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LIOUVILLE FORMS IN A NEIGHBORHOOD OF AN ISOTROPIC EMBEDDING(*)

by Frank LOOSE

1. Introduction.

Consider a symplectic manifold (X, ω). If the symplectic 2-form ω is exact, a choice of a potential, i.e., a 1-form β satisfying $-d\beta = \omega$, is called a *Liouville form* on X. Since ω is non-degenerate there is a unique vector field η on X given by $i_\eta \omega = \beta$. Here $i_\eta \omega$ denotes the contraction of the form ω by η, i.e., $\langle i_\eta \omega, \xi \rangle = \langle \omega, \eta \wedge \xi \rangle$ for all $\xi \in TX$. The vector field η is called the *associated contracting vector field.*

The importance of Liouville forms and contracting vector fields for symplectic geometry has been pointed out among others by Eliashberg and Gromov [EG]. The aim of the present paper is to investigate the flexibility of Liouville forms in a special case, i.e.: When is it possible to transform one Liouville form into another by a symplectomorphism, at least locally?

The *center* of a Liouville form β is the set $M = \{x \in X \mid \beta(x) = 0\}$. Equivalently, it is the fixed point locus of the associated contracting vector field.

As a basic example consider the standard symplectic vector space of dimension $2l$ with its canonical 2-form $\omega = \sum_{\lambda=1}^{l} dx^\lambda \wedge dy^\lambda$, (x, y) being a coordinate for $\mathbf{R}^{2l}$. A natural choice for a Liouville form is

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$\alpha = \sum_{\lambda=1}^{l} (y^\lambda dx^\lambda - x^\lambda dy^\lambda)$ (e.g., α is the unique $\mathrm{Sp}_{2l}(\mathbf{R})$-invariant potential of ω). The center is $M = \{0\}$ and the associated contracting vector field is up to the factor $-\frac{1}{2}$ the Euler vector field, $\eta = -\frac{1}{2}\sum(x^\lambda\partial/\partial x^\lambda + y^\lambda\partial/\partial y^\lambda)$.

Another basic example is given by the cotangent bundle $C = T^*M$ of a differentiable manifold (of dimension n). In fact, C carries a canonical 1-form α given by $\sum_{\nu=1}^{n} v^\nu du^\nu$, where u is a coordinate on M and (u, v) is the corresponding bundle chart of $\pi: C \to M$. The canonical symplectic structure on C is given by $\omega = -d\alpha$; thus α is a Liouville form. Obviously the center of α is given by the zero section $\sigma: M \hookrightarrow C$ of π and the associated contracting vector field is up to the factor -1 the Euler field of the vector bundle π, i.e., $\eta = -\sum_{\nu=1}^{n} v^\nu\partial/\partial v^\nu$.

The submanifolds $M = \{0\}\subseteq\mathbf{R}^{2l}$ and $\sigma: M \hookrightarrow T^*M$ are both extreme cases of the notion of an isotropic embedding. Recall that a submanifold $\iota: M \hookrightarrow X$ of a symplectic manifold (X, ω) is isotropic, if $\iota^*\omega = 0$. Its dimension $n = \dim M$ can vary between $n = 0$ (i.e., M is a point) and half of the dimension of X (i.e., M is Lagrangean). Thus we let $\dim X = 2(n+l)$ with $l \in \mathbf{Z}_+$.

Now, let X be a manifold, let β be a 1-form on X so that $\omega := -d\beta$ is non-degenerate, and let $M := \beta^{-1}(0)$ be smooth (and non-empty). It is easy to see that M must be necessarily isotropic. If one wants to find a normal form for β in a neighborhood of its center (as we do), it is natural to look first for a normal form for ω around M.

Weinstein's isotropic embedding theorem (see [We1]) gives the appropriate answer. To formulate that recall that $\iota: M \hookrightarrow X$ isotropic means that $TM_m\subseteq TM_m^\perp$. Thus $E_m := TM_m^\perp/TM_m$ is a symplectic vector space and $E = \{E_m\}_{m\in M}$ a symplectic vector bundle over M. $N(\iota) := E$ is called the symplectic normal bundle of $\iota: M \hookrightarrow X$. If $\iota_1: M \hookrightarrow (X_1, \omega_1)$ and $\iota_2: M \hookrightarrow (X_2, \omega_2)$ are isotropic embeddings and $N(\iota_1) \cong N(\iota_2)$ as symplectic vector bundles, then there exists neighborhoods $U_1\subseteq X_1$, $U_2\subseteq X_2$ of M and a diffeomorphism $f: U_1 \to U_2$ with $f|M = \mathrm{id}_M$ and $f^*\omega_2 = \omega_1$. Moreover Weinstein proved the following existence result. Given a symplectic vector bundle E over a manifold M, then there exists a canonical symplectic manifold $C = C(E)$, together with an isotropic embedding $\sigma: M \hookrightarrow C$, so that $N(\sigma) = E$.

In the Darboux case, i.e., $n = 0$ (i.e., $M = \mathrm{pt}$), the canonical model

E comes down to $\mathbf{R}^{2l}$ with its standard structure and the theorem reduces to the classical Darboux theorem. In the Lagrange case, i.e., if $l = 0$ (i.e., $\dim M = \frac{1}{2} \dim X$), the canonical model is just $C = T^*M$. In both cases we already observed that there is a canonical choice of a Liouville form. In the general case, the model $C = C(E)$ is a combination of the extreme cases $n = 0$ and $l = 0$ using the Marsden-Weinstein reduction procedure. As a general fact we will show that the reduction procedure is also valid for Liouville forms, not only for symplectic forms. (For a precise statement see section 2.) In particular, Weinstein's canonical model $C = C(E)$ for isotropic embeddings carries a canonical 1-form, i.e., a Liouville form α.

What we have discussed so far, results in the following. If one starts with a manifold X, with a 1-form β so that $-d\beta$ is non-degenerate, and with $M = \beta^{-1}(0)$ smooth, and one is interested in a normal form for β in a neighborhood of M, one may assume that $X = C(E)$ (where E is the symplectic normal bundle of $M \hookrightarrow X$), $M \hookrightarrow X$ is the standard isotropic embedding σ, and $-d\beta = \omega = -d\alpha$, where α is the canonical 1-form on C. We want to characterize those potentials β of ω which we can transform into α, i.e., those potentials for which α is a normal form.

The following argument shows that there is a necessary condition coming from considerations on the first order of β along M. Precisely, if β is a Liouville form vanishing along the isotropic submanifold $M \hookrightarrow X$, then so does the associated vector field η. The derivative of η in $m \in M \subseteq X$ is a linear transformation $L_m : TX_m \to TX_m$. Now, from the symplectic nature, one computes easily that $L_m + \frac{1}{2}\mathrm{id} \in \mathbf{sp}(TX_m)$, the symplectic linear algebra. Moreover, the subspaces TM_m and $TM_m^{\perp}$ are L_m-invariant. Thus L_m induces a linear transformation $\Lambda_m : E_m \to E_m$ which is again conformal symplectic, $\Lambda_m + \frac{1}{2}\mathrm{id} \in \mathbf{sp}(E_m)$. In conclusion one gets a bundle homomorphism $\Lambda : E \to E$ with $\Lambda + \frac{1}{2}\mathrm{id} \in \mathbf{sp}(E)$. The canonical Liouville form on $C = C(E)$ fulfills $\Lambda = -\frac{1}{2}\mathrm{id}$, as is easily shown. It is clear that this is invariant under diffeomorphisms f of C with $f|M = \mathrm{id}_M$, since the induced $f^*(\Lambda)$ coming from $f^*\alpha$ is just a conjugation of Λ. We call therefore a Liouville form β on a manifold X *special*, if the associated bundle transformation $\Lambda : E \to E$ fulfills $\Lambda = -\frac{1}{2}\mathrm{id}$. We can state now the main result of that paper.

THEOREM (Existence). — *Let $E \to M$ be a symplectic vector bundle, let C be the canonical model associated with E, and let α be the canonical Liouville form on C. If β is any potential of $\omega = -d\alpha$, vanishing along M, and being special in the above sense, then there exist* *neighborhoods U and V of M in C and a diffeomorphism $f: U \to V$ with $f|M = \mathrm{id}_M$ satisfying $f^*\beta = \alpha$.*

The theorem was proved in the Lagrangean case by Kostant-Sternberg [GuSt], chap. 5. However, to the best of our knowledge it is even unknown in the other extreme, i.e., the Darboux case $M = \mathrm{pt}$. Kostant-Sternberg also proved that the diffeomorphism f is unique. In section 3 it is shown that the canonical model C comes along with a certain fibre bundle structure over M and that the standard embedding is given by the zero section of that bundle. Moreover there is a natural sequence of fibre bundles

$$0 \longrightarrow T^*M \longrightarrow C \longrightarrow E \longrightarrow 0$$

over M. Using that we are able to state the following uniqueness result.

THEOREM (Uniqueness). — *Let $\pi: C \to M$ be the fibre bundle projection of the standard model C and assume that there are neighborhoods U and V of the zero-section $\sigma: M \hookrightarrow C$ and a diffeomorphism $f: U \to V$ satisfying $f|M = \mathrm{id}_M$ and $f^*\alpha = \alpha$, where α is the canonical 1-form on C. Then f is already (restriction of) a bundle isomorphism of C which fixes the subbundle $T^*M \subseteq C$.*

2. Proof of the existence theorem.

We start with the observation that the natural reduction procedure for Hamiltonian K-spaces is valid for Liouville forms. Recall that a Hamiltonian K-space is given by a symplectic manifold (X, ω), a (connected) Lie group K acting on X by symplectic diffeomorphisms, and a moment map $\Phi: X \to \mathbf{k}^*$ (here $\mathbf{k}$ denotes the Lie algebra of K and $\mathbf{k}^*$ its dual vector space), i.e., a K-equivariant map (with respect to the given K-action on X and the coadjoint action on $\mathbf{k}^*$) satisfying the moment condition

$$d\Phi_a = i_{a_X}\omega.$$

Here Φ_a denotes the a-th component of Φ, i.e., $\Phi_a = \langle \Phi, a \rangle$, and a_X the vector field on X associated with $a \in \mathbf{k}$.

An important case where a moment map exists is the following. Suppose (X, ω) is symplectic, suppose β is a Liouville form (i.e., $-d\beta = \omega$), and suppose that K acts by diffeomorphisms respecting β, $k^*\beta = \beta$ for all $k \in K$. Then a natural moment map is given by the formula

$$\Phi_a = \langle \beta, a_X \rangle. \tag{1}$$

Consider next the moment level $Z = \Phi^{-1}(0) \subseteq X$. Suppose now that K acts freely and properly on X. Then Z is a submanifold and the natural projection $\nu: Z \to Z/K$ gives Z the structure of a K-principal bundle over $X_0 := Z/K$. Marsden-Weinstein observed that there exists a unique 2-form ω_0 on X_0 so that $\nu^*\omega_0 = i^*\omega$, where $i: Z \hookrightarrow X$ is the inclusion map.

PROPOSITION. — *Let (X, ω) be symplectic, let β be a Liouville form, let K be a Lie group acting freely and properly and respecting β. Let Φ be the natural momentum, $i: Z \hookrightarrow X$ the inclusion and $\nu: Z \to X_0$ the natural projection. Then there exists a unique* 1*-form β_0 on X_0 satisfying $\nu^*\beta_0 = i^*\beta$.*

Proof. — Since β is K-invariant it suffices to prove that $\beta(z)$ vanishes in the fibre direction, for every $z \in Z$. A typical vector ξ tangent to the fibre of $\nu: Z \to X_0$ is given by $\xi = a_X(z)$ for some $a \in \mathbf{k}$. Thus

$$\langle \beta(z), \xi \rangle = \langle \beta(z), a_X(z) \rangle = \Phi_a(z) = 0,$$

by the definition of Z. □

Remark. — (a) By the uniqueness of the Marsden-Weinstein symplectic structure ω_0 on X it follows that $-d\beta_0 = \omega_0$, i.e., β_0 is a Liouville form on X_0.

(b) Sjamaar-Lerman [SjLe] have proved a much more general statement of a symplectic structure on the quotient space X_0, when K is acting not necessarily freely. In particular they proved that X_0 is a stratified space where on each stratum (which is a smooth manifold) there exists a unique symplectic structure compatible with the projection map. The above argument shows that in case of a Liouville form β on X, there exists a unique Liouville form β_0 on X_0 in the appropriate sense, in particular a Liouville form on each of the strata.

An application of the proposition is given by the existence of a canonical 1-form α on the standard symplectic manifold C associated to a symplectic vector bundle E (of rank $2l$) over a manifold (of dimension n) due to Weinstein [We2].

We recall the construction. Let Q be the standard symplectic vector space of dimension $2l$. Let $\pi: P \to M$ be the $\mathrm{Sp}_{2l}(\mathbf{R})$-principal bundle of symplectic frames of $E \to M$. Then Q as well as T^*P carry a natural structure of a Hamiltonian K-space, $K = \mathrm{Sp}_{2l}(\mathbf{R})$, coming from the natural Liouville forms. Therefore the product space $T^*P \times Q$ is again a Hamiltonian K-space and the moment map comes from the Liouville form (see ([1])). Moreover, since K acts freely and properly on P, the Marsden-Weinstein quotient $C := (T^*P \times Q)_0$ exists and the above proposition shows that C carries again a canonical 1-form.

Consider now a manifold X and let β be a 1-form on X such that $\omega := -d\beta$ is non-degenerate. Let $\iota: M \hookrightarrow X$ be a submanifold of X sitting in the zero level of β, i.e., $\beta | M = 0$. Further let η be the associated contracting vector field on X given by $i_\eta \omega = \beta$. Then η vanishes on M, too. For every $m \in M$ we therefore can build the derivative of η, i.e., $L_m: TX_m \to TX_m$ given by $L_m(\xi) = [\hat{\xi}, \eta](m)$ (where $\hat{\xi}$ is an arbitrary vector field on X with $\hat{\xi}(m) = \xi$).

The so-called infinitesimal conformal symplectic transformations of a symplectic vector space (V, ω) are defined by linear transformations $T: V \to V$ satisfying

$$\langle \omega, Tv_1 \wedge v_2 \rangle + \langle \omega, v_1 \wedge Tv_2 \rangle = \lambda \langle \omega, v_1 \wedge v_2 \rangle$$

for some $\lambda \in \mathbf{R}$. It is not difficult to see (cf. [GuSt], chap. 4) that our derivative $L_m: TX_m \to TX_m$ is conformal symplectic with factor $\lambda = -1$, i.e., $L_m + \frac{1}{2}\mathrm{id} \in \mathbf{sp}(TX_m)$. We conclude that the characteristic exponents of η in m are symmetric with respect to $\nu = -\frac{1}{2}$, i.e., $-\frac{1}{2} + \nu$ is an exponent if and only if $-\frac{1}{2} - \nu$ is an exponent. From

$$\langle \omega_m, L_m \xi_1 \wedge \xi_2 \rangle + \langle \omega_m, \xi_1 \wedge L_m \xi_2 \rangle = -\langle \omega_m, \xi_1 \wedge \xi_2 \rangle,$$

for all $m \in M$ and $\xi_1, \xi_2 \in TX_m$, it follows now easily that M is necessarily isotropic in $(X, -d\beta)$. In fact, since $L_m | TM_m = 0$, we have $\langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0$ for all $\xi_1, \xi_2 \in TM_m$. Furthermore $TM_m^\perp$ is an L_m-invariant subspace of TX_m. For this let $\xi_1 \in TM_m^\perp$ and $\xi_2 \in TM_m$ and compute again:

$$-\langle \omega_m, L_m \xi_1 \wedge \xi_2 \rangle = \langle \omega_m, \xi_1 \wedge L_m \xi_2 \rangle + \langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0 + 0 = 0.$$

Moreover, again using that $TM_m \subseteq \ker(L_m)$, L_m induces a transformation Λ_m on the symplectic normal vector space $E_m = TM_m^\perp / TM_m$, $\Lambda_m(\xi + TM_m) := L_m(\xi) + TM_m$, which is again an infinitesimal conformal symplectic transformation with factor -1, i.e.,

$$\Lambda_m + \frac{1}{2}\mathrm{id}_{E_m} \in \mathbf{sp}(E_m).$$

Recall that we have called β a special Liouville form with center M, if $\Lambda_m = -\frac{1}{2}\mathrm{id}$ for every $m \in M$. What we just required is, that the tangent space TX_m decomposes into the direct sum of eigenspaces of L_m corresponding to the eigenvalues $\nu = 0$, $\nu = -1$ and $\nu = -\frac{1}{2}$,

$$TX_m = \mathrm{Eig}(L_m, 0) + \mathrm{Eig}(L_m, -1) + \mathrm{Eig}\Big(L_m, -\frac{1}{2}\Big).$$

In fact, a similar computation as above using the conformal symplectic property of L_m shows that $\ker(L_m) \subseteq TM_m^{\perp}$. Thus, since $TM_m \subseteq \ker(L_m)$ and since Λ_m is in particular injective, $\mathrm{Eig}(L_m, 0) = \ker(L_m) = TM_m$. Then, using the conformal symplectic property of L_m again, one must have an eigenspace $\mathrm{Eig}(L_m, -1)$ of the same dimension $n = \dim M$. Moreover, for an infinitesimal symplectic transformation T of a symplectic vector space (V, ω) one similarly observes (cf. [GuSt], chap. 4) that ω induces a non-degenerate pairing on the pair of eigenspaces $\mathrm{Eig}(T, \nu) \times \mathrm{Eig}(T, -\nu)$ for any eigenvalue ν of T. This gives an identification of T^*M_m with $\mathrm{Eig}(L_m, -1)$. Therefore the eigenspace of L_m corresponding to $\nu = -\frac{1}{2}$ is a realization of the symplectic normal space E_m in TX_m and in particular such a special Liouville form β induces a splitting of the vector bundle sequences

$$0 \longrightarrow TM \longrightarrow TX|M \longrightarrow N_{X/M} \longrightarrow 0$$

and

$$0 \longrightarrow TM \longrightarrow TM^{\perp} \longrightarrow E \longrightarrow 0$$

over M. Here $N_{X/M} = (TX|M)/TM$ denotes the geometric normal bundle of M in X.

EXISTENCE THEOREM. — *Let X_j be a manifold and β_j a special Liouville form with center $M \subseteq X_j$ ($j = 0, 1$). Suppose that the symplectic normal bundles are isomorphic. Then there exists neighborhoods $U_j \subseteq X_j$ of M and a diffeomorphism $f: U_0 \to U_1$ with $f|M = \mathrm{id}_M$ satisfying $f^*\beta_1 = \beta_0$.*

Let us regard X_j as a germ around M. Thus we omit the notion of neighborhoods in the sequel and write, e.g., that there exists a diffeomorphism $f: X_0 \to X_1$ satisfying certain conditions, and so on.

Since $M \subseteq (X_j, -d\beta_j)$ ($j = 0, 1$) is isotropic with isomorphic symplectic normal bundle, there exists a diffeomorphism $f_j: C \to X_j$ with $f_j|M = \mathrm{id}_M$ and $f_j^*(-d\beta_j) = -d\alpha$, where α is the canonical 1-form on C, by Weinstein's isotropic embedding theorem. Moreover, Weinstein's version of the Darboux-Moser-Weinstein theorem (see [We1]) shows that one can even achieve that not only $f_j|M = \mathrm{id}_M$ but moreover $f_{j*}|(TX|M) = \mathrm{id}_{TX|M}$. Thus the Liouville forms $f_j^*\beta_j$ are again special. We may therefore assume that $X_0 = X_1 = C$, $\beta_0 = \alpha$, and $-d\beta_1 = -d\alpha = \omega$.

Proof of the theorem. — By using an appropriate bundle isomorphism one may assume that $\beta_1 =: \beta$ and $\beta_0 = \alpha$ coincide along M up to the first order (since both are special). Following the usual proof of the

Darboux-Moser-Weinstein theorem, let

$$\beta_t := \beta_0 + t(\beta_1 - \beta_0),$$

$t \in [0,1]$, be the straight line curve of 1-forms connecting β_0 with β_1. Since σ induces an isomorphism on the corresponding cohomology groups and $\sigma^*(\beta_1 - \beta_0) = 0$ it follows that there exists a smooth function H on C so that

$$dH = \beta_1 - \beta_0.$$

By using explicit integration formulas one can achieve that $H|M = 0$, $dH|M = 0$ and $\mathrm{Hess}_M(H) = 0$, since $\beta_1 - \beta_0$ vanishes of the first order along M.

We look now for a curve $t \mapsto f_t$ in $\mathrm{Diff}(C)$ with $f_0 = \mathrm{id}$ and $f_t^* \beta_t = \beta_0$ (in particular $f := f_1$ fulfills $f^*\beta = \alpha$). This is equivalent to

$$0 = \frac{d}{dt}(f_t^*\beta_t) = f_t^*\left(\mathcal{L}_{\xi_t}\beta_t + \frac{d}{dt}\beta_t\right)$$

by the Leibniz rule. Here $t \mapsto \xi_t$ denotes the corresponding curve in the vector fields of C, i.e.,

$$\xi_t(f_t(x)) = \frac{d}{dt} f_t(x),$$

and $\mathcal{L}_\xi$ denotes the Lie derivative in direction of the vector field ξ. Since $\mathcal{L}_\xi = i_\xi d + d i_\xi$ and $\frac{d}{dt}\beta_t = dH$, we end up with

$$0 = d(i_{\xi_t}\beta_t) + i_{\xi_t}(d\beta_t) + dH = d(i_{\xi_t}\beta_t) - i_{\xi_t}\omega + dH.$$

So far we have followed the familiar proof.

Now we observe that the desired curve of diffeomorphisms must satisfy $f_t \in \mathrm{Symp}(C) = \{f \mid f^*\omega = \omega\}$; thus $\xi_t \in \mathbf{symp}(C) = \{\xi \mid \mathcal{L}_\xi\omega = 0\}$. Inside $\mathbf{symp}(C)$ there are the Hamiltonians ξ_g, i.e., those which are associated to functions g on C by $i_{\xi_g}\omega = dg$, $\mathbf{ham}(C) = \{\xi_g \mid g \in \mathcal{C}^\infty(C)\}$. Therefore we make the "ansatz"

$$\xi_t := \xi_{g_t}$$

and look for an equation of the desired curve $t \mapsto g_t$ in $\mathrm{Diff}(C)$. Obviously we have $i_{\xi_t}\omega = dg_t$ and furthermore $i_{\xi_t}\beta_t = i_{\xi_t} i_{\eta_t}\omega$, where η_t is the associated contracting vector field with respect to the Liouville form β_t. But

$$i_{\eta_t}\omega = \beta_t = \beta_0 + t\,dH = i_{\eta_0 + t\xi_H}\omega,$$

and therefore $\eta_t = \eta_0 + t\xi_H$, since ω is non-degenerate and where $\eta_0 = \eta$ is the canonical vector field on C. We come down to

$$i_{\xi_t}\beta_t = i_{\xi_t} i_{\eta_t}\omega = -i_{\eta_t} i_{\xi_t}\omega = -i_{\eta_t}(dg_t) = -(\eta + t\xi_H)(g_t).$$

Our equation to solve is therefore

$$0 = -(\eta + t\xi_H)(g_t) - g_t + H$$

or

$$(\mathrm{id} + \eta + t\xi_H)(g_t) = H.$$

Observe now that ξ_H vanishes of the first order along M, because H vanishes of the second order. Therefore $T := \mathrm{id} + \eta + t\xi_H$ may be seen as a perturbation of the differential operator $\mathrm{id} + \eta$. As a consequence of the next lemma, we will prove that there exists a unique solution g_t which vanishes of the second order along M. This solution also depends differentiably on t and thus we have found our curve $t \mapsto g_t$. □

Consider now $M = \mathbf{R}^n$ linearly embedded in $X = \mathbf{R}^n \times \mathbf{R}^r$ as $M = \{(x_1, x_2) \in \mathbf{R}^{n+r} \mid x_2 = 0\}$. Denote by $\mathcal{E} = \mathcal{E}_{n,r}$ the set of germs of smooth functions around M in X. Let $\mathbf{m}$ be the ideal of (germs of) functions vanishing on M and more generally for any positive integer k let $\mathbf{m}^k$ denote the functions vanishing on M up to the $(k-1)$ -st order.

LEMMA. — *Let k be a non-negative integer and $A\colon M \to \mathrm{Mat}(r, \mathbf{R})$, $A(x_1) = (a^\rho_\sigma(x_1))_{1\le\rho,\sigma\le r}$, a matrix-valued smooth function so that $A(x_1)$ is semi-simple and for any eigenvalue $\nu(x_1)$ of $A(x_1)$ let $\mathrm{Re}(\nu(x_1)) \le -1$. Let ξ be (a germ of) a vector field along M which vanishes of the first order. Then the linear partial differential operator $T\colon \mathcal{E} \to \mathcal{E}$,*

$$T = k \cdot \mathrm{id} + \sum_{\rho,\sigma=1}^{r} a^\rho_\sigma(x_1) x^\sigma_2 \frac{\partial}{\partial x^\rho_2} + \xi$$

maps $\mathbf{m}^{k+1}$ bijectively to itself.

Let (φ^s) be the flow associated with the vector field $\eta := T - k \cdot \mathrm{id}$. Then the flow exists for all positive time s and for every $h \in \mathbf{m}^{k+1}$ the preimage under T is given by

$$g(x) = -\int_0^\infty e^{ks} h(\varphi^s x)\, ds. \tag{2}$$

Before going to the proof, let us first make some remarks concerning the existence of the integral and on its smooth dependence on x. Denoting by

$$\varphi^s(x) = (\varphi^s_1(x), \varphi^s_2(x)) \in \mathbf{R}^n \times \mathbf{R}^r$$

the components, it follows from standard results in dynamical systems (see [Ha], chap. 9, e.g.) that the flow (φ^s) converges almost as fast to its limit as its linear part does. Precisely,

$$\varphi^s_2(x) = o(e^{-(1-\varepsilon)s})$$

for every fixed x and every $\varepsilon > 0$, since the real parts of the eigenvalues of A have real part less or equal to -1. Therefore, since h vanishes up to the k-th order, we find that $h(\varphi^s x) = o(e^{-(k+1-\varepsilon)s})$ and this gives the uniform convergence of the functions $g_s(x) = -\int_0^s e^{k\sigma} h(\varphi^\sigma x)\,ds$ for $s \to \infty$. Moreover, the limit $g = \lim_{s\to\infty} g_s$ is smooth and its x-derivative, denoted in the sequel by D, can be carried out under the integral,

$$Dg(x) = -\int_0^\infty e^{ks} D(h(\varphi^s x))\,ds.$$

A similar statement for the x-derivative $D\varphi^s(x)$ of the flow and also the higher derivatives implies that g is in fact smooth.

The second remark concerns the smoothness of the solution with respect to an additional parameter in the case where the vector field η depends smoothly on some additional parameter. In particular, if

$$\eta_t = \sum_{\rho,\sigma=1}^{r} a_\sigma^\rho x_2^\sigma \frac{\partial}{\partial x_2^\rho} + t\xi$$

for $t \in \mathbf{R}$, then, again by standard results, the flow $(\varphi^s(x,t))$ depends smoothly on (s,x,t) and one can see by similar arguments as above that the solution of $(k\cdot \mathrm{id} + \eta_t)(g_t) = h$, i.e.,

$$g_t(x) = -\int_0^\infty e^{ks} h(\varphi^s(x,t))\,ds$$

is smooth in (x,t).

As a third remark, there is a version of the lemma in the manifold setting. In fact, by the uniqueness of the solution, one may assume that M is an arbitrary manifold (of dimension n) embedded as a submanifold in another manifold X (of dimension $n+r$). Denoting by $\mathcal{E} = \mathcal{E}_{X,M}$ the germs of functions around M in X and by $\mathbf{m}^k$, $k \in \mathbf{Z}_+$, its ideals as above, let η be (a germ of) a vector field on X which vanishes on M. Then η induces a derivative $L\colon TX|M \to TX|M$ along M, and moreover, since $\eta|M = 0$ and therefore $L|TM = 0$, it induces a bundle homomorphism on the normal bundle $N_{X/M}$ of M, $\Lambda\colon N_{X/M} \to N_{X/M}$. Then we make the assumption that Λ is semisimple and that the eigenvalues of Λ have real part less or equal to -1. If (φ^s) denotes the flow of η on X, and if $h \in \mathbf{m}^{k+1}$, the lemma asserts that the operator $T = k\cdot\mathrm{id} + \eta$ gives a bijection of $\mathbf{m}^{k+1}$ to itself and moreover the preimage for any $h \in \mathbf{m}^{k+1}$ is given by the formula

$$g(x) = -\int_0^\infty e^{ks} h(\varphi^s x)\,ds.$$

Since the canonical vector field η on C has exponents $\nu = -\frac{1}{2}$ and $\nu = -1$ on the geometrical normal bundle of M, applying the lemma in this version with an additional parameter t and with $k = 2$ to the equation

$$(2\mathrm{id} + 2\eta + 2t\xi_H)(g_t) = 2H,$$

we find the desired solution curve $t \mapsto g_t$ for the proof of the theorem.

Proof of the lemma. — For the uniqueness let $f \in \mathbf{m}^{k+1}$ and assume that $Tf = 0$. We have to show that $f = 0$. Let $(\cdot,\cdot)$ denote the standard inner product on $\mathbf{R}^{n+r}$ and let $a\colon \mathbf{R}^{n+r} \to \mathbf{R}^{n+r}$ be the vector-valued function describing the vector field η, i.e., $\eta(f) = (a, \mathrm{grad})(f)$. Now fix $x \in X = \mathbf{R}^{n+r}$ and consider the function $\lambda\colon [0,\infty) \to X$, $s \mapsto e^{ks} f(\varphi^s x)$. Since $f \in \mathbf{m}^{k+1}$ we have $\lim_{s\to\infty} \lambda(s) = 0$ and of course $\lambda(0) = f(x)$. But

$$\lambda'(s) = e^{ks}\left(k\cdot f(\varphi^s x) + (a(\varphi^s x), \mathrm{grad} f(\varphi^s x))\right) = e^{ks}Tf(\varphi^s x) = 0,$$

which implies $f(x) = 0$.

For the existence observe that

$$D(\varphi^s x)a(x) = a(\varphi^s x).$$

This is true for $s = 0$ and both sides give a solution of the non-autonomous linear differential equation

$$z' = Da(\varphi^s x)z,$$

where $'$ denotes differentiation with respect to s and x is fixed. In fact,

$$\frac{d}{ds}(a(\varphi^s x)) = Da(\varphi^s x)a(\varphi^s x),$$

since (φ^s) is the flow for the equation $x' = a(x)$. On the other hand, differentiating the equation $\frac{d}{ds}\varphi^s(x) = a(\varphi^s x)$ with respect to x gives

$$\frac{d}{ds}D(\varphi^s x) = Da(\varphi^s x)D\varphi^s(x).$$

So

$$\frac{d}{ds}\left(D\varphi^s(x)a(x)\right) = Da(\varphi^s x)\left(D\varphi^s(x)a(x)\right).$$

Computing directly, we have:

$$\begin{aligned}
h(x) &= -\int_0^\infty \frac{d}{ds}\left(e^{ks}h(\varphi^s x)\right) ds\\
&= -\int_0^\infty e^{ks}\left(k\cdot h(\varphi^s x) + (\mathrm{grad} h(\varphi^s x), a(\varphi^s x))\right) ds\\
&= -\int_0^\infty e^{ks}\left(k\cdot h(\varphi^s x) + (\mathrm{grad} h(\varphi^s x), D\varphi^s(x)a(x))\right) ds\\
&= -\int_0^\infty e^{ks}\left(k\cdot h(\varphi^s x) + (a(x), \mathrm{grad})\,(h(\varphi^s x))\right) ds\\
&= -\left(k + (a(x), \mathrm{grad})\right)\left(\int_0^\infty e^{ks}h(\varphi^s x)\, ds\right)\\
&= Tg(x).
\end{aligned}$$

□

Remark. — An inspection of the proof shows that we can also give a normal form for Liouville forms which are not necessarily special. Of course, a necessary condition that β_0 is a pullback of β_1 via a diffeomorphism is that the induced transformations Λ_0 and Λ_1 of E must coincide up to conjugation of a bundle isomorphism $h: E \to E$. But the proof only works, if the eigenvalues of Λ_j are in the intervall $[-\frac{2}{3}, -\frac{1}{3}]$ (i.e., the associated contracting vector field has to *contract* C to M fast enough). Otherwise, the explicit integration formula ([2]) does not necessarily converge.

3. Proof of the uniqueness theorem.

To formulate the uniqueness result we need some additional information about the standard model C associated with a symplectic vector bundle E over M. Let $\pi: P \to M$ be the associated K-principal bundle of symplectic frames of $E \to M$, $K = \mathrm{Sp}_{2l}(\mathbf{R})$, $2l = \mathrm{rank}(E)$. Denote by $\mathrm{pr}_P: T^*P \to P$ the natural projection, $\mathrm{pr}_1: T^*P \times Q \to T^*P$ the projection onto the first factor, and by $i: Z \hookrightarrow T^*P \times Q$ the inclusion of the moment level $Z = \Phi^{-1}(0)$ of $T^*P \times Q$. Since all these maps are K-equivariant (where K acts trivially on M), the composition $\pi \circ \mathrm{pr}_P \circ \mathrm{pr}_1 \circ i: Z \to M$ is K-invariant. Therefore there exists a unique map $\pi_C: C \to M$ so that $\pi_C \nu = \pi \mathrm{pr}_P \mathrm{pr}_1 i$, where $\nu: Z \to C$ is the natural projection. It is not hard to see (cf. [Lo]) that π_C gives C the structure of a fibre bundle over M with fibre $F := \mathbf{R}^n \times Q$. The structure group of π_C is described by the following. Let H be $\mathrm{GL}_n(\mathbf{R}) \times K$ and V be the linear space of homogeneous quadratic polynomials from Q to $\mathbf{R}^n$, $V = \mathrm{Sym}^2(Q, \mathbf{R}^n)$. Then H acts on V by the representation

$$(A, C).b(q) = Ab(C^{-1}q),$$

for $q \in Q$, $b \in V$ and $(A, C) \in H$. Thus we can form the semi-direct product $G := H \times V$ corresponding to that representation. Now G acts on $F = \mathbf{R}^n \times Q$ via

$$((A, C), b).(v, q) = (Av + b(Cq), Cq).$$

This is the structure group of $\pi_C: C \to M$. Essentially it results from the fact that the moment map on Q is homogeneous quadratic (the "angular momentum part," so to say) while the moment map on T^*P is linear on the fibres (the "linear momentum part," so to say). Observe further that the $\mathbf{R}_+$-action on F given by

$$t.(v, q) = (t^2 v, tq)$$

commutes with the G-action. This induces a vector field on C which turns out to be -2 times the contracting vector field η associated with the Liouville form α on C (see [Lo]). Moreover $(0,0) \in F$ is a G-fixpoint, i.e., we have a zero section $\sigma: M \hookrightarrow C$, which turns out to be the standard isotropic embedding. Finally we observe that the subspace $\mathbf{R}^n \subseteq F$ is G-invariant and G acts on $\mathbf{R}^n$ by its projection on $\mathrm{GL}_n(\mathbf{R})$. Similarly G acts on the quotient $F/\mathbf{R}^n \cong Q$ via its projection on K. Thus we have an exact sequence of G-spaces

$$0 \longrightarrow \mathbf{R}^n \longrightarrow F \longrightarrow Q \longrightarrow 0.$$

To this corresponds an exact sequence of G-fibre bundles over M with fibres $\mathbf{R}^n$, F and Q. Again a computation (see [Lo]) shows that this sequence is given by

$$0 \longrightarrow T^*M \xrightarrow{g} C \xrightarrow{h} E \longrightarrow 0$$

over M. Here g and h are defined in a natural way similar to the construction of the map $\pi_C: C \to M$.

Denote by $\mathrm{Aut}(C)$ the group of the associated bundle isomorphisms of C, i.e., $\tau \in \mathrm{Aut}(C)$, if τ fixes every fibre $C_m := \pi_C^{-1}(m) \cong F$ and every $\tau_m: C_m \to C_m$ is a transformation of F which is in G (depending differentiably on m, of course).

UNIQUENESS THEOREM. — *Let M be a manifold, $E \to M$ a symplectic vector bundle, C the standard model associated with $E \to M$ and α its canonical 1-form. Let $f: C \to C$ be a diffeomorphism with $f|M = \mathrm{id}_M$ and $f^*\alpha = \alpha$. Then f is a bundle isomorphism, $f \in \mathrm{Aut}(C)$, which fixes the subbundle T^*M, $f|T^*M = \mathrm{id}_{T^*M}$.*

Proof. — Let us first consider the case $E = 0$, i.e., $C = T^*M$. Since the diffeomorphism f respects α, it respects the associated contracting vector field η. In particular, for any $m \in M$, f respects the stable manifold $S_m = \{c \in C \mid \lim_{t\to\infty} \varphi^t(c) = m\}$, where (φ^t) is the flow associated with η. Of course, in our case η is just -1 times the Euler vector field on the vector bundle $T^*M \to M$, i.e., $S_m = T^*M_m$.

Next let us look at the derivative F of f along M, i.e., $F_m := df_m: TC_m \to TC_m$. We have $TC_m = TM_m + T^*M_m$ with its natural symplectic structure. Furthermore, due to the fact that $f|M = \mathrm{id}_M$, $F_m|TM_m = \mathrm{id}_{TM_m}$ and T^*M_m is F_m-invariant by the preceding remark. It follows immediately from the definition that a symplectic transformation T

of the symplectic vector space $V+V^*$, which is a direct sum, $T=s+t$ for $s: V \to V$ and $t: V^* \to V^*$, must satisfy $t=s^*$. Therefore $F_m = \mathrm{id}_{TC_m}$ for all $m \in M$. Now $f_m := f|T^*M_m$ is a diffeomorphism of $T^*M_m \cong \mathbf{R}^n$ which commutes with the Euler vector field and which is therefore $\mathbf{R}_+$-equivariant with respect to the natural $\mathbf{R}_+$-action. The condition $df_m(0) = \mathrm{id}$ implies that f_m fixes every orbit, which is just a straight line and therefore (using again that $df_m(0) = \mathrm{id}$) $f_m = \mathrm{id}_{T^*M_m}$, i.e., $f = \mathrm{id}_X$.

The next step is to transform as far as possible the preceding discussion to the more general case. First, the stable manifold of the canonical vector field η on C is again the fibre $C_m := \pi_C^{-1}(m) \cong F = \mathbf{R}^n \times Q$. Furthermore, inside the stable manifold C_m sits the "even more" stable manifold corresponding to the eigenvalue $1-\varepsilon$ for some $0<\varepsilon<\frac{1}{2}$, i.e.,

$$\tilde{S}_m = \{c \in S_m \mid \varphi^t(c) = O(e^{-(1-\varepsilon)t})\}$$

(see [Ha], chap. 9), which is just $T^*M_m \subseteq C_m$. Thus we see that $T^*M \subseteq C$ is f-invariant. Now the restriction of the canonical 1-form α on T^*M is just the canonical 1-form on T^*M, thereby showing that $f|T^*M = \mathrm{id}_{T^*M}$.

Let $f_m := f|C_m: C_m \to C_m$. For each $m \in M$ we now construct an element $\tau_m = ((A_m, c_m), b_m) \in G$ induced by f_m. First observe that every diffeomorphism f respects the symplectic form ω on C, i.e., $f^*\omega = \omega$, and fixing the zero-section pointwise, $f|M = \mathrm{id}_M$. Thus f induces a symplectic bundle isomorphism γ of $E \to M$. In fact, since the derivative $F_m: (TC)_m \to (TC)_m$ of f is symplectic, i.e.,

$$\langle \omega_m, F_m\xi_1 \wedge F_m\xi_2 \rangle = \langle \omega_m, \xi_1 \wedge \xi_2 \rangle$$

for all $\xi_1, \xi_2 \in (TC)_m$, and $F_m|TM_m = \mathrm{id}_{TM_m}$, the ω_m-orthogonal $TM_m^{\perp}$ is also F_m-invariant: for $\xi_1 \in TM_m^{\perp}$ and $\xi_2 \in TM_m$ compute

$$\langle \omega_m, F_m\xi_1 \wedge \xi_2 \rangle = \langle \omega_m, F_m\xi_1 \wedge F_m\xi_2 \rangle = \langle \omega_m, \xi_1 \wedge \xi_2 \rangle = 0 .$$

Therefore F_m induces a linear transformation γ_m on $E_m = TM_m^{\perp}/TM_m$ which is clearly symplectic with respect to its natural induced structure.

For our diffeomorphism f of C, which even respects α, we have already seen that $F_m|T^*M_m = \mathrm{id}_{T^*M_m}$. So we set $A_m := \mathrm{id}_{T^*M_m}$ and $c_m := \gamma_m \in \mathrm{Sp}(E_m)$. To find the element $b_m \in \mathrm{Sym}^2(E_m, T^*M_m)$, we consider the second derivative of $f_m: C_m \to C_m$ in the origin which is a symmetric bilinear map $T(C_m)_m \times T(C_m)_m \to T(C_m)_m$. By restricting this map to $E_m \times E_m \subseteq T(C_m)_m \times T(C_m)_m$ and then projecting from $T(C_m)_m$ to T^*M_m, we obtain a symmetric bilinear map $E_m \times E_m \to T^*M_m$. Here we have used the realization of E_m in $T(C_m)_m$ as the $-\frac{1}{2}$-eigenspace of

L_m. Let $b_m \in \mathrm{Sym}^2(E_m, T^*M_m)$ be the associated quadratic form. In summary, using the first and second derivatives of f, for each $m \in M$ we have $\tau_m = ((A_m, c_m), b_m) \in \mathrm{Aut}(C_m)$. These fit together to form an automorphism $\tau \in \mathrm{Aut}(C)$ with $\tau | T^*M = \mathrm{id}_{T^*M}$.

Now, in order to prove the uniqueness assertion, by composing f with τ^{-1}, we may assume that $b_m = 0$ and $\gamma_m = \mathrm{id}_{E_m}$ for all $m \in M$. We want to show that $f = \mathrm{id}_C$. Since $f_m : C_m \to C_m$ respects the canonical vector field η, it is $\mathbf{R}_+$-equivariant with respect to the action $t.(v,q) = (t^2 v, tq)$ on $C_m \cong F$. Using the equivariance and the second derivative in 0, the condition $\tau_m = \mathrm{id}$ implies that f_m must stabilize every orbit. We conclude that $f_m = \mathrm{id}_{C_m}$, i.e., $f = \mathrm{id}_C$. This finishes the proof of the uniqueness theorem. □

Remark. — (a) Although not explicitely formulated, the theorem was proved by Kostant-Sternberg in [GuSt] in the Lagrangean case, i.e., $E = (0)$. Note that this means that f is simply the identity.

(b) The proof shows that the theorem is true not only for special Liouville forms. More precisely, the proof works for Liouville forms β where the associated bundle $\Lambda : E \to E$ has its eigenvalues in the open interval $(-1, 0)$, since η has to be contracting (cf. the remark in section 2).

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