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ON INDEX THEOREMS FOR LINEAR ORDINARY DIFFERENTIAL OPERATORS

by M. LODAY-RICHAUD and G. POURCIN

Let $D = a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + a_0$ be a linear differential operator with analytic coefficients in the neighborhood of the origin in $\mathbb{C}$ (in short, D a differential operator).

In [M74] (see also [K71]), B. Malgrange proved that D has an index as a linear operator both in the vector space $\mathbb{C}[[x]]$ of formal power series at the origin, and in the subspace $\mathbb{C}\{x\}$ of the convergent ones. The values of the corresponding indices are also given. Recall that an endomorphism $D : E \to E$ of a vector space E has an index if it has finite dimensional kernel and cokernel, the index being then defined by

$$\chi(D, E) = \dim \ker D - \dim \operatorname{coker} D.$$

Later, J.-P. Ramis [R84] proved similar results in the spaces $\mathbb{C}[[x]]_s$ of power series of Gevrey order s and the subspaces $\mathbb{C}[[x]]_{s,C^+}$ and $\mathbb{C}[[x]]_{s,C^-}$ of those series of type greater or smaller than C, for all positive s and C. Recall

$$\mathbb{C}[[x]]_s = \bigcup_{C>0} \mathbb{C}[[x]]_{s,C},$$

$$\mathbb{C}[[x]]_{s,C^+} = \bigcap_{\varepsilon>0} \mathbb{C}[[x]]_{s,C+\varepsilon},$$

$$\mathbb{C}[[x]]_{s,C^-} = \bigcup_{\varepsilon>0} \mathbb{C}[[x]]_{s,C-\varepsilon}$$

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where

$$\mathbb{C}[[x]]_{s,C} = \Big\{ \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]] \mid \exists K>0,\ \forall n\geq 0,\ |a_n| \leq K(n!)^s C^{ns} \Big\}.$$

Their method relies on functional analysis, mainly limits of Banach spaces and compact perturbations of operators.

Next, P. Deligne suggested in a letter to J.-P. Ramis (see [D86]) to describe singular points –here the origin 0– of differential operators by means of a suitable space equipped with a sheaf which take into account the exponential rate of growth and decay of solutions in each direction around 0. A somewhat close viewpoint has already been used by B. Malgrange and J.-P. Ramis [MR92] to build a theory of multisummability fitting to formal solutions of linear ordinary differential equations and by J.-P. Ramis in his wild π_1 theory towards the Riemann problem.

In this paper, we investigate the sheaf of Deligne with an application to index theorems for D acting on spaces of sections of this sheaf over various sets. Our method relies on homological algebra. We first show that index theorems over small sets, small discs and narrow sectors, follow easily from normal form considerations. Over large sets, we prove index theorems using suitable Mayer-Vietoris sequences. This latter technique depends on an isomorphism theorem for D acting on cohomology groups H^1 and H^2. The proof of such an isomorphism theorem (Theorems 2.1 and 4.2) is a central part of this paper.

We obtain, in particular, the index theorems by B. Malgrange and J.-P. Ramis mentioned above and index theorems in the spaces of multisummable series. Moreover, we relate these indices to the singular big points (Definition 2.11) of D (*cf.* Tables 3.1 and 4.1).

Aknowledgments. We wish to thank J.-P. Ramis for many conversations on the subject.

Summary

1. Preliminaries.

In this section we recall some definitions and basic results of the local theory of linear differential equations. A reader familiar to this theory should begin at Section 2. Mostly we adopt the notations of [Mal] and [MR92].

We fix

$$D = a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + a_0$$

a linear differential operator with analytic coefficients in the neighborhood of the origin in $\mathbb{C}$.

It is quite often more convenient to consider differential systems instead of differential equations. Recall that the correspondence between systems and equations is meromorphic: to an equation corresponds its companion system; conversely, any system $\mathrm{d}Y/\,\mathrm{d}x = AY$ can be put in a companion form by means of a meromorphic transformation $Y \mapsto MY$, $M \in \mathrm{GL}(n, \mathbb{C}\{x\}[\frac{1}{x}])$ (*cf.* [R78], Th. 1.6.16 for instance). We shall use both viewpoints and denote by Δ a system meromorphically equivalent to the companion system of D.

Formal invariants.

To the differential system $\Delta Y = 0$ there is a normal form, a fundamental solution matrix of which reads

$$x^L \, \mathrm{e}^{Q(1/x)}$$

where L is a constant $n \times n$-matrix and $Q = \mathrm{diag}(q_1, \ldots, q_n)$ is diagonal with diagonal entries $q_1, \ldots, q_n$ which are polynomials in a fractional power $1/t = x^{-1/p}$ of x without constant term. Moreover, one can choose L and Q simultaneously decomposed into a direct sum of diagonal blocks

$$L = \bigoplus L^{(i)} \quad \text{and} \quad Q = \bigoplus Q^{(i)}$$

such that the q_j's occuring in a same block are either all zero or all of the same degree (see [BJL79], Thm I). Though the normal form is not unique, the q_j's and the blocks of a Jordan form of L are uniquely determined up to a permutation. They thus provide a complete set of formal (meromorphic) invariants.

The smallest possible p is the *degree of ramification* of Δ and D. The case when $p = 1$ is said the *unramified case*; the case when $p > 1$, the *ramified case.* The q_j's are called the *determining polynomials* of Δ or of D. The degrees of the non zero ones with respect to $1/x$ are *levels* of Δ and D. The levels are positive integers in the unramified case and positive rational numbers in the ramified case. They are precisely the positive slopes of the Newton polygon $N(D)$ of D (see [R78], I.1). We denote them by

$$k_1 < k_2 < \cdots < k_\nu.$$

By convention, if a q_j is the null polynomial, we say that it has degree $k_0 = 0$. Notice that we consider here the degrees of the q_j's themselves and not the degrees of the $(q_j - q_\ell)$'s for any pair $q_j \neq q_\ell$ like it has to be done sometimes.

To the edge of slope k_j of $N(D)$ there is a k_j-characteristic polynomial and there is a one-to-one correspondence between k_j-characteristic roots and determining polynomials of level k_j. More precisely, to each k_j-characteristic root α with multiplicity m there are m determining polynomials q of the form (see [R78])

$$q\Big(\frac{1}{x}\Big) = -\frac{\alpha}{k_j x^{k_j}}\big(1 + o(x^{-1/p})\big).$$

The numbers $c = \dfrac{|\alpha|}{k_j}$ for the different k_j-characteristic roots α are called *k_j-characteristic constants.*

Gevrey series spaces.

The spaces of series of Gevrey type are defined as $\mathbb{C}$-linear subspaces of $\mathbb{C}[[x]]$, the space of formal power series in one variable x at the origin, as follows: for $s > 0$ and $C > 0$,

$$\mathbb{C}[[x]]_{s,C} = \Big\{ \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]] \mid \exists K > 0,\ \forall n \geq 0,\ |a_n| \leq K(n!)^s C^{ns} \Big\};$$

$$\mathbb{C}[[x]]_s = \bigcup_{C>0} \mathbb{C}[[x]]_{s,C}; \qquad \mathbb{C}[[x]]_{(s)} = \bigcap_{C>0} \mathbb{C}[[x]]_{s,C};$$

$$\mathbb{C}[[x]]_{s^+} = \bigcap_{\varepsilon>0} \mathbb{C}[[x]]_{s+\varepsilon}; \qquad \mathbb{C}[[x]]_{s^-} = \bigcup_{\varepsilon>0} \mathbb{C}[[x]]_{s-\varepsilon};$$

$$\mathcal{C} = \mathbb{C}[[x]]_{\infty^-} = \bigcup_{s'>0} \mathbb{C}[[x]]_{s'};$$

$$\mathbb{C}[[x]]_{s,C^+} = \bigcap_{\varepsilon>0} \mathbb{C}[[x]]_{s,C+\varepsilon} \quad \text{and} \quad \mathbb{C}[[x]]_{s,C^-} = \bigcup_{\varepsilon>0} \mathbb{C}[[x]]_{s,C-\varepsilon}.$$

With these definitions, the equality $\mathbb{C}[[x]]_{+\infty} = \mathbb{C}[[x]]$ holds.

For all s and C, D preserves the spaces $\mathbb{C}[[x]]_s$, $\mathbb{C}[[x]]_{s^+}$, $\mathbb{C}[[x]]_{s^-}$, $\mathbb{C}[[x]]_{s,C^+}$, $\mathbb{C}[[x]]_{s,C^-}$ and $\mathcal{C}$. It does not preserve a space $\mathbb{C}[[x]]_{s,C}$ in general; however, $x^m D$ does for a suitable $m \in \mathbb{N}$.

Asymptotics.

It is well known that formal solutions of the equation $Dy = 0$ are related to actual analytic solutions through ordinary asymptotics (*cf.* [Was], Main Asymptotic Existence Theorem) and that, to better understand this relation, one must consider special kinds of asymptotics such as asymptotic expansions of Gevrey type. The sheaves $\mathcal{A}, \mathcal{A}_{(s)}, \ldots$, the definition of which we recall below, are adequate to this purpose.

Making a real blow up of 0 in $\mathbb{C}$, *i.e.*, using polar coordinates (θ, ρ) one identifies $\mathbb{C}^*$ to $\mathbb{R}/2\pi\mathbb{Z} \times]0, +\infty[$ and one replaces 0 by $S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \{0\}$.

With this identification, an open *sector* (with vertex at 0) in $\mathbb{C}$ is a set

$$\Sigma = \big\{(\theta, \rho) \mid \theta_1 < \theta < \theta_2,\ 0 < \rho < \varepsilon\big\}.$$

An open *interval* $I =]\theta_1, \theta_2[$ of S^1 defines a germ of open sector in $\mathbb{C}$. Let $|I| = |\theta_2 - \theta_1|$ denote the length of I.

One defines on S^1 the following sheaves (see [M95], [MR92], [Mal], [R78]):

- $\mathcal{A}$ is the sheaf of germs of holomorphic functions having an asymptotic expansion at 0: for all $\theta \in S^1$, a germ of $\mathcal{A}$ at θ is defined by a function f analytic on an open sector $\Sigma = I \times]0, r[$ with $\theta \in I$ and satisfying the asymptotic condition

$$\forall N \in \mathbb{N}, \qquad \lim_{\substack{x \to 0 \\ x \in \Sigma}} \left| \frac{1}{x^N} \Big(f(x) - \sum_{n=0}^{N} a_n x^n \Big) \right| = 0.$$

- For all $s > 0$, $\mathcal{A}_{(s)}$ is the subsheaf of $\mathcal{A}$ made of germs having local Gevrey asymptotics of order $k = 1/s$:

$$\exists K,\ C > 0,\ \forall N \in \mathbb{N}, \quad \left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq K (N!)^s |x|^N C^{Ns}.$$

Necessarily, $\hat{f} = \sum_{n \geq 0} a_n x^n$ belongs to $\mathbb{C}[[x]]_s$.

• $\mathcal{A}^{<0}$ is the subsheaf of $\mathcal{A}$ made of flat germs, *i.e.*, germs the asymptotic expansion of which is 0.

• For all $k > 0$, $\mathcal{A}^{\leq k}$ is the sheaf of germs of holomorphic functions having local exponential growth of order at most k at 0:

$$\exists K > 0,\ A > 0, \quad |f(x)| \leq K \exp \frac{A}{|x|^k}\cdot$$

• For all $k > 0$, $\mathcal{A}^{\leq -k}$ is the sheaf of germs of holomorphic functions having local exponential decay of order at least k at 0:

$$\exists K > 0,\ A > 0, \quad |f(x)| \leq K \exp -\frac{A}{|x|^k}\cdot$$

• We shall also use the sheaves $\mathcal{A}^{\leq k^-}$ and $\mathcal{A}^{\leq -k^+}$ defined as follows. For all $k > 0$, $\mathcal{A}^{\leq k^-}$ is the subsheaf $\varinjlim_{\varepsilon>0} \mathcal{A}^{\leq k-\varepsilon}$ of $\mathcal{A}^{\leq k}$: a germ f belongs to $\mathcal{A}^{\leq k^-}_{\theta_0}$ if $\exists\, \Sigma$ containing θ_0, $\exists\, \varepsilon > 0$, $\exists\, K$ and $C > 0$ such that

$$\forall x \in \Sigma, \quad |f(x)| \leq K \exp \frac{C}{|x|^{k-\varepsilon}}\cdot$$

The definition of $\mathcal{A}^{\leq -k^+}$ is obtained by changing k in $-k$ and ε in $-\varepsilon$.

Given a sheaf morphism $D: \sharp \to \sharp$ we denote by $\mathcal{S}o\ell(D, \sharp)$ and $\mathcal{C}oker(D, \sharp)$ the corresponding kernel and cokernel sheaves. When $\sharp = \mathcal{A}^\flat$ denotes one of the sheaves above, we denote $\mathcal{V}^\flat$ instead of $\mathcal{S}o\ell(D, \mathcal{A}^\flat)$. The following equality holds:

$$\mathcal{V}^{\leq -k} = \begin{cases} \mathcal{V}^{\leq -k_1} & \text{if } k \leq k_1, \\ \mathcal{V}^{\leq -k_{j+1}} & \text{if } k_j < k \leq k_{j+1}, \\ 0 & \text{if } k_\nu < k. \end{cases}$$

For the convenience of the reader, we state below some of the by now classical theorems which are central in our purpose. And we give references for more details. We first consider the case of Gevrey conditions relative to a given order k. Sections 2 and 3 are only concerned with this case.

From now, we use systematically the correspondence of notations

$$s = \frac{1}{k}, \quad s_j = \frac{1}{k_j}, \dots$$

Basic theorems: case of a given order k.

THEOREM OF BOREL-RITT (see [M95], Thm 1.1.4.1). — *The sequence of sheaves on* S^1

$$0 \to \mathcal{A}^{<0} \longrightarrow \mathcal{A} \xrightarrow{T} \mathbb{C}[[x]] \to 0$$

where T *denotes the Taylor map at* 0 *and, by abuse,* $\mathbb{C}[[x]]$ *denotes the constant sheaf with stalk* $\mathbb{C}[[x]]$ *is exact.*

THEOREM OF CAUCHY-HEINE (see [M95], Thm 1.3.2.1.i and ii). — *The natural map*

$$H^1(S^1, \mathcal{A}^{<0}) \longrightarrow H^1(S^1, \mathcal{A})$$

is the null map.

Consequently,

$$H^0(S^1, \mathcal{A}/\mathcal{A}^{<0}) = \mathbb{C}[[x]] \quad \text{and} \quad H^1(S^1, \mathcal{A}^{<0}) = \mathbb{C}[[x]]/\mathbb{C}\{x\}.$$

THEOREM OF BOREL-RITT WITH GEVREY CONDITIONS (see [M95], Thm 2.1.2.3, 2.1.3.1.i and 2.4.1.4). — *The sequence of sheaves on* S^1

$$0 \to \mathcal{A}^{\leq -k} \longrightarrow \mathcal{A}_{(s)} \xrightarrow{T} \mathbb{C}[[x]]_s \to 0$$

is exact.

THEOREM OF RAMIS-SIBUYA (see [M95], Thm 2.1.4.2, Cor. 2.1.4.3 and 2.1.4.4). — *For all* $k > 0$, *the natural map*

$$H^1(S^1, \mathcal{A}^{\leq -k}) \longrightarrow H^1(S^1, \mathcal{A}_{(s)})$$

is the null map.

Consequently,

$$H^0(S^1, \mathcal{A}_{(s)}/\mathcal{A}^{\leq -k}) = H^0(S^1, \mathcal{A}/\mathcal{A}^{\leq -k}) = \mathbb{C}[[x]]_s,$$

$$H^1(S^1, \mathcal{A}^{\leq -k}) = \mathbb{C}[[x]]_s/\mathbb{C}\{x\}.$$

MAIN ASYMPTOTIC EXISTENCE THEOREM (see [Mal], Append. 1, Th. 1). — *The sequences of sheaves on* S^1

$$0 \to \mathcal{V}^{<0} \longrightarrow \mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \to 0$$

and, for all $k \neq 0$,

$$0 \to \mathcal{V}^{\leq k} \longrightarrow \mathcal{A}^{\leq k} \xrightarrow{D} \mathcal{A}^{\leq k} \to 0$$

are exact.

Note however that the map $\mathcal{A} \xrightarrow{D} \mathcal{A}$ is not onto in general.

The Theorem of Borel-Ritt and the previous statement imply the Main Asymptotic Existence Theorem in its classical and its Gevrey forms (see [Was], [MR92] section 4 ii):

Given $\hat{f} \in \mathbb{C}[[x]]$ (*resp.* $\mathbb{C}[[x]]_s$) *such that* $D\hat{f} = 0$ *and* $\theta \in S^1$ *there exist an open sector* $V = I \times]0, r[$ *with* $\theta \in I$ *and a function* $f \in \mathcal{A}(V)$ (*resp.* $f \in \mathcal{A}_{(s)}(V)$) *such that*

$$Tf = \hat{f} \quad \text{and} \quad Df = 0 \text{ on } V.$$

Moreover, it is always possible to choose I *bisected by* θ *and of length* $|I| = \dfrac{\pi}{k_\nu}\cdot$

MALGRANGE LEMMA (see [Mal], Lemme 5.3).

$$\textit{For all } k > 0, \quad H^1(S^1, \mathcal{A}^{\leq k}) = 0.$$

We shall use the weaker following form of this lemma:

$$\textit{For all } k > 0, \; H^1(S^1, \mathcal{A}^{\leq k^-}) = 0.$$

We further consider the case of Gevrey conditions relative to an order k and a type c. Sections 4 and 5 are concerned with this case.

Basic theorems: case of a given type c.

One can define the sheaves $\mathcal{A}^{\leq k,c^-}$ and $\mathcal{A}^{\leq -k,c^+}$ as follows:

- a germ of $\mathcal{A}^{\leq k,c^-}$ at θ_0 is an $f \in \mathcal{A}^{\leq k}_{\theta_0}$ satisfying the condition

$\exists \Sigma = I \times]0,r[$ an open sector with $\theta_0 \in I$, $\exists \varepsilon > 0$ and $K > 0$ such that,

$$\forall x \in \Sigma, \quad \left|f(x)\right| \leq K \exp \frac{c-\varepsilon}{|x|^k};$$

- a germ of $\mathcal{A}^{\leq -k,c^+}$ at θ_0 is an $f \in \mathcal{A}^{\leq -k}_{\theta}$ satisfying the condition

$\exists \Sigma = I \times]0,r[$ an open sector with $\theta_0 \in I$, $\exists \varepsilon > 0$ and $K > 0$ such that,

$$\forall x \in \Sigma, \quad \left|f(x)\right| \leq K \exp -\frac{c+\varepsilon}{|x|^k}.$$

The basic theorems above can be extended in the following form:

PROPOSITION 1.1. — *For all $k > 0$ and $c > 0$, one has :*

(i) *the sequence of sheaves*

$$0 \to \mathcal{A}^{\leq -k,c^+} \longrightarrow \mathcal{A}_{(s),(\frac{1}{c})^-} \xrightarrow{\ T\ } \mathbb{C}[[x]]_{s,(\frac{1}{c})^-} \to 0$$

where T is the Taylor map, is exact;

(ii) *for all $k > 0$, the natural map*

$$H^1(S^1, \mathcal{A}^{\leq -k,c^+}) \longrightarrow H^1\big(S^1, \mathcal{A}_{(s),(\frac{1}{c})^-}\big)$$

is the null map;

(iii) *(Main Asymptotic Existence Theorem with a given type) for all $k > 0$, the sequences of sheaves*

$$0 \to \mathcal{V}^{\leq k,c^-} \longrightarrow \mathcal{A}^{\leq k,c^-} \xrightarrow{\ D\ } \mathcal{A}^{\leq k,c^-} \to 0,$$

$$0 \to \mathcal{V}^{\leq -k,c^+} \longrightarrow \mathcal{A}^{\leq -k,c^+} \xrightarrow{\ D\ } \mathcal{A}^{\leq -k,c^+} \to 0,$$

are exact;

(iv) $H^1\big(S^1, \mathcal{A}^{\leq k,c^-}\big) = 0.$

Proof.

(i) and (ii) are obtained by an easy adaptation of the proofs of their analogs in the ordinary Gevrey case which can be found for instance in [M95].

(iii) To prove the surjectivity of D in the first sequence let f be a germ of $\mathcal{A}^{\leq k,c^-}$ at θ_0. Let $\varepsilon > 0$ and the sector Σ containing the direction θ_0 satisfy, for some constant K,

$$\forall x \in \Sigma, \quad \big|f(x)\big| \leq K \exp \frac{c-\varepsilon}{|x|^k}.$$

There exists $\Sigma' \subset \Sigma$, Σ' containing θ_0, such that

$$\forall x \in \Sigma', \quad \Big|f(x) \exp - \frac{c - \frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}\Big| \leq K \exp \frac{-\frac{1}{2}\varepsilon}{|x|^k}$$

which proves that $f(x) \exp - \dfrac{c-\frac{1}{4}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$ belongs to $\mathcal{A}^{\leq -k}_{\theta_0}$. Let

$$D_1 = \exp - \frac{c-\frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}\ D\ \exp \frac{c-\frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}$$

be the operator conjugated to D by $\exp - \dfrac{c-\frac{1}{4}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$. The Main Asymptotic Existence Theorem in $\mathcal{A}^{\leq -k}$ applied to D_1 and $f(x) \exp - \dfrac{c-\frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}$ at θ_0 provides a germ $g \in \mathcal{A}^{\leq -k}_{\theta_0}$ such that $D_1 g(x) = f(x) \exp - \dfrac{c-\frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}$. The germ $h = g \exp \dfrac{c-\frac{1}{4}\varepsilon}{(xe^{-i\theta_0})^k}$ belongs to $\mathcal{A}^{\leq k,c^-}_{\theta_0}$ and satisfies $Dh = f$.

To prove the surjectivity of D in the second sequence, consider $f \in \mathcal{A}^{\leq -k,c^+}_{\theta_0}$ and suppose that ε, Σ and K satisfy

$$\forall x \in \Sigma, \quad \big|f(x)\big| \leq K \exp - \frac{c+\varepsilon}{|x|^k}.$$

Then, there exist $\Sigma' \subset \Sigma$, Σ' containing 0, and $K' > 0$ such that

$$\forall x \in \Sigma', \quad \Big|f(x) \exp \frac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}\Big| \leq K' \exp \frac{\frac{1}{2}\varepsilon}{|x|^k}$$

which proves that $f \exp \dfrac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$ belongs to $\mathcal{A}_{\theta_0}^{\leq k,\varepsilon^-}$. The previous result applied to this germ and to the adjoint operator

$$D_2 = \exp \frac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}\, D \exp - \frac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$$

of D provides a $g \in \mathcal{A}_{\theta_0}^{\leq k,\varepsilon^-}$ such that $D_2 g(x) = f(x) \exp \dfrac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$. The germ $h = g \exp - \dfrac{c+\frac{3}{2}\varepsilon}{(x\,\mathrm{e}^{-i\theta_0})^k}$ belongs to $\mathcal{A}_{\theta_0}^{\leq -k,c^+}$ and satisfies $Dh = f$.

(iv) The nullity of $H^1(S^1, \mathcal{A}^{\leq k})$ is proved in [Mal], Lemme 5.3. An adaption of this proof gives the nullity of $H^1(S^1, \mathcal{A}^{\leq k,c^-})$. $\square$

Irregularity after Deligne-Malgrange and Gevrey index theorems.

Recall that an endomorphism $D : E \to E$ of a vector space E has an *index* if it has finite dimensional kernel $\ker(D, E)$ and cokernel $\operatorname{coker}(D, E)$, the index being then the number

$$\chi(D, E) = \dim \ker(D, E) - \dim \operatorname{coker}(D, E).$$

The index of D is thus the Euler characteristic of the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \xrightarrow{D} E \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where D is placed in degree 0 (or even).

In [M74], B. Malgrange defines the *irregularity* $\operatorname{irr}_0(D)$ of D at 0 as being the index of D in the quotient space $\mathbb{C}[[x]]/\mathbb{C}\{x\}$. He proves that D has an index both in $\mathbb{C}[[x]]$ and in $\mathbb{C}\{x\}$, implying thus, that

$$\operatorname{irr}_0(D) = \chi\big(D, \mathbb{C}[[x]]\big) - \chi\big(D, \mathbb{C}\{x\}\big).$$

He also gives a value of these indices in terms of the coefficients of D. The proof in $\mathbb{C}[[x]]$ is elementary computational. The proof in $\mathbb{C}\{x\}$ is based on functional analysis using limits of Banach spaces and compact perturbations of operators.

Another, more algebraic proof, of these facts is due to B. Malgrange and P. Deligne (see [M79] and [D77], see also [BV89]). We add this proof in the Appendix. The analog with asymptotic conditions of exponential type provides index theorems in Gevrey series spaces. No use of these proofs is made in the following.

2. The sheaf of Deligne $\widetilde{\mathcal{F}}$.

For the local study of a differential operator D at 0 when 0 is a singularity of D, it is not sufficient to endow $\mathbb{C}$ with the sheaf of germs of holomorphic functions. In an unpublished letter to J.-P. Ramis (see [D86]), P. Deligne suggests to replace $\{0\}$ by a closed disc X and to endow X with a suitable sheaf, allowing to take into account the rate of growth or decay of the solutions of D in each direction around 0. Moreover, the set of global sections of this sheaf on X is isomorphic to $\mathbb{C}\{x\}$, that is, to the stalk at 0 of the sheaf of germs of holomorphic functions in $\mathbb{C}$.

We introduce such a sheaf step by step considering first all possible exponential orders of growth and decay (sheaf $\widetilde{\mathcal{F}}$), then all possible types relative to a given order (sheaves $\widetilde{\mathcal{F}}^k$) and, finally, types relative to finitely many orders (sheaves $\widetilde{\mathcal{F}}^{\underline{k}}$). To each step, will correspond new index theorems. Considering the types relative to infinitely many orders is not relevant here.

The closed disc X.

The space X is the topological space obtained by compactifying $\mathbb{C}$ with a circle: it is the union of $\mathbb{C}$ and $S^1\times]0,+\infty]$ where $\mathbf{C}^*$ and $S^1\times]0,+\infty[$ are identified via polar coordinates.

NOTATIONS. — For all $k>0$, we denote by $B(0,k)$ the open disc of radius k centered at 0 in X and by $\overline{B}(0,k)$ its closure. We write (k',k'') to denote an interval either open or closed at each endpoint.

The sheaf $\widetilde{\mathcal{F}}$ over X.

The sheaf $\widetilde{\mathcal{F}}$ is the sheaf corresponding to the presheaf $\mathcal{F}$ defined by the following conditions:

- for all $k>0$, $\quad \mathcal{F}(B(0,k)) = \mathbb{C}[[x]]_s$ (recall $s=1/k$);
- for all interval I of S^1 and all k',k'', $0<k'<k''$,

$$\mathcal{F}(I\times]k',k''[) = H^0(I,\mathcal{A}^{\leq k'}/\mathcal{A}^{\leq -k''}),$$
$$\mathcal{F}(I\times]k',\infty]) = H^0(I,\mathcal{A}^{\leq k'}).$$

The Theorem of Borel-Ritt makes the definition consistent.

The sheaf $\widetilde{\mathcal{F}}$ is a sheaf of $\mathbb{C}$-algebras. Its stalk $\widetilde{\mathcal{F}}_0$ at 0 is $\mathcal{C}$. In restriction to a circle $S^1\times\{k\}$, it is the quotient

$$\widetilde{\mathcal{F}}_{|S^1\times\{k\}} = \mathcal{A}^{\leq k^-}/\mathcal{A}^{\leq -k^+}.$$

Moreover, the Theorems of Cauchy-Heine and of Ramis-Sibuya imply the following equalities:

$$\begin{aligned} & H^0(X,\widetilde{\mathcal{F}}) = \mathbb{C}\{x\}, \\ \forall k = s^{-1} > 0, \quad & H^0\big(B(0,k),\widetilde{\mathcal{F}}\big) = \mathbb{C}[[x]]_{s^+}, \\ \forall k = s^{-1} > 0, \quad & H^0\big(\overline{B}(0,k),\widetilde{\mathcal{F}}\big) = \mathbb{C}[[x]]_{s^-}. \end{aligned}$$

With $\widetilde{\mathcal{F}}$ we shall prove, in particular, index theorems of D in the spaces $\mathcal{C}$, $\mathbb{C}[[x]]_{s^+}$, $\mathbb{C}[[x]]_{s^-}$ and $\mathbb{C}\{x\}$.

A typical section of the sheaf $\widetilde{\mathcal{F}}$: $\exp q\left(\frac{1}{x}\right)$.

The definition set of the exponential function $\exp q\left(\frac{1}{x}\right)$ as a section of $\widetilde{\mathcal{F}}$ when $\deg q = k$ is an integer is likely the open shadowed subset in Figure 2.1.

The disc $B(0,k)$ is partitioned into sectors: on one over two sectors $\exp q$ equals 0, on the others $\exp q$ is undefined. The support of $\exp q$ is precisely the intersection of its definition set with $X \setminus B(0,k)$. The closure of the arcs of $S^1 \times \{k\}$ limiting the sectors where $\exp q$ vanishes are the *singular big points* of q (*cf.* Definitions 2.11 and 2.12). The singular big points of the determining polynomials of D will play a central role with respect to the indices of D.

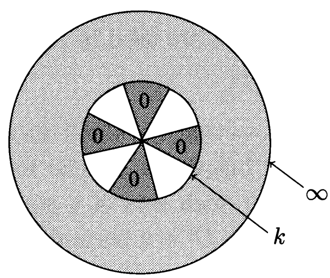


Figure 2.1. (*Here* $k = 4$)

ASSUMPTION 2.1. — From now and without further mention, we make the following assumptions:

1) Sectors (resp. annuli) are subsets of X of the form $I \times [k', k'')$ (resp. $S^1 \times [k', k'')$), *i.e.*, they are closed on the lower edge.

2) Multisectors are finite connected unions of a disc centered at 0 (either open or closed and possibly empty) and of sectors $\Sigma_j = I_j \times [k'_j, k''_j)$, satisfying the conditions $I_{j+1} \subset I_j$ and $k'_j \leq k'_{j+1} \leq k''_j < k''_{j+1}$ for all j.

3) Sectors are not multivalued. This implies for instance that, when considering $(k_1, \ldots, k_\nu)$-multisectors (see Definition 2.11), we assume $k_1 > \frac{1}{2}$. Note that, most of the time, an argument of ramified covering allows to fulfil this condition and even to assume that levels are integers.

The action of D on the cohomology of $\widetilde{\mathcal{F}}$.

The differential operator D induces a sheaf morphism on $\widetilde{\mathcal{F}}$ and then also on its cohomology groups. The remainder of this section is devoted to the proof of the following result:

THEOREM 2.1. — *The linear maps*

$$D : H^i(U, \widetilde{\mathcal{F}}) \longrightarrow H^i(U, \widetilde{\mathcal{F}}) \quad \text{for} \quad i \geq 1$$

are isomorphisms when U is a disc, a sector, a multisector or an annulus (recall that sectors and annuli are supposed to satisfy Assumption 2.1).

The proof is organized as follows: using elementary homological algebra and basic properties of the sheaf $\mathcal{C}oker(D, \widetilde{\mathcal{F}})$ we reduce the problem to the two conditions in Corollary 2.7. Next, we prove that these two conditions are satisfied using coverings which are acyclic for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$. Before we turn to the proof of Theorem 2.1, we show the existence of acyclic coverings (Proposition 2.5) which are used in the proof.

- *Acyclicity for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$.*

We are now going to prove that the small discs and the narrow sectors introduced in Definition 2.2 below are acyclic for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$. The proof is based on the existence over such sets of a sheaf isomorphism between $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ and $\mathcal{S}o\ell(D', \widetilde{\mathcal{F}})$ when D' is a normal form of D.

It is enlightening to notice the following facts about the sheaf $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$.

In restriction to the circle $S^1 \times \{+\infty\}$ at infinity in X, the sheaf $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ is the sheaf $\mathcal{V}$.

Because D: $\mathcal{A}^{\leq k} \to \mathcal{A}^{\leq k}$ is onto for all $k \neq 0$ the equality of sheaves over S^1

$$\mathcal{S}o\ell(D, \mathcal{A}^{\leq k'} / \mathcal{A}^{\leq -k''}) = \mathcal{V}^{\leq k'} / \mathcal{V}^{\leq -k''}$$

holds for all k', k'', $0 < k' < k''$ and also

$$\mathcal{S}o\ell(D, \mathcal{A}^{\leq k^-}/\mathcal{A}^{\leq -k^+}) = \mathcal{V}^{\leq k^-}/\mathcal{V}^{\leq -k^+}.$$

Then, the restriction of $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ to a circle $S^1 \times \{k\}$, when $k \neq 0$, satisfies

$$\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})_{|S^1\times\{k\}} = \begin{cases} \mathcal{V}^{\leq k_j}/\mathcal{V}^{\leq -k_{j+1}} & \text{if } k_j < k < k_{j+1}, \\ \mathcal{V}^{\leq k_{j-1}}/\mathcal{V}^{\leq -k_{j+1}} & \text{if } k = k_j, \end{cases}$$

where, by convention, $k_0 = 0$, $k_{\nu+1} = +\infty$ and $\mathcal{V}^{\leq -k_{\nu+1}} = 0$. Moreover, it is straightforward to prove that, for all interval I and all k', k'',

$$H^0\big(I \times]k', k''[, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) = H^0(I, \mathcal{V}^{\leq k'}/\mathcal{V}^{\leq -k''}),$$

$$H^0\big(I \times]k', +\infty], \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) = H^0(I, \mathcal{V}^{\leq k'}).$$

DEFINITION 2.2.

- By *small disc* we mean a disc B centered at 0, either open or closed and included in $B(0, k_1)$.

- By *narrow sector* we mean a sector $I \times [k', k'')$ contained in a sector $J \times [0, +\infty]$ where J is an open interval of length $|J| = \dfrac{\pi}{k_\nu}$.

LEMMA 2.3. — *Let U be a small disc or a narrow sector and D' be a normal form of D. Then, the sheaves $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ and $\mathcal{S}o\ell(D', \widetilde{\mathcal{F}})$ are isomorphic on $U \setminus \{0\}$.*

Notice that the isomorphism cannot hold at 0 in general since a normal form is defined up to a meromorphic transformation.

Proof. — Up to a meromorphic transformation it is equivalent to consider the case of a differential system Δ with a fundamental matrix of formal solutions $\widehat{F}Y'$ where $\widehat{F} \in \mathrm{GL}(n, \mathbb{C}[[x]])$ and where $Y' = x^L\, e^{Q(1/x)}$ is a fundamental solution of a normal form Δ' satisfying the blocks condition mentionned in Section 1 (Formal invariants).

In the case of a narrow sector U choose a determination of the argument of x in Y' and an asymptotic lift F of $\widehat{F}$ to U. The map $Y' \mapsto FY'$ provides the followed isomorphism.

In the case of a small disc U, locally, that is, on a narrow sector V the previous isomorphism $Y' \mapsto FY'$ holds. But now, the support of the ℓ^{th}

column as well in Y' as in FY' is empty as soon as $q_\ell \neq 0$. Moreover, when $q_\ell = 0$, the ℓ^{th} column $\widehat{F}^{(\ell)}$ of $\widehat{F}$ belongs to $\mathbb{C}[[x]]^n_{s_1^+}$ or, equivalently, defines a global section of $\widetilde{\mathcal{F}}^n$ over U. The map $Y' \mapsto \widehat{F}Y'$ provides the followed isomorphism.

For a proof of the fact that $\widehat{F}^{(\ell)}$ belongs to $\mathbb{C}[[x]]^n_{s_1^+}$ when $q_\ell = 0$ we refer to [MR92], 4.iv: consider a covering $\mathcal{U} = \{U_\lambda\}$ of U by narrow sectors where the intersection of any three sectors is $\{0\}$; and consider asymptotic lifts $F_\lambda^{(\ell)}$ of $\widehat{F}^{(\ell)}$. The abelian cocycle $(-F_\lambda^{(\ell)} + F_{\lambda+1}^{(\ell)})$ being made of isotropies of Δ' in flat $\exp(q_j - q_\ell)$ terms has entries in $\mathcal{A}^{\leq -k_1}$. Each $-F_\lambda^{(\ell)} + F_{\lambda+1}^{(\ell)}$ induces then a null section of $\widetilde{\mathcal{F}}^n$ over $U_\lambda \cap U_{\lambda+1}$ and the $F_\lambda^{(\ell)}$'s define a continuation of $\widehat{F}$ to U.

Nota Bene. — Applying the isomorphism over $U \setminus \{0\}$ one sees that the set $H^0(U \setminus \{0\}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ is made of the formal meromorphic solutions of D. Consequently, the subset $H^0(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ is made of the formal meromorphic solutions of D which can be continued at 0, that is, the solutions belonging to $\mathbb{C}[[x]]$.

Actually, it is well known that the $\widehat{F}^{(\ell)}$'s belong even to $\mathbb{C}[[x]]^n_{s_1}$ and that $\ker(D, \mathbb{C}[[x]]) = \ker(D, \mathbb{C}[[x]]_{s_1})$ holds (see [MR92] 4.iv for instance). The proof above in the sheaf $\widetilde{\mathcal{F}}^{k_1}$ (Section 4) would give this result. □

COROLLARY 2.4.

(i) *When $I \times [k', k'')$ is a narrow sector, the restriction map*

$$H^0\big(I \times [k', +\infty], \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) \longrightarrow H^0\big(I \times [k', k''), \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big)$$

is onto.

(ii) *When U is a small disc,* $H^0(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = \ker(D, \mathbb{C}[[x]])$.

Proof.

(i) Applying the previous isomorphism over $I \times [k', +\infty]$ one can suppose that D is a normal form in which case the assertion is clearly true.

(ii) For $s > s_1$ the equality $\ker(D, \mathbb{C}[[x]]_{s^+}) = \ker(D, \mathbb{C}[[x]])$ holds. Then assertion (ii) follows from the relation $H^0(B(0, \frac{1}{s}), \widetilde{\mathcal{F}}) = \mathbb{C}[[x]]_{s^+}$. □

Notations. — Given a sector U, we denote by $\underline{\mathbb{C}}^m$ the constant sheaf with stalk $\mathbb{C}^m$ over U; for A an open subset of U, we denote by $j_A : A \hookrightarrow U$ and $i_{U \setminus A} : U \setminus A \hookrightarrow U$ the canonical inclusions.

As usually, $j_{A!}\underline{\mathbb{C}}^m$ denotes the constant sheaf with stalk $\mathbb{C}^m$ over A continued by zero over $U\backslash A$ and $i_{(U\backslash A)*}\underline{\mathbb{C}}^m$ the direct image of the sheaf $\underline{\mathbb{C}}^m$ by the canonical inclusion $i_{U\backslash A}$.

PROPOSITION 2.5 (Acyclicity). — *Let U be a narrow sector or a small disc. Then,*

$$H^i\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) = 0 \quad \mathit{for} \quad i \geq 1.$$

Proof. — We note first that, given A an open subset of U not equal to U, the cohomology sequence corresponding to the short exact sequence (see [Ive], Prop. 6.3)

$$0 \to j_{A!}\underline{\mathbb{C}}^m \longrightarrow \underline{\mathbb{C}}^m \longrightarrow i_{(U\backslash A)*}\underline{\mathbb{C}}^m \to 0$$

implies that, when $U \setminus A$ is connected, $H^1(U, j_{A!}\underline{\mathbb{C}}^m) = 0$ and, when $U \setminus A$ is simply connected, $H^2(U, j_{A!}\underline{\mathbb{C}}^m) = 0$.

Suppose now that U is small, *i.e.*, either a small disc or a narrow sector contained in $B(0, k_1)$. Then, by Lemma 2.3, the sheaf $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ is isomorphic over U to a direct sum

$$\underline{\mathbb{C}}^{n_0} \oplus j_{(U\backslash\{0\})!}\mathcal{E}$$

where $\mathcal{E}$ is a local system and it suffices to consider the case when $n_0 = 0$. The case of a narrow sector is trivial ($\mathcal{E} \simeq \underline{\mathbb{C}}^{n_1}$ and $U \setminus \{0\}$ is connected and simply connected). In the case when U is a small disc, consider any covering $\mathcal{U} = \{U_\lambda\}$ of U by closed narrow sectors all containing 0. For any intersection N of U_λ's (and for the U_λ's themselves) $H^0(N, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$. Hence the result.

Suppose further that U is a narrow sector not contained in $B(0, k_1)$. Then, in addition to solutions with moderate growth, $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ can contain solutions of exponential type. Using Lemma 2.3 and splitting $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$ into a direct sum, it suffices to consider the case of a sheaf $\mathcal{S}$ generated by one exponential function $\exp q(\frac{1}{x})$ when its open set of definition A is a proper subset of U. Let B denote the open, possibly empty, subset of U where $\exp q(\frac{1}{x})$ vanishes.

Under the narrowness assumption $U \setminus A$ is connected, $U \setminus A$ and $U \setminus B$ are simply connected. The preliminary remark applied to the open subsets A and B of U and the constant sheaf $\underline{\mathbb{C}}$ implies that $H^2(U, j_{B!}\underline{\mathbb{C}}) = 0$ and

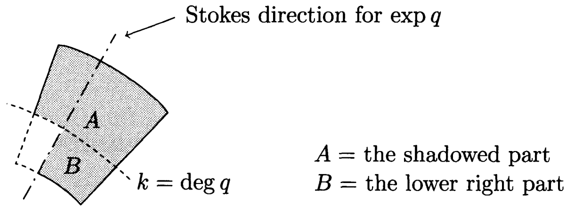


Figure 2.2

$H^i(U, j_{A!}\underline{\mathbb{C}}) = 0$ for $i \geq 1$. The result follows then from the cohomology sequence corresponding to the exact sequence of sheaves

$$0 \to j_{B!}\underline{\mathbb{C}} \longrightarrow j_{A!}\underline{\mathbb{C}} \longrightarrow \mathcal{S} \to 0.$$

This ends the proof. □

• *Proof of Theorem 2.1.*

Consider the exact sequence of sheaves

$$0 \to \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}) \longrightarrow \widetilde{\mathcal{F}} \xrightarrow{D} \widetilde{\mathcal{F}} \longrightarrow \mathcal{C}oker(D, \widetilde{\mathcal{F}}) \to 0 \tag{1}$$

and split it into the two short exact sequences

$$0 \to \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}) \longrightarrow \widetilde{\mathcal{F}} \xrightarrow{D} \mathcal{I}m(D, \widetilde{\mathcal{F}}) \to 0 \tag{2}$$

and

$$0 \to \mathcal{I}m(D, \widetilde{\mathcal{F}}) \xrightarrow{i} \widetilde{\mathcal{F}} \longrightarrow \mathcal{C}oker(D, \widetilde{\mathcal{F}}) \to 0. \tag{3}$$

There correspond the long exact sequences of cohomology

$$\begin{aligned} (4)\quad 0 &\to H^0(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) \longrightarrow H^0(U, \widetilde{\mathcal{F}}) \xrightarrow{D} H^0(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \\ &\xrightarrow{\delta} H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) \longrightarrow H^1(U, \widetilde{\mathcal{F}}) \xrightarrow{D} H^1(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \\ &\longrightarrow H^2(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) \longrightarrow H^2(U, \widetilde{\mathcal{F}}) \xrightarrow{D} H^2(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \to 0 \end{aligned}$$

and

$$\begin{aligned} (5)\quad 0 &\to H^0(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \xrightarrow{i} H^0(U, \widetilde{\mathcal{F}}) \longrightarrow H^0(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})) \\ &\longrightarrow H^1(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \xrightarrow{i} H^1(U, \widetilde{\mathcal{F}}) \longrightarrow H^1(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})) \\ &\longrightarrow H^2(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \xrightarrow{i} H^2(U, \widetilde{\mathcal{F}}) \longrightarrow H^2(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})) \to 0. \end{aligned}$$

The lemma below gathers properties of the sheaf $\mathcal{C}oker(D, \widetilde{\mathcal{F}})$.

LEMMA 2.6.

(i) $\operatorname{coker}(D, \mathcal{C}) = \operatorname{coker}(D, \mathbb{C}[[x]])$.

(ii) *The sheaf* $\mathcal{C}oker(D, \widetilde{\mathcal{F}})$ *has its support in* $\{0\}$.

(iii) *Let* $U \subset X$. *If* 0 *belongs to* U *then*

$$H^0\big(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})\big) = \operatorname{coker}\big(D, \mathbb{C}[[x]]\big).$$

(iv) *For all* $U \subset X$ *the canonical map*

$$H^0(U, \widetilde{\mathcal{F}}) \longrightarrow H^0\big(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})\big)$$

is onto.

(v) *The natural maps*

$$H^1\big(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})\big) \xrightarrow{\ i\ } H^1(U, \widetilde{\mathcal{F}})$$
$$H^2\big(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})\big) \xrightarrow{\ i\ } H^2(U, \widetilde{\mathcal{F}})$$

are isomorphisms.

Proof.

(i) By exactness of the direct limit functor,

$$\operatorname{coker}(D, \mathcal{C}) = \varinjlim_{s>0} \operatorname{coker}(D, \mathbb{C}[[x]]_s).$$

It is then enough to prove that,

$$\forall s \geq s_1, \quad \operatorname{coker}\big(D, \mathbb{C}[[x]]_s\big) = \operatorname{coker}(D, \mathbb{C}[[x]]).$$

More precisely, applying the Snake Lemma to D acting on each term of the exact sequence $0 \to \mathbb{C}[[x]]_s \to \mathbb{C}[[x]] \to \mathbb{C}[[x]]/\mathbb{C}[[x]]_s \to 0$ it is enough to prove that, for all $s \geq s_1$,

(a) $\ker(D, \mathbb{C}[[x]]/\mathbb{C}[[x]]_s) = 0$,

(b) $\operatorname{coker}(D, \mathbb{C}[[x]]/\mathbb{C}[[x]]_s) = 0$.

For this, consider the Snake Lemma exact sequence corresponding to D acting on each term of the exact sequence

$$0 \to \mathbb{C}[[x]]_s/\mathbb{C}\{x\} \longrightarrow \mathbb{C}[[x]]/\mathbb{C}\{x\} \longrightarrow \mathbb{C}[[x]]/\mathbb{C}[[x]]_s \to 0.$$

Equality (b) results trivially from the nullity of $\mathrm{coker}(D, \mathbb{C}[[x]]/\mathbb{C}\{x\})$ (see [M74], Th. 1.4.1). It is proved in [MR92], 4.iv that, for $s \geq s_1$, the natural map $\ker(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}) \to \ker(D, \mathbb{C}[[x]]/\mathbb{C}\{x\})$ is bijective. Considering the long exact sequence corresponding to the short exact sequence $0 \to \mathcal{V}^{\leq -k} \longrightarrow \mathcal{A}^{\leq -k} \xrightarrow{D} \mathcal{A}^{\leq -k} \to 0$ (Main Asymptotic Existence Theorem) and the equality $H^1(S^1, \mathcal{A}^{\leq -k}) = \mathbb{C}[[x]]_s/\mathbb{C}\{x\}$ (Theorem of Ramis-Sibuya) one sees that $\mathrm{coker}(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}) = 0$. Hence (a).

(ii) results from the Main Asymptotic Existence Theorem.

(iii) Due to (ii), $H^0(U, \mathit{Coker}(D, \widetilde{\mathcal{F}}))$ is the stalk $\mathit{Coker}(D, \widetilde{\mathcal{F}})_0$ of the sheaf $\mathit{Coker}(D, \widetilde{\mathcal{F}})$ at 0. Then the result follows from $\widetilde{\mathcal{F}}_0 = \mathcal{C}$ and Assertion (i).

(iv) Since $\mathrm{coker}(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}) = 0$ (see [M74], Théorème 1.4.1)), $\mathrm{coker}(D, \mathbb{C}[[x]])$ has representatives in $\mathbb{C}\{x\}$ hence in $H^0(U, \widetilde{\mathcal{F}})$.

(v) In the sequence (5), since the support of $\mathit{Coker}(D, \widetilde{\mathcal{F}})$ is at most a point, the terms $H^1(U, \mathit{Coker}(D, \widetilde{\mathcal{F}}))$ and $H^2(U, \mathit{Coker}(D, \widetilde{\mathcal{F}}))$ are 0. The assertion follows then from (iv). □

Taking into account Lemma 2.6 in the sequences (4) and (5) we can state:

Corollary 2.7. — *The linear maps $D : H^i(U, \widetilde{\mathcal{F}}) \to H^i(U, \widetilde{\mathcal{F}})$, for $i = 1, 2$ are isomorphisms if and only if the following conditions are satisfied*:

(i) *the coboundary map $\delta : H^0(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \to H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ is onto*;

(ii) $H^2(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$.

We are now going to check that conditions (i) and (ii) of Corollary 2.7 are satisfied.

- *The coboundary map $\delta : H^0(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})) \to H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$.*

We prove here that δ is a surjective map when U is a disc or a multisector and when U is an annulus or a large sector.

Proposition 2.8. — *When U is a disc or a multisector the coboundary map*

$$\delta : H^0\big(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})\big) \longrightarrow H^1\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big)$$

is onto.

Proof. — Let $[f]$ be a cohomology class in $H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ and $\mathcal{U} = \{U_\lambda\}_{\lambda \in \mathbb{Z}/\Lambda\mathbb{Z}}$ be a covering of U by closed narrow sectors

$U_\lambda = I_\lambda \times [0, \rho_\lambda)$ all containing 0 and such that the intersection of any three sectors is $\{0\}$. This covering being acyclic for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$, the cohomology class $[f]$ can be represented as a 1-cocycle $f = (f_{\lambda,\lambda+1})$ on $\mathcal{U}$.

To prove the proposition it suffices to find a family $g_\lambda \in H^0(U_\lambda, \widetilde{\mathcal{F}})$, $\lambda \in \mathbb{Z}/\Lambda\mathbb{Z}$, such that

(i) the local sections $h_\lambda = Dg_\lambda$ patch into a global section $h \in H^0(U, \widetilde{\mathcal{F}})$;

(ii) the 1-cocycle $(g_\lambda - g_\mu)_{(\lambda,\mu)}$ be cohomologous to the 1-cocycle f in $H^1(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$.

We claim first that f is cohomologous to a 1-cocycle $\varphi = (\varphi_{\lambda,\lambda+1})$ which is zero in restriction to $\{0\}$. Indeed, the restriction $(f_{\lambda,\lambda+1\,|\,\{0\}})$ of f to $\{0\}$ is a 1-cocycle of $H^1(\{0\}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ which is trivial. Then, there exist series $\widehat{f}_\lambda$ solutions of D such that

$$f_{\lambda,\lambda+1\,|\,\{0\}} = -\widehat{f}_\lambda + \widehat{f}_{\lambda+1} \quad \forall\lambda.$$

Because of the narrowness of U_λ, the Main Asymptotic Existence Theorem allows to lift $\widehat{f}_\lambda$ into an element f_λ of $H^0(I_\lambda \times [0, +\infty], \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ and, *a fortiori*, of $H^0(U_\lambda, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$. The cocycle $\varphi = (\varphi_{\lambda,\lambda+1})$ defined by

$$\varphi_{\lambda,\lambda+1} = f_{\lambda,\lambda+1} + f_\lambda - f_{\lambda+1} \quad \text{on} \quad U_{\lambda,\lambda+1}$$

fullfils the claim.

Now, for all λ, continue $f_{\lambda,\lambda+1}$ into an element of

$$H^0\big(I_\lambda \cap I_{\lambda+1} \times [0, +\infty], \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big)$$

(Corollary 2.4.i) and consider its restriction to $I_\lambda \cap I_{\lambda+1} \times \{+\infty\}$. This provides a 1-cocycle of $H^1(\{I_\lambda\}, \mathcal{V}^{<0})$ and, *a fortiori*, of $H^1(\{I_\lambda\}, \mathcal{A}^{<0})$. By the Cauchy-Heine Theorem, there exist $g_\lambda \in H^0(I_\lambda, \mathcal{A})$ such that

$$\varphi_{\lambda,\lambda+1} = -g_\lambda + g_{\lambda+1}.$$

Keep denoting g_λ the element of $H^0(U_\lambda, \widetilde{\mathcal{F}})$ naturally induced by g_λ and let $h_\lambda = Dg_\lambda$. This achieves the proof. □

Proposition 2.9. — *When U is an annulus or a (possibly wide) sector the coboundary map*

$$\delta : H^0\big(U, \mathcal{I}m(D, \widetilde{\mathcal{F}})\big) \longrightarrow H^1\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big)$$

is onto.

Proof. — We proceed like in the previous proof. The case when U is a disc or a sector containing 0 has been proved there.

Suppose now U is an annulus $S^1 \times [k', k'')$ and not a disc. Take a 2-finite covering $\mathcal{U}$ of U by narrow closed sectors $U_\lambda = I_\lambda \times [k'k'')$ and represent $[f]$ as a 1-cocycle of $H^1(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ (recall that 2-finite means that the intersection of any three sectors is empty). The restriction of this cocycle to $(I_\lambda \times \{+\infty\})$ defines a 1-cocycle of $H^1(S^1, \mathcal{A}^{\leq k'^-})$. However, due to Malgrange Lemma, $H^1(S^1, \mathcal{A}^{\leq k'^-})$ is trivial. Up to a refinement of $\mathcal{U}$ this allows to write, for all λ,

$$f_{\lambda,\lambda+1} = -g_\lambda + g_{\lambda+1}$$

where $g_\lambda \in H^0(I_\lambda, \mathcal{A}^{\leq k'^-})$ and then $g_\lambda \in H^0(U_\lambda, \widetilde{\mathcal{F}})$.

The case when U is a sector not containing 0 is similar. □

- *Nullity of the second cohomology group $H^2(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$.*

Proposition 2.10. — *$H^2(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$ whenever U is a disc, a multisector, a sector or an annulus.*

Proof. — We already saw that small discs and narrow sectors are acyclic for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$.

When U is a sector, a multisector or an annulus not containing 0, U admits a 2-finite covering $\mathcal{U}$ by small closed sectors. Trivially, $H^2(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{F})) = 0$ and the Theorem of Leray (see [God], Corollaire du théorème 5.2.4.) implies $H^2(U, \mathcal{S}o\ell(D, \widetilde{F})) = 0$.

When U is a disc or a multisector containing 0, take a covering $\mathcal{U}$ by small closed sectors containing 0 which is 2-finite in restriction to $U \setminus \{0\}$. The restriction to $\{0\}$ of a 2-cocycle $(f_{\lambda\mu\nu})$ of $H^2(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{F}))$ is a 2-cocycle of $H^2(\{0\}, \mathcal{S}o\ell(D, \widetilde{F}))$ which is trivial. There exist then formal solutions $(\widehat{f}_{\lambda\mu})$ of D such that

$$f_{\lambda\mu\nu} = \widehat{f}_{\lambda\nu} + \widehat{f}_{\mu\nu} + \widehat{f}_{\nu\lambda} \quad \text{on} \quad U_\lambda \cap U_\mu \cap U_\nu \quad \forall_{\lambda\mu\nu}.$$

Let $f_{\lambda\mu}$ be an asymptotic lift of $\widehat{f}_{\lambda\mu}$ on $U_\lambda \cap U_\mu$ (Main Asymptotic Existence Theorem). Thus, $(f_{\lambda\mu\nu})$ is the coboundary of the 1-cochain $(f_{\lambda\mu})$ of $H^1(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$. This proves that $H^2(\mathcal{U}, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$ and the Theorem of Leray implies that $H^2(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$. □

This ends the proof of Theorem 2.1.

Big points and some remarks.

Among the subsets of X naturally related to D important are big points and multisectors built on them. They are defined as follows:

DEFINITION 2.11.

1) A *k-big point* in X is an arc $I\times\{k\}$ where the interval I is closed and of length π/k. It is said to *support* a polynomial $q\Big(\dfrac{1}{x}\Big) = \dfrac{a}{x^k}\,(1+o(x^{-1/p}))$ of degree k if it is centered at an anti-Stokes direction α of q. In the unramified case ($p\in\mathbb{N}$), this means that $\alpha\in S^1$ is such that $a\,\mathrm{e}^{-ik\alpha}$ is real negative. In the ramified case, we previously make a choice of a determination of the argument on S^1.

2) A $(k_1,\dots,k_\nu)$-*multisector* is a multisector

$$U = B(0,k_1)\cup\bigcup_{j=1}^{\nu-1}\big(I_j\times[0,k_{j+1}[\,\big)\cup\big(I_\nu\times[0,+\infty]\,\big)$$

where, for all j, $I_j\times\{k_j\}$ is a k_j-big point. Notice that the intervals $[0,k_{j+1}[$ are chosen *open* in $[0,+\infty]$.

Like determining polynomials, singular big points of D will be considered with *multiplicity* and they are formal invariants of D. Naturally, in the ramified case, the multiplicity has to be counted with the same choice of a determination of the argument for all the q_j's and it does not depend on such a choice.

DEFINITION 2.12. — A subset U of X is said *generic for q* if it contains no big point supporting q. It is said *generic for D* if it is generic for all (non zero) determining polynomial q_j, $j=1,\dots,\nu$ of D. Otherwise, it is said *singular for q* or *for D*.

• Let $\underline{k}=(k_1,\dots,k_\nu)$ and U be a $\underline{k}$-multisector in X. A $\underline{k}$*-multisum* $(f_1,\dots,f_\nu)$ of a series $\hat{f}$ on $(I_1,\dots,I_\nu)$ in the sense of [MR92], Déf. 2.2, induces an element of $H^0(U,\widetilde{\mathcal{F}})$. However, $H^0(U,\widetilde{\mathcal{F}})$ is possibly larger than the space of the series which are $\underline{k}$-multisummable on $(I_1,\dots,I_\nu)$ since, as an element of $H^0(U,\widetilde{\mathcal{F}})$, $\hat{f}$ is only required to belong to $\mathbb{C}[[x]]_{s_1^+}$, not to $\mathbb{C}[[x]]_{s_1}$, and, the f_j's are required to belong to $\mathcal{A}/\mathcal{A}^{\leq -k_{j+1}^-}$, not to $\mathcal{A}/\mathcal{A}^{\leq -k_{j+1}}$. We shall see in Section 5 which set U and which sheaf correspond to $\underline{k}$-multisummable series on $(I_1,\dots,I_\nu)$.

• The *wild monodromy group* of D relative to a formal fundamental solution $\widehat{F}x^L\,\mathrm{e}^Q$ (*cf.* Section 1) in the sense of Ramis wild π_1 is the monodromy group defined as follows: fix a base point x_0 close to 0 in X; consider the $\mathbb{C}$-vector subspace E of $(\widetilde{\mathcal{F}}_{x_0})^n$ generated by the columns of $\widehat{F}$ and let loops act on E by "wild analytic continuation". This means continuation as sections of $(\widetilde{\mathcal{F}})^n$ along a set made of all the big points centered on the loop, in other words, continuation by $\underline{k}$-multisums. The wild monodromy group of D is generated by the actions of the loops turning once around appropriate big points including 0. The appropriate big points are not the singular big points of D but are precisely the singular big points of the differential operator $\operatorname{End} D$. For more detail on how these actions are related to the Stokes multipliers of D we refer to [L-R94], Thm III.2.14.

• We turn to *acyclicity* for $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})$.

Using the Main Asymptotic Existence Theorem we have proved that small discs and narrow sectors are acyclic for $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})$. This result is not the sharpest one but is sufficient for the application to index theorems we have in mind.

Actually, it will result from our index theorems and assuming the unicity of $\underline{k}$-multisums of solutions over generic $\underline{k}$-multisectors (*cf.* [MR92]) that generic $\underline{k}$-multisectors are acyclic for $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})$. It will also result (*cf.* Theorem 3.1 and Proposition 3.9) that singular big points are not acyclic for $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})$ in general. More precisely, when U is a big point, $H^1(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})) = \sharp\{\ell \mid q_\ell \text{ is supported by } U\}$. We call this quantity the *multiplicity* of U with respect to D.

3. Application to index theorems of D.

In this section we apply the preceding properties of the sheaf $\widetilde{\mathcal{F}}$ to prove index theorems of D acting in the spaces $H^0(U,\widetilde{\mathcal{F}})$ when U is a disc, a sector, a multisector or an annulus satisfying Assumption 2.1. This includes the case of arcs, *i.e.*, of closed sectors $I \times \{k\}$ of height 0 and, in particular, of big points (*cf.* Definition 2.11).

The computation of the indices is done as follows: we compute first the indices of D in the case of small discs and narrow sectors where the computation is easy especially because of the isomorphism in Lemma 2.3. Then, we deduce the case of general U from suitable Mayer-Vietoris sequences using the isomorphism in Theorem 2.1.

THEOREM 3.1 (General Index Theorem). — *Let U be a disc, a sector, a multisector or an annulus satisfying Assumption* 2.1. *Then, D has an index in $H^0(U,\widetilde{\mathcal{F}})$ given by*

$$\chi\big(D,H^0(U,\widetilde{\mathcal{F}})\big)=\dim H^0\big(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})\big)-\dim H^1\big(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})\big)$$
$$-\dim H^0\big(U,\mathcal{C}oker(D,\widetilde{\mathcal{F}})\big).$$

Recall that

$$H^0\big(U,\mathcal{C}oker(D,\widetilde{\mathcal{F}})\big)=\begin{cases}0 & \text{if } 0\notin U,\\ \operatorname{coker}\big(D,\mathbb{C}[[x]]\big) & \text{if } 0\in U.\end{cases}$$

Proof. — Applying Propositions 2.8 and 2.9 and Lemma 2.6 (iv) to the sequences (4) and (5) we obtain the exact sequences

$$0\to H^0\big(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})\big)\longrightarrow H^0(U,\widetilde{\mathcal{F}})\xrightarrow{D}H^0\big(U,\mathcal{I}m(D,\widetilde{\mathcal{F}})\big)$$
$$\xrightarrow{\delta}H^1\big(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})\big)\to 0,$$

$$0\to H^0\big(U,\mathcal{I}m(D,\widetilde{\mathcal{F}})\big)\xrightarrow{i}H^0(U,\widetilde{\mathcal{F}})\longrightarrow H^0\big(U,\mathcal{C}oker(D,\widetilde{\mathcal{F}})\big)\to 0.$$

The spaces $H^0(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}}))$ and $H^1(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}}))$ are finite dimensional (*cf.* Proposition 2.5). Moreover, $H^0(U,\mathcal{C}oker(D,\widetilde{\mathcal{F}}))$ is either equal to $\operatorname{coker}(D,\mathbb{C}[[x]])$ or to 0 depending on whether 0 belongs to U or not (Lemma 2.6 (ii) and (iii)). Hence the result. □

The remainder of this section is devoted to computing these indices in terms of the formal invariants of D, namely, in terms of the singular big points of D.

Recall that $k_1<k_2<\cdots<k_\nu$ denote the levels of D.

We begin by the case of small U.

PROPOSITION 3.2 (Index over small open discs). — *Let*

$$k=\frac{1}{s}\le k_1=\frac{1}{s_1}$$

and let $U=B(0,k)$ be the open disc of radius k centered at 0. *Then,*

$$H^0(U,\widetilde{\mathcal{F}})=\mathbb{C}[[x]]_{s^+}\quad\text{and}\quad\chi\big(D,\mathbb{C}[[x]]_{s^+}\big)=\chi\big(D,\mathbb{C}[[x]]\big).$$

Moreover,

$$\ker\big(D,\mathbb{C}[[x]]_{s^+}\big)=H^0\big(U,\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})\big)=\ker\big(D,\mathbb{C}[[x]]\big),$$

$$\operatorname{coker}\big(D,\mathbb{C}[[x]]_{s^+}\big)=H^0\big(U,\mathcal{C}oker(D,\widetilde{\mathcal{F}})\big)=\operatorname{coker}\big(D,\mathbb{C}[[x]]\big).$$

Proof. — In this case, $H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = 0$ (Proposition 2.5). The two sequences in the proof of the previous theorem merge into the exact sequence:

$$\begin{array}{ccccccccc} 0 \to H^0\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) \to & H^0\big(U, \widetilde{\mathcal{F}}\big) & \xrightarrow{D} & H^0(U, \widetilde{\mathcal{F}}) & \to H^0\big(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})\big) \to 0. \\ & \| & & \| & \\ & \mathbb{C}[[x]]_{s^+} & & \mathbb{C}[[x]]_{s^+} & \end{array} \tag{6}$$

Moreover,

$$H^0\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) = \ker\big(D, \mathbb{C}[[x]]\big) \qquad \text{(Lemma 2.3)},$$

$$H^0\big(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})\big) = \operatorname{coker}\big(D, \mathbb{C}[[x]]\big) \quad \text{(Lemma 2.6 (iii))}. \qquad \square$$

Similarly, considering small closed discs we can state:

Proposition 3.3 (Index over small closed discs). — *Let*

$$k = \frac{1}{s} < k_1 = \frac{1}{s_1}$$

and let $U = \overline{B}(0, k)$ *be the closed disc of radius* k *centered at* 0. *Then,*

$$H^0(U, \widetilde{\mathcal{F}}) = \mathbb{C}[[x]]_{s^-} \quad \text{and} \quad \chi\big(D, \mathbb{C}[[x]]_{s^-}\big) = \chi\big(D, \mathbb{C}[[x]]\big).$$

Moreover,

$$\ker\big(D, \mathbb{C}[[x]]_{s^-}\big) = H^0\big(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})\big) = \ker\big(D, \mathbb{C}[[x]]\big),$$

$$\operatorname{coker}\big(D, \mathbb{C}[[x]]_{s^-}\big) = H^0\big(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}})\big) = \operatorname{coker}\big(D, \mathbb{C}[[x]]\big).$$

Recall that the computation of $\chi(D, \mathbb{C}[[x]])$ can be found in [M74]. A direct identification method provides the dimensions of the kernel and the cokernel. The value of the index is up to sign the lower ordinate of points on the Newton polygon of D.

Recall also that a *k-big point* in X is an arc $I \times \{k\}$ where the interval I is closed of length π/k. It is a *singular big point of D* if it supports a non zero determining polynomial of D (*cf.* Definition 2.11).

Proposition 3.4 (Index over narrow sectors). — *Let* $U = I \times [k', k'')$, $k' > 0$, *be a narrow sector and* D' *a normal form of* D. *Then,*

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \dim H^0\big(U, \mathcal{S}o\ell(D', \widetilde{\mathcal{F}})\big).$$

More precisely,

$$\begin{aligned}\chi\big(D, H^0(U,\widetilde{\mathcal{F}})\big) &= \sharp\{j \mid \deg q_j < k'\} \\ &\qquad + \sharp\{j \mid \deg q_j \in [k', k'') \textit{ and } \exp q_j \textit{ is flat on } I\} \\ &= \sharp\{j \mid \deg q_j < k'\} \\ &\qquad + \sharp\big\{\textit{singular } k\textit{-big points of } D \\ &\qquad\qquad \textit{overlapping } U \textit{ on both sides}\big\}.\end{aligned}$$

By convention, $q = 0$ has degree $k_0 = 0$. In the ramified case, in order that this latter expression make sense, we assume that a choice of a determination of the argument of x has been done. The result does not depend on such a choice.

Proof. — In this case again $H^1(U, \mathcal{S}o\ell(D,\widetilde{\mathcal{F}})) = 0$. Moreover, $H^0(U, \mathcal{C}oker(D,\widetilde{\mathcal{F}})) = 0$ (Lemma 2.6 (ii)) and $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}})$ is isomorphic to $\mathcal{S}o\ell(D',\widetilde{\mathcal{F}})$ (Lemma 2.3). Then, Theorem 3.1 implies the first equality. The second one is obtained by counting the non zero solutions living on all of U. □

We further deduce the index of D for large U from the previous "local" ones using the following technique of Mayer-Vietoris sequences.

LEMMA 3.5 (Mayer-Vietoris Technique). — *Let $U = U_1 \cup U_2$ where U_1, U_2 are either open or closed subsets of U.*

If D is an isomorphism in $H^i(U,\widetilde{\mathcal{F}})$, $H^i(U_1,\widetilde{\mathcal{F}})$, $H^i(U_2,\widetilde{\mathcal{F}})$ and in $H^i(U_1 \cap U_2,\widetilde{\mathcal{F}})$ for $i \geq 1$ and if D has an index in $H^0(U_1,\widetilde{\mathcal{F}}), H^0(U_2,\widetilde{\mathcal{F}})$ and in $H^0(U_1 \cap U_2,\widetilde{\mathcal{F}})$ then it has an index in $H^0(U,\widetilde{\mathcal{F}})$ satisfying

$$\begin{aligned}\chi\big(D, H^0(U,\widetilde{\mathcal{F}})\big) &= \chi\big(D, H^0(U_1,\widetilde{\mathcal{F}})\big) + \chi\big(D, H^0(U_2,\widetilde{\mathcal{F}})\big) \\ &\qquad - \chi\big(D, H^0(U_1 \cap U_2,\widetilde{\mathcal{F}})\big).\end{aligned}$$

Proof. — Write the Mayer-Vietoris sequence of cohomology (see [Ive], Prop. 5.3 and 5.4):

$$\begin{aligned}0 \to H^0(U,\widetilde{\mathcal{F}}) &\longrightarrow \bigoplus_{\lambda=1,2} H^0(U_\lambda,\widetilde{\mathcal{F}}) \longrightarrow H^0(U_1 \cap U_2,\widetilde{\mathcal{F}}) \\ &\longrightarrow H^1(U,\widetilde{\mathcal{F}}) \longrightarrow \bigoplus_{\lambda=1,2} H^1(U_\lambda,\widetilde{\mathcal{F}}) \longrightarrow H^1(U_1 \cap U_2,\widetilde{\mathcal{F}}) \\ &\longrightarrow H^2(U,\widetilde{\mathcal{F}}) \longrightarrow \bigoplus_{\lambda=1,2} H^2(U_\lambda,\widetilde{\mathcal{F}}) \longrightarrow H^2(U_1 \cap U_2,\widetilde{\mathcal{F}}) \to 0.\end{aligned}$$

And apply the additivity of the Euler characteristic. □

The lemma below will also be useful to achieve the ramified case.

Consider a second copy X^t, $\widetilde{\mathcal{F}}^t, \dots$ of X, $\widetilde{\mathcal{F}}, \dots$ where the variable is t and denote by $D^t = \pi^* D$ (resp. ${D'}^t = \pi^* D'$) the pull back of D (resp. of D') by the p-fold ramified covering $\pi : X^t \longrightarrow X$, $t \mapsto x = t^p$.

LEMMA 3.6 (Ramification). — *Let U be a narrow sector not containing* 0. *Then D^t has an index on $\pi^{-1}(U)$ satisfying*

$$\chi\big(D^t, H^0(\pi^{-1}(U), \widetilde{\mathcal{F}}^t)\big) = p\,\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big).$$

Proof. — In this case, $\chi(D, H^0(U, \widetilde{\mathcal{F}})) = \dim H^0(U, \mathcal{S}o\ell(D', \widetilde{\mathcal{F}}))$. The inverse image $\pi^{-1}(U)$ of U being made of the disjoint union of p narrow sectors U^i for $i = 1, \dots, p$,

$$\chi\big(D^t, H^0(\pi^{-1}(U), \widetilde{\mathcal{F}}^t)\big) = \sum_{i=1}^{p} \dim H^0\big(U^i, \mathcal{S}o\ell({D'}^t, \widetilde{\mathcal{F}}^t)\big).$$

The result follows from the fact that π induces an isomorphism between $\mathcal{S}o\ell({D'}^t, \widetilde{\mathcal{F}}^t)_{|U^i}$ and $\mathcal{S}o\ell(D', \widetilde{\mathcal{F}})_{|U}$ for all i (choose a determination of the argument of x). □

PROPOSITION 3.7 (Index over annuli). — *Let $0 < k' \leq k'' \leq +\infty$ and let $U = S^1 \times [k', k'')$ be an annulus. Then,*

$$\begin{aligned}\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) &= \sum_{k_j \in [k', k'')} -k_j \,\sharp \{\ell \mid \deg q_\ell = k_j\} \\ &= -\sharp\big\{\textit{singular big points of } D \textit{ in } U \\ &\qquad\qquad \textit{counted with multiplicity}\big\}.\end{aligned}$$

In particular, when U is generic, $\chi(D, H^0(U, \widetilde{\mathcal{F}})) = 0$ and, when $U = S^1 \times [k, +\infty]$, then $2\,\chi(D, H^0(U, \widetilde{\mathcal{F}}))$ is the variation of the function $\theta \mapsto \dim \mathcal{V}_{\theta}^{\leq -k}$.

Proof. — Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \mathbb{Z}/\Lambda\mathbb{Z}}$ be a 2-finite covering of U by narrow sectors. The Mayer-Vietoris technique applied recursively to the U_λ's (Lemma 3.5) gives

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \sum_{\lambda \in \mathbb{Z}/\Lambda\mathbb{Z}} \big(\chi(D, H^0(U_\lambda, \widetilde{\mathcal{F}})) - \chi\big(D, H^0(U_\lambda \cap U_{\lambda+1}, \widetilde{\mathcal{F}})\big)\big). \tag{7}$$

There are as many U_λ's and $U_\lambda \cap U_{\lambda+1}$'s. Then, when U is generic, *i.e.*, when no level of D belongs to $[k', k'')$, trivially, Proposition 3.4 implies $\chi(D, H^0(U, \widetilde{\mathcal{F}})) = 0$.

When a unique level $k = k_j$ belongs to $[k', k'')$ the computation is a little more involved: suppose for the moment that k_j is an integer and consider the singular big points A_i for $i = 1, \ldots, k_j$ of a determining polynomial q_ℓ of degree k_j. For each A_i, consider the non empty intervals $I_{\lambda,i} = U_\lambda \cap A_i$. The exponential function $\exp q_\ell$ is flat on one more $I_{\lambda,i} \cap I_{\lambda+1,i}$ than $I_{\lambda,i}$. It thus contributes -1 to the value of the index at each of the k_j singular big points. Hence the result.

If, now, $k_j = \dfrac{r_j}{p}$, the Ramification Lemma 3.6 and the Formula (7) above imply that

$$\chi\big(D^t, H^0(\pi^{-1}(U), \widetilde{\mathcal{F}})\big) = p\,\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big).$$

The annulus $\pi^{-1}(U)$ is r-singular for D^t for the unique level $r = r_j$. The previous computation applies to D^t. The number of singular big points of D^t in $\pi^{-1}(U)$ is p times the number of singular big points of D in U. Recall that, in the ramified case, the singular big points are counted in a given sheet of the Riemann surface of the logarithm. Hence the result.

The general case results from Theorem 2.1 and Lemma 3.5 applied recursively to a suitable union of annuli of the previous types. □

Similarly, writing any disc as a suitable union of a disc and of annuli, Propositions 3.2 or 3.3 and 3.7 imply the following statement.

COROLLARY 3.8 (Index over discs). — *Let $0 \le k = \dfrac{1}{s} \le +\infty$ and let U be a disc either open or closed of radius k. Then,*

$$H^0(U, \widetilde{\mathcal{F}}) = \begin{cases} \mathbb{C}\{x\} & \text{if } U = X, \\ \mathbb{C}[[x]]_{s^+} & \text{if } U = B(0,k),\ 0 \ne k \ne +\infty, \\ \mathbb{C}[[x]]_{s^-} & \text{if } U = \overline{B}(0,k),\ 0 \ne k \ne +\infty, \\ \mathcal{C} & \text{if } U = \{0\}, \end{cases}$$

and

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \chi\big(D, \mathbb{C}[[x]]\big) - \sharp\{\text{singular big points of } D \text{ in } U \text{ counted with multiplicity}\}.$$

The function $s \mapsto \chi(D, \mathbb{C}[[x]]_{s^+})$ is piecewise constant, increasing and upper semicontinuous; it satisfies

$$\chi\big(D, \mathbb{C}[[x]]_{s^+}\big) = \begin{cases} \chi(D, \mathbb{C}\{x\}) & \text{if } s < s_\nu, \\ \chi(D, \mathbb{C}[[x]]_{s_j^+}) & \text{if } s_j \le s < s_{j-1} \text{ for } j = \nu, \dots, 2, \\ \chi(D, \mathbb{C}[[x]]) & \text{if } s_1 \le s, \end{cases}$$

and the discontinuities are given by

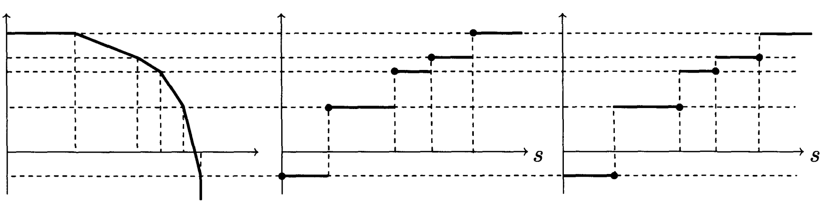
$$\chi\big(D, \mathbb{C}[[x]]_{s_j^+}\big) - \chi\big(D, \mathbb{C}[[x]]_{s_{j+1}^+}\big) = k_j \,\sharp\, \{\ell \mid \deg q_\ell = k_j\}.$$

The function $s \mapsto \chi(D, \mathbb{C}[[x]]_{s^-})$ is piecewise constant, increasing and lower semicontinuous; it satisfies

$$\chi(D, \mathbb{C}[[x]]_{s^-}) = \begin{cases} \chi(D, \mathbb{C}\{x\}) & \text{if } s \le s_\nu, \\ \chi(D, \mathbb{C}[[x]]_{s_{j-1}^-}) & \text{if } s_j < s \le s_{j-1} \text{ for } j = \nu, \dots, 2, \\ \chi(D, \mathbb{C}[[x]]) & \text{if } s_1 < s. \end{cases}$$

Moreover, $\chi(D, \mathbb{C}[[x]]_{s_{j-1}^-}) = \chi(D, \mathbb{C}[[x]]_{s_j^+})$ for $j = \nu, \dots, 2$ and then, $\chi(D, \mathbb{C}[[x]]_{s^-}) = \chi(D, \mathbb{C}[[x]]_{s^+})$ whenever $s \neq s_1, \dots, s_\nu$.

Recall that $k_1 < \cdots < k_\nu$ are the levels of D and that $s_1 = \dfrac{1}{k_1}, \dots, s_\nu = \dfrac{1}{k_\nu}$.



Symmetric of the Newton polygon of D $\qquad$ $s \mapsto \chi(D, \mathbf{C}[[x]]_{s^+})$ $\qquad$ $s \mapsto \chi(D, \mathbf{C}[[x]]_{s^-})$

Figure 3.1

Proposition 3.9 (Index over big points). — *Let $k > 0$ and let U be a k-big point $I \times \{k\}$ or a sector closely neighboring it. Let $\mathrm{mult}_D(U)$ denote the multiplicity of U as a singular big point of D. Then,*

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \sharp\{\, j \mid \deg q_j < k\} \ - \mathrm{mult}_D(U).$$

In particular, when U is generic, $\chi(D, H^0(U, \widetilde{\mathcal{F}})) = \sharp\{\, j \mid \deg q_j < k\}$.

Proof. — We proceed like in Proposition 3.7. Suppose first that k is an integer. To the covering $\mathcal{U} = \{U_\lambda\}_{\lambda=1,\dots,\Lambda}$ there is the equality

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \sum_{\lambda=1}^{\Lambda} \chi\big(D', H^0(U_\lambda, \widetilde{\mathcal{F}})\big) - \sum_{\lambda=1}^{\Lambda-1} \chi\big(D', H^0(U_\lambda \cap U_{\lambda+1}, \widetilde{\mathcal{F}})\big).$$

There is one more U_λ than $U_\lambda \cap U_{\lambda+1}$ and then, each $\exp q_j$ of degree $< k$ contributes $+1$ to the value of the index; hence the term $\sharp\{j \mid \deg q_j < k\}$. For the second term we only have to count the determining exponentials of level k which are flat on each I_λ and $I_\lambda \cap I_{\lambda+1}$ (recall the notations $U_\lambda = I_\lambda \times \{k\}$ or $U_\lambda = I_\lambda \times [k', k'')$). Now, if an exponential of level k is not supported by U, it is flat on as many I_λ's and $I_\lambda \cap I_{\lambda+1}$'s. The balance is 0. Differently, if it is supported by U it is flat on one more $I_\lambda \cap I_{\lambda+1}$ than I_λ. Hence the result.

The case when k is rational is proved like in Proposition 3.7. □

At last, writing a $(k_1, \dots, k_\nu)$-multisector as a suitable union of a disc and of k_j-sectors, $j = 1, \dots, \nu$, we prove:

COROLLARY 3.10 (Index over multisectors). — *Let $\underline{k} = (k_1, \dots, k_\nu)$ and let U be a $\underline{k}$-multisector. Then, D has an index in $H^0(U, \widetilde{\mathcal{F}})$ satisfying*

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big) = \chi\big(D, \mathbb{C}[[x]]\big) - \sharp\big\{\text{singular big points of } D \text{ in } U \text{ counted with multiplicity}\big\}.$$

In particular, when U is generic, $\chi(D, H^0(U, \widetilde{\mathcal{F}})) = \chi(D, \mathbb{C}[[x]])$.

We leave to the reader the formulation of the index of D over an arbitrary sector or multisector.

We list the values of the previous indices in the Table 3.1 next page.

We end this section by noticing that, because Lemma 3.6 is true for small discs and narrow sectors, it is also true for all the U considered above and we can state:

PROPOSITION 3.11. — *Let U be a disc, a sector, an annulus or a multisector in X. With the notations of Lemma 3.6, D^t has an index over $\pi^{-1}(U)$ satisfying*

$$\chi\big(D^t, H^0(\pi^{-1}(U), \widetilde{\mathcal{F}})\big) = p\,\chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big).$$

$U \subset X$	Index $\chi(D, H^0(U, \widetilde{\mathcal{F}}))$
In general	$\dim H^0(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) - \dim H^1(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}))$ $- \dim H^0(U, \mathcal{C}oker(D, \widetilde{\mathcal{F}}))$
Small disc	$\chi(D, \mathbb{C}[[x]])$
Narrow sector $U{=}I{\times}[k', k'')$	$\dim H^0(U, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}})) = \sharp\{j \mid \deg q_j < k'\}$ $+\sharp\{$singular big points of D overlapping U on both sides$\}$
k-big point	$\sharp\{j \mid \deg q_j < k\} - \mathrm{mult}_D(U)$
Annulus	$-\sharp\{$singular big points of D in $U\}$
Disc, Multisector	$\chi(D, \mathbb{C}[[x]]) - \sharp\{$singular big points of D in $U\}$

Table 3.1

Note. — All singular big points are counted with multiplicity.

4. The sheaf $\widetilde{\mathcal{F}}^k$ and its application to index theorems of D.

In this section, we introduce a topological space X^k and a sheaf $\widetilde{\mathcal{F}}^k$ over X^k which takes into account the exponential types of growth and decay corresponding to a given exponential order k. Then, applying the technique developed in Section 3, we prove index theorems in this space. In particular, we obtain the index theorems in the Gevrey series spaces $\mathbb{C}[[x]]_s$, $\mathbb{C}[[x]]_{s,(\frac{1}{c})^\pm}$ and $\mathbb{C}[[x]]_{(s)}$ due to J.-P. Ramis.

The spaces X^k.

Let $k > 0$ and Y^k be a copy of the annulus $S^1 \times]0, +\infty[$.

The space X^k is the topological space obtained from X by substituting the "closed" annulus $\overline{Y}^k = S^1 \times [0, +\infty]$ to the circle $S^1 \times \{k\}$. More precisely, one identifies, on one side, the boundary $S^1 \times \{k\}$ of the open disk $B(0, k)$ of X to the boundary $S^1 \times \{0\}$ of $\overline{Y}^k$, and, on the other side, the boundary $S^1 \times \{+\infty\}$ of $\overline{Y}^k$ to the boundary $S^1 \times \{k\}$ of $X \setminus B(0, k)$.

As a topological space, the space X^k is isomorphic to X. However, the forthcoming definition of the presheaf $\mathcal{F}^k$ over an open U depends on whether the open U is outside the annulus $\overline{Y}^k$, inside $\overline{Y}^k$ or across any of the two boundary circles of $\overline{Y}^k$.

Let $(\theta, \{k, \rho\})$ denote the polar coordinates in $\overline{Y}^k$ and $\pi_k : X^k \to X$ denote the canonical surjection.

For all $k' > 0$, we call *open disc of radius* k' *in* X^k, and we keep denoting $B(0, k')$, the inverse image $\pi_k^{-1}(B(0, k'))$ of the open disc of radius k' centered at 0 in X. For $c \in [0, +\infty]$, we call *open disc* (resp. *closed disc*) of "radius" $\{k, c\}$ the union of $B(0, k)$ and of $S^1 \times [0, c[$ (resp. $S^1 \times [0, c]$). The definition of a *sector* $I \times]k', \{k, c\}[$, and so on … is similar.

Open discs and open sectors form a basis of open sets in X^k.

The sheaf $\widetilde{\mathcal{F}}^k$ over X^k.

Let $k > 0$. The sheaf $\widetilde{\mathcal{F}}^k$ corresponds to the presheaf $\mathcal{F}^k$ defined as follows:

- inside $X \setminus \overline{Y}^k$, let $\mathcal{F}^k = \mathcal{F}$;
- inside Y^k, define

$$\mathcal{F}^k\big(I \times]\{k, c'\}, \{k, c''\}[\big) = H^0\big(I, \mathcal{A}^{\leq k, c'^-} / \mathcal{A}^{\leq -k, c''^+}\big);$$

- across the boundary of Y^k patch these two definitions by setting

$$\mathcal{F}^k\big(I \times]k', \{k, c\}[\big) = H^0\big(I, \mathcal{A}^{\leq k'} / \mathcal{A}^{\leq -k, c^+}\big),$$

$$\mathcal{F}^k\big(I \times]\{k, c\}, k''[\big) = H^0\big(I, \mathcal{A}^{\leq k, c^-} / \mathcal{A}^{\leq -k''}\big).$$

Proposition 1.1 (i) makes this definition consistent.

The sheaf $\widetilde{\mathcal{F}}^k$ is a sheaf of $\mathbb{C}\{x\}$-modules and no more a sheaf of $\mathbb{C}$-algebras. In restriction to the circles $S^1 \times \{k, c\}$, $0 \leq c \leq +\infty$, it satisfies

$$\widetilde{\mathcal{F}}^k{}_{|S^1 \times \{k,c\}} = \begin{cases} \mathcal{A}^{\leq k^-} / \mathcal{A}^{\leq -k} & \text{if } c = 0, \\ \mathcal{A}^{\leq k, c^-} / \mathcal{A}^{\leq -k, c^+} & \text{if } 0 < c < +\infty, \\ \mathcal{A}^{\leq k} / \mathcal{A}^{\leq -k^+} & \text{if } c = +\infty. \end{cases}$$

Moreover, Proposition 1.1 (i) (ii) implies the equalities

$$H^0\big(B(0, \{k, c\}), \widetilde{\mathcal{F}}^k\big) = \begin{cases} \mathbb{C}[[x]]_{s^+} & \text{if } c = 0, \\ \mathbb{C}[[x]]_{s,(\frac{1}{c})^+} & \text{if } 0 < c < +\infty, \\ \mathbb{C}[[x]]_{(s)} & \text{if } c = +\infty, \end{cases}$$

$$H^0\big(\overline{B}(0,\{k,c\}),\widetilde{\mathcal{F}}^k\big)=\begin{cases}\mathbb{C}[[x]]_s & \text{if } c=0,\\ \mathbb{C}[[x]]_{s,(\frac{1}{c})^-} & \text{if } 0<c<+\infty,\\ \mathbb{C}[[x]]_{s^-} & \text{if } c=+\infty,\end{cases}$$

and if I is an interval of S^1

$$H^0\big(I\times[0,\{k,c\}],\widetilde{\mathcal{F}}^k\big)=\begin{cases}H^0(I,\mathcal{A}/\mathcal{A}^{\leq -k}) & \text{if } c=0,\\ H^0(I,\mathcal{A}/\mathcal{A}^{\leq -k,c^+}) & \text{if } 0<c<+\infty,\\ H^0(I,\mathcal{A}/\mathcal{A}^{\leq -k^+}) & \text{if } c=+\infty.\end{cases}$$

With $\widetilde{\mathcal{F}}^k$ we shall prove, in particular, index theorems of D in the spaces $\mathbb{C}[[x]]_s$, $\mathbb{C}[[x]]_{(s)}$, $\mathbb{C}[[x]]_{s,(1/c)^+}$ and $\mathbb{C}[[x]]_{s,(1/c)^-}$.

NOTATIONS. — Similarly to the notations in Sections 2 and 3, we denote by $B(0,\{k,c\})$ the open disc in X^k centered at 0, union of the disc $B(0,k)$ and of the annulus $S^1\times[\{k,0\},\{k,c\}[$ and we denote by $\overline{B}(0,\{k,c\})$ its closure.

Recall that we denote by $(\theta,\{k,\rho\})$ the polar coordinates in $\overline{Y}^k$.

A typical section of the sheaf $\widetilde{\mathcal{F}}^k$: $\exp q(\frac{1}{x})$ when $\deg q = k$.

The definition set of the exponential function $\exp q\Big(\dfrac{1}{x}\Big)$ as a section of $\widetilde{\mathcal{F}}^k$ when $\deg q = k$ is an integer is likely the open shadowed subset in Figure 4.1.

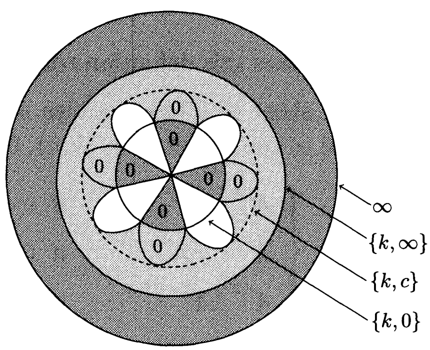


Figure 4.1 (*Here* $k=4$)

Sectors are now replaced by petals. The complement of the support is likely a daisy and $\exp q$ equals 0 on one petal over two. The closure of the pieces of petals in $\overline{Y}^k$ where $\exp q$ vanishes are the *singular big points* of q in X^k. Like in the case of $(X, \widetilde{\mathcal{F}})$ they will play a central role with respect to the indices of D.

ASSUMPTION 4.1. — From now and without further mention we assume that sectors and multisectors in $(X^k, \widetilde{\mathcal{F}}^k)$ satisfy conditions similar to those of Assumption 2.1 in $(X, \widetilde{\mathcal{F}})$.

DEFINITION 4.1.

1) A *k-big point in X^k* is the closure of a connected component of the set where an exponential function $\exp -\dfrac{A\,\mathrm{e}^{i\alpha}}{x^k}$, $A > 0$ vanishes in $\overline{Y}^k$. It is then said to *support the polynomials* $q\Big(\dfrac{1}{x}\Big) = -\dfrac{A\,\mathrm{e}^{i\alpha}}{x^k} + o\Big(\dfrac{1}{x^k}\Big)$. We also consider the limit case when A tends to 0 and, in the ramified case, we choose a determination $\widetilde{\alpha}$ of the argument α.

A k-big point has a polar equation in $\overline{Y}^k$ of the form

$$\rho \le A\cos\big(k(\alpha-\theta)\big), \quad k(\alpha-\theta) \in \big[-\tfrac{1}{2}\pi + 2\ell\pi, \tfrac{1}{2}\pi + 2\ell\pi\big], \ \ell \in \mathbb{Z}.$$

When $k' \neq k$ we define *k'-big points in X^k* like in X (Definition 2.11).

2) The top point $a = (\alpha, \{k, A\}) \in \overline{Y}^k$ of a k-big point supporting q is called a *k-anti-Stokes point of q*. We denote by $\operatorname{sing}^k(q)$ the set of all k-anti-Stokes points of q. If $\deg q \neq k$ then $\operatorname{sing}^k(q) = \emptyset$.

3) A subset U of X^k is said *generic for q* if it contains no big point supporting q. It is said *generic for D* if it is generic for all (non zero) determining polynomial q_j for $j = 1, \dots, \nu$ of D. Otherwise, it is said *singular for q* or *for D*.

The action of D on the cohomology of $\widetilde{\mathcal{F}}^k$.

The differential operator D induces a sheaf morphism on $\widetilde{\mathcal{F}}^k$ and then also on its cohomology groups. The statements in Section 2 have their analogs in $(X^k, \widetilde{\mathcal{F}}^k)$; in particular, the following isomorphism theorem holds:

THEOREM 4.2. — *The linear maps*

$$D : H^i(U, \widetilde{\mathcal{F}}^k) \longrightarrow H^i(U, \widetilde{\mathcal{F}}^k) \quad \textit{for } i \ge 1$$

are isomorphisms when U is a disc, a sector, a multisector or an annulus (satisfying Assumption 4.1).

The proofs are similar to those in Section 2 by changing, when necessary, the basic theorems relative to a given order by Proposition 1.1 relative to a given type. We only point out some specificity.

- To prove that narrow sectors are acyclic for $\mathcal{S}o\ell(D,\widetilde{\mathcal{F}}^k)$ (*cf.* Proposition 2.5) Figure 2.2 has to be changed into Figure 4.2; however, again, $U \setminus A$ is connected, $U \setminus A$ and $U \setminus B$ are simply connected.

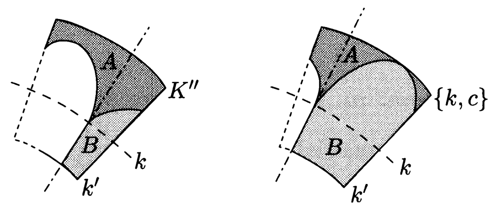


Figure 4.2

- To prove that the coboundary map

$$\delta : H^0\big(U, \mathcal{I}m(D,\widetilde{\mathcal{F}}^k)\big) \longrightarrow H^1\big(U, \mathcal{S}o\ell(D,\widetilde{\mathcal{F}}^k)\big)$$

is onto when U is an annulus or a sector not containing 0 we require the triviality of $H^1(S^1, \mathcal{A}^{\leq k,c^-})$ if $c \in\,]0,+\infty[$ (Proposition 1.1 (iv)) and the triviality of $H^1(S^1, \mathcal{A}^{\leq k})$ if $c = +\infty$ (Malgrange Lemma).

Index theorems for D in $(X^k, \widetilde{\mathcal{F}}^k)$.

The General Index Theorem (Theorem 3.1) has its analog in $(X^k, \widetilde{\mathcal{F}}^k)$:

THEOREM 4.3 (General Index Theorem in $(X^k, \widetilde{\mathcal{F}}^k)$). — *Let U^k be a disc, a sector, a multisector or an annulus satisfying Assumption* 4.1. *Then, D has an index in $H^0(U^k, \widetilde{\mathcal{F}}^k)$ given by*

$$\chi\big(D, H^0(U^k,\widetilde{\mathcal{F}}^k)\big) = \dim H^0\big(U^k, \mathcal{S}o\ell(D,\widetilde{\mathcal{F}}^k)\big) - \dim H^1\big(U^k, \mathcal{S}o\ell(D,\widetilde{\mathcal{F}}^k)\big) - \dim H^0\big(U^k, \mathcal{C}oker(D,\widetilde{\mathcal{F}}^k)\big)$$

where

$$H^0\big(U^k, \mathcal{C}oker(D,\widetilde{\mathcal{F}}^k)\big) = \begin{cases} 0 & \text{if } 0 \notin U^k, \\ \operatorname{coker}(D, \mathbb{C}[[x]]) & \text{if } 0 \in U^k. \end{cases}$$

Moreover, Theorem 4.2 implies that the Mayer-Vietoris Technique is valid in $(X^k, \widetilde{\mathcal{F}}^k)$ allowing thus to deduce index theorems over large discs, sectors, ..., from index theorems over small discs and narrow sectors like in $(X, \widetilde{\mathcal{F}})$.

Over small discs, here, discs of radius less than k and k_1, the sheaf $\widetilde{\mathcal{F}}^k$ is $\widetilde{\mathcal{F}}$. Over narrow sectors, the analog of Proposition 3.4 holds:

PROPOSITION 4.4 (Index over narrow sectors of X^k). — *Let U^k be a narrow sector in X^k and D' be a normal form of D. Then,*

$$\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) = \dim H^0\big(U^k, \mathcal{S}o\ell(D', \widetilde{\mathcal{F}}^k)\big).$$

However, the computation of $\dim H^0(U^k, \mathcal{S}o\ell(D', \widetilde{\mathcal{F}}^k))$ when $U^k = I \times [K', \{k, c\})$ and c is a k-characteristic constant of D requires some attention: let $\mathcal{S}$ be the subsheaf of $\widetilde{\mathcal{F}}^k$ generated by one $\exp q_\ell$ with $\deg q_\ell = k$ and suppose that a singular point $a \in \operatorname{sing}^k(q_\ell)$ (*cf.* Definition 4.1) belongs to the interior of the boundary arc $I \times \{k, c\}$ as shown in Figure 4.3.

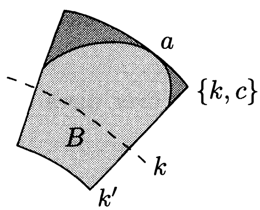


Figure 4.3

Notice that, because of the narrowness of I, such a singular point is necessarily unique. Then, $U^k \setminus B$ has one connected component when $a \in U^k$ (case U^k closed for instance) and two when $a \notin U^k$. If, in addition, we assume that U^k is very narrow ($|I| < \pi/2k_\nu$) then, we can state:

LEMMA 4.5. — *Under the preceding conditions*

$$\dim H^0(U^k, \mathcal{S}) = \begin{cases} 1 & \text{if } U^k = I \times [k', \{k, c\}], \\ 2 & \text{if } U^k = I \times [k', \{k, c\}[. \end{cases}$$

This phenomenon does not occur when c is not a k-characteristic constant of D.

We now formulate the values of the indices of D corresponding to large U^k.

PROPOSITION 4.6 (Index over annuli of X^k). — *Let $U^k = S^1 \times [K', K'')$ be an annulus in X^k.*

(i) *Let $0 \leq c' \leq c'' \leq +\infty$ and $K' = \{k, c'\}$, $K'' = \{k, c''\}$. Then,*

$$\begin{aligned}\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) &= -\sum_{\ell=1}^{n} \sharp\,\{U^k \cap \operatorname{sing}^k(q_\ell)\}\\ &= -\sharp\big\{k\text{-anti-Stokes points of } D \text{ in } U^k\\ &\qquad\qquad \text{counted with multiplicity}\big\}.\end{aligned}$$

(ii) *For general K' and K'',*

$$\begin{aligned}&\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big)\\ &\quad = -\sharp \bigcup_{\tilde{k}\neq k} \{\text{singular } \tilde{k}\text{-big points in } U^k \text{ with multiplicity}\}\\ &\qquad\qquad - \sharp\{k\text{-anti-Stokes points in } U^k \text{ with multiplicity}\}.\end{aligned}$$

PROPOSITION 4.7 (Index over discs of X^k). — *Let U^k be a disc of radius K in X^k.*

(i) *Let $K = \{k, c\}$. Then,*

$$H^0(U^k, \widetilde{\mathcal{F}}^k) = \begin{cases} \mathbb{C}[[x]]_{s^+} & \text{if } U^k = B(0, \{k, 0\}),\\ \mathbb{C}[[x]]_{s} & \text{if } U^k = \overline{B}(0, \{k, 0\}),\\ \mathbb{C}[[x]]_{s,(1/c)^+} & \text{if } U^k = B(0, \{k, c\}),\ 0 \neq c \neq +\infty,\\ \mathbb{C}[[x]]_{s,(1/c)^-} & \text{if } U^k = \overline{B}(0, \{k, c\}),\ 0 \neq c \neq +\infty,\\ \mathbb{C}[[x]]_{(s)} & \text{if } U^k = B(0, \{k, +\infty\}),\\ \mathbb{C}[[x]]_{s^-} & \text{if } U^k = \overline{B}(0, \{k, +\infty\}),\end{cases}$$

and

$$\begin{aligned}\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) &= \chi\big(D, \mathbb{C}[[x]]_{s^+}\big)\\ &\quad - \sharp\big\{\text{singular } k\text{-big points of } D \text{ in } U^k \text{ with multiplicity}\big\}\\ &= \chi\big(D, \mathbb{C}[[x]]\big)\\ &\quad - \sharp\big\{\text{singular big points of } D \text{ in } U^k \text{ with multiplicity}\big\}.\end{aligned}$$

In particular,

- $\chi\big(D, \mathbb{C}[[x]]_{s^+}\big) = \chi\big(D, \mathbb{C}[[x]]_s\big)$
$\qquad = \chi\big(D, \mathbb{C}[[x]]_{s,(1/c)^\pm}\big)$ *when c is small;*

- $\chi\big(D, \mathbb{C}[[x]]_{s,(1/c)^+}\big) = \chi\big(D, \mathbb{C}[[x]]_{s,(1/c)^-}\big)$ *when c is not a k-characteristic constant of D;*

- $\chi\big(D, \mathbb{C}[[x]]_{(s)}\big) = \chi\big(D, \mathbb{C}[[x]]_{s^-}\big)$

$$= \chi\big(D, \mathbb{C}[[x]]_{s,(1/c)^\pm}\big) \text{ when } c \text{ is large.}$$

The function $C = \dfrac{1}{c} \mapsto \chi(D, \mathbb{C}[[x]]_{s,C^+})$ is increasing and upper semicontinuous.

The function $C = \dfrac{1}{c} \mapsto \chi(D, \mathbb{C}[[x]]_{s,C^-})$ is increasing and lower semicontinuous.

(ii) *Let $K < k$ or $K > k$ and let $U = \pi_k^{-1}(U^k)$ denote the disc of radius K induced by U^k in X. Then,*

$$\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) = \chi\big(D, H^0(U, \widetilde{\mathcal{F}})\big).$$

PROPOSITION 4.8 (Index over k-big points). — *Let U^k be a k-big point and $\check{U}^k$ be the smallest closed sector containing U. Let* $\mathrm{mult}_D(U^k)$ *denote the multiplicity of U^k as a singular k-big point of D in X^k. Then,*

$$\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) = \chi\big(D, H^0(\check{U}^k, \widetilde{\mathcal{F}}^k)\big)$$
$$= \sharp\big\{j \mid \deg q_j < k\big\} - \mathrm{mult}_D(U^k).$$

In particular, when U^k is generic,

$$\chi\big(D, H^0(U^k, \widetilde{\mathcal{F}}^k)\big) = \sharp\big\{j \mid \deg q_j < k\big\}.$$

We end this section extending Lemma 4.5 to various arcs and summarizing the results in Figure 4.4.

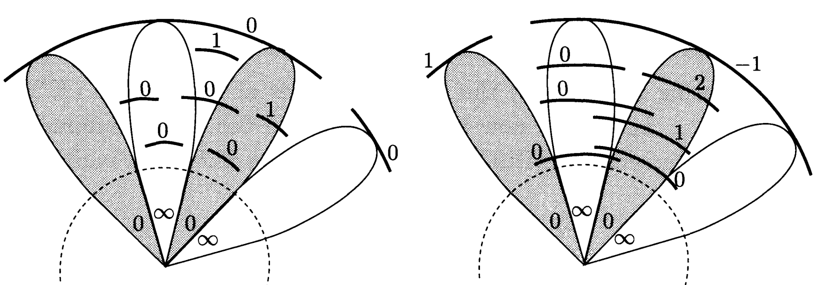


Figure 4.4

The petals are those of a determining exponential $\exp q_\ell$. The number beside each arc represents the contribution of $\exp q_\ell$ to the value of the index.

$U^k \subset X^k$	Index $\chi(D, H^0(U^k, \widetilde{\mathcal{F}}^k))$
In general	$\dim H^0(U^k, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}^k)) - \dim H^1(U^k, \mathcal{S}o\ell(D, \widetilde{\mathcal{F}}^k))$ $- \dim H^0(U^k, \mathcal{C}oker(D, \widetilde{\mathcal{F}}^k))$
Small disc	$\chi(D, \mathbb{C}[[x]])$
k-big point	$\sharp\{j \mid \deg q_j < k\} - \mathrm{mult}_D(U^k)$
Annulus	$-\sharp \bigcup_{\tilde{k} \neq k} \{\text{singular } \tilde{k}\text{-big points in } U^k\}$ $-\sharp\{k\text{-anti-Stokes points in } U^k\}$
Disc, Multisector	$\chi(D, \mathbb{C}[[x]]) - \sharp\{\text{singular big points of } D \text{ in } U^k\}$

Table 4.1

Note. — All singular big points are counted with multiplicity.

5. The sheaf $\widetilde{\mathcal{F}}^{\underline{k}}$, multisummability and index theorems of D.

In this section, we denote by $\underline{k}$ a multi-index $\underline{k} = (k_1, \ldots, k_\nu)$ satisfying

$$0 < k_1 < k_2 < \cdots < k_\nu < +\infty.$$

Repeating at $k_1, \ldots, k_\nu$ the construction made from $(X, \widetilde{\mathcal{F}})$ at k to build $(X^k, \widetilde{\mathcal{F}}^k)$ gives rise to a topological space $X^{\underline{k}}$ and to a sheaf $\widetilde{\mathcal{F}}^{\underline{k}}$ which takes into account the exponential types of growth and decay corresponding to the finitely many exponential orders $k_1, \ldots, k_\nu$.

Gathering in $(X^{\underline{k}}, \widetilde{\mathcal{F}}^{\underline{k}})$ the local properties of the sheaves $(X^k, \widetilde{\mathcal{F}}^k)$ for $k = k_1, \ldots, k_\nu$ we are now able to formulate the $\underline{k}$-multisummability condition in the sense of [MR92] in terms of sections of $\widetilde{\mathcal{F}}^{\underline{k}}$ over multisectors in $X^{\underline{k}}$.

Definition 5.1. — A *$\underline{k}$-multisector U in $X^{\underline{k}}$* is a finite union

$$U = \overline{B}(0, \{k_1, 0\}) \cup \bigcup_{j=1}^{\nu} \left(I_j \times [0, \{k_{j+1}, 0\}]\right)$$

where, for $j = 1, \ldots, \nu$, I_j is a closed interval of length $|I_j| = \pi/k_j$ and $I_1 \supset \cdots \supset I_\nu$.

By convention, $\{k_{\nu+1}, 0\} = +\infty$.

Notice that a multisector U in $X^{\underline{k}}$ is a closed subset of $X^{\underline{k}}$.

The definition of *k-big points in X^k* is the obvious one.

With the previous definition and the same notations one has the following correspondence:

PROPOSITION 5.2. — *Let U be a $\underline{k}$-multisector in $X^{\underline{k}}$. Then, $H^0(U, \widetilde{\mathcal{F}}^{\underline{k}})$ is the space of the series which are $\underline{k}$-multisummable on $(I_1, \dots, I_\nu)$ in the sense of* [MR92], *Déf.* 2.2.

Moreover, the previous properties of $\widetilde{\mathcal{F}}^k$ can be brought to $\widetilde{\mathcal{F}}^{\underline{k}}$. Thus, using again the technique of Mayer-Vietoris sequences developed in Sections 3 and 4, we are able to prove index theorems in $H^0(U, \widetilde{\mathcal{F}}^{\underline{k}})$ when U is a disc, a multisector, an annulus, a sector or a big point. Let formulate the following one:

THEOREM 5.3. — *Let U be a $\underline{k}$-multisector in $X^{\underline{k}}$. Then, D has an index in the space of multisummable series $H^0(U, \widetilde{\mathcal{F}}^{\underline{k}})$ and*

$$\begin{aligned}&\chi(D, H^0(U, \widetilde{\mathcal{F}}^{\underline{k}}))\\ &\quad = \chi\big(D, \mathbb{C}[[x]]\big) - \sharp\{\textit{singular big points of } D \textit{ in } U \textit{ with multiplicity}\}\end{aligned}$$

In particular, if U is generic, i.e., if U contains no singular big point of D, then

$$\chi\big(D, H^0(U, \widetilde{\mathcal{F}}^{\underline{k}})\big) = \chi\big(D, \mathbb{C}[[x]]\big).$$

This result shows in particular that, in case a Stokes multiplier of D vanishes, the cokernel of D increases as much as the kernel. It shows that generic k-multisectors are acyclic for $\mathcal{S}o\ell(D, \widetilde{\mathcal{F}})$. Compare to Theorem 3.1 and Theorem 4.3.

We end in noticing that the index formulæ stated in Sections 3, 4 and 5 prove that all the indices considered here are, modulo $\chi(D, \mathbb{C}[[x]])$, formal meromorphic invariants of D.

6. Conclusion.

Using a general procedure we have stated index theorems for a differential operator D over large sets from index theorems over small ones in the spaces $(X, \widetilde{\mathcal{F}})$, $(X^k, \widetilde{\mathcal{F}}^k)$ and $(X^{\underline{k}}, \widetilde{\mathcal{F}}^{\underline{k}})$.

The index of D over a small disc centered at 0 is proved to be its index in the space of formal power series $\mathbb{C}[[x]]$ and this latter index has been computed by B. Malgrange in [M74]. Over a small sector, the index of D is the index of a normal form D' of D and equals the number of solutions of D' defined on all of the sector.

The procedure applied over discs in the space $(X, \widetilde{\mathcal{F}})$ provides, in particular, the index of D in the spaces $\mathbb{C}[[x]]_{s^+}$, $\mathbb{C}[[x]]_{s^-}$ and $\mathbb{C}\{x\}$. The index in $\mathbb{C}\{x\}$ has already received (at least) two proofs, one quite different and based on functional analysis by B. Malgrange (see [M74]) and another one by P. Deligne and B. Malgrange (*cf.* Appendix) which is somewhat similar to ours. The indices in $\mathbb{C}[[x]]_{s^+}$ and $\mathbb{C}[[x]]_{s^-}$, although not explicitely formulated by J.-P. Ramis in [R84], are an easy consequence of the index theorems in Gevrey spaces stated there.

The procedure applied over discs in the spaces $(X^k, \widetilde{\mathcal{F}}^k)$ provides, in particular, the index of D in the spaces $\mathbb{C}[[x]]_s$, $\mathbb{C}[[x]]_{(s)}$, $\mathbb{C}[[x]]_{s,(1/c)^+}$, $\mathbb{C}[[x]]_{s,(1/c)^-}$. These indices are those stated by J.-P. Ramis in [R84] using the quite different approach of functional analysis initiated by B. Malgrange in [M74]. Let us mention that a differential operator D does not have an index in $\mathbb{C}[[x]]_{s,1/c}$ in general. Take, for instance, the Euler operator $D = x^2\dfrac{\mathrm{d}}{\mathrm{d}x} - 1$. It is an operator in $\mathbb{C}[[x]]_{1,1}$ and the series $\sum\limits_{n>1} (n-1)!\, n^\alpha x^n$ for $0 < \alpha < 1$ generate an infinite dimensional subspace of $\operatorname{coker}(D, \mathbb{C}[[x]]_{1,1})$. Thus, D does not have an index in $\mathbb{C}[[x]]_{1,1}$.

The procedure applied over $\underline{k}$-multisectors in the spaces $(X^{\underline{k}}, \widetilde{\mathcal{F}}^{\underline{k}})$ where $\underline{k} = (k_1, \ldots, k_\nu)$ such that $0 < k_1 < \cdots < k_\nu$ provides index theorems for D acting in the spaces of $\underline{k}$-multisummable series.

The procedure has also been applied over annuli, large sectors and multisectors and big points.

The values of these indices have been computed in terms of formal meromorphic invariants of D and, especially, in terms of singular big points of D.

7. Appendix.

We give here the computation of the irregularity $\mathrm{irr}_0(D)$ after Deligne (letter to Malgrange [D77]) and Malgrange ([M79]). See also [BV89], Prop. 3.3.1, 3.3.2 and 3.3.3.

PROPOSITION 7.1 (Deligne-Malgrange). — *Let $N(\theta)$ denote the dimension of the stalk $\mathcal{V}_\theta^{<0}$ of $\mathcal{V}^{<0}$ at $\theta \in S^1$ and* $\mathrm{var}N$ *denote the variation of the function $\theta \mapsto N(\theta)$ on S^1. Then,*

(i) $\mathrm{irr}_0(D) = \dim H^1(S^1, \mathcal{V}^{<0})$,

(ii) $\dim H^1(S^1, \mathcal{V}^{<0}) = \frac{1}{2}\mathrm{var}N$.

Proof.

(i) Writing the long exact sequence of cohomology corresponding to

$$0 \to \mathcal{A}^{<0} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}^{<0} \to 0$$

and using the Theorems of Borel-Ritt and Cauchy-Heine give an exact diagram

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & H^0(S^1,\mathcal{A}) \arrow[r] \arrow[d, equal] & H^0(S^1,\mathcal{A}/\mathcal{A}^{<0}) \arrow[r] \arrow[d, equal] & H^1(S^1,\mathcal{A}^{<0}) \arrow[r] \arrow[dr] & H^1(S^1,\mathcal{A}) \\ & \mathbb{C}\{x\} & \mathbb{C}[[x]] & & 0 \arrow[u] \end{tikzcd}$$

Therefore, $H^1(S^1, \mathcal{A}^{<0}) = \mathbb{C}[[x]]/\mathbb{C}\{x\}$.

Then, the long exact sequence of cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^1(S^1,\mathcal{V}^{<0}) \arrow[r] & H^1(S^1,\mathcal{A}^{<0}) \arrow[r, "D"] \arrow[d, equal] & H^1(S^1,\mathcal{A}^{<0}) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{C}[[x]]/\mathbb{C}\{x\} & \mathbb{C}[[x]]/\mathbb{C}\{x\} & \end{tikzcd}$$

corresponding to the short exact sequence $0 \to \mathcal{V}^{<0} \to \mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \to 0$ implies

$$\ker\big(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}\big) = H^1\big(S^1, \mathcal{V}^{<0}\big),$$

$$\mathrm{coker}\big(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}\big) = 0.$$

(ii) The computation of $\dim H^1(S^1, \mathcal{V}^{<0})$ by P. Deligne relies on the fact that $\mathcal{V}^{<0}$ is piecewise constant (discontinuities occur at the finitely many Stokes directions) and that the function $\theta \mapsto N(\theta)$ is lower semicontinuous.

Let α_ℓ, with $\ell \in \mathbb{Z}/\mu\mathbb{Z}$, denote the Stokes directions of D in a cyclic order and let $i_\ell : \{\alpha_\ell\} \hookrightarrow S^1$ and $j_\ell :\ ]\alpha_\ell, \alpha_{\ell+1}[\ \hookrightarrow S^1$ denote the natural inclusions. The sequence

$$0 \longrightarrow \bigoplus_\ell j_{\ell!}\mathcal{V}^{<0} \longrightarrow \mathcal{V}^{<0} \longrightarrow \bigoplus_\ell i_{\ell*}\mathcal{V}^{<0} \longrightarrow 0$$

is an exact sequence of sheaves on S^1. This implies in between the Euler characteristics on S^1 the relation

$$\chi(\mathcal{V}^{<0}) = \sum_\ell \big(\chi(j_{\ell!}\mathcal{V}^{<0}) + \chi(i_{\ell*}\mathcal{V}^{<0})\big).$$

By definition, $\chi(\mathcal{V}^{<0}) = \dim H^0(S^1, \mathcal{V}^{<0}) - \dim H^1(S^1, \mathcal{V}^{<0})$ and $H^0(S^1, \mathcal{V}^{<0}) = 0$. Then,

$$\dim H^1(S^1, \mathcal{V}^{<0}) = -\sum_\ell \big(\chi(j_{\ell!}\mathcal{V}^{<0}) + \chi(i_{\ell*}\mathcal{V}^{<0})\big).$$

Moreover,

$$\begin{aligned}
\chi(j_{\ell!}\mathcal{V}^{<0}) &= -\dim H^1(S^1, j_{\ell!}\mathcal{V}^{<0}) && \begin{array}{l}(H^0 = 0 \text{ since the support}\\ \quad]\alpha_\ell, \alpha_{\ell+1}[\text{ is not closed in } S^1)\end{array}\\
&= -\dim H^0(]\alpha_\ell, \alpha_{\ell+1}[, \mathcal{V}^{<0}) && (\mathcal{V}^{<0} \text{ is constant on }]\alpha_\ell, \alpha_{\ell+1}[)\\
&= -\dim \mathcal{V}^{<0}_{\alpha'} && \forall \alpha' \in]\alpha_\ell, \alpha_{\ell+1}[
\end{aligned}$$

and

$$\begin{aligned}
\chi(i_{\ell*}\mathcal{V}^{<0}) &= \dim H^0(S^1, i_{\ell*}\mathcal{V}^{<0}) && \begin{array}{l}(H^1 = 0 \text{ since the support}\\ \quad \{\alpha_\ell\} \text{ is reduced to one point})\end{array}\\
&= \dim \mathcal{V}^{<0}_{\alpha_\ell}.
\end{aligned}$$

Now, using the fact that $\mathcal{V}^{<0}$ is locally isomorphic to the sheaf $\mathcal{V}'^{<0}$ of flat solutions of a normal form D' of D (Main Asymptotic Existence Theorem) its results that

$$\dim \mathcal{V}^{<0}_{\alpha'_\ell} - \dim \mathcal{V}^{<0}_{\alpha_\ell} = \dim \mathcal{V}'^{<0}_{\alpha'_\ell} - \dim \mathcal{V}'^{<0}_{\alpha_\ell}$$

and the claim follows from the fact that at any point $\alpha \in S^1$, $\dim \mathcal{V}'^{<0}_{\alpha}$ is the number of exponentials $\exp q_j$ which are flat at α. The $\frac{1}{2}$ coefficient comes from the fact that an exponential $\exp q_j$ increases $\dim \mathcal{V}'^{<0}_{\alpha'_\ell} - \dim \mathcal{V}'^{<0}_{\alpha_\ell}$ by one only when α_ℓ is a Stokes direction of q_j such that $\exp q_j$ be flat on $]\alpha_\ell, \alpha_{\ell+1}[$; thus, it contributes at only one over two of its Stokes directions. $\square$

Replacing $\mathcal{A}^{<0}$ by $\mathcal{A}^{\leq -k}$, $\mathcal{V}^{<0}$ by $\mathcal{V}^{\leq -k}$ and so on and using the Theorem of Ramis-Sibuya instead of the Theorem of Borel-Ritt give the similar result with Gevrey conditions.

PROPOSITION 7.2. — *Let $N^{\leq -k}(\theta)$ denote the dimension of the stalk $\mathcal{V}_\theta^{\leq -k}$ of $\mathcal{V}^{\leq -k}$ at $\theta \in S^1$. Recall $s = 1/k$. Then, D has an index in $\mathbb{C}[[x]]_s/\mathbb{C}\{x\}$ satisfying*

$$\chi\big(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}\big) = \dim H^1\big(S^1, \mathcal{V}^{\leq -k}\big) = \tfrac{1}{2}\,\mathrm{var} N^{\leq -k}.$$

Notice that, again, the function $\theta \mapsto N^{\leq -k}(\theta)$ is lower semicontinuous.

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