q}\to\mathcal{L}_\mathrm{Z}\otimes_{\mathcal{O}_\mathrm{Z}}\mathrm{E}^2_{p,q-2}$ and $\alpha_{\mathcal{F},\mathrm{R}\pi_*\mathcal{G},\mathrm{X/S}} : \mathrm{E}_r\to\mathcal{L}_\mathrm{Z}\otimes_{\mathcal{O}_\mathrm{Z}}\mathrm{E}_{r-2}$. Thus, it suffices to apply Lemma 3.3.1 to show the equality (3.3.5.1). □

3.4. *Excess intersection formula.* — We prove the excess intersection formula Theorem 3.4.3 and the self-intersection formula Corollary 3.4.4. First, we study the excess conormal complex.

Lemma **3.4.1.** — *Let* $\mathrm{V}\to\mathrm{X}$ *be a closed immersion of schemes over* S *and*

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{i_\mathrm{T}} & \mathrm{W} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{V} & \xrightarrow[i_\mathrm{V}]{} & \mathrm{X} \end{array}$$

be a cartesian diagram of schemes over S*. Assume that* X *is locally a hypersurface of virtual relative dimension* $n-1$*,* V *is locally of complete intersection of virtual relative dimension* $n-c$ *and that the immersion* $i_\mathrm{T}:\mathrm{T}\to\mathrm{W}$ *is a regular immersion of codimension* c'*.*

Let $i:\mathrm{Z}\to\mathrm{X}$ *be the closed subscheme defined by the ideal* $\mathrm{Ann}\Omega^n_{\mathrm{X/S}}$ *and put* $\mathcal{L}_\mathrm{Z} = \mathrm{L}^1 i^*\mathrm{L}_{\mathrm{X/S}}$*. Let* $\mathrm{M}'_{\mathrm{V/X,W}}$ *be the excess conormal complex. Then,*

1. The complex $\mathrm{M}'_{\mathrm{V/X,W}}$ *of* $\mathcal{O}_\mathrm{T}$*-modules satisfies the condition* $(\mathrm{L}(c-c'))$ *in Section 2.4.*

2. On the complement $\mathrm{T}-\mathrm{Z}_\mathrm{T}$ *of* $\mathrm{Z}_\mathrm{T}=\mathrm{T}\times_\mathrm{X}\mathrm{Z}$*, the canonical map* $\mathrm{M}'_{\mathrm{V/X,W}}|_{\mathrm{T}-\mathrm{Z}_\mathrm{T}}\to\mathrm{N}'_{\mathrm{V}-\mathrm{Z}_\mathrm{V}/\mathrm{X}-\mathrm{Z},\mathrm{W}-\mathrm{Z}_\mathrm{W}}$ *is an isomorphism and the excess conormal sheaf* $\mathrm{N}'_{\mathrm{V}-\mathrm{Z}_\mathrm{V}/\mathrm{X}-\mathrm{Z},\mathrm{W}-\mathrm{Z}_\mathrm{W}}$ *is a locally free* $\mathcal{O}_{\mathrm{T}-\mathrm{Z}_\mathrm{T}}$*-module of rank* $c-c'-1$*.*

3. Assume $p>0$ *or* $q\geq c-c'$*. Then, the* $\mathcal{O}_\mathrm{T}$*-module* $\mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$ *is an* $\mathcal{O}_{\mathrm{Z}_\mathrm{T}}$*-module and the map* $\lambda_{\mathrm{M}'_{\mathrm{V/X,W}}}$ (2.4.2.1) *defines an isomorphism*

$$\textbf{(3.4.1.1)}\qquad \lambda_{\mathrm{V/X/S,W}} : \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}} \longrightarrow \mathcal{L}_\mathrm{Z}\otimes_{\mathcal{O}_\mathrm{Z}}\mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$$

of $\mathcal{O}_{\mathrm{Z}_\mathrm{T}}$*-modules. Let* P *be a smooth scheme over* S *and* $\mathrm{X}\to\mathrm{P}$ *be a regular immersion over* S*. Then, the isomorphisms* $\lambda_{\mathrm{V/X/S,W}}$ *and* $\lambda_{\mathrm{V/X/P,W}} : \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}}\to\mathrm{N}_{\mathrm{X/P}}\otimes_{\mathcal{O}_\mathrm{X}}\mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$

(1.7.2.1) *form a commutative diagram*

(3.4.1.2)
$$\begin{array}{ccc} \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}} & \xrightarrow{\lambda_{\mathrm{V/X/S,W}}} & \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathrm{L}^p\Lambda^q \mathrm{M}'_{\mathrm{V/X,W}} \\ \| & & \downarrow{\scriptstyle \nu_{\mathrm{X/P/S}}} \\ \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}} & \xrightarrow[\lambda_{\mathrm{V/X/P,W}}]{} & \mathrm{N}_{\mathrm{X/P}}\otimes_{\mathscr{O}_{\mathrm{X}}} \mathrm{L}^p\Lambda^q \mathrm{M}'_{\mathrm{V/X,W}}. \end{array}$$

Proof. — 1. The assertion is local on T. Hence, we may assume there exists a smooth scheme P of relative dimension n over S and a regular immersion $\mathrm{X} \to \mathrm{P}$ of codimension 1 over S. Then, we have a distinguished triangle $\to (i \circ g)^*\mathrm{N}_{\mathrm{X/P}} \to \mathrm{N}'_{\mathrm{V/P,W}} \to \mathrm{M}'_{\mathrm{V/X,W}} \to$ (1.7.1.2). Since the excess conormal sheaf $\mathrm{N}'_{\mathrm{V/P,W}}$ is locally free of rank $c - c'$, the complex $\mathrm{M}'_{\mathrm{V/X,W}}$ satisfies the condition $(\mathrm{L}(c - c'))$.

2. The map $\mathrm{X} \to \mathrm{S}$ is smooth on the complement of Z. Hence the immersion $\mathrm{V} \to \mathrm{X}$ is a regular immersion of codimension $c - 1$ on the complement of Z_{V}. Thus the assertion follows from Proposition 1.6.4.2.

3. Let $i' : \mathrm{Z}' \to \mathrm{T}$ be the closed immersion defined by $\mathrm{Ann}\Lambda^{c-c'}\mathrm{N}'_{\mathrm{V/X,W}}$. We show that Z' is a closed subscheme of Z_{T} and that the canonical map $\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{L}g^*\mathrm{M}_{\mathrm{V/X}} \to \mathrm{L}(i_{\mathrm{V}} \circ g)^*\mathrm{L}_{\mathrm{X/S}}$ induces an isomorphism $\mathrm{L}^1 i'^*\mathrm{M}'_{\mathrm{V/X,W}} \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathscr{O}_{\mathrm{Z}'}$ of invertible $\mathscr{O}_{\mathrm{Z}'}$-modules. The question is local on T. The inverse image $\mathrm{Z}_{\mathrm{T}} \subset \mathrm{T}$ is defined by the ideal $\mathrm{Ann}(i_{\mathrm{V}} \circ g)^*\Lambda^n\Omega^1_{\mathrm{X/S}}$. Let the notation be as in the proof of 1. Then, the claim follows from the map

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{N}_{\mathrm{X/P}}\otimes_{\mathscr{O}_{\mathrm{X}}}\mathscr{O}_{\mathrm{T}} & \longrightarrow & \mathrm{N}'_{\mathrm{V/P,W}} & \longrightarrow & \mathrm{M}'_{\mathrm{V/X,W}} & \longrightarrow \\ & \| & & \downarrow & & \downarrow & \\ \longrightarrow & \mathrm{N}_{\mathrm{X/P}}\otimes_{\mathscr{O}_{\mathrm{X}}}\mathscr{O}_{\mathrm{T}} & \longrightarrow & \Omega^1_{\mathrm{P/S}}\otimes_{\mathscr{O}_{\mathrm{P}}}\mathscr{O}_{\mathrm{T}} & \longrightarrow & (i_{\mathrm{V}}\circ g)^*\mathrm{L}_{\mathrm{X/S}} & \longrightarrow \end{array}$$

of distinguished triangles.

By Lemma 2.4.2.1, $\mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$ is an $\mathscr{O}_{\mathrm{Z}'}$-module and hence is an $\mathscr{O}_{\mathrm{Z}_{\mathrm{T}}}$-module for $p > 0$. By the isomorphism $\mathrm{L}^1 i'^*\mathrm{M}'_{\mathrm{V/X,W}} \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathscr{O}_{\mathrm{Z}'}$, the isomorphism $\lambda_{\mathrm{M}'_{\mathrm{V/X,W}}} : \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}} \to \mathrm{L}^1 i'^*\mathrm{M}'_{\mathrm{V/X,W}} \otimes_{\mathscr{O}_{\mathrm{Z}'}} \mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$ defines an isomorphism $\lambda_{\mathrm{V/X/S,W}}$. The commutative diagram (3.4.1.2) is clear from the commutative diagram (2.4.1.2). □

We relate the localized intersection product with the derived exterior power of the excess conormal complex.

Proposition **3.4.2.** — *Let* S *be a scheme and* $\mathrm{V} \to \mathrm{X}$ *be a closed immersion of schemes over* S. *Assume that* X *is locally a hypersurface of virtual relative dimension* $n - 1$ *over* S *and* V *is locally of complete intersection of virtual relative dimension* $n - c$ *over* S. *Let*

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{i_{\mathrm{T}}} & \mathrm{W} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{V} & \xrightarrow[i_{\mathrm{V}}]{} & \mathrm{X} \end{array}$$

be a cartesian diagram of schemes over S. *Assume that* W *is a noetherian scheme and that the immersion* $i_{\mathrm{T}} : \mathrm{T} \to \mathrm{W}$ *is a regular immersion of codimension* c'. *Let* $i : \mathrm{Z} \to \mathrm{X}$ *be the closed subscheme defined by the ideal* $\mathrm{Ann}\Omega^n_{\mathrm{X/S}}$ *and put* $\mathscr{L}_{\mathrm{Z}} = \mathrm{L}^1 i^* \mathrm{L}_{\mathrm{X/S}}$. *Let* $\mathrm{M}'_{\mathrm{V/X,W}}$ *be the excess conormal complex. We put* $[\mathrm{L}\Lambda^{c-c'}\mathrm{M}'_{\mathrm{V/X,W}}] = \sum_{p=0}^{c-c'}(-1)^p[\mathrm{L}^p\Lambda^{c-c'}\mathrm{M}'_{\mathrm{V/X,W}}]$ *in* $\mathrm{G}(\mathrm{Z_T})$.

Then, we have an equality

(3.4.2.1) $$[[\mathrm{V},\mathrm{W}]]_{\mathrm{X}} = (-1)^{c-c'}\big[\mathrm{L}\Lambda^{c-c'}\mathrm{M}'_{\mathrm{V/X,W}}\big].$$

in $\mathrm{G}(\mathrm{Z_T})_{/\mathscr{L}_{\mathrm{Z}}}$. *In particular, if* $\mathrm{W} = \mathrm{T}$ *is a scheme over* V, *we have*

(3.4.2.2) $$[[\mathrm{V},\mathrm{W}]]_{\mathrm{X}} = (-1)^{c}\big[\mathrm{L}\Lambda^{c}\mathrm{L}g^*\mathrm{M}_{\mathrm{V/X}}\big]$$

in $\mathrm{G}(\mathrm{Z_W})_{/\mathscr{L}_{\mathrm{Z}}}$.

Proof. — Proof is similar to Propositions 3.3.2 and 3.3.5. Let E be the spectral sequence $\mathrm{E}^1_{p,q} = \mathrm{L}^{2p+q}\Lambda^{-p}\mathrm{M}'_{\mathrm{V/X,W}} \Rightarrow \mathrm{E}_{p+q} = \mathscr{T}or^{\mathscr{O}_{\mathrm{X}}}_{p+q}(\mathscr{O}_{\mathrm{V}}, \mathscr{O}_{\mathrm{W}})$ (1.6.4.3). We have $(-1)^{c-c'}[\mathrm{L}\Lambda^{c-c'}\mathrm{M}'_{\mathrm{V/X,W}}] = \sum_q(-1)^{c-c'+q}[\mathrm{E}^1_{-(c-c'),q}]$. Since $\mathrm{L}^p\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}} = 0$ except for $\max(0, q-(c-c')) \le p \le q$, we have $\mathrm{E}_1^{p,q} = 0$ except for $-(c-c') \le 3p+q \le 0$. We have $[\mathrm{E}^1_{p,q}] = [\mathrm{E}^1_{p+1,q-3}]$ for $p \le -(c-c')$ by the isomorphism $\lambda_{\mathrm{V/X/S,W}} : \mathrm{L}^{p+1}\Lambda^{q+1}\mathrm{M}'_{\mathrm{V/X,W}} \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathrm{L}^q\Lambda^q\mathrm{M}'_{\mathrm{V/X,W}}$ (3.4.1.1) for $q \ge c-c'$. Hence, it is further equal to $\sum_{p+q=r,r+1}(-1)^{p+q}[\mathrm{E}^1_{p,q}]$ for sufficiently large r. On the other hand, we have $[[\mathrm{V},\mathrm{W}]]_{\mathrm{X}} = (-1)^r[\mathrm{E}_r] + (-1)^{r+1}[\mathrm{E}_{r+1}]$ for sufficiently large r. Thus it suffices to show that the assumption of Lemma 3.3.1 is satisfied with $t = 1$.

We have the isomorphisms $\alpha_{\mathscr{O}_{\mathrm{V}},\mathscr{O}_{\mathrm{W}},\mathrm{X/S}} : \mathrm{E}_r = \mathscr{T}or^{\mathscr{O}_{\mathrm{X}}}_r(\mathscr{O}_{\mathrm{V}}, \mathscr{O}_{\mathrm{W}}) \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathscr{T}or^{\mathscr{O}_{\mathrm{X}}}_{r-2}(\mathscr{O}_{\mathrm{V}}, \mathscr{O}_{\mathrm{W}})$ (3.1.3.1) and $\lambda_{\mathrm{V/X/S,W}} : \mathrm{E}^1_{p,q} = \mathrm{L}^{2p+q}\Lambda^{-p}\mathrm{M}'_{\mathrm{V/X,W}} \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathrm{L}^{2p+q-1}\Lambda^{-p-1}\mathrm{M}'_{\mathrm{V/X,W}}$ (3.4.1.1). Let $\mathrm{U} \subset \mathrm{X}$ be an open subscheme, P be a smooth scheme of relative dimension n over S and $\mathrm{U} \to \mathrm{P}$ be a regular immersion of codimension 1 over S. Then, we have a map of spectral sequences $\alpha_{\mathrm{U/P}} : \mathrm{E}|_{\mathrm{U}} \to \mathrm{N}_{\mathrm{U/P}} \otimes \mathrm{E}|_{\mathrm{U}}[-1, 3]$ (1.7.2.2). By the commutative diagrams (3.1.3.3) and (3.4.1.2), the map $\alpha_{\mathrm{U/P}}$ is compatible with $\alpha_{\mathscr{O}_{\mathrm{V}},\mathscr{O}_{\mathrm{W}},\mathrm{X/S}} : \mathrm{E}_r \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathrm{E}_{r-2}$ and $\lambda_{\mathrm{V/X/S,W}} : \mathrm{E}^1_{p,q} \to \mathscr{L}_{\mathrm{Z}} \otimes_{\mathscr{O}_{\mathrm{Z}}} \mathrm{E}^1_{p+1,q-3}$. Thus it suffices to apply Lemma 3.3.1. □

To state the excess intersection formula, Theorem 3.4.3, we introduce further notation. We keep the notation in Proposition 3.4.2. We assume further that the regular noetherian scheme S is equidimensional of finite dimension. If the conormal complex $M_{V/X}$ satisfies the condition (G) in Section 2.4, then the excess conormal complex $M'_{V/X,W}$ also satisfies the condition (G) and the localized Chern class $c_{c-c'}{}^{T}_{Z_T}(M'_{V/X,W}) \in CH^{c-c'}(Z_T \to T)$ is defined.

We briefly recall the localized intersection product defined by Abbes in [1] Definition 4.4 after slight modification. Let W be a scheme of finite type over X. Assume W is integral and is of dimension p. We put $T = V \times_X W$ and $Z_T = Z \times_X T$. If $T \subsetneq W$, let $\pi : W' \to W$ be the blow-up at T and $T' = T \times_W W'$ be the exceptional divisor. Since the immersion $T' \to W'$ is a regular immersion of codimension 1, the localized Chern class $c_{c-1}{}^{T'}_{Z_{T'}}(M'_{V/X,W'}) \in CH^{c-1}(Z_{T'} \to T')$ of the excess conormal complex $M'_{V/X,W'}$ is defined. Then the localized intersection product $(V, W)_{X,\mathrm{loc}} \in CH_{p-c}(Z_T)$ is defined by

$$\textbf{(3.4.3.1)} \qquad (V, W)_{X,\mathrm{loc}} = \begin{cases} (-1)^c c_c{}^{V}_{Z_V}(M_{V/X}) \cap [W] & \text{if } T = W \\ \pi_{Z_T *}\big((-1)^{c-1} c_{c-1}{}^{T'}_{Z_{T'}}(M'_{V/X,W'}) \cap [T']\big) & \text{if } T \subsetneq W. \end{cases}$$

If the closed immersion $T \to W$ is a regular immersion of codimension c' and $M'_{V/X,W}$ denotes the excess conormal complex, we have

$$\textbf{(3.4.3.2)} \qquad (V, W)_{X,\mathrm{loc}} = (-1)^{c-c'} c_{c-c'}{}^{T}_{Z_T}(M'_{V/X,W}) \cap [T].$$

The equality (3.4.3.2) is called the localized excess intersection formula (cf. [1] Proposition 4.11).

For an integer $p \geq 0$, let $Z_p(W)$ be the free abelian group generated by the classes of integral closed subscheme of dimension p. Thus we obtain a collection of maps $(V, \ )_{X,\mathrm{loc}} : Z_p(W) \to CH_{p-c}(T)$ sending the closed integral subscheme W' to $(V, W')_{X,\mathrm{loc}}$ for morphisms $W \to X$ of finite type over S. The localized intersection product $(V, \ )_{X,\mathrm{loc}}$ is characterized by the localized excess intersection formula (3.4.3.2) and the projection formula $(V, \pi_* W)_{X,\mathrm{loc}} = \pi_*(V, W)_{X,\mathrm{loc}}$.

Let $T = V \times_X W$ and $F_p(G(Z_T)_{/\mathscr{L}_Z})$ denote the filtration on $G(Z_T)_{/\mathscr{L}_Z}$ induced by the topological filtration on $G(Z_T)$.

Theorem **3.4.3.** — *Let* X *be locally a hypersurface of virtual relative dimension* $n-1$ *over a equidimensional regular noetherian scheme* S *of finite dimension and* $j : V \to X$ *be a closed subscheme of* X. *Let* Z *be the closed subscheme of* X *defined by the ideal* $\mathrm{Ann}\ \Omega^n_{X/S}$. *Assume that* V *is locally of complete intersection over* S *of relative dimension* $n-c$.

Let W *be a scheme over* X *and assume* W *is of finite type over a regular noetherian scheme of finite dimension. We put* $T = V \times_X W \to W$.

1. The localized intersection product $[[V, \ ]]_X : G(W) \to G(Z_T)_{/\mathscr{L}_Z}$ *sends the topological filtration* $F_pG(W)$ *to* $F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$ *for* $p \geq 0$.

2. Assume further that the conormal complex $M_{V/X}$ *satisfies the condition* (G) *in Section 2.4. Then the map induced by* $[[V,\ ]]_X$ *on the graded quotients sits in the commutative diagram*

$$\begin{array}{ccc} Z_p(W) & \xrightarrow{(V,\ )_{X,\mathrm{loc}}} & CH_{p-c}(Z_T) \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathrm{Gr}^F_p G(W) & \xrightarrow{[[V,\ ]]_X} & \mathrm{Gr}^F_{p-c}(G(Z_T)_{/\mathscr{L}_Z}). \end{array}$$

Proof. — The proof is similar to those of Propositions 2.2.2 and 2.4.4. We use the notation of the proof of Proposition 2.2.2. By the same argument as loc.cit. and by the projection formula Proposition 3.3.5 and [1] Proposition 4.6 (a), it suffices to show the following: Assume that W is of dimension p and that either T is equal to W or T is a Cartier divisor of W. Then, the localized intersection product $[[V,W]]_X$ is in $F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$ and, if $M_{V/X}$ satisfies the condition (G), the class of $[[V,W]]_X$ in $\mathrm{Gr}^F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$ is equal to the image of $(V,W)_{X,\mathrm{loc}} \in CH_{p-c}(Z_T)$

First, we assume $T = W$. Then by (3.4.2.2), we have $[[V,W]]_X = (-1)^c[L\Lambda^c Lg^* M_{V/X}]$ in $G(Z_W)_{/\mathscr{L}_Z}$. Hence, by Proposition 2.4.4, $[[V,W]]_X$ is in $F_{p-c}(G(Z_W)_{/\mathscr{L}_Z})$ and, if $M_{V/X}$ satisfies the condition (G), the class of $[[V,W]]_X$ in $\mathrm{Gr}^F_{p-c}(G(Z_W)_{/\mathscr{L}_Z})$ is equal to the image of $(-1)^c c_{c\,Z_V}^{\ V}(M_{V/X}) \cap [W]$. Thus the assertion follows from the first equality in (3.4.3.1) in this case.

Next, we consider the case where T is a Cartier divisor of W. Then by (3.4.2.1), we have $[[V,W]]_X = (-1)^{c-1}[L\Lambda^{c-1}M'_{V/X,W}]$ in $G(Z_T)_{/\mathscr{L}_Z}$. Hence, by Proposition 2.4.4, $[[V,W]]_X$ is in $F_{(p-1)-(c-1)}(G(Z_T)_{/\mathscr{L}_Z})$ and, if $M_{V/X}$ satisfies the condition (G), the class of $[[V,W]]_X$ in $\mathrm{Gr}^F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$ is equal to the image of $(-1)^{c-1}c_{c-1\,Z_T}^{\ T}(M'_{V/X,W}) \cap [T]$. Thus the assertion follows from the excess intersection formula in (3.4.3.2) in this case. □

Corollary **3.4.4.** — *Let the notation be the same as in Theorem 3.4.3. Assume* W *is of dimension* p *and that the closed immersion* $T \to W$ *is a regular immersion of codimension* c'. *Assume also that the conormal complex* $M_{V/X}$ *satisfies the condition* (G).

Then for the class of $[[V,W]]_X \in F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$ *and for the image of* $(-1)^{c-c'}c_{c-c'\,Z_T}^{\ T}(M'_{V/X,W}) \cap [T] \in CH_{p-c}(Z_T)$, *we have an equality*

$$[[V,W]]_X = (-1)^{c-c'}c_{c-c'\,Z_T}^{\ T}(M'_{V/X,W}) \cap [T] \qquad \textbf{(3.4.4.1)}$$

in $\mathrm{Gr}^F_{p-c}(G(Z_T)_{/\mathscr{L}_Z})$.

If W *is a scheme over* V, *the class of* $[[V,W]]_X$ *in* $\mathrm{Gr}^F_{p-c}(G(Z_W)_{/\mathscr{L}_Z})$ *is equal to the image of* $(-1)^c c_{c\,Z_V}^{\ V}(M_{V/X}) \cap [W] \in CH_{p-c}(Z_W)$. *In particular, if* $V = W$, *we have an equality*

$$[[V,V]]_X = (-1)^c c_{c\,Z_V}^{\ V}(M_{V/X}) \cap [V] \qquad \textbf{(3.4.4.2)}$$

in $\mathrm{Gr}^F_{p-c}(G(Z_V)_{/\mathscr{L}_Z})$.

We call the equality (3.4.4.1) the localized excess intersection formula and the equality (3.4.4.2) the localized self-intersection formula

Proof. — Similarly as the proof of Theorem 3.4.3 above in the case T is a Cartier divisor, the excess intersection formula (3.4.4.1) follows from Proposition 3.4.2.3 and Proposition 2.4.4. The case $W = T$ is proved in the proof above. □

Corollary **3.4.5.** — *Let* X *be locally a* flat *hypersurface of virtual relative dimension* $n-1$ *over a scheme* S *and* V *be a closed subscheme of* X. *Let* $i : Z \to X$ *be the closed immersion defined by the ideal* $\mathrm{Ann}\,\Omega^n_{X/S}$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$*-module* $L^1 i^* L_{X/S}$ *as in Theorem 3.1.3. We consider the self-product* $X \times_S X$ *as a scheme over* X *with respect to the second projection.*

1. *The scheme* $X \times_S X$ *is locally a hypersurface of virtual relative dimension* $n-1$. *Let* $\tilde{i} : \tilde{Z} \to X \times_S X$ *be the closed subscheme defined by the ideal* $\mathrm{Ann}\,\Omega^n_{X\times_S X/X}$. *Then the intersection* $\tilde{Z} \times_{X\times_S X} X$ *with the diagonal* $\Delta : X \to X \times_S X$ *is* $Z \subset X$ *and the pull-back of the invertible* $\mathscr{O}_{\tilde{Z}}$*-module* $L^1 \tilde{i}^* L_{X\times_S X/X}$ *is* $\mathscr{L}_Z$. *There is a canonical isomorphism* $L_{X/S} \to M_{X/X\times_S X}$.
2. *Further if* S *is equidimensional regular noetherian and of dimension* d, *we have an equality*

$$[[X, X]]_{X\times_S X} = (-1)^n c_{n\,Z}^{\;X}(L_{X/S}) \cap [X]$$

in $\mathrm{Gr}^F_{d-1}(G(Z)_{/\mathscr{L}_Z})$.

Proof. — 1. We obtain an isomorphism $M_{X/X\times_S X} \to L\Delta^* L_{X\times_S X/X} \to L_{X/S}$ by the distinguished triangle $\to L\Delta^* L_{X\times_S X/X} \to L_{X/X} \to L_{X/X\times_S X} \to$. The rest follows immediately from Lemma 3.2.4.

2. It suffices to apply Corollary 3.4.4. □

The image of $[[V, W]]_X$ in $\mathrm{Gr}^F_{p-c}(G(T)_{/\mathscr{L}_Z})$ may be computed using the Segre classes. For a perfect complex $\mathscr{K}$, we put $c(\mathscr{K})^* = c(\mathscr{K}^*) = \sum_i (-1)^i c_i(\mathscr{K})$ as usual.

Corollary **3.4.6.** — *Let* $V \subset X \to S$ *and* $T = V \times_X W \subset W \to X$ *be as in Theorem 3.4.3. Assume* W *is an integral scheme of dimension* p *of finite type over a regular noetherian scheme of finite dimension and* $T \neq W$. *Let* $g : T \to V$ *be the natural map, let* $G(T)_{/\mathscr{L}_Z}$ *denote the cokernel of the map* $[\mathscr{L}_Z] - 1 : G(Z_T) \to G(T)$ *and let* $F_\bullet(G(T)_{/\mathscr{L}_Z})$ *denote the filtration induced by the topological filtration. Then the class of the localized intersection product* $[[V, W]]_X$ *in* $\mathrm{Gr}^F_{p-c}(G(T)_{/\mathscr{L}_Z})$ *is equal to the image of*

$$\{c(Lg^* M_{V/X})^* \cap s(T, W)\}_{\dim p-c} = \sum_{i=0}^{c-1} (-1)^i c_i(Lg^* M_{V/X}) s_{c-i}(T, W)$$

$\in \mathrm{CH}_{p-c}(T)$.

Proof. — Let $\pi : W' \to W$ be the blow-up at T and $D = \pi^{-1}(T) = W' \times_W T$ be the inverse image of T as above. By Proposition 3.3.5.1, we have $[[V, W]]_X = \pi_*[[V, W']]_X$. Since D is a Cartier divisor of W′, by Theorem 3.4.3.2, the class of $[[V, W']]_X$ in $\mathrm{Gr}^F_{p-c}(G(D)_{/\mathscr{L}_Z})$ is equal to the image of $(-1)^{c-1}c_{c-1}(M'_{V/X,W'}) \cap [D] = \{c(M_{V/X})^* c(N_{D/W'})^{*-1} \cap [D]\}_{\dim p-c} \in CH_{p-c}(D)$. Hence the class of $[[V, W]]_X$ in $\mathrm{Gr}^F_{p-c}(G(T)_{/\mathscr{L}_Z})$ is equal to the image of $\{c(Lg^* M_{V/X})^* \pi_*(c(N_{D/W'})^{*-1} \cap [D])\}_{\dim p-c}$. Since $N_{D/W'} = \mathscr{O}_D(-D)$, we have an equality $\pi_*(c(N_{D/W'})^{*-1} \cap [D]) = s(T, W)$. Thus we obtain the required equality. □

4. Logarithmic products

We define and study logarithmic products. In 4.1, after recalling generalities on log schemes, we define a functor [P] on the category of log schemes for an fs-monoid P and introduce the notion of frames. We define log products in Definition 4.2.4 and establish basic properties in 4.2. We study generality on properties of morphisms of log schemes in 4.3 as an application of log products. In 4.4, we study morphisms log locally of complete intersection.

For generalities on log schemes such as the definitions of log smooth morphisms, exact immersions etc., we refer to [23], [25] and [20].

4.1. *Frames.* — We define a functor [P] for an fs-monoid P on the category of fs-log schemes and introduce the notion of frames as a preliminary for the definition of the logarithmic product in the next subsection. It is closely related to the toric stack studied in [21] and [32]. First, we briefly recall generalities on log schemes. Basic references are [23], [25] and [28] Section 1.

In this paper, a monoid means a commutative monoid. For a monoid P, P^{gp} denotes the associated commutative group and $P^\times$ denotes the subgroup of invertible elements. A monoid P is called integral if the canonical map $P \to P^{gp}$ is injective. We will identify an integral monoid P with its image in P^{gp}. A monoid P is called saturated if it is integral and if it is equal to the saturation $P^{sat} = \{x \in P^{gp} | x^n \in P \text{ for some } n \geq 1\}$. A monoid is called an fs-monoid if it is finitely generated and saturated. We regard $\mathscr{O}_X$ as a sheaf of monoids on the etale site of X with respect to the multiplication. An fs-log structure on a scheme X is a morphism $\alpha : M_X \to \mathscr{O}_X$ of sheaves of monoids on the etale site of X satisfying the following conditions (1) and (2).

(1) The induced map $\alpha^{-1}(\mathscr{O}_X^\times) \to \mathscr{O}_X^\times$ is an isomorphism.
(2) For each geometric point $\bar{x}$, there exist an etale neighboorhood U, an fs-monoid P and a morphism of monoids $\beta : P \to \Gamma(U, M_X)$ such that the diagram

$$\begin{array}{ccc} \beta^{-1}\left(M_X^\times|_U\right) & \xrightarrow{\subset} & P_U \\ \downarrow & & \downarrow \beta \\ M_X^\times|_U & \xrightarrow{\subset} & M_X|_U \end{array}$$

is co-cartesian in the category of sheaves of monoids. Here P_U denotes the constant sheaf.

A morphism $\beta : P \to \Gamma(U, M_X)$ of monoids satisfying the condition (2) above is called a chart of M_X on U. The log structure on $M_X|_U$ on U is called the log structure associated to $P \to \Gamma(U, M_X)$. A scheme with an fs-log structure is called an fs-log scheme. In this paper, we only consider fs-log schemes and fs-log structures and we simply call them log schemes and log structures respectively. The condition (1) implies $M_X^\times = \alpha^{-1}(\mathcal{O}_X^\times)$ and that the map $M_X^\times \to \mathcal{O}_X^\times$ is an isomorphism. The log structure $M_X = \mathcal{O}_X^\times$ is called the trivial log structure.

For a monoid P, let $\bar{P}$ denote the quotient $P/P^\times$. The quotient $\bar{P}$ of an fs-monoid P is also an fs-monoid. For a log scheme X, we put $\bar{M}_X = M_X/M_X^\times$. The sheaf M_X is the inverse image of $\bar{M}_X$ by $M_X^{gp} \to \bar{M}_X^{gp}$. For a log scheme X, the monoid $\Gamma(X, M_X)$ is integral and saturated. For a geometric point $\bar{x}$ of X, the stalk $\bar{M}_{X,\bar{x}}$ is an fs-monoid and there exists a section $\bar{M}_{X,\bar{x}} \to M_{X,\bar{x}}$ inducing an isomorphism $\bar{M}_{X,\bar{x}} \times M_{X,\bar{x}}^\times \to M_{X,\bar{x}}$. We say a morphism $f : X \to Y$ of log schemes is *strict* if the induced map $f^*\bar{M}_Y \to \bar{M}_X$ is an isomorphism. If $X \to Y$ is strict, we say that the log structure M_X on X is the pull-back of the log structure M_Y on Y.

A typical example of log scheme is given by a divisor with normal crossings on a regular locally noetherian scheme. Let X be a regular locally noethrian scheme. Recall that we say a divisor D on X has simple normal crossings if its irreducible components are regular and if they meet transversally. More precisely, let $D_i, i \in I$ be the irreducible components of D. Then for any finite subset $J = \{i_1, ..., i_s\} \subset I$, the intersection $D_J = \bigcap_{i\in J} D_i = D_{i_1} \times_X \cdots \times_X D_{i_s}$ is a regular subscheme of codimension s. In other words, for each $x \in X$, there exist a regular system $t_1, ..., t_l$ of parameters of the regular local ring $\mathcal{O}_{X,x}$ and an integer $0 \le r \le l$ such that the divisor D is defined by $\prod_{i=1}^r t_i$ in a neighborhood of x. We say D has normal crossings if, etale locally on X, the divisor D has simple normal crossings. A divisor D with normal crossings has simple normal crossings if and only if each of its irreducible components is regular. If X is a regular noetherian scheme, D is a divisor with normal crossings and $j : U \to X$ is the open immersion of the complement of D, we call the log structure $M_X = \mathcal{O}_X \cap j_*\mathcal{O}_U^\times$ the standard log structure on X defined by D.

For an fs-monoid P, let

$$\mathbf{S}[P] = \operatorname{Spec} \mathbf{Z}[P]$$

denote the log scheme with the log structure associated to $P \to \mathbf{Z}[P]$. For a log scheme X, maps $P \to \Gamma(X, M_X)$ of monoids correspond bijectively with maps

$X \to \mathbf{S}[P]$ of log schemes. In other words, the log scheme $\mathbf{S}[P]$ represents the functor associating the set $\mathrm{Hom}_{\mathrm{monoid}}(P, \Gamma(X, M_X))$ of morphisms of monoids to a log scheme X. A map $P \to \Gamma(X, M_X)$ is a chart on X if and only if the corresponding map $X \to \mathbf{S}[P]$ is strict. By abuse of terminology, we call a strict map $X \to \mathbf{S}[P]$ a chart. We call a pair of a log scheme X and a chart $P \to \Gamma(X, M_X)$ a *charted log scheme* and will abbreviate it as (X, P). For charted log schemes (X, P) and (Y, Q), we call a pair of a morphism $X \to Y$ of log schemes and a morphism $Q \to P$ of fs-monoids such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbf{S}[P] & \longrightarrow & \mathbf{S}[Q] \end{array}$$

is commutative a morphism of charted log schemes and will abbreviate it as $(X, P) \to (Y, Q)$.

For maps of log schemes $X \to S$ and $Y \to S$, we let $X \times^{\log}_S Y$ denote the fiber product in the category of fs-log schemes. For maps $f : N \to P$ and $g : N \to Q$ of fs-monoids, the saturation $P \oplus^{\mathrm{sat}}_N Q$ of the image of $P \oplus Q$ in $P^{\mathrm{gp}} \oplus_{N^{\mathrm{gp}}} Q^{\mathrm{gp}} = \mathrm{Coker}(f - g : N^{\mathrm{gp}} \to P^{\mathrm{gp}} \oplus Q^{\mathrm{gp}})$ is the amalgamate sum of P and Q over N in the category of fs-monoids. The canonical map $\mathbf{S}[P \oplus^{\mathrm{sat}}_N Q] \to \mathbf{S}[P] \times^{\log}_{\mathbf{S}[N]} \mathbf{S}[Q]$ is an isomorphism. If $(X, P) \to (S, N)$ and $(Y, Q) \to (S, N)$ are morphisms of charted log schemes, we have

$$X \times^{\log}_S Y = (X \times_S Y) \times_{\mathbf{S}[P\oplus Q]} \mathbf{S}\big[P \oplus^{\mathrm{sat}}_N Q\big] \tag{4.1.0.1}$$

and $X \times^{\log}_S Y$ is strict over $\mathbf{S}[P \oplus^{\mathrm{sat}}_N Q]$.

Definition **4.1.1.** — *Let* P *be an fs-monoid.*

1. Let $[P]$ *denote the functor on the category of log schemes associating to a log scheme* X *the set*

$$[P](X) = \mathrm{Hom}_{\mathrm{monoid}}(P, \Gamma(X, \bar{M}_X))$$

of monoid homomorphisms. We identify a map $P \to \Gamma(X, \bar{M}_X)$ *of monoids with a map* $X \to [P]$ *of functors.*

2. Let $\mathbf{S}[P] \to [P]$ *be the map induced by the tautological map* $P \to \Gamma(\mathbf{S}[P], M_{\mathbf{S}[P]})$. *If a map* $X \to [P]$ *is the composition of* $X \to \mathbf{S}[P]$ *and the map* $\mathbf{S}[P] \to [P]$, *we say the map* $X \to \mathbf{S}[P]$ *is a* lifting *of* $X \to [P]$.

3. We say a map $Q \to P$ *of fs-monoids is a* quasi-isomorphism *if* $\bar{Q} = Q/Q^{\times} \to \bar{P} = P/P^{\times}$ *is an isomorphism.*

Lemma **4.1.2.** — *Let* $\varphi : \mathrm{Q} \to \mathrm{P}$ *be a morphism of fs-monoids.*

1. If $\varphi : \mathrm{Q} \to \mathrm{P}$ *is a quasi-isomorphism, the induced map* $[\mathrm{Q}] \to [\mathrm{P}]$ *of functors is an isomorphism.*

2. Let P' *be the inverse image of* P *by the map* $\mathrm{P}^{\mathrm{gp}} \oplus \mathrm{Q}^{\mathrm{gp}} \to \mathrm{P}^{\mathrm{gp}}$ *sending* (a, b) *to* $a + \varphi(b)$. *Then the map* $\mathrm{P}^{\mathrm{gp}} \oplus \mathrm{Q}^{\mathrm{gp}} \to \mathrm{P}^{\mathrm{gp}} \oplus \mathrm{Q}^{\mathrm{gp}}$ *sending* (a, b) *to* $(a + \varphi(b), b)$ *induces an isomorphism* $\mathrm{P}' \to \mathrm{P} \oplus \mathrm{Q}^{\mathrm{gp}}$. *Hence the map* $\mathrm{P} \to \mathrm{P}'$ *defined by* $a \mapsto (a, 0)$ *and the map* $\mathrm{P}' \to \mathrm{P}$ *induced by* $(a, b) \mapsto a + \varphi(b)$ *are quasi-isomorphisms.*

3. Let $\mathrm{Q} \to \mathrm{Q}'$ *be a quasi-isomorphism of fs-monoids. Then the map* $\mathrm{P} \to \mathrm{P} \oplus^{\mathrm{sat}}_{\mathrm{Q}} \mathrm{Q}'$ *is a quasi-isomorphism.*

4. Let $(\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim} \subset \mathrm{P}^{\mathrm{gp}} \oplus_{\mathrm{Q}^{\mathrm{gp}}} \mathrm{P}^{\mathrm{gp}}$ *be the inverse image of* P *by the map* $\mathrm{P}^{\mathrm{gp}} \oplus_{\mathrm{Q}^{\mathrm{gp}}} \mathrm{P}^{\mathrm{gp}} \to \mathrm{P}^{\mathrm{gp}}$ *sending* (a, b) *to* $a + b$. *Then the map* $\mathrm{P} \oplus \mathrm{P} \oplus \mathrm{P}^{\mathrm{gp}} \to (\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim}$ *sending* (a, b, c) *to* $(a + c, b - c)$ *induce a surjection* $\mathrm{P} \oplus \mathrm{P} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}})) \to (\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim}$. *Further the monoid* $(\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim}$ *is identified with the quotient of* $\mathrm{P} \oplus \mathrm{P} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}}))$ *by the equivalence relation generated by* $(a, 0, 0) \sim (0, a, \bar{a})$ *for* $a \in \mathrm{P}$.

Proof. — 1. Clear from the definition.

2. Clear.

3. It is reduced to the case $\mathrm{Q}' = \bar{\mathrm{Q}} = \mathrm{Q}/\mathrm{Q}^{\times}$. Then $\mathrm{P}' = \mathrm{P} \oplus^{\mathrm{sat}}_{\mathrm{Q}} \mathrm{Q}' = \mathrm{P}/\mathrm{Im}\ \mathrm{Q}^{\times}$ and $\bar{\mathrm{P}} \to \bar{\mathrm{P}}'$ is an isomorphism.

4. The map $\mathrm{P} \oplus \mathrm{P} \to \mathrm{P} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}})) : (a, b) \mapsto (a + b, a)$ induces an isomorphism $\mathrm{P}^{\mathrm{gp}} \oplus_{\mathrm{Q}^{\mathrm{gp}}} \mathrm{P}^{\mathrm{gp}} \to \mathrm{P}^{\mathrm{gp}} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}}))$ of abelian groups. Hence, it induces an isomorphism $(\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim} \to \mathrm{P} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}}))$. The composition $\mathrm{P} \oplus \mathrm{P} \oplus \mathrm{P}^{\mathrm{gp}} \to (\mathrm{P} \oplus_{\mathrm{Q}} \mathrm{P})^{\sim} \to \mathrm{P} \oplus (\mathrm{P}^{\mathrm{gp}}/\varphi(\mathrm{Q}^{\mathrm{gp}}))$ maps (a, b, c) to $(a + b, \overline{a + c})$. Now the assertion is clear. □

Definition **4.1.3.** — *Let* X *be a log scheme and* P *be an fs-monoid.*

1. We say a map $\mathrm{X} \to [\mathrm{P}]$ *is* strict *if, for each geometric point* $\bar{x}$, *there exist an etale neighborhood* U *of* $\bar{x}$ *and a strict morphism* $\mathrm{U} \to \mathbf{S}[\mathrm{P}]$ *lifting the restriction* $\mathrm{U} \to [\mathrm{P}]$.

2. We call a strict map $\mathrm{X} \to [\mathrm{P}]$ *a* frame. *We call a pair of a log scheme* X *and a frame* $\mathrm{X} \to [\mathrm{P}]$ *a* framed log scheme *and, by abuse of notation, let it denoted by* $(\mathrm{X}, [\mathrm{P}])$. *For framed log schemes* $(\mathrm{X}, [\mathrm{P}])$ *and* $(\mathrm{Y}, [\mathrm{Q}])$, *we call a pair of a morphism* $\mathrm{X} \to \mathrm{Y}$ *of log schemes and a morphism* $\mathrm{Q} \to \mathrm{P}$ *of fs-monoids such that the diagram*

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathrm{Y} \\ \big\downarrow & & \big\downarrow \\ [\mathrm{P}] & \longrightarrow & [\mathrm{Q}] \end{array}$$

is commutative a morphism of framed log scheme and will abbreviate it as $(\mathrm{X}, [\mathrm{P}]) \to (\mathrm{Y}, [\mathrm{Q}])$.

The functor $[\mathrm{P}]$ is in fact a sheaf with respect to the classical etale topology. In [21] and [32], the "toric stack" $\mathscr{S}_{\mathrm{P}}$ and a stack $\mathscr{S}_{\mathrm{P}}^{\log}$ associated to it are introduced for a fine monoid P. For an fs-monoid P, the stack $\mathscr{S}_P^{\log}$ is identified with the sheaf [P] by [32] Proposition 5.17. Moreover, a map to $\mathscr{S}_P$ is strict if and only if the corresponding map to [P] is strict in the sense of Definition 4.1.3.1 by loc.cit. Remark 5.18.

By definition, a map $X \to [P]$ is a frame if and only if it is etale locally lifted to a chart. A typical example of frames is given by a divisor with simple normal crossings on a regular locally noetherian scheme.

Lemma **4.1.4.** — *Let* X *be a log regular* ([25] Definition (2.1)) *locally noetherian log scheme and* U *be the maximum open subscheme of* X *where the log structure* M_X *is trivial.*

1. The following conditions are equivalent.

(1) The underlying scheme X *is regular, the open subscheme* U *is the complement of a divisor* D *with normal crossings and* M_X *is the standard log structure defined by* D.

(2) Etale locally on X*, there exist a chart* $X \to \mathbf{S}[\mathbf{N}^m]$ *for some integer m.*

2. If X *is quasi-compact, the following conditions are equivalent.*

(1) The underlying scheme X *is regular, the open subscheme* U *is the complement of a divisor* D *with simple normal crossings and* M_X *is the standard log structure defined by* D.

(2) There exist a frame $X \to [\mathbf{N}^m]$ *for some integer m.*

Proof. — 1. Clear from the definition ([25] Definition (2.1)).

2. (1)⇒(2). Let $D_1, \ldots, D_m$ be the irreducible components of D. Then, the monoid $P = \Gamma(X, \bar{M}_X)$ is isomorphic to $\mathbf{N}^m$. The tautological map $X \to [P]$ is strict.

(2)⇒(1). It follows from 1 (2)⇒(1) that X is regular, U is the complement of a divisor with normal crossings and M_X is the standard log structure defined by D. We show that each irreducible component of X is regular. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbf{N}^m$. For $i = 1, \ldots, m$, we define a closed subscheme D_i of X by the image of e_i in $\mathscr{O}_X$ by etale locally lifting the frame $X \to [\mathbf{N}^m]$ to a chart. Then, $D_1, \ldots, D_m$ are regular. Since an irreducible component of D is an irreducible component of one of D_i, the assertion follows. □

We call the frame $X \to [P]$ in the proof of Lemma 4.1.4.2 (1)⇒(2) the standard frame on X defined by D.

Lemma **4.1.5.** — *Let* X *be a log scheme,* $\bar{x}$ *be a geometric point of* X *and* $P \to \bar{M}_{X,\bar{x}}$ *be a map of fs-monoids.*

1. There exist an etale neighborhood U *of* $\bar{x}$ *and a map* $P \to \Gamma(U, M_X)$ *inducing* $P \to \bar{M}_{X,\bar{x}}$.

2. Let $\varphi : Q \to P$ *be a map of fs-monoids and* $X \xrightarrow{f} Y \to \mathbf{S}[Q]$ *be morphisms of log schemes such that the diagram*

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ \Gamma(Y, M_Y) & \longrightarrow & \bar{M}_{X,\bar{x}} \end{array}$$

is commutative. Let $P' \subset P^{gp} \oplus Q^{gp}$ *be the inverse image of* P *as in Lemma 4.1.2.2. Then there exist an etale neighborhood* U *of* $\bar{x}$ *and a map* $P' \to \Gamma(U, M_X)$ *such that the diagram*

$$\begin{array}{ccccc} Q & \longrightarrow & P' & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(Y, M_Y) & \longrightarrow & \Gamma(U, M_X) & \longrightarrow & \bar{M}_{X,\bar{x}} \end{array}$$

is commutative.

Proof. — 1. We may assume $P = \bar{M}_{X,\bar{x}}$. Since there exists a section $\bar{M}_{X,\bar{x}} \to M_{X,\bar{x}}$, the assertion follows.

2. We take an etale neighborhood U and a map $P \to \Gamma(U, M_X)$ as in 1. Let $Q \to \Gamma(Y, M_Y) \to \Gamma(U, M_X)$ be the composition. Then, we have a commutative diagram

$$\begin{array}{ccc} P \oplus Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ \Gamma(U, M_X) & \longrightarrow & \bar{M}_{X,\bar{x}}. \end{array}$$

Since $M_{X,\bar{x}}$ is the inverse image of $\bar{M}_{X,\bar{x}}$ by the canonical map $M^{gp}_{X,\bar{x}} \to \bar{M}^{gp}_{X,\bar{x}}$, the composition $P \oplus Q \to \Gamma(U, M_X) \to M_{X,\bar{x}}$ is extended to a map $P' \to M_{X,\bar{x}}$. Hence shrinking U if necessary, we get the assertion. □

Corollary **4.1.6.** — *Let* X *be a log scheme and* $\bar{x}$ *be a geometric point of* X.

1. Let $X \to [P]$ *be a map. Then there exist an etale neighborhood* U *of* $\bar{x}$ *and a map* $U \to \mathbf{S}[P]$ *lifting the restriction* $U \to [P]$.

2. Let $Q \to P$ *be a map of fs-monoids,* $X \to Y$ *be a map of log schemes and*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ [P] & \longrightarrow & [Q] \end{array} \tag{4.1.6.1}$$

be a commutative diagram. Let $P' \subset P^{gp} \oplus Q^{gp}$ *be the inverse image of* P *as in Lemma 4.1.5.2. Then there exist etale neighborhoods* U *of* $\bar{x}$ *and* V *of* $\bar{y} = f(\bar{x})$ *and a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbf{S}[P'] & \longrightarrow & \mathbf{S}[Q] \end{array} \tag{4.1.6.2}$$

lifting the restriction of (4.1.6.1).

Proof. — Clear from Lemma 4.1.5. □

Lemma **4.1.7.** — *Let* X *be a log scheme,* P *be an fs-monoid and* $X \to [P]$ *be a map.*

1. For a morphism $f : X \to \mathbf{S}[P]$ *of log schemes lifting* $X \to [P]$*, the map* $X \to [P]$ *is strict if and only if* $X \to \mathbf{S}[P]$ *is strict.*

2. Let $P' \to P$ *be a quasi-isomorphism of fs-monoids. Then the map* $X \to [P]$ *is strict if and only if the composition* $X \to [P] \to [P']$ *is strict.*

3. There exist a log structure M'_X*, a map* $X \to X' = (X, M'_X)$ *of log schemes and a strict map* $X' \to [P]$ *such that* $X \to [P]$ *is the composition.*

Proof. — 1. The if part is trivial. We show the only if part. Since the question is etale local, we may assume there exists a strict map $g : X \to \mathbf{S}[P]$ lifting $X \to [P]$. Then the difference of the two maps $P \to \Gamma(X, M_X)$ is a map to $\Gamma(X, M_X^\times)$ and the assertion follows.

2. We may assume $P = \bar{P}'$. Then P' is isomorphic to $P \times P'^\times$ and $\mathbf{S}[P'] \to \mathbf{S}[P]$ is strict. Hence the assertion follows.

3. If there exists a map $X \to \mathbf{S}[P]$ lifting $X \to [P]$, it is sufficient to define a log structure M'_X on X by the chart $P \to M_X \to \mathscr{O}_X$. If there are 2 such maps $X \to \mathbf{S}[P]$, the difference of the maps $P \to \Gamma(X, M_X)$ is a map $P \to \Gamma(X, \mathscr{O}_X^\times)$ and the log structure M'_X on X is indepenent of the choice of lifting. In general, we obtain the log structure M'_X by patching by Lemma 4.1.5.1. □

Corollary **4.1.8.** — *1. Let* P *be an fs-monoid and* X *be a log scheme. Let* $X \to [P]$ *be a map and* $\bar{x}$ *be a geometric point of* X*. If the composition* $P \to \Gamma(X, \bar{M}_X) \to \bar{M}_{X,\bar{x}}$ *is a quasi-isomorphism, there exists an etale neighborhood* U *of* $\bar{x}$ *such that the restriction* $U \to [P]$ *is strict.*

2. Let $X \to Y$ *be a map of log schemes,* $\bar{x}$ *be a geometric point of* X *and* $Y \to [Q]$ *be a frame. We put* $P = \bar{M}_{X,\bar{x}}$*. Then there exist an etale neighborhood* U *of* $\bar{x}$ *and a frame* $U \to [P]$ *such that the composition* $Q \to \Gamma(Y, \bar{M}_Y) \to \Gamma(X, \bar{M}_X) \to P = \bar{M}_{X,\bar{x}}$ *defines a map* $(U, [P]) \to (Y, [Q])$ *of framed log schemes.*

3. Let $f : X \to Y$ *be a map of log schemes and* $\bar{x}$ *be a geometric point of* X*. We put* $P = \bar{M}_{X,\bar{x}}$*,* $\bar{y} = f(\bar{x})$ *and* $Q = \bar{M}_{Y,\bar{y}}$*. Then there exist etale neighborhoods* U *of* $\bar{x}$ *and* V *of* $\bar{y}$ *and frames* $U \to [P]$ *and* $V \to [Q]$ *inducing the identities* $P \to M_{X,\bar{x}}$ *and* $Q \to M_{Y,\bar{y}}$ *and a map* $(U, [P]) \to (V, [Q])$ *of framed log schemes.*

4. Let $(X, [P]) \to (Y, [Q])$ *be a map of framed log schemes and* $\bar{x}$ *be a geometric point of* X*. Then the commutative diagram* (4.1.6.2) *in Corollary 4.1.6.2 defines a map* $(U, P') \to (V, Q)$ *of charted log schemes lifting the restriction of* $(X, [P]) \to (Y, [Q])$*.*

5. Let $Y \to [P]$ *be a strict map. Then a map* $X \to Y$ *of log schemes is strict if and only if the composition* $X \to Y \to [P]$ *is strict.*

6. Let $N \to P$ *and* $N \to Q$ *be morphisms of fs-monoids and let*

$$\text{(4.1.8.1)}\qquad \begin{array}{ccccc} X & \xrightarrow{f} & S & \xleftarrow{g} & Y \\ \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] & \longleftarrow & [Q] \end{array}$$

be a commutative diagram. Then, the vertical maps induce a map $X \times_S^{\log} Y \to [P \oplus_N^{\mathrm{sat}} Q]$*. If the vertical arrows are strict, the induced map* $X \times_S^{\log} Y \to [P \oplus_N^{\mathrm{sat}} Q]$ *is also strict.*

Proof. — 1. Replacing P by $P/P^\times$, we may assume $P = \bar{M}_{X,\bar{x}}$. There exist an etale neighborhood U of $\bar{x}$ and a chart $P \to \Gamma(U, M_X)$ on U such that the diagram

$$\begin{array}{ccc} P & \longrightarrow & \Gamma(U, M_X) \\ \downarrow & & \downarrow \\ \Gamma(X, \bar{M}_X) & \longrightarrow & \bar{M}_{X,\bar{x}} \end{array}$$

is commutative. Shrinking U, we may assume that the diagram

$$\begin{array}{ccc} P & \longrightarrow & \Gamma(U, M_X) \\ \downarrow & & \downarrow \\ \Gamma(X, \bar{M}_X) & \longrightarrow & \Gamma(U, \bar{M}_X) \end{array}$$

is commutative. Hence the assertion follows from Lemma 4.1.7.1.

2. By 1, there exist an etale neighborhood U and a frame $U \to [P]$. Shrinking U, if necessary, we obtain a map $(U, [P]) \to (Y, [Q])$ of framed log schemes.

3. By 1, there exist an etale neighborhood V and a frame $V \to [Q]$. Hence it suffices to apply 2.

4. It follows from Lemma 4.1.7.1.

5. Since the question is etale local on Y, we may assume there is a map $Y \to \mathbf{S}[P]$ lifting $Y \to [P]$ by Corollary 4.1.6.1. Then the assertion follows from Lemma 4.1.7.1.

6. Since $\bar{M}_{X \times_S^{\log} Y}$ is saturated, the map $P \oplus_N Q \to \Gamma(X \times_S^{\log} Y, \bar{M}_{X \times_S^{\log} Y})$ induces a map $P \oplus_N^{\mathrm{sat}} Q \to \Gamma(X \times_S^{\log} Y, \bar{M}_{X \times_S^{\log} Y})$.

We show that the induced map $X \times_S^{\log} Y \to [P \oplus_N^{\mathrm{sat}} Q]$ is strict assuming that the vertical arrows in the diagram (4.1.8.1) are strict. The question is etale local on X, Y and S. Let P' be the inverse image of P by the map $P^{gp} \oplus N^{gp} \to P^{gp}$ and Q' be the inverse image of Q by the map $Q^{gp} \oplus N^{gp} \to Q^{gp}$ as in Lemma 4.1.5.2. The canonical surjections $P' \to P$ and $Q' \to Q$ are quasi-isomorphism and hence the maps $[P] \to [P']$ and $[Q] \to [Q']$ are isomorphisms. By Lemma 4.1.5.2, we may assume there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & S & \xleftarrow{g} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{S}[P'] & \longrightarrow & \mathbf{S}[N] & \longleftarrow & \mathbf{S}[Q'] \end{array}$$

lifting the diagram (4.1.8.1). By Lemma 4.1.7.1, the verical maps are stricts. Hence the map $X \times_S^{\log} Y \to [P' \oplus_N^{\mathrm{sat}} Q']$ is strict. By Lemma 4.1.2.3, the map $P \oplus_N^{\mathrm{sat}} Q \to P' \oplus_N^{\mathrm{sat}} Q'$ is a quasi-isomorphism and the assertion follows. □

4.2. *Logarithmic products.* — Let X be a log scheme, $Q \to P$ be a map of fs-monoids and $X \to [Q]$ be a map. Then, let $X \times_{[Q]}^{\log} [P]$ denote the functor associating to a log scheme T the set $X(T) \times_{[Q](T)} [P](T)$.

Proposition **4.2.1.** — *Let* $Q \to P$ *be a map of fs-monoids and assume that the map* $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ *is surjective. Then,*

1. The map $[P] \to [Q]$ *is relatively representable, log etale and affine. Namely, if* X *is a log scheme and if* $X \to [Q]$ *is a map, the functor* $X \times_{[Q]}^{\log} [P]$ *is represented by a log scheme log etale and affine over* X.

2. Let $X \to \mathbf{S}[Q]$ *be a map of log schemes and let* $P^{\sim}$ *denote the inverse image of* P *by the surjection* $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$. *Then the log scheme* $X \times_{\mathbf{S}[Q]}^{\log} \mathbf{S}[P^{\sim}]$ *is log etale over* X *and represents the functor* $X \times_{[Q]}^{\log} [P]$.

Proof. — 1. We reduce the assertion 1 to the assertion 2. Let $P^{\sim} \subset Q^{\mathrm{gp}}$ denote the inverse image of P by the map $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$. Since $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is surjective, the map $P^{\sim} \to P$ is a quasi-isomorphism and hence $[P^{\sim}] \to [P]$ is an isomorphism by Lemma 4.1.2.1. Thus, by replacing P by $P^{\sim}$, we may assume $Q \subset P \subset Q^{\mathrm{gp}} = P^{\mathrm{gp}}$.

For an fs-log scheme T, a map $T \to [P]$ is determined by the induced map $P^{\mathrm{gp}} = Q^{\mathrm{gp}} \to \Gamma(T, \bar{M}_T^{\mathrm{gp}})$, since the monoid $\Gamma(T, \bar{M}_T) \subset \Gamma(T, \bar{M}_T^{\mathrm{gp}})$ is integral. Hence, for a log scheme X, the base change $X \times_{[Q]}^{\log} [P]$ is the subfunctor of X associating to a log scheme T the set $\{T \to X|$ the composition $Q \to \Gamma(X, \bar{M}_X) \to \Gamma(T, \bar{M}_T)$ is extended to $P \to \Gamma(T, \bar{M}_T)\}$. Thus the assertion is etale local on X. By Lemma 4.1.5.1, we may assume that there exists a map $X \to \mathbf{S}[Q]$ lifting $X \to [Q]$. Thus the assertion 1 is reduced to the assertion 2.

2. Similarly as above, we may assume $P = P^{\sim}$ and $Q^{\mathrm{gp}} = P^{\mathrm{gp}}$. Further, it is sufficient to prove the case $X = \mathbf{S}[Q]$. By the proof of 1, $\mathbf{S}[Q] \times_{[Q]}^{\log} [P]$ is the functor associating to a log scheme T the set $\{Q \to \Gamma(T, M_T) | Q \to \Gamma(T, \bar{M}_T)$ is extended to $P \to \Gamma(T, \bar{M}_T)\}$. A map $Q \to \Gamma(T, M_T)$ is extended to $P \to \Gamma(T, \bar{M}_T)$ if and only if it is extended to $P \to \Gamma(T, M_T)$ since M_T is the inverse image of $\bar{M}_T$ by $M_T^{\mathrm{gp}} \to \bar{M}_T^{\mathrm{gp}}$. Thus the functor $\mathbf{S}[Q] \times_{[Q]}^{\log} [P]$ is represented by $\mathbf{S}[P]$ and the assertion follows. □

We let $X \times^{\log}_{[Q]} [P]$ denote the log scheme representing the functor $X \times^{\log}_{[Q]} [P]$. The log etaleness of the map $X \times^{\log}_{[Q]} [P] \to X$ in Proposition 4.2.1.1 is a special case of the log etaleness of the map of toric stacks induced by a map of fs-monoid, [32] Corollary 5.29. The following Corollary 4.2.2.2 is a variant of the local exactification in [23] Proposition (4.10).

Corollary **4.2.2.** — *Let* $Q \to P$ *be a map of fs-monoids such that* $Q^{gp} \to P^{gp}$ *is surjective.*

1. Let $Q \to Q'$ *be a map of fs-monoids. Let* X *be a log scheme and* $X \to [Q']$ *be a frame. Then the map* $X \times^{\log}_{[Q]} [P] \to [Q' \oplus^{\text{sat}}_{Q} P]$ *is a frame. In particular, if* $X \to [Q]$ *is a frame, then the map* $X \times^{\log}_{[Q]} [P] \to [P]$ *is also a frame.*

2. Let $(X, [P]) \to (Y, [Q])$ *be a map of framed log schemes. Then* $X \to Y$ *is the composition of the strict map* $X \to Y \times^{\log}_{[Q]} [P]$ *and the log etale map* $Y \times^{\log}_{[Q]} [P] \to Y$.

Proof. — 1. Since the assertion is etale local on X, we may assume there exists a chart $X \to \mathbf{S}[Q']$ lifting the frame $X \to [Q']$. Then the assertion follows from Proposition 4.2.1.2 and Corollary 4.1.8.6.

2. The map $Y \times^{\log}_{[Q]} [P] \to [P]$ is strict by 1. Hence the map $X \to Y \times^{\log}_{[Q]} [P]$ is strict by the assumption that $X \to [P]$ is strict and by Corollary 4.1.8.5. By Proposition 4.2.1, the map $Y \times^{\log}_{[Q]} [P] \to Y$ is log etale. □

To define logarithmic products, we introduce notations. Let X and Y be log schemes over a log scheme S, let P be an fs-monoid and let $X \to [P] \leftarrow Y$ be maps. Then, let $X \times^{\log}_{S,[P]} Y$ denote the functor associating to a log scheme T the set $X(T) \times_{(S(T) \times [P](T))} Y(T)$. For a map $N \to P$ of fs-monoids and a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & S & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] & \longleftarrow & [P], \end{array}$$

let $X \times^{\log}_{S \times_{[N]} [P]} Y$ denote the functor associating to a log scheme T the set $X(T) \times_{(S(T) \times_{[N](T)} [P](T))} Y(T)$. Since $S(T) \times_{[N](T)} [P](T)$ is a subset of $S(T) \times [P](T)$, the natural map

$$X \times^{\log}_{S \times_{[N]} [P]} Y \longrightarrow X \times^{\log}_{S,[P]} Y$$

is an isomorphism. If $P = 0$, we have $X \times^{\log}_{S,[0]} Y = X \times^{\log}_{S} Y$.

Proposition **4.2.3.** — *Let* X *and* Y *be log schemes over a log scheme* S. *Let* P *be an fs-monoid and* $X \to [P] \leftarrow Y$ *be maps. Then, the log scheme* $(X \times^{\log}_{S} Y) \times^{\log}_{[P \oplus P]} [P]$ *is log etale over* $X \times^{\log}_{S} Y$ *and represents the functor* $X \times^{\log}_{S,[P]} Y$.

Proof. — Clear from Proposition 4.2.1. □

Definition **4.2.4.** — *We let* $X \times^{\log}_{S,[P]} Y$ *denote the log scheme* $(X \times^{\log}_{S} Y) \times^{\log}_{[P\oplus P]} [P]$ *representing the functor* $X \times^{\log}_{S,[P]} Y$ *and call it the* log product *of* X *and* Y *over* S *and* [P].

Example. — Let $m \geq 1$ be an integer and $(\mathbf{N}^m \oplus \mathbf{N}^m)^\sim$ be the submonoid $\{(a_1, ..., a_m, b_1, ..., b_m) \in \mathbf{Z}^{2m} | a_i + b_i \geq 0 \text{ for all } 1 \leq i \leq m\}$. Then, we have $\mathbf{S}[\mathbf{N}^m] \times^{\log}_{\mathrm{Spec}\mathbf{Z},[\mathbf{N}^m]} \mathbf{S}[\mathbf{N}^m] = \mathbf{S}[(\mathbf{N}^m \oplus \mathbf{N}^m)^\sim]$. In other words, we have

$$\begin{aligned} &\mathrm{Spec}\mathbf{Z}[X_1, ..., X_m] \times^{\log}_{\mathrm{Spec}\mathbf{Z},[\mathbf{N}^m]} \mathrm{Spec}\mathbf{Z}[Y_1, ..., Y_m] \\ &= \mathrm{Spec}\mathbf{Z}\big[X_1, ..., X_m, Y_1, ..., Y_m, (X_1/Y_1)^{\pm 1}, ..., (X_m/Y_m)^{\pm 1}\big]. \end{aligned}$$

Corollary **4.2.5.** — *Let* X *and* Y *be log schemes over a log scheme* S, $N \to P$ *be a map of fs-monoids and*

$$\begin{array}{ccccc} X & \longrightarrow & S & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] & \longleftarrow & [P] \end{array}$$

be a commutative diagram. Assume $S \to [N]$ *is strict.*

1. *Let* $P \to P'$ *and* $P \to P''$ *be maps of fs-monoids and let* $X \to [P']$ *and* $Y \to [P'']$ *be strict maps inducing* $X \to [P]$ *and* $Y \to [P]$ *respectively. Then, the induced map* $X \times^{\log}_{S,[P]} Y \to [P' \oplus^{\mathrm{sat}}_{P} P'']$ *is strict. In particular, if* $P \to P'$ *is a quasi-isomorphism (resp. if* $P \to P'$ *and* $P \to P''$ *are quasi-isomorphisms), the induced map* $X \times^{\log}_{S,[P]} Y \to [P'']$ *(resp.* $X \times^{\log}_{S,[P]} Y \to [P]$*) is strict.*

2. *If* $(X, [P]) \to (S, [N])$ *is a map of framed log schemes, then the projection* $X \times^{\log}_{S,[P]} Y \to Y$ *is strict.*

Proof. — 1. The map $X \times^{\log}_{S} Y \to [P' \oplus^{\mathrm{sat}}_{N} P'']$ is strict by Corollary 4.1.8.6. Hence $X \times^{\log}_{S,[P]} Y = (X \times^{\log}_{S} Y) \times^{\log}_{[P\oplus P]} [P]$ is strict over $[(P' \oplus^{\mathrm{sat}}_{N} P'') \oplus^{\mathrm{sat}}_{P\oplus P} P] = [P' \oplus^{\mathrm{sat}}_{P} P'']$ by Corollary 4.2.2.1. The rest of assertion follows from Lemma 4.1.2.3.

2. Since the question is etale local on Y, we may assume there exist a map $(Y, [P']) \to (S, [N])$ of framed log schemes by Corollary 4.1.8.2. Hence the assertion follows from 1 and Corollary 4.1.8.5. □

The log product may be explicitly computed as follows.

Corollary **4.2.6.** — *Let* $\varphi : N \to P$ *be a map of fs-monoid and*

(4.2.6.1)
$$\begin{array}{ccccc} X & \longrightarrow & S & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{S}[P] & \longrightarrow & \mathbf{S}[N] & \longleftarrow & \mathbf{S}[P] \end{array}$$

be a commutative diagram of log schemes. Let $\alpha_X : P \to \Gamma(X, \mathscr{O}_X)$ *and* $\alpha_Y : P \to \Gamma(Y, \mathscr{O}_Y)$ *be the maps induced by* $X \to \mathbf{S}[P] \leftarrow Y$. *Let* $(P \oplus_N P)^\sim$ *denote the inverse image of* P *by the surjection* $(P \oplus_N^{\rm sat} P)^{\rm gp} = P^{\rm gp} \oplus_{N^{\rm gp}} P^{\rm gp} \to P^{\rm gp}$ *sending* (a, b) *to* $a + b$ *as in Lemma 4.1.2.5.*

1. We have

$$X \times^{\log}_{S,[P]} Y = \left(X \times^{\log}_S Y\right) \times^{\log}_{\mathbf{S}[P\oplus^{\rm sat}_N P]} \mathbf{S}[(P \oplus_N P)^\sim].$$

2. Assume the vertical arrows in (4.2.6.1) *are strict. Then, we have*

$$X \times^{\log}_{S,[P]} Y = (X \times_S Y) \times_{\mathbf{S}[P\oplus P]} \mathbf{S}[(P \oplus_N P)^\sim].$$

On the right hand side, the underlying scheme is identified with the closed subscheme of $(X \times_S Y) \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{S}[P^{\rm gp}/\varphi(Q^{\rm gp})]$ *defined by the ideal* $(\alpha_X(a) \otimes 1 - \alpha_Y(a) \otimes \bar{a} : a \in P)$ *and the log structure is the pull-back of that of* $\mathbf{S}[(P \oplus_N P)^\sim]$.

Proof. — 1. It is clear from Propositions 4.2.1.2 and 4.2.3.

2. By 1 and $X \times^{\log}_S Y = (X \times_S Y) \times_{\mathbf{S}[P\oplus P]} \mathbf{S}[P \oplus^{\rm sat}_N P]$ (4.1.0.1), we have $X \times^{\log}_{S,[P]} Y = (X \times^{\log}_S Y) \times^{\log}_{\mathbf{S}[P\oplus^{\rm sat}_N P]} \mathbf{S}[(P \oplus_N P)^\sim] = (X \times_S Y) \times_{\mathbf{S}[P\oplus P]} \mathbf{S}[(P \oplus_N P)^\sim]$. The assertion on the underlying scheme follows from this and Lemma 4.1.2.4. □

We give a global example where the closed immersion $X \times^{\log}_{S,[P]} Y \to (X \times_S Y) \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{S}[P^{\rm gp}/\varphi(Q^{\rm gp})]$ in Corollary 4.2.6.2 is an isomorphism. We prepare some notations. Let P and N be fs-monoids and $(S, [P\oplus N]) \to (S', [N])$ be a map of framed log schemes. Assume that the map $S \to S'$ of underlying schemes is the identity. Assume further that $P^\times = \{1\}$ and that the composition $P \to \bar{M}_S \to \mathscr{O}_S/\mathscr{O}_S^\times$ sends $P \setminus \{1\}$ to 0. The assumptions imply that, etale locally on S', there exists an isomorphism $M_{S'} \times P \to M_S$ inducing the map $P \to \bar{M}_S$ defining $S \to [P]$. Thus the map $P \to \bar{M}_S$ induces an isomorphism $P \to \bar{M}_S/\bar{M}_{S'} = M_S/M_{S'}$. For a log scheme $f : T \to S'$ over S', the set $S(T)$ of log schemes $T \to S$ over S' is identified with the set

$$\{\varphi : f^{-1}M_S \to M_T \mid \text{the composition } P \to f^{-1}\bar{M}_S \to \mathscr{O}_T/\mathscr{O}_T^\times \text{ sends } P \setminus \{1\} \text{ to } 0 \text{ and the composition } f^{-1}M_{S'} \to f^{-1}M_S \to M_T \text{ underlies the map } T \to S'\}.$$

Let G be the torus $\operatorname{Hom}(P^{\rm gp}, \mathbf{G}_m)$. We define an action of G on S over S' as follows. Namely, we define a functorial action of $G(T) = \operatorname{Hom}(P, \Gamma(T, \mathscr{O}_T^\times))$ on $S(T)$ for a log scheme $f : T \to S'$ over S'. For $u : P \to \mathscr{O}_T^\times$ and $\varphi : f^{-1}M_S \to M_T$, let $u\varphi : f^{-1}M_S \to M_T$ denote the product of $\varphi : f^{-1}M_S \to M_T$ and the composition $f^{-1}M_S \to f^{-1}M_S/M_{S'} \to P \xrightarrow{u} \mathscr{O}_T^\times \to M_T$. Then it is easy to see that, for $u \in G(T) = \operatorname{Hom}(P, \Gamma(T, \mathscr{O}_T^\times))$ and $\varphi \in S(T)$, the product $u\varphi$ is in $S(T)$ and that the maps $G(T) \times S(T) \to S(T)$ sending (u, φ) to $u\varphi$ define an action of G on S over S'. This action is also compatible with the map $S \to [P \oplus N]$.

Lemma **4.2.7.** — *Let* $S \to S' \leftarrow X$ *be maps of log schemes and* P *and* N *be fs-monoids. Let*

$$\textbf{(4.2.7.1)}\qquad \begin{array}{ccccc} S & \longrightarrow & S' & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ [P \oplus N] & \longrightarrow & [N] & \longleftarrow & [P \oplus N] \end{array}$$

be a commutative diagram of maps. Assume the vertical arrows $S' \to [N]$ *and* $S \to [P \oplus N]$ *are strict and that the map* $S \to S'$ *of underlying schemes is the identity. Assume further that* $P^\times = \{1\}$ *and that the compositions* $P \to \bar{M}_S \to \mathscr{O}_S/\mathscr{O}_S^\times$ *and* $P \to \bar{M}_X \to \mathscr{O}_X/\mathscr{O}_X^\times$ *send* $P \setminus \{1\}$ *to* 0.

Then, the log product $S \times^{\log}_{S',[P]} X$ *is strict over* X. *Further, the action of the torus* $G = \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{G}_m)$ *on* S *induces an action on* $S \times^{\log}_{S',[P]} X$ *over* X *and* $S \times^{\log}_{S',[P]} X$ *is a* G*-torsor over* X.

Proof. — The map $S \times^{\log}_{S',[P]} X \to X$ is strict by Corollary 4.2.5.2. Since the action of G on S is compatible with the maps $S \to S'$ and $S \to [P]$, the action of G on $S \times^{\log}_{S',[P]} X$ is defined. To show that $S \times^{\log}_{S',[P]} X$ is a G-torsor over X, first we show that the map $G \times S \to S \times^{\log}_{S',[P]} S$ is an isomorphism. Let $f : T \to S'$ be a log scheme over S' and $\varphi, \psi : T \to S$ be maps over S and over $[P]$. Then, since the maps $P \to f^{-1}\bar{M}_S \to \bar{M}_T$ induced by φ and ψ are equal, there exists a unique map $u : P \to \mathscr{O}_T^\times$ such that $\psi = u\varphi$. Thus, the map $G \times S \to S \times^{\log}_{S',[P]} S$ is an isomorphism.

We show that $S \times^{\log}_{S',[P]} X$ is a G-torsor over X. By the assumption that $P \setminus \{1\}$ is sent to 0 in $\mathscr{O}_X/\mathscr{O}_X^\times$, there exists a commutative diagram

$$\begin{array}{ccc} S' & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathbf{S}[N] & \longleftarrow & \mathbf{S}[P \oplus N] \end{array}$$

lifting the right square in (4.2.7.1) etale locally on S' and on X. Hence there exists a map $X \to S$ over S' and over $[P \oplus N]$ etale locally on X. Thus, etale locally on X, the scheme $S \times^{\log}_{S',[P]} X$ is the pull-back of $S \times^{\log}_{S',[P]} S$ by $X \to S$ and has a section over X. Thus the assertion is proved. □

We define the log diagonal map and study the relation with the sheaf of logarithmic differentials. Recall that, for a morphism $f : (X, M_X) \to (S, M_S)$ of log schemes, the $\mathscr{O}_X$-module $\Omega^1_{(X,M_X)/(S,M_S)}$ is defined in [23] (1.7). It is canonically isomorphic to

$$\left(\Omega^1_{X/S} \oplus \mathscr{O}_X \otimes_{\mathbf{Z}} \left(M_X^{\mathrm{gp}}/f^*M_S^{\mathrm{gp}}\right)\right)/((d\alpha(m), -\alpha(m) \otimes m) : m \in M_X).$$

For $m \in M_X$, its image is denoted by $d \log m$.

Corollary **4.2.8.** — *Let* $X \to S$ *be a map of log schemes,* P *be an fs-monoid and* $X \to [P]$ *be a map.*

1. The diagonal map $X \to X \times_S^{\log} X$ *is uniquely decomposed as the composition of an immersion*

$$\Delta : X \longrightarrow X \times_{S,[P]}^{\log} X$$

and the log etale map

$$X \times_{S,[P]}^{\log} X \longrightarrow X \times_S^{\log} X.$$

2. Let $(X, [P]) \to (S, [N])$ *be a map of framed log schemes. Then, the immersion* $\Delta : X \to X \times_{S,[P]}^{\log} X$ *is an exact immersion. Let* $N_{X/X\times_{S,[P]}^{\log}X}$ *be the conormal sheaf of the exact immersion* $\Delta : X \to X \times_{S,[P]}^{\log} X$*. Then, there is a canonical isomorphism*

(4.2.8.1) $$N_{X/X\times_{S,[P]}^{\log}X} \longrightarrow \Omega^1_{(X,M_X)/(S,M_S)}.$$

Proof. — 1. Clear from Proposition 4.2.3.1.

2. Since the projection $X \times_{S,[P]}^{\log} X \to X$ is strict by Corollary 4.2.5.2, the immersion $\Delta : X \to X \times_{S,[P]}^{\log} X$ is an exact immersion. Hence the immersion $\Delta : X \to X \times_{S,[P]}^{\log} X$ is an exactification of the diagonal map $X \to X \times_S^{\log} X$. Thus, taking it as Z in [23] (5.6), we obtain an isomorphism (4.2.8.1) as a special case of loc.cit. (5.8.1). Here, we give more detail. We regard $X \times_{S,[P]}^{\log} X$ as a scheme over X by the second projection $p_2 : X \times_{S,[P]}^{\log} X \to X$. The canonical map $X \times_{S,[P]}^{\log} X \to X \times_S^{\log} X$ is log etale and the projection $p_2 : X \times_{S,[P]}^{\log} X \to X$ is strict by Corolllary 4.2.5.2. Hence we have canonical isomorphisms $p_1^*\Omega^1_{(X,M_X)/(S,M_S)} \to \Omega^1_{(X\times_{S,[P]}^{\log}X, M_{X\times_{S,[P]}^{\log}X})/(X,M_X)} \to \Omega^1_{X\times_{S,[P]}^{\log}X/X}$ and $\Omega^1_{(X,M_X)/(S,M_S)} \to \Delta^*\Omega^1_{X\times_{S,[P]}^{\log}X/X}$. Since the canonical map $N_{X/X\times_{S,[P]}^{\log}X} \to \Delta^*\Omega^1_{X\times_{S,[P]}^{\log}X/X}$ is an isomorphism, the assertion follows. □

Definition **4.2.9.** — *Let* $f : X \to S$ *be a morphism of log schemes,* P *be an fs-monoid and* $X \to [P]$ *be a map. We call the immersion* $\Delta : X \to X \times_{S,[P]}^{\log} X$ *the* log diagonal map.

We may describe the modification associated to a subdivision using the construction above in the following way (cf. [25] Proposition (9.9)). Let P be an fs-monoid and $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$ be the dual monoid. We say a submonoid $N' \subset N$ is a *face* of N if there exists $a \in P$ such that $N' = \{f \in N | f(a) = 0\}$.

Lemma **4.2.10.** — *Let* P *be an fs-monoid and* N' *be a face of* N*. Let* X *be a log scheme and* $X \to [P]$ *be a map. Then*

1. The monoid $P' = \{x \in P^{gp} | f(x) \geq 0 \text{ for } f \in N'\}$ *is an fs-monoid and the canonical map* $N' \to \mathrm{Hom}_{\mathrm{monoid}}(P', \mathbf{N})$ *is an isomorphism. The natural map* $X \times^{\log}_{[P]} [P'] \to X$ *is an open immersion.*

2. Let N'' *be another face of* N. *Then the intersection* $N''' = N' \cap N''$ *is a face of* N. *We define* $P'', P''' \subset P^{gp}$ *similarly as in 1. Then the natural map* $X \times^{\log}_{[P]} [P'''] \to (X \times^{\log}_{[P]} [P']) \times_X (X \times^{\log}_{[P]} [P''])$ *is an isomorphism.*

Proof. — 1. Assume $N' = \{f \in N | f(a) = 0\}$ for $a \in P$. Then, we have $P' = \langle P, a^{-1} \rangle \subset P^{gp}$ and P' is an fs-monoid. The isomorphism $N' \to \mathrm{Hom}_{\mathrm{monoid}}(P', \mathbf{N})$ is clear.

We show that the map $X \times^{\log}_{[P]} [P'] \to X$ is an open immersion. Since the question is etale local on X, we may assume there is a map $X \to \mathbf{S}[P]$ lifting $X \to [P]$. Since $\mathbf{Z}[P'] = \mathbf{Z}[P][a^{-1}]$, we have $X \times^{\log}_{[P]} [P'] = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P][a^{-1}]$ and the assertion follows.

2. Assume $N' = \{f \in N | f(a) = 0\}$ for $a \in P$ and $N'' = \{f \in N | f(a') = 0\}$ for $a' \in P$. Then $N''' = \{f \in N | f(aa') = 0\}$ is a face. Since $P''' = P' \oplus^{\mathrm{sat}}_P P''$, the isomorphism follows. □

We say a sub fs-monoid $N' \subset N$ is saturated in N if $N' = \{x \in N | x^n \in N' \text{ for some } n \geq 1\}$. A sub fs-monoid N' is saturated in N if and only if N'^{gp} is a direct summand of the free abelian group N^{gp}. We identify a sub fs-monoid N' saturated in N with the dual $\mathrm{Hom}_{\mathrm{monoid}}(P', \mathbf{N})$ of $P' = \{x \in P^{gp} | f(x) \geq 0 \text{ for } f \in N'\}$. We say a finite set Σ of submonoids of N is a *subdivision* of N if the following conditions 1.–3. are satisfied:

1. If N' is in Σ, N' is saturated in N.
2. If $N' \in \Sigma$ and N'' is a face of N', then $N'' \in \Sigma$.
3. If $N', N'' \in \Sigma$, the intersection $N' \cap N''$ is a face of N' and of N'' and hence is in Σ.

We call an element $\sigma \in \Sigma$ a face in Σ. If a subdivision Σ further satisfies the following condition 4 (resp. 5), we say Σ is *proper* (resp. *regular*).

4. $N = \bigcup_{\sigma \in \Sigma} N_\sigma$.
5. There exists an isomorphism $N_\sigma \to \mathbf{N}^{r(\sigma)}$ for each $\sigma \in \Sigma$.

Let P be an fs-monoid and Σ be a subdivision of the dual monoid $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$. In the following, we write $\Sigma = \{N_\sigma | \sigma \in \Sigma\}$. Let X be a log scheme and $X \to [P]$ be a map. Then we define a log scheme X_Σ log etale over X as follows. For $\sigma \in \Sigma$, we put $P_\sigma = \{x \in P^{gp} | f(x) \geq 0 \text{ for } f \in N_\sigma\}$. Then the log scheme $X_\sigma = X \times^{\log}_{[P]} [P_\sigma]$ log etale over X is defined. For $\sigma \subset \tau$, we have an open immersion $X_\sigma \to X_\tau$ by Lemma 4.2.10.1. Patching X_σ for $\sigma \in \Sigma$, we define a log scheme $X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$ log etale over X.

For a face τ in Σ, a closed subscheme $V_\tau \subset X_\Sigma$ is defined by patching the closed subschemes $V_\tau \cap X_\sigma$ of X_σ defined by the ideal generated by $P_\sigma \setminus \{x \in P_\sigma | f(x) = 0$ for all $f \in N_\tau\}$ for $\sigma \supset \tau$.

Lemma **4.2.11.** — *Let* P *be an fs-monoid and* Σ *be a subdivision of the dual monoid* N. *Let* X *be a log scheme and* $X \to [P]$ *be a map.*

1. ([25] Proposition (9.11)) *If* Σ *is proper, the map* $X_\Sigma \to X$ *is proper.*

2. Assume X *is log regular* ([25] Definition (2.1)) *locally noetherian,* $X \to [P]$ *is a frame and the subdivision* Σ *is regular. Then, the scheme* X_Σ *is regular and the log structure on* X_Σ *is defined by a divisor with simple normal crossings.*

3. Let σ *and* σ' *be faces in* Σ. *If there exists* $\tau \in \Sigma$ *such that* $\sigma, \sigma' \subset \tau$, *the intersection* $V_\sigma \cap V_{\sigma'}$ *is equal to* V_τ *for the smallest* τ *satisfying* $\sigma, \sigma' \subset \tau$. *If there exists no such* $\tau \in \Sigma$, *the intersection* $V_\sigma \cap V_{\sigma'}$ *is empty.*

Proof. — 2. Since the map $X_\Sigma \to X$ is log etale, the log scheme X_Σ is log regular. Hence it follows from Lemma 4.1.4.2.

3. Clear from the definition. □

Lemma **4.2.12.** — *Let* X *be a regular locally noetherian scheme of dimension* n *and* D *be a divisor with normal crossings. Let* $\bar{D}$ *be the normalization of* D *and* V_i *be the closed subset* $\{x \in X | \deg_x \bar{D}_x \geq n - i\}$ *with the reduced closed subscheme structure. We put* $X_0 = X$ *and, for* $0 \leq i \leq n-2$, *define* $X_{i+1} \to X_i$ *inductively to be the blow-up at the proper transform* V'_i *of* V_i. *Then,*

1. The scheme X_i *is regular. The reduced inverse image* D_i *of* D *in* X_i *is a divisor with normal crossings. The subscheme* V'_i *is regular for* $0 \leq i \leq n-1$.

2. The divisor D_{n-1} *has simple normal crossings.*

Proof. — 1. Since the assertion is etale local, we may assume that the divisor D has simple normal crossings. Let M_X be the standard log structure of X and put $P = \Gamma(X, \bar{M}_X)$. Let $D_1, \ldots, D_r$ be the irreducible components of D and we identify $P = \mathbf{N}^r$. We describe the blow-up $X_i \to X$ in terms of a partial barycentric subdivision of a simplex as follows.

We regard $\Delta = \{1, \ldots, r\}$ as the set of vertices $\{f_1, \ldots, f_r\}$ of the simplex $|\Delta|$ spanned by the standard basis $f_1, \ldots, f_r$ of $\mathbf{R}^r$. We define a subdivision of $|\Delta|$ as follows. For a subset $\tau \subset \Delta$, let $b_\tau = \sum_{j \in \tau} f_j / \mathrm{Card}\, \tau$ be the barycenter of the face spanned by $f_j, j \in \tau$. For each $0 \leq i < n$, let $\Delta_i = \Delta \amalg \{b_\tau | \tau \subset \Delta, \#\tau > n - i\}$ be the set of vertices of $|\Delta|$ together with the barycenters of faces with dimension $\geq n - i$. We say a subset $\sigma \subset \Delta_i$ is a face of Δ_i if the following condition is satisfied: There exists a sequence $\sigma_0 \subsetneq \ldots \subsetneq \sigma_k$ such that $\mathrm{Card}\sigma_0 \leq n - i$, $\mathrm{Card}\sigma_1 > n - i$ and $\sigma = \sigma_0 \amalg \{b_{\sigma_1}, \ldots, b_{\sigma_k}\}$. Let Σ_i be the set of faces of Δ_i. We define a regular and proper subdivision Σ_i of the dual monoid $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$. Let $e_1, \ldots, e_r$ be the standard basis of $P = \mathbf{N}^r$

and $f_1, \ldots, f_r$ be the dual basis of N. For a subset $\tau \subset \Delta$, we put $f_\tau = \sum_{j\in\tau} f_j \in \mathrm{N}$. For a face σ in Σ_i, we put $\mathrm{N}_\sigma = \langle f_\tau | \tau \in \sigma\rangle$. Then $(\mathrm{N}_\sigma)_{\sigma\in\Sigma_i}$ is a regular proper subdivision of N. We have $\mathrm{X}_i = \mathrm{X}_{\Sigma_i}$. By Lemma 4.2.11.2, X_i is regular and the divisor D_i has simple normal crossings.

For a subset $\tau \subset \Delta$, let D_τ be the intersection $\bigcap_{i\in\tau} \mathrm{D}_i$. We have $\mathrm{V}_i = \bigcup_{\#\tau=n-i} \mathrm{D}_\tau$. For a subset $\tau \subset \Delta$ satisfying $\#\tau = n-i$, the proper transform of D_τ in X_i is the closed subscheme V_τ of X_{Σ_i} defined by the face $\tau \in \Sigma_i$. Since V_τ is regular and $\mathrm{V}_\tau \cap \mathrm{V}_{\tau'} = \emptyset$ if $\tau \neq \tau'$ by Lemma 4.2.11.3, the closed subscheme $\mathrm{V}'_i = \coprod_{\#\tau=n-i} \mathrm{V}_\tau$ is regular.

2. By 1, V'_{n-1} is a regular divisor. Since the exceptional divisors are also regular, every irreducible components of the divisor D_{n-1} is regular. Therefore D_{n-1} has simple normal crossings. □

4.3. *Log products and properties of morphisms of log schemes.* — In [32], for a property $\mathscr{P}$ of morphisms of algebraic spaces, Olsson gives a definition for a morphism of log schemes to have property log $\mathscr{P}$, using algebraic stacks. We give an interpretation of the definition without using algebraic stack under the condition (P1) below, after briefly recalling the main result and the definition in [32].

For a log scheme S, a stack $\mathscr{L}og_\mathrm{S}$ over S is defined. An object of $\mathscr{L}og_\mathrm{S}$ is a log scheme X over S and a morphism is a strict morphism over S. The natural map $\mathscr{L}og_\mathrm{S} \to \mathrm{S}$ is defined by sending a log scheme X to the underlying scheme. The main result, Theorem 1.1, of [32] asserts that the stack $\mathscr{L}og_\mathrm{S}$ is an algebraic stack locally of finite presentation over S. In the following, we identify an object X of $\mathscr{L}og_\mathrm{S}$ with the induced morphism $\mathrm{X} \to \mathscr{L}og_\mathrm{S}$. The identity of S defines a section $\mathrm{S} \to \mathscr{L}og_\mathrm{S}$. The section $\mathrm{S} \to \mathscr{L}og_\mathrm{S}$ is an open immersion (loc. cit. Proposition 3.19 (ii)). A map $\mathrm{X} \to \mathrm{S}$ of log schemes induces a natural map $\mathscr{L}og_\mathrm{X} \to \mathscr{L}og_\mathrm{S}$. The map $\mathscr{L}og_\mathrm{X} \to \mathscr{L}og_\mathrm{S}$ is relatively representable. Namely for an arbitrary object $\mathrm{T} \to \mathscr{L}og_\mathrm{S}$, the fiber product $\mathscr{L}og_\mathrm{X} \times_{\mathscr{L}og_\mathrm{S}} \mathrm{T}$ is representable by an algebraic space.

For a property $\mathscr{P}$ of morphisms of algebraic spaces, we say a morphism $\mathrm{X} \to \mathrm{S}$ of log schemes is log $\mathscr{P}$ (resp. weakly log $\mathscr{P}$) if the induced morphism $\mathscr{L}og_\mathrm{X} \to \mathscr{L}og_\mathrm{S}$ (resp. the composition $\mathrm{X} \to \mathscr{L}og_\mathrm{X} \to \mathscr{L}og_\mathrm{S}$) of algebraic stacks is $\mathscr{P}$. Namely for an arbitrary object $\mathrm{T} \to \mathscr{L}og_\mathrm{S}$, the base change $\mathscr{L}og_\mathrm{X} \times_{\mathscr{L}og_\mathrm{S}} \mathrm{T} \to \mathrm{T}$ (resp. the composition $\mathrm{X} \times_{\mathscr{L}og_\mathrm{S}} \mathrm{T} \to \mathscr{L}og_\mathrm{X} \times_{\mathscr{L}og_\mathrm{S}} \mathrm{T} \to \mathrm{T}$) is $\mathscr{P}$ (loc. cit. Definition 4.1). Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition:

(P1) Let $(\mathrm{U}_i \to \mathrm{X})_{i\in\mathrm{I}}$ be an etale covering of X. Then $\mathrm{X} \to \mathrm{S}$ is $\mathscr{P}$ if and only if the compositions $\mathrm{U}_i \to \mathrm{X} \to \mathrm{S}$ are $\mathscr{P}$ for all $i \in \mathrm{I}$.

Then we say a morphism $\mathrm{X} \to \mathrm{S}$ of algebraic spaces is $\mathscr{P}$ if, for any scheme U etale over X, the composition $\mathrm{U} \to \mathrm{X} \to \mathrm{S}$ is $\mathscr{P}$. Thus, for a morphism of log schemes, we have the following.

Lemma **4.3.1.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satifying the condition* (P1). *Then, for a morphism $\mathrm{X} \to \mathrm{S}$ of log schemes, the following conditions are equivalent.*

(1) $X \to S$ *is log* $\mathscr{P}$ *(resp. weakly log* $\mathscr{P}$*).*

(2) *For an arbitrary commutative diagram*

$$\begin{array}{ccc} W & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array} \tag{4.3.1.1}$$

of log schemes, if $W \to X \times_S^{\log} T$ *is log etale and if* $W \to T$ *is strict (resp. and if* $W \to T$ *and* $W \to X$ *are strict), then the underlying map* $W \to T$ *is* $\mathscr{P}$*.*

Proof. — First, we show the assertion for log $\mathscr{P}$. By the definition, an object of $\mathscr{L}og_X \times_{\mathscr{L}og_S} T$ is a commutative diagram (4.3.1.1) of log schemes such that $W \to T$ is strict. Thus, it is sufficient to show that, for a scheme W over $\mathscr{L}og_X \times_{\mathscr{L}og_S} T$, the map $W \to \mathscr{L}og_X \times_{\mathscr{L}og_S} T$ of algebraic spaces is etale if and only if the map $W \to X \times_S^{\log} T$ of log schemes is log etale. The algebraic space $\mathscr{L}og_X \times_{\mathscr{L}og_S} T$ is naturally endowed with the pull-back log structure of that on T. Then, it suffices to show that the map $\mathscr{L}og_X \times_{\mathscr{L}og_S} T \to X \times_S^{\log} T$ is log etale. The underlying map $\mathscr{L}og_X \times_{\mathscr{L}og_S} T \to X \times_S^{\log} T$ is locally of finite presentation by the main result of [32]. Hence, it is sufficient to show that the map $\mathscr{L}og_X \times_{\mathscr{L}og_S} T \to X \times_S^{\log} T$ is formally log etale by loc.cit. Theorem 4.6. We consider a commutative diagram

$$\begin{array}{ccc} W_0 & \longrightarrow & \mathscr{L}og_X \times_{\mathscr{L}og_S} T \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \times_S^{\log} T \end{array}$$

of log schemes such that the map $W_0 \to \mathscr{L}og_X \times_{\mathscr{L}og_S} T$ is strict and that the map $W_0 \to W$ is a nilpotent exact closed immersion. Then, since $W_0 \to T$ is strict, the map $W \to T$ is also strict. Thus, there exists a unique map $W \to \mathscr{L}og_X \times_{\mathscr{L}og_S} T$ making the two triangles commutative. Hence the map $\mathscr{L}og_X \times_{\mathscr{L}og_S} T \to X \times_S^{\log} T$ is formally log etale and is log etale further by loc.cit. Theorem 4.6. Thus the assertion is proved.

Similarly, an object of $X \times_{\mathscr{L}og_S} T$ is a commutative diagram (4.3.1.1) of log schemes such that $W \to T$ and $W \to X$ are strict. Since $X \to \mathscr{L}og_X$ is an open immersion, the composition $X \times_{\mathscr{L}og_S} T \to \mathscr{L}og_X \times_{\mathscr{L}og_S} T \to X \times_S^{\log} T$ is log etale. Thus the assertion for weakly log $\mathscr{P}$ is proved similarly. □

By Lemma 4.3.1, for a property $\mathscr{P}$ of morphisms of schemes satisfying the condition (P1), we may regard the condition (2) in Lemma 4.3.1 as a definition for a morphism of log schemes to be log $\mathscr{P}$. By [32] Theorem 4.6, we recover the definition of log etale, log smooth and log flat in the literature by taking $\mathscr{P}$ to be etale, smooth and flat respectively.

We also consider the following conditions on a property $\mathscr{P}$ of morphisms of schemes:

(P2) If $X \to S$ is $\mathscr{P}$, its base change $X' = X \times_S S' \to S'$ is also $\mathscr{P}$ for an arbitrary map $S' \to S$.

(P3) Let $X \to S'$ be a map and $S' \to S$ be an etale morphism. Then the composition $X \to S$ is $\mathscr{P}$ if and only if $X \to S'$ is $\mathscr{P}$.

(P4) Let $X \to S$ be a morphism of schemes and $S' \to S$ be a faithfully flat map. Then $X \to S$ is $\mathscr{P}$ if the base change $X' \to S'$ is $\mathscr{P}$.

(P5) If $f : X \to Y$ and $g : Y \to S$ are $\mathscr{P}$, the composition $g \circ f : X \to S$ is $\mathscr{P}$.

(P6) If $X \to S$ is $\mathscr{P}$, its base change $X' = X \times_S S' \to S'$ is also $\mathscr{P}$ for a flat map $S' \to S$.

The following is clear from Lemma 4.3.1.

Corollary **4.3.2.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition* (P1). *Let $f : X \to S$ be a morphism of log schemes.*

1. Assume $\mathscr{P}$ satisfies (P2). *If $f : X \to S$ is log $\mathscr{P}$, its base change $f' : X' \to S'$ is also log $\mathscr{P}$ for an arbitrary morphism of log schemes $S' \to S$.*

2. If $X \to S$ is log $\mathscr{P}$ and if $U \to X$ is log etale, the composition $U \to S$ is log $\mathscr{P}$.

3. Assume $\mathscr{P}$ satisfies (P2) *and $f : X \to S$ is strict. Then f is log $\mathscr{P}$ (resp. weakly log $\mathscr{P}$) if and only if the underlying morphism is $\mathscr{P}$.*

4. Assume $\mathscr{P}$ satisfies (P3). *Then, the following conditions are equivalent.*

(1) The map $f : X \to S$ is log $\mathscr{P}$ (resp. weakly log $\mathscr{P}$).

(2) There exist an etale covering $(U_i \to X)_{i \in I}$ of X, etale maps $V_i \to S$ and log $\mathscr{P}$ (resp. weakly log $\mathscr{P}$) maps $g_i : U_i \to V_i$ such that the diagrams

$$\begin{array}{ccc} U_i & \xrightarrow{g_i} & V_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

are commutative for $i \in I$.

We give a criterion for a morphism of log schemes to be log $\mathscr{P}$ using log products and Lemma 4.3.1.

Proposition **4.3.3.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition* (P1). *Let $(X, [P]) \to (S, [N])$ be a map of framed log schemes. We consider the conditions:*

(1) $f : X \to S$ is log $\mathscr{P}$.

(2) $f : X \to S$ is weakly log $\mathscr{P}$.

(1′) (resp. (2′)) For an arbitrary map $T \to S$ of log schemes and an arbitrary map (resp. an arbitrary strict map) $T \to [P]$ such that the diagram

(4.3.3.1)
$$\begin{array}{ccc} T & \longrightarrow & S \\ \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] \end{array}$$

is commutative, the strict map $X \times^{\log}_{S,[P]} T \to T$ *is* $\mathscr{P}$.

We have (1)⇒(2) ⇒(1′) ⇔(2′). If $\mathscr{P}$ *further satisfies the condition* (P3), *the four conditions are equivalent.*

Proof. — (1)⇒(2) and (1′)⇒(2′). Clear.

(2)⇒(2′) We consider the commutative diagram

(4.3.3.2)
$$\begin{array}{ccc} X \times^{\log}_{S,[P]} T & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & S. \end{array}$$

Since $X \to [P]$ is strict, the map $X \times^{\log}_{S,[P]} T \to T$ is strict by Corollary 4.2.5.2. If further $T \to [P]$ is strict, the map $X \times^{\log}_{S,[P]} T \to X$ is also strict by Corollary 4.2.5.2. Since $X \times^{\log}_{S,[P]} T \to X \times^{\log}_{S} T$ is log etale, (2) implies (2′) by Lemma 4.3.1.

(2′)⇒(1′). We consider the commutative diagram (4.3.3.2). Assuming (2′), we show the map $X \times^{\log}_{S,[P]} T \to T$ is $\mathscr{P}$. Let T' be the log scheme as in Lemma 4.1.7.3 such that the map $T \to [P]$ is the composition of a strict map $T' \to [P]$ and a map $T \to T'$ whose underlying map is the identity of T. The diagram (4.3.3.1) with T replaced by T' is commutative. Since $X \times^{\log}_{S,[P]} T = (X \times^{\log}_{S,[P]} T') \times^{\log}_{T'} T$ and the maps $X \times^{\log}_{S,[P]} T \to T$ and $X \times^{\log}_{S,[P]} T' \to T'$ are strict by Corollary 4.2.5.2, the underlying morphism $X \times^{\log}_{S,[P]} T \to T$ of schemes is the same as that of $X \times^{\log}_{S,[P]} T' \to T'$. Since $T' \to [P]$ is strict, the map $X \times^{\log}_{S,[P]} T' \to T'$ is $\mathscr{P}$ by (2′). Thus (2′) implies (1′).

(1′)⇒(1). We consider the commutative diagram (4.3.1.1). We assume $W \to X \times^{\log}_{S} T$ is log etale and $W \to T$ is strict and we show $W \to T$ is $\mathscr{P}$. Since we assume (P1) and (P3), the question is etale local on W and on T by Corollary 4.3.2.4. Let $\bar{w}$ be a geometric point of W and put $P' = \bar{M}_{W,\bar{w}}$. The composition $W \to X \to [P]$ induces a map $P \to P'$ of fs-monoids. Replacing T by an etale neighborhood of the image $\bar{t}$ of $\bar{w}$, we may assume there exists a strict map $T \to [P']$ such that the composition $W \to T \to [P']$ induces the identity $P' \to \bar{M}_{W,\bar{w}}$ since $\bar{M}_{T,\bar{t}} \to \bar{M}_{W,\bar{w}}$ is an isomorphism. We define a map $T \to [P]$ as the composite $T \to [P'] \to [P]$.

We may assume the diagram (4.3.3.1) is commutative by shrinking T if necessary. Shrinking W if necessary, we may assume that the two compositions $W \to X \to [P]$ and $W \to T \to [P]$ are equal. Hence, we obtain a map $W \to X \times^{\log}_{S,[P]} T$ of log schemes log etale over $X \times^{\log}_{S} T$. Thus the map $W \to X \times^{\log}_{S,[P]} T$ is log etale.

The map $X \times^{\log}_{S,[P]} T \to T$ is strict by Corollary 4.2.5.2 and the map $W \to T$ is strict by the assumption. Hence the map $W \to X \times^{\log}_{S,[P]} T$ is also strict and hence is etale. By (1′), the map $X \times^{\log}_{S,[P]} T \to T$ is $\mathscr{P}$. Hence by (P1), the map $W \to T$ is $\mathscr{P}$. Thus the assertion follows by Lemma 4.3.1. □

Corollary **4.3.4.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition* (P1).

1. Assume $\mathscr{P}$ satisfies (P3). *Then a morphism $f : X \to S$ of log schemes is log $\mathscr{P}$ if and only if it is weakly log $\mathscr{P}$.*

2. Assume $\mathscr{P}$ satisfies (P3) *and* (P5). *Then, if morphisms $f : X \to Y$ and $g : Y \to S$ of log schemes are log $\mathscr{P}$, the composition $g \circ f : X \to S$ is also log $\mathscr{P}$.*

3. Assume $\mathscr{P}$ satisfies (P2) *and* (P5). *Let $X \to Y$ and $X' \to Y'$ be maps of log schemes over a log scheme S, $N \to Q \to P$ be maps of fs-monoids and*

$$\begin{array}{ccccccccc}
X & \longrightarrow & Y & \longrightarrow & S & \longleftarrow & Y' & \longleftarrow & X' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[P] & \longrightarrow & [Q] & \longrightarrow & [N] & \longleftarrow & [Q] & \longleftarrow & [P]
\end{array}$$

be a commutative diagram. Assume $X \to [P]$, $Y \to [Q]$ and $S \to [N]$ are strict, $X \to Y$ is log $\mathscr{P}$ and the underlying map of $X' \to Y'$ is $\mathscr{P}$. Then the underlying map of $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} Y'$ is $\mathscr{P}$.

Proof. — 1. By Corollary 4.3.2.4, the assertion is etale local on X and S. Hence we may assume there exists a morphism $(X, [P]) \to (S, [N])$ of framed log schemes by Corollary 4.1.6.2. Thus the assertion follows from the equivalence (1)⇔(2) in Proposition 4.3.3.

2. Since the question is etale local, we may assume that there exist maps $(X, [P]) \to (Y, [Q]) \to (S, [N])$ of framed log schemes. Let $(T, [P]) \to (S, [N])$ be a map of framed log schemes. We consider the diagram (4.3.3.2) and show that the strict map $X \times^{\log}_{S,[P]} T \to T$ is $\mathscr{P}$. By the assumption and Proposition 4.3.3 (1)⇒(2′), the strict maps $X \times^{\log}_{Y,[P]} (Y \times^{\log}_{S,[Q]} T) \to Y \times^{\log}_{S,[Q]} T$ and $Y \times^{\log}_{S,[Q]} T \to T$ are $\mathscr{P}$. Since $X \times^{\log}_{Y,[P]} (Y \times^{\log}_{S,[Q]} T) = X \times^{\log}_{S,[P]} T$, the assertion follows by (P5) and Proposition 4.3.3 (2′)⇒(1).

3. We show the maps $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} X'$ and $Y \times^{\log}_{S,[Q]} X' \to Y \times^{\log}_{S,[Q]} Y'$ are $\mathscr{P}$. In the diagram

(4.3.4.1)
$$\begin{array}{ccc}
X \times^{\log}_{S,[P]} X' & \longrightarrow & Y \times^{\log}_{S,[Q]} X' \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y,
\end{array}$$

the top arrow $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} X'$ is strict since $X \times^{\log}_{S,[P]} X'$ and $Y \times^{\log}_{S,[Q]} X'$ are strict over X' by Corollary 4.2.5.2. The log scheme $X \times^{\log}_{S,[P]} X'$ is log etale over $X \times^{\log}_{Y} (Y \times^{\log}_{S,[Q]} X') = X \times^{\log}_{S,[Q]} X'$. Since $X \to Y$ is log $\mathscr{P}$, the strict map $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} X'$ is $\mathscr{P}$.

In the diagram

$$\begin{array}{ccc} Y \times^{\log}_{S,[Q]} X' & \longrightarrow & Y \times^{\log}_{S,[Q]} Y' \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y', \end{array} \tag{4.3.4.2}$$

the vertical arrows are strict since $Y \to [Q]$ is strict. Hence the diagram of underlying scheme is cartesian. Since the underlying map of $X' \to Y'$ is $\mathscr{P}$, the underlying map of $Y \times^{\log}_{S,[Q]} X' \to Y \times^{\log}_{S,[Q]} Y'$ is $\mathscr{P}$ by (P2). Thus we conclude by (P5). □

In particular, for log flat morphisms, we have the following.

Corollary **4.3.5.** — *1.* (cf. [32] Corollary 4.12 (i)) *If* $X \to S$ *is log flat and* $S' \to S$ *is a map of log schemes, the base change* $X \times^{\log}_{S} S' \to S'$ *is log flat.*

2. (cf. [32] Corollary 4.12 (ii)) *If* $X \to Y$ *is log flat and* $Y \to S$ *is log flat, the composition* $X \to S$ *is log flat.*

3. If X *and* Y *are log flat log schemes over* S*, the log fiber product* $X \times^{\log}_{S} Y$ *is log flat over* S.

4. Let X *and* Y *be log schemes over* S *and* $N \to P$ *be a map of fs-monoids. Let*

$$\begin{array}{ccccc} X & \longrightarrow & S & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] & \longleftarrow & [P] \end{array}$$

be a commutative diagram and assume $X \to [P]$ *and* $S \to [N]$ *are strict. If* $X \to S$ *is log flat, the strict map* $X \times^{\log}_{S,[P]} Y \to Y$ *is flat.*

5. Let $X \to Y$ *and* $X' \to Y'$ *be maps of log schemes over a log scheme* S *and let* $N \to Q \to P$ *be maps of fs-monoids. Let*

$$\begin{array}{ccccccccc} X & \longrightarrow & Y & \longrightarrow & S & \longleftarrow & Y' & \longleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [Q] & \longrightarrow & [N] & \longleftarrow & [Q] & \longleftarrow & [P] \end{array}$$

be a commutative diagram and assume $X \to [P]$, $Y \to [Q]$ *and* $S \to [N]$ *are strict. If* $X \to Y$ *is log flat and if the underlying map of* $X' \to Y'$ *is flat, the underlying map of* $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} Y'$ *is flat.*

Proof. — 1 and 2. It suffices to apply Corollaries 4.3.2.1 and 4.3.4.2 respectively.
3. It follows from 1 and 2.
4. It suffices to apply Proposition 4.3.3 (1)⇒(1′).
5. It follows from Corollary 4.3.4.3. □

In Section 4.4, we define morphisms log locally of complete intersection as a special case of the following definition.

Definition **4.3.6.** — *Let* $\mathscr{P}$ *be a property of morphisms of schemes satisfying the condition* (P1). *We say a morphism of log schemes* $X \to S$ *is* very weakly log $\mathscr{P}$ *if the following condition is satisfied.*

For an arbitrary commutative diagram

$$\text{(4.3.6.1)}\qquad \begin{array}{ccc} W & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of log schemes, if $T \to S$ *is log flat,* $W \to X \times_S^{\log} T$ *is log etale and if* $W \to T$ *and* $W \to X$ *are strict, then the underlying map* $W \to T$ *is* $\mathscr{P}$.

For a property $\mathscr{P}$ satisfying (P1), a weakly log $\mathscr{P}$ morphism is very weakly log $\mathscr{P}$.

Similarly as in Corollary 4.3.2.4, if $\mathscr{P}$ satisfies (P1) and (P3), the following conditions are equivalent.

(1) The map $f : X \to S$ is very weakly log $\mathscr{P}$.

(2) There exist an etale covering $(U_i \to X)_{i\in I}$ of X, etale maps $V_i \to S$ and very weakly log $\mathscr{P}$ maps $g_i : U_i \to V_i$ such that the diagrams

$$\begin{array}{ccc} U_i & \xrightarrow{g_i} & V_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

are commutative for $i \in I$.

The following lemma is useful in the study of very weakly log $\mathscr{P}$ morphisms.

Lemma **4.3.7.** — *Let* $N \to P$ *be an injection of fs-monoids. Then the induced map* $\mathbf{S}[P] \to \mathbf{S}[N]$ *of log schemes is log flat. More precisely, for an arbitrary log schemes* T *over* $\mathbf{S}[N]$ *and an arbitrary strict map* $T \to [P]$ *such that the diagram*

$$\begin{array}{ccc} T & \longrightarrow & \mathbf{S}[N] \\ \downarrow & & \downarrow \\ [P] & \longrightarrow & [N] \end{array}$$

is commutative, the strict map $T \times^{\log}_{\mathbf{S}[N],[P]} \mathbf{S}[P] \to T$ *is faithfully flat.*

Proof. — Since flatness satisfies (P1) and (P3), it is sufficient to show the second assertion by Proposition 4.3.3 (2′)⇒(1). The assertion is etale local on T. Hence by Corollary 4.1.6.2, we may assume there exists a map $(T, P') \to (\mathbf{S}[N], N)$ of charted log schemes where $P' \subset P^{gp} \oplus N^{gp}$ is the inverse image of P as in Lemma 4.1.5.2. Thus it is reduced to the case $T = \mathbf{S}[P']$. In this case, we have $T \times^{\log}_{\mathbf{S}[N],[P]} \mathbf{S}[P] = \mathbf{S}[(P' \oplus_N P)^\sim]$ where $(P' \oplus_N P)^\sim \subset P'^{gp} \oplus_{N^{gp}} P^{gp}$ is the inverse image of P. The isomorphism $P^{gp} \oplus N^{gp} \to P^{gp} \oplus N^{gp} : (a, b) \mapsto (a + \varphi(b), b)$ induces an isomorphism $P' \to P \oplus N^{gp}$ and the isomorphism $P'^{gp} \oplus_{N^{gp}} P^{gp} = (P^{gp} \oplus N^{gp}) \oplus_{N^{gp}} P^{gp} \to P^{gp} \oplus P^{gp} : ((a, b), c) \mapsto (a + \varphi(b) + c, \varphi(b) + c)$ induces an isomorphism $(P' \oplus_N P)^\sim \to P \oplus P^{gp}$. These isomorphisms make a commutative diagram

$$\begin{array}{ccc} P' & \longrightarrow & (P' \oplus_N P)^\sim \\ \downarrow & & \downarrow \\ P \oplus N^{gp} & \xrightarrow{(1,\varphi^{gp})} & P \oplus P^{gp}. \end{array}$$

Since $\varphi^{gp} : N^{gp} \to P^{gp}$ is injective, the map $\mathbf{Z}[N^{gp}] \to \mathbf{Z}[P^{gp}]$ is faithfully flat. Thus the map $T \times^{\log}_{\mathbf{S}[N],[P]} \mathbf{S}[P] = \mathbf{S}[(P' \oplus_N P)^\sim] \to \mathbf{S}[P']$ is faithfully flat. □

Proposition **4.3.8.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition* (P1). *Let $f : (X, [P]) \to (S, [N])$ be a morphism of framed log schemes. We consider the conditions:*

(3) $f : X \to S$ is very weakly log $\mathscr{P}$.

(3′) For an arbitrary map $(T, [P]) \to (S, [N])$ of framed log schemes such that $T \to S$ is log flat, the strict map $X \times^{\log}_{S,[P]} T \to T$ is $\mathscr{P}$.

1. We have (3)⇒(3′). If $\mathscr{P}$ satisfies the condition (P3), *the two conditions are equivalent.*

2. Let $S \to \mathbf{S}[N]$ be a chart lifting the frame $S \to [N]$. Assume $N \to P$ is injective. We consider the condition:

(3″) For $T_P = S \times_{\mathbf{S}[N]} \mathbf{S}[P]$, the strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is $\mathscr{P}$.

Then we have (3′)⇒(3″). If $\mathscr{P}$ satisfies the conditions (P4) *and* (P6), *we have (3″)⇔(3′). If $\mathscr{P}$ satisfies the conditions* (P2) *and* (P4), *the condition (3″) implies the condition (2′) in Proposition 4.3.3.*

Proof. — The proof is similar to that of Proposition 4.3.3. The implications (3)⇒(3′)⇒(3″) are clear. The proof of (3′)⇒(3) is the same as that of (1′)⇒(1) except that here we need to notice that the constructed map $T \to [P]$ is strict after shrinking T if necessary.

We show (3″)⇒(3′). Let $(T, [P]) \to (S, [N])$ be a map of framed log schemes such that $T \to S$ is log flat. We show that the strict map $X \times^{\log}_{S,[P]} T \to T$ is $\mathscr{P}$. We consider the cartesian diagram

$$
\textbf{(4.3.8.1)}\qquad
\begin{array}{ccccc}
X \times^{\log}_{S,[P]} T & \longleftarrow & X \times^{\log}_{S,[P]} T \times^{\log}_{S,[P]} T_P & \longrightarrow & X \times^{\log}_{S,[P]} T_P \\
\downarrow & & \downarrow & & \downarrow \\
T & \longleftarrow & T \times^{\log}_{S,[P]} T_P & \longrightarrow & T_P
\end{array}
$$

of strict morphisms. By (3″), the right vertical map $X \times^{\log}_{S,[P]} T_P \to T_P$ is $\mathscr{P}$. The strict map $T \times^{\log}_{S,[P]} T_P \to T_P$ is flat since $T \to S$ is assumed log flat. Hence by (P6), the middle vertical map $X \times^{\log}_{S,[P]} T \times^{\log}_{S,[P]} T_P \to T \times^{\log}_{S,[P]} T_P$ is $\mathscr{P}$. Since $T \times^{\log}_{S,[P]} T_P \to T$ is faithfully flat by Lemma 4.3.7, the assertion follows by (P4).

The implication (3″)⇒(2′) is proved similarly by replacing (P6) by (P2). □

Corollary **4.3.9.** — *Let $\mathscr{P}$ be a property of morphisms of schemes satisfying the condition* (P1).

1. Assume $\mathscr{P}$ satisfies (P2), (P3) *and* (P4). *Then a morphism $f : X \to S$ of log schemes is log $\mathscr{P}$ if and only if it is very weakly log $\mathscr{P}$.*

2. Assume $\mathscr{P}$ satisfies (P3) *and* (P5). *Then, if morphisms $f : X \to Y$ and $g : Y \to S$ of log schemes are very weakly log $\mathscr{P}$, the composition $g \circ f : X \to S$ is also very weakly log $\mathscr{P}$.*

3. Assume $\mathscr{P}$ satisfies (P6) *and* (P5). *Let $X \to Y$ and $X' \to Y'$ be maps of log schemes over a log scheme S, $N \to Q \to P$ be maps of fs-monoids and*

$$
\begin{array}{ccccccccc}
X & \longrightarrow & Y & \longrightarrow & S & \longleftarrow & Y' & \longleftarrow & X' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[P] & \longrightarrow & [Q] & \longrightarrow & [N] & \longleftarrow & [Q] & \longleftarrow & [P]
\end{array}
$$

be a commutative diagram. Assume $X \to [P]$, $Y \to [Q]$ and $S \to [N]$ are strict, $X \to Y$ is very weakly log $\mathscr{P}$, the underlying map of $X' \to Y'$ is $\mathscr{P}$ and $X' \to S$ and $Y \to S$ are log flat. Then the underlying map of $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} Y'$ is $\mathscr{P}$.

Proof. — 1. It is sufficient to show that a very weakly log $\mathscr{P}$ morphism $X \to S$ is log $\mathscr{P}$. By (P3) and Corollary 4.1.8.2, we may assume there is a map $(X, [P]) \to (S, [N])$ of framed log schemes. By replacing P by the inverse image $P' \subset P^{gp} \oplus N^{gp}$ of P as in Lemma 4.1.5.2, we may assume that the map $N \to P$ is injective. Hence the assertion follows from Proposition 4.3.8 (3″)⇒(2′) and Proposition 4.3.3 (2′)⇒(1).

2. The proof is similar to that of Corollary 4.3.4.2. We only indicate the points where a modification is required. Let $(T, [P]) \to (S, [N])$ be a *log flat* map of framed log schemes. Then, the projection $(Y \times^{\log}_{S,[Q]} T, [P]) \to (Y, [Q])$ is also log flat. Hence, by the assumption and Proposition 4.3.8 (3)⇒(3′), the strict maps $X \times^{\log}_{Y,[P]} (Y \times^{\log}_{S,[Q]} T) \to Y \times^{\log}_{S,[Q]} T$ and $Y \times^{\log}_{S,[Q]} T \to T$ are $\mathscr{P}$. Thus we conclude by (P5) and Proposition 4.3.8 (3′)⇒(3).

3. The proof is similar to that of Corollary 4.3.4.3. We only indicate the points where a modification is required. In the diagram (4.3.4.1), since further $Y \times^{\log}_{S,[Q]} X' \to Y$ is log flat, the strict map $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} X'$ is $\mathscr{P}$. In the diagram (4.3.4.2), since further the strict map $Y \times^{\log}_{S,[Q]} Y' \to Y'$ is flat, the map $Y \times^{\log}_{S,[Q]} X' \to Y \times^{\log}_{S,[Q]} Y'$ is $\mathscr{P}$ by (P6). Thus we conclude by (P5). □

For log flat morphisms, we have the following criterion.

Proposition **4.3.10** ([32] Theorem 4.6). — *For a morphism* $f : X \to S$ *of log schemes, the following conditions are equivalent.*

(1) $f : X \to S$ *is log flat.*

(2) For an arbitrary commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of log schemes, if $W \to X \times^{\log}_{S} T$ *is log etale and* $W \to T$ *is strict, then the underlying map* $W \to T$ *is flat.*

(3) $f : X \to S$ *is very weakly log flat.*

(4) For an arbitrary point $x \in X$*, there exist an* injection $N \to P$ *of fs-monoids and a commutative diagram*

$$\textbf{(4.3.10.1)} \qquad \begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & \mathbf{S}[P] \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & V & \longrightarrow & \mathbf{S}[N] \end{array}$$

of log schemes satisfying the following conditions: The map $U \to X$ *is strict and flat, the image of* $U \to X$ *contains an open neighborhood of* x*,* $V \to S$ *is an open immersion, the maps* $U \to \mathbf{S}[P]$ *and* $V \to \mathbf{S}[N]$ *are strict and the strict map* $U \to V \times^{\log}_{\mathbf{S}[N]} \mathbf{S}[P]$ *is flat.*

Here, we give a proof using Proposition 4.3.8.

Proof. — (1)⇔(2). Since flatness satisfies the condition (P1), it is clear from Lemma 4.3.1.

(1)⇔(3). Since flatness further satisfies the conditions (P2), (P3) and (P4), it is clear from Corollary 4.3.9.1.

(3)⇒(4). Assume $X \to S$ is very weakly log flat. We show that $X \to S$ satisfies the condition (4). The question is etale local on X and S. Hence by Corollary 4.1.8.3 and 4, we may assume there exist an injection $N \to P$ of fs-monoids and a map

$(X, P) \to (S, N)$ of charted log schemes since the map $Q \to P'$ loc.cit. is injective. We put $U = X \times^{\log}_{\mathbf{S}[N],[P]} \mathbf{S}[P]$ and consider the commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & \mathbf{S}[P] \\ \downarrow & & \downarrow & & \downarrow \\ S & = & S & \longrightarrow & \mathbf{S}[N]. \end{array}$$

By Lemma 4.3.7, the strict map $U \to X$ is faithfully flat. We show that the strict map $U \to T_P = S \times^{\log}_{\mathbf{S}[N]} \mathbf{S}[P]$ is flat. We consider the commutative diagram

$$\begin{array}{ccc} U = X \times^{\log}_{S,[P]} T_P & \longrightarrow & T_P \\ \downarrow & & \downarrow \\ X & \longrightarrow & S. \end{array}$$

Then, since $T_P \to S$ is log flat by Lemma 4.3.7 and $X \to S$ is very weakly log flat by the assumption, the strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is flat. Hence the assertion follows.

(4)⇒(3). We assume $X \to S$ satisfies the condition (4) and show that the map $X \to S$ is very weakly log flat. We assume there exist an injection $N \to P$ of fs-monoids and a commutative diagram (4.3.10.1) satisfying the condition in (4). Since the question is etale local on X, we may further assume that the map $U \to X$ is faithfully flat and $V = S$. Then we obtain a commutative diagram

$$\textbf{(4.3.10.2)} \qquad \begin{array}{ccc} U & \longrightarrow & T_P \\ \downarrow & & \downarrow \\ X & \longrightarrow & S. \end{array}$$

The map $U \to X$ is strict and faithfully flat and the map $U \to T_P$ is strict and flat. Since $U \to X$ is strict and surjective, by shrinking them if necessary, we may assume there is a strict map $X \to [P]$ such that the diagram

$$\textbf{(4.3.10.3)} \qquad \begin{array}{ccc} U & \longrightarrow & T_P \\ \downarrow & & \downarrow \\ X & \longrightarrow & [P] \end{array}$$

is commutative.

We show the condition (3″) in Proposition 4.3.8 is satisfied. Namely, we show that the strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is flat. We consider the commutative diagram

$$\textbf{(4.3.10.4)} \qquad \begin{array}{ccc} U \times^{\log}_{S,[P]} T_P & \longrightarrow & T_P \times^{\log}_{S,[P]} T_P \\ \downarrow & & \downarrow \\ X \times^{\log}_{S,[P]} T_P & \longrightarrow & T_P \end{array}$$

induced by the diagrams (4.3.10.2) and (4.3.10.3). The strict map $U \times^{\log}_{S,[P]} T_P \to T_P \times^{\log}_{S,[P]} T_P$ is flat since it is a base change of the strict and flat map $U \to T_P$. By Lemma 4.3.7, the strict map $T_P \times^{\log}_{S,[P]} T_P \to T_P$ is flat. The strict map $U \times^{\log}_{S,[P]} T_P \to X \times^{\log}_{S,[P]} T_P$ is faithfully flat since it is a base change of the strict and faithfully flat map $U \to X$. Hence the strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is flat. □

For a morphism $f : X \to S$ locally of finite presentation of schemes and $x \in X$, we put $s = f(x)$ and

$$\dim_x f^{-1}(f(x)) = \dim \mathcal{O}_{X_s,x} + \mathrm{tr.deg}\ \kappa(x)/\kappa(f(x)).$$

The fiber dimension $\dim_x f^{-1}(f(x))$ at x is equal to the maximum of the dimensions of components of the fiber $X_s = f^{-1}(f(x))$ containing x. We also define a log version. Let $f : X \to S$ be a map of log schemes whose underlying map is locally of finite presentation. For $x \in X$, we put

$$\begin{aligned} &\dim^{\log}_x f^{-1}(f(x)) \\ &= \dim \mathcal{O}_{X_s,\bar{x}}/\big(\alpha\big(M_{X,\bar{x}} \setminus \mathcal{O}^{\times}_{X,\bar{x}}\big)\big) + \mathrm{tr.deg}\ \kappa(x)/\kappa(s) + \mathrm{rank}\ \bar{M}^{gp}_{X,\bar{x}}/\bar{M}^{gp}_{S,\bar{s}} \end{aligned}$$

by taking geometric points $\bar{x}$ and $\bar{s}$ above x and $s = f(x)$.

Proposition **4.3.11** (cf. [3] Lemma 3.10). — *Let $f : X \to S$ be a morphism of log schemes such that the map of underlying schemes is locally of finite presentation. Let*

$$\begin{array}{ccc} W & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

be a commutative diagram of log schemes such that $W \to X \times^{\log}_S T$ is log etale and $W \to T$ and $W \to X$ are strict. Then, for $w \in W$ and its image $x \in X$, we have

$$\dim^{\log}_x f^{-1}(f(x)) = \dim_w g^{-1}(g(w)).$$

Proof. — By replacing S and T by geometric points on the images $s = f(x)$ and $t = g(w)$, we may assume S and T are the spectrums of algebraically closed fields with the pull-back log structures. We put $N = \bar{M}_{S,\bar{s}}$ and $P = \bar{M}_{X,\bar{x}}$. Let $P' \subset P^{gp} \oplus N^{gp}$ be the inverse image of P as in Lemma 4.1.5.2. Since the question is etale local on X, by replacing X by an etale neighborhood of $\bar{x}$, we define a map of charted log schemes $(X, P') \to (S, N)$ as in Corollary 4.1.8.3 and 4. The chart $X \to P'$ induces a chart $W \to P'$ and hence a chart $T \to P'$. Since the question is etale local on W and the strict map $W \to X \times^{\log}_{S,[P']} T$ is etale, we may assume $W = X \times^{\log}_{S,[P']} T$. By replacing S by T with the pull-back log structure of that of S, we may assume the underlying map $T \to S$ is the identity.

By Proposition 4.2.3.3, we have $X \times^{\log}_{S,[P']} T = X \times_{\mathbf{S}[P' \oplus P']} \mathbf{S}[(P' \oplus_N P')^\sim]$. Let $\alpha : P' \to \Gamma(X, \mathscr{O}_X)$ and $\alpha_t : P' \to \kappa(t)$ denote the maps defining the charts $X \to P'$ and $T \to P'$. Then by Corollary 4.2.6.2, the underlying scheme of $X \times^{\log}_{S,[P']} T$ is identified with the closed subscheme of $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{S}[P'^{gp}/N^{gp}]$ defined by the ideal $I = ((\alpha(a) - \alpha_t(a)) \otimes \bar{a}; a \in P')$. Since $\alpha_t(a) = 0$ for $a \notin P'^\times$, the ideal I is the sum of $I_1 = (\alpha(a) \otimes 1; a \in P' \setminus P'^\times)$ and $I_2 = (1 \otimes \bar{a} - (\alpha_t(a^{-1})\alpha(a)) \otimes 1; a \in P'^\times)$. Since $P = P'/P'^\times$, the closed subscheme of $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{S}[P'^{gp}/N^{gp}]$ defined by the ideal I_2 is identified with $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{S}[P^{gp}/N^{gp}]$. Hence $X \times^{\log}_{S,[P']} T$ is identified with the closed subscheme of $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{S}[P^{gp}/N^{gp}]$ defined by the image of the ideal $I_1 = (\alpha(a) \otimes 1; a \in P' \setminus P'^\times)$. Thus the assertion follows. □

4.4. *Log locally of complete intersection morphisms.* — We briefly recall the definition and some facts on morphisms locally of complete intersection. Let $X \to S$ be a morphism locally of finite presentation of schemes. As we have recalled in Definition 1.6.1, we say X is locally of complete intersection over S if, for each $x \in X$, there exist an open neighborhood U of x in X, a smooth scheme P over S and a regular immersion $U \to P$ over S. Assume X is locally of complete intersection over S. For $x \in X$, the difference $d_x = \mathrm{rank}\Omega^1_{P/S,x} - \mathrm{rank}N_{U/P,x}$ in the notation above is independent of $U \to P \to S$ ([17] Exp. VIII Proposition 1.8) and is called the virtual relative dimension at x. If d_x is a constant d on X, we say X is of virtual relative dimension d over S. The function d_x is locally constant on X and is different from $\dim_S x$ in Section 2.1. We have the following criterion for a locally of complete intersection morphism to be flat in terms of a relation between d_x and $\dim_x f^{-1}(f(x))$.

We give a criterion for a locally of complete intersection morphism to be flat in terms of the relative dimension. A flat and locally of complete intersection morphism is called a *syntomic* morphism.

Proposition **4.4.1.** — *Let $f : X \to S$ be a locally of complete intersection morphism of virtual relative dimension d. Then, the following conditions are equivalent.*

(1) $f : X \to S$ is flat.

(2) For each point $x \in X$, we have $\dim_x f^{-1}(f(x)) = d$.

Proof. — Since the question is local on X, we may assume there exist a smooth scheme Y over S purely of relative dimension n and a regular immersion $X \to Y$ of codimension $c = n - d$. Let x be a point of X and $(g_1, \ldots, g_c)$ be a regular sequence of $\mathscr{O}_{Y,x}$ generating the ideal defining the immersion $X \to Y$ at x. By [15] Théorème (11.3.8) b)⇔c), the condition (1) at x is equivalent to that the image $(\bar{g}_1, \ldots, \bar{g}_c)$ is a regular sequence of $\mathscr{O}_{Y_{f(x)},x}$. Since $\mathscr{O}_{Y_{f(x)},x}$ is of Cohen-Macaulay, it is further equivalent to that $\dim \mathscr{O}_{X_{f(x)},x} = \dim \mathscr{O}_{Y_{f(x)},x} - c$ by [15] Chap. 0 Corollaire (16.5.6). Since $n = \dim \mathscr{O}_{Y_{f(x)},x} + \operatorname{tr.deg} \kappa(x)/\kappa(f(x))$, the assertion follows. □

Following Definition 4.3.6, we make the following definition. Note that morphisms locally of complete intersection satisfy the properties (P1) and (P3)–(P6) in Section 4.3.

Definition **4.4.2.** — *We say a morphism of log schemes* $X \to S$ *is* log locally of complete intersection *(resp. log locally of complete intersection of* virtual relative dimension *d) if the underlying map is locally of finite presentation and if the following condition is satisfied.*

For an arbitrary commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array} \tag{4.4.2.1}$$

of log schemes, if $T \to S$ *is log flat,* $W \to X \times_S^{\log} T$ *is log etale and if* $W \to T$ *and* $W \to X$ *are strict, then the underlying map* $W \to T$ *is locally of complete intersection (resp. locally of complete intersection of virtual relative dimension d).*

Let $X \to S$ be a log smooth map. Then we say X is purely of relative dimension d, if, for an arbitrary commutative diagram (4.4.2.1) of log schemes such that $W \to X \times_S^{\log} T$ is log etale and $W \to T$ is strict, the underlying smooth map $W \to T$ is purely of relative dimension d. A log smooth scheme X is purely of relative dimension d if and only if the locally free $\mathscr{O}_X$-module $\Omega^1_{(X,M_X)/(S,M_S)}$ is of constant rank d.

Lemma **4.4.3.** — *1. A log smooth morphism (resp. purely of dimension d) is log locally of complete intersection (resp. of virtual relative dimension d).*

2. The composition of log locally of complete intersection morphisms (resp. of virtual relative dimension d and d′) is log locally of complete intersection (resp. of virtual relative dimension $d + d'$*).*

Proof. — 1. If $\mathscr{P}$ is the property "smooth", the property $\log \mathscr{P}$ is "log smooth" by [32] Theorem 4.6. Hence the assertion follows by Lemma 4.3.1.

2. Clear from the corresponding property ([17] Exp. VIII Propositions 1.5 and 1.10) in the non-log case and Corollary 4.3.9.2. □

Proposition **4.4.4.** — *Let* $X \to S$ *be a map of log schemes and assume the underlying map is locally of finite presentation.*

1. The following conditions are equivalent.

(1) $X \to S$ *is log locally of complete intersection.*

(2) For an arbitrary geometric point $\bar{x}$ *of* X*, there exist an etale neighborhood* U *and a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

of log schemes such that $V \to S$ *is log smooth and* $U \to V$ *is an exact and regular closed immersion.*

2. Let $Y \to S$ *be a log smooth morphism of relative dimension* n *and* $X \to Y$ *be an exact closed immersion. Then the following conditions are equivalent.*

(1) $X \to S$ *is log locally of complete intersection of virtual relative dimension* d.

(2) $X \to Y$ *is a regular immersion of codimension* $n - d$.

Proof. — 1. We reduce the assertion 1 to 2. Let $X \to S$ be a morphism of log schemes whose underlying map is locally of finite presentation and $\bar{x}$ be a geometric point of X. It is sufficient to show that there exist an etale neighborhood U of $\bar{x}$, a log smooth log scheme Y over S and an exact closed immersion $U \to Y$ over S. By Corollary 4.1.8.3 and 4, shrinking X and S if necessary, we may assume there exist a map $N \to P$ of fs-monoids such that N^{gp} is a direct summand of P^{gp} and a map $(X, P) \to (S, N)$ of charted log schemes. Then, we obtain a strict map $X \to T_P = S \times^{\log}_{\mathbf{S}[N]} \mathbf{S}[P]$. Since T_P is log smooth over S, by replacing S by T_P, it is reduced to the case $X \to S$ is strict. Now the assertion is clear.

2. The question is etale local on X and on S. By Corollary 4.1.8.3 and 4, shrinking Y and S if necessary, we may assume there exist a map $N \to P$ of fs-monoids such that N^{gp} is a direct summand of P^{gp} and a map $(Y, P) \to (S, N)$ of charted log schemes. Let $T_P = S \times^{\log}_{\mathbf{S}[N]} \mathbf{S}[P]$ be as in Proposition 4.3.8. We consider the commutative diagram

(4.4.4.1)
$$\begin{array}{ccccc} X \times^{\log}_{S,[P]} T_P & \longrightarrow & Y \times^{\log}_{S,[P]} T_P & \longrightarrow & T_P \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & S \end{array}$$

and the condition:

(1′) The strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is locally of complete intersection of virtual relative dimension d.

By Proposition 4.3.8.2, the condition (1) is equivalent to (1′). Hence it is sufficient to show that (1′) is equivalent to (2).

(1′)⇒(2). Since the strict map $Y \times^{\log}_{S,[P]} T_P \to T_P$ is smooth purely of relative dimension n, the immersion $X \times^{\log}_{S,[P]} T_P \to Y \times^{\log}_{S,[P]} T_P$ is a regular immersion of codimension $n-d$. Since the left square of (4.4.4.1) is cartesian and the middle vertical arrow $Y \times^{\log}_{S,[P]} T_P \to Y$ is faithfully flat by Lemma 4.3.7, the immersion $X \to Y$ is a regular immersion of codimension $n-d$.

(2)⇒(1′). Since the middle vertical arrow $Y \times^{\log}_{S,[P]} T_P \to Y$ is flat, the immersion $X \times^{\log}_{S,[P]} T_P \to Y \times^{\log}_{S,[P]} T_P$ is a regular immersion of codimension $n-d$. Hence the strict map $X \times^{\log}_{S,[P]} T_P \to T_P$ is locally of complete intersection of virtual relative dimension d. □

Corollary **4.4.5.** — *1. Let* $f : X \to S$ *be a log locally of complete intersection morphism of log schemes and* $Y \to S$ *be a log flat morphism of log schemes. Let* $N \to P$ *be a map of fs-monoids and* $(X,[P]) \to (S,[N])$ *and* $(Y,[P]) \to (S,[N])$ *be maps of framed log schemes. Then, the strict map* $X \times^{\log}_{S,[P]} Y \to Y$ *is locally of complete intersection.*

2. Let $X \to Y$ *and* $X' \to Y'$ *be maps of log schemes over a log scheme* S *and let* $N \to Q \to P$ *be maps of fs-monoids. Let*

$$\begin{array}{ccccccccc} X & \longrightarrow & Y & \longrightarrow & S & \longleftarrow & Y' & \longleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [P] & \longrightarrow & [Q] & \longrightarrow & [N] & \longleftarrow & [Q] & \longleftarrow & [P] \end{array}$$

be a commutative diagram and assume $X \to [P]$, $Y \to [Q]$ *and* $S \to [N]$ *are strict. Assume* $X \to Y$ *is log locally of complete intersection, the underlying map of* $X' \to Y'$ *is locally of complete intersection and* $X' \to S$ *and* $Y \to S$ *are log flat. Then the underlying map of* $X \times^{\log}_{S,[P]} X' \to Y \times^{\log}_{S,[Q]} Y'$ *is locally of complete intersection.*

Proof. — It suffices to apply Proposition 4.3.8.2 (3)⇒(3′) and Corollary 4.3.9.3 respectively. □

Similarly to Proposition 4.4.1, we have a criterion for a log locally of complete intersection morphism to be log flat.

Proposition **4.4.6** (cf. [3] Lemma 3.10). — *Let* $X \to S$ *be a log locally of complete intersection morphism of virtual relative dimension* d*. Then, the following conditions are equivalent.*

(1) The map $f : X \to S$ *is log flat.*

(2) For each $x \in X$*, we have an equality* $\dim^{\log}_x f^{-1}(f(x)) = d$.

Proof. — By Propositions 4.4.1 and 4.3.11, the condition (2) is equivalent to the condition that the map $f : X \to S$ is very weakly log flat. Hence the assertion follows by Corollary 4.3.9.1. □

Corollary **4.4.7.** — *Let* X *and* S *be regular noetherian schemes and* D_X *and* D_S *be divisors with simple normal crossings. Let* $f: X \to S$ *be a morphism of finite type and assume we have an inclusion* $f^{-1}(D_S) \subset D_X$ *of the underlying sets. Let* X *and* S *also denote the log schemes with the standard log structures and* $f: X \to S$ *be the induced map of log schemes. Then*

1. (cf. [3] Lemma 3.9) *The map* $f: X \to S$ *is log locally of complete intersection.*

2. We put $U = S \setminus D_S$ *and* $D_{1,U}, \ldots, D_{m,U}$ *be the irreducible component of* $D_X \cap f^{-1}(U)$. *Assume* $\dim S = 1$, *the underlying map* $X \to S$ *is flat and the irreducible components* $D_{1,U}, \ldots, D_{m,U}$ *and their intersections* $D_{i_1,U} \cap \cdots \cap D_{i_k,U}$ *for* $1 \leq i_1 \leq \ldots \leq i_k \leq m$ *are flat over* U. *Then the map* $f: X \to S$ *is log flat.*

Proof. — 1. We put $N = \Gamma(S, \bar{M}_S)$ and $P = \Gamma(X, \bar{M}_X)$. The assertion is etale local on X and on S. Shrinking them, we may assume there exists a map of charted log schemes $(X, P') \to (S, N)$ by Corollary 4.1.8.3 and 4 where $P' \subset P^{gp} \oplus N^{gp}$ is as in Lemma 4.1.5.2. The map $S' = S \otimes^{\log}_{\mathbf{Z}[N]} \mathbf{Z}[P'] \to S$ is log smooth and the map $X \to S$ is the composition $X \to S' \to S$. Since P' is isomorphic to $P \oplus N^{gp}$ and S' is log regular, the underlying scheme S' is regular and the log structure is the standard one defined by a divisor with simple normal crossings. Thus it is reduced to the case where $X \to S$ is strict. Now the assertion is well-known.

2. We may assume X and S are connected. Let d be the relative dimension of X over S. It is sufficient to show that $\dim^{\log}_x f^{-1}(f(x)) = d$ for each $x \in X$. We put $\operatorname{rank} \bar{M}^{gp}_{X,x} = r$ and let $D_1, \ldots, D_r$ be the irreducible component of D containing x. We put $V = D_1 \cap \cdots \cap D_r$ and put $s = f(x)$.

First, we consider the case $s \in D_S$. Then, V is in $f^{-1}(s)$ and we have $\mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) = \mathcal{O}_{V,x}$. Hence, we have $\dim \mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) + \operatorname{tr.deg} \kappa(x)/\kappa(s) = \dim V = \dim X - r = d + 1 - r$ and $\operatorname{rank} \bar{M}^{gp}_{S,s} = 1$. Next, we assume s is a closed point not in D_S. Then V is flat over S and we have $\mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) = \mathcal{O}_{V_s,x}$. Hence we have $\dim \mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) + \operatorname{tr.deg} \kappa(x)/\kappa(s) = \dim V - 1 = \dim X - r - 1 = d - r$ and $\operatorname{rank} \bar{M}^{gp}_{S,s} = 0$. Finally, we assume s is the generic point of S. Then we have $\mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) = \mathcal{O}_{V_s,x}$, $\dim \mathcal{O}_{X_s,x}/(\alpha(M_{X,x} - \mathcal{O}^\times_{X,x})) + \operatorname{tr.deg} \kappa(x)/\kappa(s) = d - r$ and $\operatorname{rank} \bar{M}^{gp}_{S,s} = 0$. In each case, we obtain $\dim^{\log}_x f^{-1}(f(x)) = d$ as required. □

5. Localized intersection product on schemes over a discrete valuation ring

We study localized intersection theory for regular schemes over a discrete valuation ring and its logarithmic version. In 5.1, we study the non-logarithmic case. We define and study the logarithmic localized intersection product in 5.4. We prove the crucial property Proposition 5.4.3 that it is factored through the generic fiber. As a preliminary, we study the log self-products and the sheaves of logarithmic 1-forms in 5.2 and 5.3 respectively.

In this section, K denotes a discrete valuation field with perfect residue field F, S denotes Spec $\mathcal{O}_K$, $s \in S$ denotes the closed point and π denotes a prime element of K.

5.1. *Non-logarithmic case.* — We study non-logarithmic localized intersection product. In this subsection, X denotes a scheme over $S = \operatorname{Spec} \mathcal{O}_K$ satisfying the following condition:

(R(n)) X is a regular and flat equidimensional scheme of finite type over $\mathcal{O}_K$ of relative dimension $n-1$. The generic fiber X_K is smooth.

Lemma **5.1.1.** — *Let* X *be a scheme over* $\mathcal{O}_K$ *satisfying the condition* (R(n)) *and* x *be a point of* X *in the closed fiber. Then there exist an open neighborhood* U *of* x *and a regular immersion* $U \to P$ *of codimension 1 into a smooth scheme* P *of relative dimension* n *over* $\mathcal{O}_K$. *Namely,* X *is locally a hypersurface of virtual relative dimension* $n-1$ *over* $\mathcal{O}_K$.

Proof. — Let $t_1, \ldots, t_m \in \mathcal{O}_{X,x}$ be a minimal system of generators of the maximal ideal m_x of the local ring $\mathcal{O}_{X,x}$. Let $t_{m+1}, \ldots, t_n \in \mathcal{O}_{X,x}$ be a lifting of a transcendental basis of the residue field $\kappa(x)$ over F such that $\kappa(x)$ is a finite separable extension of $F(t_{m+1}, \ldots, t_n)$. We take an open neighborhood U of x and define a map $U \to \mathbf{A}^n_{\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[T_1, \ldots, T_n]$ by sending T_i to t_i. Then we have $\Omega^1_{U/\mathbf{A}^n_{\mathcal{O}_K}, x} = 0$. By shrinking U if necessary, we may assume $\Omega^1_{U/\mathbf{A}^n_{\mathcal{O}_K}} = 0$, namely $U \to \mathbf{A}^n_{\mathcal{O}_K}$ is unramified. By [15] Corollaire (18.4.7), further shrinking U if necessary, there exist a closed immersion $U \to P$ and an etale morphism $P \to \mathbf{A}^n_{\mathcal{O}_K}$ such that the composition is the map $U \to \mathbf{A}^n_{\mathcal{O}_K}$. The scheme P is smooth over $\mathcal{O}_K$ of relative dimension n. Hence it is regular of dimension $n+1$. Therefore the immersion $U \to P$ is regular of codimension 1. □

We give a local description of the sheaf $\Omega^1_{X/S}$ using an immersion as in Lemma 5.1.1.

Corollary **5.1.2.** — *Let* X *be a scheme over* $\mathcal{O}_K$ *satisfying the condition* (R(n)). *Then*

1. The canonical map $L_{X/S} \to \Omega^1_{X/S}$ *is an isomorphism.*

2. Let $U \to P$ *be an immersion as in Lemma 5.1.1. Then we have an exact sequence*

$$0 \longrightarrow N_{U/P} \longrightarrow \Omega^1_{P/S} \otimes_{\mathcal{O}_P} \mathcal{O}_U \longrightarrow \Omega^1_{U/S} \longrightarrow 0. \tag{5.1.2.1}$$

The $\mathcal{O}_U$*-module* $\Omega^1_{P/S} \otimes_{\mathcal{O}_P} \mathcal{O}_U$ *is locally free of rank* n *and the conormal sheaf* $N_{U/P}$ *is invertible.*

3. The cotangent complex $L_{X/S}$ *satisfies the conditions* (L(n)) *and* (G) *in Section 2.4.*

Proof. — 1. It follows immediately from Lemma 1.6.2.3 and from the assertion 2.

2. For the exact sequence (5.1.2.1), it is sufficient to show the injectivity of $N_{U/P} \to \Omega^1_{P/S} \otimes_{\mathcal{O}_P} \mathcal{O}_U$. Since the generic fiber is smooth, it is injective there. Since X is normal, the map is injective. The rest of assertion is clear.

3. It follows from 2 and Lemma 2.1.1. □

Lemma **5.1.3.** — *Let* X *be a scheme over* $\mathcal{O}_K$ *satisfying the condition* (R(n)). *Let* $i : Z \to X$ *be the closed immersion defined by the ideal* Ann $\Omega^n_{X/S}$ *and* $\mathcal{L}_Z$ *be the invertible* $\mathcal{O}_Z$*-module* $L^1 i^* \Omega^1_{X/S}$.

1. Let W *be a normal scheme of finite type over* s = Spec F *and* $\varphi : W \to Z$ *be a morphism over* S. *Then, there exists a canonical isomorphism* $\varphi^* \mathcal{L}_Z = L^1(i \circ \varphi)^* \Omega^1_{X/S} \to N_{s/S} \otimes \mathcal{O}_W \simeq \mathcal{O}_W$ *of invertible* $\mathcal{O}_W$*-modules.*

2. The bivariant Chern class $c_1(\mathcal{L}_Z) \in CH^1(Z \to Z)$ *is* 0.

3. For a scheme T *of finite type over* Z, *the map* $\cdot \mathcal{L}_Z : G(T) \to G(T)$ *sending* $[\mathcal{F}]$ *to* $[\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{L}_Z]$ *is the identity. The canonical map* $G(T) \to G(T)_{/\mathcal{L}_Z} = \mathrm{Coker}(1 - \cdot \mathcal{L}_Z : G(T) \to G(T))$ *is an isomorphism.*

Proof. — 1. The $\mathcal{O}_W$-module $\varphi^* \mathcal{L}_Z = L^1(i \circ \varphi)^* \Omega^1_{X/S}$ is invertible by Corollary 5.1.2.2. Therefore, to define an isomorphism $L^1(i \circ \varphi)^* \Omega^1_{X/S} \to N_{s/S} \otimes \mathcal{O}_W$ of invertible $\mathcal{O}_W$-modules, we may shrink W to an open subset containing all the points of codimension 1. Shrinking W, we may assume W is smooth over s. The distinguished triangle (1.4.0.1) gives us distinguished triangles

(5.1.3.1) $$\to L(i \circ \varphi)^* \Omega^1_{X/S} \longrightarrow L_{W/S} \longrightarrow L_{W/X} \longrightarrow$$

and $\to L_{s/S} \otimes \mathcal{O}_W \to L_{W/S} \to L_{W/s} \to$. Since $L_{s/S} = N_{s/S}[1]$ and $L_{W/s} = \Omega^1_{W/s}$, we have $\mathcal{H}_0(L_{W/S}) = \Omega^1_{W/s}$ and $\mathcal{H}_1(L_{W/S}) = N_{s/S} \otimes_F \mathcal{O}_W$. Taking the cohomology sheaves $\mathcal{H}_1$ of the distinguished triangle (5.1.3.1), we obtain an exact sequence

$$0 \longrightarrow L^1(i \circ \varphi)^* \Omega^1_{X/S} \xrightarrow{a} N_{s/S} \otimes_F \mathcal{O}_W \xrightarrow{b} \mathcal{H}_1(L_{W/X}).$$

We show that the map a is an isomorphism. Since W is locally of complete intersection over X, the $\mathcal{O}_W$-module $\mathcal{H}_1(L_{W/X})$ is locally a subsheaf of a locally free $\mathcal{O}_W$-module and hence is torsion free. On the other hand, since a is injective, the cokernel of a is torsion. Hence the map b is 0 and a is an isomorphism.

2. For a scheme T of finite type over Z, the Chow group $CH_i(T)$ is generated by $\pi_*[W]$ where $\pi : W \to T$ runs through the normalization of integral closed subschemes of T of dimension i. By 1, we have $c_1(\mathcal{L}_Z) \cap \pi_*[W] = \pi_*(c_1(\pi^* \mathcal{L}_Z) \cap [W]) = 0$ and the assertion follows.

3. For a scheme T of finite type over Z, the K-group G(T) is generated by $\pi_*[\mathcal{O}_W]$ where $\pi : W \to T$ runs through the normalization of integral closed subschemes of T. By 1, we have $\mathcal{L}_Z \cdot \pi_*[\mathcal{O}_W] = \pi_*[\mathcal{L}_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_W] = \pi_*[\mathcal{O}_W]$ and the assertion follows. □

Proposition **5.1.4.** — *Let* X *be a scheme over* $S = \mathrm{Spec}\ \mathcal{O}_K$ *satisfying the condition* (R(n)) *and let* $Z \subset X$ *be the closed subscheme defined by the ideal* $\mathrm{Ann}\ \Omega^n_{X/S}$.

Then the spectral sequence

$$E^1_{p,q} = L^{2p+q}\Lambda^{-p}\Omega^1_{X/S} \Rightarrow \mathcal{T}or^{\mathcal{O}_{X\times_S X}}_{p+q}(\mathcal{O}_X, \mathcal{O}_X)$$

(1.6.4.3) *degenerates at* E^1*-terms. It defines an increasing filtration* $F_\bullet$ *on* $\mathcal{T}or^{\mathcal{O}_{X\times_S X}}_n(\mathcal{O}_X, \mathcal{O}_X)$ *satisfying* $F_n = \mathcal{T}or^{\mathcal{O}_{X\times_S X}}_n(\mathcal{O}_X, \mathcal{O}_X)$ *and* $F_{-1} = 0$ *and isomorphisms* $L^p\Lambda^q\Omega^1_{X/S} \to \mathrm{Gr}^F_p \mathcal{T}or^{\mathcal{O}_{X\times_S X}}_n(\mathcal{O}_X, \mathcal{O}_X)$ *for* $p + q = n$. *The* $\mathcal{O}_X$*-modules* $L^p\Lambda^q\Omega^1_{X/S}$ *are* $\mathcal{O}_Z$*-modules for* $p > 0$.

Proof. — We have an isomorphism $M_{X/X\times_S X} \to \Omega^1_{X/S}$ by Corollaries 5.1.2 and 3.4.5. By applying Proposition 1.6.7 to the diagonal embedding $X \to X \times_S X$, we see that the spectral sequence (1.6.4.3) degenerates at E^1-terms. It defines a filtration $F_\bullet$ satisfying the condition up to decalage. The $\mathcal{O}_X$-modules $L^p\Lambda^q\Omega^1_{X/S}$ are $\mathcal{O}_Z$-modules for $p > 0$ by Lemma 2.4.2.1. □

We define the non-logarithmic localized intersection product. Let X be a scheme over $\mathcal{O}_K$ satisfying the condition (R(n)) as above. Let $i : Z \to X$ be the closed immersion defined by the ideal $\mathrm{Ann}\ \Omega^n_{X/S}$ and $\mathcal{L}_Z$ be the invertible $\mathcal{O}_Z$-module $L^1 i^*\Omega^1_{X/S}$ as in Lemma 5.1.3. Then, by Lemmas 5.1.1 and 3.2.4, the projection $pr_2 : X \times_S X \to X$ is locally a hypersurface of virtual relative dimension $n-1$ over X and the closed subscheme of $X \times_S X$ defined by the ideal $\mathrm{Ann}\Omega^n_{X\times_S X/X}$ is the pull-back $Z \times_X (X \times_S X)$ of $Z \subset X$ by the first projection. Let W be a noetherian scheme over $X \times_S X$ and let V be a closed subscheme of $X \times_S X$. We put $T = V \times_{X\times_S X} W$ and $Z_T = Z \times_X T$ be the pull-back by the composition $T \to X \times_S X \to X$ with the first projection. By Lemma 5.1.3.3, we have $G(Z_T)_{/\mathcal{L}_Z} = G(Z_T)$. Thus, the localized intersection product (3.2.2.1) defines a map $[[\ ,\ ]]_{X\times_S X} : G(V) \times G(W) \to G(Z_T)$. Since the generic fiber is smooth, the subscheme Z is supported on the closed fiber X_s and we have a natural map $G(Z_T) \to G(T_s)$.

Definition **5.1.5.** — *Let* X *be a scheme over* $S = \mathrm{Spec}\ \mathcal{O}_K$ *satisfying the condition* (R(n)) *and* $Z \to X$ *be the closed subscheme defined by the ideal* $\mathrm{Ann}\ \Omega^n_{X/S}$. *For a closed subscheme* V *of* $X \times_S X$ *and a noetherian scheme* W *over* $X \times_S X$, *we put* $T = V \times_{X\times_S X} W$ *and call the composition*

(5.1.5.1) $$G(V) \times G(W) \xrightarrow{[[\ ,\ ]]_{X\times_S X}} G(Z_T)_{/\mathcal{L}_Z} = G(Z_T) \longrightarrow G(T_s)$$

the localized intersection product. We also define

(5.1.5.2) $$[[\ , W]]_{X\times_S X} : G(X \times_S X) \longrightarrow G(W_s)$$

as the localized intersection product with the class $[\mathcal{O}_W] \in G(W)$ *by taking* $V = X \times_S X$. *If* $V = X \to X \times_S X$ *is the diagonal map, we call the* localized intersection product

$$\textbf{(5.1.5.3)} \qquad [[X,\ ]]_{X\times_S X} : G(W) \longrightarrow G(T_s)$$

with the class $[\mathcal{O}_X] \in G(X)$ *the localized intersection product with the diagonal.*

By Theorem 3.4.3.1, the map $[[X,\ ]]_{X\times_S X} : G(W) \to G(T_s)$ induces

$$\textbf{(5.1.5.4)} \qquad \begin{aligned} F_pG(W) &\longrightarrow F_{p-n}G(T_s), \\ \mathrm{Gr}^F_pG(W) &\longrightarrow \mathrm{Gr}^F_{p-n}G(T_s). \end{aligned}$$

By abuse of notation, we use the same notation $[[X,\ ]]_{X\times_S X}$ for them. For $W = X \times_S X$, we have

$$\textbf{(5.1.5.5)} \qquad [[X,\ ]]_{X\times_S X} : G(X \times_S X) \longrightarrow G(X_s).$$

For the self-intersection, we have an equality

$$\textbf{(5.1.5.6)} \qquad [[X, X]]_{X\times_S X} = (-1)^n c^X_{n\,Z}\big(\Omega^1_{X/S}\big) \cap [X] = (\Delta_X, \Delta_X)_S$$

in $\mathrm{Gr}^F_0G(X_s)$ by Corollaries 5.1.2.1 and 3.4.5.

The localized Chern class $c^X_{n\,X_F}(\Omega^1_{X/S}) \cap [X] \in CH_0(X_F)$ is computed explicitly as follows.

Lemma **5.1.6.** — *Let* X *be a scheme over* $\mathcal{O}_K$ *satisfying the condition* (R(*n*)) *and let* Z *be the closed subscheme defined by the ideal* Ann $\Omega^n_{X/S}$ *as in Lemma 5.1.3. Let* $\pi : X' \to X$ *be the blow-up at* Z *and* $D = Z \times_X X'$ *be the exceptional divisor.*

Then the pull-back $\pi^*\Omega^1_{X/S}$ *is an extension of a locally free* $\mathcal{O}_{X'}$*-module* $\mathcal{E}'$ *of rank* $n-1$ *by an invertible* $\mathcal{O}_D$*-module and we have*

$$c^X_{n\,Z}\big(\Omega^1_{X/S}\big) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}') \cap [D]).$$

Another computation of $\deg(\Delta_X, \Delta_X)_S$ in terms of the torsion parts of $\Omega^q_{X/S}$ is given in [39].

Example. — Let the notation be as in Lemma 5.1.6. Assume $x \in X$ is an isolated non-degenerate quadratic singularity of the map $X \to S$ and assume $X - \{x\}$ is smooth over S. Then $Z = \{x\}$ with reduced scheme structure, $D \simeq \mathbf{P}^{n-1}_x$ is the exceptional divisor and $\mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$ is a quotient of $\mathcal{O}^n_D$ by $\mathcal{O}_D(-1)$. Hence $c_{n-1}(\mathcal{E}') \cap [D]$ is the class $[x']$ of a $\kappa(x)$-rational point x' of D and $c^X_{n\,Z}(\Omega^1_{X/S}) \cap [X] = \pi_*[x'] = [x]$.

Proof. — By Corollary 5.1.2.3, we may apply Corollary 2.4.5. The assertion follows by Lemma 5.1.3.2. □

We prove a K-theoretic version of the projection formula conjectured in [1] Section 6 formula (20).

Lemma **5.1.7.** — *Let* X *and* Y *be schemes over* $\mathcal{O}_K$ *satisfying the condition* (R(n)) *and* $f : X \to Y$ *be a morphism over* $\mathcal{O}_K$. *Then, for a closed subscheme* Γ *of* $X \times_S X$ *of dimension* n, *we have an equality*

$$[[\Gamma, (f \times f)^* \Delta_Y]]_{X \times_S X} = [[Y, \Gamma]]_{Y \times_S Y}$$

in $F_0 G((X \times_Y X)_s)$.

Proof. — We apply Corollary 3.3.4.3 by taking $Y \leftarrow Y \times_S Y \leftarrow X \times_S X = X \times_S X \to X$, $[\Delta_Y] \in G(Y \times_S Y)$ and $\Gamma \subset X \times_S X$ as $S \leftarrow X \leftarrow W \to X' \to S'$, $\Gamma \in G(X)$ and $V' \subset X'$. Then, since the map $X \times_S X \to Y \times_S Y$ is locally of complete intersection, it is of finite tor-dimension. Thus the assumption of Corollary 3.3.4.3 is satisfied and we obtain the equality in $G((X \times_Y X)_s)$.

We show the right hand side is in $F_0 G((X \times_Y X)_s)$. Since $\dim \Gamma = n$, we have $[\mathcal{O}_\Gamma] \in F_n G(X \times_S X)$. Thus the assertion follows from Theorem 3.4.3.1. □

5.2. *Logarithmic self-products.* — We keep the notation that K is a discrete valuation field with perfect residue field. In this subsection, X denotes a scheme over $\mathcal{O}_K$ satisfying the following condition:

(S′(n)) X is a regular and flat equidimensional scheme over $\mathcal{O}_K$ of finite type of relative dimension $n-1$. The reduced closed fiber $X_{s,\mathrm{red}}$ is a divisor with simple normal crossings.

For a regular and flat equidimensional scheme X over $\mathcal{O}_K$ of relative dimension $n-1$, the condition (S′(n)) is equivalent to the following condition:

> For each closed point x in the closed fiber X_s, there exist a minimal system $(t_1, \ldots, t_n)$ of generaters of the maximal ideal m_x of the local ring $\mathcal{O}_{X,x}$, a unit $u \in \mathcal{O}_{X,x}^\times$ and integers $l_1, \ldots, l_n \geq 0$ such that $\pi = u \prod_i t_i^{l_i}$ for a prime element π of K.

We consider a scheme X satisfying (S′(n)) as a log scheme with the standard log structure M_X defined by the reduced closed fiber. Unless we say otherwise, we also consider $S = \operatorname{Spec} \mathcal{O}_K$ as a log scheme with the standard log structure M_S defined by the closed point. We put $P = \Gamma(X, \bar{M}_X)$ and let $X \to [P]$ denote the standard frame. If $D_1, \ldots, D_m$ are the irreducible components of $X_s = \sum_{i=1}^m l_i D_i$, the monoid

$P = \Gamma(X, \bar{M}_X)$ is identified with $\mathbf{N}^m$. We identify $\Gamma(S, \bar{M}_S) = \mathbf{N}$. The canonical map $\mathbf{N} \to P = \mathbf{N}^m$ sends 1 to $(l_1, \ldots, l_m)$. We define the log self-product $(X \times_S X)^\sim$ to be $X \times^{\log}_{S,[P]} X$ defined in Definition 4.2.4. For schemes X and Y over S satisfying the condition $(S'(n))$, a morphism $f : X \to Y$ over S induces a morphism $(f \times f)^\sim : (X \times_S X)^\sim \to (Y \times_S Y)^\sim$. In the following, we regard the log product $(X \times_S X)^\sim$ as a scheme over X with respect to the second projection.

Lemma **5.2.1.** — *Let* X *be a scheme over* S *satisfying the condition* $(S'(n))$.

1. The map $X \to S$ *is log flat and log locally of complete intersetion. The projection* $(X \times_S X)^\sim \to X$ *is strict and flat.*

2. Let X *and* Y *be schemes over* S *satisfying the condition* $(S'(n))$ *and* $f : X \to Y$ *be a morphism over* S. *Let* $(f \times f)^\sim : (X \times_S X)^\sim \to (Y \times_S Y)^\sim$ *be the map induced by* f. *Then, the underlying map* $(X \times_S X)^\sim \to (Y \times_S Y)^\sim$ *is locally of complete intersection.*

3. Further assume $X \to Y$ *is log flat and its underlying map is flat. Then, the underlying map of* $(X \times_S X)^\sim \to (Y \times_S Y)^\sim$ *is flat.*

Proof. — 1. The map $X \to S$ is log flat and log locally of complete intersetion by Corollary 4.4.7. The map $(X \times_S X)^\sim \to X$ is strict by Corollary 4.2.5.2. Since $X \to S$ is log flat, the strict map $(X \times_S X)^\sim \to X$ is flat by Corollary 4.3.5.4.

2 and 3. It suffices to apply Corollaries 4.4.5.2 and 4.3.5.5 respectively. □

We study the closed fiber of log self-product $(X \times_S X)^\sim$. An irreducible component D_i of the closed fiber X_s is smooth of dimension $n-1$ over the residue field F. We consider two log structures on D_i and introduce two log self-products. Let M_{D_i} be the pull-back log structure on D_i of M_X and let M'_{D_i} be the log structure defined by the divisor $\bigcup_{j \neq i}(D_j \cap D_i)$ with simple normal crossings. Let D_i denote the log scheme (D_i, M_{D_i}) and D'_i denote the log scheme (D_i, M'_{D_i}). There is a canonical map $D_i \to D'_i$ of log schemes. Similarly, let s denote the log point SpecF with the pull-back log structure from S and let s' denote SpecF with the *trivial* log structure. The canonical map $P = \Gamma(X, \bar{M}_X) \to \Gamma(D_i, \bar{M}_{D_i})$ defines a frame $D_i \to [P]$. We identify $P = \mathbf{N}^m$ and let $P_i \subset P = P_i \oplus N_i$ be the submonoid obtained by omitting the i-th component N_i. Then, we have a frame $D'_i \to [P_i]$. We consider the log self-products $D_i \times^{\log}_{s,[P]} D_i$ and $D'_i \times^{\log}_{s',[P_i]} D'_i$. The canonical map $D_i \to D'_i$ induces a map $D_i \times^{\log}_{s,[P]} D_i \to D'_i \times^{\log}_{s',[P_i]} D'_i$.

The following lemma will be used in the proof of Theorem 5.4.3.

Lemma **5.2.2.** — *Let* X *be a scheme over* S *satisfying the condition* $(S'(n))$. *Let* D_i *be an irreducible component of* X_s *and* l_i *be the multiplicity of* D_i *in* X_s. *Then,*

1. The map $D_i \times^{\log}_{s,[P]} D_i \to X \times^{\log}_{S,[P]} X = (X \times_S X)^\sim$ *is a closed immersion and induces an isomorphism to the inverse image* $(X \times_S X)^\sim \times_X D_i$ *of* D_i *by the projection* $(X \times_S X)^\sim \to X$.

2. The underlying scheme $D_i \times^{\log}_{s,[P]} D_i$ *is a* μ_{l_i}*-torsor over* $D'_i \times^{\log}_{s',[P_i]} D'_i$.

Proof. — 1. Since the map $D_i \to X$ is strict, the inverse image $(X \times_S X)^\sim \times_X D_i$ is equal to the log product $X \times^{\log}_{S,[P]} D_i$. The log product $X \times^{\log}_{S,[P]} D_i$ represents the functor sending a log scheme T over S to the set $\{(f : T \to X, g : T \to D_i) | f$ and g are maps over S and induce the same map $P = \Gamma(X, \bar{M}_X) \to \Gamma(T, \bar{M}_T)\}$. The condition that f and g induce the same map $P \to \Gamma(T, \bar{M}_T)$ implies that the map $f : T \to X$ factors through D_i. Thus the canonical map $D_i \times^{\log}_{s,[P]} D_i \to X \times^{\log}_{S,[P]} D_i$ is an isomorphism and the assertion follows.

2. Since the projections $D_i \times^{\log}_{s,[P]} D_i \to D_i$ and $D'_i \times^{\log}_{s',[P_i]} D'_i \to D'_i$ are strict by Corollary 4.2.5.2, it is sufficient to show that $D_i \times^{\log}_{s,[P]} D_i$ is a μ_{l_i}-torsor over $(D'_i \times^{\log}_{s',[P_i]} D'_i) \times^{\log}_{D'_i} D_i = D'_i \times^{\log}_{s',[P_i]} D_i$. We consider the commutative diagram

$$\begin{array}{ccccc} D_i \times^{\log}_{s,[P]} D_i & \longrightarrow & D_i \times^{\log}_{s',[P]} D_i & \longrightarrow & D'_i \times^{\log}_{s',[P_i]} D_i \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & s \times^{\log}_{s',[\mathbf{N}]} s & \longrightarrow & s. \end{array}$$

We have $D_i \times^{\log}_{s',[P]} D_i = D_i \times^{\log}_{D'_i,[N_i]} (D'_i \times^{\log}_{s',[P_i]} D_i)$. Hence by applying Lemma 4.2.7 to $D_i \to D'_i \leftarrow D'_i \times^{\log}_{s',[P_i]} D_i$, we see that $D_i \times^{\log}_{s',[P]} D_i$ is a $\mathrm{Hom}(N_i^{gp}, \mathbf{G}_m)$-torsor over $D'_i \times^{\log}_{s',[P_i]} D_i$. Similarly, we see that $s \times^{\log}_{s',[\mathbf{N}]} s$ is a $\mathrm{Hom}(\mathbf{N}^{gp}, \mathbf{G}_m)$-torsor over s. Further, it is easy to see that the middle vertical map $D_i \times^{\log}_{s',[P]} D_i \to s \times^{\log}_{s',[\mathbf{N}]} s$ is compatible with the map $\mathrm{Hom}(N_i^{gp}, \mathbf{G}_m) \to \mathrm{Hom}(\mathbf{N}^{gp}, \mathbf{G}_m)$ induced by the composition $\mathbf{N} \to P \to N_i$. Namely, it is compatible with the l_i-th power map $\mathbf{G}_m = \mathrm{Hom}(N_i^{gp}, \mathbf{G}_m) \to \mathbf{G}_m = \mathrm{Hom}(\mathbf{N}^{gp}, \mathbf{G}_m)$. Since the left square is cartesian, the assertion follows. □

We construct a compactification of log products of strictly semi-stable schemes. A scheme X locally of finite type over the integer ring $\mathcal{O}_K$ is said to be *strictly semi-stable*, if the following conditions 1–3 are satisfied.

1. X is regular and flat over S.
2. The generic fiber X_K is smooth.
3. The closed fiber is a divisor with simple normal crossings.

A scheme X is strictly semi-stable over S, if and only if Zariski locally it is etale over Spec $\mathcal{O}_K[T_1, ..., T_n]/(T_1 \cdot \cdots \cdot T_r - \pi)$ for some $1 \le r \le n$. For a scheme over S satisfying the condition $(S'(n))$, the condition 3 is equivalent to that the closed fiber is reduced. The standard log structure on a strictly semi-stable scheme X over S is that defined by the closed fiber.

Lemma **5.2.3.** — *1. For a log smooth scheme* X *of finite type over* S, *the following conditions are equivalent.*

(1) X *is strictly semi-stable and the log structure is the standard log structure.*

(2) There exist a map $(X, [P]) \to (S, [\mathbf{N}])$ *of framed log schemes and a quasi-isomorphism* $P \to \mathbf{N}^r$ *such that the composition* $\mathbf{N} \to P \to \mathbf{N}^r$ *sends* 1 *to* $(1, ..., 1)$.

2. Let X *and* Y *be strictly semi-stable schemes with the standard log structures and let* $(X, [P]) \to (S, [\mathbf{N}])$ *and* $(Y, [P]) \to (S, [\mathbf{N}])$ *be maps of framed log schemes. Then the log product* $X \times^{\log}_{S,[P]} Y$ *is strictly semi-stable. The projections* $X \times^{\log}_{S,[P]} Y \to X$ *and* $X \times^{\log}_{S,[P]} Y \to Y$ *are smooth. When* $X = Y$ *and* $[P] = \Gamma(X, \bar{M}_X)$*, the log diagonal map* $X \to (X \times_S X)^{\sim}$ *is a regular immersion.*

Proof. — 1. (1) ⇒ (2), It is sufficient to take the standard frame.

(2) ⇒ (1). By Lemma 4.1.7.2, we may replace P by $\bar{P} = P/P^{\times}$ and hence we may assume $P = \mathbf{N}^r$. Since X is log regular, it follows from Lemma 4.1.4.2 that the underlying scheme X is regular, the open subscheme U is the complement of a divisor D with simple normal crossings and M_X is the standard log structure defined by D. By the assumption that 1 is sent to (1, ..., 1), the divisor D is equal to the closed fiber. Since X is log smooth and the log structure is trivial on the generic fiber, the generic fiber is smooth.

2. The projections are strict and log smooth. Hence the underlying map is smooth. Since $X \times^{\log}_{S,[P]} Y$ is smooth over a strictly semi-stable scheme, it is also strictly semi-stable. The log diagonal map is a section of a smooth map and is a regular immersion. □

Let $\mathbf{N} \to \mathbf{N}^r$ be the map sending 1 to (1, ..., 1) and $P = \mathbf{N}^r \oplus_{\mathbf{N}} \mathbf{N}^r$ be the amalgamate sum. We define a regular proper subdivision of the dual monoid $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$ as follows. We regard $\Delta = \{1, ..., r\} \times \{1, ..., r\}$ as a partially ordered set with the product order. We identify an element $(i, j) \in \Delta$ with an element $f_{i,j} \in N$ characterized by $f_{i,j}(e_{i'}) = \delta_{ii'}$ and $f_{i,j}(e'_{j'}) = \delta_{jj'}$ where $e_{i'}$ and $e'_{j'}$ denote the images of the standard basis of $\mathbf{N}^r$ and δ denotes Kronecker's delta. We say a subset σ of Δ is a face if it is a totally ordered subset. Let Σ be the set of faces of Δ. For a face σ, let N_σ be the submonoid $\langle f_{i,j}, (i, j) \in \sigma \rangle$ of N. The family $(N_\sigma)_{\sigma \in \Sigma}$ is a regular proper subdivision of N.

Lemma **5.2.4** (cf. [41] Lemma 1.2.2). — *Let* X *and* Y *be strictly semi-stable schemes over* S. *Let* $\mathbf{N} \to \mathbf{N}^r$ *be the map sending* 1 *to* (1, ..., 1) *and* $(X, [\mathbf{N}^r]) \to (S, [\mathbf{N}])$ *and* $(Y, [\mathbf{N}^r]) \to (S, [\mathbf{N}])$ *be maps of framed schemes. Let* $P = \mathbf{N}^r \oplus_{\mathbf{N}} \mathbf{N}^r$ *be the amalgamate sum and* $X \times_S Y \to [P]$ *be the induced frame. Let* Σ *be the subdivision of the dual* $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$ *defined above and* $(X \times_S Y)_\Sigma$ *be the associated modification. For* $i = 1, ..., r$*, let* e_i *(resp.* e'_i*) be the image in* P *of the i-th standard basis of the first (resp. second) factor* $\mathbf{N}^r$ *and* $\mathscr{I}_i$ *(resp.* $\mathscr{I}'_i$*) be the ideal locally generated by a lifting of the image of* e_i *(resp.* e_i*) in* $\bar{M}_{X \times_S Y}$.

Then the underlying scheme of $(X \times_S Y)_\Sigma$ *is strictly semi-stable and equal to the blow-up of* $X \times_S Y$ *by the ideal* $\prod_{1 \le i, i' \le r} (\prod_{1 \le j \le i} \mathscr{I}_j + \prod_{1 \le j' \le i'} \mathscr{I}'_{j'})$*. There is an open immersion* $X \times^{\log}_{S,[\mathbf{N}^r]} Y \to (X \times_S Y)_\Sigma$.

Proof. — To show that $(X \times_S Y)_\Sigma$ is strictly semi-stable, it is sufficient to show that $(X \times_S Y) \times_{[P]} [P_\sigma]$ is strictly semi-stable for each face σ. There is an isomorphism

$\mathbf{N}^k \to \mathrm{N}_\sigma$ for $k = \mathrm{Card}\sigma$ and the composition $\mathbf{N}^k \to \mathrm{N}_\sigma \to \mathbf{N} = \mathrm{Hom}(\mathbf{N}, \mathbf{N})$ sends each element of the standard basis to 1. It induces a quasi-isomorphism $\mathrm{P}_\sigma \to \mathbf{N}^k$ such that the composition $\mathbf{N} \to \mathrm{P}_\sigma \to \mathbf{N}^k$ maps 1 to $(1, ..., 1)$. Hence by Lemma 5.2.3.1, the underlying scheme $(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y}) \times^{\log}_{[\mathrm{P}]} [\mathrm{P}_\sigma]$ is strictly semi-stable.

For the proof of the isomorphism from $(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})_\Sigma$ to the blow-up, we refer to [41] Lemma 1.2.2. For the face $\sigma_0 = \{(i, i) | i = 1, ..., r\}$, the monoid P_{σ_0} is the inverse image $(\mathbf{N}^r \oplus_{\mathbf{N}} \mathbf{N}^r)^\sim$ of $\mathbf{N}^r$ as in Proposition 4.2.3.2 and $(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y}) \times^{\log}_{[\mathrm{P}]} [\mathrm{P}_{\sigma_0}] = \mathrm{X} \times^{\log}_{\mathrm{S}, [\mathbf{N}^r]} \mathrm{Y}$ is an open subscheme of $(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})_\Sigma$. □

5.3. *Differentials with log poles.* — We keep the notation that K is a discrete valuation field with perfect residue field. In this subsection, X denotes a scheme over $\mathscr{O}_{\mathrm{K}}$ satisfying the following condition:

(S(n)) X satisfies the condition (R(n)) in Section 5.1 and the condition (S′(n)) in Section 5.2.

We consider a scheme X satisfying (S(n)) as a log scheme with the standard log structure M_{X} defined by the reduced closed fiber. Let M_{S} be the standard log structure on S defined by the closed point.

Lemma **5.3.1.** — *Let* X *be a scheme over* $\mathscr{O}_{\mathrm{K}}$ *satisfying the condition* (S(n)) *and let* x *be a point of* X *in the closed fiber. We consider* X *as a log scheme with the standard log structure* M_{X}. *Let* $\mathrm{D}_1, ..., \mathrm{D}_r$ *be the irreducible components of the closed fiber of* X *containing* x *and* $l_1, ..., l_r$ *be the multiplicities of* $\mathrm{D}_1, ..., \mathrm{D}_r$ *in the closed fiber* X_s.

1. We consider $\mathrm{S} = \mathrm{Spec}\ \mathscr{O}_{\mathrm{K}}$ *as a log scheme with the standard log structure* M_{S}. *We define a ring homomorphism* $\mathbf{Z}[\mathbf{N}] \to \mathscr{O}_{\mathrm{K}}$ *by sending* 1 *to* π *and a map* $\mathbf{N} \to \mathbf{N}^n \times \mathbf{Z}$ *of monoids by sending 1 to* $(l_1, ..., l_r, 0, ..., 0, 1)$. *We define a log smooth scheme* Y_0 *over* $\mathscr{O}_{\mathrm{K}}$ *by* $\mathrm{Y}_0 = \mathrm{Spec}\ \mathscr{O}_{\mathrm{K}} \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[\mathbf{N}^n \times \mathbf{Z}] = \mathrm{Spec}\ \mathscr{O}_{\mathrm{K}}[\mathrm{T}_1, ..., \mathrm{T}_n, \mathrm{W}^{\pm 1}]/(\pi - \mathrm{W} \prod_{i=1}^r \mathrm{T}_i^{l_i})$ *with the log structure defined by the chart* $\mathbf{N}^r \to \mathscr{O}_{\mathrm{K}} \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[\mathbf{N}^n \times \mathbf{Z}]$ *sending the standard basis* e_i *to* T_i *for* $1 \leq i \leq r$.

Then there exist an open neighborhood U *of* x *and a regular immersion* $\mathrm{U} \to \mathrm{Y}$ *of codimension 1 into a log scheme* Y *etale over* Y_0 *such that the divisor* D_i *is defined by the image* $t_i \in \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$ *of* T_i *for* $1 \leq i \leq r$. *The map* $\mathrm{X} \to \mathrm{S}$ *is log flat and log locally of complete intersection.*

2. We consider $\mathrm{S} = \mathrm{Spec}\ \mathscr{O}_{\mathrm{K}}$ *as a log scheme with the* trivial *log structure* $\mathscr{O}_{\mathrm{S}}^\times$. *We regard* $\mathbf{A}^n_{\mathrm{S}} = \mathrm{Spec}\ \mathscr{O}_{\mathrm{K}}[\mathrm{T}_1, ..., \mathrm{T}_n]$ *as a log smooth log scheme over* $\mathscr{O}_{\mathrm{K}}$, *with the log structure defined by the chart* $\mathbf{N}^r \to \mathscr{O}_{\mathrm{K}}[\mathrm{T}_1, ..., \mathrm{T}_n]$ *sending the standard basis* e_i *to* T_i *for* $1 \leq i \leq r$.

Then there exist an open neighborhood U *of* x, *a regular immersion* $\mathrm{U} \to \mathrm{V}$ *of codimension 1 into a log scheme* V *etale over* $\mathbf{A}^n_{\mathrm{S}}$ *and a unit* $v \in \Gamma(\mathrm{V}, \mathscr{O}_{\mathrm{V}}^\times)$ *such that the divisor* D_i *is defined by the image* $t_i \in \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$ *of* T_i *for* $1 \leq i \leq r$ *and the closed subscheme* $\mathrm{U} \to \mathrm{V}$ *is the divisor defined by* $\pi - v \prod_{i=1}^r \mathrm{T}_i^{l_i}$. *The map* $\mathrm{X} \to \mathrm{S}$ *is log locally of complete intersection.*

Proof. — 1. Let t_i be an element of $\mathscr{O}_{X,x}$ defining D_i at x for $1 \leq i \leq r$. We define a unit $w \in \mathscr{O}^\times_{X,x}$ by $\pi = w\prod_{i=1}^r t_i^{l_i}$. Let $t_1, \ldots, t_m$ be a minimal system of generators of the maximal ideal m_x extending $t_1, \ldots, t_r$ and let $t_{m+1}, \ldots, t_n \in \mathscr{O}_{X,x}$ be a lifting of a transcendental basis of the residue field $\kappa(x)$ over F such that $\kappa(x)$ is a finite separable extension of $F(t_{m+1}, \ldots, t_n)$. We take an open neighborhood U of x and define a map $U \to Y_0$ by sending T_i to t_i and W to w. Shrinking U if necessary, we define a regular immersion $U \to Y$ of codimension 1 and an etale morphism $Y \to Y_0$ as in the proof of Lemma 5.1.1. The map $X \to S$ is log flat and log locally of complete intersection by Corollary 4.4.7.

2. Let $t_1, \ldots, t_n \in \mathscr{O}_{X,x}$ and $w = \pi/\prod_{i=1}^r t_i^{l_i} \in \mathscr{O}^\times_{X,x}$ be as in the proof of 1. We take an open neighborhood U of x and define a map $U \to \mathbf{A}^n_{\mathscr{O}_K}$ by sending T_i to t_i. Shrinking U if necessary, we define a regular immersion $U \to V$ of codimension 1 and an etale morphism $V \to \mathbf{A}^n_{\mathscr{O}_K}$ as in the proof of Lemma 5.1.1. Shrinking U and V if necessary, we take a unit $v \in \Gamma(V, \mathscr{O}_V^\times)$ lifting w. Then the function $f = \pi - v\prod_{i=1}^r T_i^{l_i}$ vanishes in $\mathscr{O}_{X,x}$. Since f is not in $m^2_{P,x}$, we have $\mathscr{O}_{X,x} = \mathscr{O}_{V,x}/(f)$. Hence shrinking U and V if necessary, the subscheme U of V is defined by the equation $f = 0$. The map $X \to S$ is log locally of complete intersection by Corollary 4.4.7.1. □

Let $\Omega^1_{X/S}(\log)$ and $\Omega^1_{X/S}(\log/\log)$ denote the $\mathscr{O}_X$-modules $\Omega^1_{(X,M_X)/(S,\mathscr{O}_S^\times)}$ and $\Omega^1_{(X,M_X)/(S,M_S)}$ respectively. The $\mathscr{O}_X$-module $\Omega^1_{X/S}(\log)$ is canonically isomorphic to

$$\big(\Omega^1_{X/S} \oplus \big(\mathscr{O}_X \otimes_{\mathbf{Z}} j_*\mathscr{O}^\times_{X_K}\big)\big)/\big(da - a \otimes a : a \in \mathscr{O}_X \cap j_*\mathscr{O}^\times_{X_K}, 1 \otimes b : b \in K^\times\big)$$

and we have an exact sequence

$$\mathscr{O}_{X_s} \cdot d\log\pi \longrightarrow \Omega^1_{X/S}(\log) \longrightarrow \Omega^1_{X/S}(\log/\log) \longrightarrow 0$$

for a prime element π of K. The canonical maps $\Omega^1_{X/S} \to \Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log)$ induce isomorphisms $\Omega^1_{X_K/K} = \Omega^1_{X/S}|_{X_K} \to \Omega^1_{X/S}(\log)|_{X_K} \to \Omega^1_{X/S}(\log/\log)|_{X_K}$ on the generic fiber.

We give a local description of $\Omega^1_{X/S}$, $\Omega^1_{X/S}(\log)$ and $\Omega^1_{X/S}(\log/\log)$ using immersions as in Lemma 5.3.1.2.

Corollary **5.3.2.** — *Let* X *be a scheme over* $\mathscr{O}_K$ *satisfying the condition* (S(*n*)). *Let* $U \to V$ *be an immersion as in Lemma 5.3.1.2. Then we have a commutative diagram of exact sequences*

$$\textbf{(5.3.2.1)}\quad \begin{array}{ccccccccc} 0 \to & N_{U/V} & \to & \Omega^1_{V/S} \otimes_{\mathscr{O}_V} \mathscr{O}_U & \to & \Omega^1_{U/S} & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & N_{U/V} & \to & \Omega^1_{V/S}(\log) \otimes_{\mathscr{O}_V} \mathscr{O}_U & \to & \Omega^1_{U/S}(\log) & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & N_{U/V} \otimes_{\mathscr{O}_K} m_K^{-1} & \to & \Omega^1_{V/S}(\log) \otimes_{\mathscr{O}_V} \mathscr{O}_U & \to & \Omega^1_{U/S}(\log/\log) & \to 0. \end{array}$$

The $\mathcal{O}_U$*-modules* $\Omega^1_{V/S}\otimes_{\mathcal{O}_V}\mathcal{O}_U$ *and* $\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$ *are locally free of rank n and* $N_{U/V}$ *and* $N_{U/V}\otimes_{\mathcal{O}_K} m_K^{-1}$ *are invertible.*

2. The $\mathcal{O}_X$*-modules* $\Omega^1_{X/S}(\log/\log)$ *and* $\Omega^1_{X/S}(\log)$ *satisfy the conditions* (L(n)) *and* (G) *in Section 2.4.*

Proof. — 1. The top line is the same as in Corollary 5.1.2. The exactness of the middle line is proved similarly as in Corollary 5.1.2. To get the bottom exact sequence, we show that the map $N_{U/V}\to\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$ is extended uniquely to an injection $N_{U/V}\otimes_{\mathcal{O}_K} m_K^{-1}\to\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$. The generator $\pi - v\prod_i T_i^{l_i}$ of $N_{U/V}$ is mapped to $d(v\prod_i T_i^{l_i}) = \pi\cdot(v^{-1}dv+\sum_i l_i d\log T_i)$ in $\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$. Since it is divisible by π, the map $N_{U/V}\to\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$ is uniquely extended to an injection $N_{U/V}\otimes_{\mathcal{O}_K} m_K^{-1}\to\Omega^1_{V/S}(\log)\otimes_{\mathcal{O}_V}\mathcal{O}_U$ sending the generator $(\pi - v\prod_i T_i^{l_i})/\pi$ to $v^{-1}dv+\sum_i l_i d\log T_i$. Since the image of $v^{-1}dv+\sum_i l_i d\log T_i$ in $\Omega^1_{U/S}(\log)$ is $d\log\pi$, the lower sequence is also exact. The rest of assertion is clear.

2. It follows from 1 and Lemma 2.1.1 immediately. □

We study relations between $\Omega^1_{X/S}$, $\Omega^1_{X/S}(\log)$ and $\Omega^1_{X/S}(\log/\log)$. We use the following generalization of the Poincaré residue map [9] II (3.7.2).

Lemma **5.3.3.** — *Let* X *be a locally noetherian regular scheme,* D *be a divisor of* X *with simple normal crossings and* M_X *be the standard log structure on* X *defined by* D. *Let* D_i, $(i\in I)$ *be the irreducible components of* D. *Then, the map* $d\log:\mathcal{O}_X\otimes\bar{M}_X^{gp}\to\Omega^1_{(X,M_X)/(X,\mathcal{O}_X^\times)}$ *induces an isomorphism*

$$\textbf{(5.3.3.1)}\qquad \bigoplus_{i\in I}\mathcal{O}_{D_i}\longrightarrow\Omega^1_{(X,M_X)/(X,\mathcal{O}_X^\times)}.$$

Proof. — The map $d\log: M_X\to\Omega^1_{(X,M_X)/(X,\mathcal{O}_X^\times)}$ induces an isomorphism $\mathcal{O}_X\otimes_{\mathbf{Z}_X}\bar{M}_X^{gp}\to\Omega^1_{(X,M_X)/(X,\mathcal{O}_X^\times)}$. Since $\bar{M}_X^{gp}=\bigoplus_{i\in I}\mathbf{Z}_{D_i}$, we obtain an isomorphism $\bigoplus_{i\in I}\mathcal{O}_{D_i}\to\Omega^1_{(X,M_X)/(X,\mathcal{O}_X^\times)}$. □

Lemma **5.3.4.** — *Let* X *be a scheme over* $\mathcal{O}_K$ *satisfying the condition* (S(n)). *Let* $D_1,\dots,D_m$ *be the irreducible components of the reduced closed fiber* $X_{s,\mathrm{red}}$ *and* l_i *be the multiplicity of* D_i *in* X_s. *Then,*

1. We identify $\Omega^1_{(X,M_X)/X}$ *with* $\bigoplus_{i=1}^m\mathcal{O}_{D_i}$ *by the isomorphism* (5.3.3.1). *Then, the exact sequence* $\Omega^1_{X/S}\to\Omega^1_{(X,M_X)/S}\to\Omega^1_{(X,M_X)/X}\to 0$ *gives an exact sequence*

$$\textbf{(5.3.4.1)}\qquad 0\longrightarrow\Omega^1_{X/S}\longrightarrow\Omega^1_{X/S}(\log)\longrightarrow\bigoplus_{i=1}^m\mathcal{O}_{D_i}\longrightarrow 0.$$

2. We identify $\Omega^1_{(S,M_S)/S}$ *with* F *by the isomorphism* (5.3.3.1). *The exact sequence* $\mathcal{O}_X\otimes_{\mathcal{O}_K}\Omega^1_{(S,M_S)/S}\to\Omega^1_{(X,M_X)/S}\to\Omega^1_{(X,M_X)/(S,M_S)}\to 0$ *gives an exact sequence*

$$\textbf{(5.3.4.2)}\qquad 0\longrightarrow\mathcal{O}_{X_s}\longrightarrow\Omega^1_{X/S}(\log)\longrightarrow\Omega^1_{X/S}(\log/\log)\longrightarrow 0.$$

3. The kernel and cokernel of the map $\Omega^1_{X/S} \to \Omega^1_{X/S}(\log/\log)$ *are isomorphic respectively to the kernel and cokernel of the map* $\mathscr{O}_{X_s} \to \bigoplus_{i=1}^m \mathscr{O}_{D_i}$ *sending 1 to* $(l_1, \ldots, l_m)$.

Proof. — 1. By Lemma 5.3.3, it is sufficient to show the injectivity of $\Omega^1_{X/S} \to \Omega^1_{X/S}(\log)$. The question is local on X. Let $U \to V$ be as in Lemma 5.3.1.2. Then the assertion follows from the injectivity of the upper middle vertical arrow $\Omega^1_{V/S} \otimes_{\mathscr{O}_V} \mathscr{O}_U \to \Omega^1_{V/S}(\log) \otimes_{\mathscr{O}_V} \mathscr{O}_U$ in (5.3.2.1) by the snake lemma.

2. Similarly, by Lemma 5.3.3, it suffices to show that the surjection $\mathscr{O}_{X_s} \to \mathrm{Ker}(\Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log))$ is an isomorphism. Hence, it is reduced to showing that $\mathrm{Ker}(\Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log))$ is an invertible $\mathscr{O}_{X_s}$-module. The question is local on X. The assertion follows from the lower half of the commutative diagram (5.3.2.1) by the snake lemma.

3. The image of 1 by the composition $\mathscr{O}_{X_s} \to \Omega^1_{X/S}(\log) \to \bigoplus_{i=1}^m \mathscr{O}_{D_i}$ is $(l_1, \ldots, l_m)$. The assertion 3 follows from this and the assertions 1 and 2 by the snake lemma. □

Lemma **5.3.5.** — *Let* X *be a scheme over* $\mathscr{O}_K$ *satisfying the condition* (S(n)). *Let* $i : Z \to X$ *be the closed immersion defined by the ideal* $\mathrm{Ann}\, \Lambda^n \Omega^1_{X/S}(\log/\log)$ *and let* $\mathscr{L}_Z = L^1 i^* \Omega^1_{X/S}(\log/\log)$. *Let* $\bar{Z} = Z_{\mathrm{red}}$ *and* $\bar{i} : \bar{Z} \to X$ *be the immersion.*

1. There is a canonical isomorphism $\mathscr{L}_Z \otimes_{\mathscr{O}_Z} \mathscr{O}_{\bar{Z}} = L^1 \bar{i}^* \Omega^1_{X/S}(\log/\log) \to \mathscr{O}_{\bar{Z}}$ *of invertible* $\mathscr{O}_{\bar{Z}}$*-modules.*

2. The bivariant Chern class $c_1(\mathscr{L}_Z) \in \mathrm{CH}^1(Z \to Z)$ *is* 0.

3. For a scheme T *of finite type over* Z, *the map* $\cdot\mathscr{L}_Z : G(T) \to G(T)$ *sending a class* $[\mathscr{F}]$ *to* $[\mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{L}_Z]$ *is the identity. The canonical map* $G(T) \to G(T)_{/\mathscr{L}_Z} = \mathrm{Coker}(1 - \cdot\mathscr{L}_Z : G(T) \to G(T))$ *is an isomorphism.*

Proof. — 1. Applying $L\bar{i}^*$ to the exact sequence (5.3.4.2), we obtain a long exact sequence

$$\begin{aligned} 0 \to \mathscr{O}_{\bar{Z}} \longrightarrow L^1 \bar{i}^* \Omega^1_{X/S}(\log) \longrightarrow L^1 \bar{i}^* \Omega^1_{X/S}(\log/\log) \longrightarrow \\ \mathscr{O}_{\bar{Z}} \longrightarrow \Omega^1_{X/S}(\log) \otimes_{\mathscr{O}_X} \mathscr{O}_{\bar{Z}} \longrightarrow \Omega^1_{X/S}(\log/\log) \otimes_{\mathscr{O}_X} \mathscr{O}_{\bar{Z}} \to 0. \end{aligned}$$

It follows from the lower half of the commutative diagram (5.3.2.1) that the map $\Omega^1_{X/S}(\log) \otimes_{\mathscr{O}_X} \mathscr{O}_{\bar{Z}} \to \Omega^1_{X/S}(\log/\log) \otimes_{\mathscr{O}_X} \mathscr{O}_{\bar{Z}}$ is an isomorphism and the map $L^1 \bar{i}^* \Omega^1_{X/S}(\log) \to L^1 \bar{i}^* \Omega^1_{X/S}(\log/\log)$ is the 0-map. Hence the boundary map $L^1 \bar{i}^* \Omega^1_{X/S}(\log/\log) \to \mathscr{O}_{\bar{Z}}$ is an isomorphism.

2 and 3. Similarly as in the proof of Lemma 5.1.3, it follows from 1. □

Similarly, we have the following analogue for $\Omega^1_{X/S}(\log)$.

Lemma **5.3.6.** — *Let* X *be a scheme over* $\mathscr{O}_K$ *satisfying the condition* (S(n)), $D_1, \ldots, D_m$ *be the irreducible components of* $D = (X_F)_{\mathrm{red}}$ *and let* $J \subset \{1, \ldots, m\}$ *be a non-empty subset* *of the index set of the irreducible components of the closed fiber. We put* $D_J = \bigcap_{i\in J} D_i$ *and let* $i_J : D_J \to X$ *denote the closed immersion.*

1. The scheme D_J *is smooth over* F *of dimension* $n-\#J$ *and the divisor* $B_J = D_J \cap \bigcup_{i\notin J} D_i$ *has simple normal crossings.*

2. The $\mathcal{O}_{D_J}$*-module* $i_J^*\Omega^1_{X/S}(\log) = \Omega^1_{X/S}(\log)\otimes_{\mathcal{O}_X}\mathcal{O}_{D_J}$ *is locally free of rank n and we have an exact sequence*

$$\textbf{(5.3.6.1)}\qquad 0 \to \Omega^1_{D_J/F}(\log B_J) \longrightarrow i_J^*\Omega^1_{X/S}(\log) \longrightarrow \bigoplus_{i\in J}\mathcal{O}_{D_J} \to 0.$$

3. The first map in the exact sequence (5.3.4.2) *induces an isomorphism* $\mathcal{O}_{D_J} \simeq L^1 i_J^*\mathcal{O}_{X_s} \to L^1 i_J^*\Omega^1_{X/S}(\log)$. *We have* $L^q i_J^*\Omega^1_{X/S}(\log) = 0$ *for* $q \neq 0, 1$.

Proof. — 1. Clear.

2. Let M'_{D_J} be the standard log structure on D_J defined by B_J and M_{D_J} be the pull-back log structure of M_X. First, we show that the exact sequence $\Omega^1_{(D_J,M'_{D_J})/F} \to \Omega^1_{(D_J,M_{D_J})/F} \to \Omega^1_{(D_J,M_{D_J})/(D_J,M'_{D_J})} \to 0$ gives an exact sequence

$$\textbf{(5.3.6.2)}\qquad 0 \to \Omega^1_{(D_J,M'_{D_J})/F} \longrightarrow \Omega^1_{(D_J,M_{D_J})/F} \longrightarrow \bigoplus_{i\in J}\mathcal{O}_{D_J} \to 0.$$

A canonical isomorphism $\bigoplus_{i\in J}\mathcal{O}_{D_J} \to \Omega^1_{(D_J,M_{D_J})/(D_J,M'_{D_J})}$ is defined similarly as in Lemma 5.3.3. Hence, it is sufficient to show that the canonical map $\Omega^1_{(D_J,M'_{D_J})/F} \to \Omega^1_{(D_J,M_{D_J})/F}$ is injective. Locally on D_J, the log scheme (D_J, M_{D_J}) is isomorphic to the product of (D_J, M'_{D_J}) with the log point F with the chart $\mathbf{N}^J \to F$ sending the non-0 elements to 0. Thus we obtain a locally splitting exact sequence (5.3.6.2).

We have $\Omega^1_{D_J/F}(\log B_J) = \Omega^1_{(D_J,M'_{D_J})/F}$ and $\Omega^1_{D_J/F}(\log B_J)$ is locally free of rank $n-\#J$ by 1. Hence $\Omega^1_{(D_J,M'_{D_J})/F}$ is locally free of rank n. Since $\Omega^1_{X/S}(\log)$ satisfies the condition (L(n)) in Section 2.4 by Corollary 5.3.2.2, the pull-back $i_J^*\Omega^1_{X/S}(\log)$ is locally generated by n-sections. Hence the canonical surjection $i_J^*\Omega^1_{X/S}(\log) \to \Omega^1_{(D_J,M'_{D_J})/F}$ is an isomorphism and the assertion follows.

3. Since $\Omega^1_{X/S}(\log)$ satisfies the condition (L(n)) in Section 2.4, we have $L^q i_J^*\Omega^1_{X/S}(\log) = 0$ for $q \neq 0, 1$. Further, since $i_J^*\Omega^1_{X/S}(\log)$ is locally free of rank n, the $\mathcal{O}_{D_J}$-module $L^1 i_J^*\Omega^1_{X/S}(\log)$ is invertible. By the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X_s} \to 0$, we obtain an isomorphism $\mathcal{O}_{D_J} \to L^1 i_J^*\mathcal{O}_{X_s}$. We show the map $L^1 i_J^*\mathcal{O}_{X_s} \to L^1 i_J^*\Omega^1_{X/S}(\log)$ is an isomorphism. By the exact sequence (5.3.4.2), we get an exact sequence

$$0 \longrightarrow L^1 i_J^*\mathcal{O}_{X_s} \longrightarrow L^1 i_J^*\Omega^1_{X/S}(\log) \longrightarrow L^1 i_J^*\Omega^1_{X/S}(\log/\log).$$

The first two $\mathcal{O}_{D_J}$-modules are invertible. The last one is locally a submodule of an invertible $\mathcal{O}_{D_J}$-module and is torsion free. Hence the cokernel of the injection $\mathcal{O}_{D_J} \simeq L^1 i_J^*\mathcal{O}_{X_s} \to L^1 i_J^*\Omega^1_{X/S}(\log)$ is 0 and the map is an isomorphism. □

The relation between the localized Chern classes $c_{n\mathrm{X}_\mathrm{F}}^{\mathrm{X}}(\Omega^1_{\mathrm{X}/\mathrm{S}}) \cap [\mathrm{X}]$ and $c_{n\mathrm{X}_\mathrm{F}}^{\mathrm{X}}(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)) \cap [\mathrm{X}]$ is as follows.

Corollary **5.3.7.** — *Let* X *be a scheme over* S *satisfying the condition* (S(*n*)). *Then we have an equality*

$$\begin{aligned}
&\left(c_{n\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}\right) - c_{n\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)\right)\right) \cap [\mathrm{X}] \\
&= c_{n-1}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)\right) \cap [\mathrm{X}_s] \\
&\quad + \sum_{r=1}^{n} \sum_{\mathrm{J}\subset\{1,\ldots,m\},\#\mathrm{J}=r} (-1)^r c_{n-r}\left(\Omega^1_{\mathrm{D}_\mathrm{J}/\mathrm{F}}(\log \mathrm{B}_\mathrm{J})\right) \cap [\mathrm{D}_\mathrm{J}]
\end{aligned} \tag{5.3.7}$$

in $\mathrm{CH}_0(\mathrm{X}_s)$.

Proof. — We have equalities

$$\begin{aligned}
c_{\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}\right) &= c_{\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log)\right) \prod_{i=1}^{m} c_{\mathrm{X}_s}^{\mathrm{X}}(\mathscr{O}_{\mathrm{D}_i})^{-1}, \\
c_{\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)\right) &= c_{\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log)\right) c_{\mathrm{X}_s}^{\mathrm{X}}(\mathscr{O}_{\mathrm{X}_s})^{-1}
\end{aligned}$$

in $\mathrm{CH}^*(\mathrm{X}_s \to \mathrm{X})^{(n)}$ by the exact sequences (5.3.4.1) and (5.3.4.2) and by Lemma 2.3.1.4. Further we have

$$\prod_{i=1}^{m} c_{\mathrm{X}_s}^{\mathrm{X}}(\mathscr{O}_{\mathrm{D}_i})^{-1} \cap [\mathrm{X}] = \prod_{i=1}^{m}(1 - [\mathrm{D}_i]) = \sum_{r=0}^{n} \sum_{\mathrm{J}\subset\{1,\ldots,m\},\#\mathrm{J}=r} (-1)^r [\mathrm{D}_\mathrm{J}]$$

and $c_{\mathrm{X}_s}^{\mathrm{X}}(\mathscr{O}_{\mathrm{X}_s})^{-1} \cap [\mathrm{X}] = [\mathrm{X}] - [\mathrm{X}_s]$ by Corollary 2.3.3. Hence we have an equality

$$\begin{aligned}
&\left(c_{n\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}\right) - c_{n\mathrm{X}_s}^{\mathrm{X}}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)\right)\right) \cap [\mathrm{X}] \\
&= c_{n-1}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log / \log)\right) \cap [\mathrm{X}_s] \\
&\quad + \sum_{r=1}^{n} \sum_{\mathrm{J}\subset\{1,\ldots,m\},\#\mathrm{J}=r} (-1)^r c_{n-r}\left(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log)\right) \cap [\mathrm{D}_\mathrm{J}]
\end{aligned}$$

in $\mathrm{CH}_0(\mathrm{X}_s)$. We have $c(\Omega^1_{\mathrm{X}/\mathrm{S}}(\log)) \cap [\mathrm{D}_\mathrm{J}] = c(\Omega^1_{(\mathrm{D}_\mathrm{J},\mathrm{M}_{\mathrm{D}_\mathrm{J}})/\mathrm{F}}) \cap [\mathrm{D}_\mathrm{J}]$ by Lemma 5.3.6. Thus the assertion follows. □

5.4. *Logarithmic localized intersection product.* — We define logarithmic localized intersection product for a scheme X over $\mathscr{O}_\mathrm{K}$ satisfying the condition (S(*n*)) in the last subsection. We prove that the logarithmic localized intersection product has an advantage that it is factored through the generic fiber in Theorem 5.4.3.

Lemma **5.4.1.** — *Let* X *be a scheme over* $\mathscr{O}_K$ *satisfying the condition* (S(n)). *Let* $i : Z \to X$ *be the closed immersion defined by the ideal* Ann $\Omega^n_{X/S}(\log/\log)$ *and* $\mathscr{L}_Z$ *be the invertible* $\mathscr{O}_Z$-*module* $L^1 i^* \Omega^1_{X/S}(\log/\log)$. *Let* $L_{(X\times_S X)^\sim/X}$ *be the cotangent complex,* $\Delta : X \to (X \times_S X)^\sim$ *be the log diagonal map and* $M_{X/(X\times_S X)^\sim}$ *be the conormal complex. Then,*

1. The projection $pr_2 : (X \times_S X)^\sim \to X$ *is flat and locally a hypersurface of virtual relative dimension* $n-1$ *over* X. *The canonical map* $L_{(X\times_S X)^\sim/X} \to \Omega^1_{(X\times_S X)^\sim/X}$ *is an isomorphism.*

2. The canonical maps $M_{X/(X\times_S X)^\sim} \to L\Delta^* L_{(X\times_S X)^\sim/X} \to \Omega^1_{X/S}(\log/\log)$ *are isomorphisms. The composition induces the isomorphism* $N_{X/(X\times_S X)^\sim} \to \Omega^1_{X/S}(\log/\log)$ (4.2.8.1).

3. The closed subscheme $\tilde{i} : \tilde{Z} \to (X \times_S X)^\sim$ *defined* Ann $\Omega^n_{(X\times_S X)^\sim/X}$ *is equal to the pull-back of* Z *by the first projection* $(X \times_S X)^\sim \to X$. *The invertible* $\mathscr{O}_{\tilde{Z}}$-*module* $\tilde{\mathscr{L}}_{\tilde{Z}} = L^1 \tilde{i}^* \Omega^1_{(X\times_S X)^\sim/X}$ *is equal to the pull-back of* $\mathscr{L}_Z$.

Proof. — 1. Let $X \to [P]$ be the standard frame. By Lemma 5.3.1.1 and by Corollaries 4.3.5.4 and 4.4.5.1, the strict map $(X \times_S X)^\sim = X \times^{\log}_{S,[P]} X \to X$ is flat and locally of complete intersection of virtual relative dimension $n-1$. Let x be a point in the closed fiber and $U \to Y$ be an exact regular immersion as in Lemma 5.3.1.1. Shrinking Y if necessary, we obtain a frame $Y \to [P]$ lifting the restriction $U \to [P]$. Then, since the strict map $Y \times^{\log}_{S,[P]} X \to X$ is smooth of relative dimension n, the strict map $U \times^{\log}_{S,[P]} X \to Y \times^{\log}_{S,[P]} X$ is a regular immersion of codimension 1 by Proposition 4.4.4.2. Since $U \times^{\log}_{S,[P]} X$ for each x gives a covering of the closed fiber of $(X \times_S X)^\sim = X \times^{\log}_{S,[P]} X$ and the generic fiber is assumed smooth, the scheme $(X \times_S X)^\sim$ is locally a hypersurface of virtual relative dimension $n-1$ over S.

We show $L_{(X\times_S X)^\sim/X} \to \Omega^1_{(X\times_S X)^\sim/X}$ is an isomorphism. Since $(X \times_S X)^\sim \to X$ is locally of complete intersection, it is sufficient to show that $\mathscr{H}_1 L_{(X\times_S X)^\sim/X} = 0$. The restriction of $\mathscr{H}_1 L_{(X\times_S X)^\sim/X}$ on the generic fiber is 0 since the generic fiber is smooth. Since $(X \times_S X)^\sim$ is flat over X, it is flat over S. Since $\mathscr{H}_1 L_{(X\times_S X)^\sim/X}$ is locally a subsheaf of locally free module, it is π-torsion free and the assertion follows.

2. We obtain an isomorphism $M_{X/(X\times_S X)^\sim} \to L\Delta^* L_{(X\times_S X)^\sim/X}$ by the distinguished triangle $\to L\Delta^* L_{(X\times_S X)^\sim/X} \to L_{X/X} \to L_{X/(X\times_S X)^\sim} \to$. Since $(X \times_S X)^\sim \to X \times^{\log}_S X$ is log etale, the canonical map $p_2^* \Omega^1_{X/S}(\log/\log) \to \Omega^1_{(X\times_S X)^\sim/X}$ is an isomorphism. Similarly as in 1, we see that it induces an isomorphism $Lp_2^* \Omega^1_{X/S}(\log/\log) \to \Omega^1_{(X\times_S X)^\sim/X}$ by using the assumption that the generic fiber is smooth. By the isomorphism in 1, it induces an isomorphism $L\Delta^* L_{(X\times_S X)^\sim/X} \to \Omega^1_{X/S}(\log/\log)$. The assertion on the composition is clear from the definition.

3. It follows from the isomorphism $Lp_2^* \Omega^1_{X/S}(\log/\log) \to \Omega^1_{(X\times_S X)^\sim/X}$ in the proof of 2. □

We define the logarithmic localized intersection product. Let X be a scheme over S satisfying the condition (S(n)). Let $i : Z \to X$ be the closed immersion and $\mathscr{L}_Z$ be the invertible modules as in Lemma 5.4.1. Let W be a noetherian scheme over

$(X \times_S X)^\sim$ and let V be a closed subscheme of $(X \times_S X)^\sim$. We put $T = V \times_{(X\times_S X)^\sim} W$ and $Z_T = Z \times_X T$ be the pull-back by the composition $T \to (X \times_S X)^\sim \to X$ with the first projection. By Lemmas 5.4.1.3 and 5.3.5.3, we have $Z_T = T \times_{(X\times_S X)^\sim} \tilde{Z}$ and $G(Z_T)_{/\mathscr{L}_{\tilde{Z}}} = G(Z_T)$ in the notation loc.cit. Thus, the localized intersection product (3.2.2.1) defines a map $[[\ ,\ ]]_{(X\times_S X)^\sim} : G(V) \times G(W) \to G(Z_T)$. Since the generic fiber is smooth, the subscheme Z is supported on the closed fiber X_s and we have a natural map $G(Z_T) \to G(T_s)$.

Definition **5.4.2.** — *Let* X *be a scheme over* $S = \operatorname{Spec} \mathscr{O}_K$ *satisfying the condition* (S(n)) *and* $Z \to X$ *be the closed subscheme defined by the ideal* $\operatorname{Ann} \Omega^n_{X/S}(\log/\log)$. *For a closed subscheme* V *of* $(X\times_S X)^\sim$ *and a noetherian scheme* W *over* $(X\times_S X)^\sim$, *we put* $T = V \times_{(X\times_S X)^\sim} W$ *and we call the composition*

$$\textbf{(5.4.2.1)} \qquad G(V) \times G(W) \xrightarrow{[[\ ,\ ]]_{(X\times_S X)^\sim}} G(Z_T)_{/\mathscr{L}_Z} = G(Z_T) \longrightarrow G(T_s)$$

the logarithmic localized intersection product. *We define*

$$\textbf{(5.4.2.2)} \qquad [[\ , W]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \longrightarrow G(W_s)$$

as the logarithmic localized intersection product with the class $[\mathscr{O}_W] \in G(W)$ *by taking* $V = (X \times_S X)^\sim$. *If* $V = X \to (X \times_S X)^\sim$ *is the log diagonal map, we call the log localized intersection product*

$$\textbf{(5.4.2.3)} \qquad [[X,\ ]]_{(X\times_S X)^\sim} : G(W) \longrightarrow G(T_s)$$

with the class $[\mathscr{O}_X] \in G(X)$ *the logarithmic localized intersection product with the log diagonal.*

By Theorem 3.4.3.1, the map $[[X,\ ]]_{(X\times_S X)^\sim} : G(W) \to G(T_s)$ induces maps

$$\textbf{(5.4.2.4)} \qquad F_p G(W) \longrightarrow F_{p-n} G(T_s).$$

By abuse of notation, we use the same notation $[[X,\ ]]_{(X\times_S X)^\sim}$ for them. If there is no fear of confusion, we drop the suffix $_{(X\times_S X)^\sim}$. For $W = (X \times_S X)^\sim$, we have

$$\textbf{(5.4.2.5)} \qquad [[X,\ ]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \longrightarrow G(X_s).$$

For the self-intersection, we have an equality

$$\textbf{(5.4.2.6)} \qquad [[X, X]]_{(X\times_S X)^\sim} = (-1)^n c_{n\,Z}^{\ X}\big(\Omega^1_{X/S}(\log/\log)\big) \cap [X]$$

in $\operatorname{Gr}^F_0 G(X_s)$ by Lemma 5.4.1.2 and Corollary 3.4.5.

The advantage of the logarithmic localized intersection product against the non-logarithmic one is the following Theorem 5.4.3. It claims that the logarithmic localized intersection product is factored through the generic fiber. The non-logarithmic product does not share this property in general.

Theorem **5.4.3.** — *Let* $\mathscr{O}_K$ *be a discrete valuation ring with perfect residue field and* X *be a scheme over* $S = \operatorname{Spec} \mathscr{O}_K$ *satisfying the condition* (S(*n*)). *Then the map* $[[X,\ ]]_{(X\times_S X)^\sim}$: $G((X \times_S X)^\sim) \to G(X_s)$ *is factored by the surjection* $G((X \times_S X)^\sim) \to G(X_K \times_K X_K)$.

Proof. — Let $D_1, \ldots, D_m$ be the irreducible components of X_s. Let $E_i = (X\times_S X)^\sim \times_X D_i$ be the inverse image of D_i by the second projection $(X\times_S X)^\sim \to X$. Since the open subscheme $X_K \times_K X_K$ of $(X\times_S X)^\sim$ is the complement of the union $\bigcup_{i=1}^m E_i$, we have an exact sequence $\bigoplus_{i=1}^m G(E_i) \to G((X\times_S X)^\sim) \to G(X_K \times_K X_K) \to 0$. Hence it is reduced to showing that the composition $G(E_i) \to G((X \times_S X)^\sim) \xrightarrow{[[X,\ ]]} G(X_s)$ is the 0-map for each i. The projection $(X \times_S X)^\sim \to X$ is flat by Lemma 5.2.1.1. Hence by applying Corollary 3.2.5 to $D_i \to X \to (X \times_S X)^\sim \to X \leftarrow D_i$ as $T \to V \to X \to S \leftarrow S'$ loc.cit., we obtain a commutative diagram

$$\begin{array}{ccc} G((X \times_S X)^\sim) & \xrightarrow{[[X,\ ]]} & G(X_s) \\ \uparrow & & \uparrow \\ G(E_i) & \xrightarrow[{[[D_i,\ ]]_{E_i}}]{} & G(D_i) \end{array}$$

where the vertical arrows are the push-forward. Thus it is reduced to showing that the localized intersection product $[[D_i,\ ]]_{E_i} : G(E_i) \to G(D_i)$ is the 0-map.

By Lemma 5.2.2, the scheme $E_i = D_i \times^{\log}_{s,[P]} D_i$ is a μ_{l_i}-torsor over $E'_i = D'_i \times^{\log}_{s',[P_i]} D'_i$. Let $D_i \to E'_i$ be the log diagonal map. Since the log diagonal map $D_i \to E_i$ gives a section $D_i \to E_i \times_{E'_i} D_i$ of the μ_{l_i}-torsor $E_i \times_{E'_i} D_i$ over D_i, we obtain an isomorphism $\mu_{l_i,D_i} \to E_i \times_{E'_i} D_i$. We identify $\mu_{l_i,D_i} = E_i \times_{E'_i} D_i$ in the following.

We show that the immersion $j_i : \mu_{l_i,D_i} = E_i \times_{E'_i} D_i \to E_i$ is a regular immersion. Since the projection $E'_i \to D_i$ is log smooth and strict, it is smooth. Since the log diagonal map $D_i \to E'_i$ is a section, it is a regular immersion. Since the μ_{l_i}-torsor E_i is flat over E'_i, the immersion $E_i \times_{E'_i} D_i \to E_i$ is also a regular immersion.

The localized intersection product $[[D_i,\ ]]_{\mu_{l_i,D_i}} : G(\mu_{l_i,D_i}) \to G(D_i)$ is defined and is the 0-map by Lemma 3.2.6. To complete the proof, it is sufficient to show that the map $[[D_i,\ ]]_{E_i} : G(E_i) \to G(D_i)$ is equal to the composition

$$G(E_i) \xrightarrow{j_i^*} G(E_i \times_{E'_i} D_i) = G(\mu_{l_i,D_i}) \xrightarrow{[[D_i,\ ]]_{\mu_{l_i,D_i}}} G(D_i).$$

We apply Corollary 3.3.4.3 by taking $D_i \leftarrow E_i \leftarrow E_i \times_{E'_i} D'_i \to D'_i$ and the log diagonals $D_i \to E_i$ and $D'_i \to E_i \times_{E'_i} D_i$ as $S \leftarrow X \leftarrow W = X' \to S'$, $V \to X$ and $V' \to X'$ in Corollary 3.3.4.3. Then, since the immersion $j_i : E_i \times_{E'_i} D'_i \to E_i$ is a regular immersion, the assumption is satisfied. Hence $[[D_i,\ ]]_{E_i} : G(E_i) \to G(D_i)$ is the composition $G(E_i) \to G(E_i \times_{E'_i} D_i) \to G(D_i)$ and is the 0-map. □

Lemma **5.4.4.** — *Let* X *and* Y *be schemes over* S *satisfying the condition* (S(n)) *and let* $f : X \to Y$ *be a morphism over* S. *Then we have a commutative diagram*

$$\begin{array}{ccc} G(Y_K \times_K Y_K) & \xrightarrow{[[Y,\ ]]} & G(Y_s) \\ {\scriptstyle (f_K \times f_K)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ G(X_K \times_K X_K) & \xrightarrow[{[[X,\ ]]}]{} & G(X_s). \end{array}$$

Proof. — The map $(f \times f)^{\sim} : (X \times_S X)^{\sim} \to (Y \times_S Y)^{\sim}$ is locally of complete intersection by Lemma 5.2.1.1. Hence it is of finite tor-dimension and the map $(f \times f)^{\sim *} : G((Y \times_S Y)^{\sim}) \to G((X \times_S X)^{\sim})$ is defined. Similarly, $f : X \to Y$ is locally of complete intersection and the map $f^* : G(Y_s) \to G(X_s)$ is defined. By Theorem 3.2.1.4, we have $[[X,\ ]] = [[\ ,X]]$ and $[[Y,\ ]] = [[\ ,Y]]$. Hence it is enough to show that the diagram

$$\begin{array}{ccc} G((Y \times_S Y)^{\sim}) & \xrightarrow{[[\ ,Y]]} & G(Y_s) \\ {\scriptstyle (f \times f)^{\sim *}}\downarrow & & \downarrow{\scriptstyle f^*} \\ G((X \times_S X)^{\sim}) & \xrightarrow[{[[\ ,X]]}]{} & G(X_s) \end{array}$$

is commutative since $G((Y \times_S Y)^{\sim}) \to G(Y_K \times_K Y_K)$ is surjective.

We show that both of the compositions are equal to $[[\ ,X]]_{(Y \times_S Y)^{\sim}}$ by applying Corollary 3.3.4. First, we consider the composition via the upper right. We apply Corollary 3.3.4.1 by taking $X \to Y \to (Y \times_S Y)^{\sim} \to Y$ and the log diagonal $Y \to (Y \times_S Y)^{\sim}$ as $W' \to W \to X \to S$ and $V \to X$ in Corollary 3.3.4.1. Since f is of finite tor-dimension, the assumption of Corollary 3.3.4.1 is satisfied. Thus the composition $f^* \circ [[\ ,Y]]$ is equal to $[[\ ,X]]_{(Y \times_S Y)^{\sim}}$. Next, we consider the composition via the lower left. We apply Corollary 3.3.4.3 by taking $X \leftarrow (X \times_S X)^{\sim} \to (Y \times_S Y)^{\sim} \to Y$ and the log diagonals $Y \to (Y \times_S Y)^{\sim}$ and $X \to (X \times_S X)^{\sim}$ as $S' \leftarrow X' = W \to X \to S$, $V \to X$ and $V' \to X'$ in Corollary 3.3.4.3. Since $(f \times f)^{\sim}$ and f are of finite tor-dimension, the assumption of Corollary 3.3.4.3 is satisfied. Thus the composition $[[\ ,X]] \circ (f \times f)^{\sim *}$ is also equal to $[[\ ,X]]_{(Y \times_S Y)^{\sim}}$. Hence the diagram is commutative. □

Lemma **5.4.5.** — *Let* X *be a scheme over* S *satisfying the condition* (S(n)).

1. The logarithmic self-intersection product $[[X, X]]_{(X \times_S X)^{\sim}} \in F_0G(X_s)$ *is equal to the image of the logarithmic self-intersection cycle* $(\Delta_X, \Delta_X)_S^{\log} = (-1)^n c_n{}_Z^X(\Omega^1_{X/S}(\log / \log)) \cap [X] \in CH_0(X_s)$*:*

$$\textbf{(5.4.5.1)} \qquad [[X, X]]_{(X \times_S X)^{\sim}} = (-1)^n c_n{}_Z^X\big(\Omega^1_{X/S}(\log / \log)\big) \cap [X] = (\Delta_X, \Delta_X)_S^{\log}.$$

2. Let n be the dimension of X. *Then the map* $[[X,\ ]] : G((X\times_S X)^{\sim}) \to G(X_s)$ *sends the topological filtration* $F_pG((X\times_S X)^{\sim})$ *into* $F_{p-n}G(X_s)$.

3. Let $d = n-1$ *be the dimension of* X_K. *Then the induced map* $[[X,\ ]] : G(X_K\times_K X_K) \to G(X_s)$ *sends the topological filtration* $F_pG(X_K\times_K X_K)$ *into* $F_{p-d}G(X_s)$.

Proof. — 1. Applying Corollary 3.4.4.1 to the log diagonal map $X \to (X\times_S X)^{\sim}$, we obtain $[[X,X]]_{(X\times_S X)^{\sim}} = (-1)^n c_{n}{}_Z^X(M_{X/(X\times_S X)^{\sim}}) \cap [X]$ in $F_0G(X_s)$. Thus it follows by the isomorphism $M_{X/(X\times_S X)^{\sim}} \to \Omega^1_{X/S}(\log / \log)$ in Lemma 5.4.1.2.

2. It suffices to apply Theorem 3.4.3.1 to the map $[[X,\ ]]_{(X\times_S X)^{\sim}} : G((X\times_S X)^{\sim}) \to G(Z)$.

3. Clear from 2. □

The induced map $\mathrm{Gr}^F_pG(X_K\times_K X_K) \to \mathrm{Gr}^F_{p-d}G(X_s)$ is also denoted by $[[X,\ ]]$.

Lemma **5.4.6.** — *Let* X *be a scheme over* S *satisfying the condition* $(S(n))$ *and* $Z \subset X$ *be the closed subscheme defined by* $\mathrm{Ann}\Lambda^n\Omega^1_{X/S}(\log / \log)$ *as in Lemma 5.4.1. Let* $\pi : X' \to X$ *be the blow-up at* Z *and* $D = Z\times_X X'$ *be the exceptional divisor. Then the pull-back* $\pi^*\Omega^1_{X/S}(\log / \log)$ *is an extension of a locally free* $\mathcal{O}_{X'}$*-module* $\mathcal{E}'$ *of rank* $n-1$ *by an invertible* $\mathcal{O}_D$*-module and we have*

$$c_{n}{}_Z^X\big(\Omega^1_{X/S}(\log / \log)\big) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}') \cap [D]).$$

Proof. — The proof is the same as that of Lemma 5.1.6 except that we use Corollary 5.1.2.3 and Lemma 5.3.5.2 in place of Corollary 5.3.2.2, Lemma 5.1.3.2. □

Definition **5.4.7.** — *Let* X *be a scheme over* S *satisfying the condition* $(S(n))$ *and* σ *be an automorphism of* X *over* S. *Then, we say* σ *is* admissible *if the following condition is satisfied.*

For each irreducible component D_i *of the reduced closed fiber* $X_{s,\mathrm{red}}$, *we have either* $\sigma(D_i) = D_i$ *or* $\sigma(D_i) \cap D_i = \emptyset$.

For an admissible automorphism σ of X over S, the localized intersection product $[[X,\Gamma_\sigma]]$ is computed using the Segre classes as follows.

Lemma **5.4.8.** — *Let* X *be a scheme over* S *satisfying the condition* $(S(n))$ *and* σ *be an admissible automorphism of* X *over* S. *Let* $D_1, \ldots, D_m$ *be the irreducible components of* X_s *and put* $U = X - \bigcup_{i:\sigma(D_i)\cap D_i=\emptyset} D_i$. *Then,*

1. The pair $(1,\sigma) : U \to X$ *of maps induces a closed immersion* $U \to (X\times_S X)^{\sim}$.

2. Let Γ_σ *denote* U *regarded as a closed subscheme of* $(X\times_S X)^{\sim}$ *by the immersion in 1 and let* $\Delta_U \subset (U\times_S U)^{\sim}$ *denote the log diagonal. Define the logarithmic fixed part* $X^\sigma_{\log}$ *by* $X^\sigma_{\log} = X \times_{(X\times_S X)^{\sim}} \Gamma_\sigma$. *Then we have* $X^\sigma_{\log} = \Delta_U \times_{(U\times_S U)^{\sim}} \Gamma_\sigma$.

3. Assume that σ does not have a fixed point in the generic fiber X_K. Then the localized intersection product $[[X, \Gamma_\sigma]]_{(X\times_S X)^\sim} \in F_0 G(X^\sigma_{\log})$ *is equal to the image of*

$$\begin{aligned}&\left\{c\left(\Omega^1_{X/S}(\log/\log)\right)^* \cap s\left(X^\sigma_{\log}, X\right)\right\}_{\dim 0}\\ &= \sum_{i=0}^{n-1}(-1)^i c_i\left(\Omega^1_{X/S}(\log/\log)\right) s_{n-i}\left(X^\sigma_{\log}, X\right).\end{aligned}$$

In particular, if the logarithmic fixed part $X^\sigma_{\log}$ *is a Cartier divisor of* X*, we have*

$$[[\Gamma_\sigma, X]]_{(X\times_S X)^\sim} = \left\{c\left(\Omega^1_{X/S}(\log/\log)\right)^* \cap \left(1 + X^\sigma_{\log}\right)^{-1} \cap \left[X^\sigma_{\log}\right]\right\}_{\dim 0}.$$

Proof. — 1. We set $(X\times_S X)^0 = X\times_S X - \bigcup_{(i,j):D_i\cap D_j=\emptyset} D_i\times D_j$. By the definition of $(X\times_S X)^\sim$, we have $pr_1^{-1}(D_i) = pr_2^{-1}(D_i)$ in $(X\times_S X)^\sim$. Hence $(X\times_S X)^\sim$ is a scheme over $(X\times_S X)^0$. By the definition of U, it is the inverse image of $(X\times_S X)^0 \subset X\times_S X$ by the map $(1,\sigma) : X \to X\times_S X$. Hence the map $U \to (X\times_S X)^0$ is a closed immersion. Since σ is admissible, the map $(1,\sigma) : X \to X\times_S X$ induces a map $U \to (X\times_S X)^\sim$. Since $U \to (X\times_S X)^0$ is a closed immersion, the induced map $U \to (X\times_S X)^\sim$ is also a closed immersion.

2. Since U is stable under σ, Γ_σ is a subscheme of $(U\times_S U)^\sim \subset (X\times_S X)^\sim$. The assertion follows from $\Delta_U = X \times_{(X\times_S X)^\sim} (U\times_S U)^\sim$.

3. By the assumption that σ does not have a fixed point in the generic fiber X_K, the underlying set of $X^\sigma_{\log}$ is a subset of the closed fiber X_s. We apply Corollary 3.4.6, by taking $X \to (X\times_S X)^\sim \to X$ to be $V \to X \to S$ in Corollary 3.4.6 and $X^\sigma_{\log} \to \Gamma_\sigma \to (X\times_S X)^\sim$ to be $T \to W \to X$. Since $M_{X/(X\times_S X)^\sim} = \Omega^1_{X/S}(\log/\log)$, we obtain $[[X,\Gamma_\sigma]]_{(X\times_S X)^\sim} = \{c(\Omega^1_{X/S}(\log/\log))^* \cap s(X^\sigma_{\log}, \Gamma_\sigma)\}_{\dim 0}$. By the automorphism $(x, y) \mapsto (y, \sigma(x))$ of $(U\times_S U)^\sim$, the closed subschemes Δ_U and Γ_σ are switched. Hence by 2, we have $s(X^\sigma_{\log}, \Gamma_\sigma) = s(X^\sigma_{\log}, \Delta_U) = s(X^\sigma_{\log}, X)$. Thus the assertion is proved. □

Lemma **5.4.9.** — *Let* K *be a discrete valuation field with perfect residue field and* X *be a scheme over* S *satisfying the condition* (S(n))*. Let* K′ *be a discrete valuation field with perfect residue field. Assume that* K′ *is an extension of* K*, the valuation of* K′ *is an extension of that of* K *and that a prime element of* K *is a prime element of* K′*. Put* $S' = \operatorname{Spec} \mathcal{O}_{K'}$ *and let* s' *be the closed point of* S′*. Then,*

1. $X' = X\times_S S'$ *is regular and the reduced closed fiber* $X'_{s',\mathrm{red}}$ *has simple normal crossings.*

2. We have a commutative diagram

$$\begin{array}{ccc} G((X\times_S X)^\sim) & \xrightarrow{[[X,\ ]]_{(X\times_S X)^\sim}} & G(X_s)\\ \downarrow & & \downarrow\\ G((X'\times_{S'} X')^\sim) & \xrightarrow[{[[X',\ ]]_{(X'\times_{S'} X')^\sim}}]{} & G(X'_{s'}) \end{array}$$

where the vertical arrows are the pull-backs.

Proof. — The assertion 1 is checked easily using Lemma 5.3.1.2. We show 2. We have $(X' \times_{S'} X')^\sim = (X \times_S X)^\sim \times_S S'$ and the vertical arrows are defined. We show that the both compositions are equal to the map $[[X', \;]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \to G(X'_{s'})$ by applying Corollary 3.3.4. For the composition via $G(X_s)$, it suffices to apply Corollary 3.3.4.1 by taking $(X \times_S X)^\sim \leftarrow X \leftarrow X'$ as $X \leftarrow W \leftarrow W'$ in Corollary 3.3.4.1. For the composition via $G((X' \times_{S'} X')^\sim)$, we take $(X \times_S X)^\sim \leftarrow (X' \times_{S'} X')^\sim$ as $X \leftarrow W \to X'$ in Corollary 3.3.4.3. Then since $(X' \times_{S'} X')^\sim = (X\times_S X)^\sim \times_S S' \to (X\times_S X)^\sim$ is flat and hence of finite tor-dimension, the assumption in Corollary 3.3.4.3 is satisfied. Hence the assertion follows. □

6. Conductor formula

We recall the precise formulation of the conductor formula and give the exact statements of the main result, Theorem 6.2.3, and its log version, Theorem 6.2.5, in 6.2. We state a generalization, Theorem 6.3.1, of Theorem 6.2.5 to an algebraic correspondence in 6.3. We recall the definition of conductor and give an interpretation Lemma 6.1.1 in terms of localized intersection product in 6.1.

The proof of Theorem 6.3.1 is given in 6.4 and 6.5. The both sides of the equality in Theorem 6.3.1 is computed using an alteration in 6.4. In the final subsection 6.5, we complete the proof of Theorem 6.3.1 by combining the computations with the logarithmic Lefschetz trace formula, Theorem 6.5.1.

6.1. *Artin and Swan conductors.* — We recall generalities on conductor. Basic references are [36] Chapitres IV, VI and [37] Partie III §3.4.

Let K be a discrete valuation field with perfect residue field F. Let ℓ be a prime number different from the characteristic p of F and $G_K \to GL_{\mathbf{Q}_\ell}(V)$ be a continuous ℓ-adic representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$. We recall the definition of the Artin conductor Art(V) and the Swan conductor Sw(V) of V.

In this subsection, L denotes a finite separable extension of K and we assume that the integral closure $\mathscr{O}_L$ of $\mathscr{O}_K$ is a discrete valuation ring. Let E be the residue field of L. Assume that L is a finite Galois extension of K of Galois group $G_{L/K}$. The Artin character $a_{L/K}$ and the Swan character $sw_{L/K}$ of $G_{L/K}$ are defined by

$$a_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_L/\mathscr{O}_K} & \text{if } \sigma = 1, \\ -\mathrm{length}_{\mathscr{O}_K} \mathscr{O}_L/(\sigma(x) - x : x \in \mathscr{O}_L) & \text{if } \sigma \neq 1, \end{cases}$$

$$sw_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_L/\mathscr{O}_K} - ([L:K] - [E:F]) & \text{if } \sigma = 1, \\ -\mathrm{length}_{\mathscr{O}_K} \mathscr{O}_L/\left(\frac{\sigma(x)}{x} - 1 : x \in L^\times\right) & \text{if } \sigma \neq 1 \end{cases}$$

for $\sigma \in G_{L/K}$. We call the p-Sylow subgroup $P_{L/K}$ of the inertia subgroup $I_{L/K}$ of $G_{L/K}$ the wild inertia subgroup. If $\sigma \in I_{L/K}$ and π_L is a prime element of L, the ideals $(\sigma(x) - x, x \in \mathcal{O}_L)$ and $(\sigma(x)/x - 1 : x \in L^\times)$ are generated by $\sigma(\pi) - \pi$ and by $\sigma(\pi)/\pi - 1$ respectively. Hence we have $a(\sigma) = -\mathrm{ord}_L(\sigma(\pi) - \pi)$ and $\mathrm{sw}(\sigma) = -\mathrm{ord}_L(\sigma(\pi)/\pi - 1)$ for $\sigma \neq 1, \in I_{L/K}$. For $\sigma \in G_{L/K}$, the condition $-\mathrm{sw}_{L/K}(\sigma) > 0$ is equivalent to $\sigma \in P_{L/K} - \{1\}$ and the condition $-a_{L/K}(\sigma) > 0$ is equivalent to $\sigma \in I_{L/K} - \{1\}$.

We give an interpretation, Lemma 6.1.1, of the Artin and Swan characters as a localized intersection product, which plays a crucial role in the proof of the conductor formula. Let L be a finite separable extension of K such that the integral closure $\mathcal{O}_L$ is a discrete valuation ring. We put $S = \mathrm{Spec}\ \mathcal{O}_K$ and $T = \mathrm{Spec}\ \mathcal{O}_L$ and regard them as log schemes with the standard log structures. We define the log self-product $(T\times_S T)^\sim$ and the log diagonal map $T \to (T\times_S T)^\sim$ as in Section 5.2. On a neighborhood of the log diagonal $T \subset (T\times_S T)^\sim$, the log self-product $(T\times_S T)^\sim$ is isomorphic to the blow-up of $T \times_S T$ at the image of the closed point of T. We also consider the diagonal map $T \to T \times_S T$. We introduce further notation assuming L is a Galois extension. For $\sigma \in G_{L/K}$, let $T = T_\sigma \to T \times_S T$ be the graph of $\sigma : T \to T$. It is defined by the surjection $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_L : a\otimes b \mapsto a\sigma(b)$. Let $T = \tilde{T}_\sigma \to (T\times_S T)^\sim$ be the map defined by the pair $(\mathrm{id} : T \to T, \sigma^* : T \to T)$. If $\sigma = 1$, the immersion $T_1 \to T \times_S T$ is the diagonal map and $\tilde{T}_1 \to (T \times_S T)^\sim$ is the log diagonal map.

Lemma **6.1.1.** — *Let* K *be a discrete valuation field with perfect residue field and* L *be a finite separable extension of* K *such that the integral closure* $\mathcal{O}_L$ *of* $\mathcal{O}_K$ *is a discrete valuation ring. Regard* $S = \mathrm{Spec}\ \mathcal{O}_K$ *and* $T = \mathrm{Spec}\ \mathcal{O}_L$ *as log schemes with the standard log structures. Let* $f : T \to S$ *be the canonical map and* s *and* t *denote the closed points of* S *and* T *respectively. We identify* $G(s) = \mathbf{Z}$ *and* $G(t) = \mathbf{Z}$*. The push-forward map* $f_* : G(t) = \mathbf{Z} \to G(s) = \mathbf{Z}$ *is the multiplication by the residual degree* $[E : F]$*. Then,*

1. We have

$$[[T, T]]_{T\times_S T} = -\mathrm{length}_{\mathcal{O}_T} \Omega_{\mathcal{O}_L/\mathcal{O}_K},$$
$$[[T, T]]_{(T\times_S T)^\sim} = -\mathrm{length}_{\mathcal{O}_T} \Omega_{\mathcal{O}_L/\mathcal{O}_K}(\log / \log).$$

2. Assume L *is a Galois extension of* K*. Then for an element* $\sigma \in G_{L/K}$ *of the Galois group, we have*

$$a_{L/K}(\sigma) = -f_*[[T, T_\sigma]]_{T\times_S T} \quad \text{and} \quad \mathrm{sw}_{L/K}(\sigma) = -f_*[[T, \tilde{T}_\sigma]]_{(T\times_S T)^\sim}$$

in $G(s) = \mathbf{Z}$*. If* $\sigma \in G_{L/K} - P_{L/K}$*, the intersection* $\tilde{T}_\sigma \cap T$ *in* $(T \times_S T)^\sim$ *is empty.*

3. ([36] Chapitre IV Proposition 3) *Further, let* $M \subset L$ *be a sub Galois extension over* K*. Then for an element* $\sigma \in G_{M/K}$*, we have*

$$[L : M]a_{M/K}(\sigma) = \sum_{\tau\mapsto\sigma} a_{L/K}(\tau) \quad \text{and} \quad [L : M]\mathrm{sw}_{M/K}(\sigma) = \sum_{\tau\mapsto\sigma} \mathrm{sw}_{L/K}(\tau).$$

Proof. — 1. It is a special case of the equalities (5.1.3.1) and (5.4.5.1).

2. If $\sigma = 1$, it follows from 1 and Lemma 5.3.4.3.

We assume $\sigma \neq 1$. Then the intersection $D_\sigma = T \times_{T\times_S T} T_\sigma$ is a divisor of T and we have $\mathcal{O}_{D_\sigma} = \mathcal{O}_L/(\sigma(x) - x : x \in \mathcal{O}_L)$. Hence, by Theorem 3.4.3, we have $[[T, T_\sigma]]_{T\times_S T} = \text{length}_{\mathcal{O}_T}\mathcal{O}_{D_\sigma} = -a_{L/K}(\sigma)$. Since the log self-product $(T \times_S T)^\sim$ is isomorphic to the blow-up of $T \times_S T$ at the closed point on a neighborhood of the log diagonal $T \subset (T \times_S T)^\sim$, similarly as above, the intersection $\tilde{D}_\sigma = T \times_{(T\times_S T)^\sim} \tilde{T}_\sigma$ is a divisor of T and we have $\mathcal{O}_{\tilde{D}_\sigma} = \mathcal{O}_L/(\sigma(x)/x - 1 : x \in L^\times)$. By Theorem 3.4.3, we have $[[T, T_\sigma]]_{(T\times_S T)^\sim} = \text{length}_{\mathcal{O}_T}\mathcal{O}_{\tilde{D}_\sigma} = -sw_{L/K}(\sigma)$. If $\sigma \notin P_{L/K}$, we have $sw_{L/K}(\sigma) = 0$ and hence $\tilde{D}_\sigma = T \cap \tilde{T}_\sigma$ is empty.

3. We put $U = \text{Spec}\ \mathcal{O}_M$ and let $g : T \to U$ be the induced map. Let $u \in U$ be the closed point. Since the maps $g : T \to U$ and $g \times g : T \times_S T \to U \times_S U$ are locally of complete intersection, they are of finite tor-dimension and the pull-back maps $g^* : G(u) \to G(t)$ and $(g \times g)^* : G(U \times_S U) \to G(T \times_S T)$ are defined. We have an equality $(g \times g)^*[U_\sigma] = \sum_{\tau\mapsto\sigma}[T_\tau]$ in $\text{Gr}_1^F G(T \times_S T)$. We apply Proposition 3.3.3 by taking $U \subset U \times_S U \leftarrow T \times_S T = T \times_S T \supset T_\sigma$ as $V \subset X \leftarrow W \to X' \supset V'$. Then we obtain $[[U_\sigma, T]]_{U\times_S U} = [[T, (g \times g)^* U_\sigma]]_{T\times_S T} = \sum_{\tau\mapsto\sigma}[[T, T_\tau]]_{T\times_S T}$ in $F_0G(t) = G(t)$. By the projection formula, Proposition 3.3.5, we have $g_*[[U_\sigma, T]]_{U\times_S U} = [L : M][[U_\sigma, U]]_{U\times_S U}$. Thus the assertion follows from 2.

For the equality for the Swan character, we replace $g \times g : T \times_S T \to U \times_S U$ in the above proof by $(g \times g)^\sim : (T \times_S T)^\sim \to (U \times_S U)^\sim$. Since the map $(g \times g)^\sim$ is also of finite tor-dimension by Lemma 5.2.1.2, the same argument as above proves the equality. □

Let K' be the completion of K. Taking an embedding $\bar{K} \to \bar{K}'$ we identify the absolute Galois group $G_{K'}$ with a subgroup of G_K. Let $I_K = \text{Gal}(\bar{K}/K'^{ur}) \subset G_K$ be the inertia group of K corresponding to the maximum unramified extension K'^{ur} of K'. We call the pro-p Sylow subgroup $P_K = \text{Gal}(\bar{K}/K'^{tr}) \subset I_K$ the wild inertia group of K. It corresponds to the maximum tamely ramified extension $K'^{tr} = K'^{ur}(\pi^{1/m}; p \nmid m)$ of K' where π is a prime element of K.

Let $G_K \to GL_{\mathbf{Q}_\ell}(V)$ be an ℓ-adic representation. The image of the wild inertia P_K is finite. Let L be a finite Galois extension of the completion K' such that $P_L = P_K \cap \text{Gal}(\bar{K}/L)$ acts trivially on V. We identify $P_{L/K'} = P_K/P_L$ as a subgroup of the Galois group $G_{L/K'}$. The action of $P_{L/K'}$ on V is well-defined by the assumption on L. The Swan conductor Sw(V) is defined as the intertwining number

$$\text{Sw}(V) = \frac{1}{[L : K']} \sum_{\sigma\in P_{L/K'}} sw_{L/K'}(\sigma)\text{Tr}(\sigma : V).$$

Note that $sw_{L/K'}(\sigma) = 0$ unless $\sigma \in P_{L/K'}$ and the sum is taken over the subgroup $P_{L/K'} \subset G_{L/K'}$. It is a theorem that Sw(V) is a non-negative integer. It is 0 if and only if the action of P_K is trivial. The Artin conductor is defined by the equality $\mathrm{Art}(V) = \dim V - \dim V^I + \mathrm{Sw}(V)$ where V^I denotes the I-fixed part. The fact that the right hand side is independent of the choice of L is a consequence of Lemma 6.1.1.3.

For an endomorphism $f : V \to V$ of an ℓ-adic representation of G_K, we define the Swan conductor $\mathrm{Sw}(f : V)$ as follows. Take a finite Galois extension L of the completion K' such that P_L acts trivially on V as above. Then we put

$$\mathrm{Sw}(f : V) = \frac{1}{[L : K']} \sum_{\sigma \in P_{L/K'}} \mathrm{sw}_{L/K'}(\sigma)\mathrm{Tr}(f \circ \sigma : V).$$

It also follows from Lemma 6.1.1.3. that the right hand side is independent of the choice of L. For $f = \mathrm{id}$, we have $\mathrm{Sw}(V) = \mathrm{Sw}(\mathrm{id} : V)$.

6.2. *Conductor formula.* — Let K be a discrete valuation field with perfect residue field F. In the rest of the paper, S will denote Spec $\mathcal{O}_K$ and $s =$ Spec F denotes the closed point. Let X be a proper scheme over $\mathcal{O}_K$ satisfying the condition (R(n)) in Section 5.1. We define the conductors of X. Let $d = n - 1$ be the dimension of the generic fiber X_K. The Swan conductor is defined to be the alternating sum

$$\mathrm{Sw}(X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw} H^q(X_{\bar{K}}, \mathbf{Q}_\ell).$$

The cohomology in the right hand side is the ℓ-adic etale cohomology for a prime ℓ different from the characteristic p of F. It is known that the alternating sum is independent of the choice of ℓ [30]. The Artin conductor $\mathrm{Art}(X/\mathcal{O}_K)$ is defined by

$$\mathrm{Art}(X/\mathcal{O}_K) = \chi(X_{\bar{K}}) - \chi(X_{\bar{F}}) + \mathrm{Sw}(X_K/K).$$

In the right hand side, χ denotes the ℓ-adic Euler number which is known to be independent of ℓ as a consequence of the Weil conjecture.

Recall that the localized self-intersection class $(\Delta_X, \Delta_X)_S \in \mathrm{CH}_0(X_F)$ is defined as the localized Chern class $(-1)^n c_{n\,X_F}^{\ X}(\Omega^1_{X/\mathcal{O}_K}) \cap [X]$. We consider its image $\deg(\Delta_X, \Delta_X)_S \in \mathbf{Z}$ by the degree map $\deg : \mathrm{CH}_0(X_F) \to \mathrm{CH}_0(F) = \mathbf{Z}$.

Conjecture **6.2.1** ([6] Conjecture). — *Let* K *be a discrete valuation field with perfect residue field* F *and let* X *be a proper scheme over* $\mathcal{O}_K$ *satisfying the condition* (R(n)) *in Section 5.1. Then we have*

(6.2.1) $$\mathrm{Art}(X/\mathcal{O}_K) = -\deg(\Delta_X, \Delta_X)_S.$$

The formula (6.2.1) is called the conductor formula for X. The conductor formula in the case $\dim X = 1$ is the classical conductor-discriminant formula. In the case $\dim X = 2$, it is proved by Bloch in the same paper [6].

Proposition **6.2.2.** — *Let* X *be a proper scheme over* S *satisfying the condition* (R(n)) *in Section 5.1. Let* C *be a regular closed subscheme of* X *supported in the closed fiber* X_s *and* $\pi : X' \to X$ *be the blow-up at* C. *Then, the conductor formula* (6.2.1) *for* X *is equivalent to that for* X′.

Proof. — Let $E = X' \times_X C$ be the exceptional divisor. Then we have

$$-(\mathrm{Art}(X'/S) - \mathrm{Art}(X/S)) = \chi(X'_{\bar{s}}) - \chi(X_{\bar{s}}) = \chi(E_{\bar{s}}) - \chi(C_{\bar{s}}).$$

Since E is a $\mathbf{P}^{c-1}$-bundle over C, we have $\chi(E_{\bar{s}}) = c\chi(C_{\bar{s}})$. On the other hand, by Lemma 2.3.4, we have

$$\begin{aligned} &\pi_*\big(c^{X'}_{X'_s}\big(\Omega^1_{X'/S}\big) \cap [X']\big) - c^{X}_{X_s}\big(\Omega^1_{X/S}\big) \cap [X] \\ &= c^{X}_{X_s}\big(\Omega^1_{X/S}\big)\pi_{E*}\big(\big(c^{X'}_{E}\big(\Omega^1_{X'/X}\big) - 1\big) \cap [X']\big) \\ &= (-1)^c(c-1)c\big(\Omega^1_{X/S}\big)c(N_{C/X})^{-1} \cap [C] \end{aligned}$$

where $\pi_E : E \to C$ denotes the restriction of $\pi : X' \to X$. Let $i : C \to X$ denote the immersion and $f_s : C \to s$ denote the canonical map. By the distinguished triangles $\to Li^*\Omega^1_{X/S} \to L_{C/S} \to N_{C/X}[1] \to$ and $\to Lf_s^*N_{s/S}[1] \to \Omega^1_{C/F} \to L_{C/S} \to 0$, we have $c(\Omega^1_{X/S})c(N_{C/X})^{-1} \cap [C] = c(L_{C/S}) \cap [C] = c(\Omega^1_{C/F}) \cap [C]$. Thus it follows form the Lefschetz trace formula $\chi(C_{\bar{s}}) = \deg(-1)^{n-c}c_{n-c}(\Omega^1_{C/F}) \cap [C]$. □

Our first main result is the following.

Theorem **6.2.3.** — *Let* $\mathcal{O}_K$ *be a discrete valuation ring with perfect residue field* F *and let* X *be a proper scheme over* $\mathcal{O}_K$ *satisfying the following condition*

(N(n)) X *satisfies the condition* (R(n)) *in Section 5.1 and the reduced closed fiber* $(X_F)_{\mathrm{red}}$ *is a divisor with normal crossings.*

Then we have

$$\mathrm{Art}(X/\mathcal{O}_K) = -\deg(\Delta_X, \Delta_X)_S.$$

By Proposition 6.2.2 and Lemma 4.2.12, Theorem 6.2.3 is equivalent to the following weaker version.

Corollary **6.2.4.** — *Let* K *be a discrete valuation field with perfect residue field* F *and let* X *be a proper scheme over* $\mathcal{O}_K$ *satisfying the condition* (S(n)) *in Section 5.3. Then we have*

$$\mathrm{Art}(X/\mathcal{O}_K) = -\deg(\Delta_X, \Delta_X)_S.$$

We show that Corollary 6.2.4 is equivalent to the following logarithmic version.

Theorem **6.2.5.** — *Let the assumption be the same as in Corollary 6.2.4. Then we have*

$$\mathrm{Sw}(X_K/K) = -\deg(\Delta_X, \Delta_X)_S^{\log}.$$

Proof of equivalence of Corollary 6.2.4 and Theorem 6.2.5. — The proof of equivalence is similar to that of the conductor formula in the tame case in [4]. Let $D_1, \ldots, D_m$ be the irreducible components of $D = X_{s,\mathrm{red}}$. For a subset $J \subset \{1, \ldots, m\}$, let D_J be the intersection $\bigcap_{i\in J} D_i$ and B_J be the divisor $\bigcup_{i\notin J} D_i \cap D_J$ with simple normal crossings as in Lemma 5.3.6. By the definition of Artin conductor and Corollary 5.3.7, it is sufficient to show the equalities

$$\chi(X_{\bar K}) = (-1)^{n-1} \deg c_{n-1}\big(\Omega^1_{X/S}(\log / \log)\big) \cap [X_s], \tag{6.2.5.1}$$

$$\chi(X_{\bar F}) = (-1)^n \sum_{r=1}^{n} \sum_{J\subset\{1,\ldots,m\},\#J=r} (-1)^r \deg c_{n-r}\big(\Omega^1_{D_J/F}(\log B_J)\big) \cap [D_J]. \tag{6.2.5.2}$$

Since $\deg c_{n-1}(\Omega^1_{X_K/K}) \cap [X_K] = \deg c_{n-1}(\Omega^1_{X/S}(\log / \log)) \cap [X_s]$, the equality (6.2.5.1) follows from the Lefschetz trace formula $(-1)^{n-1} \deg c_{n-1}(\Omega^1_{X_K/K}) \cap [X_K] = \chi(X_{\bar K})$. Since $\chi(X_{\bar F}) = \sum_{r=1}^n \sum_{J\subset\{1,\ldots,m\},\#J=r} \chi((D_J - B_J)_{\bar s})$, the equality (6.2.5.2) is reduced to the equalities

$$\chi((D_J - B_J)_{\bar s}) = (-1)^{n-r} \deg c_{n-r}\big(\Omega^1_{D_J/F}(\log B_J)\big) \cap [D_J], \tag{6.2.5.3}$$

for a subset $J \subset \{1, \ldots, m\}$ of cardinality r. Thus it suffices to show the following lemma.

Lemma **6.2.6.** — *Let* V *be a proper smooth scheme of dimension n over a perfect field* F *and* D *be a divisor of* V *with simple normal crossings. Then we have*

$$\chi(V_{\bar F} - D_{\bar F}) = \deg(-1)^n c_n\big(\Omega^1_{V/F}(\log D)\big).$$

Proof. — Let $D_1, \ldots, D_r$ be the irreducible components of the divisor D and $\mathrm{res}_i : \Omega^1_{V/F}(\log D) \to \mathcal{O}_{D_i}$ be the residue map. For a subset $J \subset \{1, \ldots, r\}$, we define $B_J \subset D_J \subset V$ as above. Then we have an exact sequence

$$0 \longrightarrow \Omega^1_{V/F} \longrightarrow \Omega^1_{V/F}(\log D) \xrightarrow{\oplus_i \mathrm{res}_i} \bigoplus_{i=1}^r \mathcal{O}_{D_i} \longrightarrow 0.$$

Hence we have

$$\begin{aligned} c\big(\Omega^1_{V/F}\big) \cap [V] &= c\big(\Omega^1_{V/F}(\log D)\big) \prod_{i=1}^r c(\mathcal{O}_{D_i})^{-1} \cap [V] \\ &= c\big(\Omega^1_{V/F}(\log D)\big) \prod_{i=1}^r (1 - [D_i]) \cap [V] \\ &= \sum_{m=0}^{n} \sum_{J\subset\{1,\ldots,r\},\#J=m} (-1)^m c\big(\Omega^1_{V/F}(\log D)\big) \cap [D_J]. \end{aligned}$$

By the exact sequence

$$0 \longrightarrow \Omega^1_{D_J/F}(\log B_J) \longrightarrow \Omega^1_{V/F}(\log D)|_{D_J} \longrightarrow \bigoplus_{i\in J} \mathcal{O}_{D_J} \longrightarrow 0,$$

we have $c(\Omega^1_{V/F}(\log D)) \cap [D_J] = c(\Omega^1_{D_J/F}(\log B_J)) \cap [D_J]$. Hence we have

$$\begin{aligned} &(-1)^n c_n\big(\Omega^1_{V/F}\big) \cap [V] \\ &= \sum_{m=0}^{n} \sum_{J\subset\{1,\dots,r\},\#J=m} (-1)^{n-m} c_{n-m}\big(\Omega^1_{D_J/F}(\log B_J)\big) \cap [D_J]. \end{aligned} \tag{6.2.6.1}$$

On the other hand, we have

$$\chi(V_{\bar F}) = \sum_{m=0}^{n} \sum_{J\subset\{1,\dots,r\},\#J=m} \chi((D_J - B_J)_{\bar F}). \tag{6.2.6.2}$$

By the Lefschetz trace formula $\chi(V_{\bar F}) = (-1)^n c_n(\Omega^1_{V/F}) \cap [V]$, the left hand sides of the equalities (6.2.6.1) and (6.2.6.2) are equal. Hence the assertion follows by induction on $\dim V$. □

We prove Theorem 6.2.5 together with its generalization Theorem 6.3.1 in Sections 6.4 and 6.5.

By Proposition 6.2.2, Theorem 6.2.3 has the following consequence.

Corollary **6.2.7.** — *Let* X *be as in Conjecture 6.2.1. Assume there exists a sequence of blowing-ups* $X' = X_m \to \cdots \to X_0 = X$ *at regular closed subschemes supported in the closed fibers such that* X′ *satisfies the condition* (S(n)) *in Section 5.3. Then Conjecture 6.2.1 is true for* X.

By Corollary 6.2.7, if the reduced closed fiber $(X_F)_{red}$ has an embedded resolution in a strong sense, Conductor formula for X is true. In particular when $\dim X = 2$, the assumption of Corollary 6.2.7 is satisfied and hence we obtain a new proof of Conjecture 6.2.1 in this case.

6.3. *Correspondences.* — We formulate a generalization, Theorem 6.3.1, of Theorem 6.2.5 for an algebraic correspondence. To state it, we prepare some terminology and notations on the cycle map and algebraic correspondences.

Let X_K be a proper smooth scheme over a field K and ℓ be a prime number different from the characteristic of K. Then, for an integer $r \geq 0$, we have a cycle map $cl : CH^r(X_K) \to H^{2r}(X_{\bar K}, \mathbf{Q}_\ell(r))$. For $\Gamma \in CH^r(X_K)$, the image $cl(\Gamma)$ is also denoted by $[\Gamma]$. It is compatible with the product and the pull-back. It also makes the degree map $\deg : CH_0(X_K) \to \mathbf{Z}$ compatible with the trace map. Its composition with the Chern character map $ch : Gr^r_F K(X_K) \to CH^r(X_K)_{\mathbf{Q}}$ is the Chern character map $ch : Gr^r_F K(X_K) \to H^{2r}(X_{\bar K}, \mathbf{Q}_\ell(r))$.

Let Y_K be another proper smooth schemes over a field K and assume X_K and Y_K are purely of dimension d. We call an element $\Gamma \in CH_d(X_K \times_K Y_K)$ an algebraic correspondence from X_K to Y_K. An algebraic correspondence $\Gamma \in CH_d(X_K \times_K Y_K)$ defines a G_K-equivariant map $\Gamma^* : H^*(Y_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ as the composition

$$H^*(Y_{\bar{K}}, \mathbf{Q}_\ell) \xrightarrow{pr_2^*} H^*(X_{\bar{K}} \times_{\bar{K}} Y_{\bar{K}}, \mathbf{Q}_\ell) \xrightarrow{[\Gamma]\cup} H^{*+2d}(X_{\bar{K}} \times_{\bar{K}} Y_{\bar{K}}, \mathbf{Q}_\ell(d)) \xrightarrow{pr_{1*}} H^*(X_{\bar{K}}, \mathbf{Q}_\ell).$$

When $X_K = Y_K$, an algebraic correspondence Γ on X_K defines an endomorphism Γ^* of the ℓ-adic representation $H^q(X_{\bar{K}}, \mathbf{Q}_\ell)$ of G_K.

Assume K is a discrete valuation field with perfect residue field F and ℓ is different from the characteristic of F. We put

$$\mathrm{Sw}(\Gamma, X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw}\big(\Gamma^* : H^q(X_{\bar{K}}, \mathbf{Q}_\ell)\big).$$

For an endomorphism $f : X_K \to X_K$ over K, similarly we put

$$\mathrm{Sw}(f, X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw}\big(f^* : H^q(X_{\bar{K}}, \mathbf{Q}_\ell)\big).$$

If $\Gamma_f \in CH_d(X_K \times_K X_K)$ denotes the class of the graph of f, we have $\mathrm{Sw}(f, X_K/K) = \mathrm{Sw}(\Gamma_f, X_K/K)$. In particular, for $f = \mathrm{id}$ and $\Gamma_f = \Delta_{X_K}$, we have $\mathrm{Sw}(\mathrm{id}, X_K/K) = \mathrm{Sw}(X_K/K)$.

As in the last subsection, let K be a discrete valuation field with perfect residue field F, $S = \mathrm{Spec}\, \mathcal{O}_K$ and $s = \mathrm{Spec}\, F$ be the closed point of S. Let X be a proper and flat regular scheme over $S = \mathrm{Spec}\, \mathcal{O}_K$ satisfying the condition (S(n)) in Section 5.3. For $\Gamma \in CH_d(X_K \times_K X_K)$, let $[[X, \Gamma]] \in \mathrm{Gr}_0^F G(X_s)$ be the image by the composition map $CH_d(X_K \times_K X_K) \to \mathrm{Gr}_d^F G(X_K \times_K X_K) \xrightarrow{[[X,\ ]]} \mathrm{Gr}_0^F G(X_s)$. We define the degree map $\deg_{X_s} : G(X_s) \to G(s) = \mathbf{Z}$ to be the push-forward for $X_s \to s$.

Theorem **6.3.1.** — *Let $\mathcal{O}_K$ be a discrete valuation ring with perfect residue field and ℓ be a prime number different from the characteristic of the residue field. Let X_K be a proper smooth scheme over K of dimension d. Let $\Gamma \in CH_d(X_K \times_K X_K)$ be an algebraic correspondence on X_K. Then,*

1. $\mathrm{Sw}(\Gamma, X_K/K)$ *is a rational number independent of ℓ.*

2. Let X be a proper scheme over S satisfying the condition (S(n)) *in Section 5.3 such that $X \otimes_{\mathcal{O}_K} K = X_K$. Then we have an equality of integers*

$$\mathrm{Sw}(\Gamma, X_K/K) = -\deg_{X_s}[[X, \Gamma]].$$

Proof will be completed in Section 6.5. Theorem 6.2.5, which is shown to be equivalent to Theorem 6.2.3, is the special case of the following Corollary where $f = \mathrm{id}$, by Lemma 5.4.5. Theorem 6.3.1.1 also follows from [41] Theorem 0.1.

Corollary **6.3.2.** — *Let* K, X_K *and* ℓ *be as in Theorem 6.3.1. Let* $f : X_K \to X_K$ *be an endomorphism over* K. *Then,*

1. $\mathrm{Sw}(f, X_K/K)$ *is a rational number independent of* ℓ.

2. Let X *be a proper scheme over* S *satisfying the condition* (S(*n*)) *in Section 5.3 such that* $X \otimes_{\mathscr{O}_K} K = X_K$. *Let* $\Gamma_f \in \mathrm{CH}_d(X_K \times_K X_K)$ *be the class of the graph of* f. *Then we have an equality of integers*

$$\mathrm{Sw}(f, X_K/K) = -\deg_{X_s}[[X, \Gamma_f]].$$

Proof. — It is enough to apply Theorem 6.3.1 to Γ_f. □

If the relative dimension of X over S is 1 and if f is an automorphism of X over S, analogous formula is proved in [1].

Corollary **6.3.3.** — *Let* X *be a proper scheme over* S *satisfying the condition* (S(*n*)) *in Section 5.3 and* σ *be an admissible automorphism of* X *over* S. *Assume that* σ *does not have a fixed point in the generic fiber* X_K. *Then we have*

$$\begin{aligned}\mathrm{Sw}(\sigma, X_K/K) &= -\deg\left\{c\left(\Omega^1_{X/S}(\log/\log)\right)^* \cap s\left(X^\sigma_{\log}, X\right)\right\}_{\dim 0}\\ &= -\deg\sum_{i=0}^{n-1}(-1)^i c_i\left(\Omega^1_{X/S}(\log/\log)\right)s_{n-i}\left(X^\sigma_{\log}, X\right).\end{aligned}$$

In particular, if the logarithmic fixed part $X^\sigma_{\log} = X \times_{(X\times_S X)^\sim} \Gamma_\sigma$ *is a Cartier divisor of* X, *we have*

$$\mathrm{Sw}(\sigma, X_K/K) = -\deg\left\{c\left(\Omega^1_{X/S}(\log/\log)\right)^* \cap \left(1 + X^\sigma_{\log}\right)^{-1} \cap \left[X^\sigma_{\log}\right]\right\}_{\dim 0}.$$

Proof. — It follows from Theorem 6.3.1.2 and Lemma 5.4.8. □

We show that Theorem 6.3.1 is reduced to the case where K is complete.

Corollary **6.3.4.** — *Let* X, K *and* Γ *be as in Theorem 6.3.1 and let* K' *be the completion of* K. *Then Theorem 6.3.1 for* X *and* Γ *is equivalent to that for* $X' = X \otimes_{\mathscr{O}_K} \mathscr{O}_{K'}$ *and the pull-back* Γ' *of* Γ *to* $X'_{K'} \times_{K'} X'_{K'}$.

Proof. — We have $\mathrm{Sw}(\Gamma, X_K/K) = \mathrm{Sw}(\Gamma', X_{K'}/K')$. By Lemma 5.4.9, we have $\deg_{X_s}[[X, \Gamma]] = \deg_{X'_{s'}}[[X', \Gamma']]$. □

6.4. *Alteration.* — To prove the main result, Theorem 6.3.1, we compute the Swan conductor $\mathrm{Sw}(\Gamma, X_K/K)$ and the logarithmic localized intersection product $[[X, \Gamma]]_{(X\times_S X)^\sim}$ using an alteration. First, we recall results on alteration.

Theorem **6.4.1.** — *Let* K *be a complete discrete valuation field.*

1. ([27]) *Let* X_K *be a separated scheme of finite type over* K. *Then there exist a proper scheme* X *over* $\mathscr{O}_K$ *and an open immersion* $X_K \to X$ *over* $\mathscr{O}_K$.

2. ([8] Theorem 6.5) *Let* X *be a flat integral and separated scheme of finite type over* $\mathscr{O}_K$. *Then there exist a finite extension* L *of* K, *a projective, strictly semi-stable and geometrically connected scheme* $\bar{W}$ *over the integer ring* $\mathscr{O}_L$, *an open subscheme* $W \subset \bar{W}$ *and a proper, surjective and generically finite morphism* $f : W \to X$ *over* $\mathscr{O}_K$.

Lemma **6.4.2** ([41] Lemma 1.2.4). — *Let* L *be a finite extension of* K *and* W *be a strictly semi-stable scheme of finite type over the integer ring* $\mathscr{O}_L$. *Let* L' *be a finite extension of* L. *Then there exist a strictly semi-stable scheme* W' *of finite type over the integer ring* $\mathscr{O}_{L'}$ *and a projective and surjective morphism* $W' \to W$ *over* $\mathscr{O}_L$ *such that the induced map* $W'_{L'} = W' \otimes_{\mathscr{O}_{L'}} L' \to W_L = W \otimes_{\mathscr{O}_L} L$ *is an isomorphism.*

By Lemma 6.4.2, Theorem 6.4.1 has the following consequence.

Corollary **6.4.3.** — *Let* K *be a complete discrete valuation field.*

1. Let X_K *be a proper irreducible scheme over* K. *Then there exist a finite normal extension* L *of* K, *a projective, strictly semi-stable and geometrically connected scheme* W *over the integer ring* $\mathscr{O}_L$ *and a proper, surjective and generically finite morphism* $W_L \to X_K$ *over* K.

2. Let X *be a proper and flat irreducible scheme over* K. *Then there exist a finite normal extension* L *of* K, *a projective, strictly semi-stable and geometrically connected scheme* W *over the integer ring* $\mathscr{O}_L$ *and a proper, surjective and generically finite morphism* $W \to X$ *over* $\mathscr{O}_K$.

We compute the trace using an alteration. We introduce some notation. Let K be an arbitrary field for the moment. Let X_K be a proper smooth scheme purely of dimension d over a field K, $\sigma \in G_K$ be an element of the absolute Galois group and $\Gamma \in \mathrm{CH}^d(X_K \times_K X_K)$ be an algebraic correspondence. We assume X_K is irreducible. Let $L \supset K$ be a finite normal extension of K, W_L be a proper, smooth and geometrically irreducible scheme over L and $f : W_L \to X_K$ be a proper, surjective and generically finite morphism over K.

We fix an embedding $\bar{K} \to \bar{L}$ of separable closures and extend σ to automorphisms of $\bar{L}$ and of L. For an automorphism $\tau \in \mathrm{Aut}_K L$, let $W_L^\tau = W_L \times_{L \swarrow \tau^*} L$ be the base change by τ and f_τ denote the composition $f \times 1 : W_L^\tau \to X_K$. For $\tau \in \mathrm{Aut}_K L$, let $\Gamma_{\tau,\sigma\tau} \in \mathrm{CH}^n(W_L^\tau \times_L W_L^{\sigma\tau})$ be the pull-back $(f_\tau \times f_{\sigma\tau})^*\Gamma$ of Γ by $f_\tau \times f_{\sigma\tau} : W_L^\tau \times_L W_L^{\sigma\tau} \to X_K \times_K X_K$. It induces a homomorphism $\Gamma^*_{\tau,\sigma\tau} : H^*(W_{\bar{L}}^{\sigma\tau}, \mathbf{Q}_\ell) \to H^*(W_{\bar{L}}^\tau, \mathbf{Q}_\ell)$. If $\tau = \mathrm{id}$, we put $\Gamma^*_\sigma = \Gamma^*_{\mathrm{id},\sigma} : H^*(W_{\bar{L}}^\sigma, \mathbf{Q}_\ell) \to H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$. The isomorphism $\sigma^* = 1 \times \sigma^* : W^{\sigma\tau}_{\bar{L}} \to W^{\tau}_{\bar{L}}$ induces an isomorphism $\sigma_* = (\sigma^*)^* : H^*(W^{\tau}_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W^{\sigma\tau}_{\bar{L}}, \mathbf{Q}_\ell)$. The composition $\Gamma^*_{\tau,\sigma\tau} \circ \sigma_*$ is an endomorphism of $H^*(W^{\tau}_{\bar{L}}, \mathbf{Q}_\ell)$.

Lemma **6.4.4** ([41] Lemma 3.3). — *Let* X_K *be a proper and smooth irreducible scheme of dimension* d *over a field* K, $\sigma \in G_K$ *be an element of the absolute Galois group and* $\Gamma \in CH^d(X_K \times_K X_K)$ *be an algebraic correspondence. Let* L *be a finite normal extension of* K *of inseparable degree* q, W_L *be a proper, smooth and geometrically irreducible scheme over* L *and* $f : W_L \to X_K$ *be a proper, surjective and generically finite morphism of degree* $[W_L : X_K]$ *over* K.

Then, we have an equality

$$\begin{aligned}&[W_L : X_K] \cdot \mathrm{Tr}\left(\Gamma^* \circ \sigma_* : H^r(X_{\bar{K}}, \mathbf{Q}_\ell)\right)\\ &= q \cdot \sum_{\tau \in \mathrm{Aut}_K L} \mathrm{Tr}\left(\Gamma^*_{\tau,\sigma\tau} \circ \sigma_* : H^r\left(W^{\tau}_{\bar{L}}, \mathbf{Q}_\ell\right)\right).\end{aligned}$$

Now we assume K is a discrete valuation field and compute the Swan conductor $\mathrm{Sw}\,(\Gamma^*, X_K/K)$ using an alteration as in Corollary 6.4.3.1.

Corollary **6.4.5.** — *Let* X_K *be a proper and smooth irreducible scheme of dimension* d *over a complete discrete valuation field* K *and* $\Gamma \in CH^d(X_K \times_K X_K)$ *be an algebraic correspondence. Let* L *be a finite normal extension of* K *of inseparable degree* q, W *be a proper, strictly semi-stable and irreducible scheme over* $\mathcal{O}_L$ *and* $f : W_L = W \otimes_{\mathcal{O}_L} L \to X_K$ *be a proper, surjective and generically finite morphism of degree* $[W_L : X_K]$ *over* K. *Then,*

1. The restriction to the wild inertia subgroup $P_L \subset G_L$ *of the action of* G_K *on* $H^r(X_{\bar{K}}, \mathbf{Q}_\ell)$ *is trivial.*

2. Let L_0 *be the separable closure of* K *in* L, $G_0 = \mathrm{Gal}(L_0/K)$ *be the Galois group and* $P_0 \subset G_0$ *be the wild inertia subgroup. Then we have an equality*

$$\begin{aligned}&[W_L : X_K] \cdot \mathrm{Sw}\left(\Gamma^*, X_K/K\right)\\ &= q \cdot \sum_{\sigma \in P_0} \mathrm{sw}(\sigma) \cdot \mathrm{Tr}\left(\Gamma^*_{\sigma} \circ \sigma_* : H^r(W_{\bar{L}}, \mathbf{Q}_\ell)\right).\end{aligned}$$

Proof. — 1. We identify $G_0 = \mathrm{Gal}(L_0/K)$ with $\mathrm{Aut}_K L$. For $\sigma \in G_0$, the conjugate $W^\sigma = W \otimes_{\mathcal{O}_L \nearrow \sigma^*} \mathcal{O}_L$ is also strictly semi-stable over $\mathcal{O}_L$. Hence the wild inertia $P_L \subset G_L$ acts trivially on $H^*(W^{\sigma}_{\bar{L}}, \mathbf{Q}_\ell)$ for $\sigma \in G_0$. Since the composition $f_* \circ f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to \bigoplus_{\sigma \in G_0} H^*(W^{\sigma}_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ is the multiplication by $[W_L : X_K]$, the G_L-equivariant map $f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to \bigoplus_{\sigma \in G_0} H^*(W^{\sigma}_{\bar{L}}, \mathbf{Q}_\ell)$ is injective. Hence the action of P_L on $H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ is also trivial.

2. For $\sigma \in P_K$, the action σ_* on $H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ depends only on the image in $P_0 = P_K/P_L$ by 1. By the definition of Swan conductor and Lemma 6.4.4, we have

$$[W_L : X_K]\mathrm{Sw}\big(\Gamma^*, X_K/K\big) \\ = \frac{q}{|G_0|} \cdot \sum_{\sigma \in P_0} \sum_{\tau \in \mathrm{Aut}_K L} \mathrm{sw}(\sigma)\mathrm{Tr}\ \big(\Gamma^*_{\tau,\sigma\tau} \circ \sigma_* : H^r\big(W^\tau_{\bar{L}}, \mathbf{Q}_\ell\big)\big).$$

Since $\mathrm{sw}(\sigma) = \mathrm{sw}(\tau\sigma\tau^{-1})$ and $\mathrm{Tr}\ (\Gamma^*_{\tau,\sigma\tau} \circ \sigma_* : H^r(W^\tau_{\bar{L}}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (\Gamma^*_{\tau^{-1}\sigma\tau} \circ \tau^{-1}\sigma\tau_* : H^r(W_{\bar{L}}, \mathbf{Q}_\ell))$, the assertion follows. □

We compute the logarithmic localized intersection product $[[X, \Gamma]]$ using an alteration as in Corollary 6.4.3.2. To state it, we introduce some notation. Let K be a complete discrete valuation field and X be a proper scheme over $\mathcal{O}_K$ satisfying the condition (S(n)) in Section 5.3 and $\Gamma \in CH^{n-1}(X_K \times_K X_K)$ be an algebraic correspondence. Let L be a finite normal extension of K and t be the closed point of $T = \mathrm{Spec}\ \mathcal{O}_L$. Let W be a proper, strictly semi-stable and geometrically irreducible scheme over $T = \mathrm{Spec}\ \mathcal{O}_L$ and $f : W \to X$ be a proper, surjective and generically finite morphism. Let $P_0 \subset G_0 = \mathrm{Gal}(L_0/K)$ be the wild inertia subgroup of the Galois group of the separable closure L_0 in L.

We regard W and W^σ as log schemes with the standard log structures defined by the closed fiber. For $\sigma \in P_0$, we have a canonical isomorphism $W_t \to W^\sigma_t$ of log schemes. We identify $\Gamma(W^\sigma, M_{W^\sigma})$ with $P = \Gamma(W, M_W)$ by the isomorphism $\Gamma(W, M_W) \to \Gamma(W_t, M_W) \to \Gamma(W^\sigma_t, M_{W^\sigma}) \to \Gamma(W^\sigma, M_{W^\sigma})$. We define the log product $(W \times_T W^\sigma)^\sim$ to be $W \times_{T,[P]} W^\sigma$. Since W is strictly semi-stable, W^σ is also strictly semi-stable over $\mathcal{O}_L$ and the projection $(W \times_T W^\sigma)^\sim \to W$ is strict and smooth. The canonical isomorphism $W_t \to W^\sigma_t$ induces a map $\Delta_{W_t} : W_t \to (W \times_T W^\sigma)^\sim_t$. Since $\Delta_{W_t} : W_t \to (W \times_T W^\sigma)^\sim_t$ is a section of the smooth map $(W \times_T W^\sigma)^\sim_t \to W_t$, it is a regular immersion.

We have a map $CH^{n-1}(X_K \times_K X_K) \to \mathrm{Gr}^F_{n-1}G(X_K \times_K X_K)$ by Lemma 2.1.4.2. By Lemma 5.4.5.2, the logarithmic localized intersection product defines a map $\mathrm{Gr}^F_{n-1}G(X_K \times_K X_K) \to F_0G(X_s)$. Let $\sigma \in P_0$. By Corollary 2.2.3, the pull-back map $(f \times f_\sigma)^* : G(X_K \times_K X_K) \to G(W_L \times_L W^\sigma_L)$ induces a map $\mathrm{Gr}^F_{n-1}G(X_K \times_K X_K) \to \mathrm{Gr}^F_{n-1}G(W_L \times_L W^\sigma_L)$. By Corollary 2.2.4, the reduction map $G(W_L \times_L W^\sigma_L) \to G((W \times_T W^\sigma)^\sim_t)$ induces a map $\mathrm{Gr}^F_{n-1}G(W_L \times_L W^\sigma_L) \to \mathrm{Gr}^F_{n-1}G((W \times_T W^\sigma)^\sim_t)$. Since the immersion $\Delta_{W_t} : W_t \to (W \times_T W^\sigma)^\sim_t$ is a regular immersion, the pull-back $\Delta^*_{W_t} : G((W \times_T W^\sigma)^\sim_t) \to G(W_t)$ is defined. By Proposition 2.2.2, it induces a map $\mathrm{Gr}^F_{n-1}G((W \times_T W^\sigma)^\sim_t) \to F_0G(W_t)$.

Proposition **6.4.6.** — *Let* K *be a complete discrete valuation field,* X *be a proper scheme over* $\mathcal{O}_K$ *satisfying the condition* (S(n)) *in Section 5.3 and* $\Gamma \in CH^{n-1}(X_K \times_K X_K)$ *be an algebraic correspondence. Let* L *be a finite normal extension of* K *of inseparable degree* q *and* t *be the closed* *point of* $T = \operatorname{Spec} \mathcal{O}_L$. *Let* W *be a proper, strictly semi-stable and irreducible scheme over* $T = \operatorname{Spec} \mathcal{O}_L$ *and* $f : W \to X$ *be a proper, surjective and generically finite morphism of degree* $[W : X]$ *over* $\mathcal{O}_K$.

Let $[[X, \ ]] : CH^{n-1}(X_K \times_K X_K) \to Gr^F_{n-1}G(X_K \times_K X_K) \to F_0G(X_s)$ *denote the logarithmic localized intersection product. For an element* $\sigma \in P_0 \subset G_0 = \mathrm{Gal}(L_0/K)$ *of the wild inertia subgroup of the separable closure* L_0, *let* $\Gamma_{\sigma,t} \in Gr^F_{n-1}G((W \times_T W^\sigma)^\sim_t)$ *denote the reduction of the pull-back* $\Gamma_\sigma = (f \times f_\sigma)^*\Gamma \in Gr^F_{n-1}G(W_L \times_L W^\sigma_L)$ *and* $\Delta^*_{W_t} : Gr^F_{n-1}G((W \times_T W^\sigma)^\sim_t) \to F_0G(W_t)$ *denote the pull-back by the regular immersion* $\Delta_{W_t} : W_t \to (W \times_T W^\sigma)^\sim_t$. *Then, we have an equality*

$$\textbf{(6.4.6.1)} \qquad [W : X] \deg[[X, \Gamma]] = -q \cdot \sum_{\sigma \in P_0} \mathrm{sw}(\sigma) \cdot \deg_{W_t} \Delta^*_{W_t}(\Gamma_{\sigma,t}).$$

Proof. — Since the map $F_nG((X \times_S X)^\sim) \to F_{n-1}G(X_K \times_K X_K)$ is surjective, we may assume the image of Γ in $Gr^F_{n-1}G(X_K \times_K X_K)$ is the image of an element $\tilde{\Gamma} \in F_nG((X \times_S X)^\sim)$. By abuse of notation, we drop $\tilde{}$ and write $\Gamma \in F_nG((X \times_S X)^\sim)$.

Since $f_* \circ f^* : F_0G(X_s) \to F_0G(W_t) \to F_0G(X_s)$ is the multiplication by the degree $[W : X]$ by Corollary 2.2.3, it is sufficient to show the equality

$$\textbf{(6.4.6.2)} \qquad q \cdot f^*[[X, \Gamma]] = -q^2 \cdot \sum_{\sigma \in P_0} \mathrm{sw}(\sigma) \cdot \Delta^*_{W_t}(\Gamma_{\sigma,t})$$

in $G(W_t)$ for $\Gamma \in G((X \times_S X)^\sim)$.

We have $[[X, \Gamma]] = [[\Gamma, X]]$ by Theorem 3.2.1.4. We show the equalities

$$\textbf{(6.4.6.3)} \qquad f^*[[\Gamma, X]]_{(X \times_S X)^\sim} = [[\Gamma, W]]_{(X \times_S X)^\sim} = \Delta^*_W[[\Gamma, (W \times_T W)^\sim]]_{(X \times_S X)^\sim}$$

by applying the associativity, Corollary 3.3.4.1. In the middle and the right, $[[\ , W]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \to G(W_t)$ and $[[\ , (W \times_T W)^\sim]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \to G((W \times_T W)^\sim_t)$ denote the localized intersection product respectively. In the right hand side, $\Delta^*_W : G((W \times_T W)^\sim_t) \to G(W_t)$ denotes the pull-back by the regular immersion $W \to (W \times_T W)^\sim$. Since $(W \times_T W)^\sim$ is flat over W by Lemma 5.2.1.1, the map $\Delta^*_W : G((W \times_T W)^\sim_t) \to G(W_t)$ is the same as the pull-back by the regular immersion $\Delta_{W_t} : W_t \to (W \times_T W)^\sim_t$. For the first equality, we apply Corollary 3.3.4.1 by taking $W \to X \to (X \times_S X)^\sim \to X$ as $W' \to W \to X \to S$. Since W and X are regular, the map $f : W \to X$ is of finite tor-dimension. Hence the assumption of Corollary 3.3.4.1 is satisfied and the first equality is proved. For the second equality, we apply the same Corollary 3.3.4.1 by taking $W \to (W \times_T W)^\sim \to (X \times_S X)^\sim \to X$ as $W' \to W \to X \to S$. Since W is strictly semi-stable over T, the map $(W \times_T W)^\sim \to W$ is smooth. Hence $(W \times_T W)^\sim$ is regular and the log diagonal map $\Delta : W \to (W \times_T W)^\sim$ is of finite tor-dimension. Thus the assumption of Corollary 3.3.4.1 is also satisfied and the second equality follows.

Since $\mathcal{O}_{W_t} = \mathcal{O}_{(W\times_T W)^\sim_t} \otimes^L_{\mathcal{O}_{(W\times_T W)^\sim}} \mathcal{O}_W$, we further have

$$\textbf{(6.4.6.4)} \qquad \Delta^*_W[[\Gamma, (W \times_T W)^\sim]]_{(X\times_S X)^\sim} = \Delta^*_{W_t}[[\Gamma, (W \times_T W)^\sim]]_{(X\times_S X)^\sim}.$$

Hence, it is reduced to showing the equality

$$\textbf{(6.4.6.5)} \qquad q \cdot [[\Gamma, (W \times_T W)^\sim]]_{(X\times_S X)^\sim} = -q^2 \cdot \sum_{\sigma\in P_0} \mathrm{sw}(\sigma)\Gamma_{\sigma,t}$$

in $G((W \times_T W)^\sim_t)$.

To go from $G((W \times_T W)^\sim_t)$ to $G((W \times_{T_0} W)^\sim_{t_0})$, we use the following lemma. If L is separable over K, we have $T_0 = T$ and this step is trivial. Since the action of $\sigma \in P_0$ on the log point t is trivial, we naturally identify W^σ_t and $(W \times_T W^\sigma)^\sim_t$ with W_t and $(W \times_T W)^\sim_t$ for $\sigma \in P_0$ respectively.

Lemma **6.4.7.** — *1. The immersion* $(W \times_T W)^\sim \to (W \times_{T_0} W)^\sim$ *induces an isomorphism* $G((W \times_T W)^\sim_t) \to G((W \times_{T_0} W)^\sim_{t_0})$.

2. We identify $G((W \times_T W)^\sim_t)$ *and* $G((W \times_{T_0} W)^\sim_{t_0})$ *by the isomorphism in 1. Then, for* $\Gamma \in G((X \times_S X)^\sim)$, *we have the equality*

$$q \cdot [[\Gamma, (W \times_T W)^\sim]]_{(X\times_S X)^\sim} = [[\Gamma, (W \times_{T_0} W)^\sim]]_{(X\times_S X)^\sim}$$

of the localized intersection products $[[\ \ , (W \times_T W)^\sim]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \to G((W \times_T W)^\sim_t)$ *and* $[[\ , (W \times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \to G((W \times_{T_0} W)^\sim_{t_0})$.

3. For $\Gamma \in G(X_K \times_K X_K)$ *and* $\sigma \in P_0$, *let* $\Gamma_{\sigma,t} \in G((W \times_T W)^\sim_t)$ *and* $\Gamma_{\sigma,t_0} \in G((W\times_{T_0} W)^\sim_{t_0})$ *be the images by the compositions* $G(X_K \times_K X_K) \xrightarrow{(f_K\times f_{\sigma,K})^*} G(W_L \times_L W^\sigma_L) \xrightarrow{(\ ,t)_T} G((W \times_T W)^\sim_t)$ *and* $G(X_K \times_K X_K) \xrightarrow{(f_K\times f_{\sigma,K})^*} G(W_L \times_{L_0} W^\sigma_L) \xrightarrow{(\ ,t_0)_{T_0}} G((W \times_{T_0} W)^\sim_{t_0})$ *respectively. Then, we have*

$$q^2 \cdot \Gamma_{\sigma,t} = \Gamma_{\sigma,t_0}.$$

Proof. — 1. The diagram

$$\textbf{(6.4.7.1)} \qquad \begin{array}{ccc} (W \times_{T_0} W)^\sim & \longleftarrow & (W \times_T W)^\sim \\ \downarrow & & \downarrow \\ (T \times_{T_0} T)^\sim & \longleftarrow & T \end{array}$$

is cartesian. The purely inseparable extension L of L_0 is generated by the q-th root π_L of a prime element π_0 of L_0. The map $\mathcal{O}_L[x]/(x^q) \to (\mathcal{O}_L \otimes_{\mathcal{O}_{L_0}} \mathcal{O}_L)^\sim : x \mapsto 1 - \frac{1\otimes\pi_L}{\pi_L\otimes 1}$ is an isomorphism. Hence the immersion $T \to (T \times_{T_0} T)^\sim$ is a nilpotent immersion.

Thus the closed immersion $(W \times_T W)^\sim \to (W \times_{T_0} W)^\sim$ induces an isomorphism on the K-groups of coherent sheaves.

2. Let I be the kernel of the surjection $(\mathscr{O}_L \otimes_{\mathscr{O}_{L_0}} \mathscr{O}_L)^\sim \to \mathscr{O}_L$. Then, in $G((T \times_{T_0} T)^\sim)$, we have $[\mathscr{O}_{(T\times_{T_0}T)^\sim}] = \sum_{i=0}^{q-1}[I^i/I^{i+1}] = q[\mathscr{O}_T]$. The vertical arrows of the diagram (6.4.7.1) are flat by Lemma 5.2.1.3. Hence we have $[\mathscr{O}_{(W\times_{T_0}W)^\sim}] = q[\mathscr{O}_{(W\times_T W)^\sim}]$. Thus the assertion follows by Theorem 3.2.1.3.

3. Similarly, we have $[\mathscr{O}_{W_L\times_{L_0}W_L^\sigma}] = q[\mathscr{O}_{W_L\times_L W_L^\sigma}]$ in $G(W_L \times_{L_0} W_L^\sigma)$. Further for a coherent $\mathscr{O}_{(W\times_{\mathscr{O}_L}W^\sigma)^\sim}$-module $\mathscr{F}$, we have $[\mathscr{F} \otimes^L_{\mathscr{O}_{L_0}} \mathscr{O}_{L_0}/m_{L_0}] = [\mathscr{F} \otimes^L_{\mathscr{O}_L} \mathscr{O}_L/m_L^q] = q[\mathscr{F} \otimes^L_{\mathscr{O}_L} \mathscr{O}_L/m_L]$. Thus the assertion follows. □

By Lemma 6.4.7, the equality (6.4.6.5) is equivalent to

(6.4.6.6)
$$[[\Gamma, (W \times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} = -\sum_{\sigma\in P_0} \mathrm{sw}(\sigma)\Gamma_{\sigma,t_0}$$

in $G((W \times_{T_0} W)^\sim_{t_0})$.

We show the equality

(6.4.6.7)
$$[[\Gamma, (W \times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} = [[T_0, (f \times f)^{\sim *}\Gamma]]_{(T_0\times_S T_0)^\sim}.$$

by applying the associativity, Corollary 3.3.4.3. We take $X \leftarrow (X\times_S X)^\sim \leftarrow (W\times_S W)^\sim \to (T_0 \times_S T_0)^\sim \stackrel{\Delta}{\leftarrow} T_0$ to be $S \leftarrow X \leftarrow W \to X' \leftarrow V'$ in Corollary 3.3.4.3. We verify that the assumption in Corollary 3.3.4.3 is satisfied. The map $(W \times_S W)^\sim \to (T_0 \times_S T_0)^\sim$ is flat by Lemma 5.2.1.3 and the map $(f \times f)^\sim : (W \times_S W)^\sim \to (X \times_S X)^\sim$ is of finite tor-dimension by Lemma 5.2.1.2. The subscheme W' in loc.cit. is $(W \times_S W)^\sim \times_{(T_0\times_S T_0)^\sim} T_0 = (W \times_{T_0} W)^\sim$. The closed subsets $Z_{W'}$ and $Z'_{W'}$ in loc.cit. are $(W \times_{T_0} W)^\sim \times_X Z$ and $(W \times_{T_0} W)^\sim \times_{T_0} t_0$ respectively. Since the closed subscheme $Z \subset X$ is supported on the closed fiber, the condition that $Z_{W'}$ is $Z'_{W'}$ set-theoretically a subset in loc.cit. is satisfied. Further by Lemma 5.3.5.3, the condition $G(Z_{W'})_{/\mathscr{L}_Z} = G(Z_{W'})$ and $G(Z'_{W'})_{/\mathscr{L}'_{Z'}} = G(Z'_{W'})$ is satisfied. Hence the assumption in Corollary 3.3.4.3 is satisfied. Since $(W \times_S W)^\sim \times_{(T_0\times_S T_0)^\sim} T_0 = (W \times_{T_0} W)^\sim$, applying Corollary 3.3.4.3, we obtain the equality.

Remark. — If L is assumed separable over K and hence if $T = T_0$, there is an alternative proof of the equality (6.4.6.7). By Corollary 3.3.4.3, we have equalities

$$\begin{aligned}[[\Gamma, (W \times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} &= [[(W \times_{T_0} W)^\sim, (f \times f)^{\sim *}\Gamma]]_{(W\times_S W)^\sim} \\ &= [[T_0, (f \times f)^{\sim *}\Gamma]]_{(T_0\times_S T_0)^\sim}\end{aligned}$$

and the equality (6.4.6.7) follows.

By (6.4.6.7), the equality (6.4.6.6) is equivalent to

(6.4.6.8) $$[[\mathrm{T}_0, (f\times f)^{\sim *}\Gamma]]_{(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim} = -\sum_{\sigma\in \mathrm{P}_0} \mathrm{sw}(\sigma)\Gamma_{\sigma,t_0}$$

in $\mathrm{G}((\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W})^\sim_{t_0})$. Hence it suffices to apply the following lemma to $(f\times f)^{\sim *}\Gamma \in \mathrm{G}((\mathrm{W}\times_\mathrm{S}\mathrm{W})^\sim)$.

Lemma **6.4.8.** — *Let* L_0 *be the separable closure of* K *in* L *and let* t_0 *be the closed point of* $\mathrm{T}_0 = \mathrm{Spec}\ \mathcal{O}_{\mathrm{L}_0}$. *Let* P_0 *be the wild inertia subgroup of the Galois group* $\mathrm{G}_0 = \mathrm{Gal}(\mathrm{L}_0/\mathrm{K})$. *For* $\Gamma \in \mathrm{G}((\mathrm{W}\times_\mathrm{S}\mathrm{W})^\sim)$ *and* $\sigma\in \mathrm{P}_0$, *let* $\Gamma_\sigma \in \mathrm{G}(\mathrm{W}_\mathrm{L}\times_{\mathrm{L}_0}\mathrm{W}^\sigma_\mathrm{L})$ *be the restriction and* $\Gamma_{\sigma,t_0}\in \mathrm{G}((\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W})^\sim_{t_0})$ *be the reduction of* Γ_σ. *Then, we have*

$$[[\mathrm{T}_0, \Gamma]]_{(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim} = -\sum_{\sigma\in \mathrm{P}_0} \mathrm{sw}(\sigma)\Gamma_{\sigma,t_0}$$

in $\mathrm{G}((\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W})^\sim_{t_0})$.

Proof. — The map $\coprod_{\sigma\in\mathrm{G}_0}\mathrm{T}_{0,\sigma}\to \mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0$ is surjective and $(\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W}^\sigma)^\sim = (\mathrm{W}\times_\mathrm{S}\mathrm{W})^\sim\times_{(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim}\mathrm{T}_{0,\sigma}$. Hence the map $\coprod_{\sigma\in\mathrm{G}_0}(\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W}^\sigma)^\sim\to(\mathrm{W}\times_\mathrm{S}\mathrm{W})^\sim$ is surjective and consequently the sum of the push-forward map $\bigoplus_{\sigma\in\mathrm{G}_0}\mathrm{G}((\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W}^\sigma)^\sim)\to \mathrm{G}((\mathrm{W}\times_\mathrm{S}\mathrm{W})^\sim)$ is surjective. Thus it is sufficient to show the equality

$$[[\mathrm{T}_0, \Gamma]]_{(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim} = \begin{cases} -\mathrm{sw}(\sigma)\Gamma_{\sigma,t_0} & \text{if } \sigma\in\mathrm{P}_0\\ 0 & \text{if } \sigma\in \mathrm{G}_0\setminus\mathrm{P}_0\end{cases}$$

for $\sigma\in\mathrm{G}_0$ and $\Gamma\in\mathrm{G}((\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W}^\sigma)^\sim)$.

In Corollary 3.3.4.2, we take $\mathrm{T}_0\xrightarrow{\Delta}(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim\leftarrow\mathrm{T}_{0,\sigma}\leftarrow(\mathrm{W}\times_{\mathrm{T}_0}\mathrm{W}^\sigma)^\sim$ as $\mathrm{V}\to\mathrm{X}\leftarrow\mathrm{W}\leftarrow\mathrm{W}'$. Since $\mathrm{T}_{0,\sigma}=\mathrm{T}_0$ is regular, the assumption of Corollary 3.3.4.2 is satisfied. By Lemma 6.1.1.2, we have $[[\mathrm{T}_0,\mathrm{T}_{0,\sigma}]]_{(\mathrm{T}_0\times_\mathrm{S}\mathrm{T}_0)^\sim} = -\mathrm{sw}(\sigma)\in\mathrm{G}(t_0)=\mathbf{Z}$ for $\sigma\in\mathrm{P}_0$ and $\mathrm{T}_{0,\sigma}\cap\mathrm{T}_0=\emptyset$ for $\sigma\in\mathrm{G}_0-\mathrm{P}_0$. Hence the equality follows. □

6.5. *Log Lefschetz trace formula.* — We state and prove logarithmic Lefschetz trace formula. To state it, we fix some notations. Let K be a complete discrete valuation field with perfect residue field. Let L be a finite extension of K and σ be an automorphism L over K. We assume that σ acts trivially on the residue field E and that the order of σ is a power of the characteristic p of E. In other words, the action of σ on the log point $t=\mathrm{Spec}\ \mathrm{E}$ is trivial. We extend σ to an element $\tilde\sigma\in\mathrm{P}_\mathrm{K}$.

Let W be a projective and strictly semi-stable scheme purely of relative dimension d over $\mathrm{T}=\mathrm{Spec}\ \mathcal{O}_\mathrm{L}$. The conjugate $\mathrm{W}^\sigma\to\mathrm{T}$ is defined as the base change $pr_2:\mathrm{W}\times_{\mathrm{T}\swarrow\sigma^*}\mathrm{T}\to\mathrm{T}$. For a prime number ℓ different from $p=\mathrm{char}\ \mathrm{E}$, we define a map $\sigma_*:\mathrm{H}^*(\mathrm{W}_{\bar{\mathrm{L}}},\mathbf{Q}_\ell)\to\mathrm{H}^*(\mathrm{W}^\sigma_{\bar{\mathrm{L}}},\mathbf{Q}_\ell)$ to be the pull-back by the map

$1 \times \tilde{\sigma}^* : W^\sigma_{\bar{L}} = W \times_{L \swarrow \sigma^*} L \times_L \bar{L} = W \times_{L \swarrow \sigma^*} \bar{L} \to W_{\bar{L}}$. Since we assume W is proper and strictly semi-stable and $\ell \neq p$, the action of the wild inertia $P_{L'}$ is trivial on $H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$. Hence the map $\tilde{\sigma}_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W^\sigma_{\bar{L}}, \mathbf{Q}_\ell)$ depends only on σ and is independent of the choice of a lifting $\tilde{\sigma}$.

We put $P = \Gamma(W, \bar{M}_W)$ and $N = \Gamma(T, \bar{M}_T) = \mathbf{N}$. Then the map $P = \Gamma(W, \bar{M}_W) \to \Gamma(W^\sigma, \bar{M}_{W^\sigma})$ defines a frame and the canonical map $N \to P$ defines maps $(W, [P]) \to (T, [N])$ and $(W^\sigma, [P]) \to (T, [N])$ of framed log schemes. We put $(W \times_T W^\sigma)^\sim = W \times_{T,[P]} W^\sigma$. Since σ is the identity on the log point t, we have $W^\sigma_t = W_t$ as log schemes over t. Hence the closed fiber $(W \times_T W^\sigma)^\sim_t = (W \times_T W^\sigma)^\sim \times_T t$ is canonically identified with $(W \times_T W)^\sim_t$.

For an algebraic correspondence $\Gamma \in CH_d(W_L \times_L W^\sigma_L)$, let Γ also denote its image in $Gr^F_d G(W_L \times_L W^\sigma_L)$ by abuse of notation and let $\Gamma_t \in Gr^F_d G((W \times_T W)^\sim_t)$ denote the specialization $(\Gamma, t)_T$. Since the immersion $\Delta_{W_t} : W_t \to (W \times_T W)^\sim_t$ is a regular immersion by Lemma 5.2.3.2, the pull-back $\Delta^*_{W_t}(\Gamma_t) \in Gr^F_0 G(W_t)$ is defined. We define the degree map $\deg_{W_t} : G(W_t) \to G(t) = \mathbf{Z}$ to be the push-forward for $W_t \to t$.

Theorem **6.5.1.** — *Let* L *be a discrete valuation field with perfect residue field* E *of characteristic* p *and* $\ell \neq p$ *be a prime number. Let* σ *be an automorphism of* $\mathcal{O}_L$ *of order a power of* p *which induces the identity on the residue field* E. *Let* W *be a projective and strictly semi-stable scheme of relative dimension* d *over* $T = \operatorname{Spec} \mathcal{O}_L$.

Then for an algebraic correspondence $\Gamma \in CH_d(W_L \times_L W^\sigma_L)$, *we have an equality of integers*

$$\textbf{(6.5.1.1)} \qquad \operatorname{Tr}(\Gamma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) = \deg_{W_t} \Delta^*_{W_t}(\Gamma_t).$$

Proof. — We show the formula (6.5.1.1) by using log-etale cohomology of the closed fiber. Basic references for log-etale cohomology are [12], [28], [29] and [20].

We regard t as a log scheme with the log structure induced by the standard one on T. The assumption on σ means that σ acts trivially on the log point t. Let $\bar{t}$ be a log geometric point over the log point t and $W_{\bar{t}}$ be the geometric closed fiber. Let $H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$ be the log-etale cohomology. By [29] Proposition (4.2), there is a canonical isomorphism $H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$.

We fix an isomorphism $\mathbf{N}^r \to \Gamma(W, \bar{M}_W)$. It induces an isomorphism $\mathbf{N}^r \to \Gamma(W^\sigma, \bar{M}_{W^\sigma})$. We put $P = \mathbf{N}^r \oplus_{\mathbf{N}} \mathbf{N}^r$ and let Σ be the subdivision of the dual monoid $N = \mathrm{Hom}_{\mathrm{monoid}}(P, \mathbf{N})$ as in Lemma 5.2.4. Let $(W \times_T W^\sigma)^-$ be the log blow-up $(W \times_T W^\sigma)_\Sigma$ of $W \times_T W^\sigma$ studied loc.cit. It contains $(W \times_T W^\sigma)^\sim$ as an open subscheme.

We reduce Theorem 6.5.1 to a statement, Lemma 6.5.2 below, for an element in $Gr^d_F K((W \times_T W^\sigma)^-)$. Since W_L and W^σ_L are projective and smooth, the Chern character map $ch : Gr^d_F K(W_L \times_L W^\sigma_L)_{\mathbf{Q}} \to CH_d(W_L \times_L W^\sigma_L)_{\mathbf{Q}}$ is an isomorphism by Lemma 2.1.4.3. Since $(W \times_T W^\sigma)^-$ is regular by Lemma 5.2.3.2, the canonical map $K((W\times_T W^\sigma)^-) \to G((W\times_T W^\sigma)^-)$ is an isomorphism. Hence the maps $K((W\times_T W^\sigma)^-) \to K(W_L\times_L W_L^\sigma)$ and $\mathrm{Gr}_F^d K((W\times_T W^\sigma)^-) \to \mathrm{Gr}_F^d K(W_L\times_L W_L^\sigma)$ are surjective. Thus, there exists an element $\tilde\Gamma \in \mathrm{Gr}_F^d K((W\times_T W^\sigma)^-)_{\mathbf{Q}}$ such that the image of Γ in $\mathrm{CH}_d(W_L\times_L W_L^\sigma)_{\mathbf{Q}}$ is equal to $ch(\tilde\Gamma|_{W_L\times_L W_L^\sigma})$. Since the equality (6.5.1.1) is an equality in $\mathbf{Q}_\ell$, we may assume that the image of Γ in $\mathrm{CH}_d(W_L\times_L W_L^\sigma)_{\mathbf{Q}}$ is the images of $\tilde\Gamma \in \mathrm{Gr}_F^d K((W\times_T W^\sigma)^-)$ by replacing Γ by a multiple.

The diagram

$$\begin{array}{ccccc}
\mathrm{Gr}_F^d K\big(W_L\times_L W_L^\sigma\big) & \xrightarrow{ch} & \mathrm{CH}_d\big(W_L\times_L W_L^\sigma\big)_{\mathbf{Q}} & \to & \mathrm{Gr}_d^F G\big(W_L\times_L W_L^\sigma\big)_{\mathbf{Q}} \\
\uparrow & & & & \downarrow{\scriptstyle(\ ,t)} \\
\mathrm{Gr}_F^d K\big((W\times_T W^\sigma)^-\big) & \underset{(\ ,t)}{\longrightarrow} & \mathrm{Gr}_F^d K\big((W\times_T W^\sigma)_t^-\big) & \overset{\text{canores}}{\to} & \mathrm{Gr}_d^F G\big((W\times_T W^\sigma)_t^\sim\big)_{\mathbf{Q}} \\
 & & {\scriptstyle\Delta_{W_t}^*}\downarrow & & \downarrow{\scriptstyle\Delta_{W_t}^*} \\
 & & \mathrm{Gr}_F^d K(W_t) & \overset{\text{can}}{\to} & \mathrm{Gr}_0^F G(W_t)_{\mathbf{Q}}
\end{array}$$

is commutative, since the composition of the top horizontal arrows is the canonical map by Lemma 2.1.4.3. Hence the image of $\Delta_{W_t}^*(\Gamma_t) \in \mathrm{Gr}_0^F G(W_t)_{\mathbf{Q}}$ is the image of $\Delta_{W_t}^*(\tilde\Gamma_t) \in \mathrm{Gr}_F^d K(W_t)$ where $\tilde\Gamma_t \in \mathrm{Gr}_F^d K((W\times_T W^\sigma)_t^-)$ is the reduction of $\tilde\Gamma$.

Thus Theorem 6.5.1 is reduced to the following lemma. Let $\deg : \mathrm{Gr}_F^d K(W_t) \to \mathbf{Z}$ denote the composition map $\mathrm{Gr}_F^d K(W_t) \to \mathrm{Gr}_0^F G(W_t) \overset{\deg}{\to} \mathbf{Z}$.

Lemma **6.5.2.** — *Let* $\tilde\Gamma$ *be an element of* $\mathrm{Gr}_F^d K((W\times_T W^\sigma)^-)$. *Let* $\Gamma \in \mathrm{CH}_d(W_L\times_L W_L^\sigma)_{\mathbf{Q}}$ *be the Chern character* $ch(\tilde\Gamma|_{W_L\times_L W_L^\sigma})$ *of the restriction and let* $\Delta_{W_t}^*(\tilde\Gamma_t) \in \mathrm{Gr}_F^d K(W_t)$ *be the pull-back of the reduction* $\tilde\Gamma_t \in \mathrm{Gr}_F^d K((W\times_T W^\sigma)_t^-)$ *of* $\tilde\Gamma$. *Then we have an equality of integers*

$$\mathrm{Tr}\ (\Gamma^*\circ\sigma_* : H^*(W_{\bar L}, \mathbf{Q}_\ell)) = \deg_{W_t}\Delta_{W_t}^*(\tilde\Gamma_t).$$

We show that $\tilde\Gamma_t \in \mathrm{Gr}_F^d K((W\times_T W^\sigma)_t^-)$ defines an endomorphism of $H_{\log}^*(W_{\bar t}, \mathbf{Q}_\ell)$ corresponding to $\Gamma^*\circ\sigma_*$ on $H^*(W_{\bar L}, \mathbf{Q}_\ell)$. We define an endomorphism $\tilde\Gamma_t^*$ of $H_{\log}^*(W_{\bar t}, \mathbf{Q}_\ell)$ as follows. The Chern character map $ch : K((W\times_T W^\sigma)_t^-) \to H_{\log}^{2d}((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d))$ induces a map $ch : \mathrm{Gr}_F^d K((W\times_T W^\sigma)_t^-) \to H_{\log}^{2d}((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d))$. It is the composition of the Chern character map $ch : \mathrm{Gr}_F^d K((W\times_T W^\sigma)_t^-) \to H^{2d}((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d))$ with the canonical map $H^{2d}((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d)) \to H_{\log}^{2d}((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d))$.

First, we show that the projections $(W\times_T W^\sigma)^- \to W$, $(W\times_T W^\sigma)^- \to W^\sigma$ and the cup-product induce an isomorphism $\bigoplus_{p+q=r} H_{\log}^p(W_{\bar t}, \mathbf{Q}_\ell(d)) \otimes H_{\log}^q(W_{\sigma,\bar t}, \mathbf{Q}_\ell) \to H_{\log}^r((W\times_T W^\sigma)_{\bar t}^-, \mathbf{Q}_\ell(d))$. Since $(W\times_T W^\sigma)^-$, W and W^σ are semi-stable, the log etale cohomology of the closed fibers are canonically isomorphic to the etale cohomology of the generic fibers by [29] Proposition (4.2). Since the canonical isomorphism is compatible with the pull-back and the cup-product, it is reduced to the Künneth formula for the generic fibers.

Recall that we have $W_{\sigma,t} = W_t$ as log schemes over t. By Poincaré duality loc.cit. Theorem (7.5) for log-etale cohomology, we have a canonical isomorphism $\bigoplus_q \mathrm{End}(H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell)) \to H^{2d}_{\log}((W \times_T W^\sigma)^-_t, \mathbf{Q}_\ell(d))$. Taking the composition of the maps, we obtain a map $\mathrm{Gr}^d_F K((W \times_T W^\sigma)^-_t) \to \bigoplus_q \mathrm{End}(H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell))$. Thus an element $\tilde\Gamma_t \in \mathrm{Gr}^d_F K((W \times_T W^\sigma)^-_t)$ defines an endomorphism $\tilde\Gamma^*_t$ of $H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell)$. It is the composition of

$$\begin{aligned} H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell) = H^q_{\log}\big(W^\sigma_{\bar t}, \mathbf{Q}_\ell\big) &\xrightarrow{p_2^*} H^q_{\log}\big((W \times_T W^\sigma)^-_{\bar t}, \mathbf{Q}_\ell\big) \xrightarrow{\cup ch(\tilde\Gamma_t)} \\ H^{2d+q}_{\log}\big((W \times_T W^\sigma)^-_{\bar t}, \mathbf{Q}_\ell(d)\big) &\xrightarrow{p_{1*}} H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell). \end{aligned}$$

We show that the endomorphism $\Gamma^* \circ \sigma_*$ of $H^*(W_{\bar L}, \mathbf{Q}_\ell)$ corresponds to the endomorphism Γ^*_t on $H^*_{\log}(W_{\bar t}, \mathbf{Q}_\ell)$.

Lemma **6.5.3.** — *Let the notation be the same as in Lemma 6.5.2. Let* $\tilde\Gamma^*_t$ *be the endomorphism of* $H^q_{\log}(W_{\bar t}, \mathbf{Q}_\ell)$ *defined above and let* $ch(\Delta^*_{W_t}(\tilde\Gamma_t)) \in H^{2d}_{\log}(W_{\bar t}, \mathbf{Q}_\ell(d))$ *be the Chern character of the pull-back* $\Delta^*_{W_t}(\tilde\Gamma_t) \in \mathrm{Gr}^d_F K(W_t)$. *Then,*

1. The diagram

(6.5.3.1)
$$\begin{array}{ccc} H^*(W_{\bar L}, \mathbf{Q}_\ell) & \xrightarrow{\Gamma^*\circ\sigma_*} & H^*(W_{\bar L}, \mathbf{Q}_\ell) \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ H^*_{\log}(W_{\bar t}, \mathbf{Q}_\ell) & \xrightarrow[\Gamma^*_t]{} & H^*_{\log}(W_{\bar t}, \mathbf{Q}_\ell) \end{array}$$

is commutative and we have an equality

(6.5.3.2) $$\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(W_{\bar L}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (\tilde\Gamma^*_t : H^*_{\log}(W_{\bar t}, \mathbf{Q}_\ell)).$$

2. We have an equality

$$\mathrm{Tr}\ (\tilde\Gamma^*_t : H^*_{\log}(W_{\bar t}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (ch(\Delta^*_{W_t}(\tilde\Gamma_t))).$$

Proof. — 1. For the commutative diagram (6.5.3.1), it is sufficient to show the commutativity of the diagram

$$\begin{array}{ccccc}
\mathrm{H}^q(\mathrm{W}_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell) & \xrightarrow{\sigma_*} & \mathrm{H}^q\big(\mathrm{W}^\sigma_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell\big) & \xrightarrow{p_2^*} & \mathrm{H}^q\big(\mathrm{W}_{\bar{\mathrm{L}}} \times_{\bar{\mathrm{L}}} \mathrm{W}^\sigma_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^q_{\log}\big(\mathrm{W}_{\bar{t}}, \mathbf{Q}_\ell\big) & = & \mathrm{H}^q_{\log}\big(\mathrm{W}^\sigma_{\bar{t}}, \mathbf{Q}_\ell\big) & \xrightarrow{p_2^*} & \mathrm{H}^q_{\log}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell\big)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\cup[\Gamma]} & \mathrm{H}^{2d+q}\big(\mathrm{W}_{\bar{\mathrm{L}}} \times_{\bar{\mathrm{L}}} \mathrm{W}^\sigma_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell(d)\big) & \xrightarrow{p_{1*}} & \mathrm{H}^q(\mathrm{W}_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell) \\
 & \downarrow & & \downarrow \\
\xrightarrow{\cup ch(\tilde{\Gamma}_t)} & \mathrm{H}^{2d+q}_{\log}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)\big) & \xrightarrow{p_{1*}} & \mathrm{H}^q_{\log}(\mathrm{W}_{\bar{t}}, \mathbf{Q}_\ell).
\end{array}$$

The vertical maps are the canonical isomorphisms. The commutativity of the first two squares is the functoriality of the canonical isomorphisms. The commutativity of the last square follows from the functoriality and the compatibility with the Poincaré duality. We show the remaining square is also commutative. The diagram

$$\begin{array}{ccc}
\mathrm{Gr}^d_{\mathrm{F}}\mathrm{K}\big((\mathrm{W}_{\mathrm{T}} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-\big) & \longrightarrow & \mathrm{Gr}^d_{\mathrm{F}}\mathrm{K}\big(\mathrm{W}_{\mathrm{L}} \times_{\mathrm{L}} \mathrm{W}^\sigma_{\mathrm{L}}\big) \\
\downarrow & & \downarrow{\scriptstyle ch} \\
\mathrm{Gr}^d_{\mathrm{F}}\mathrm{K}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_t\big) & & \mathrm{CH}^d\big(\mathrm{W}_{\mathrm{L}} \times_{\mathrm{L}} \mathrm{W}^\sigma_{\mathrm{L}}\big)_{\mathbf{Q}} \\
{\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle cl} \\
\mathrm{H}^{2d}_{\log}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)\big) & \longleftarrow & \mathrm{H}^{2d}\big(\mathrm{W}_{\bar{\mathrm{L}}} \times_{\bar{\mathrm{L}}} \mathrm{W}^\sigma_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell(d)\big)
\end{array}$$

is commuatitive, since the composition of the right vertical arrows is the Chern character map. Hence it follows from the compatiblity of the canonical isomorphism with the cup-product.

The equality (6.5.3.2) is an immediate consequence of the commutative diagram (6.5.3.1).

2. By the functoriality of the Chern character map, Künneth formula and Poincaré duality, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}^d_{\mathrm{F}}\mathrm{K}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_t\big) & \xrightarrow{\Delta^*} & \mathrm{Gr}^d_{\mathrm{F}}\mathrm{K}(\mathrm{W}_t) \\
{\scriptstyle ch}\downarrow & & {\scriptstyle ch}\downarrow \\
\mathrm{H}^{2d}_{\log}\big((\mathrm{W} \times_{\mathrm{T}} \mathrm{W}^\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)\big) & \xrightarrow{\Delta^*} & \mathrm{H}^{2d}_{\log}(\mathrm{W}_{\bar{t}}, \mathbf{Q}_\ell(d)) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\
\bigoplus_q \mathrm{End}\big(\mathrm{H}^q_{\log}(\mathrm{W}_{\bar{t}}, \mathbf{Q}_\ell)\big) & \xrightarrow{\sum_q (-1)^q \mathrm{Tr}} & \mathbf{Q}_\ell.
\end{array}$$

The equality follows from this immediately. □

To complete the proof of theorem, we compare the trace map with the degree map.

Lemma **6.5.4.** — *Let* Γ *be an element in* $\mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\mathrm{W}_t)$ *and let* $ch(\Gamma)$ *be the image by the Chern character map* $ch : \mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\mathrm{W}_t) \to \mathrm{H}^{2d}_{\log}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell(d))$. *Then we have* $\mathrm{Tr}\ (ch(\Gamma)) = \deg \Gamma$. *In other words, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\mathrm{W}_t) & \xrightarrow{ch} & \mathrm{H}^{2d}_{\log}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell(d)) \\ {\scriptstyle \deg}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q}_\ell . \end{array}$$

Proof. — Let $\pi : \bar{\mathrm{W}}_t \to \mathrm{W}_t$ be the normalization of W_t. The scheme $\bar{\mathrm{W}}_t$ is projective and smooth over t. We show that the diagram

$$\textbf{(6.5.4.1)} \qquad \begin{array}{ccccc} \mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\mathrm{W}_t) & \xrightarrow{ch} & \mathrm{H}^{2d}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell) & \longrightarrow & \mathrm{H}^{2d}_{\log}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell) \\ {\scriptstyle \pi^*}\downarrow & & {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\bar{\mathrm{W}}_t) & \xrightarrow[ch]{} & \mathrm{H}^{2d}(\bar{\mathrm{W}}_{\bar t}, \mathbf{Q}_\ell) & \xrightarrow[\mathrm{Tr}]{} & \mathbf{Q}_\ell \end{array}$$

is commutative. Let $\mathrm{W}^\circ_{\bar t}$ denote the smooth locus of $\mathrm{W}_{\bar t}$. Then the canonical map $\mathrm{H}^{2d}_c(\mathrm{W}^\circ_{\bar t}, \mathbf{Q}_\ell) \to \mathrm{H}^{2d}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell)$ is an isomorphism. The composition $\mathrm{H}^{2d}_c(\mathrm{W}^\circ_{\bar t}, \mathbf{Q}_\ell) \to \mathrm{H}^{2d}(\bar{\mathrm{W}}_{\bar t}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$ is the trace map for $\mathrm{W}^\circ_{\bar t}$. The other composition $\mathrm{H}^{2d}_c(\mathrm{W}^\circ_{\bar t}, \mathbf{Q}_\ell) \to \mathrm{H}^{2d}_{\log}(\mathrm{W}_{\bar t}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$ is also equal to the trace map for $\mathrm{W}^\circ_{\bar t}$ by the definition of the trace map for log etale cohomology in [28] Proof of Proposition (7.8.2). Hence the right square is commutative. The left square is commutative by the functoriality of the Chern character map.

We show the equality $\mathrm{Tr}\ (ch(\Gamma)) = \deg \Gamma$. Since the composition of the upper line of the commutative diagram (6.5.4.1) is the Chern character map, we have $\mathrm{Tr}\ (ch(\Gamma)) = \mathrm{Tr}\ (\pi^*(ch(\Gamma)))$. On the other hand, we have $\Gamma = \pi_*\pi^*\Gamma \in \mathrm{Gr}^{\mathrm{F}}_0 \mathrm{G}(\mathrm{W}_t)$ since $\pi_*[\mathscr{O}_{\bar{\mathrm{W}}_t}] = [\mathscr{O}_{\mathrm{W}_t}] \bmod \mathrm{F}_{d-1}\mathrm{G}(\mathrm{W}_t)$. Hence we have $\deg_{\mathrm{W}_t}\Gamma = \deg_{\mathrm{W}_t}\pi_*\pi^*\Gamma = \deg_{\bar{\mathrm{W}}_t}\pi^*\Gamma$. Thus it is reduced to the well-known equality $\mathrm{Tr}\ (ch(\pi^*\Gamma)) = \deg_{\bar{\mathrm{W}}_t}\pi^*\Gamma$ for the projective smooth scheme $\bar{\mathrm{W}}_t$. □

We complete the proof of theorem. We have $\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : \mathrm{H}^*(\mathrm{W}_{\bar{\mathrm{L}}}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ ch(\Delta^*_{\mathrm{W}_t}(\tilde\Gamma_t))$ by Lemma 6.5.3. Further, applying Lemma 6.5.4 to $\Delta^*_{\mathrm{W}_t}(\tilde\Gamma_t) \in \mathrm{Gr}_{\mathrm{F}}^d \mathrm{K}(\mathrm{W}_t)$, we obtain an equality $\mathrm{Tr}\ ch(\Delta^*_{\mathrm{W}_t}(\tilde\Gamma_t)) = \deg \Delta^*_{\mathrm{W}_t}(\tilde\Gamma_t)$. □

Proof of Theorem 6.3.1. — By Corollary 5.4.9, we may assume K is complete. We may further assume X_{K} is irreducible. By Corollary 6.4.3, we have an alteration W as in loc. cit. By the computation, Corollary 6.4.5.2, and the log Lefschetz trace formula, Theorem 6.5.1, we have

$$[\mathrm{W_L} : \mathrm{X_K}] \cdot \mathrm{Sw}\ (\Gamma^*, \mathrm{X_K}/\mathrm{K}) = q \cdot \sum_{\sigma \in \mathrm{P}_0} \mathrm{sw}(\sigma) \cdot \deg_{\mathrm{W}_t} \Delta^*_{\mathrm{W}_t}(\Gamma_{\sigma,t}).$$

Thus the assertion 1 follows. The assertion 2 follows from this equality and Proposition 6.4.6. □

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