Journées *Équations aux dérivées partielles*
Plestin-les-grèves, 5–8 juin 2002
GDR 1151 (CNRS)

Dynamics of Singularity Surfaces for Compressible Navier-Stokes Flows in Two Space Dimensions

David Hoff

Abstract

We prove the global existence of solutions of the Navier-Stokes equations of compressible, barotropic flow in two space dimensions with piecewise smooth initial data. These solutions remain piecewise smooth for all time, retaining simple jump discontinuities in the density and in the divergence of the velocity across a smooth curve, which is convected with the flow. The strengths of these discontinuities are shown to decay exponentially in time, more rapidly for larger acoustic speeds and smaller viscosities.

The Navier-Stokes equations describe the conservation of mass and the balance of momentum:

$$\rho_t + \mathrm{div}(\rho u) = 0 \tag{1}$$

$$(\rho u^j)_t + \mathrm{div}(\rho u^j u) + P(\rho)_{x_j} = \varepsilon \Delta u^j + \lambda \mathrm{div}\, u_{x_j}. \tag{2}$$

Here $t \geq 0$ is time, $x \in \mathbb{R}^2$ is the spatial coordinate, and $\rho(x,t)$, $P = P(\rho)$, and $u(x,t) = (u^1(x,t), u^2(x,t))$ are the fluid density, pressure, and velocity. $\varepsilon > 0$ and $\lambda \geq 0$ are viscosity constants, and div and Δ are the usual spatial divergence and Laplace operators.

Specifically, we fix a positive, constant reference density $\tilde{\rho}$, and we assume that Cauchy data (ρ_0, u_0) is given for which $\rho_0 - \tilde{\rho}$ is small in $L^2 \cap L^\infty$, u_0 is small in H^β for some arbitrary but positive β (the L^2-norms must be weighted slightly), and that ρ_0 is piecewise C^α $(0 < \alpha < \beta)$, having simple jump discontinuities across a $C^{1+\alpha}$ curve $\mathcal{C}(0)$. We then show that there is a global weak solution (ρ, u) for which $\rho(\cdot, t)$ and $\mathrm{div}\, u(\cdot, t)$ are piecewise C^α, having simple jump discontinuities across a $C^{1+\alpha}$ curve $\mathcal{C}(t)$, which is the transport of $\mathcal{C}(0)$ by the velocity field u, and that certain other features of the solution concerning its singularities, readily obtainable from heuristic jump conditions, hold in a strict, pointwise sense.

This research was supported in part by the NSF under Grant No. DMS-9986658.
MSC 2000 : 35Q30, 76N10.
Keywords : Navier-Stokes equations, compressible flow, singularity surfaces.

Extensions of the results of the present paper to three space dimensions and to nonbarotropic flows, in which the pressure depends as well on the temperature, and the energy-balance equation is appended, will be presented elsewhere.

We begin by reviewing certain heuristic considerations regarding the propagation of singularities in solutions of (1)–(2) (see for example Hoff [3, 5], Serre [9], and even Duhem [1]). Suppose then that (ρ, u) is a smooth solution of (1) except across a hypersurface S in $\mathbb{R}^2 \times [0, \infty)$, and that ρ, u, and ∇u have one-sided limits at each point of S. (Of course, it is the existence of such a solution that is our goal here.) Let (n, n_0) be the normal to S at such a point, where n_0 denotes the scalar time-component and n the vector x-component. Then (ρ, u) will be a distribution solution of (1) if and only if, at each point of S,

$$n_0[\rho] + n \cdot [\rho u] = 0. \tag{3}$$

Here $[\,\cdot\,]$ denotes the difference between the limit in a given quantity from one side of S to the other, and (3) results by applying the divergence theorem in the definition of weak solution. Now, we expect that, due to the diffusion terms in (2), u should become continuous for $t > 0$, so that $[u] = 0$. It then follows from (3) that, if $[\rho] \neq 0$,

$$n_0 + n \cdot u = 0, \tag{4}$$

that is, that the normal $\left[\begin{smallmatrix} n \\ n_0 \end{smallmatrix}\right]$ to S is perpendicular to the vector $\left[\begin{smallmatrix} u \\ 1 \end{smallmatrix}\right]$. The latter vector is tangent to the space-time trajectory of a fluid particle, so that, at least at this heuristic level, S is the union of these particle trajectories, and singularities are convected with the flow.

Before computing the jump condition for the momentum (2), we rewrite this equation as follows:

$$\begin{aligned}(\rho u^j)_t + \mathrm{div}(\rho u^j u) &= \big((\varepsilon + \lambda) u^k_{x_k} - P(\rho)\big)_{x_j} + \varepsilon (u^j_{x_k} - u^k_{x_j})_{x_k} \\ &= \big((\varepsilon + \lambda)\mathrm{div} u - P(\rho)\big)_{x_j} + \varepsilon \omega^{j,k}_{x_k},\end{aligned} \tag{5}$$

where $\omega^{j,k} = u^j_{x_k} - u^k_{x_j}$ is the vorticity, and summation over k is understood in (5). Then (ρ, u) will be a weak solution of (2) if and only if, at each point of S, and for $j = 1, 2$,

$$n_0[\rho u^j] + n \cdot [\rho u^j u] = n_j\big[(\varepsilon + \lambda)\mathrm{div}\, u - P(\rho)\big] + n_k[\varepsilon \omega^{j,k}]. \tag{6}$$

The left-hand side of (6) is zero, by (4). Multiplying by n_j and summing over j, we then find that

$$\big[(\varepsilon + \lambda)\mathrm{div}\, u - P(\rho) + \tilde{P}\big] = 0, \tag{7}$$

by the skew-symmetry of ω. Here $\tilde{P} = P(\tilde{\rho})$ is a constant, positive reference pressure. Evidently, although u is continuous for $t > 0$, ∇u is not, nor is ρ; but the jumps in $(\varepsilon + \lambda)\mathrm{div}\, u$ and $P(\rho)$ should exactly cancel, so that the combination

$$F \equiv (\varepsilon + \lambda)\mathrm{div}\, u - P(\rho) + \tilde{P} \tag{8}$$

should be continuous. Returning to (6), we then find that ω will be continuous as well, so that

$$[F] = [\omega] = 0. \tag{9}$$

To summarize, we expect that a curve $\mathcal{C}(t)$ of singularity of the solution (ρ, u) should be transported by the velocity field u, and that at each time t, u, F, and ω should be continuous functions of x, but that ρ, P, and $\operatorname{div} u$ will be discontinuous across $\mathcal{C}(t)$.

The quantity F, sometimes referred to as the "effective viscous flux," plays an enormously important role in the overall analysis. A differential equation for F can be derived by writing the momentum equation (5) in the form

$$\rho \dot{u}^j = F_{x_j} + \varepsilon \omega^{j,k}_{x_k}, \tag{10}$$

where $\dot{u}^j = \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) u^j$ is the usual convective derivative. (10) thus gives a Helmholtz decomposition of the acceleration density $\rho\dot{u}$, and, again because of the skew symmetry of ω,

$$\Delta F = \operatorname{div}(\rho \dot{u}). \tag{11}$$

We remark that, for incompressible flow, $\operatorname{div} u = 0$, $F = \tilde{P} - P$, and (11) becomes the well-known elliptic equation for the incompressible pressure.

These heuristic observations were made precise in a sequence of papers [3] – [5] in both two and three space dimensions, and for the non-barotropic Navier-Stokes system as well, in which $P = P(\rho, e)$, where e is specific internal energy, and a third equation, expressing the balance of energy, is appended to (1)–(2). The following gives a version of these results adapted to the present case:

Theorem 1. *Assume that $P(\rho) = K\rho^\gamma$ where $K > 0$ and $\gamma \geq 1$, and that $\varepsilon > 0$ and $\lambda \geq 0$. Fix positive but arbitrarily small constants $\tilde{\rho}, b$, and β, and let initial data (ρ_0, u_0) for (1)–(2) be given for which*

$$C_0 \equiv |\rho_0 - \tilde{\rho}|^2_{L^\infty} + \int_{\mathbb{R}^2} \Big[(\rho_0(x) - \tilde{\rho})^2 + |u_0(x)|^2\Big](1 + |x|^2)^b dx + |u_0|^2_{H^\beta}$$

is sufficiently small (H^β is the usual Sobolev space of functions with β derivatives in L^2). Then the system (1)–(2) has a global weak solution $(\rho - \tilde{\rho}, u) \in C\big([0,\infty);\ H^{-1}(\mathbb{R}^2)\big) \times C\big((0,\infty);\ L^2(\mathbb{R}^2)\big)$ for which

$$\sup_t \Bigg[|\rho(\cdot,t) - \tilde{\rho}|^2_{L^\infty} + |u(\cdot,t)|^2_{H^\beta} + \int_{\mathbb{R}^2} \big(|\rho(x,t) - \tilde{\rho}|^2 + |u(x,t)|^2 + \sigma(t)^{1-\beta}|\nabla u|^2 + \sigma^{2-\beta}(|\dot{u}(x,t)|^2 + |\nabla\omega(x,t)|^2)\big) dx \Bigg] + \int_0^\infty \int_{\mathbb{R}^2} \big[|\nabla u|^2 + \sigma^{1-\beta}(|\dot{u}|^2 + |\nabla \omega|^2) + \sigma^{2-\beta}|\nabla \dot{u}|^2\big] dx\, dt \ \leq\ CC_0\ , \tag{12}$$

where $\sigma(t) = \min\{1, t\}$; and

$$u, F, \text{ and } \omega \text{ are locally Hölder continuous in } \{t > 0\}. \tag{13}$$

Theorem 1 is essentially the result of [3], the major difference being that here $u_0 \in H^\beta$, whereas in [3] we assumed that $u_0 \in L^2 \cap L^4$. This small degree of extra regularity in the initial velocity is required for the derivation of the slightly more favorable smoothing rates for u indicated in (12), and these in turn will be needed to resolve certain initial–layer effects. The assumption that $u_0 \in H^\beta$ also enables us to avoid a restriction on λ/ε that was imposed in [3]. The conclusion (13) concerning the regularity of F and ω is somewhat stronger than that given in [3].

The latter conclusion, (13), also gives a rather satisfactory rigorous expression of the jump conditions (9) that we derived above at the heuristic level. The questions which are not addressed by Theorem 1, and which we set out to answer here, are those described at the beginning: if ρ_0 is piecewise continuous, having simple jump discontinuities across a curve $\mathcal{C}(0)$, will $\rho(\cdot,t)$ and $\operatorname{div} u(\cdot,t)$ be piecewise continuous, with simple discontinuities across $\mathcal{C}(t)$, the u-transport of $\mathcal{C}(0)$? will the jump conditions (9) hold in a strict, pointwise sense? what can be said about the evolution in time of the strengths $|[\rho]|$ and $|[\operatorname{div} u]|$ of these discontinuities? and finally, how regular can $\mathcal{C}(t)$ be, given that it is transported by a velocity field u which is not C^1, the discontinuities in its derivatives being concentrated on $\mathcal{C}(t)$ itself?

Before stating a rigorous result in this direction, we return to the heuristic discussion preceding the statement of Theorem 1 in order to gain some insight into the required analysis. Thus suppose that (ρ, u) is a weak solution of (1)–(2), and, with benefit of hindsight, that $\rho(\cdot,t)$ is piecewise C^α, having simple discontinuities across a $C^{1+\alpha}$ curve $\mathcal{C}(t) : \{y(s,t) : s \in I \subseteq \mathbb{R}\}$, where I is an open interval, and that $\mathcal{C}(t)$ is the u-transport of $\mathcal{C}(0)$:

$$y(s,t) = y(s,0) + \int_0^t u\big(y(s,\tau),\tau\big)\, d\tau. \tag{14}$$

(Again, it is the existence of such a solution that is our goal here.) We shall attempt to estimate $\big|y(\cdot,t)\big|_{C^{1+\alpha}}$ and

$$|\rho(\cdot,t)\,|_{C^\alpha_{pw}} = \sup_{x_1 \neq x_2} \frac{|\rho(x_2,t) - \rho(x_1,t)|}{|x_2 - x_1|^\alpha}\ ,$$

where the sup is taken over points x_1, x_2 on the same side of $\mathcal{C}(t)$. We shall see shortly that these norms can be bounded in terms of $e^{\int_0^t |\nabla u(\cdot,\tau)|_\infty d\tau}$ and $|u(y(\cdot,\tau),\tau)|_{C^{1+\alpha}}$, and that these latter two quantities are in turn bounded in terms of $|\rho(\cdot,t)|_{C^\alpha_{pw}}$ and $|y(\cdot,t)|_{C^{1+\alpha}}$. The resulting four estimates will be coupled in a *nonlinear* way, and, as we shall see, a certain dissipative effect will be required to effect their uncoupling.

To begin, we see immediately from (14) that

$$|y(\cdot,t)|_{C^{1+\alpha}} \le C_0 + \int_0^t |u(y(\cdot,\tau),\tau)|_{C^{1+\alpha}} d\tau\ , \tag{15}$$

where C_0 will now denote a generic positive constant determined by the initial data. To estimate $|\rho(\cdot,t)|_{C^\alpha_{pw}}$, we fix two particle trajectories $x_1(t)$ and $x_2(t)$ (thus $\dot{x}_j(t) = u(x_j,t)$) and compute from (1) and (8) that

$$\frac{d}{dt}\left[\log\rho(x_2,t) - \log\rho(x_1,t)\right] = -(\varepsilon+\lambda)^{-1}\left[F(x_2,t) - F(x_1,t) + P(x_2,t) - P(x_1,t)\right]$$

so that

$$\frac{d}{dt}\left[\log\rho(x_2,t)-\log\rho(x_1,t)\right]+A(t)\left[\log\rho(x_2,t)-\log\rho(x_1,t)\right] \\ =-(\varepsilon+\lambda)^{-1}\left[F(x_2,t)-F(x_1,t)\right] \ , \quad (16)$$

where $A=\rho P'(\rho)$ for some ρ is evidently strictly positive. We therefore expect that

$$|\rho(x_2(t),t)-\rho(x_1(t),t)|\le Ce^{-C_1t}|\rho_0(x_2(0))-\rho_0(x_1(0))| \\ +C\int_0^t e^{C_1(\tau-t)}|F(x_2(\tau),\tau)-F(x_1(\tau),\tau)|d\tau, \quad (17)$$

where C is a generic positive constant, and C_1 is a constant bounded away from zero, depending on pointwise bounds for ρ. The relation $\dot{x}_j=u(x_j,t)$ implies that

$$|x_2(t)-x_1(t)|\le e^{\int_0^t|\nabla u(\cdot,\tau)|_\infty d\tau}|x_1(0)-x_2(0)|,$$

so that, taking x_2 and x_1 on the same side of $\mathcal{C}$, we find from (17) that

$$|\rho(\cdot,t)|_{C^\alpha_{pw}}\le CC_0e^{\int_0^t(|\nabla u|_\infty-C_1)d\tau}+C\int_0^t e^{\int_\tau^t(|\nabla u|_\infty-C_1)}|F(\cdot,\tau)|_{C^\alpha}d\tau.$$

A bound for $|F(\cdot,t)|_{C^\alpha}$ can be obtained by applying the bounds in (12) for $\dot{u}$ and $\nabla\dot{u}$ in the fundamental relation (11). The result is that

$$|\rho(\cdot,t)|_{C^\alpha_{pw}}\le CC_0e^{\int_0^t(|\nabla u|_\infty-C_1)d\tau}+CC_0\int_0^t e^{\int_\tau^t(|\nabla u|_\infty-C_1)}\sigma(\tau)^{\beta-\alpha-1}d\tau, \quad (18)$$

where again $\sigma=\min\{1,t\}$. Clearly, we shall have to assume that $\alpha<\beta$.

The bounds for $e^{\int_0^t|\nabla u|_\infty d\tau}$ and $|u(y(\cdot,\tau),\tau)|_{C^{1+\alpha}}$ required to close the estimates in (15) and (18) are more subtle. First, the relation

$$\Delta u^j=u^j_{x_kx_k}=(u^k_{x_k})_{x_j}+(u^j_{x_k}-u^k_{x_j})_{x_k},$$

together with (8), shows that we may write

$$\begin{aligned} u &= \nabla\Gamma * LC(F,\omega)+(\varepsilon+\lambda)^{-1}\nabla\Gamma*(P(\rho)-\tilde{P}), \\ &\equiv u^F+u^P, \end{aligned} \quad (19)$$

where Γ is the fundamental solution of the Laplace operator on $\mathbb{R}^2$ and $LC(F,\omega)$ denotes a linear combination of F and components of ω. Now, (13) implies that u^F will be globally $C^{1+\alpha}$ at positive times; u^P will be less regular, however, since $P(\rho(\cdot,t))$ is discontinuous. Indeed, letting Ω_- and Ω_+ be the two sides of $\mathcal{C}(t)$, we might write

$$u^P(x,t)=(\varepsilon+\lambda)^{-1}\int_{\Omega_+\cup\Omega_-}\nabla\Gamma(x-y)[P(\rho(y,t))-\tilde{P}]dy \quad (20)$$

and apply a standard result of harmonic analysis to the effect that, if $P(\rho(\cdot,t))\in C^\alpha(\overline{\Omega}_+)$, say, and if $\mathcal{C}=\partial\Omega_+$ is $C^{2+\alpha}$, then $\int_{\Omega_+}$ will be in $C^{1+\alpha}(\overline{\Omega}_+)$. This is far from adequate for our purposes, however, first because (15) shows only that $\mathcal{C} \in C^{1+\alpha}$ if $u \in C^{1+\alpha}$, and the estimates would therefore not close, and second because it does not address the question of regularity of $\int_{\Omega_+}$ in Ω_-, or its regularity across the boundary $\mathcal{C} = \partial\Omega_\pm$.

A more refined analysis, however, shows that, if $\mathcal{C} = \partial\Omega_\pm \in C^{1+\alpha}$, if $P(\rho(\cdot,t))$ is C^α separately in Ω_+ and in Ω_-, and if u^P is as defined in (20), then $\nabla u^P(\cdot,t) \in L^\infty$ and $u(y(\cdot,t),t) \in C^{1+\alpha}$. This is the maximum that can be obtained, and the minimum that our analysis requires. (We have been unable to locate these results in the literature, and will therefore present their proofs in an appendix to [6].) Specifically, we show that, with these assumptions,

$$|\nabla u^P(\cdot,t)|_{L^\infty},\ |u^P(y(\cdot,t),t)|_{C^{1+\alpha}} \le C\big[C_0 + |\rho(\cdot,t)|_{C^\alpha_{pw}} + |[\rho(\cdot,t)]|_{L^\infty(\mathcal{C}(t))}|y(\cdot,t)|^{2+1/\alpha}_{C^{1+\alpha}}\big] \tag{21}$$

(plus lower-order terms of no consequence). Observe the superlinear dependence on $|y|_{C^{1+\alpha}}$. Of course, if $[\rho(\cdot,t)] \equiv 0$, then P is globally C^α, and the dependence on $|y|_{C^{1+\alpha}}$ drops out.

Can the four estimates in (15), (18), and (21) be closed? Ignoring completely the effect of u^F (recall (19)) we see very roughly from (15) and (21) that

$$\begin{aligned}\frac{d}{dt}|y(\cdot,t)|_{C^{1+\alpha}} &\sim |u(y(\cdot,t),t)|_{C^{1+\alpha}}\\ &\sim C|[\rho(\cdot,t)]|_{L^\infty}|y(\cdot,t)|^{2+1/\alpha}_{C^{1+\alpha}}.\end{aligned} \tag{22}$$

It thus appears that $|y(\cdot,t)|_{C^{1+\alpha}}$ may blow up in finite time, and that no smallness condition could prevent this blow-up. The analysis is saved, however, by the following fact, which reflects the dissipative effect referred to earlier, and which shows that the coefficient in (22) decays exponentially in time:

Lemma. $\big|[\rho(\cdot,t)]\big| \le CC_0 e^{-C_1 t}$.

The (heuristic) proof is obtained from (17) by letting x_1 and x_2 approach $\mathcal{C}(t)$ from different sides and by applying the Hölder continuity (13) of F.

Assuming then that $C_0/C_1 \ll 1$ (recall that C_0 and C_1 are generic positive constants, C_0 measuring the size of $(\rho_0 - \tilde{\rho}, u_0)$ and C_1 giving a lower bound away from zero for ρ), we can then close the four estimates in (15), (18), and (21) to obtain that

$$|\nabla u^P(\cdot,t)|_{L^\infty},\ |u^P(y(\cdot,t),t)|_{C^{1+\alpha}},\ |\rho(\cdot,t)|_{C^\alpha_{pw}} \le CC_0 \tag{23}$$

and

$$|y(\cdot,t)|_{C^{1+\alpha}} \le Ce^{C_1 t} \tag{24}$$

These considerations form the basis of the proof of the following result:

Theorem 2. *In addition to the hypotheses of Theorem 1, assume that $P(\rho) = K\rho$ (i.e., that $\gamma = 1$), that ρ_0 is piecewise C^α, where $0 < \alpha < \beta$, having simple jump discontinuities across a $C^{1+\alpha}$ curve $\mathcal{C}(0)$, and that $|\rho_0|_{C^\alpha_{pw}}$ is sufficiently small. Let (ρ, u) be the corresponding solution of (1)–(2) given by Theorem 1. Then: $\rho(\cdot,t)$ is* *piecewise C^α, having simple jump discontinuities across a $C^{1+\alpha}$ curve $\mathcal{C}(t)$, which is the u-transport of $\mathcal{C}(0)$, as in (14); the estimates in (23) and (24) hold; for $t > 0$, $\operatorname{div} u(\cdot,t)$ has one-sided limits on $\mathcal{C}(t)$, and the jump conditions (9) hold in a strict, pointwise sense; and $\big|[\rho(\cdot,t)]\big|_{L^\infty}$ and $|[\operatorname{div} u(\cdot,t)]|_{L^\infty}$ decay exponentially in time, as in the Lemma.*

Complete details of the proof will be presented in [6]. (There is an additional technical hypothesis on the global structure of $\mathcal{C}(0)$ which must be imposed, and which we have omitted here for the sake of brevity.)

Notice that the heuristic discussion given above, leading to the bounds (23)–(24), is based on an analysis of a solution (ρ, u) which is assumed to exist, and which is assumed to have all the qualitative properties described in Theorem 2. In fact, no such solution was known to exist heretofore, even locally in time; indeed, it is precisely the existence of such a solution that must be established in the proof of Theorem 2.

We deal with this issue in [6] as follows. First, let $(\overline{\rho}, \overline{u})$ be the solution of Theorem 1, which *is* known to exist, and let $\overline{F}$ and $\overline{u}^F$ be the corresponding quantities defined in (8) and (19). Then by (1) and (19), $(\rho, u) = (\overline{\rho}, \overline{u})$ is a solution of the system

$$\begin{cases} u = \overline{u}^F + (\varepsilon+\lambda)^{-1}\nabla\Gamma * (P(\rho) - \tilde{P}) \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{25}$$

We construct approximate solutions (ρ^a, u^a) of (25) as follows. Let $\eta = \eta(r)$ be a C^∞ function on $[0,\infty)$ which is zero on $[0,1)$, one on $[2,\infty)$, and increasing, and define an operator $\nabla\Gamma^a *$ by

$$(\nabla\Gamma^a * w)(x) = \int \eta\left(\frac{|x-y|}{a}\right)\nabla\Gamma(x-y)w(y)dy.$$

We then seek approximations (ρ^a, u^a) to $(\overline{\rho}, \overline{u})$ as solutions to

$$\begin{cases} u^a = \overline{u}^F + (\varepsilon+\lambda)^{-1}\nabla\Gamma^a * (P(\rho^a) - \tilde{P}) \\ \rho^a_t + \operatorname{div}(\rho^a u^a) = (\varepsilon+\lambda)^{-1}\rho^a\big[(\varepsilon+\lambda)\operatorname{div} u^a - P(\rho^a) + \tilde{P} - \overline{F}\,\big], \\ \rho^a(x,0) = \rho_0(x). \end{cases} \tag{26}$$

(Actually, there is a further regularization involved, but this rather technical point need not detain us here.) Observe that u^a is now globally $C^{1+\alpha}$, and that, since ρ^a satisfies a first-order equation, its singularities are the singularities in its initial data ρ_0 convected with characteristic speed u^a. The heuristic analysis leading up to (23) and (24) then applies to (ρ^a, u^a) in a completely rigorous way.

The proof of Theorem 2 then consists of four parts: First, we show by an iteration argument that (26) has a solution (ρ^a, u^a) in a suitable sense, defined for all time. We then apply the analysis given in the above heuristic discussion to show that the approximations $\{(\rho^a, u^a)\}$ satisfy the bounds (23) and (24), independently of a, as well as all the geometrical considerations described in Theorem 2 relating to the propagation of singularities. This enables us to extract a sequence converging as $a \to 0$ to a limit (ρ, u) which also satisfies (23) and (24), whose singularities propagate as described in Theorem 2, and which is a solution of (25). The final step is to show that solutions of (25) are unique, so that (ρ, u) is precisely $(\overline{\rho}, \overline{u})$, the solution of (1)–(2) of Theorem 1, and therefore that $(\overline{\rho}, \overline{u})$ satisfies all the conclusions of Theorem 2. It is only for this very last step, the uniqueness of solutions of (25), that the assumption $P(\rho) = K\rho$ is required.

We conclude with two remarks concerning the exponential decay of singularities asserted in the Lemma. First, a closer examination of the argument leading to (18) shows that the constant C_1 measuring the rate of exponential decay of singularities is essentially

$$C_1 \approx \tilde{\rho} P'(\tilde{\rho})/(\varepsilon + \lambda) \ ,$$

that is, density times the square of the sound speed divided by the viscosity. We thus conclude that the singularities studied here actually *result from* the presence of viscosity, that they decay more rapidly with smaller viscosity, and therefore that they should disappear altogether in the inviscid limit, that is, in solutions of the Euler equations of compressible flow. Indeed, while solutions of the Euler equations may exhibit shock-wave singularities, these shock waves propagate at acoustic speeds rather than particle speeds, and the singularities examined here for solutions of the Navier-Stokes equations do not occur.

The second remark is that, for fixed viscosity, singularities disappear as well in the large–time limit $t \to \infty$, thereby giving a sort of "asymptotic compactness" for the fluid density and the velocity gradient. This compactifying effect evidently results from the fact that the pressure P is increasing (that is, from the positivity of A in (16)), and this is equivalent to the hyperbolicity of the underlying Euler equations for inviscid, compressible flow. It is instructive to contrast the parabolic smoothing, which takes (ρ_0, u_0) from $L^2_{loc} \cap L^\infty \times L^2$ into $L^2_{loc} \cap L^\infty \times H^1$ (see (12)), and which occurs instantaneously in time, with this "hyperbolic smoothing," which appears to take (ρ_0, u_0) from $L^2_{loc} \cap L^\infty \times L^2$ into $H^1_{loc} \cap H^2$, but which occurs only in *infinite* time. These observations are made precise for one-dimensional flows, for which a complete well-posedness theory is available, in Hoff–Ziane [7] and [8], where the existence of a "global attractor" is established. The global attractor, which is determined by an external force, attracts all solutions, even with singular initial data, and is contained in $H^1_{loc} \times H^2$. These results thus give a precise and rigorous expression, at least in one space dimension, to the observation that singularites in ρ and in ∇u disappear in the time-asymptotic limit, and that the solution operator is compact in infinite time. See also Feireisl [2] for related results.

References

[1] P. Duhem, Recherches sur l'hydrodynamique, Ann. Toulouse 2 (1901–03).

[2] Eduard Feireisl, Global attractors for the Navier-Stokes equations of three-dimensional compressible flow, to appear.

[3] David Hoff, Global solutions of the Navier-Stokes equations for multidimensional, compressible flow with discontinuous initial data, J. Diff. Eqns. 120, no. 1 (1995), 215-254.

[4] David Hoff, Strong convergence to global solutions for compressible, viscous, multidimensional flow, with polytropic equations of state and discontinuous initial data, Arch. Rational Mech. Ana. 132 (1995), 1-14.

[5] David Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional, heat conducting flow, Archive Rational Mech. Ana. 139 (1997), 303–354.

[6] David Hoff, Dynamics of Singularity Surfaces for Solutions of the Navier-Stokes Equations of Compressible Flow in Two Space Dimensions, in preparation.

[7] David Hoff and Mohammed Ziane, Compact attractors for the Navier-Stokes equations of one dimensional, compressible flow, C.R. Acad. Sci. Série I (1999), 239–244.

[8] David Hoff and Mohammed Ziane, The global attractor and finite determining nodes for the Navier-Stokes equations of compressible flow with singular initial data, Indiana Univ. Math. J. 49, no. 3 (2000), 843–889.

[9] Denis Serre, Variations de grande amplitude pour la densite d'un fluide visqueux compressible, Physica D 48 (1991), 113-128.

Department of Mathemtics
Indiana University–Rawles Hall
Bloomington, IN 47405 USA
hoff@indiana.edu