Changing blow-up time
in nonlinear Schrödinger equations

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Abstract

Solutions to nonlinear Schrödinger equations may blow up in finite time. We study the influence of the introduction of a potential on this phenomenon. For a linear potential (Stark effect), the blow-up time remains unchanged, but the location of the collapse is altered. The main part of our study concerns isotropic quadratic potentials. We show that the usual (confining) harmonic potential may anticipate the blow-up time, and always does when the power of the nonlinearity is L^2–critical. On the other hand, introducing a "repulsive" harmonic potential prevents finite time blow-up, provided that this potential is sufficiently "strong". For the L^2–critical nonlinearity, this mechanism is explicit: according to the strength of the potential, blow-up is first delayed, then prevented.

1. Introduction

Consider the nonlinear Schrödinger equation with power-like nonlinearity in $\mathbb{R}^n$,

$$\begin{cases} i\partial_t v + \dfrac{1}{2}\Delta v = \lambda |v|^{2\sigma} v \,, \\ v_{|t=0} = u_0 \,. \end{cases} \tag{1.1}$$

Many results are available on the local and global existence issues for this initial value problem (see e.g. [10]). If $u_0 \in H^1(\mathbb{R}^n)$, $\lambda < 0$, $\sigma \geq 2/n$ and $\sigma < 2/(n-2)$ when $n \geq 3$, (1.1) has a unique solution, which is defined locally in time, $v \in C([0,T]; H^1(\mathbb{R}^n))$. It may not be global. Define

$$\Sigma := \left\{ f \in H^1(\mathbb{R}^n) \ ; \ |x| f \in L^2(\mathbb{R}^n) \right\} \,.$$

If we assume moreover $u_0 \in \Sigma$, and

$$E(u_0) := \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{\lambda}{\sigma + 1}\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2} < 0 \,, \tag{1.2}$$

then v blows up in finite time, that is there exists $T > 0$ such that

$$\lim_{t \to T} \|\nabla_x v(t)\|_{L^2} = +\infty\,.$$

This is proven by the general approach of Zakharov-Glassey ([16], [10]). In some particular cases, the blow-up phenomenon is well understood; see for instance [18], [1], [21], [19], [20].

The introduction of a stochastic white noise in (1.1) may amplify or prevent blow-up formation (see [14, 13], and references therein); we consider a deterministic framework.

It is shown in [12] that if the initial datum u_0, such that (1.2) holds, is replaced by $u_0(x)e^{-ib|x|^2}$ with $b > 0$ sufficiently large, then the blow-up time is anticipated (and is $O(b^{-1})$). On the other hand, if u_0 is replaced by $u_0(x)e^{ib|x|^2}$ with $b > 0$ sufficiently large, then no blow-up occurs. We want to obtain similar results without modifying the initial datum, but by introducing a potential,

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = V(x)u + \lambda|u|^{2\sigma}u\,, \\[2mm] u_{|t=0} = u_0\,. \end{cases} \tag{1.3}$$

Our results are suggested by the semi–classical régime for linear Schrödinger equation with potential. Consider the initial value problem

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{1}{2}\varepsilon^2\Delta u^\varepsilon = V(x)u^\varepsilon\,, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n\,, \\[2mm] u^\varepsilon_{|t=0} = u^\varepsilon_0\,, \end{cases} \tag{1.4}$$

where $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\varepsilon \in\,]0,1]$. In the semi–classical limit $\varepsilon \to 0$, the energy of the solution u^ε is carried by bicharacteristics, which are the integral curves associated to the classical Hamiltonian

$$p(t,x,\tau,\xi) = \tau + \frac{|\xi|^2}{2} + V(x)\,.$$

If the energy tends to concentrate in this case, one can expect that for (1.3) with a focusing nonlinearity ($\lambda < 0$), blow-up in finite time, which corresponds to the concentration of the mass, may occur. Bicharacteristic curves solve

$$\dot{t} = 1\,;\quad \dot{x} = \xi\,;\quad \dot{\tau} = 0\,;\quad \dot{\xi} = -\nabla V(x)\,.$$

Rays of geometric optics, which are the projection of bicharacteristic curves on (t,x)–space, are of the form $x = x(t)$, with

$$\ddot{x} + \nabla V(x) = 0\,;\ x(0) = x_0\,,\ \dot{x}(0) = \xi_0\,. \tag{1.5}$$

If the initial datum is of the form $u^\varepsilon_0(x) = f(x)e^{i\varphi(x)/\varepsilon}$, then $\xi_0 = \nabla\varphi(x_0)$. We give four examples which correspond to cases where (1.5) can easily be solved.
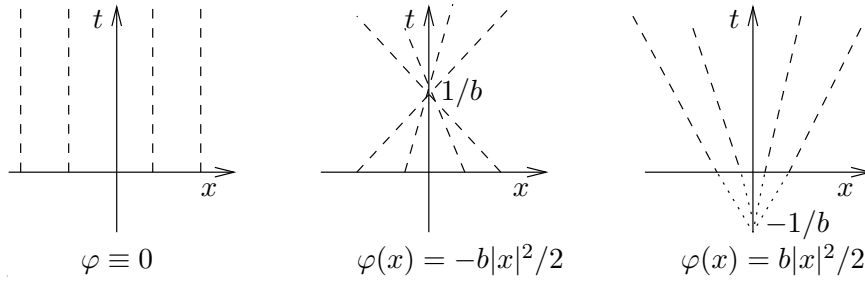
Figure 1: Geometry of rays: case $V \equiv 0$.

Example 1. Suppose $V \equiv 0$. Then the solutions of (1.5) are

$$x(t) = x_0 + t\nabla\varphi(x_0).$$

If $\varphi \equiv 0$, rays are parallel. More interesting is the case of quadratic oscillations (see also [8]). If $\varphi(x) = -b|x|^2/2$ with $b > 0$, then rays are given by $x(t) = x_0(1 - bt)$, and meet at the origin at time $t = 1/b$ (see Figure 1). There is focusing, which suggests that in a nonlinear setting, such oscillations may cause wave collapse. If $\varphi(x) = b|x|^2/2$ with $b > 0$, then rays are given by $x(t) = x_0(1 + bt)$, and met at the origin at time $t = -1/b$ (in the past). In particular, they are spread out for positive times, which suggests that in a nonlinear setting, such oscillations may prevent wave collapse. The intuition described on the last two cases is confirmed by the results of Cazenave and Weissler [12].

Example 2. If $V(x) = E \cdot x$ is a Stark potential ($E \in \mathbb{R}^n$ is constant), then

$$x(t) = x_0 + t\nabla\varphi(x_0) - \frac{1}{2}Et^2.$$

If $\varphi \equiv 0$, rays do not meet (see Figure 2). If $\varphi(x) = -b|x|^2/2$ with $b > 0$, then rays
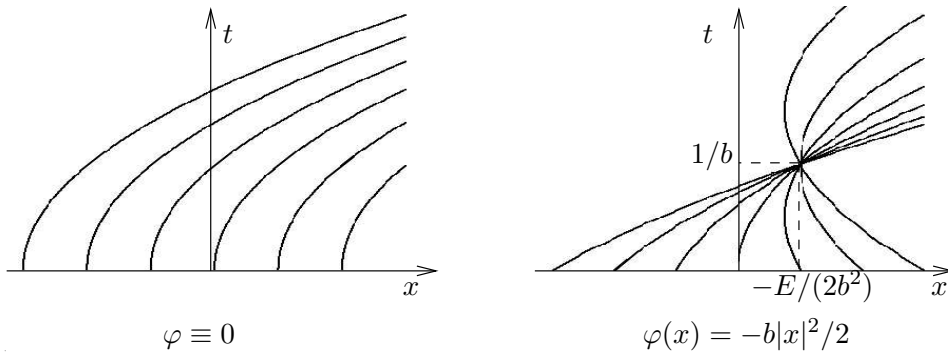

Figure 2: Geometry of rays: case $V(x) = E \cdot x$.

meet at the point $-E/(2b^2)$ at time $t = 1/b$ (see Figure 2). Compare this with the second case of the first example. This suggests that in a nonlinear setting, a linear potential does not affect the blow-up time, but the location when this phenomenon occurs.

Example 3. Suppose $V(x) = \omega^2|x|^2/2$, with $\omega > 0$. In the case $\varphi \equiv 0$, rays are given by $x(t) = x_0\cos(\omega t)$, and meet at the origin at time $t = \pi/(2\omega)$ (see Figure 3). The first example suggests that blow-up may happen more easily than when $V \equiv 0$. This phenomenon is reinforced by the case $\varphi(x) = -\omega\tan(\omega t_0)|x|^2/2$, $|t_0| < \pi/(2\omega)$, where

$$x(t) = \frac{x_0}{\cos(\omega t_0)}\cos\omega(t + t_0).$$

Rays meet at the origin at time $t = \pi/(2\omega) - t_0$. If $t_0 > 0$, focusing is anticipated, while if $t_0 < 0$, it is delayed (but in no case prevented).
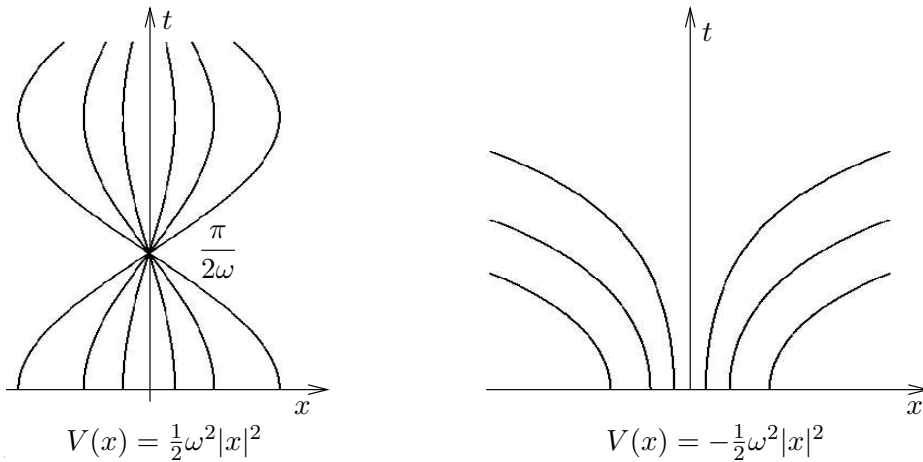

$$V(x) = \tfrac{1}{2}\omega^2|x|^2 \qquad\qquad V(x) = -\tfrac{1}{2}\omega^2|x|^2$$

Figure 3: Geometry of rays, with $\varphi \equiv 0$.

Example 4. Suppose $V(x) = -\omega^2|x|^2/2$, with $\omega > 0$. In the case $\varphi \equiv 0$, rays are given by $x(t) = x_0\cosh(\omega t)$, and are strongly dispersed for positive times (see Figure 3). This geometry is to be compared with the third case of the first example; rays are scattered, but go to infinity exponentially fast, instead of algebraically. This is a hint that such potentials may prevent blow-up.

We prove that in the last three examples, our heuristic approach agrees with rigorous results.

Proposition 1.1 (Stark potential, [9]). *Let $E \in \mathbb{R}^n$, and $V(x) = E \cdot x$. Then the solutions u and v to* (1.3) *and* (1.1) *are related through the formula*

$$u(t,x) = v\left(t, x + \frac{t^2}{2}E\right)e^{-i\left(tE\cdot x + \frac{t^3}{6}|E|^2\right)}. \tag{1.6}$$

In particular, if v blows up at time $t = T$, then so does u, but the set where this phenomenon occurs is shifted by a factor $-T^2E/2$.

Remark. In the linear case $\lambda = 0$, the above formula is the well–known Avron-Herbst formula (see e.g. [11], Chapter 7). It yields interesting results in the nonlinear setting, as shown by the above proposition.

We now turn to the case of isotropic quadratic potentials. Let $\omega > 0$. Introduce the solutions u_+^ω and u_-^ω to the initial value problems

$$
\begin{cases}
i\partial_t u_\pm^\omega + \dfrac{1}{2}\Delta u_\pm^\omega = \pm\omega^2\dfrac{|x|^2}{2}u_\pm^\omega + \lambda|u_\pm^\omega|^{2\sigma}u_\pm^\omega\,, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
\qquad u_{\pm|t=0}^\omega = u_0\,.
\end{cases}
\tag{1.7}
$$

Remark. The plus case corresponds to the current model used to analyze Bose–Einstein condensation. The harmonic potential is due to a magnetic trap, its confining properties are used to form a condensate of atoms. The nonlinear term corresponds to the interaction of particles. The real λ may be positive or negative, according to the chemical element considered (see e.g. [3], [2], [23]). The power of the nonlinearity is $\sigma = 2$ if $n = 1$, $\sigma = 1$ if $n = 2$ or 3 (see e.g. [17]).

Theorem 1.2 ([4], [7]). *Let $u_0 \in \Sigma$, $\lambda < 0$, $\sigma \geq 2/n$ and $\sigma < 2/(n-2)$ if $n \geq 3$.*
1. Assume that (1.2) holds. Then v blows up in finite time T, and u_+^ω blows up at time $T_+^\omega \leq \pi/(2\omega)$. Let $\omega_0 = \pi/(2T)$; then for any $\omega \geq \omega_0$, u_+^ω blows up before time T.
2. If the initial datum u_0 satisfies

$$
\frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2} < -\frac{\omega^2}{2}\|xu_0\|_{L^2}^2\,,
$$

then u_-^ω blows up in finite time, in the future or in the past.
3. If the initial datum u_0 satisfies

$$
\frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u_0\|_{L^{2\sigma+2}}^{2\sigma+2} < -\frac{\omega^2}{2}\|xu_0\|_{L^2}^2 - \omega\left|\operatorname{Im}\int \overline{u_0}x\cdot\nabla_x u_0\right|\,,
$$

then u_-^ω blows up in finite time, in the future and *in the past.*
4. There exists $\omega_1 > 0$ such that for any $\omega \geq \omega_1$, u_-^ω is globally defined, $u_-^\omega \in C(\mathbb{R}, \Sigma)$.

In the particular L^2–critical case, we can give more precise results.

Theorem 1.3 ([5], [7]). *Let $u_0 \in \Sigma$, $\lambda < 0$, $\sigma = 2/n$.*
Assume that the solution v to (1.1) blows up at time $T > 0$.

- *For any $\omega > 0$, u_+^ω blows up at time $\arctan(\omega T)/\omega < T$.*

- *If $\omega < 1/T$, then u_-^ω blows up at time $\operatorname{arg\,tanh}(\omega T)/\omega > T$.*

- *If $\omega \geq 1/T$, then u_-^ω is globally defined in the future, $u_-^\omega \in C(\mathbb{R}_+, \Sigma)$.*

In Section 2, we prove the first point of Theorem 1.2, and we sketch the proof of the last point in Section 3. The second and third points can be proved by using the general approach of Zakharov-Glassey (see [7]). In Section 4, we show why Theorem 1.3 holds.

2. Confining harmonic potential

Assume $u_0 \in \Sigma$, $\sigma \geq 2/n$ and $\sigma < 2/(n-2)$ if $n \geq 3$. Assume also $\lambda \in \mathbb{R}$. Only later will we restrict to the case $\lambda < 0$.

Consider the solution u_+^ω, which is known to exist in Σ, locally in time (see [10]). The following quantities are independent of time,

$$\text{Mass: } M = \|u_+^\omega(t)\|_{L^2} \,,$$

$$\text{Energy: } E_+^\omega = \frac{1}{2}\|\nabla_x u_+^\omega(t)\|_{L^2}^2 + \frac{\omega^2}{2}\|x u_+^\omega(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u_+^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \,.$$

It is well known that if $E_+^\omega < 0$, then u_+^ω blows up in finite up (see [10]). Notice that the assumption (1.2) is weaker.

The tools we use are suggested by geometric optics. The solution to (1.4) with $V(x) = \omega^2|x|^2/2$ is given explicitly by Mehler's formula (see e.g. [15]), for $|t| < \pi/(2\omega)$,

$$u^\varepsilon(t,x) = e^{-in\frac{\pi}{4}\operatorname{sgn} t} \left|\frac{\omega}{2\pi\varepsilon \sin\omega t}\right|^{n/2} \int_{\mathbb{R}^n} e^{\frac{i\omega}{\varepsilon \sin\omega t}\left(\frac{|x|^2+|y|^2}{2}\cos\omega t - x\cdot y\right)} u_0^\varepsilon(y)\,dy\,. \qquad (2.1)$$

If $u_0^\varepsilon = f$ is independent of ε, then stationary phase formula yields, for $0 < t < \pi/2\omega$,

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \frac{1}{(\cos\omega t)^{n/2}} f\left(\frac{x}{\cos\omega t}\right) e^{-i\omega\frac{|x|^2}{2\varepsilon}\tan\omega t}\,.$$

With the study of a nonlinear problem in mind (especially in a semi–classical régime, see [6]), this suggests to introduce the operator

$$\cos(\omega t) \times e^{-i\omega\frac{|x|^2}{2\varepsilon}\tan\omega t}\nabla_x\left(\cdot\, e^{i\omega\frac{|x|^2}{2\varepsilon}\tan\omega t}\right) = i\omega\frac{x}{\varepsilon}\sin\omega t + \cos\omega t\nabla_x\,.$$

The above operator is well-known in the linear theory; this is an Heisenberg observable (see e.g. [22] p. 108). The factorization by which we retrieve this operator is of particular interest to treat nonlinear problems. Now set $\varepsilon = 1$, and introduce the operators

$$J_+^\omega(t) := -\omega x \sin\omega t + i\cos\omega t\nabla_x\,;\quad H_+^\omega(t) := x\cos\omega t + i\frac{\sin\omega t}{\omega}\nabla_x\,.$$

We can split the energy into two parts,

$$E_1(t) := \frac{1}{2}\|J_+^\omega(t)u_+^\omega\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\cos^2\omega t\|u_+^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\,,$$

$$E_2(t) := \frac{\omega^2}{2}\|H_+^\omega(t)u_+^\omega\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\sin^2\omega t\|u_+^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\,.$$

We check that $E_1(t) + E_2(t) \equiv E_+^\omega$. It is proved in [4] that the evolution of E_1 is given by

$$\frac{dE_1}{dt} = -\frac{dE_2}{dt} = \frac{\omega\lambda}{2\sigma+2}(n\sigma-2)\sin 2\omega t\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\,. \qquad (2.2)$$

The first point of Theorem 1.2 follows easily. Assume that $E(u_0) \leq 0$, which is even a weaker assumption than (1.2), and suppose that u_+^ω exists in Σ up to time $\pi/(2\omega)$. Then $E_1(0) = E(u_0) \leq 0$, and from (2.2), if $\sigma \geq 2/n$ and $\lambda < 0$,

$$\frac{dE_1}{dt} \leq 0 , \quad \forall t \in \left[0, \frac{\pi}{2\omega}\right] .$$

This implies

$$E_1\left(\frac{\pi}{2\omega}\right) \leq 0 .$$

But from the definition of E_1,

$$E_1\left(\frac{\pi}{2\omega}\right) = \frac{\omega^2}{2} \left\| x u_+^\omega \left(\frac{\pi}{2\omega}\right) \right\|_{L^2}^2 ,$$

so this leads to a contradiction (unless $u_+^\omega \equiv 0$, which means that $u_0 \equiv 0$). Therefore, u_+^ω does not remains in Σ up to time $\pi/(2\omega)$. It is easy to conclude that this is so because there exists $T \leq \pi/(2\omega)$ such that

$$\lim_{t \to T} \left\| \nabla_x u_+^\omega(t) \right\|_{L^2} = \infty .$$

3. Repulsive harmonic potential

The most important step to study u_-^ω is the analysis of the Cauchy problem.

Formally, (1.7) with the plus sign is turned into (1.7) with the minus sign by replacing ω with $i\omega$ (or $-i\omega$). Therefore, define

$$J_-^\omega(t) := \omega x \sinh \omega t + i \cosh \omega t \nabla_x ; \quad H_-^\omega(t) := x \cosh \omega t + i \frac{\sinh \omega t}{\omega} \nabla_x .$$

Introduce the evolution group associated to the linear equation ($\lambda = 0$),

$$U_-^\omega(t) := \exp\{-it/2(-\Delta - \omega^2|x|^2)\} .$$

• The first thing to notice is that $U_-^\omega(t)$ can be computed explicitly (essentially, trigonometric functions in (2.1) are turned into hyperbolic functions), and satisfies a dispersion estimate which is *independent* of $\omega > 0$,

$$\left\| U_-^\omega(t) \right\|_{L^1 \to L^\infty} \leq \left| \frac{\omega}{2\pi \sinh \omega t} \right|^{n/2} \leq \frac{1}{|2\pi t|^{n/2}} .$$

The group $U_-^\omega(t)$ thus yields the same Strichartz inequalities as in the case $\omega = 0$, with constants independent of $\omega > 0$.

• Duhamel's principle for u_-^ω is

$$u_-^\omega(t) = U_-^\omega(t)u_0 - i\lambda \int_0^t U_-^\omega(t-s)|u_-^\omega|^{2\sigma} u_-^\omega(s) ds .$$

The operators $J_-^\omega(t)$ and $H_-^\omega(t)$ satisfy the same properties as those which are used to study (1.1) in Σ (which are ∇_x and $x + it\nabla_x$), namely:

- They commute with the linear part of the equation.

- They act on the nonlinear term like derivatives.

- They yield weighted Gagliardo–Nirenberg inequalities.

We make the last statement precise. Because $J_-^\omega(t)$ can be factorized in a similar fashion as $J_+^\omega(t)$, Gagliardo–Nirenberg inequalities imply

$$\|f\|_{L^r} \leq \frac{C_r}{(\cosh \omega t)^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|J_-^\omega(t)f\|_{L^2}^{\delta(r)} \ ,$$

where

$$\delta(r) := n \left(\frac{1}{2} - \frac{1}{r} \right) \ , \quad \text{and } 2 \leq r < \frac{2n}{n-2} \ .$$

Since $\cosh \omega t \geq 1$, the above estimate imply a Gagliardo–Nirenberg inequality involving only $\|f\|_{L^2}$ and $\|J_-^\omega(t)f\|_{L^2}$, with a weight independent of $\omega > 0$.

• We infer that there exist $t_0 > 0$ *independent* of $\omega > 0$, and a unique solution u_-^ω defined for $|t| < 2t_0$ with values in Σ, to (1.7) with the minus sign. Moreover, there exists C_0 depending only on λ, n, σ and $\|u_0\|_\Sigma$ such that

$$\sum_{A \in \{Id, J_-^\omega, H_-^\omega\}} \sup_{|t| \leq t_0} \|A(t)u_-^\omega\|_{L^2} \leq C_0 \ . \tag{3.1}$$

• Define

$$E_1(t) := \frac{1}{2} \|J_-^\omega(t)u_-^\omega\|_{L^2}^2 + \frac{\lambda}{\sigma+1} \cosh^2 \omega t \|u_-^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \ .$$

It solves, so long as u_-^ω is defined in Σ,

$$\frac{dE_1}{dt} = \frac{\omega \lambda}{2\sigma+2} (2 - n\sigma) \sinh 2\omega t \|u_-^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \ . \tag{3.2}$$

• This evolution law, its analog for $H_-^\omega(t)u_-^\omega$, along with the properties of $J_-^\omega(t)$ and $H_-^\omega(t)$, the algebraic relation

$$\|J_-^\omega(t)u_-^\omega\|_{L^2}^2 - \omega^2 \|H_-^\omega(t)u_-^\omega\|_{L^2}^2 = \|\nabla_x u_-^\omega(t)\|_{L^2}^2 - \omega^2 \|x u_-^\omega(t)\|_{L^2}^2 \ ,$$

and the conservations of mass and energy, imply the following crucial result.

Corollary 3.1. *Let $\lambda \in \mathbb{R}$, $\sigma, \omega > 0$, with $\sigma < 2/(n-2)$ if $n \geq 3$.*
1. Let $u_0 \in \Sigma$, and u_-^ω solve (1.7) on some time interval I containing 0. For any $I \ni t > 0$, the following properties are equivalent:

- $\nabla_x u_-^\omega \in L^\infty([0,t]; L^2)$.

- $u_-^\omega \in L^\infty([0,t]; \Sigma)$.

- $J_-^\omega(s)u_-^\omega$ *or* $H_-^\omega(s)u_-^\omega$ *is in* $L^\infty([0,t]; L^2)$.

- $J_-^\omega(s)u_-^\omega$ *and* $H_-^\omega(s)u_-^\omega$ *are in* $L^\infty([0,t]; L^2)$.

2. There exists a unique maximal solution, defined on $]-T_*, T^*[$. If $T^* < \infty$, then $\|\nabla_x u_-^\omega(t)\|_{L^2} \to \infty$ as $t \uparrow T^*$.

Remark. Thanks to this result and to (3.2), we can prove that when $\lambda > 0$, then u_-^ω is global, and a complete scattering theory is available (with no long range effect, see [7]). This result, if expected (the nonlinearity is repulsive), is not obvious from the usual conservations of mass and energy. The energy,

$$E_-^\omega = \frac{1}{2}\|\nabla_x u_+^\omega(t)\|_{L^2}^2 - \frac{\omega^2}{2}\|x u_+^\omega(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u_+^\omega(t)\|_{L^{2\sigma+2}}^{2\sigma+2} ,$$

is not the sum of positive terms, so we cannot conclude at this stage. On the other hand, Corollary 3.1 and (3.2) easily yield global existence.

• We can now prove the last point of Theorem 1.2. For $t \geq t_0$, integrate (3.2) between time t_0 and time t. Since $\lambda(2 - n\sigma) \geq 0$,

$$E_1(t) \leq E_1(t_0) + C\omega \int_{t_0}^t \sinh 2\omega s \|u_-^\omega(s)\|_{L^{2\sigma+2}}^{2\sigma+2} ds$$

$$\leq C(C_0) + C\omega \int_{t_0}^t \frac{\sinh 2\omega s}{(\cosh \omega s)^{n\sigma}}\|J_-^\omega(s)u_-^\omega\|_{L^2}^{n\sigma} ds .$$

As noticed before, the constants in the last estimate do not depend on ω. Define

$$y(t) := \sup_{t_0 \leq s \leq t} \|J(s)u\|_{L^2}^2 .$$

We have

$$E_1(t) \leq C(C_0) + Cy(t)^{n\sigma/2} \int_{t_0}^t \frac{\sinh 2\omega s}{(\cosh \omega s)^{n\sigma}} ds \leq C(C_0) + \frac{C}{(\cosh \omega t_0)^{n\sigma-2}} y(t)^{n\sigma/2} .$$

We finally have

$$y(t) \leq C(C_0) + \frac{C}{(\cosh \omega t_0)^{n\sigma-2}} y(t)^{n\sigma/2} .$$

If $\sigma > 2/n$, we conclude by a bootstrap argument, since the constants and t_0 do not depend on $\omega > 0$, and

$$\cosh \omega t_0 \underset{\omega \to +\infty}{\longrightarrow} +\infty .$$

This yields a uniform bound for $\|J(t)u\|_{L^2}$. The last point of Theorem 1.2 stems from Corollary 3.1.

When $\sigma = 2/n$, the approach is different, as shown in the last section.

4. The L^2–critical case

Theorem 1.3 is the consequence of two explicit changes of variables. In [5], we noticed that if u_+^ω is defined by

$$u_+^\omega(t, x) = \frac{1}{(\cos \omega t)^{n/2}} e^{-i\frac{\omega}{2}|x|^2 \tan \omega t} v\left(\frac{\tan \omega t}{\omega}, \frac{x}{\cos \omega t}\right) , \tag{4.1}$$

then u_+^ω solves (1.7) with the plus sign and $\sigma = 2/n$. We also proved that v blows up at time $T > 0$ if and only if u_+^ω blows up at time $\arctan(\omega T)/\omega$. This yields the first point of Theorem 1.3.

Replacing ω by $\pm i\omega$, define

$$u_-^\omega(t,x) = \frac{1}{(\cosh \omega t)^{n/2}} e^{i\frac{\omega}{2}|x|^2 \tanh \omega t} v\left(\frac{\tanh \omega t}{\omega}, \frac{x}{\cosh \omega t}\right). \qquad (4.2)$$

Then u_-^ω solves (1.7) with the minus sign and $\sigma = 2/n$. Now assume that $\lambda < 0$, and that v blows up at some finite time $T > 0$.

It is easy to see that

$$\|J_-^\omega(t)u_-^\omega\|_{L^2} = \left\| \nabla_x v \left(\frac{\tanh \omega t}{\omega}\right) \right\|_{L^2}. \qquad (4.3)$$

Since $\tanh(\mathbb{R}_+) = [0, 1[$, if $\omega \geq 1/T$, then the function of the right hand side of (4.2) does not "see" the time T, and u_-^ω does not blow up in finite time.

If $\omega < 1/T$, then u_-^ω blows up at time

$$T_\omega = \frac{\arg\tanh \omega T}{\omega},$$

which completes the proof of Theorem 1.3.

More precise results on the influence of ω on u_-^ω are proved in [7].

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