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L. CATTABRIGA

L. ZANGHIRATI

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GLOBAL ANALYTIC AND GEVREY SURJECTIVITY OF THE MIZOHATA
OPERATOR $D_2 + ix_2^{2k}D_1$ (*)

L. Cattabriga (**)

L. Zanghirati (***)

Global analytic surjectivity of all linear constant coefficient partial differential operators on \mathbb{R}^2 has been proved by E. De Giorgi and L. Cattabriga [5], where counterexamples for the case of operators on \mathbb{R}^3 are also indicated (see also E. De Giorgi [4] and for other results for operators on open sets of \mathbb{R}^n : T.Kawai [10] and L.Hörmander [8]). Global Gevrey surjectivity of all linear constant coefficient partial differential operators on \mathbb{R}^2 was subsequently proved by L. Cattabriga [3] in the case of Gevrey spaces with rational index (see also L. Cattabriga [2], G. Zampieri [15], R.W. Braun-R. Meise-D. Vogt [1] for the case of operators on \mathbb{R}^n). As for the case of linear operators with variable coefficients results on global Gevrey surjectivity are given by L.Ehrenpreis [6] and in the analytic case by T.Kawai [11].

Here we consider, as a simple example of an operator on \mathbb{R}^2 with variable coefficients, the Mizohata operator $D_2 + ix_2^{2k}D_1$, where k is a positive integer and $D_j = -i\partial_j$, $j = 1, 2$, and prove the following

Theorem. Let $\mathcal{E}^{\{s\}}(\mathbb{R}^n)$, $s \geq 1$, be the space of all C^∞ functions f on \mathbb{R}^n such that for every compact subset K of \mathbb{R}^n there exists a constant $A > 0$ such that

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(**) Dipartimento Matematico, Università di Bologna, Piazza di Porta S. Donato 5, I-40127 Bologna.

(***) Dipartimento Matematico, Università di Ferrara, Via Machiavelli 35, I-44100 Ferrara.

$$\sup_{x \in K} \sup_{\alpha \in \mathbb{Z}^n} A^{-|\alpha|} |\alpha|^{-s} |\partial_x^\alpha f(x)| < +\infty.$$

Then for every $s \geq 1$

$$i) (D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^2) = \mathcal{E}^{\{s\}}(\mathbb{R}^2),$$

$$ii) (D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^3) \subsetneq \mathcal{E}^{\{s\}}(\mathbb{R}^3).$$

Proof. Let $f \in \mathcal{E}^{\{s\}}(\mathbb{R}^2)$, $s \geq 1$, and let $\sigma > \max\{s, 1\}$. Since $f \in \mathcal{E}^{\{\sigma\}}(\mathbb{R}^2)$, from a result by H. Komatsu [12] it follows that there exists $v_1 \in \mathcal{E}^{\{\sigma\}}(\mathbb{R}^2)$ such that

$$(1) \quad \begin{cases} v_1(x_1, 0) = 0 \\ \partial_x^\gamma [(D_2 + ix_2^{2k} D_1)v_1 - f](x_1, 0) = 0 \end{cases} \quad \forall \gamma \in \mathbb{Z}_+^2, x_1 \in \mathbb{R}.$$

Let

$$(2) \quad h(x_1, x_2) = f(x_1, x_2) - (D_2 + ix_2^{2k} D_1)v_1(x_1, x_2),$$

and with the new variables in \mathbb{R}^2

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2^{2k+1}/(2k+1), \end{cases}$$

set

$$g(y_1, y_2) = \begin{cases} h(y_1, [(2k+1)y_2]^{1/(2k+1)}) [(2k+1)y_2]^{-2k/(2k+1)} & \text{for } y_2 \neq 0 \\ 0 & \text{for } y_2 = 0. \end{cases}$$

In view of (1), $g \in C^\infty(\mathbb{R}^2)$. Hence, by the global $C^\infty(\mathbb{R}^2)$ -surjectivity of all linear partial differential operators with constant coefficients, there exists $w \in C^\infty(\mathbb{R}^2)$ such that

$$(D_2 + i D_1)w(y_1, y_2) = g(y_1, y_2) \quad \text{on } \mathbb{R}^2.$$

Letting

$$v_2(x_1, x_2) = w(x_1, x_2^{2k+1}/(2k+1)), \quad (x_1, x_2) \in \mathbb{R}^2,$$

it follows that

$$v_2 \in C^\infty(\mathbb{R}^2)$$

and

$$(3) \quad (D_2 + ix_2^{2k} D_1) v_2(x_1, x_2) = x_2^{2k} g(x_1, x_2^{2k+1}/(2k+1)) = h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2.$$

Thus from (2) and (3)

$$(D_2 + ix_2^{2k} D_1)(v_1 + v_2) = f \quad \text{on } \mathbb{R}^2,$$

where

$$v = v_1 + v_2 \in C^\infty(\mathbb{R}^2).$$

Recalling now that the Mizohata operator $(D_2 + ix_2^{2k} D_1)$ is (analytic and) s -Gevrey hypoelliptic on \mathbb{R}^2 for every $s \geq 1$ (see for example S.Mizohata [13], L.Rodino [14], and also L.Hörmander [9]), we conclude that v is in fact in the same space $\mathcal{E}^{\{s\}}(\mathbb{R}^2)$ as f is. This proves part i) of the theorem.

To prove part ii), let $f \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$, $s \geq 1$, and define

$$g(x_1, x_2, x_3) = f(x_1, x_2^{2k+1}/(2k+1), x_3) x_2^{2k}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Since $g \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$, if $(D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^3) = \mathcal{E}^{\{s\}}(\mathbb{R}^3)$, there exists

$w \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$ such that $(D_2 + ix_2^{2k} D_1)w(x_1, x_2, x_3) = g(x_1, x_2, x_3)$ on \mathbb{R}^3 .

Hence $v(x_1, x_2, x_3) = w(x_1, [(2k+1)x_2]^{1/(2k+1)}, x_3)$ would be a $\mathcal{D}'(\mathbb{R}^3)$ solution of $(D_2 + iD_1)v = f$. Since the operator $D_2 + iD_1$ is partially hypoelliptic in \mathbb{R}^3 with respect to the (x_1, x_2) variables, from the corollary of Theorem 4.1.2 of [7], it follows that $v \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$. So the operator $D_2 + iD_1$ on \mathbb{R}^3 would be s -surjective, what is false as it has been proven in [3] and [4].

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