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The Feynman path integral as an improper integral over the Sobolev space

By

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§ 1. Introduction.

Let $L(\dot{x},x)=\frac{1}{2}|\dot{x}|^2 - V(x)$, be the Lagrangian with the potential $V(x)$. For the sake of simplicity of notations, we assume that the configuration space is of dimension 1. But the same results hold for n-dimensional case. The action of a path γ: $[0,t] \longrightarrow \mathbb{R}$, $\gamma(0)=y$ and $\gamma(t)=x$, with $\dot{\gamma} \in L^2(0,t)$ is the functional

$$S(\gamma)=\int_0^t L(\dot{\gamma},\gamma)\, ds.$$

The Feynman path integral is a formal integral over the path space Ω :

$$\int_{\Omega} e^{ih^{-1}S(\gamma)} \mathcal{D}[\gamma].$$

Among various proposals to give mathematically rigorous meaning to the Feynman path integral, we adopt Itô's formulation [4] and [1].

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We denote by γ_0 the straight line segment joining y and x, i.e.,

$$\gamma_0(s) = \frac{s}{t} x + \left(1 - \frac{s}{t}\right) y.$$

We denote the Sobolev space $H_0^1(0,t) = \{\gamma ; \dot{\gamma} \in L^2(0,t),\ \gamma(0) = \gamma(t) = 0\}$ by $\mathcal{H}$, which is equipped with the inner product

$$(\gamma_1, \gamma_2)_{\mathcal{H}} = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s)\, ds.$$

Let $Q(\gamma) = \sum_j \lambda_j\ e_j \otimes e_j$ be a positive quadratic form such that $\{e_j\}_j$ is a complete orthonormal system (c.o.n.s.) and $\{\lambda_j\}_j$ is a positive summable series. Let b be an arbitrary vector in $\mathcal{H}$. Let $N(d\gamma, b, Q)$ be the Gaussian measure on $\mathcal{H}$ with the mean b and the variance Q. Itô defined that

$$(1) \quad \int_{\Omega} e^{ih^{-1}S(\gamma)}\ \mathcal{D}[\gamma] =$$
$$= \left(\frac{1}{2\pi i h t}\right)^{1/2} \lim_{n \to \infty} \prod_{j=1}^{\infty} \left(1 + \frac{n\lambda_j}{hi}\right)^{1/2} \int_{\mathcal{H}} e^{ih^{-1}S(\gamma_0 + \gamma)}\ N(d\gamma, b, nQ),$$

if the right hand side converges.

Unfortunately, existence of the limit on the right hand side of (1) was proved only for potentials V(x) of the following two types:

1 $\quad V(x) = ax^2 + bx + c.$

2 $\quad V(x) = \int e^{ix\xi}\, d\mu,$ where μ is a signed measure of finite total variation on R.

On the other hand Pauli [5] discussed the Feynman path integrals for potentials satisfying

$$|\operatorname{grad} V(x)| = O(|x|) \quad \text{as} \quad x \longrightarrow \infty.$$

We assume in the present note a little stronger condition for the potential:

$$(2) \qquad |V^{(j)}(x)| < C_j, \qquad \text{if} \quad 2 \leq j.$$

We use the special quadratic form

$$Q(\gamma) = \sum_{j=1}^{\infty} \lambda^{-j} |(\gamma, e_j)_{\mathcal{H}}|^2 ,$$

where $\lambda > 1$ is a constant and $\{e_j\}_j$ is the c.o.n.s. consisting of indefinite integrals of Haars' functions. Our main results is

Theorem. Let Q be as above and b be any element of $\mathcal{H}$. We assume that V satisfy the condition (2). If $|t|$ is sufficiently small, then the right hand side of (1) converges to the limit independent of b. The limit K(t,x,y) is the fundamental solution of the Schrödinger equation:

$$\{(\frac{h}{i}\frac{\partial}{\partial t}) - (\frac{h}{i}\frac{\partial}{\partial x})^2 - V(x)\} K(t,x,y) = 0,$$

$$K(0,x,y) = \delta(x-y).$$

Here h is a small positive constant.

In the next section we will give more accurate statement of the result as well as outline of the proof.

§ 2. Outline of the proof.

Indefinite integrals of Haars' functions.

Let $q=2^{-n}(2k+1)$, $n=1, 2, 3, \ldots$ and $k=0, 1, 2, \ldots, 2^{n-1}-1$, be any finite binary fraction. We denote n by $n(q)$ and k by $k(q)$. We set $m(q)=2^{n(q)-1}+k(q)$. For each such q, we define $\delta_q=2^{-n(q)}t$ and

$$e_q(s) = \begin{cases} 0, & \text{for } |qt-s| \geq \delta_q, \\ (2\delta_q)^{-1/2}(\delta_q - |s-qt|) & \text{for } |s-qt| \leq \delta_q. \end{cases}$$

The system $\{\frac{d}{ds} e_q(s)\}_q$ is Haars' c.o.n.s. in $L^2(0,t)$. Hence $\{e_q\}$ forms a c.o.n.s. in $\mathcal{H}$. Note that

$$\|e_q\|_{L^\infty} = 2^{-1/2}\delta_q^{1/2}, \quad \|e_q\|_{L^1} = 2^{-1/2}\delta_q^{3/2}, \quad \|e_q\|_{L^2} = 3^{-1/2}\delta_q$$

The Quadratic form Q. Let $\lambda > 1$. Then

$$Q(\gamma) = \sum_q \lambda^{-m(q)} y_q^2, \qquad \text{where } y_q = (\gamma, e_q)_{\mathcal{H}}.$$

Splitting of $\mathcal{H}$. Let $N_0 > \lambda+2$ be so large that

(2.1) $$2^N \lambda^{-2^{N-10}} < 1 \quad \text{for any } N \geq N_0.$$

We choose n so large that

(2.2) $$n^{-1} \lambda^{2^{N_0}} < 1.$$

Let N_n be the positive integer such that

(2.3) $$n^{-1} \lambda^{2^{N_n}} \leq 1 < n^{-1} \lambda^{2^{N_n+1}},$$

and m_n be the integer such that

(2.4) $$n^{-1} \lambda^{m_n} < 2^{-N_n/4} < n^{-1} \lambda^{m_n+1}.$$

We can easily prove from (2.1),(2.3) and (2.4) that

(2.5) $$2^{N_n-1} < m_n < 2^{N_n+1}.$$

Let $\mathcal{H}_1$= span of $\{e_q: m(q)<m_n\}$ and $\mathcal{H}_2$=span of $\{e_q:m(q)\geq m_n\}$. Then we have orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

which reduces the quadratic form Q. For any γ in $\mathcal{H}$, we denote the corresponding decomposition

$$\gamma = \gamma_1 + \gamma_2.$$

For any b in $\mathcal{H}$, the Gaussian measure $N(d\gamma,b,nQ)$ coincides with the product measure

$$N(d\gamma,b,nQ) = N(d\gamma_1,b_1,nQ_1) \times N(d\gamma_2,b_2,nQ_2).$$

New coordinates in $\mathcal{H}_1$. The subspace $\mathcal{H}_1$ is of dimension m_n-1. We arrange points $\{qt\}_{m(q)<m_n}$ of $[0,t]$ in its order of magnitude. Then we obtain a division of $[0,t]$ $0 < t_1 < t_2 < \dots < t_{m_1-1} < t$. We set $t_0=0$ amd $t_{m_n}=t$. The subspace $\mathcal{H}_1$ coincides with the space of all piecewise linear paths with vertices at $s=t_j$, $j=0, 1, \dots, m_n$. So we introduce a new coordinates $x_j=\gamma(t_j)$ of γ with respect to the basis $\{w_j\}_j$, which is given by

$$w_j(s) = \begin{cases} 0 & \text{if } s\le t_{j-1} \text{ or } t_{j+1}\le s, \\ 1 & \text{if } s=t_j, \\ \text{linear} & \text{for } t_{j-1}\le s\le t_j \text{ and } t_j\le s\le t_{j+1}. \end{cases}$$

Any γ_1 in $\mathcal{H}_1$ can be written into two forms:

$$\gamma_1 = \sum_{m(q)<m_n} y_q e_q = \sum_{j=1}^{m_n-1} x_j\, w_j.$$

<u>Evaluation of Itô's integral.</u>

Let $b=b_1+b_2$ be decomposition of b into $\mathcal{H}_1$ and $\mathcal{H}_2$. Using any $\gamma = \gamma_1 + \gamma_2$, we make a new path $\gamma_1 + b_2$. The action of it is

$$S(\gamma_0+\gamma_1+b_2)=\frac{1}{2}\int_0^t |\dot{\gamma}_0|^2\, ds + \|b_2\|_{\mathcal{H}}^2 + \|\gamma_1\|_{\mathcal{H}}^2 - \int_0^t V(\gamma_0+\gamma_1+b_2)\, ds.$$

We can write this as

$$S(\gamma_0+\gamma_1+b_2)= \frac{1}{2}\ \frac{(x-y)^2}{t}+\|b_2\|_{\mathcal{H}}^2 +\sum_{j=1}^{m_n} S_j(x_j,x_{j-1};b_2),$$

where

$$S_j(x_j,x_{j-1};b_2)= \frac{|x_j-x_{j-1}|^2}{2\Delta t_j} -\Delta t_j\ \omega_j(x_j,x_{j-1};b_2)$$

with $\Delta t_j= t_j-t_{j-1}$ and

$$\omega_j(x_j,x_{j-1};b_2)=(\Delta t_j)^{-1}\int_{t_{j-1}}^{t_j} V(x_jw_j+x_{j-1}w_{j-1}+\gamma_0+b_2)\, ds.$$

The Hessian Hess(S) of $S(\gamma_0+\gamma_1+b_2)$ with respect to $x_1,\ldots x_{m_n-1}$ equals $\mathrm{Hess}(S) = H - W = H\,(I-H^{-1}W)$, where

$$H=\begin{pmatrix} \frac{1}{\Delta t_1}+\frac{1}{\Delta t_2}, & -\frac{1}{\Delta t_2}, & 0, & 0 & \cdots\cdots \\ -\frac{1}{\Delta t_2}, & \frac{1}{\Delta t_2}+\frac{1}{\Delta t_3}, & -\frac{1}{\Delta t_3}, & 0,, & \cdots \\ 0, & -\frac{1}{\Delta t_3}, & \frac{1}{\Delta t_3}+\frac{1}{\Delta t_4}, & -\frac{1}{\Delta t_4}, & \\ 0, & 0, & \cdots\cdots & & \end{pmatrix}$$

and W is the Hessian of $\sum_{j=1}^{m_n} t_j\,\omega_j(x_j,x_{j-1};b_2)$.

If $|t|$ is small $(I-H^{-1}W)$ is invertible. Hence the functional $\gamma_1 \longrightarrow S(\gamma_0+\gamma_1+b_2)$ has only one critical point at $\gamma_1=\gamma_1^*$ $=\sum_j x_j^*\,w_j$. which is given by

$$\partial_{x_j} S_j(x_j^*,x_{j-1}^*)+\partial_{x_j} S_{j+1}(x_{j+1}^*,x_j^*)=0, \qquad 1\le j\le m_n-1,$$

where $x_{m_n}^* = x_0^*=0$.

For any finite binary fraction q with $m(q)>m_n$, we define

$$\varepsilon_q(\gamma_0+\gamma_1+b_2)= \int_0^t V'(\gamma_0(s)+\gamma_1(s)+b_2(s))\, e_q(s)\, ds$$

and $\varepsilon_q^*(\gamma_0+b_2) = \varepsilon_q(\gamma_0+\gamma_1^*+b_2)$. The splitting $\mathcal{H}= \mathcal{H}_1+\mathcal{H}_2$ reduces the quadratic form Q into Q_1 and Q_2 and it reduces the

Gaussian measure $N(d\gamma, b, nQ)$:

$$N(d\gamma, b, nQ) = N(d\gamma_1, b_1, nQ_1) \times N(d\gamma_2, b_2, nQ_2).$$

It is clear that

$$N(d\gamma_1, b_1, nQ_1) = \prod_{j=1}^{m_n-1} \left(\frac{1}{2\pi\Delta t_j}\right)^{1/2} \exp\left(-\frac{1}{2n} \sum_{m(q)<m_n} \lambda^{m(q)} (y_q - b_q)^2\right) \prod_{j=1}^{m_n-1} dx.$$

The variance of this Gaussian measure is nQ_1, which is large. So the density of this with respect to the Lebesgue measure can be regarded as an amplitude of an oscillatory integral. On the other hand $N(d\gamma_2, b_2, nQ_2)$ has variance relatively small. The delta measure concentrated at $\gamma_2 = b_2$ is a good approximation to $N(d\gamma_2, b_2, nQ_2)$. Thus we have

<u>Theorem 2.1.</u> If $|t|$ is sufficiently small,

$$(2\pi i h)^{-1/2} \prod_q \left(1 + \frac{n\lambda^{-m(q)}}{ih}\right)^{1/2} \int_{\mathcal{H}} e^{ih^{-1} S(\gamma_0 + \gamma)} N(d\gamma, nQ, b)$$

$$= \exp\left(-\frac{1}{2} \sum_{m(q)\geq m_n} \frac{n\lambda^{-m(q)}}{h^2 + inh\lambda^{-m(q)}} (\xi_q^* - b_q)^2 - \sum_{m(q)<m_n} \frac{\lambda^{m(q)}}{2n} (y_q^* - b_q)^2\right)$$

$$\cdot \prod_{j=1}^{m_n} \left(\frac{1}{2\pi i h \Delta t_j}\right)^{1/2} \int_{R^{m_n-1}} e^{ih^{-1} S(\gamma_0 + \gamma_1 + b_2)} \prod_{j=1}^{m_n-1} dx_j$$

$$+ \left(\frac{1}{2\pi i t h}\right)^{1/2} e^{ih^{-1} S(\gamma_0 + \gamma_1^* + b_2)} r_n(t, x, y),$$

where $y_q^* = (\gamma_1^*, e_q)_{\mathcal{H}}$. We have the estimate

$$\left|\partial_x^\alpha \partial_y^\beta r_n(t, x, y)\right| \leq C_{\alpha\beta}(h) \, 2^{-N_n/8} t.$$

The last term of the right hand side of Theorem goes to 0 as n goes to ∞. While the exponential factor of the first term tends to 1 as n goes to ∞. In fact we have

Theorem 2.2. Let $\delta_n = 2^{-N_n} t$. Then we have the following estimates:

$$\left| \partial_x^{\alpha} \partial_y^{\beta} \xi_q^* \right| \leq C_{\alpha\beta} \delta_n^{3/2} \, (1+|x|+|y|)^{(1-\alpha-\beta)^+},$$

$$\left| \sum_{m(q)\geq m_n} \frac{n\lambda^{-m(q)}}{h^2+inh\lambda^{-m(q)}} (\xi_q^* - b_q)^2 \right| < C(1+|x|^2+|y|^2+\|b_2\|_{\mathcal{H}}^2)\, \delta_n^{3/2}.$$

$$\left| \partial_x^{\alpha} \partial_y^{\beta} y_q^* \right| < C_{\alpha\beta}(1+|x|+|y|+\|b\|_{\mathcal{H}})^{(1-\alpha-\beta)^+} \delta_n^{3/2}.$$

$$\sum_{m(q)<m_n} \frac{\lambda^{m(q)}}{n} (y_q^* - b_q)^2 < C(1+|x|^2+|y|^2+\|b_2\|_{\mathcal{H}}^2)\, \delta_n.$$

Here a^+ denotes $\max(a,0)$.

Theorem 2.1 and Theorem 2.2 imply that the decisive role is played by the oscillatory integral

$$J_n = \prod_{j=1}^{m_n} \left(\frac{1}{2\pi i h \Delta t_j}\right)^{1/2} \int_{R^{m_n-1}} e^{ih^{-1}S(\gamma_0+\gamma_1+b_2)} \prod_{j=1}^{m_n-1} dx_j .$$

Similar integrals are treated previously in [3]. We have

Theorem 2.3. If $|t|$ is small enough, we can rewrite

$$J_n = \left(\frac{1}{2\pi i h t}\right)^{1/2} e^{ih^{-1}S(\gamma_0+\gamma_1^*+b_2)} \, (1+k_n(t,x,y)),$$

For any $K>0$, there exists C_K such that we have

$$\left|\partial_x^{\alpha}\partial_y^{\beta} k_n(t,x,y)\right| \leq \prod_{j=1}^{m_n} (1+C_K \Delta t_j|t|) -1,$$

if α and β are smaller than K.

Convergence as n goes to ∞ . In order to discuss convergence of the oscillatry integral J_n. We compare $S(\gamma_0+\gamma_1+b_2)$ with the action of piecewise classical orbit. Let

$$\Delta;\ 0=t_0 < t_1 < \ldots < t_{m_n-1} < t_{m_n}=t$$

be the division of the $[0, t]$ as above. Let γ_Δ be the piecewise classical orbit

$$\gamma_\Delta(s) + \partial_x V(\gamma_\Delta) = 0 \qquad \text{for } t_{j-1}<s<t_j,$$

$$\gamma_\Delta(t_j)=x_j+\gamma_0(t_j), \qquad \text{for } j=0,1,\ldots,m_n,$$

with $x_0 = x_{m_n}=0$. The action along γ_Δ is

$$S_{cl}(t,x,x_{m_n-1},\ldots,x_1,y)=\sum_{j=1}^{m_n-1} S_j(t_j,x_j,x_{j-1}),$$

where

$$S_j(t_j,x_j,x_{j-1}) = \int_{t_{j-1}}^{t_j} L(\gamma_\Delta(s))\, ds.$$

The critical point of $S_{cl}(t,x,x_{m_n-1},x_{m_n-2},\ldots,x_1,y)$ with respect to vertices $x_{m_n-1}, x_{m_n-2},\ldots,x_1$ is unique if $|t|$ is small. The critical level of it equals the classical action

$$S_{cl}(t,x,y) = \int_0^t L(\gamma_{cl}(s))\, ds,$$

where γ_{cl} is the classical orbit joininig $\gamma(0)=y$ and $\gamma(t)=x$.

Theorem 2.4. If $|t|$ is small, we have

$$\prod_{j=1}^{m_n}\left(\frac{1}{2\pi i h \Delta t_j}\right)^{1/2}\int_{R^{m_n-1}} e^{ih^{-1}S_{cl}(t,x,x_{m_n-1},\dots,x_1,y)}\prod_{j=1}^{m_n-1} dx_j.$$

$$=\left(\frac{1}{2\pi i h t}\right)^{1/2} e^{ih^{-1}S_{cl}(t,x,y)}(1+k^{\#}(\Delta,t,x,y)).$$

For any $K>0$, There exists a constant $C_K>0$ such that

$$\left|\partial_x^{\alpha}\partial_y^{\beta} k^{\#}(\Delta,t,x,y)\right| < \prod_{j=1}^{m_n}(1+C_K\,\Delta t_j|t|)-1,$$

if α and $\beta < K$.

The next theorem compares $k^{\#}(\Delta,t,x,y)$ with $k_n(t,x,y)$.

Theorem 2.5. For any $K>0$, there exists a constant C_K such that

$$\left|\partial_x^{\alpha}\partial_y^{\beta}(k^{\#}(\Delta,t.x.y)-k_n(t,x,y))\right| < C\,\delta_n^{1/8}(1+|x|+|y|+\|b_2\|_{\mathcal{H}}).$$

If we make the division of interval finer, $k^{\#}(\Delta,t,x,y)$ forms a Cauchy sequence. More precisely, we have

Theorem 2.6. Let Δ_1 be a subdivision of Δ, then

$$\left|\partial_x^{\alpha}\partial_y^{\beta}(k^{\#}(\Delta,t,x,y)-k^{\#}(\Delta_1,t,x,y))\right| \le C_K\,|t|\cdot|\Delta|,$$

where $|\Delta| = \max_{1<j<m_n} |\Delta t_j|$. We set

$$k(t,x,y) = \lim_{|\Delta| \to 0} k^{\#}(\Delta, t,x,y).$$

Then $k(t,x,y)$ satisfies

$$\left| \partial_x^{\alpha} \partial_y^{\beta} k(t,x,y) \right| < C\, t.$$

Combining all these, we can prove

Theorem 2.7.

$$\lim_{n \to \infty} (2\pi i h)^{-1/2} \prod_q \left(1+\frac{n\lambda^{-m(q)}}{ih}\right)^{1/2} \int_{\mathcal{H}} e^{ih^{-1}S(\gamma_0+\gamma)} N(d\gamma, nQ, b)$$

$$= \left(\frac{1}{2\pi i t h}\right)^{1/2} e^{ih^{-1}S_{cl}(t,x,y)} (1+k(t,x,y)),$$

which is the fundamental solution of the Schrödinger equation.

Appendix

In proving above results, we use the following stationary phase method in a space of large dimension. We consider the following oscillatory integrals

$$I(S,a,\nu) = \prod_{k=1}^{L} \left(\frac{\nu}{2\pi i t_k}\right)^{1/2} \int_{\mathbb{R}^{L-1}} e^{i\nu S(x_L,\dots,x_0)} a(x_L,\dots,x_0) \prod_{j=1}^{L-1} dx_j.$$

The phase function $S(x_L,\ldots,x_0)$ is a real valued function of the form

$$S(x_L,x_{L-1},\ldots,x_0) = \sum_{j=1}^{L-1} S_j(x_j,x_{j-1}),$$

with

$$S_j(x_j,x_{j-1}) = \frac{1}{2t_j}(x_j-x_{j-1})^2 + t_j\,\omega_j(x_j,x_{j-1}), \qquad j=1,2,\ldots L.$$

We assume further that

$$\sup_{x,y} \left|\partial_x^\alpha\, \partial_y^\beta\, \omega_j(x,y)\right| \leq C_{\alpha\beta},$$

if $0 \leq \alpha, \beta \leq K$ and $\alpha+\beta \geq 2$.

If $T=t_1+t_2+\ldots+t_L$ is small enough, the critical point $(x_{L-1}^*,x_{L-2}^*,\ldots,x_1^*)$ of the phase is the unique solution of

$$\partial_{x_j} S_{j+1}(x_{j+1}^*,x_j^*) + \partial_{x_j} S_j(x_j^*,x_{j-1}^*)=0, \qquad j=1,2,\ldots,L-1,$$

where $x_L^*=x_L$ and $x_0^*=x_0$. We use the following notation

$$a(\overline{x_L,x_0}) = a(x_L,x_{L-1}^*,\ldots,x_1^*,x_0).$$

Let $k < l$ and $x_{k+1}^*,\ldots,x_{l-1}^*$ be the partial critical point, i,e.,

$$\partial_{x_j} S_{j+1}(x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(x_j^*,x_{j-1}^*)=0, \qquad \text{for } j=k+1,\ldots, l-1,$$

where $x_k^*=x_k$ and $x_l^*=x_l$. Then we set

$$a(x_L,\ldots,\overline{x_l,x_k},\ldots,x_0) = a(x_L,\ldots,x_l,x_{l-1}^*,\ldots,x_{k+1}^*,x_k,\ldots,x_0).$$

We assume that for any sequence of positive integers

$$j_0=0 < j_1 < j_2 < \dots < j_r < L$$

and positive K we have the estimate

$$\left|\partial_{x_0}^{\alpha_0}\ \partial_{x_{j_1}}^{\alpha_1} \dots \partial_{x_{j_r}}^{\alpha_r}\ \partial_{x_L}^{\alpha_L}\ a(\overline{x_L, x_{j_s}}, \overline{x_{j_s-1}, x_{j_{s-1}}}, \dots \overline{x_{j_1}, x_0})\right| < A_K,$$

if each $\alpha_j \leq K$, $j=0,1,\dots .r$ and L.

<u>Theorem.</u> Under the assumptions above there is a positive constant δ such that if $T=t_1+t_2+\dots+t_L < \delta$

$$I(S,a,\nu)=\left(\frac{\nu}{2\pi i T}\right)^{1/2} e^{i\nu S(\overline{x_L, x_0})} \det(I+H^{-1}W)^{-1/2} \quad (a(\overline{x_L, x_0})+r(x_L,x_0)),$$

where $r(x_L,x_0)$ satisfies the estimate:

for any $\alpha_0, \alpha_L \leq K$,

$$\left|\partial_{x_0}^{\alpha_0}\ \partial_{x_L}^{\alpha_L}\ r(x_L,x_0)\right| \leq A_{3K+10}\left(\prod_{j=1}^{L}(1+ C_k\, \nu^{-1}|t_j' T|) - 1\right),$$

Constants δ and C_K are independent of the dimension L of the space. H is the matrix

$$H=\begin{pmatrix} \frac{1}{t_1}+\frac{1}{t_2}, & -\frac{1}{t_2}, & 0, & 0, & \dots\dots 0 \\ -\frac{1}{t_2}, & \frac{1}{t_2}+\frac{1}{t_3}, & -\frac{1}{t_3}, & 0, & \dots .0 \\ 0, & -\frac{1}{t_3}, & \frac{1}{t_3}+\frac{1}{t_4}, & -\frac{1}{t_4} & \\ 0, & 0, & \dots & & \end{pmatrix}$$

and W is the Hessian matrix of $\sum_j t_j\, \omega_j(x_j,x_{j-1})$ at the critical point of $S(x_L, x_{L-1}, \ldots x_1, x_0)$ with respect to $x_{L-1}, x_{L-2}, \ldots, x_1$.

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