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ASYMPTOTIC DISTRIBUTION OF NEGATIVE EIGENVALUES FOR TWO DIMENSIONAL PAULI OPERATORS WITH NONCONSTANT MAGNETIC FIELDS

by A. IWATSUKA & H. TAMURA

1. Introduction.

The present paper is a continuation to [10] where we have studied the asymptotic distribution of negative eigenvalues near the origin for two and three dimensional Pauli operators perturbed by electric fields falling off at infinity. In the previous work, the strength of magnetic fields (not necessarily constant) has been assumed to be uniformly bounded from below. The special emphasis here is placed on the case that Pauli operators have magnetic fields falling off at infinity.

The Pauli operator describes the motion of a particle with spin in a magnetic field and it acts on the space $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$. The unperturbed Pauli operator without electric field is given by

$$H_p = (-i\nabla - A)^2 - \sigma \cdot B$$

under a suitable normalization of units, where $A : \mathbf{R}^3 \to \mathbf{R}^3$ is a magnetic potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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is the vector of 2×2 Pauli matrices and $B = \nabla \times A$ is a magnetic field. We write $(x, z) = (x_1, x_2, z)$ for the coordinates over the three dimensional space $\mathbf{R}^3 = \mathbf{R}_x^2 \times \mathbf{R}_z$. We now assume that the magnetic field B has a constant direction. For brevity, the field is assumed to be directed along the positive z axis, so that B takes the form

$$B(x) = (0, 0, b(x)).$$

Since the magnetic field B is a closed two form, it is easily seen that B is independent of the z variable. In this work, we exclusively work in the two dimensional space $\mathbf{R}_x^2$. We identify $B(x)$ with the function $b(x)$. Let $A(x) = (a_1(x), a_2(x))$, $a_j \in C^1(\mathbf{R}^2)$, be a magnetic potential associated with $b(x)$. Then

$$b(x) = \nabla \times A = \partial_1 a_2 - \partial_2 a_1, \quad \partial_j = \partial/\partial x_j,$$

and the Pauli operator takes the simple form

$$H_p = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

in two dimensions, where

$$H_\pm = (-i\nabla - A)^2 \mp b = \Pi_1^2 + \Pi_2^2 \mp b, \quad \Pi_j = -i\partial_j - a_j.$$

The magnetic field b is represented as the commutator $b = i[\Pi_2, \Pi_1]$ and hence $H_\pm$ can be rewritten as

$$H_\pm = (\Pi_1 \pm i\Pi_2)^*(\Pi_1 \pm i\Pi_2). \tag{1.1}$$

This implies that $H_\pm \geq 0$ is nonnegative. If, in particular, $b(x) \geq 0$ is nonnegative, then it is known ([1], [4], [12]) that H_+ has zero as an eigenvalue and its essential spectrum begins at zero for a fairly large class of magnetic fields. We are going to refer to several basic spectral properties of $H_\pm$ in Section 2.

We now write H for H_+ and consider the Pauli operator

$$H(V) = H - V, \quad H = (-i\nabla - A)^2 - b,$$

perturbed by electric field $V(x)$. As stated above, the essential spectrum of unperturbed operator $H = H_+$ begins at zero. If the electric field $V(x)$ falling off at infinity is added to this operator as a perturbation, then the above operator $H(V)$ has negative discrete eigenvalues accumulating at the origin. The aim of the present paper is to study the asymptotic distribution near the origin of such negative eigenvalues.

We formulate the obtained results precisely. Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. We first make the following assumptions on $b(x)$ and $V(x)$:

(b) $b(x) \in C^1(\mathbf{R}^2)$ is a positive function and

$$\langle x\rangle^{-d}/C \le b(x) \le C\langle x\rangle^{-d}, \quad |\nabla b(x)| \le C\,\langle x\rangle^{-d-1}, \quad C > 1,$$

for some $d \ge 0$.

(V) $V(x) \in C^1(\mathbf{R}^2)$ is a real function and

$$|V(x)| \le C\,\langle x\rangle^{-m}, \quad |\nabla V(x)| \le C\,\langle x\rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$.

Under these assumptions, the operator $H(V)$ formally defined above admits a unique self-adjoint realization in $L^2 = L^2(\mathbf{R}^2)$ with natural domain $\{u \in L^2 : Hu \in L^2\}$, where Hu is understood in the distributional sense. We denote by the same notation $H(V)$ this self-adjoint realization and by $N(H(V) < -\lambda)$, $\lambda > 0$, the number of negative eigenvalues less than $-\lambda$. We now formulate a series of theorems obtained here.

THEOREM 1.1. — *Let the notations be as above. Assume that (b) and (V) are fulfilled. If d and m satisfy*

$$0 \le d < 2, \quad d < m, \tag{1.2}$$

then, for any $\delta > 0$ small enough, there exists $\lambda_\delta > 0$ such that

$$N(H(V) < -\lambda) \le (2\pi)^{-1}\int_{V(x)>(1-\delta)\lambda} b(x)\,dx + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

THEOREM 1.2. — *Keep the same notations and assumptions as above. Then*

$$N(H(V) < -\lambda) \ge (2\pi)^{-1}\int_{V(x)>\lambda} b(x)\,dx - (2\pi)^{-1}\int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} b(x)\,dx - \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

The asymptotic formula as $\lambda \to 0$ for $N(H(V) < -\lambda)$ can be easily obtained by combining these two theorems. We further assume $V(x)$ to satisfy

$$\liminf_{\lambda\to 0} \lambda^{2/m}\int_{V(x)>\lambda} dx > 0, \tag{1.3}$$

so that

$$\liminf_{\lambda\to 0} \lambda^{2/m} \int_{\lambda<V(x)<M\lambda} dx > 0, \quad M \gg 1,$$

by assumption (V). This, together with assumptions (b) and (V), implies that

$$\liminf_{\lambda\to 0} \lambda^{(2-d)/m} \int_{V(x)>\lambda} b(x)\, dx > 0.$$

Thus we have the following theorem as an immediate consequence of Theorems 1.1 and 1.2.

THEOREM 1.3. — *Assume again the same assumptions as in Theorem 1.1. If, in addition to (1.3), $V(x)$ fulfills*

$$\limsup_{\lambda\to 0} \lambda^{(2-d)/m} \int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} \langle x\rangle^{-d}\, dx = o(1), \quad \delta \to 0,$$

then

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\, dx(1+o(1)), \quad \lambda \to 0.$$

Remarks.

(1) If $\lim_{|x|\to\infty} |x|^2 b(x) = \infty$, it is known that the bottom, zero, of essential spectrum of $H = H_+$ is an eigenvalue with infinite multiplicities $\dim \operatorname{Ker} H = \infty$ ([12, Theorem 3.4]). On the other hand, if $b(x) = O(|x|^{-d})$ as $|x| \to \infty$ for some $d > 2$, then it follows that $\dim \operatorname{Ker} H < \infty$ (see Remark 4.1). We point out that no decay condition on the derivatives of $b(x)$ is assumed in these results.

(2) The assumption $d < m$ means that magnetic fields are stronger than electric fields at infinity. In the last section (Section 11), we will briefly discuss the case $m < d$, $0 < m < 2$, when electric fields are stronger than magnetic fields. This case is much easier to deal with and $N(H(V) < -\lambda)$ is shown to obey the classical Weyl formula. Roughly speaking, it behaves like $N(H(V) < -\lambda) \sim \lambda^{(m-2)/m}$ as $\lambda \to 0$. If $d > 2$ and $m > 2$, then the number of negative eigenvalues is expected to be finite, but it seems that the problem has not yet been established.

(3) Under the same assumptions as in Theorem 1.1, we can prove that

$$N(H_-(V) < -\lambda) = O(\lambda^{-\varepsilon})$$

for any $\varepsilon > 0$ small enough, where $H_-(V) = H_- - V$. This follows from Theorem 1.1 at once, if we take account of the form inequality

$$H_-(V) = H_+ + 2b - V \geq H_+ - c_N \langle x \rangle^{-N}, \quad c_N > 0,$$

for any $N \gg 1$ large enough. Thus the number $N(H_p(V) < -\lambda)$ of negative eigenvalues less than $-\lambda$ of the perturbed Pauli operator

$$H_p(V) = H_p - V = \begin{pmatrix} H_+ - V & 0 \\ 0 & H_- - V \end{pmatrix} \quad \text{on} \quad L^2(\mathbf{R}^2) \otimes \mathbf{C}^2$$

obeys the same asymptotic formula as in Theorem 1.3.

There are a lot of works on the problem of spectral asymptotics for magnetic Schrödinger operators. An extensive list of literatures can be found in the survey [11]. The problem of asymptotic distribution of discrete eigenvalues below the bottom of essential spectrum has been studied by [13], [15] when $b(x) = b$ is a uniform magnetic field. Both the works make an essential use of the uniformity of magnetic fields and the argument there does not extend directly to the case of nonconstant magnetic fields. Roughly speaking, the difficulty arises from the fact that magnetic potentials which actually appear in Pauli operators undergo nonlocal changes even under local changes of magnetic fields. This makes it difficult to control nonconstant magnetic fields by a local approximation of uniform magnetic fields. As stated at the beginning of the section, we have recently studied the case of nonconstant magnetic fields in [10], and all the above theorems have been already obtained under the assumption that the magnetic field $b(x)$ satisfies (b) with $d = 0$. However the case of magnetic fields falling off at infinity remains untouched. The special emphasis here is laid on the case $d > 0$. If the magnetic field $b(x) > c > 0$ is uniformly positive, then the unperturbed operator $H = H_+$ is known to have zero as an isolated eigenvalue with infinite multiplicities. On the other hand, if $b(x)$ falls off at infinity, the essential spectrum of H occupies the interval $[0, \infty)$ (see [4]). This difference in spectral structure produces some serious difficulties. The argument in [10] makes an essential use of the fact that the bottom of essential spectrum of H is an isolated eigenvalue. Thus, in order to prove the theorems for the case $d > 0$, some new devices are further required in many states of the proof, although the basic idea is in principle the same as in the previous work. Much attention is now paid on the Lieb-Thirring estimate on the sum of negative eigenvalues of Pauli operators with nonconstant magnetic fields in relation to the magnetic Thomas-Fermi theory ([5], [6], [7], [8], [14]). The present work is motivated by these works.

We end the section by making a comment on the asymptotic formula in three dimensions. In [10], we have derived such a formula under the assumption that a magnetic field $B(x) = (0,0,b(x))$, $(x,z) \in \mathbf{R}^2 \times \mathbf{R}$, is directed along the positive z axis and its strength $b(x)$ satisfies (b) with $d = 0$. The derivation is based on the asymptotic formula in two dimensions. The argument there seems to extend to the case $0 < d < 2$ without any essential changes, if we make use of the two dimensional formula obtained in Theorem 1.3. The matter will be discussed in details elsewhere.

2. Preliminaries.

In this section we summarize two basic facts required for proving the main theorems. One is concerned with the spectral properties of unperturbed Pauli operators without electric fields and the other is with the perturbation theory for singular numbers of compact operators.

We consider the following operators

$$\tilde{H}_\pm = (-i\nabla - \tilde{A})^2 \mp \tilde{b} = \tilde{\Pi}_1^2 + \tilde{\Pi}_2^2 \mp \tilde{b} \quad \text{on} \quad L^2 = L^2(\mathbf{R}^2),$$

where $\tilde{A}(x) = (\tilde{a}_1(x), \tilde{a}_2(x))$, $\tilde{\Pi}_j = -i\partial_j - \tilde{a}_j$ and $\tilde{b}(x) = \nabla \times \tilde{A}$. As stated in the previous section, these operators can be rewritten as

$$\tilde{H}_\pm = (\tilde{\Pi}_1 \pm i\tilde{\Pi}_2)^*(\tilde{\Pi}_1 \pm i\tilde{\Pi}_2)$$

and hence they become nonnegative operators. If, in particular, $\tilde{b}$ satisfies

$$\tilde{b}(x) > \tilde{c} > 0, \tag{2.1}$$

then $\tilde{H}_- \geq \tilde{c}$ becomes a strictly positive operator. On the other hand, it is known ([1], [12]) that $\tilde{H}_+$ has zero as an eigenvalue with infinite multiplicities. If we choose the magnetic potential $\tilde{A}(x)$ in the divergenceless form

$$\tilde{A}(x) = (\tilde{a}_1(x), \tilde{a}_2(x)) = (-\partial_2\varphi, \partial_1\varphi)$$

for some real function $\varphi \in C^2(\mathbf{R}^2)$ obeying $\Delta\varphi = \tilde{b}$, then we have

$$\tilde{\Pi}_1 + i\tilde{\Pi}_2 = -ie^{-\varphi}(\partial_1 + i\partial_2)e^{\varphi}.$$

This implies that the zero eigenspace just coincides with the subspace

$$K_\varphi = \{u \in L^2 : u = he^{-\varphi} \quad \text{with} \quad h \in A(\mathbf{C})\},$$

where $A(\mathbf{C})$ denotes the class of analytic functions over the complex plane $\mathbf{C}$. Let $P_\varphi : L^2 \to L^2$ be the orthogonal projection on the zero eigenspace

K_φ of $\tilde{H}_+$. We write Q_φ for $\mathrm{Id} - P_\varphi$, Id being the identity operator. We also know ([4]) that the non-zero spectra of $\tilde{H}_\pm$ coincide with each other. Hence it follows that

$$Q_\varphi \tilde{H}_+ Q_\varphi \geq \tilde{c} Q_\varphi$$

in the form sense, $\tilde{c} > 0$ being as in (2.1).

We use the perturbation theory for singular numbers of compact operators as another basic tool to prove the theorems. We shall briefly explain several basic properties of singular numbers. We refer to [9] for details.

We denote by $N(S > \lambda)$ and $N(S < \lambda)$ the number of eigenvalues more and less than λ of self-adjoint operator S, respectively. Let $T : X \to X$ be a compact (not necessarily self-adjoint) operator acting on a separable Hilbert space X. We write $|T|$ for $\sqrt{TT^*}$. The singular number $\{s_n(T)\}$, $n \in \mathbf{N}$, of compact operator T is defined as the non-increasing sequence of eigenvalues of $|T|$ and it has the following properties: $s_n(T) = s_n(T^*)$ and

$$s_{n+m-1}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2) \tag{2.2}$$

for two compact operators T_1 and T_2. We now write

$$N(|T| > \lambda) = \#\{n \in \mathbf{N} : s_n(T) > \lambda\}, \quad \lambda > 0,$$

according to the above notation. If $T : X \to X$ is a compact self-adjoint operator, then

$$N(|T| > \lambda) = N(T > \lambda) + N(T < -\lambda), \quad \lambda > 0.$$

If, in particular, $T \geq 0$, it follows that $N(|T| > \lambda) = N(T > \lambda)$. The next proposition is repeatedly used throughout the entire discussion.

PROPOSITION 2.1. — *Assume that T_1 and T_2 are compact operators. Let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 + \lambda_2 = \lambda$. Then*

$$N(|T_1 + T_2| > \lambda) \leq N(|T_1| > \lambda_1) + N(|T_2| > \lambda_2).$$

If, in particular, $T_1, T_2 \geq 0$, then

$$N(T_1 + T_2 > \lambda) \leq N(T_1 > (1-\delta)\lambda) + N(T_2 > \delta\lambda),$$
$$N(T_1 - T_2 > \lambda) \geq N(T_1 > (1+\delta)\lambda) - 2N(T_2 > \delta\lambda)$$

for any $\delta > 0$ small enough.

Proof. — We prove only the third inequality. The other inequalities are immediate consequences of (2.2). To prove this, we decompose T_1 into $T_1 = (T_1 - T_2) + T_2$ and use (2.2) again. Then we have

$$\begin{aligned} N(T_1 > (1+\delta)\lambda) &\leq N(|T_1 - T_2| > \lambda) + N(T_2 > \delta\lambda) \\ &= N(T_1 - T_2 > \lambda) + N(T_1 - T_2 < -\lambda) + N(T_2 > \delta\lambda). \end{aligned}$$

Since $N(T_1 - T_2 < -\lambda) \le N(T_2 > \lambda)$, the inequality in question follows at once. $\square$

3. Upper bound; proof of Theorem 1.1.

In this section we complete the proof of Theorem 1.1 by reducing it to the proof of the three lemmas formulated below.

Before going into the proof, we first fix several notations used throughout the proof. We use a nonnegative smooth partition $\{\psi_0, \psi_\infty\}$ of unity normalized by $\psi_0(x)^2 + \psi_\infty(x)^2 = 1$, where $\psi_0 \in C_0^\infty(\mathbf{R}^2)$ has the property

$$\psi_0(x) = 1 \quad \text{for} \quad |x| \le 1, \qquad \psi_0(x) = 0 \quad \text{for} \quad |x| \ge 2.$$

Let d and m satisfy the assumption (1.1) in Theorem 1.1. Then we can take α as

$$\max(1/2,\ 1/m) < \alpha < 1/d. \tag{3.1}$$

We choose the magnetic potential $A(x) = (a_1(x), a_2(x))$ of the unperturbed operator $H = H_+$ in the divergenceless form

$$a_1(x) = -\partial_2 \varphi_0(x), \quad a_2(x) = \partial_1 \varphi_0(x),$$

where $\varphi_0 \in C^2(\mathbf{R}^2)$ satisfies the equation $\Delta \varphi_0 = b$. It is shown in the next section (see Proposition 4.1) that there exists a solution $\varphi_0(x)$ obeying the bound

$$|\varphi_0(x)| \le C_\rho \langle x \rangle^{2-\rho}$$

for any $\rho < d$. Set $g_0(x) = r^2/4$, $r = |x|$. Then g_0 satisfies

$$\Delta g_0 = g_0'' + g_0'/r = 1.$$

We define $\varphi_\alpha(x) = \varphi_\alpha(x; \lambda)$ by

$$\varphi_\alpha(x) = \varphi_0(x) + \eta \lambda^{\alpha d} \psi_\infty(\lambda^\alpha x) g_0(x) \tag{3.2}$$

for $\eta > 0$, and $A_\alpha(x) = A_\alpha(x; \lambda)$ and $b_\alpha(x) = b_\alpha(x; \lambda)$ by

$$A_\alpha(x) = (-\partial_2 \varphi_\alpha, \partial_1 \varphi_\alpha), \quad b_\alpha(x) = \Delta \varphi_\alpha = \nabla \times A_\alpha.$$

By definition, it follows that $A_\alpha(x) = A(x)$ for $|x| < \lambda^{-\alpha}$, and by assumption (b), we can take $\eta > 0$ so small that

$$b_\alpha(x) \ge c_\alpha \lambda^{\alpha d}, \quad c_\alpha > 0. \tag{3.3}$$

We are now in a position to prove the first main theorem.

Proof of Theorem 1.1. — Throughout the proof, δ, $0 < \delta \ll 1$, is fixed arbitrarily and λ, $0 < \lambda < \lambda_\delta \ll 1$, is assumed to be small enough.

To prove the theorem, we introduce the auxiliary operator $H_\alpha(V)$ as

$$H_\alpha(V) = H_\alpha - V, \quad H_\alpha = H_\alpha(\lambda) = (-i\nabla - A_\alpha)^2 - b_\alpha.$$

Let $\{\psi_{\alpha 0}, \psi_\alpha\}$ be the smooth partition of unity defined by

$$\psi_{\alpha 0}(x) = \psi_0(\lambda^\alpha x/2), \quad \psi_\alpha(x) = \psi_\infty(\lambda^\alpha x/2).$$

Then a simple computation yields the relation

$$H(V) = \psi_{\alpha 0}(H(V) - \Psi_\alpha)\psi_{\alpha 0} + \psi_\alpha(H(V) - \Psi_\alpha)\psi_\alpha$$

in the form sense, where

$$\Psi_\alpha(x) = |\nabla\psi_{\alpha 0}(x)|^2 + |\nabla\psi_\alpha(x)|^2 = O(\lambda^{2\alpha}) = o(\lambda), \quad \lambda \to 0.$$

This relation is often called the IMS localization formula ([4]). Recall that $A(x) = A_\alpha(x)$ for $|x| < \lambda^{-\alpha}$. Hence

$$H(V) = \psi_{\alpha 0}(H_\alpha(V) - \Psi_\alpha)\psi_{\alpha 0} + \psi_\alpha(H(V) - \Psi_\alpha)\psi_\alpha.$$

If $x \in \operatorname{supp}\psi_\alpha$, then $V(x) = O(\lambda^{m\alpha}) = o(\lambda)$ by (3.1). Thus we obtain

$$N(H(V){<}-\lambda) \le N(H_{\alpha D}(V){<}-(1-\delta)\lambda) \le N(H_\alpha(V){<}-(1-\delta)\lambda), \tag{3.4}$$

where $H_{\alpha D}(V)$ denotes the operator $H_\alpha(V)$ over the domain $\{|x| < \lambda^{-\alpha}\}$ with zero Dirichlet conditions. We can also obtain

$$N(H_\alpha(V) < -(1+\delta)\lambda) \le N(H(V) < -\lambda) \tag{3.5}$$

by making use of a similar relation. By (3.3), H_α has zero as an isolated eigenvalue with infinite multiplicities. We denote by $P_\alpha = P_\alpha(\lambda) : L^2 \to L^2$ the orthogonal projection on the zero eigenspace of H_α and we write $Q_\alpha = \mathrm{Id} - P_\alpha$. Then we have

$$Q_\alpha H_\alpha Q_\alpha \ge c_\alpha \lambda^{\alpha d} Q_\alpha, \quad c_\alpha > 0.$$

If we use the simple form inequality

$$P_\alpha V Q_\alpha + Q_\alpha V P_\alpha \le c P_\alpha V^2 P_\alpha + Q_\alpha/c, \quad c > 0,$$

then we get

$$H_\alpha(V) \ge P_\alpha(-V - c\lambda^{-\alpha d}V^2)P_\alpha + Q_\alpha(H_\alpha(V) - \lambda^{\alpha d}/c)Q_\alpha$$

for any $c > 0$ and hence it follows from (3.4) that

$$\begin{aligned} N(H(V) < -\lambda) \le{}& N(P_\alpha(V + c\lambda^{-\alpha d}V^2)P_\alpha > (1-\delta)\lambda) \\ &+ N(Q_\alpha(H_\alpha(V) - \lambda^{\alpha d}/c)Q_\alpha < -(1-\delta)\lambda). \end{aligned}$$

We can take $c \gg 1$ so large that

$$Q_\alpha(H_\alpha(V) - \lambda^{\alpha d}/c)Q_\alpha \geq Q_\alpha(H_\alpha/2 - V + c_2\lambda^{\alpha d})Q_\alpha$$

for some $c_2 > 0$. Since $\alpha d < 1$ strictly, this implies that

$$N(Q_\alpha(H_\alpha(V) - \lambda^{\alpha d}/c)Q_\alpha < -(1-\delta)\lambda) \leq N(H_\alpha - 2V < -2c_3\lambda^{\alpha d})$$

for some $c_3 > 0$, and hence we have

$$N(H_\alpha - 2V < -2c_3\lambda^{\alpha d}) \leq N(H(2V) < -c_3\lambda^{\alpha d})$$

by the same argument as used to prove (3.5), where $H(2V) = H - 2V$. Thus we obtain

$$\begin{aligned}(3.6)\quad & N(H(V) < -\lambda) \\ & \leq N(H(2V) < -c_3\lambda^{\alpha d}) + N(P_\alpha(V + c\lambda^{-\alpha d}V^2)P_\alpha > (1-\delta)\lambda)\end{aligned}$$

for some $c,\ c_3 > 0$.

We here denote by $\mathcal{V}$ the set of all real functions $V(x)$ satisfying the assumption (V), $m > 0$ being fixed in assumption (V). We now accept the following three lemmas as proved.

LEMMA 3.1. — *Let $0 < \sigma < 1/m$. Assume that $U(x) = U(x;\lambda) \geq 0$ is a bounded function (uniformly in λ) with support in $\{|x| < \lambda^{-\sigma}\}$. Then*

$$N(P_\alpha U P_\alpha > \lambda^L) = o(\lambda^{(d-2)/m}), \quad \lambda \to 0,$$

for any $L > 0$.

LEMMA 3.2. — *Assume that $W \in \mathcal{V}$. Let $H(W) = H - W$. Then*

$$N(H(W) < -\lambda) = O(\lambda^{(d-2)/m}), \quad \lambda \to 0.$$

LEMMA 3.3. — *Assume that $W \in \mathcal{V}$. Then*

$$N(P_\alpha W P_\alpha > \lambda) \leq (2\pi)^{-1}\int_{W > (1-\delta)\lambda} b(x)\,dx + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

We apply these three lemmas to complete the proof of the theorem. Since $\alpha d < 1$ strictly, we first obtain

$$N(H(2V) < -c_3\lambda^{\alpha d}) = o(\lambda^{(d-2)/m})$$

by Lemma 3.2. Next we choose σ as $(\alpha d + 1)/2m < \sigma < 1/m$, so that

$$\lambda^{-\alpha d}V(x)^2 = O(\lambda^{-\alpha d + 2\sigma m}) = o(\lambda)$$

for $|x| > \lambda^{-\sigma}$. Hence

$$N(P_\alpha V^2 P_\alpha > \lambda^{\alpha d+1}/c) = o(\lambda^{(d-2)/m}) \tag{3.7}$$

by Lemma 3.1. Thus, by Proposition 2.1 and Lemma 3.3, the theorem follows from (3.6). □

The three lemmas above remain unproved. We prove Lemma 3.1 in Section 4, and Lemmas 3.2 and 3.3 in Section 7.

4. Proof of Lemma 3.1.

The aim of this section is to prove Lemma 3.1. The proof is based on the stationary phase method and it requires two preparatory results. The first result can be easily proved by use of the Fourier series expansion. We will give its proof at the end of this section.

PROPOSITION 4.1. — *Let $0 \le d < 2$. Assume that $b \in C^1(\mathbf{R}^2)$ satisfies $|b(x)| = O(r^{-d})$ as $r = |x| \to \infty$. Then there exists a real solution $\varphi_0 \in C^2(\mathbf{R}^2)$ to equation $\Delta\varphi_0 = b$ with bound*

$$\varphi_0(x) = \begin{cases} O(r^{2-d}), & 0 < d < 2, \quad d \ne 1, \\ O(r^{2-d}\log r), & d = 0, 1. \end{cases}$$

Remark. — We do not assume the decay condition for the first derivatives of $b(x)$ but the C^1-smoothness of $b(x)$ is required to obtain $\varphi_0(x) \in C^2(\mathbf{R}^2)$.

Before formulating the second preparatory result, we recall the notations

$$\tilde{H}_+ = (-i\nabla - \tilde{A})^2 - \tilde{b}, \quad \tilde{A} = (-\partial_2\varphi, \partial_1\varphi), \quad \tilde{b} = \nabla \times \tilde{A}$$

in Section 2. We denote by $P_\varphi : L^2 \to L^2$ the orthogonal projection on the zero eigenspace of $\tilde{H}_+$, which coincides with

$$K_\varphi = \{u \in L^2 : u = he^{-\varphi} \ \text{ with } \ h \in A(\mathbf{C})\},$$

where $A(\mathbf{C})$ is the class of analytic functions over the complex plane.

LEMMA 4.1. — *Assume that $U(x) \ge 0$ is a bounded function with compact support. Let $\varphi_j \in C^2(\mathbf{R}^2)$, $1 \le j \le 2$, be real functions. Write P_j for P_{φ_j} and K_j for K_{φ_j}. If $\varphi_1(x) \le \varphi_2(x)$, then*

$$N(P_1 U P_1 > \mu) \le N(P_2 U P_2 > \mu/\gamma), \quad \mu > 0,$$

where

$$\gamma = \max_{x \in \operatorname{supp} U} \exp(2\omega(x)), \quad \omega(x) = \varphi_2(x) - \varphi_1(x) \geq 0.$$

Proof. — This lemma has been already verified in [10] (Lemma 2.12). However, for completeness, we here prove the lemma by repeating the argument there. The proof is done by a simple application of the min-max principle. Let $\mathcal{L}_n$, $n \geq 0$, be the totality of subspaces in L^2 with dimension not exceeding n. Each element of $\mathcal{L}_n$ is expressed as a subspace $F_n = S(f_1, \ldots, f_n)$ spanned by the linear hulls of $f_j \in L^2$, $1 \leq j \leq n$. We define the injective mapping $\tau : \mathcal{L}_n \to \mathcal{L}_n$ by

$$\tau(F_n) = G_n = S(g_1, \ldots, g_n)$$

with $g_j = \exp(-\omega) f_j \in L^2$ and we denote the range of such a mapping as $\mathcal{M}_n = \tau(\mathcal{L}_n)$. Let $w = h \exp(-\varphi_1) \in K_1$ with $h \in A(\mathbf{C})$. If w belongs to the orthogonal complement $G_n^\perp$ of $G_n = \tau(F_n)$ with $F_n = S(f_1, \ldots, f_n)$, then $\tilde{w} = h \exp(-\varphi_2) \in K_2$ with the same $h \in A(\mathbf{C})$ belongs to $F_n^\perp$, and also it follows that $(\tilde{w}, \tilde{w}) \leq (w, w)$ and

$$(Uw, w)/(w, w) \leq \gamma\, (U\tilde{w}, \tilde{w})/(\tilde{w}, \tilde{w}),$$

where (,) stands for the scalar product in the space L^2. Thus we have

$$\min_{G_n \in \mathcal{M}_n} \max_{u \in K_1 \cap G_n^\perp} (Uu, u)/(u, u) \leq \gamma \min_{F_n \in \mathcal{L}_n} \max_{v \in K_2 \cap F_n^\perp} (Uv, v)/(v, v). \tag{4.1}$$

We now denote by $\lambda_n(T)$ the n-th eigenvalue of nonnegative compact operator T. By the min-max principle,

$$\lambda_{n+1}(P_j U P_j) = \min_{F_n \in \mathcal{L}_n} \max_{u \in K_j \cap F_n^\perp} (Uu, u)/(u, u)$$

for $n \geq 0$. Hence, if we take account of the inclusion relation $\mathcal{M}_n \subset \mathcal{L}_n$, the lemma follows from (4.1). □

We shall prove Lemma 3.1 in question.

Proof of Lemma 3.1. — For the sake of brevity, we prove the lemma only for the case $d > 0$. Let $0 < \sigma < 1/m$ and let $U(x) \geq 0$ be as in the lemma. We may assume that $U = U(r)$ is a function of $r = |x|$ only. Let $\varphi_\alpha(x)$ be defined by (3.2). We take ρ, $0 < \rho < d$, as $(2-\rho)\sigma < (2-d)/m$. It follows from Proposition 4.1 and (3.2) that

$$\varphi_\alpha(x) \leq \varphi(r) \equiv c\Big((1 + r^2)^{(2-\rho)/2} + \lambda^{\alpha d} r^2\Big)$$

for some $c > 0$. Hence we have

$$N(P_\alpha U P_\alpha > \lambda^L) \leq N(P_\varphi U P_\varphi > \lambda^L/\gamma),$$

by Lemma 4.1, where $\gamma = \exp(c_1\lambda^{-(2-\rho)\sigma})$ with some $c_1 > 0$.

Let (r, θ) be the polar coordinate system over the plane. Since $U(r)$ and $\varphi(r)$ are both spherically symmetric, the compact operator $P_\varphi U P_\varphi : L^2 \to L^2$ has

$$u_n(x) = (x_1 + ix_2)^n \exp(-\varphi(r)) = r^n \exp(in\theta) \exp(-\varphi(r)), \quad n \in \mathbf{N} \cup \{0\},$$

as an orthogonal system of eigenfunctions (complete in the subspace K_φ), and the corresponding eigenvalue λ_n is given by $\lambda_n = s_n/e_n$, where

$$e_n = (u_n, u_n) = 2\pi \int_0^\infty r^{2n+1} \exp(-2\varphi(r))\, dr$$

$$s_n = (Uu_n, u_n) = 2\pi \int_0^\infty U(r) r^{2n+1} \exp(-2\varphi(r))\, dr.$$

We now choose ν as

$$(2-\rho)\sigma < \nu < (2-d)/m < (2-\rho)\alpha d/\rho.$$

We assert that

$$\gamma\lambda^{-L}\lambda_n = \exp(c_1\lambda^{-(2-\rho)\sigma})\lambda^{-L}\lambda_n < 1 \tag{4.2}$$

for $n > \lambda^{-\nu}$. If this is verified, then it follows that

$$N(P_\alpha U P_\alpha > \lambda^L) = O(\lambda^{-\nu}) = o(\lambda^{(d-2)/m}), \quad \lambda \to 0,$$

and hence the lemma is obtained.

The assertion above is proved by use of the stationary phase method. We write e_n as

$$e_n = 2\pi \int_0^\infty \exp(2f_n(r))\, dr,$$

where

$$f_n(r) = (n + 1/2) \log r - \varphi(r).$$

We define $q(r)$ and $p(r)$ as

$$q(r) = \Delta\varphi = \varphi''(r) + \varphi'(r)/r, \quad p(r) = r\varphi'(r) = \int_0^r tq(t)\, dt.$$

By a direct calculation, we see that $q(r) > 0$ is strictly positive and it behaves like

$$q(r) \sim r^{-\rho} + \lambda^{\alpha d}, \quad r \to \infty,$$

uniformly in λ, where the notation $f(r) \sim g(r)$ means that $f(r)/g(r)$ and $g(r)/f(r)$ are both bounded uniformly in $r \gg 1$. Hence $p(r)$ behaves like

$$p(r) \sim r^{2-\rho} + \lambda^{\alpha d} r^2, \quad r \to \infty.$$

We look at the stationary point of $f_n(r)$ with

$$f_n'(r) = ((n+1/2) - p(r))/r = 0.$$

Since $p(r)$ is a strictly increasing function, there exists a unique root r_n to equation

$$p(r_n) = \int_0^{r_n} tq(t)\,dt = n + 1/2.$$

The value of $f_n''(r)$ at stationary point $r = r_n$ is calculated as

$$f_n''(r_n) = -p'(r_n)/r_n = -q(r_n) < 0.$$

Thus the function $f_n(r)$ attains its maximum at $r = r_n$. When $n \gg 1$ is large enough, r_n behaves like

$$r_n \sim \begin{cases} n^{1/(2-\rho)}, & \lambda^{-\nu} < n \le \lambda^{-(2-\rho)\alpha d/\rho}, \\ \lambda^{-\alpha d/2} n^{1/2}, & n > \lambda^{-(2-\rho)\alpha d/\rho}. \end{cases}$$

We estimate e_n from below and s_n from above when $n > \lambda^{-\nu}$. The value e_n is evaluated as

$$e_n \ge 2\pi \exp(2f_n(r_n)) \int_{r_n}^{r_n+1} \exp(2(f_n(r) - f_n(r_n)))\,dr.$$

If $r_n < r < r_n + 1$, then we have

$$f_n(r) - f_n(r_n) = \int_{r_n}^{r} t^{-1}(p(r_n) - p(t))\,dt \ge -c_2 q(r_n) \ge -c_2$$

for some $c_2 > 0$. Hence

$$e_n \ge c_3 \exp(2f_n(r_n)), \quad c_3 > 0.$$

Next we evaluate s_n from above. We write it as

$$s_n = 2\pi \exp(2f_n(r_n)) \int_0^{\lambda^{-\sigma}} U(r) \exp(2(f_n(r) - f_n(r_n)))\,dr.$$

If $0 < r < \lambda^{-\sigma}$, then $r \ll r_n$ for $n > \lambda^{-\nu}$ and

$$f_n(r) - f_n(r_n) = -\int_r^{r_n} t^{-1}(p(r_n) - p(t))\,dt \le -\int_{\eta r_n/3}^{\eta r_n/2} t^{-1}(p(r_n) - p(t))\,dt$$

for $\eta > 0$ small enough. We can take $\eta > 0$ so small that

$$f_n(r) - f_n(r_n) \le -(p(r_n) - p(\eta r_n/2)) \log 3/2 \le -c_4 n$$

for some $c_4 > 0$. Thus we have

$$s_n \le c_5 \lambda^{-\sigma} \exp(-c_4 n) \exp(2f_n(r_n))$$

and hence

$$\lambda_n = s_n/e_n \leq c_6\lambda^{-\sigma}\exp(-c_4 n).$$

Since $n > \lambda^{-\nu} > \lambda^{-(2-\rho)\sigma}$, it follows that

$$\gamma\lambda^{-L}\lambda_n \leq c_6\exp(c_2\lambda^{-(2-\rho)\sigma})\exp(-c_4 n)\lambda^{-(\sigma+L)} \to 0, \quad \lambda \to 0.$$

This proves the assertion (4.2) and the proof of the lemma is complete. □

In the previous section, we have used the auxiliary operator H_α, $\alpha > 1/2$ (chosen close enough to $1/2$) for spatial localization, so that (3.4) holds true. By (3.3), the operator H_α has at least $O(\lambda^{\alpha d})$ as a spectral gap above zero eigenvalue. However this width is not wide enough to prove the remaining two lemmas (Lemmas 3.2 and 3.3). We here introduce another auxiliary operator with a wider spectral gap (this gap property will be used in Section 6 to control the commutator of the orthogonal projection onto its zero eigenspace and multiplication operators). The precise definition requires some new notations. By assumption (1.2), we can take β so close to $1/m$ that

$$1/m < \beta < \min(\alpha,\ 2/m,\ (3d+2)/4md) < 2/md. \tag{4.3}$$

We define $\varphi_\beta(x) = \varphi_\beta(x;\lambda)$ by

$$\varphi_\beta(x) = \varphi_0(x) + \eta\lambda^{\beta d}\psi_\infty(\lambda^\beta x)g_0(x)$$

in a way similar to $\varphi_\alpha(x)$ (see (3.2)). The auxiliary operator in question is defined by

$$H_\beta = (-i\nabla - A_\beta)^2 - b_\beta, \tag{4.4}$$

where

$$A_\beta(x) = (-\partial_2\varphi_\beta, \partial_1\varphi_\beta), \quad b_\beta(x) = \Delta\varphi_\beta = \nabla\times A_\beta.$$

We can take the same $\eta > 0$ as in (3.2) so small that

$$b_\beta(x) \geq c_\beta\lambda^{\beta d}, \quad c_\beta > 0. \tag{4.5}$$

This means that H_β has zero as an isolated eigenvalue with infinite multiplicities and that it has at least $O(\lambda^{\beta d})$ as a spectral gap, which is wider than that of H_α. Since β may be less than $1/2$, the estimate of type (3.4), in turn, does not hold true in general for H_β. Lemma 4.2 below enables us to evaluate the upper bound instead.

We now denote by P_β the orthogonal projection onto the zero eigenspace of H_β and write Q_β for $\mathrm{Id} - P_\beta$. By definition, φ_β satisfies $\varphi_\alpha(x) \leq \varphi_\beta(x)$ and

$$\varphi_\alpha(x) = \varphi_\beta(x), \quad |x| < \lambda^{-\beta}.$$

Thus we obtain the following lemma as an immediate consequence of Lemma 4.1, which plays an important role in proving Lemmas 3.2 and 3.3. In fact, the lemma reduces the proof of these two lemmas to evaluating the quantity $N(P_\beta W P_\beta > \lambda)$ in place of $N(P_\alpha W P_\alpha > \lambda)$.

LEMMA 4.2. — *If $U(x) \geq 0$ is a bounded function with support in $\{|x| < \lambda^{-\beta}\}$, then*

$$N(P_\alpha U P_\alpha > \mu) \leq N(P_\beta U P_\beta > \mu)$$

for any $\mu > 0$.

The next lemma is proved in exactly the same way as in the proof of Lemma 3.1.

LEMMA 4.3. — *Let $0 < \sigma < 1/m$. Assume again that $U(x) = U(x;\lambda) \geq 0$ is a bounded function (uniformly in λ) with support in $\{|x| < \lambda^{-\sigma}\}$. Then*

$$N(P_\beta U P_\beta > \lambda^L) = o(\lambda^{(d-2)/m}), \quad \lambda \to 0,$$

for any $L > 0$.

We end the section by proving Proposition 4.1.

Proof of Proposition 4.1. — For the sake of brevity, we prove the proposition only for the case $d > 0$. If $f \in C_0^1(\mathbf{R}^2)$ has compact support, then

$$\psi(x) = (2\pi)^{-1} \int \log|x-y|\, f(y)\, dy$$

satisfies $\Delta\psi = f$ and obeys the bound $\psi(x) = O(\log r)$ as $r \to \infty$. Thus we may assume that $b(x)$ vanishes in a neighborhood of the origin. We expand $b(x)$ as

$$b(x) = \sum_{n=-\infty}^{\infty} b_n(r) \exp(in\theta)$$

by the Fourier series. By assumption, the coefficients $b_n(r)$ satisfy $b_n(r) = O(r^{-d})$ uniformly in n. We also expand the solution $\varphi_0(x)$ as

$$\varphi_0(x) = \sum_{n=-\infty}^{\infty} g_n(r) \exp(in\theta). \tag{4.6}$$

Then $g_n(r)$ obeys the equation

$$g_n'' + g_n'/r - n^2 g_n/r^2 = b_n. \tag{4.7}$$

If $n = 0$, then $(rg_0')' = rb_0$ and hence the equation is easily seen to have a solution behaving like $g_0(r) = O(r^{2-d})$. If $n \geq 1$, the homogeneous equation has $f_{\pm n}(r) = r^{\pm n}$ as a linearly independent solution and the Wronskian $w_n(r) = W(f_{+n}, f_{-n})$ is calculated as $w_n(r) = -2nr^{-1}$. If we set

$$g_1(r) = \int_0^r \frac{f_{+1}(s)b_1(s)}{w_1(s)}\, ds f_{-1}(r) - \int_0^r \frac{f_{-1}(s)b_1(s)}{w_1(s)}\, ds f_{+1}(r), \quad 0 < d \leq 1,$$

$$g_1(r) = \int_0^r \frac{f_{+1}(s)b_1(s)}{w_1(s)}\, ds f_{-1}(r) + \int_r^\infty \frac{f_{-1}(s)b_1(s)}{w_1(s)}\, ds f_{+1}(r), \quad 1 < d < 2,$$

and

$$g_n(r) = \int_0^r \frac{f_{+n}(s)b_n(s)}{w_n(s)}\, ds\, f_{-n}(r) + \int_r^\infty \frac{f_{-n}(s)b_n(s)}{w_n(s)}\, ds\, f_{+n}(r)$$

for $n \geq 2$, then $g_n(r)$, $n \geq 1$, yield a solution to equation (4.7) and obey the bounds

$$g_1(r) = \begin{cases} O(r^{2-d}), & 0 \leq d < 2,\ d \neq 1, \\ O(r \log r), & d = 1, \end{cases}$$

and $g_n(r) = n^{-2}O(r^{2-d})$ uniformly in $n \geq 2$. Similar solutions can be also constructed for the case $n \leq -1$. Thus the function $\varphi_0(x)$ defined by series (4.6) is absolutely convergent and it belongs to $C^2(\mathbf{R}^2)$ by the regularity property for solutions to elliptic equations. Thus we can obtain the desired solution and the proof is complete. □

Remark 4.1. — If $b(x)$ satisfies the assumption in Proposition 4.1 for $d > 2$, then the solution φ_0 can be shown to obey $\varphi_0 = O(\log r)$ and hence it follows that $\exp(\varphi_0(x))$ is at most of polynomial growth. This implies that

$$\dim \operatorname{Ker} H = \dim K_{\varphi_0} < \infty.$$

The result is known as the Ahanorov-Casher theorem ([1]) when $b(x)$ has compact support.

5. Min-Max principle.

The present and next sections are devoted to several preparatory lemmas required for proving Lemmas 3.2 and 3.3. The proof of these lemmas is based on the min-max principle. However this principle does not directly apply to the present problem. The difficulty comes from the fact that the field $b(x)$ and the potential $V(x)$ have the decaying order different from each other. This makes it difficult to control the field $b(x)$ by local approximation of constant field. In fact, let Q be a cube of side R, R being fixed, with center y_0, $|y_0| \gg 1$. If we approximate $b(x)$ over Q by the value $b(y_0)$ at the center, then we have the error order $O(|y_0|^{-d-1})$ dominant to $V(y_0) \sim |y_0|^{-m}$, provided that $m > d+1$. On the other hand, both the field $b(x)$ and the potential $V(x)$ make a contribution to the leading term of the asymptotic formula in Theorem 1.3. Thus the local approximation by uniform magnetic fields works only for potentials decaying slightly faster than $b(x)$. The aim of the present section is to establish the upper bound (Lemma 5.2) for such a special class of potentials by use of the result due to Colin de Verdière [3] (Proposition 5.1 below) about the number of eigenvalues in a cube with constant field.

Let H_β be defined by (4.4). We write $H_\beta(W) = H_\beta - W$. By (4.3), we can take q as

$$\beta d < q < (3d+2)/4m < 2/m, \tag{5.1}$$

so that $W(x)^q \sim |x|^{-mq}$, $mq > d$ falls off slightly faster than $|x|^{-d}$ for $W \in \mathcal{V}$. We further set

$$\sigma_0 = \max\Big(q/2,\ \beta d/mq,\ 3q/(2+2mq),\ 4q/(3d+2)\Big). \tag{5.2}$$

Then σ_0 satisfies $\sigma_0 < 1/m$ by (5.1). The next lemma plays an important role in the future argument.

LEMMA 5.1. — *Let q be as (5.1) and let σ be such that*

$$\sigma_0 < \sigma < 1/m, \tag{5.3}$$

for σ_0 as above. Assume that $U(x) = U(x;\lambda)$, $U \in C^1(\mathbf{R}^2)$, is a real function with support in $\{|x| > \lambda^{-\sigma}\}$ and it satisfies

$$|U(x)| \le C\langle x\rangle^{-mq}, \quad |\nabla U(x)| \le C\langle x\rangle^{-mq-1}$$

for some $C > 0$ independent of $\lambda > 0$. Then, for any $\delta > 0$ small enough, there exists λ_δ such that

$$N(H_\beta(U) < -\lambda^q) \le (2\pi)^{-1}\int_{U>(1-\delta)\lambda^q} b(x)\,dx + \delta\lambda^{(d-2)/m}$$

$$N(H_\beta(U) < -\lambda^q) \ge (2\pi)^{-1}\int_{U>(1+\delta)\lambda^q} b(x)\,dx - \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

The proof of the lemma uses the min-max principle and is based on the following proposition due to [3].

PROPOSITION 5.1. — *Let Q_R be a cube with side R and let*

$$S_B = (-i\nabla - \hat{A})^2, \quad \hat{A}(x) = (-Bx_2/2, Bx_1/2),$$

be the Schrödinger operator with constant magnetic field $B = \nabla \times \hat{A} > 0$. Consider S_B as an operator acting on the space $L^2(Q_R)$ under zero Dirichlet conditions and denote by $N_D(S_B < \mu; Q_R)$, $\mu > 0$, the number of eigenvalues less than μ. Then there exists $c > 0$ independent of μ, R and Λ, $0 < \Lambda < R/2$, such that:

(1) $N_D(S_B < \mu; Q_R) \le (2\pi)^{-1} B\,|Q_R|\,F(\mu/B)$

(2) $N_D(S_B < \mu; Q_R) \ge (2\pi)^{-1}(1-\Lambda/R)^2 B\,|Q_R|\,F((\mu - c\Lambda^{-2})/B)$,

where $|Q_R| = R^2$ is the measure of cube Q_R and

$$F(\mu) = \#\{n \in \mathbf{N} \cup \{0\} : 2n+1 \le \mu\}.$$

Proof of Lemma 5.1. — The proof is divided into several steps. Throughout the proof, δ, $0 < \delta \ll 1$, is again fixed arbitrarily and λ, $0 < \lambda < \lambda_\delta \ll 1$, is assumed to be small enough.

(1) We first prove the upper bound. Let q and σ be as in the lemma. We set

$$\nu_0 = \min\Big(\sigma,\ (mq+1)\sigma - q,\ (d+1)\sigma - q,\ (3d+2)\sigma/4 - q/2\Big).$$

Then ν_0 satisfies $\nu_0 > q/2$ by (5.3). Hence we can choose ν so that

$$q/2 < \nu < \nu_0. \tag{5.4}$$

Let $\{Q_k\}$, $k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$, be a family of disjoint open cubes with side $\lambda^{-\nu}$, which covers the whole plane $\mathbf{R}^2 = \cup_k \overline{Q}_k$, $\overline{Q}_k$ being the closure of Q_k. We denote by $y_k \in \mathbf{R}^2$ the center of cube Q_k. We further introduce another family of cubes $\tilde{Q}_k$, $\tilde{Q}_k \supset Q_k$, with the same center y_k and side $(1+\delta)\lambda^{-\nu}$. We restrict the operator $H_\beta(U)$ to each cube domain $\tilde{Q}_k$. Let $H_{kD}(U)$ be the self-adjoint realization in $L^2(\tilde{Q}_k)$ of $H_\beta(U)$ under zero Dirichlet conditions and denote the number of negative eigenvalues less than $-\mu$ of $H_{kD}(U)$ as

$$N_k(\mu) = N(H_{kD}(U) < -\mu), \quad \mu > 0.$$

Since $\nu > q/2$ by (5.4), we have by use of the IMS localization formula that

$$N(H_\beta(U) < -\lambda^q) \le \sum_k N_k((1-\delta)\lambda^q). \tag{5.5}$$

(2) We again write $A_\beta(x)$ for the magnetic potential of H_β and denote by $b_k = b_\beta(y_k)$ the value of field $b_\beta(x) = \nabla \times A_\beta(x)$ at center $y_k = (y_{1k}, y_{2k})$ of cube Q_k. We set

$$r_k(x) = (r_{1k}(x), r_{2k}(x)) = A_\beta(x) - \hat{A}_k(x), \quad \hat{A}_k(x) = (-b_k x_2/2, b_k x_1/2),$$

for $x \in \tilde{Q}_k$. Then r_k satisfies the relation

$$\nabla \times r_k = b_\beta(x) - b_k.$$

Hence $A_\beta(x)$ takes the form

$$A_\beta(x) = \hat{A}_k(x) + r_k(x) = \hat{A}_k(x) + \nabla g_k(x) + e_k(x), \quad x \in \tilde{Q}_k,$$

with some real function $g_k \in C^1(\mathbf{R}^2)$, where $e_k = (e_{1k}(x), e_{2k}(x))$ is given by

$$e_{1k}(x) = -\int_0^1 s(x_2 - y_{2k})\Big(b_\beta(y_k + s(x - y_k)) - b_k\Big)\, ds$$

$$e_{2k}(x) = \int_0^1 s(x_1 - y_{1k})\Big(b_\beta(y_k + s(x - y_k)) - b_k\Big)\, ds.$$

We define the operator S_{kD} by

$$S_{kD} = (-i\nabla - \hat{A}_k)^2, \quad \hat{A}_k(x) = (-b_k x_2/2, b_k x_1/2), \tag{5.6}$$

and $T_{kD}(U)$ by

$$T_{kD}(U) = \exp(-ig_k) H_{kD}(U) \exp(ig_k) = (-i\nabla - \hat{A}_k - e_k)^2 - b_\beta - U.$$

These operators act on $L^2(\tilde{Q}_k)$ under zero Dirichlet conditions. By definition, S_{kD} has the constant magnetic field $b_k = \nabla \times \hat{A}_k$, while $T_{kD}(U)$ has the magnetic potential $\hat{A}_k + e_k$ and is unitarily equivalent to $H_{kD}(U)$, so that

$$N_k(\mu) = N(H_{kD}(U) < -\mu) = N(T_{kD}(U) < -\mu).$$

(3) Let σ be as in the lemma. We set

$$X = \{k \in \mathbf{Z} \times \mathbf{Z} : \tilde{Q}_k \cap G \neq \emptyset\},$$

where $G = \{x \in \mathbf{R}^2 : \lambda^{-\sigma} < |x| < \lambda^{-\beta}/2\}$. We can easily see that

$$N_k((1-\delta)\lambda^q) = 0, \quad k \notin X. \tag{5.7}$$

We further divide X into two sets

$$X_1 = \{k \in X : U_k > (1-2\delta)\lambda^q\}, \quad X_2 = \{k \in X : U_k \leq (1-2\delta)\lambda^q\},$$

where $U_k = U(y_k)$ denotes the value of $U(x)$ at the center y_k of cube Q_k. By (5.4), we have

$$|U(x) - U_k| = O(\lambda^{(mq+1)\sigma})|x - y_k| = O(\lambda^{(mq+1)\sigma-\nu}) = o(\lambda^q)$$

for $x \in \tilde{Q}_k$ and hence $U(x) < (1-\delta)\lambda^q$ for $x \in \tilde{Q}_k$ with $k \in X_2$. Thus

$$N_k((1-\delta)\lambda^q) = 0, \quad k \in X_2. \tag{5.8}$$

If $k \in X_1$, then

$$\tilde{Q}_k \subset \{x \in \mathbf{R}^2 : \lambda^{-\sigma}/c < |x| < c\lambda^{-1/m}\}$$

for some $c > 1$. Hence $b_\beta(x)$ coincides with $b(x)$ on $\tilde{Q}_k$ by definition, and we have

$$|\nabla b_\beta(x)| = |\nabla b(x)| = O(\lambda^{(d+1)\sigma})$$

by assumption (b). This, together with (5.4), implies that

$$|b(x) - b_k| = O(\lambda^{(d+1)\sigma})|x - y_k| = O(\lambda^{(d+1)\sigma-\nu}) = o(\lambda^q)$$
$$|e_k(x)| = O(\lambda^{(d+1)\sigma})|x - y_k|^2 = O(\lambda^{(d+1)\sigma-2\nu}).$$

Thus $T_{kD}(U)$ obeys the form inequality

$$T_{kD}(U) \geq (1-\varepsilon)S_{kD} - b_k - U_k - \delta\lambda^q - \varepsilon^{-1}O(\lambda^{2(d+1)\sigma-4\nu}), \quad k \in X_1,$$

for any $\varepsilon > 0$ small enough, where S_{kD} is defined by (5.6) and it has the constant magnetic field b_k. We take ε as $\varepsilon = \delta\lambda^{q-\sigma d} \ll 1$, $q > \beta d > \sigma d$, so that

$$\varepsilon^{-1}O(\lambda^{2(d+1)\sigma-4\nu}) = O(\lambda^{(3d+2)\sigma-4\nu-q}) = o(\lambda^q)$$

by (5.4) again. Hence we have

$$N_k((1-\delta)\lambda^q) \leq N(S_{kD} < \tau_k(\lambda)),$$

where

$$\tau_k(\lambda) = (b_k + U_k - \lambda^q + 2\delta\lambda^q)/(1 - \delta\lambda^{q-\sigma d}).$$

When $k \in X_1$, the values b_k and U_k satisfy $\lambda^{\beta d}/c < b_k < c\lambda^{\sigma d}$, $c > 1$, and $U_k = O(\lambda^{mq\sigma})$. Since $\beta d < mq\sigma < q$ by (5.3), it follows that

$$\tau_k(\lambda) = b_k + U_k - \lambda^q + \delta O(\lambda^q) < 2b_k.$$

This, together with Proposition 5.1 (1), yields

$$N_k((1-\delta)\lambda^q) \leq (2\pi)^{-1}b_k\,|\tilde{Q}_k| = (2\pi)^{-1}b_k\,|Q_k|(1 + O(\delta))$$

for $k \in X_1$. As is easily seen,

$$Q_k \subset \{x \in \mathbf{R}^2 : U(x) > (1-3\delta)\lambda^q\}$$

for $k \in X_1$, and also we have

$$\int_{U>\lambda^q} b(x)\,dx = O(\lambda^{(d-2)/m}).$$

Thus we combine (5.5), (5.7) and (5.8) to obtain the desired upper bound

$$N(H_\beta(U) < -\lambda^q) \le (2\pi)^{-1} \int_{U>(1-3\delta)\lambda^q} b(x)\,dx + \delta O(\lambda^{(d-2)/m}).$$

(4) To prove the lower bound, we again apply the min-max principle by restricting the operator $H_\beta(U)$ to cube domain Q_k. If we use Proposition 5.1 (2) with $\Lambda = \delta\lambda^{-\nu}$, then the lower bound can be obtained in almost the same way as the upper one. Thus the proof of the lemma is completed. □

LEMMA 5.2. — *Let q be as in (5.1). Assume that $W \in \mathcal{V}$ satisfies $W(x) \ge C\langle x\rangle^{-m}$ for some $C > 0$. Then*

$$N(P_\beta W^q P_\beta > \lambda^q) \le (2\pi)^{-1} \int_{W>(1-\delta)\lambda} b(x)\,dx + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — We choose σ as in (5.3). Note that $0 < \sigma < 1/m$. By assumption, we can decompose $W(x)^q$ into

$$W(x)^q = W_0(x) + U(x),$$

where W_0, $U \in C^1(\mathbf{R}^2)$ have support in $\{|x| < 2\lambda^{-\sigma}\}$ and in $\{|x| > \lambda^{-\sigma}\}$, respectively, and $U(x)$ satisfies the assumption in Lemma 5.1. By decomposition, $U(x)$ equals $W(x)^q$ on $\{|x| > 2\lambda^{-\sigma}\}$. By Lemma 4.3, we have

$$N(P_\beta W_0 P_\beta > \lambda^q) = o(\lambda^{(d-2)/m})$$

and the min-max principle shows that

$$N(P_\beta U P_\beta > \lambda^q) \le N(H_\beta(U) < -\lambda^q).$$

Thus the lemma follows from Proposition 2.1 and Lemma 5.1. □

6. Commutator calculus.

This is also a preparatory section. In the previous section, we have shown the bound on $N(P_\beta W^q P_\beta > \lambda)$. We here establish the upper bound on $N(P_\beta W P_\beta > \lambda)$ for $W(x) > 0$, $W \in \mathcal{V}$. To do this, we make use of the relation

$$\begin{aligned} N(P_\beta W P_\beta > \lambda) &= N((P_\beta W P_\beta)^q > \lambda^q) \\ &= N(P_\beta W^q P_\beta > \lambda^q) + \text{ error term,} \end{aligned}$$

and we show that the error term arising from the commutator $[P_\beta, W]$ is negligible. The following two facts are essential to evaluate the commutator (Lemma 6.3 below): (1) The contribution from the region $\{x : |x| < \lambda^{-\sigma}\}$, $\sigma < 1/m$, is negligible (Lemma 3.1), and $|\nabla W(x)| = O(\lambda^{-(m+1)\sigma})$ over $\{x : |x| > \lambda^{-\sigma}\}$. (2) The operator H_β has at least $O(\lambda^{\beta d})$ as a spectral gap above zero eigenvalue, and we can choose $\sigma < 1/m$ so close to $1/m$ that $\sigma > \beta d/2 \sim d/2m$, because $d < 2$ by assumption.

Throughout the section, the notations δ and λ_δ are used with the same meanings as in the previous sections. The aim here is to prove the following lemma.

LEMMA 6.1. — *Assume again that $W \in \mathcal{V}$ satisfies $W(x) \geq C\langle x\rangle^{-m}$ for some $C > 0$. Then*

$$N(P_\beta W P_\beta > \lambda) \leq (2\pi)^{-1} \int_{W>(1-\delta)\lambda} b(x)\, dx + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

As an immediate consequence, we obtain the following lemma, which is used for the proof of Lemma 3.2.

LEMMA 6.2. — *Assume that $W \in \mathcal{V}$. Then*

$$N(P_\beta W P_\beta > \lambda) = O(\lambda^{(d-2)/m}), \quad \lambda \to 0.$$

The proof of Lemma 6.1 is done through a series of lemmas. The basic tool used for the proof is a simple commutator calculation.

LEMMA 6.3. — *Assume that $U(x) = U(x;\lambda) \in C^1(\mathbf{R}^2)$ has support in $\{|x| > \lambda^{-\kappa}\}$ for some $\kappa > 0$ and it obeys*

$$|U(x)| \leq C\langle x\rangle^{-n}, \quad |\nabla U(x)| \leq C\langle x\rangle^{-n-1}$$

uniformly in λ for some $n > 0$. Then

$$\|P_\beta U Q_\beta\| = O(\lambda^\eta), \quad \lambda \to 0,$$

with $\eta = (n+1)\kappa - \beta d/2$, where $\| \ \|$ denotes the norm of bounded operators acting on $L^2 = L^2(\mathbf{R}^2)$.

Proof. — The operator under consideration is represented as

$$P_\beta U Q_\beta = P_\beta(UQ_\beta - Q_\beta U) = P_\beta[U, Q_\beta] = P_\beta[P_\beta, U].$$

Recall that zero is the isolated eigenvalue of H_β. Let Γ be a circle around the origin with radius $c\lambda^{-\beta d}$ in the complex plane, $c>0$ being small enough. We may assume that H_β has no eigenvalues in the interior of closed curve Γ except for zero. Then the eigenprojection P_β is represented by the line integral

$$P_\beta = (2\pi i)^{-1}\oint_\Gamma (\zeta - H_\beta)^{-1}\,d\zeta$$

along Γ. We now write H_β as

$$H_\beta = (-i\nabla - A_\beta)^2 - b_\beta = \Pi_1^2 + \Pi_2^2 - b_\beta = \Pi_\beta^*\Pi_\beta$$

with $\Pi_\beta = \Pi_1 + i\Pi_2$, and we calculate the commutator

$$[(\zeta - H_\beta)^{-1}, U] = (\zeta - H_\beta)^{-1}[H_\beta, U](\zeta - H_\beta)^{-1}.$$

If we write $U_j(x)$ for $\partial_j U(x) = \partial U(x)/\partial x_j$, then U_j obeys the bound $U_j(x) = O(\lambda^{(n+1)\kappa})$ by assumption, and we have

$$[H_\beta, U] = \Pi_\beta^*[\Pi_\beta, U] + [\Pi_\beta^*, U]\Pi_\beta = \Pi_\beta^*[\Pi_\beta, U] - i(U_1 - iU_2)\Pi_\beta.$$

Since $P_\beta\Pi_\beta^* = (\Pi_\beta P_\beta)^* = 0$, we obtain

$$P_\beta[P_\beta, U] = -iP_\beta(U_1 - iU_2)B_\lambda,$$

where

$$B_\lambda = (2\pi i)^{-1}\oint_\Gamma \zeta^{-1}\Pi_\beta(\zeta - H_\beta)^{-1}\,d\zeta.$$

As is easily seen,

$$\|\Pi_\beta(H_\beta - \zeta)^{-1}\| \le \|H_\beta^{1/2}(H_\beta - \zeta)^{-1}\| = O(\lambda^{-\beta d/2}), \quad \zeta \in \Gamma.$$

This proves the lemma. □

LEMMA 6.4. — *Assume that $W \in \mathcal{V}$, $W(x) \ge 0$, is nonnegative. If $l \in \mathbf{N}$ is an integer, then*

$$N(P_\beta W P_\beta > \lambda) \le N(P_\beta W^l P_\beta > (1-\delta)\lambda^l) + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — Since $\beta d < 2/m$ by (4.3), we can take σ as

$$(l + \beta d/2)/(ml+1) < \sigma < 1/m \tag{6.1}$$

for given $l \in \mathbf{N}$. We decompose $W(x)$ into

$$W(x) = W_0(x) + U(x), \quad W_0 \ge 0, \quad U \ge 0, \tag{6.2}$$

where $W_0 \in C_0^1(\mathbf{R}^2)$ has support in $\{|x| < 2\lambda^{-\sigma}\}$, and $U(x)$ has support in $\{|x| > \lambda^{-\sigma}\}$ and satisfies the assumption in Lemma 6.3 with $n = m$. By Lemma 6.3, we have

$$\left(P_\beta U P_\beta\right)^l = P_\beta U^l P_\beta + R_\lambda,$$

where $R_\lambda : L^2 \to L^2$ satisfies

$$\|R_\lambda\| = O(\lambda^{(ml+1)\sigma - \beta d/2}) = o(\lambda^l)$$

by (6.1). This, together with Proposition 2.1 and Lemma 4.3, completes the proof. □

LEMMA 6.5. — *Assume that $W \in \mathcal{V}$ satisfies the same assumption as in Lemma 6.1. If $q = l/k > 0$ is a rational number, then*

$$N(P_\beta W P_\beta > \lambda) \le N(P_\beta W^q P_\beta > (1-\delta)\lambda^q) + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — We again take σ as in (6.1). By assumption, $W(x)^q$ is a C^1-smooth function and it satisfies

$$|\nabla W(x)^q| = O(|x|^{-(mq+1)}), \quad |x| \to \infty.$$

Hence we can decompose $W(x)^l$ into

$$W(x)^l = W_0(x) + U(x)^k, \quad W_0 \ge 0, \quad U \ge 0,$$

where $W_0 \in C_0^1(\mathbf{R}^2)$ has support in $\{|x| < 2\lambda^{-\sigma}\}$, and $U(x)$ has support in $\{|x| > \lambda^{-\sigma}\}$ and satisfies the assumption in Lemma 6.3 with $n = mq$. By decomposition, $U(x)$ equals $W(x)^q$ on $\{|x| > 2\lambda^{-\sigma}\}$ and we have

$$P_\beta U^k P_\beta = \left(P_\beta U P_\beta\right)^k + R_\lambda$$

by Lemma 6.3, where R_λ obeys the bound $\|R_\lambda\| = o(\lambda^l)$. This, together with Proposition 2.1 and Lemma 4.3 again, implies that

$$N(P_\beta W^l P_\beta > \lambda^l) \le N(P_\beta W^q P_\beta > (1-\delta)\lambda^q) + \delta\lambda^{(d-2)/m}.$$

Thus the lemma follows from Lemma 6.4 at once. □

We conclude the section by proving Lemma 6.1 in question.

Proof of Lemma 6.1. — We take a rational number $q = l/k$ to satisfy (5.1). Then the lemma is immediately obtained from Lemmas 5.2 and 6.5. □

7. Proof of Lemmas 3.2 and 3.3.

In this section we prove Lemmas 3.2 and 3.3 which remain unproved. The proof of Lemma 3.2 is based on the following lemma.

LEMMA 7.1. — *Assume that $W \in \mathcal{V}$. Then*

$$N(P_\alpha W P_\alpha > \lambda) = O(\lambda^{(d-2)/m}), \quad \lambda \to 0.$$

Proof. — The lemma is easy to prove. We may assume that $W \in \mathcal{V}$ is nonnegative. Since $W(x) = o(\lambda)$ for $|x| > \lambda^{-\beta}$, it follows from Lemma 4.2 that

$$N(P_\alpha W P_\alpha > \lambda) \leq N(P_\beta W P_\beta > (1-\delta)\lambda)$$

for $\delta > 0$ small enough. This, together with Lemma 6.2, completes the proof. □

Proof of Lemma 3.2. — It suffices to prove the lemma for nonnegative potential $W \in \mathcal{V}$. The proof uses (3.7). It should be noted that this relation has been obtained without using Lemma 3.2 (see the proof of Theorem 1.1). By Lemma 7.1, it follows from (3.6) and (3.7) that

$$N(H(W) < -\lambda) \leq O(\lambda^{(d-2)/m}) + N(H(2W) < -c\lambda^{\alpha d})$$

for some $c > 0$. Fix ε so small that $0 < \varepsilon < (2-d)/2m\alpha$. Since $\alpha d < 1$ strictly, we repeat the same argument as above to obtain that

$$N(H(W) < -\lambda) \leq O(\lambda^{(d-2)/m}) + N(H(cW) < -\lambda^{\varepsilon}).$$

We here recall the notations

$$H(W) = H - W, \quad H = (-i\nabla - A)^2 - b, \quad b = \nabla \times A.$$

Let $G_\varepsilon = \{x \in \mathbf{R}^2 : |x| < \lambda^{-\alpha\varepsilon}\}$ and let S_D be the self-adjoint realization in $L^2(G_\varepsilon)$ of $S_0 = (-i\nabla - A)^2$ under zero Dirichlet conditions. Then we have by the IMS localization formula that

$$N(H(cW) < -\lambda^{\varepsilon}) \leq N(S_D < c_1)$$

for some $c_1 > 0$ independent of λ. By the Feynman-Kac-Ito formula ([2]), the integral kernel of $\exp(-tS_D)$, $t > 0$, obeys the bound

$$0 < \exp(-tS_D)(x,x) \leq \exp(t\Delta)(x,x) = (4\pi t)^{-1}.$$

If we use the relation

$$\mathrm{Tr}\,(\exp(-tS_D)) = \int_{G_\varepsilon} \exp(-tS_D)(x,x)\,dx,$$

then the above inequality with $t = 1/c_1$ implies that

$$N(S_D < c_1) = O(\lambda^{-2\alpha\varepsilon}) = O(\lambda^{(d-2)/m})$$

and hence the proof is complete. □

The proof of Lemma 3.3 requires the following lemma.

LEMMA 7.2. — *Assume that $W \in \mathcal{V}$. Set $W_+(x) = \max(W(x), 0)$. Then*

$$N(P_\beta W_+ P_\beta > \lambda) \le (2\pi)^{-1} \int_{W > (1-\delta)\lambda} b(x)\,dx + \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — We approximate $W_+(x)$ by a smooth potential. We can construct $W_\varepsilon(x) \in \mathcal{V}$ in such a way that $W_\varepsilon(x) \ge c_\varepsilon \langle x\rangle^{-m}$ for some $c_\varepsilon > 0$ and

$$W_+(x) \le W_\varepsilon(x) \le W_+(x) + \varepsilon\langle x\rangle^{-m}$$

for any $\varepsilon > 0$ small enough. Hence it follows from Lemma 6.1 that

$$N(P_\beta W_+ P_\beta > \lambda) \le N(P_\beta W_\varepsilon P_\beta > \lambda) \le (2\pi)^{-1} \int_D b(x)\,dx + \delta\lambda^{(d-2)/m},$$

where

$$D = \{x \in \mathbf{R}^2 : W_\varepsilon(x) > (1-\delta)\lambda\}.$$

We now take $\varepsilon = \delta^2$ and divide D into two domains

$$D_1 = \{x \in D : W_+(x) > \delta\langle x\rangle^{-m}\}, \quad D_2 = \{x \in D : W_+(x) \le \delta\langle x\rangle^{-m}\}.$$

If $x \in D_2$, then

$$(1-\delta)\lambda < W_\varepsilon(x) \le W_+(x) + \delta^2\langle x\rangle^{-m} \le (\delta + \delta^2)\langle x\rangle^{-m}$$

with $\varepsilon = \delta^2$, so that $|x| \le \delta^{1/m} O(\lambda^{-1/m})$. Thus we have

$$\int_{D_2} b(x)\,dx = \delta^{(2-d)/m} O(\lambda^{(d-2)/m}).$$

On the other hand, if $x \in D_1$, then

$$(1-\delta)\lambda < W_\varepsilon(x) \le W_+(x) + \delta^2\langle x\rangle^{-m} \le (1+\delta)W_+(x).$$

This implies that $W(x) > (1-2\delta)\lambda$ for $x \in D_1$, and hence

$$\int_{D_1} b(x)\,dx \le \int_{W > (1-2\delta)\lambda} b(x)\,dx.$$

Thus the proof is complete. □

Proof of Lemma 3.3. — As in the proof of Lemma 7.1, we have

$$N(P_\alpha W P_\alpha > \lambda) \le N(P_\alpha W_+ P_\alpha > \lambda) \le N(P_\beta W_+ P_\beta > (1-\delta)\lambda).$$

Hence the lemma follows from Lemma 7.2 at once. □

8. Auxiliary operator with wider spectral gap.

The remaining sections are devoted to proving the lower bound (Theorem 1.2). For the same reason as explained at the beginning of Section 5, the proof based on the min-max principle does not directly apply to the lower bound estimate. It also seems that the trial function method does not work for it. Indeed, it would be natural to take the zero eigenstates $(x_1 + ix_2)^l e^{-\varphi(x)}$, $\Delta\varphi = b$, of the unperturbed operator $H = H_+$ as a candidate for trial functions. However this family of eigenfunctions does not necessarily form an orthogonal system if $\varphi(x)$ is not spherically symmetric. Hence it is not simple to look carefully at the dependence on $l \gg 1$, when the family is orthogonalized by the Schimidt method. In addition, since these eigenfunctions are not spatially localized, this also makes it difficult to evaluate the quantity $(Wu, u)_{L^2}$ for a linear combination u of the eigenfunctions. Thus to prove Theorem 1.2, we again employ the same idea as developed in the previous sections.

By (3.5), we have

$$(8.1)\quad N(H(V) < -\lambda) \geq N(H_\alpha(V) < -(1+\delta)\lambda) \geq N(P_\alpha V P_\alpha > (1+\delta)\lambda).$$

However Lemma 4.2 is not useful for the lower bound estimate, even if the lower bound on the quantity $N(P_\beta V P_\beta > \lambda)$ is obtained. Thus we take a slightly different approach to prove the theorem. The idea is to introduce a new auxiliary operator different from H_β, which still has the same zero eigenprojection P_α as the original operator H_α. The construction of such an operator is very simple.

We represent H_α in the form $H_\alpha = \Pi_\alpha^* \Pi_\alpha$ like (1.1), so that

$$\Pi_\alpha \Pi_\alpha^* = \Pi_\alpha^* \Pi_\alpha + 2b_\alpha, \quad b_\alpha = \nabla \times A_\alpha.$$

We now set

$$K_\alpha = K_\alpha(\lambda) = \Gamma_\alpha^* \Gamma_\alpha, \quad \Gamma_\alpha = g_\alpha \Pi_\alpha,$$

with some positive bounded function $g_\alpha(x) = g_\alpha(x;\lambda) \in C^\infty(\mathbf{R}^2)$, g_α being not necessarily bounded uniformly in λ. By definition, $K_\alpha \geq 0$ is nonnegative and it has the same zero eigenprojection P_α as H_α. We also have the relation

$$\Gamma_\alpha \Gamma_\alpha^* = g_\alpha \Pi_\alpha \Pi_\alpha^* g_\alpha \geq 2b_\alpha g_\alpha^2$$

in the form sense. Since $b_\alpha(x) \geq c(\langle x\rangle^{-d} + \lambda^{\alpha d})$, we can choose $g_\alpha(x)$ in such a way that $b_\alpha(x) g_\alpha(x)^2 \geq c_\gamma \lambda^{\beta d}$, $c_\gamma > 0$, and

$$(8.2)\qquad g_\alpha(x) = 1, \quad |x| \leq \lambda^{-\beta} \ll \lambda^{-\alpha}.$$

By construction, the operator K_α coincides with H_β over $\{|x| < \lambda^{-\beta}\}$ and it obeys the form inequality

$$Q_\alpha K_\alpha Q_\alpha \geq c_\gamma \lambda^{\beta d} Q_\alpha, \quad Q_\alpha = \mathrm{Id} - P_\alpha. \tag{8.3}$$

This is the desired operator and it has spectral properties similar to those of operator H_β. We formulate these properties in a series of lemmas below.

We use the notation $K_\alpha(W)$ to denote the operator $K_\alpha(W) = K_\alpha - W$, and we again take q and σ as in Section 5 (see (5.1) $\sim$ (5.3));

$$\beta d < q < (3d+2)/4m < 2/m, \quad \sigma_0 < \sigma < 1/m,$$

where

$$\sigma_0 = \max\Big(q/2,\ \beta d/mq,\ 3q/(2+2mq),\ 4q/(3d+2)\Big).$$

Lemma 8.1. — *Let q and σ be as above. Assume that $U(x) = U(x;\lambda)$, $U \in C^1(\mathbf{R}^2)$, is a real function with support in $\{\lambda^{-\sigma} < |x| < \lambda^{-\beta}\}$ and it satisfies*

$$|U(x)| \leq C\langle x\rangle^{-mq}, \quad |\nabla U(x)| \leq C\langle x\rangle^{-mq-1}$$

for some $C > 0$ independent of $\lambda > 0$. Then

$$N(K_\alpha(U) < -\lambda^q) \geq (2\pi)^{-1}\int_{U>(1+\delta)\lambda^q} b(x)\,dx - \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — The proof is based on the min-max principle. We first note that

$$\int_{|x|<\lambda^{-\sigma}} b(x)\,dx = o(\lambda^{(d-2)/m}).$$

By (8.2), $g_\alpha(x) = 1$ on the support of U and hence $H_\beta(U) = K_\alpha(U)$ over the region $\{|x| < \lambda^{-\beta}\}$. To prove the lower bound, we need not take account of the contribution from the exterior region $\{|x| > \lambda^{-\beta}\}$. Thus the desired lower bound is obtained in exactly the same way as in the proof of Lemma 5.1, if we make use of Proposition 5.1 (2). □

Lemma 8.2. — *Let q be again as above. Assume that $W \in \mathcal{V}$ satisfies $W(x) \geq C\langle x\rangle^{-m}$ for some $C > 0$. Then*

$$N(P_\alpha W^q P_\alpha > \lambda^q) \geq (2\pi)^{-1}\int_{W>(1+\delta)\lambda} b(x)\,dx - \delta\lambda^{(d-2)/m}$$

for $0 < \lambda < \lambda_\delta \ll 1$.

Proof. — Let σ_0 be as above. We take σ as

$$\beta d/mq < \max(\sigma_0,\ (q+\beta d)/2mq) < \sigma < 1/m.$$

By assumption, there exists a nonnegative function $U \in C^1(\mathbf{R}^2)$ with support in $\{\lambda^{-\sigma} < |x| < \lambda^{-\beta}\}$ such that

$$U(x) = W(x)^q, \quad 2\lambda^{-\sigma} \le |x| \le \lambda^{-\beta}/2,$$

and $W(x)^q \ge U(x) \ge 0$ over $\mathbf{R}^2$ and that $U(x)$ satisfies the assumption in Lemma 8.1. The quantity $N(K_\alpha(U) < -\lambda^q)$ is majorized by the sum

$$N(P_\alpha(U + c\lambda^{-\beta d}U^2)P_\alpha > \lambda^q) + N(Q_\alpha(K_\alpha - U - \lambda^{\beta d}/c)Q_\alpha < -\lambda^q).$$

As is easily seen, $mq\sigma > \beta d$ and $2mq\sigma - \beta d > q$, so that

$$U(x) = O(\lambda^{mq\sigma}) = o(\lambda^{\beta d}), \quad \lambda^{-\beta d}U(x)^2 = O(\lambda^{2mq\sigma-\beta d}) = o(\lambda^q)$$

on the support of U. By (8.3),

$$Q_\alpha(K_\alpha - U - \lambda^{\beta d}/c)Q_\alpha \ge 0$$

for $c \gg 1$. Thus it follows that

$$N(P_\alpha W^q P_\alpha > \lambda^q) \ge N(P_\alpha U P_\alpha > \lambda^q) \ge N(K_\alpha(U) < -(1+\delta)\lambda^q).$$

This, together with Lemma 8.1, proves the lemma. □

LEMMA 8.3. — *Let σ be such that*

$$(1+\beta d/2)/(m+1) < \sigma < 1/m < \beta.$$

If $U \in C^1(\mathbf{R}^2)$ has support in $\{\lambda^{-\sigma} < |x| < \lambda^{-\beta}\}$ and obeys

$$|U(x)| \le C\langle x\rangle^{-m}, \quad |\nabla U(x)| \le C\langle x\rangle^{-m-1},$$

then one has

$$\|P_\alpha U Q_\alpha\| = O(\lambda^{1+\eta}), \quad \lambda \to 0,$$

with $\eta = (m+1)\sigma - \beta d/2 - 1 > 0$.

Proof. — The lemma can be proved in almost the same way as in the proof of Lemma 6.3. By (8.3), the zero eigenvalue of K_α has at least $c_\gamma\lambda^{\beta d}$, $c_\gamma > 0$, as a spectral gap. Since $g_\alpha(x) = 1$ on the support of U, we have

$$[\Gamma_\alpha^*, U] = [\Pi_\alpha^*, U] = O(\lambda^{(m+1)\sigma}).$$

If we regard P_α as the zero eigenprojection of K_α, this relation makes it possible to repeat the same argument as used in the proof of Lemma 6.3 and the proof is complete. □

9. Form inequality.

We further prepare a simple proposition on a form inequality, which is also used in proving Theorem 1.2.

PROPOSITION 9.1. — *Let $A \geq 0$ be a bounded nonnegative operator acting on a Hilbert space X and let $P : X \to X$ be an orthogonal projection. Then*

$$\left(PAP\right)^{\theta} \geq PA^{\theta}P, \quad 0 < \theta < 1,$$

in the form sense, and hence

$$N(\left(PAP\right)^{\theta} > \lambda) \geq N(PA^{\theta}P > \lambda), \quad \lambda > 0.$$

The proof requires the following simple lemma.

LEMMA 9.1. — *Let the notations be as above. Then one has*

$$\|(A+s)^{1/2}P(PAP+s)^{-1/2}P\| \leq 1, \quad s > 0,$$

as a bounded operator on X, and hence

$$\|(A+s)^{1/2}P(PAP+s)^{-1}P(A+s)^{1/2}\| \leq 1.$$

Proof. — We denote by $(\ ,\ )$ and $\|\ \|$ the scalar product and norm in X, respectively. Let $Y =$ Ran P denote the range of P. We calculate

$$\|(A+s)^{1/2}Pv\|^2 = ((PAP+s)Pv, Pv) = \|(PAP+s)^{1/2}Pv\|^2$$

for $v \in X$. If we set $u = (PAP+s)^{1/2}Pv \in Y$, then u ranges over Y and we have

$$\|(A+s)^{1/2}(PAP+s)^{-1/2}u\| = \|u\|, \quad u \in Y.$$

Thus it follows that

$$\|(A+s)^{1/2}(PAP+s)^{-1/2}Pw\| = \|Pw\| \leq \|w\|, \quad w \in X,$$

and the lemma is obtained. □

Proof of Proposition 9.1. — As is well known, the fractional power is represented by the formula

$$\left(PAP\right)^{\theta} = c_{\theta}\int_0^{\infty} s^{\theta-1}(PAP+s)^{-1}PAP\,ds,$$

where
$$1/c_\theta = \int_0^\infty s^{\theta-1}(s+1)^{-1}\,ds.$$
Similarly we have
$$PA^\theta P = c_\theta \int_0^\infty s^{\theta-1}P(A+s)^{-1}AP\,ds.$$
The operators in the integrands are rewritten as
$$(PAP+s)^{-1}PAP = P - sP(PAP+s)^{-1}P$$
$$P(A+s)^{-1}AP = P - sP(A+s)^{-1}P.$$
Thus it suffices to show that

$$P(PAP+s)^{-1}P \le P(A+s)^{-1}P, \quad s>0, \tag{9.1}$$

in the form sense. By Lemma 9.1,
$$((A+s)^{1/2}P(PAP+s)^{-1}P(A+s)^{1/2}v,v) \le (v,v), \quad v\in X.$$
If we set $v=(A+s)^{-1/2}Pw$ for $w\in X$, then
$$(P(PAP+s)^{-1}Pw,w) \le (P(A+s)^{-1}Pw,w).$$
This yields (9.1) and the proof is complete. □

10. Lower bound; proof of Theorem 1.2.

In this section we complete the proof of Theorem 1.2 in question. Throughout the proof, δ, $0<\delta\ll 1$, is again fixed arbitrarily and λ, $0<\lambda<\lambda_\delta\ll 1$, is assumed to be small enough. By (8.1), we have
$$N(H(V)<-\lambda) \ge N(H_\alpha(V) < -(1+\delta)\lambda) \ge N(P_\alpha VP_\alpha > (1+\delta)\lambda).$$
Thus the proof of the theorem is reduced to evaluating $N(P_\alpha WP_\alpha > \lambda)$ from below for $W\in\mathcal{V}$. We do this through a series of lemmas below.

LEMMA 10.1. — *If $W\in\mathcal{V}$ satisfies $W(x)\ge C\langle x\rangle^{-m}$ for some $C>0$, then*
$$N(P_\alpha W^2P_\alpha > \lambda^2) \ge (2\pi)^{-1}\int_{W>(1+\delta)\lambda} b(x)\,dx - \delta\lambda^{(d-2)/m}$$
for $0<\lambda<\lambda_\delta\ll 1$.

Proof. — The lemma is obtained as a simple application of Proposition 9.1. If $U\in C^1(\mathbf{R}^2)$ satisfies the assumption in Lemma 8.3, then
$$P_\alpha U^2P_\alpha = \Big(P_\alpha UP_\alpha\Big)^2 + R_\lambda,$$

where $R_\lambda : L^2 \to L^2$ obeys the bound $\|R_\lambda\| = o(\lambda^2)$. Thus we combine Proposition 2.1, Lemmas 3.1 and 8.3 to obtain that

$$N(P_\alpha W^2 P_\alpha > \lambda^2) \geq N\left(\left(P_\alpha W P_\alpha\right)^2 > (1+\delta)\lambda^2\right) - \delta\lambda^{(d-2)/m}$$

and hence it follows that

$$N(P_\alpha W^2 P_\alpha > \lambda^2) \geq N(P_\alpha W P_\alpha > (1+\delta)\lambda) - \delta\lambda^{(d-2)/m}.$$

We now apply Lemma 8.2 to evaluate the first term on the right side. Let q be as in Lemma 8.2. Since $\beta d < \alpha d < 1$, we may assume that $\beta d < q < 1$. Thus the lemma follows from Proposition 9.1 at once. □

The lemma above may be obtained by repeated use of commutator calculations as in Section 6. However it is more convenient to use Proposition 9.1 for the proof and the proposition seems to have another application. Thus we have proved this lemma in a different way.

LEMMA 10.2. — *Let $W \in \mathcal{V}$ and set $W_-(x) = \max(0, -W(x))$. Then*

$$N(P_\alpha W P_\alpha < -\lambda) \leq N(P_\alpha W_- P_\alpha > \lambda) \leq (2\pi)^{-1} \int_{W<-(1-\delta)\lambda} b(x)\, dx + \delta\lambda^{(d-2)/m}.$$

Proof. — By Lemma 4.2,

$$N(P_\alpha W_- P_\alpha > \lambda) \leq N(P_\beta W_- P_\beta > (1-\delta/2)\lambda)$$

and hence the lemma is proved in exactly the same way as in the proof of Lemma 7.2. □

LEMMA 10.3. — *Let $W \in \mathcal{V}$. Then*

$$\limsup_{\lambda \to 0} \lambda^{(2-d)/m} N(P_\alpha W^2 P_\alpha > \lambda^2/\delta^2) = O(\delta^{(2-d)/m}), \quad \delta \to 0.$$

Proof. — We may assume that $W(x) \geq C\langle x\rangle^{-m}$, $C > 0$. By the argument used in the proof of Lemma 10.1, we can show that

$$N(P_\alpha W^2 P_\alpha > \lambda^2/\delta^2) \leq N(P_\alpha W P_\alpha > \lambda/2\delta) + c\delta^{(2-d)/m}\lambda^{(d-2)/m}$$

for some $c > 0$. We further obtain

$$N(P_\alpha W P_\alpha > \lambda/2\delta) \leq N(H_\alpha(W) < -\lambda/2\delta) \leq N(H(W) < -\lambda/3\delta)$$

by repeating the same argument as used to derive (3.5). Thus the lemma follows from Lemma 3.2 at once. □

We now evaluate $N(P_\alpha W P_\alpha > \lambda)$ from below for $W \in \mathcal{V}$. We set

$$W_\delta(x) = \left(W(x)^2 + \delta^3 U_m(x)^2\right)^{1/2}, \quad U_m(x) = \langle x\rangle^{-m}.$$

Then it follows from Proposition 2.1 that

$$N(P_\alpha W_\delta^2 P_\alpha > (1+2\delta)\lambda^2) \le N(P_\alpha W^2 P_\alpha > (1+\delta)\lambda^2) + N(P_\alpha U_m^2 P_\alpha > \lambda^2/\delta^2).$$

Since $W_\delta(x) \ge |W(x)|$, the term on the left side satisfies

$$N(P_\alpha W_\delta^2 P_\alpha > (1+2\delta)\lambda^2) \ge (2\pi)^{-1} \int_{|W|>(1+3\delta)\lambda} b(x)\, dx - \delta\lambda^{(d-2)/m}$$

by Lemma 10.1. Next we estimate the two terms on the right side from above. By Lemma 10.3, the second term satisfies

$$\limsup_{\lambda\to 0} \lambda^{(2-d)/m} N(P_\alpha U_m^2 P_\alpha > \lambda^2/\delta^2) = O(\delta^{(2-d)/m}) \to 0, \quad \delta \to 0.$$

If we again repeat the same argument as used in the proof of Lemma 10.1, the first term obeys

$$N(P_\alpha W^2 P_\alpha > (1+\delta)\lambda^2) \le N(\left(P_\alpha W P_\alpha\right)^2 > \lambda^2) + \delta\lambda^{(d-2)/m},$$

so that we have

$$N(P_\alpha W P_\alpha > \lambda) \ge N(P_\alpha W^2 P_\alpha > (1+\delta)\lambda^2) - N(P_\alpha W P_\alpha < -\lambda) - \delta\lambda^{(d-2)/m}.$$

Thus it follows from Lemma 10.2 that

$$\begin{aligned} &N(P_\alpha W P_\alpha > \lambda) \\ &\ge (2\pi)^{-1} \int_{W>\lambda} b(x)\, dx - (2\pi)^{-1} \int_{(1-\delta)\lambda<|W|<(1+\delta)\lambda} b(x)\, dx - \delta\lambda^{(d-2)/m}. \end{aligned}$$

This yields the desired lower bound and the proof of Theorem 1.2 is now complete.

We end the section by some comments on the proof of Theorems 1.1 and 1.2. As previously mentioned, the remarkable spectral property of Pauli operator is that its essential spectrum occupies the positive half-line $[0,\infty)$ and the bottom (origin) of essential spectrum is an eigenvalue with infinite multiplicities for a fairly large class of magnetic fields falling off not too rapidly at infinity. The important ingredient in the proof of both the theorems is how to make a spectral gap with suitable width in a neighborhood of the bottom of essential spectrum. To do this, we have introduced two auxiliary operators H_β and K_α. The first method using H_β is useful only for the upper bound estimate. On the other hand, the second method using K_α may apply to the upper bound estimate as well as the lower bound estimate under several modifications and hence the two theorems can be verified in a unified way. However the comparison lemma (Lemma 4.2), which has played a basic role in the first method, seems to have an interest of its own. Thus we have proved the upper and lower bounds in two slightly different methods, although both the methods are in principle based on the same idea.

11. Concluding remarks.

We end the paper by making some comments on the asymptotic distribution of negative eigenvalues in the case that electric fields are stronger than magnetic fields at infinity. The case is much easier to deal with. We can obtain the following theorem.

THEOREM 11.1. — *Assume that (b) and (V) are fulfilled. Let $0 < m < d < 2$. If $V(x)$ satisfies (1.3), then*

$$N(H(V) < -\lambda) = (4\pi)^{-1} \int_{V>\lambda} (V(x) - \lambda)\, dx + o(\lambda^{(m-2)/m}), \quad \lambda \to 0.$$

Remarks.

(1) The asymptotic formula above can be rewritten as

$$N(H(V) < -\lambda) = (2\pi)^{-2} \operatorname{vol}\Big(\{(x,\xi) \in \mathbf{R}^2 \times \mathbf{R}^2 : H(x,\xi) < -\lambda\}\Big)(1+o(1)),$$

where $H(x,\xi) = |\xi - A(x)|^2 - b(x) - V(x)$. Thus $N(H(V) < -\lambda)$ obeys the classical Weyl formula.

(2) As is easily seen from the proof below, the theorem remains true without assuming $b(x)$ to be strictly positive, and also it is still valid for the case when $d \geq 2$ and $0 < m < 2$.

Sketch of proof. — We give only a sketch for the proof. The proof uses the min-max principle. We cover the plane $\mathbf{R}^2$ by a family of disjoint cubes Q_j with side r_j, where r_j satisfies

$$(1 + |y_j|)^{d/2}/c \leq r_j \leq c(1 + |y_j|)^{d/2}, \quad c > 1,$$

for the center $y_j = (y_{1j}, y_{2j}) \in \mathbf{R}^2$ of cube Q_j. Let $b_j = b(y_j)$ be the value of $b(x)$ at the center y_j. Then the magnetic potential $A(x)$ takes the form

$$A(x) = \hat{A}_j(x) + \nabla g_j(x) + e_j(x)$$

on each cube Q_j, where

$$\hat{A}_j(x) = (-b_j(x_2 - y_{2j})/2, b_j(x_1 - y_{1j})/2).$$

When $|y_j| \gg 1$ is large enough, $\hat{A}_j(x)$ and $e_j(x)$ obey the bounds

$$|\hat{A}_j(x)| = O(|y_j|^{-d/2}), \quad |e_j(x)| = O(|y_j|^{-1}), \quad |y_j| \to \infty.$$

By assumption, $m < d$, and hence the operator $H(V)$ is approximated as $H(V) \sim -\Delta - V_j$ on each cube Q_j by gauge transformation, where $V_j = V(y_j)$. This yields

$$N(H(V) < -\lambda) \sim (2\pi)^{-2} \sum_j \mathrm{vol}\Big(\{(x,\xi) \in Q_j \times \mathbf{R}^2 : |\xi|^2 - V(x) < -\lambda\}\Big)$$

and the asymptotic formula in the theorem is obtained. □

Finally we discuss the case $d = m$ when magnetic and electric fields are comparable to each other. We get the following theorem. The theorem can be verified in almost the same way as in the proof of Theorem 11.1, if we make use of Proposition 5.1. We skip its proof.

THEOREM 11.2. — *Let (b) and (V) be fulfilled and let $0 < d = m < 2$. Assume that $V(x)$ satisfies (1.3). Set $V_n(x) = V(x) - 2n\,b(x)$ for $n \in \mathbf{N} \cup 0$. If $V_n(x)$ satisfies*

$$\limsup_{\lambda \to 0} \lambda^{(2-m)/m} \int_{(1-\delta)\lambda < |V_n| < (1+\delta)\lambda} \langle x \rangle^{-m}\, dx = o(1), \quad \delta \to 0,$$

then

$$N(H(V) < -\lambda) = (2\pi)^{-1} \sum_{n=0}^{\infty} \int_{V_n > \lambda} b(x)\, dx + o(\lambda^{(m-2)/m}), \quad \lambda \to 0,$$

where the summation is actually a finite sum.

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