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VARIATONAL CONSTRUCTION OF CONNECTING ORBITS

by John N. MATHER (*)

To Bernard Malgrange on his 65^{th} *Birthday*

Introduction.

In Hamiltonian mechanics, various questions and results concern whether there exists an orbit which in the infinite past tends to one region of phase space and in the infinite future tends to another region of phase space. Other questions and results concern the possibility of finding an orbit which visits a prescribed sequence of regions of phase space in turn.

In [Ma5], I obtained results of this type for a class $\mathcal{P}^1$ of C^1 diffeomorphisms of the infinite cylinder $\mathbb{T}\times\mathbb{R}$ (where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$). This is the class of diffeomorphisms which can be represented as $f_1 \dots f_n$, where each f_i is an exact area preserving positive monotone twist diffeomorphism of the infinite cylinder which preserves the ends and twists each end infinitely. (See [Ma5, §1] for a detailed definition.)

It is well known, by KAM theory, that invariant simple closed curves for such diffeomorphims often exist. Of course, any such invariant curve divides the cylinder into two regions and any orbit must stay in one region or the other. Thus, the existence of invariant curves limits the possibility of the construction of orbits of the type we seek.

Simple closed curves (i.e., Jordan curves, or homeomorphs of the circle) in the cylinder come in two varieties : those which separate the

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top end of the cylinder from the bottom end, and those which don't. These two varieties may also be described as the homotopically non-trivial and the homotopically trivial Jordan curves in the cylinder.

G.D. Birkhoff proved that when $f \in \mathcal{P}^1$, any f-invariant homotopically non-trivial Jordan curve Γ in the infinite cylinder is the graph of a Lipschitz function $u : \mathbb{T} \longrightarrow \mathbb{R}$. Moreover, his argument provides an *a priori* upper bound for the Lipschitz constant of u, which depends only on f. Explicitly, if both f and f^{-1} twist the vertical by an angle of at least θ, then $\cot\theta$ is an upper bound for the Lipschitz constant of u. (See, e.g., the Appendix of [Ma5].)

Let $K = K_f$ denote the union of all f-invariant homotopically non-trivial Jordan curves Γ in the infinite cyclinder. From Birkhoff's *a priori* bound, it follows that K is closed. For a generic f, each component of the complement of K is topologically an annulus, which goes around the cylinder. (The only exceptions occur when there are two f-invariant homotopically non-trivial Jordan curves of the same rotation number. In such cases, the rotation number is rational. Such cases do not occur for generic f.) A component of the complement of K which is topologically an annulus is called a *Birkhoff region of instability.*

The results in [Ma5] concern the existence of orbits in a Birkhoff region of instability. To state these results, it is necessary to introduce the notion of the average action (or average Poincaré- Cartan invariant) of an invariant probability measure. The notion of the action (or Poincaré-Cartan invariant) for a periodic orbit of a Hamiltonian system is well known [Cart]. Dividing the action by the period, we obtain the average action of an orbit. This notion has an obvious generalization to invariant measures. (See §1 for the definition.)

It is also necessary to introduce the notion of the rotation number of an invariant probability measure μ. This is the average advance in the $\mathbb{T}$ coordinate of a point of the cylinder under iteration by f, the average being taken with respect to μ. For what we will do, it is necessary to define the rotation number as a real number (not a number mod. 1). For this, it is necessary to choose a lift $\tilde{f}$ of f to the universal cover $\mathbb{R}^2$ of the cylinder. Replacing the lift $\tilde{f}$ with another lift $\tilde{f} + (k, 0)$, $k \in \mathbb{Z}$, adds k to the rotation number of every f-invariant probability measure, and thus does not change anything important in the subsequent discussion. In the sequel, we will suppose that a lift $\tilde{f}$ of f has been chosen once and for all. Then for any invariant probability measure μ whose average action $A(\mu)$ is finite, the rotation number $\rho(\mu)$ is a well defined real number. (See §1 for precise definitions.)

The set $\{(\rho(\mu), A(\mu)\colon \mu$ is an f-invariant probability measure with $A(\mu) < +\infty\}$ is a convex subset of $\mathbb{R}^2$, since the set of f-invariant probability measures of finite action is convex, and $\rho(\mu)$ and $A(\mu)$ are affine functions of μ. In fact, this set is the epigraph of a convex real valued function $\beta = \beta_f$ of a real variable. We call $\beta(\omega)$ the *minimal average action* associated to the rotation number ω (and the diffeomorphism f). By definition, $A(\mu) \geqslant \beta(\rho(\mu))$, for any invariant probability measure μ for which $A(\mu) < +\infty$. We say that an invariant probability measure μ is *action minimizing* if $A(\mu) = \beta(\rho(\mu))$.

If Γ is an f-invariant homotopically non-trivial Jordan curve in the cylinder, then any invariant measure supported in Γ is action minimizing [Ma5, Prop. 2.8]. There are two cases. When the rotation number of Γ is irrational, then $f|\Gamma$ is uniquely ergodic, i.e., there is just one invariant probability measure with support in Γ. This is part of the well known Denjoy theory. (See, e.g., [Her1].)

When the rotation number of Γ is rational, each periodic orbit in Γ supports a unique ergodic invariant probability measure. In either case, these are all the action minimizing ergodic probability measure of f whose rotation number ω is the same as that of $f|\Gamma$.

These results generalize to arbitrary real numbers ω (not necessarily the rotation number of an invariant curve), as long as $f \in \mathcal{P}^1$. Thus, if ω is irrational, there is a unique action minimizing f-invariant probability measure μ_ω of rotation number ω, and this measure is ergodic. If ω is rational, $\omega = p/q$ in lowest terms, then every periodic orbit of period q and rotation number p/q carries a unique ergodic probability measure. If the periodic orbit minimizes the action over orbits of period q and rotation number p/q, then the corresponding measure is action minimizing. All ergodic action minimizing measures are obtained in this way. In particular, one has the existence, and, for generic f, the uniqueness of action minimizing measures of rotation number p/q. These results were proved in [Ma4] and again in [Ma6] by a different method and provide a slight refinement (in the sense that the action minimizing measures are unique) of previous results in the theory developed by Aubry and myself (independently).

Let M_ω denote the union of the supports of all the action minimizing measures μ of rotation number ω. In the case that ω is irrational, there is just one such measure μ_ω and $M_\omega = \text{supp}\ \mu_\omega$. In this case, M_ω is called the *Aubry-Mather set* of rotation number ω, provided that there is no f-invariant homotopically non-trivial Jordan curve of rotation number ω. If, to the contrary, there is such a curve, it contains a unique minimal set (in the sense of topological dynamics), by Denjoy theory (see, e.g., [Her1]), and M_ω is that set. In the case that ω is rational, M_ω is the union of all action minimizing periodic orbits of period q and rotation number p/q.

Now we may state the main results of [Ma5]. Let R be a Birkhoff region of instability bounded by f-invariant homotopically non-trivial Jordan curves Γ_- and Γ_+ with $\rho(\Gamma_-) < \rho(\Gamma_+)$.

THEOREM A. — *Suppose $\rho(\Gamma_-) < \alpha, \omega < \rho(\Gamma_+)$. Then there is an orbit of f whose α-limit set lies in M_α and whose ω-limit set lies in M_ω. Furthermore, if $\rho(\Gamma_-)$ (resp. $\rho(\Gamma_+)$) is irrational, then this conclusion still holds with the weaker hypothesis $\rho(\Gamma_-) \leqslant \alpha, \omega$(resp. $\alpha, \omega \leqslant \rho(\Gamma_+)$).*

THEOREM B. — *Consider for each $i \in \mathbb{Z}$ a real number $\rho(\Gamma_-) \leqslant \omega_i \leqslant \rho(\Gamma_+)$ and a positive number ϵ_i. Then there exists an orbit $(\ldots, P_j, \ldots)$ and an increasing bi-infinite sequence of integers $j(i)$ such that dist. $(P_{j(i)}, M_{\omega(i)}) < \epsilon_i$.*

These are Theorems 4.1 and 4.2 of [Ma5]. Our purpose in this paper is to state and prove a version of these results in more degrees of freedom, specifically to the setting considered in [Ma5]. Our results are far less than what one might hope for in more degrees of freedom. We regard this paper as (hopefully) the first step in a program which could lead to interesting results in more degrees of freedom. We will state a conjecture in §13 to indicate what we have in mind.

We state our results in §9. They are too technical to state in this introduction, as they depend on a generalization of Peierls's barrier to more degrees of freedom (§6). This generalization perhaps deserves attention as a new idea.

In [Ma5], the orbits we constructed were constrained minima. The main technical difficulty was to construct the constraints so that the constrained minima do not bump up against the constraints. The proof that the constrained minima do not bump up against the constraints depended heavily on order properties of the real line. For the generalization, no order properties are available, and we have had to find a new method. When we originally planned this paper, we planned to construct appropriate constraints and show that constrained minima do not bump up against the constraints, as in [Ma5], but in a way that did not depend on the order properties of the real line. In writing the proof down, we found that it was simpler to introduce a new variational principle and show that the action minimizing configurations for the new variational principle have the required properties. Although we use no constraints, the main technical difficulty is similar to that of [Ma5]. Here, the main technical difficulty is to show that the action minimizing configurations for the new variational principle correspond to trajectories of the Euler-Lagrange flow of the original variational principle. In view of the way that the new variational principle is constructed, this amounts to showing that the action minimizing configurations are restricted to appropriate regions of the configuration space M. This is analogous to showing that the constrained minima do not bump up against the constraints.

As a historical remark, I would like to point out that the methods of [Ma5] extend those of [Ma1], and that I spoke of the results of [Ma5] in Oberwolfach in 1985. Although the results of [Ma1] are very different from the results of [Ma5] and this paper, the methods have something in common, and may perhaps be taken as an indication that the methods of [Ma1], [Ma5] and this paper have many possibilities, which are yet to be exploited.

As a further historical remark, I would like to mention that Bernstein and Katok [B-K] were the first to use the general sort of variational method which we discuss in this paper in more degrees of freedom. They proved the existence of periodic orbits near invariant tori. I also note the article by Katok [Kat] which contains results about minimal orbits in more degrees of freedom, and the article by Bangert [Ban2], in which he studied minimal (or "class A") geodesics on higher dimensional manifolds.

The results of Herman [Her2] give a very important complement to the results of [Ma6]. He gives examples showing that the Lipschitz graph property of invariant tori holds only for positive (or negative) definite invariant tori, thus showing that the positive definiteness condition in not just a convenience for the proof, but actually makes a difference in the dynamics.

In a recent paper [Bol], Bolotin constructs connecting orbits by a variational method similar to the one we use, but under very different hypotheses.

Since this paper is aimed at a general audience, I have included a great deal of expository material. §§1-5 is a summary of previous work I have done, with occasional small modifications. The new material begins in §6, where I explain how to generalize Peierls's barrier to more degrees of freedom. In §7, I show that this generalization reduces to Peierls's barrier in the case that $M = \mathbb{T}$. For the statement of the main theorem of this paper, I needed a variant on the barrier introduced in §6. I define this in §8, where I also discuss the nature of both barriers in the twist map case.

In §9, I state the main theorems and discuss their application to the twist map case. I prove these theorems in §§10, 11. In §12, I state generalizations of these theorems. In §13, I state a conjecture. This is intended to suggest what I hope to do with this theory.

1. The setting.

To generalize Peierls's barrier to more degrees of freedom and state our main results, we use the setting of [Ma6]. In this section, we recall this setting.

We consider a smooth compact manifold M, and a C^2 real valued function L defined on $TM \times \mathbb{T}$, where TM denotes the tangent bundle of M and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In the usual terminology, L is a periodic Lagrangian (of period one) on M. For our methods to apply, we need the following two hypotheses :

Positive Definiteness. For each $m \in M$ and each $\theta \in \mathbb{T}$, the restriction of L to $TM_m \times \theta$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

Superlinear Growth. Let $\| \ \|$ denote a Riemannian metric on M. Then

$$L(v, \theta)/\|v\| \longrightarrow +\infty, \qquad \text{as } \|v\| \longrightarrow \infty,$$

where v ranges over TM and $\theta \in \mathbb{T}$.

In other words, for every $C > 0$, there exists $C_1 > 0$ such that $\|v\| \geqslant C_1$ implies $L(v, \theta) \geqslant C\|v\|$. This condition is plainly independent of the choice of Riemannian metric, since M is compact.

Under these conditions, the Legendre transformation $\mathcal{L}$ is defined : if $m \in M$, $v \in TM_m$, $\theta \in \mathbb{T}$, then $\mathcal{L}(v, \theta) = (d_v(L|TM_m \times \theta), \theta) \in T^*M_m \times \theta$, where T^*M denotes the cotangent bundle of M and T^*M_m denotes the fiber over m. If L is C^r $(r \geqslant 2)$, then $\mathcal{L}$ is a C^{r-1} diffeomorphism of $TM \times \mathbb{T}$ onto $T^*M \times \mathbb{T}$ which commutes with the projections on $M \times \mathbb{T}$. We will write $\mathcal{L}_{m,\theta}$ for the restriction of $\mathcal{L}$ to the fiber $TM_m \times \theta$.

The Hamiltonian H: $T^*M \times \mathbb{T} \longrightarrow \mathbb{R}$ is defined by $H(m, \xi, \theta) = \langle \xi, \mathcal{L}_{m,\theta}^{-1}(\xi) \rangle - L(\mathcal{L}_{m,\theta}^{-1}(\xi))$. If we introduce local coordinates $x = (x_1, \dots, x_n)$ in M and let $(x, \dot{x}) = (x_1, \dots, x_n, \dot{x}_1, \dots \dot{x}_n)$ denote the corresponding local coordinates in TM, and $(q,p) = (x,p) = (q_1, \dots, q_n, p_1, \dots, p_n) = (x_1, \dots, x_n, p_1, \dots, p_n)$ the corresponding local coordinates in T^*M, then we may express the Hamiltonian in its classical form

$$H(q,p) = p \cdot \dot{x} - L(x, \dot{x}),$$

where $(x, \dot{x})$ and (q,p) are related by the Legendre transformation :

$$q = x, \qquad\qquad p = L_{\dot{x}}.$$

One easily computes

$$H_q = -L_x, \qquad\qquad H_p = \dot{x}.$$

If L is $C^r(r \geqslant 2)$, then $\mathcal{L}$ is C^{r-1}. The equations above show that the first derivatives of H are C^{r-1}. Consequently H is C^r. Similarly if H is $C^r(r \geqslant 2)$, then L is C^r.

In the Lagrangian formulation of classical mechanics, the evolution of the system is described by the flow of the Euler-Lagrange vector field E_L associated to L. Its trajectories correspond to the solutions of the variational equation (fixed endpoint problem) :

$$\delta \int_a^b L\big(\gamma(t),\, d\gamma(t)/dt,\, t\big)\, dt = 0.$$

In other words, a curve in $TM \times \mathbb{T}$ is a trajectory of E_L if and only if it can be represented in the form

$$t \longrightarrow \big(\gamma(t),\, d\gamma(t)/dt,\, t \text{ (mod. } 1)\big),$$

where γ is a curve on M which satisfies the variational equation.

In local coordinates, the Euler-Lagrange vector field is defined by the Euler-Lagrange equations

$$dx/dt = \dot{x}, \qquad\qquad d(L_{\dot{x}})/dt = L_x.$$

The Euler-Lagrange vector field is $\mathcal{L}$-related to the symplectic gradient of H, defined by Hamilton's equations

$$dq/dt = H_p, \qquad dp/dt = -H_q.$$

Caratheodory [Cara, p. 207] made the following remark concerning differentiability classes. If L is C^r ($r \geqslant 2$), then, as we have seen, H is C^r, so the corresponding Hamiltonian vector field is C^{r-1}, and so is the flow that it generates. Since the Legendre transformation is C^{r-1}, the flow generated by E_L is also C^{r-1}, even though E_L itself may be only C^{r-2}. This applies even in the case $r = 2$, and we obtain that the conclusions of the fundamental existence and uniqueness theorem for ordinary differential equations holds for E_L, even though it may be only C^0.

Since a trajectory $t \to \big(\gamma(t),\, d\gamma(t)/dt,\, t \ (\text{mod. } 1)\big)$ of E_L is C^{r-1}, the curve γ on M is C^r.

In the classical theory of the calculus of variations, one also has the following basic result concerning the boundary value problem, under the two hypotheses that we have imposed above.

Tonelli Theorem. — *Let $a < b \in \mathbb{R}$, and let $m_0, m_1 \in M$. Among the absolutely continuous curves γ: $[a,b] \to M$ such that $\gamma(a) = m_0$ and $\gamma(b) = m_1$, there is one which minimizes the action $\int_a^b L\big(\gamma(t),\, d\gamma(t)/dt,\, t\big)dt$.*

As Mañé pointed out [Mañ1], it is not necessary to assume compactness of M, if the superlinear growth condition is satisfied with respect to some complete Riemannian metric on M.

A curve which minimizes $\int_a^b L\big(\gamma(t),\, d\gamma(t)/dt,\, t\big)dt$ subject to the fixed endpoint condition $\gamma(a) = m_0$, $\gamma(b) = m_1$, is called a *Tonelli minimizer*. Ball and Mizel [Bal] have constructed examples of Tonelli minimizers which are not C^1, even though L satisfies the hypotheses we have stated above. A Tonelli minimizer which is C^1 is C^r (if L is C^r) and satisfies the Euler-Lagrange equation, by the usual elementary arguments in the calculus of variations, together with Caratheodory's remark on differentiability.

The Ball and Mizel examples may be excluded by imposing the following additional hypothesis :

Completeness of the Euler-Lagrange Flow. Every maximal trajectory of E_L is defined for all time.

The fundamental existence and uniqueness theorem for ordinary differential equations says that for any initial condition $\gamma(t_0) = m_0$, $d\gamma(t_0)/dt = v_0$, there is a unique maximal trajectory $\gamma : (a,b) \longrightarrow M$, where $-\infty \leqslant a < b \leqslant +\infty$. The completeness hypothesis is that $a = -\infty$ and $b = +\infty$, for any initial condition.

Even without the completeness hypothesis, a Tonelli minimizer is C^1 on an open and dense set of full measure in the interval in which it is defined, and the velocity goes to infinity on the exceptional set. In view of this, the completeness hypothesis implies that a Tonelli minimizer is C^1 (and hence C^r).

For every E_L-invariant probability measure μ on $TM \times \mathbb{T}$ we may define its *average action*

$$A(\mu) = \int L d\mu \,.$$

Since L is bounded below, this integral is defined, although it may be $+\infty$. If $A(\mu) < +\infty$, we may also associate to μ its *rotation vector* $\rho(\mu) \in H_1(M, \mathbb{R})$. This may be uniquely characterized as follows. Consider a cohomology class $c \in H^1(M, \mathbb{R})$ and let λ_c be a closed smooth 1-form on M whose de Rham cohomology class is c. Usually one thinks of λ_c as a section of T^*M, but one may also think of λ_c as a real valued function on TM which is linear on the fibers. One may then pull back λ_c to $TM \times \mathbb{T}$ by composing with the projection on the first factor. The rotation vector $\rho(\mu)$ is uniquely characterized by the following equation :

$$\langle c, \rho(\mu)\rangle = \int \lambda_c \, d\mu \ .$$

The bracket on the left is the canonical pairing of $H^1(M, \mathbb{R})$ and $H_1(M, \mathbb{R})$. The convergence of the integral on the right follows from the assumption $A(\mu) < +\infty$ and the superlinear growth hypothesis. It is elementary to show that addition of an exact one form to λ_c does not change the integral on the right. (See [Ma6].) Since this integral is clearly linear in c, the equation above defines $\rho(\mu) \in H_1(M, \mathbb{R})$.

The idea of such a rotation vector is classical, going back to Schwartzman's asymptotic cycles [Sch].

2. The basic theory.

Throughout this paper, we let M be a fixed smooth compact manifold. In all examples that interest us, M is a torus, but the theorem we will state in this section is true without any restriction on the topology of M. We let L be a C^2 real valued function defined on $TM \times \mathbb{T}$, satisfying the three hypotheses given in §1 : positive definiteness, superlinear growth, and completeness of the Euler-Lagrange flow. We will also fix L throughout the discussion. We call L *the Lagrangian.*

We let $U_L = \{(\rho(\mu), A(\mu)) : \mu$ is an E_L-invariant probability measure on $TM \times \mathbb{T}$ satisfying $A(\mu) < +\infty\}$. Clearly, U_L is a convex subset of $H_1(M,\mathbb{R}) \times \mathbb{R}$: the set of invariant probability measures is convex and A and ρ are linear functions on it.

Moreover, the projection of U_L on $H_1(M,\mathbb{R})$ is surjective. This is the content of the Proposition on p. 178 of [Ma6]. Here, we briefly outline the proof and refer to [Ma6] for details.

We let $\widetilde{M}$ be the covering space of M defined by $\pi_1(\widetilde{M}) = \ker\big(\mathcal{H} : \pi_1(M) \to H_1(M,\mathbb{R})\big)$ where $\mathcal{H}$ denotes the Hurewicz homomorphism. For example, if $M = \mathbb{T}^n$ then $\widetilde{M} = \mathbb{R}^n$. The group of deck transformations of this covering space is

$$\begin{aligned}\mathcal{D} &= \operatorname{im}\big(\mathcal{H} : \pi_1(M) \longrightarrow H_1(M, \mathbb{R})\big)\\ &= \operatorname{im}\big(H_1(M, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{R})\big).\end{aligned}$$

It is a lattice in the finite dimensional vector space $H_1(M, \mathbb{R})$, i.e. , it is discrete and spans $H_1(M, \mathbb{R})$. For example, if $M = \mathbb{T}^n$, then $\mathcal{D} = \mathbb{Z}^n$.

Choose $h \in H_1(M, \mathbb{R})$. Let $T_1, \dots, T_n, \dots$ be a sequence of deck transformation such that

$$n^{-1} T_n \longrightarrow h \in H_1(M, \mathbb{R}), \qquad \text{as } n \longrightarrow +\infty.$$

Let $\widetilde{x}_0 \in \widetilde{M}$. Let $\widetilde{x}_n = T_n \widetilde{x}_0$. Let $\widetilde{\alpha}_n$: $[0, n] \longrightarrow \widetilde{M}$ minimize $\int_0^n L\big(d\alpha_n(t), t\big)dt$ subject to the boundary conditions $\widetilde{\alpha}_n(0) = \widetilde{x}_0$ and $\widetilde{\alpha}_n(n) = \widetilde{x}_n$, where α_n is the projection of $\widetilde{\alpha}_n$ on M. The existence of $\widetilde{\alpha}_n$ follows from a version of Tonelli's theorem on $\widetilde{M}$ (cf. [Ma6]). Moreover, $\widetilde{\alpha}_n$ is C^1, by the completeness hypothesis.

To obtain an invariant measure, we use a method which Kryloff and Bogoliuboff used [KB] to show that any flow on a compact metric space has an invariant measure.

For this purpose, it is useful to extend the Euler-Lagrange flow from $TM \times \mathbb{T}$ to its one point compactification $(TM \times \mathbb{T})^*$ by letting the point at infinity be a rest point. We extend the definition of the average action $A(\mu)$ of an invariant measure μ, by letting $A(\mu) = +\infty$, if the point at infinity has positive μ-mass.

We let $\gamma_n(t) = \big(d\,\alpha_n(t), t \text{ mod. } 1\big) \in TM \times \mathbb{T}$. We let μ_n denote the probability measure evenly distributed along γ_n and let μ be a point of accumulation of μ_n, with respect to the vague topology on measures on $(TM \times \mathbb{T})^*$. The Kryloff-Bogoliuboff argument shows that μ is an invariant measure.

It is easy to see that there exists $C > 0$ and, for each positive integer n, a curve $\widetilde{\beta}_n$: $[0, n] \longrightarrow \widetilde{M}$ such that $\widetilde{\beta}_n(0) = \widetilde{x}_0$, $\widetilde{\beta}_n(n) = \widetilde{x}_n$ and $n^{-1} \int_0^n L\big(d\,\beta_n(t),\, t\big)dt \leqslant C$, where β_n is the projection of $\widetilde{\beta}_n$ on M. Consequently,

$$\int L\,d\mu_n = n^{-1} \int_0^n L\,\big(d\,\alpha_n(t),\, t\big)dt \leqslant n^{-1} \int_0^n L\,\big(d\,\beta_n(t),\, t\big)dt \leqslant C,$$

and it follows that $\int L\,d\,\mu \leqslant C$ and the point at infinity has zero measure with respect to μ.

Thus μ is an E_L-invariant probability measure on $TM \times \mathbb{T}$. It is easily seen that $\rho(\mu) = \lim_{n \to +\infty} n^{-1}\, T_n = h \in H_1(M, \mathbb{R})$. This completes our outline of the proof that the projection of U_L on $H_1(M, \mathbb{R})$ is surjective.

Clearly, L is bounded below : there exists $B \in \mathbb{R}$ such that $L \geqslant B$. It follows from the definition that $U_L \subset H_1(M, \mathbb{R}) \times [B, \infty)$. Therefore, U_L is the epigraph of a convex function $\beta = \beta_L$: $H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$. For $h \in H_1(M, \mathbb{R})$, we will call $\beta\,(h)$ the *minimal average action* of the rotation vector h. This generalizes the notion of minimal average action discussed in the introduction.

It is easy to see that β has superlinear growth, i.e., $\beta(h)/\|h\| \longrightarrow +\infty$ as $\|h\| \longrightarrow \infty$, where we may choose $\|\ \|$ to be any norm on the finite dimensional vector space $H_1(M, \mathbb{R})$.

Let $\alpha : H^1(M, \mathbb{R}) \longrightarrow \mathbb{R}$ be the conjugate of β in the sense of convex analysis. (See, e.g., [Roc].) In other words,

$$\alpha(c) = \max\big\{ < c, h > -\beta(h):\ h \in H_1(M, \mathbb{R})\big\},$$

for $c \in H^1(M, \mathbb{R})$. Since β has superlinear growth, the maximum is achieved and α is everywhere defined. Since β is everywhere defined, α has superlinear growth.

We set $A_c(\mu) = A(\mu) - < c, \rho(\mu) >$, for $c \in H^1(M, \mathbb{R})$. We will say that an E_L-invariant measure μ is *c-minimal* if it minimizes $A_c(\mu)$ over all E_L-invariant measures. We will say that it is *minimal* or *action minimizing* if it is c-minimal for some $c \in H^1(M, \mathbb{R})$. We let $\mathcal{M}_c$ denote the family of all c-minimal measures. We let supp $\mathcal{M}_c \subset TM \times \mathbb{T}$ denote the closure of the union of the supports of μ for $\mu \in \mathcal{M}_c$. For brevity, we set $M_c = \text{supp } \mathcal{M}_c$. Let π: $TM \times \mathbb{T} \longrightarrow M \times \mathbb{T}$ denote the projection.

The principle result of [Ma6] may be stated as follows :

THEOREM 2.2. — *M_c is a compact, non-empty subset of $TM \times \mathbb{T}$. The restriction of π to M_c is injective. The inverse mapping π^{-1}: $\pi(M_c) \longrightarrow M_c$ is Lipschitz.*

Here, we have combined Proposition 4 and Theorems 1 and 2 of [Ma6].

Let $\Sigma_c = \pi(M_c) \subset M \times \mathbb{T}$. It follows from this theorem that the flow on Σ_c which corresponds to the Euler-Lagrange flow on M_c is Lipschitz, and is generated by a Lipschitz vector field.

From the point of view of existence theory, this theorem tells us nothing. For each $h \in H_1(M, \mathbb{Z})$, there exists, by Tonelli's theorem, a closed curve γ in M, of period 1, which minimizes the action $\int L\big(d\alpha(t), t\big)dt$ among the closed curves α of period 1 whose homology class is h. By the completeness hypothesis, γ is C^1. Then γ satisfies the Euler-Lagrange equation, and so is a periodic orbit of the Euler-Lagrange flow. Thus, one already has results which imply the existence of many compact invariant sets.

Our belief that this theorem should prove interesting is based on other considerations. These relate to results about twist maps which we believe should generalize to more degrees of freedom. Theorems A and B of the introduction are examples of one way that the basic theory described in this section may be used in the context of twist maps. In this paper, we make a beginning towards generalizing Theorems A and B to more degrees of freedom.

A few words on the proof of Theorem 2.2 may be helpful. The fact that $\mathcal{M}_c$ is non-empty is contained in Theorem 1 of [Ma6], specifically the fact that the minimum is achieved in the formula given there for $-\alpha(c)$. We may give a version of the proof given there, as follows : Let λ_c be a closed smooth 1-form on M whose de Rham cohomology class is c. As in the end of §1, we think of λ_c as a function on $TM \times \mathbb{T}$ (independent of the $\mathbb{T}$ variable). The function $L - \lambda_c$ is a Lagrangian, i.e., it satisfies the conditions of positive definiteness, superlinear growth, and completeness we assumed at the beginning. Moreover, its Euler–Lagrange flow is the same as that of L. For an E_L-invariant measure, we have

$$A_c(\mu) = \int (L - \lambda_c) d\mu.$$

The statement that $M_c \neq \varnothing$ amounts to the assertion that there exists a c-minimal measure μ. The existence of such a measure may be proved by letting α_n: $[0, n] \longrightarrow M$ be a curve which minimizes $\int (L - \lambda_c)\,(d\alpha_n(t),\, t) dt$ for the free endpoint problem. As before, one sets $\alpha_n(t) = (d\,\alpha_n(t),\, t \text{ (mod. } 1)) \in TM \times \mathbb{T}$. One lets μ_n be the probability measure uniformly distributed along α_n and lets μ be a point of accumulation of the μ_n with respect to the vague topology. By Kryloff-Bogoliuboff, μ is E_L-invariant. It is not at all difficult to see that μ minimizes A_c, for example by the argument used to prove Proposition 1 in [Ma6].

The fact that M_c is compact is Proposition 4 in [Ma6]. By definition, M_c is closed in $TM \times \mathbb{T}$. Hence if M_c is not compact, $\|\xi\|$ is unbounded as $\big(\xi,\, t \text{(mod. } 1)\big)$ ranges over M_c. From this, we were able [Ma6, pp. 185-186] to construct an incomplete trajectory of the Euler-Lagrange flow, a contradiction.

The fact that π: $M_c \longrightarrow M \times \mathbb{T}$ is injective and its inverse is Lipschitz is Theorem 2 of [Ma6]. The intuitive idea of the proof is simple. There is a well known elementary *curve shortening* lemma in Riemannian geometry, as follows. Let α, β be curves on a Riemannian manifold joining points P, P' and Q, Q' resp. Suppose α and β cross. Then there exist curves a, joining P and Q, and b joining P' and Q' such that

$$\text{length } (a) + \text{ length } (b) < \text{ length } (\alpha) + \text{ length } (\beta)\,.$$

(See Fig. 1.) In our setting, one may decrease the action in the same way. (See the lemma used in the proof of Theorem 2 in [Ma6].) If π were not injective on M_c, or its inverse were not Lipschitz, it would be possible to construct an E_L-invariant measure μ on $TM \times \mathbb{T}$ for which $A_c(\mu) < \alpha(c)$, a contradiction. This would be done by cutting and pasting trajectories using the curve shortening lemma. Then the Tonelli theorem and the Kryloff-Bogoliuboff argument would supply the required measure. See [Ma6] for the details, which are not simple.

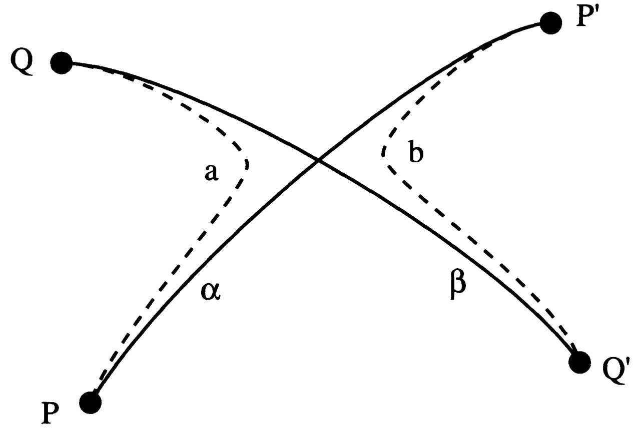

Fig. 1

3. Twist mappings.

We define the *Poincaré map* $f = f_L$: $TM \longrightarrow TM$, associated to the Lagrangian L, as follows. Let $\xi \in TM$ and let γ: $\mathbb{R} \longrightarrow TM \times \mathbb{T}$ denote the trajectory of E_L with initial condition $\gamma(0) = (\xi, 0 \pmod{1})$. We define f by $\gamma(1) = \big(f(\xi), 0 \pmod{1}\big)$.

Such a mapping is an example of an *optical mapping* in the sense of [Arn1]. In the case the $M = \mathbb{T}$, such mappings include *twist mappings*. To be precise, if $f \in \mathcal{P}^1$, Moser showed [Mo] that there is a Lagrangian on $T\mathbb{T} \times \mathbb{T}$ whose Poincaré map is f. Following an idea of Moser, Denzler [Den] showed how the basic results of Aubry-Mather theory can be obtained for Lagrangians on $T\mathbb{T} \times \mathbb{T}$. Our paper [Ma6] may be regarded as a generalization of [Den], with the idea of taking action minimizing measures as the basic notion being the new idea. This idea seems essential for the higher dimensional generalization.

In [Ma6, §6], we showed how the basic results of Aubry-Mather theory can be obtained from the basic theory described in §2. Here, we recall briefly the arguments.

In the case $M = \mathbb{T}$, we have $H_1(M, \mathbb{R}) = \mathbb{R}$, of course, so the minimal average action may be regarded as a function β: $\mathbb{R} \longrightarrow \mathbb{R}$. The first point is that this function is *strictly* convex, i.e., its graph has no flat parts. For, suppose that graph β intersects a line l in $\mathbb{R}^2$ in a segment σ (not reduced to a point). Let $(h_i, \beta(h_i))$, $i = 0, 1$ be the endpoints of σ. Because these endpoints are extremal points of the epigraph of β, there exist action minimizing ergodic measures μ_i, $i = 0, 1$ such that $\rho(\mu_0) = h_0$ and $\rho(\mu_1) = h_1$. Each has its support in M_c, where $c \in H^1(M, \mathbb{R}) = \mathbb{R}$ is the slope of l. By Theorem 2.2, the projection π of M_c on the torus $\mathbb{T}^1 \times \mathbb{T}^1$ in injective. But this leads to a contradiction : since μ_0 and μ_1 have different rotation numbers, trajectories in π (supp μ_0) cross trajectories in π (supp μ_1), contradicting the injectivity of $\pi|M_c$. Thus, we have shown that β is strictly convex. (For more detail, see [Ma6, Proposition 6].)

Let $h \in H_1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$, let $l \subset H^1(\mathbb{T}, \mathbb{R}) \times \mathbb{R}$ be a supporting hyperplane of the epigraph of β which touches the epigraph of β at h, and let c be the slope of l. Let $M_h^0 = (T\,\mathbb{T}\times 0) \cap M_c$. Note that M_h^0 is independent of the choice of l : it is the support of the set of action minimizing invariant measures of Poincaré map f of rotation number h.

The second point is that the projection π of $M_h^0 \subset T\mathbb{T}$ on $\mathbb{T}$ is injective and has Lipschitz inverse. This is immediate from Theorem 2.2.

Of course, $\pi(M_h^0)$ inherits a cyclic order from $\mathbb{T}$. The third point is that $f|M_h^0$ preserves this cyclic order. For, otherwise, the projection of M_c on $\mathbb{T} \times \mathbb{T}$ would not be injective.

It is a well known result in the Denjoy theory of orientation preserving homeomorphisms of the circle that if $g\colon \mathbb{T} \longrightarrow \mathbb{T}$ is an orientation preserving homeomorphism of irrational rotation number, then g is uniquely ergodic, i.e. there is exactly one g-invariant measure on $\mathbb{T}$. In the same way we may prove the fourth point : $f|M_h^0$ is uniquely ergodic when h is irrational. This follows from the cyclic order preserving property of $f|M_h^0$.

It is easy to check that if $g\colon \mathbb{T} \longrightarrow \mathbb{T}$ is an orientation preserving homeomorphism of rational rotation number, then every ergodic measure is supported on a periodic orbit. Similarly, we have the fifth point : if h is rational, say $h = p/q$ lowest terms, then $f|M_h^0$ is periodic of period q. This follows from the cyclic order preserving property of $f|M_h^0$, together with the definition of M_h^0 as the support of a set of invariant measures.

4. The variational principle.

As in §2, we let $\widetilde{M}$ be the covering space of M such that $\pi_1(\widetilde{M}) = \ker\big(\mathcal{H}\colon \pi_1(M) \longrightarrow H_1(M, \mathbb{R})\big)$.

We may define a continuous function $h = h_L\colon \widetilde{M} \times \widetilde{M} \longrightarrow \mathbb{R}$, called the *variational principle associated to* L, as follows. For $\widetilde{m}, \widetilde{m}' \in \widetilde{M}$, let

$$h(\widetilde{m}, \widetilde{m}') = \min \int_0^1 L\left(d\gamma(t), t\right)dt$$

where the minimum is taken over all curves $\widetilde{\gamma}$: $[0, 1] \longrightarrow \widetilde{M}$ such that $\widetilde{\gamma}(0) = \widetilde{m}$, $\widetilde{\gamma}(1) = \widetilde{m}'$, and γ denotes the projection of $\widetilde{\gamma}$ on M. By Tonelli's theorem, the minimum is achieved, and

$$(H_0) \qquad h \text{ is continuous.}$$

Moreover,

$$(H_1) \qquad h(T\widetilde{m}, T\widetilde{m}') = h(\widetilde{m}, \widetilde{m}'), \quad \text{if } T \in \mathcal{D}.$$

(Recall that $\mathcal{D}$ denotes the group of deck transformations of $\widetilde{M}$.) If we provide M with a Riemannian metric, lift it to $\widetilde{M}$, and let d denote the corresponding metric on $\widetilde{M}$, then we have

$$(H_2) \qquad h(\widetilde{m}, \widetilde{m}') \to +\infty, \quad \text{as } d(\widetilde{m}, \widetilde{m}') \to \infty.$$

Actually, it follows from the superlinear growth condition that

$$h(\widetilde{m}, \widetilde{m}')/d(\widetilde{m}, \widetilde{m}') \to +\infty \text{ as } d(\widetilde{m}, \widetilde{m}') \to \infty.$$

However, the condition above is strong enough for the applications which have been given in [Ban1], [Ma2], and [Ma3].

In the case that $M = \mathbb{T}$, we have that there exists a positive continuous function ρ on $\mathbb{R}^2$ such that

$$(H_5) \qquad \partial_{12}\, h\,(x, x') \leqslant -\rho\,(x, x') \qquad (\mathcal{D}).$$

Here, $(\mathcal{D})$ stands for *in the sense of distributions*, and ∂_{12} denotes the mixed second partial derivative. In general, the function h need not be differentiable, so this inequality makes sense only if it is understood in the distributional sense. See, for example, [Ma2] for a proof in the case $f \in \mathcal{P}^1$: the proof in general works the same way. We do not know of any generalization of (H_5) to manifolds of higher dimension. (See, however, [B-P] for some progress on related questions.)

In the case that $M = \mathbb{T}$, we also have that there exists on positive continuous function θ on $\mathbb{R}^2$ such that

$$(H_6) \qquad \partial_{11}\, h\,(x, x') \leqslant \theta\,(x, x'), \qquad \partial_{22}\, h\,(x, x') \leqslant \theta\;(x, x')(\mathcal{D}).$$

See [Ma2] for a proof in the case $f \in \mathcal{P}^1$ (when θ can be taken to be constant) : the proof in general works the same way, although θ cannot be taken to be constant in general.

Until recently, Aubry-Mather theory for twist maps was based on the study of minimal configurations in $\mathbb{R}$ for a variational principle h satisfying suitable conditions. The basic theory was developed in [Ban1] under conditions $(H_0) - (H_4)$ which are implied by our conditions (H_0), (H_1), (H_2), (H_5) and (H_6). The subsequent development in [Ma2] and [Ma3] was based on these latter conditions.

We may define minimal configurations in complete generality, not just in the case $M = \mathbb{T}$. A *configuration* is a bi-infinite sequence $(\cdots, \widetilde{m}_i, \cdots)$, $\widetilde{m}_i \in \widetilde{M}$. A *segment* of a configuration is a finite sequence $(\widetilde{m}_a, \cdots, \widetilde{m}_i, \cdots, \widetilde{m}_b)$, $\widetilde{m}_i \in \widetilde{M}$, $a < b \in \mathbb{Z}$. For such a segment, we set $h(\widetilde{m}_a, \cdots, \widetilde{m}_b) = \sum_{i=a}^{b-1} h(\widetilde{m}_i, \widetilde{m}_{i+1})$. Such a segment is said to be *minimal* if

$$h(\widetilde{m}_a, \cdots, \widetilde{m}_b) \leqslant h(\widetilde{m}'_a, \cdots, \widetilde{m}'_b)$$

for any other segment $(\widetilde{m}'_a, \cdots, \widetilde{m}'_i, \cdots, \widetilde{m}'_b)$ such that $\widetilde{m}'_a = \widetilde{m}_a$ and $\widetilde{m}'_b = \widetilde{m}_b$ (but not necessarily $\widetilde{m}'_i = \widetilde{m}_i$ for $a < i < b$). A configuration is said to be *minimal* if every segment of it is minimal.

Given a segment of a minimal configuration $(\widetilde{m}_a, \cdots, \widetilde{m}_b)$, we may construct a Tonelli minimizer $\widetilde{\gamma}$: $[a, b] \longrightarrow \widetilde{M}$ by letting $\widetilde{\gamma}(t)$, $i \leqslant t \leqslant i+1$, be a Tonelli minimizer satisfying the boundary condition $\widetilde{\gamma}(i) = \widetilde{m}_i$, $\widetilde{\gamma}(i+1) = \widetilde{m}_{i+1}$. Conversely, if $\widetilde{\gamma}$: $[a, b] \longrightarrow \widetilde{M}$ is a Tonelli minimizer, $a < b \in \mathbb{Z}$, then $(\widetilde{m}_a, \cdots, \widetilde{m}_b)$ is a segment of a minimal configuration, if $\widetilde{m}_i = \widetilde{\gamma}(i)$. These assertions follow immediately from the definitions.

We will say that $\widetilde{\gamma}$: $\mathbb{R} \longrightarrow \widetilde{M}$ is a Tonelli minimizer if the restriction of it to each finite interval is a Tonelli minimizer. From the above discussion, it follows that there is a one-one correspondence between mappings $\mathbb{R} \longrightarrow \widetilde{M}$ which are Tonelli minimizers and minimal configurations.

5. Minimizers and minimal measures.

In what follows, we identify a curve γ in M (or $\widetilde{M}$) with the curve $t \longrightarrow (\gamma(t), t \bmod. 1)$ in $M \times \mathbb{T}$ (or $\widetilde{M} \times \mathbb{T}$).

We say that a curve in M is an $\widetilde{M}$-minimizer if a lift of it to $\widetilde{M}$ is a Tonelli minimizer. There is a close relationship between $\widetilde{M}$-minimizers and minimal measures, which is stated as Propositions 2 and 3 in [Ma6, §3]. In this section, we recall Proposition 2 and state a slightly more precise version of Proposition 3, which may be proved in the same way.

Let $\gamma\colon \mathbb{R} \longrightarrow M$ be a C^1 curve and let $\zeta(t) = \big(d\gamma(t),\ t \text{ mod. } 1\big) \in TM \times \mathbb{T}$. Let μ be a probability measure on the one point compactification $(TM \times \mathbb{T})^*$ of $TM \times \mathbb{T}$. We say that μ is a *limit measure* of γ (or ζ) if there is a sequence $[a_i,\, b_i]$ of closed intervals in $\mathbb{R}$ with $b_i - a_i$ tending to ∞, such that μ_i tends vaguely to μ, where μ_i is the probability measure evenly distributed along $\gamma \mid [a_i,\, b_i]$.

Let d denote the metric on $\widetilde{M}$ associated to the lift of a smooth Riemannian metric on M.

PROPOSITION 5.1. — *Let $\gamma\colon \mathbb{R} \to M$ be an $\widetilde{M}$-minimizer and suppose that*

$$\lim_{b \to +\infty} \inf_{a \to -\infty} d\big(\widetilde{\gamma}(a),\, \widetilde{\gamma}(b)\big)/(b-a) < \infty,$$

where $\widetilde{\gamma}$ denotes a lift of γ to $\widetilde{M}$. Then there exists $c \in H^1(M, \mathbb{R})$ such that every limit measure of γ minimizes A_c.

In particular, the point at infinity in $(TM \times \mathbb{T})^*$ has zero mass with respect to such a limit measure.

This is [Ma6, Proposition 2].

To state our refinement of [Ma6, Propositon 3], we introduce the following notion. We say a curve γ in M is a *c*-minimizer (where $c \in H^1\big(M, \mathbb{R}\big)$) if it satisfies the following condition. For any interval $[a,\, b]$ and any curve $\gamma_1\colon\ [c,\, d] \to M$ such that $c - a \in \mathbb{Z}$ and $d - b \in \mathbb{Z}$, we have

$$\int_a^b (L - \eta_c - \alpha(c))\,(d\gamma(t),\, t)dt \leqslant \int_c^d (L - \eta_c - \alpha(c))\,(d\gamma_1(t),\, t)dt,$$

where α is as defined in §2, i.e., the conjugate of β (the minimal average action).

We emphasize that segments of the curve minimize for curves in M (*not* $\widetilde{M}$). Moreover, we have replaced the fixed endpoint problem by the requirement that the endpoints differ by an integer.

Note that if we replaced L by $L - \eta_c - \alpha(c)$ in the definition of $\widetilde{M}$ minimizer, we would not change the class of curves we get. Subtraction of the closed one form η_c would make no difference, by Stokes's theorem. Subtraction of the constant $\alpha(c)$ would make no difference for a fixed endpoint problem. However, in the definition of c-minimizer, we no longer have a fixed endpoint problem, and the constant $-\alpha(c)$ is important.

It also follows from Stokes's theorem that the notion of c-minimizer is independent of the choice of η_c representing c. Clearly, any c-minimizer is an $\widetilde{M}$-minimizer. On the other hand, any limit measure of a c-minimizer minimizes A_c, as may be seen from the proof of [Ma6, Proposition 2].

By Theorem 2.2, π: $M_c \longrightarrow \Sigma_c \left(= \pi\left(M_c\right) \subset M \times \mathbb{T}\right)$ is a bi-Lipschitz homeomorphism, for any $c \in H^1(M, \mathbb{R})$. The restriction of the Euler-Lagrange vector field to M_c induces a Lipschitz vector field E_L^c on Σ_c. This vector field generates a flow on Σ_c which we will call the c-Euler-Lagrange flow.

PROPOSITION 5.2. — *Any trajectory of the c-Euler-Lagrange flow is a c-minimizer.*

This is a slight refinement of [Ma6, Propositon 3] and may be proved in the same way.

6. A barrier.

In this section, we will define a partial generalization of Peierls's barrier to several degrees of freedom.

Throughout this section, we fix $c \in H^1(M, \mathbb{R})$. Recall that $\Sigma_c \subset M \times \mathbb{T}$ and that there is a Lipschitz flow on Σ_c, which we called the c-Euler-Lagrange flow. We set $\Sigma_c^0 = \Sigma_c \cap (M \times 0)$ and let f_c: $\Sigma_c^0 \longrightarrow \Sigma_c^0$ be the time one map associated to the c-Euler-Lagrange flow. We call f_c the c-Poincaré map. Clearly, f_c is Lipschitz.

We define a function $h = h_c$: $M \times M \longrightarrow \mathbb{R}$, as follows. We choose a closed smooth one form η_c on M whose de Rham cohomology class is c. For $m, m' \in M$, we let

$$h_c(m, m') = \min \int_0^1 (L - \eta_c)(d\gamma(t), t)dt - \alpha(c),$$

where α: $H^1(M, \mathbb{R}) \rightarrow \mathbb{R}$ is as defined in §2, and the minimum is taken over all curves γ: $[0, 1] \longrightarrow M$ such that $\gamma(0) = m$ and $\gamma(1) = m'$. This is similar to the variational principle defined in §4, but differs in two respects : L is replaced by $L - \eta_c$ and h_c is defined on M, whereas the variational principle of §4 was defined on $\widetilde{M}$. In addition, we have subtracted $\alpha(c)$.

Note that h_c depends on the choice of η_c. If $\eta'_c = \eta_c + du$, where u is a smooth function on M, then $h'_c(m, m') = h_c(m, m') + u(m') + u(m)$.

In this section, we will consider configurations in M rather than in $\widetilde{M}$. Thus, we use the same definitions as in §4, but now the configurations are bi-infinite sequences $(\cdots, m_i, \cdots)$, with $m_i \in M$. When we wish to distinguish the two notions, we will refer to $\widetilde{M}$-configurations or M-configurations. We will say that a segment $(m_a, \cdots, m_b)$ of an M-configuration is *c-minimal* if

$$h_c(m_a, \cdots, m_b) \leqslant h_c(m'_c, \cdots, m'_d)$$

for any other segment $(m'_c, \cdots, m'_d)$ such that $m_a = m'_c$ and $m_b = m'_d$. Note that we do not require $d - c = b - a$, in contrast to the definition of minimal $\widetilde{M}$-configurations given in §4. We will say that an M-configuration is *c-minimal* if every segment of it is c-minimal.

In analogy to the one-one correspondence between $\widetilde{M}$-minimizers and minimal $\widetilde{M}$-configurations described at the end of §4, there is a one-one correspondence between c-minimizers and c-minimal M-configurations : starting with a c-minimal configuration $(\cdots, m_i, \cdots)$ one connects m_i to m_{i+1} by a curve γ which minimizes $\int_i^{i+1}(L - \eta_c)\big(d\gamma(t), t\big)dt$. In this way, one obtains a c-minimizer.

Any orbit of the c-Poincaré map f_c: $\Sigma_c^0 \longrightarrow \Sigma_c^0$ is a c-minimal M-configuration, by Proposition 5.2.

The n-fold conjunction h_c^n of h_c with itself (in the sense of [Ma2, §5]) is defined by the formula

$$h_c^n\,(\xi, \eta) = \min\Big\{\sum_{i=0}^{n-1} h_c\,(m_i, m_{i+1})\colon\ m_0 = \xi,\ m_n = \eta \\ \text{and } m_i \in M \text{ for } 0 \leqslant i \leqslant n\Big\}.$$

We set

$$h_c^\infty\,(\xi, \eta) = \liminf_{n\to\infty}\ h_c^n\,(\xi, \eta),$$

for $\xi, \eta \in M$. We set $B_c(\xi) = h_c^\infty\,(\xi, \xi)$. Note that B_c is independent of the choice of η_c representing c. We call B_c the *barrier* (or *c-barrier*). We will show in §7 that when $M = \mathbb{T}$, this function reduces in many cases to Peierls's barrier.

The barrier B_c is a Lipschitz non-negative function on M which vanishes identically on Σ_c^0. In fact, we added the normalizing summand $-\alpha(c)$ in the definition of $h_c(m, m')$ to obtain this property.

To show that B_c is non-negative, we first note that $\int (L - \eta_c)\, d\mu - \alpha(c) = 0$, for any c-minimal measure μ, by the definition of α. From this, it follows that $h_c^n(\xi, \xi) \geqslant 0$. For, otherwise, it would be possible to construct an invariant measure μ such that $\int (L-\eta_c)\, d\mu - \alpha(c) < 0$, by using Tonelli's theorem and the Kryloff-Bogoliuboff argument as in [Ma6]. But this would contradict the definition of α. From $h_c^n(\xi, \xi) \geqslant 0$, it follows that B_c is non-negative.

Let $\xi_0 \in \Sigma_c^0$. To show that $B_c(\xi_0) = 0$, we suppose the contrary. Then there exists a large positive integer N and a small positive number δ such that $h_c^n(\xi_0, \xi_0) \geqslant \delta$, for all $n \geqslant N$. Note that a Lipschitz constant for h_c is also a Lipschitz constant for h_c^n, for all $n \geqslant 1$. Consequently, there is a neighborhood U of ξ_0 in M such that $h_c^n(\xi, \eta) \geqslant \delta/2$, for all $\xi, \eta \in U$ and all $n \geqslant N$.

By the definition of Σ_c^0 and Proposition 5.2, c-minimal measures correspond one-one to f_c-invariant measures on Σ_c^0. In particular, Σ_c^0 is the closure of the union of the supports of all such measures. Thus, there exists an ergodic f_c-invariant measure μ whose support meets U. From the Birkhoff ergodic theorem, it follows that there exists $\xi \in U$ such that $f_c^i(\xi)$ returns to U with positive frequency and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} h_c\big(f_c^i(\xi), f_c^{i+1}(\xi)\big) = \int h_c\left(\xi, f_c(\xi)\right) d\mu(\xi) = 0.$$

The last equation is a consequence of the fact that μ corresponds to a c-minimal measure.

Since $f_c^i(\xi)$ returns to U with positive frequency, there exist $n_1 \geqslant N$, $n_2 \geqslant N, \ldots$ such that $f_c^{n_1}(\xi) \in U$, $f_c^{n_1+n_2}(\xi) \in U, \ldots$, and there exists a constant C such that $\sum_{k=0}^{K-1} n_k \leqslant CK$ for all $K \geqslant 1$. Since $h_c^n(\xi, \eta) \geqslant \delta/2$ for all $\xi, \eta \in U$ and $n \geqslant N$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} h_c\big(f_c^i(\xi), f_c^{i+1}(\xi)\big) \geqslant \frac{K}{n}\frac{\delta}{2} \geqslant \frac{\delta}{2C},$$

when $n = n_1 + \cdots + n_K$. This contradicts the previous equation. This contradiction proves that $B_c(\xi_0) = 0$.

It is obvious that h_c^∞ is Lipschitz as a function on $M \times M$ and that a Lipschitz constant for h_c is also a Lipschitz constant for h_c^∞. In particular, the barrier B_c is a Lipschitz function on M. Clearly, $h_c^{m+n}(\xi, \nu) \leqslant h_c^m(\xi, \eta) + h_c^n(\eta, \nu)$; hence, $h_c^\infty(\xi, \nu) \leqslant h_c^\infty(\xi, \eta) + h_c^\infty(\eta, \nu)$, for all $\xi, \eta, \nu \in M$.

For $\xi, \eta \in M$, we set $d_c(\xi, \eta) = h_c^\infty(\xi, \eta) + h_c^\infty(\eta, \xi)$. Obviously, $d_c(\xi, \xi) = 2B_c(\xi) \geqslant 0$, and $d_c(\xi, \nu) \leqslant d_c(\xi, \eta) + d_c(\eta, \nu)$ for all $\xi, \eta, \nu \in M$. We let $\Sigma_c^{0'}$ be the set where B_c vanishes. Thus, $\Sigma_c^0 \subset \Sigma_c^{0'}$. Clearly, the restriction of d_c to $\Sigma_c^{0'}$ is a pseudo-metric, i.e., $d_c(\xi, \xi) = 0$, $d_c(\xi, \eta) = d_c(\eta, \xi)$, and the triangle inequality holds.

Clearly,

$$h_c^\infty(\xi, \eta) + h_c^\infty(\eta, \nu) \leqslant \min\left(d_c(\xi, \eta), d_c(\eta, \nu)\right) + h_c^\infty(\xi, \nu).$$

Thus, $h_c^\infty(\xi,\nu) = h_c^\infty(\xi,\eta) + h_c^\infty(\eta,\nu)$ if either $d_c(\xi,\eta) = 0$ or $d_c(\eta,\nu) = 0$.

If $(\cdots, m_i, \cdots)$ is a c-minimal M-configuration, it follows from Proposition 5.1 that any limit measure of it has support in Σ_c^0. In particular, there exist $\alpha, \omega \in \Sigma_c^0$ such that α is an α-limit point of $(\cdots, m_i, \cdots)$ and ω is an ω-limit point of $(\cdots, m_i, \cdots)$, i.e., there exist $i_k \to -\infty$ such that α is the limit of m_{i_k} and $j_k \to +\infty$ such that ω is the limit of m_{j_k}.

Note that $h_c^{k-i}(m_i, m_k) = h_c^{j-i}(m_i, m_j) + h_c^{k-j}(m_j, m_k)$, for $i < j < k$. Since $(\cdots, m_i, \cdots)$ is c-minimal, we have $h_c^{k-i}(m_i, m_k) = \inf_l h_c^l(m_i, m_k)$ and similarly for j in place of i. Hence, $\inf_l h_c^l(m_i, m_k) = h_c^{j-i}(m_i, m_j) + \inf_l h_c^l(m_j, m_k)$. By passing to the limit, we have $h_c^\infty(m_i, \omega) = h_c^{j-i}(m_i, m_j) + h_c^\infty(m_j, \omega)$, for $i < j$. Furthermore, $h_c^\infty(\omega, m_j) \leqslant h_c^\infty(\omega, m_i) + h_c^{j-i}(m_i, m_j)$. Adding, we obtain $d_c(m_j, \omega) \leqslant d_c(m_i, \omega)$, for $i < j$. In other words, $d_c(m_i, \omega)$ is a monotonically decreasing sequence. Since we already know that its lim inf is zero, we obtain

$$\lim_{i \to +\infty} d_c(m_i, \omega) = 0.$$

Similarly, $d_c(m_i, \alpha)$ is a monotonically increasing sequence and

$$\lim_{i \to -\infty} d_c(m_i, \alpha) = 0.$$

From the triangle inequality, it follows that

$$d_c(\alpha, \omega) = \lim_{i \to +\infty} d_c(\alpha, m_i) = \lim_{i \to -\infty} d_c(m_i, \omega).$$

Also, if ω' is a second ω-limit point of $(\cdots, m_i, \cdots)$, we have $d_c(\omega, \omega') \leqslant d_c(\omega, m_i) + d_c(m_i, \omega')$, so $d_c(\omega, \omega') = 0$, by passage to the limit. Likewise, if α' is a second α-limit point of $(\cdots, m_i, \cdots)$, then $d_c(\alpha, \alpha') = 0$.

If $d_c(\alpha, \omega) = 0$, we will say that $(\cdots, m_i, \cdots)$ is a *regular* c-minimal configuration. In this case, $d_c(\alpha, m_i) = d_c(\omega, m_i) = 0$, for all i, since these are monotonic bi-infinite sequences converging to zero at both ends. Consequently $d_c(m_i, m_i) \leqslant d_c(m_i, \omega) + d_c(\omega, m_i) = 0$, so $m_i \in \Sigma_c^{0'}$. By the triangle inequality, $d_c(m_i, m_j) \leqslant d_c(m_i, \alpha) + d_c(\alpha, m_j) = 0$, so we have $d_c(m_i, m_j) = 0$.

Conversely, for any $\xi \in \Sigma_c^{0'}$, there exists a unique regular c-minimal configuration $(\cdots, m_i, \cdots)$ such that $\xi = m_0$. This may be shown as follows.

Existence. Choose an increasing sequence $n_1, n_2, \cdots, n_k, \cdots$ of positive integers such that $h_c^{n_k}(\xi, \xi) \to B_c(\xi) = 0$. For each k, choose a segment $m_0^k, \cdots, m_i^k, \cdots m_{n_k}^k$ of an M-configuration such that $\xi = m_0^k = m_{n_k}^k$ and $h_c^{n_k}(\xi, \xi) = \sum_{i=0}^{n_k-1} h_c(m_i^k, m_{i+1}^k)$. For every integer j, set $m_j^k = m_i^k$, where $0 \leqslant i < n_k$ is the remainder obtained by dividing j by n_k. By passing to a subsequence, we may suppose that $m_j^k \longrightarrow m_j$. The resulting M-configuration $(\cdots, m_j, \cdots)$ is easily seen to be c-minimal and satisfy $m_0 = \xi$.

To show that $(\cdots, m_j, \cdots)$ is regular, we observe that for $i>0$, $d_c(\xi, m_i) = \lim_{k\to\infty} d_c(\xi, m_i^k)$. Moreover, $d_c(\xi, m_i^k) = h_c^\infty(\xi, m_i^k) + h_c^\infty(m_i^k, \xi)$. Since $h_c^\infty(\xi, \xi) = B_c(\xi) = 0$, we have $h_c^\infty(\xi, m_i^k) \leqslant h_c^\infty(\xi, \xi) + h_c^i(\xi, m_i^k) = h_c^i(\xi, m_i^k)$. Likewise, $h_c^\infty(m_i^k, \xi) \leqslant h_c^{n_k - i}(m_i^k, \xi)$. Therefore,

$$d_c(\xi, m_i^k) \leqslant h_c^i(\xi, m_i^k) + h_c^{n_k-i}(m_i^k, \xi) = h_c^{n_k}(\xi, \xi).$$

Since the last quantity tends to zero as k goes to infinity, it follows that $d_c(\xi, m_i) = 0$, for $i > 0$. A similar argument shows that we have this also for $i < 0$. Passing to the limit, we obtain $d_c(\alpha, \omega) = 0$.

Uniqueness. This is again the curve shortening argument. Given two regular c-minimal configurations $(\cdots, m_i, \cdots)$ and $(\cdots, m_i', \cdots)$ with $m_0 = m_0' = \xi$, we may use the curve shortening Lemma of [Ma6, §4] to construct two new configurations $(\cdots, m_i'', \cdots)$ and $(\cdots, m_i''', \cdots)$ such that $m_i'' = m_i$, for $i < 0$, $m_i'' = m_i'$, for $i > 0$, $m_i''' = m_i'$, for $i < 0$, $m_i''' = m_i$, for $i > 0$, and

$$\begin{aligned} h_c(m_{-1}''', m_0''') + h_c(m_0''', m_1''') + h_c(m_{-1}'', m_0'') + h_c(m_0'', m_1'') \\ < h_c(m_{-1}, m_0) + h_c(m_0, m_1) + h_c(m_{-1}', m_0') + h_c(m_0', m_1'). \end{aligned}$$

It follows that $d_c(\alpha''', \omega''') + d_c(\alpha'', \omega'') < d_c(\alpha, \omega) + d_c(\alpha', \omega')$, where α, α', etc. are the α-limit points of $(\cdots, m_i, \cdots)$, $(\cdots, m_i', \cdots)$ etc. and ω, ω', etc. are the corresponding ω-limit points. However, $d_c(\alpha, \omega) = d_c(\alpha', \omega')$, so we have obtained a contradiction. This contradiction proves uniqueness.

We may extend the c-Poincaré map f_c to $\Sigma_c^{0'}$, as follows. Given $\xi \in \Sigma_c^{0'}$, we let $(\cdots, m_i, \cdots)$ be the unique regular c-minimal configuration such that $m_0 = \xi$. We set $f_c(\xi) = m_1$. It is clear that this extends the previously defined f_c. Moreover, $f_c : \Sigma_c^{0'} \longrightarrow \Sigma_c^{0'}$ is a bi-Lipschitz homeomorphism. This follows from the proof of the uniqueness of the regular c-minimal configuration $(\cdots, m_i, \cdots)$ such that $\xi = m_0$: the curve shortening Lemma of [Ma6, §4] contains the requisite Lipschitz result.

Now we may introduce some more terminology, using the one-one correspondence between c-minimal trajectories of the Euler-Lagrange flow and c-minimal configurations. We will say that a c-minimal trajectory of the Euler-Lagrange flow is *regular* if the corresponding M-configuration is regular. We let $M_c' \subset TM \times \mathbb{T}$ denote the union of all c-minimal trajectories. The assertions of Theorem 2.2 extend to M_c'.

THEOREM 6.1. — *M_c' is a compact, non-empty subset of $TM \times \mathbb{T}^1$ containing M_c. The restriction of π to M_c' is injective. The inverse mapping $\pi^{-1} : \pi(M_c') \to M_c'$ is Lipschitz.*

The proofs of all these assertions follow from the discussion above.

We set $\Sigma_c' = \pi(M_c') \subset M \times \mathbb{T}$. It follows from this theorem that the flow on Σ_c' which corresponds to the Euler-Lagrange flow on M_c' is Lipschitz, and is generated by a Lipschitz vector field. Clearly, $\Sigma_c^{0'} = \Sigma_c' \cap (M \times 0)$ and $f_c : \Sigma_c^{0'} \to \Sigma_c^{0'}$ is the time one map of this flow.

7. Peierls's barrier.

In this section, we specialize to the case $M = \mathbb{T}$. We will show that if $(c, \alpha(c))$ is an extremal point of the epigraph of α, then the barrier B_c (defined in the previous section) is the same as Peierls's barrier P_ω, for a suitable rotation symbol ω. Peierls's barrier was defined originally in [A-LD-A] and again in [Ma1] and [Ma2]. See also [Ma5], which is perhaps the most convenient reference for this notion, and also for the notion of rotation symbol. Here we recall only that a rotation symbol is a Dedekind cut of

$\mathbb{Q}$. Thus, to each irrational number, there corresponds a unique rotation symbol. To each rational number, there correspond three rotation symbols, denoted $p/q-$, p/q and $p/q+$.

To be more explicit, we have to recall some properties of the minimal average action $\beta : H_1(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$ and of its conjugate $\alpha : H^1(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$, which were proved independently by the author [Ma7] and Bangert [Ba3], and discussed much earlier from a physicist's viewpoint by Aubry [Aub]. In terms of the identification $H_1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$, we have that β is differentiable at all irrational numbers. Moreover, if p/q is a rational number in lowest terms, then β is differentiable at p/q if and only if there exists a homotopically non-trivial f_L-invariant curve $\Gamma \subset T\mathbb{T} \,(= \mathbb{T} \times \mathbb{R})$ of rotation number p/q, consisting entirely of periodic orbits of period q.

Of course, these results may be reinterpreted in terms of the conjugate function α: $H^1(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$. We use the identification $H^1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$ and thus think of α as a real valued function of a real variable. The fact that β is strictly convex translates to the fact that α is differentiable. The fact the β is differentiable at irrational numbers translates to the fact that every flat piece of graph α has rational slope.

PROPOSITION 7.1. — *When $\omega = \alpha'(c)$ is irrational, we have $B_c = P_\omega$.*

Strictly speaking, in [Ma1] and [Ma2], we defined P_ω to be a real valued function of a real variable. However, it is periodic of period one. Thus, we may think of it as a real valued function on $\mathbb{T}$. The equation $B_c = P_\omega$ above means equality of functions on $\mathbb{T}$.

For the proof, we recall that P_ω was defined to be identically zero on Σ_c^0 in [Ma1] and [Ma2]. In the previous section, we showed that B_c is identically zero on Σ_c^0. Thus, it is enough to consider $a \in \mathbb{T} \backslash \Sigma_c^0$ and show that $B_c(a) = P_\omega(a)$. The component of $\mathbb{T} \backslash \Sigma_c^0$ which contains a is a segment whose endpoints we denote by a_- and a_+. Since $a_\pm \in \Sigma_c^0$, there exist unique c-minimal configuration $\xi_\pm = (\cdots, \xi_{i\pm}, \cdots)$ such that $\xi_{0\pm} = a_\pm$. We choose lifts $\widetilde{a}$, $\widetilde{a}_\pm$, $\widetilde{\xi}_{i\pm}$ of a, $a_\pm$, $\xi_{i\pm}$ to $\mathbb{R}$ such that $\widetilde{a}_- < \widetilde{a} < \widetilde{a}_+ < \widetilde{a}_- + 1$, $\widetilde{\xi}_{0\pm} = \widetilde{a}_\pm$, and such that $\widetilde{\xi}_\pm = (\cdots, \widetilde{\xi}_{i\pm}, \cdots)$ is a minimal configuration. Peierls's barrier is defined as follows ([Ma1], [Ma2]) :

$$P_\omega(a) = \min \sum_{i \in \mathbb{Z}} \widetilde{h}\,(\widetilde{\xi}_i, \, \widetilde{\xi}_{i+1}) - \widetilde{h}\,(\widetilde{\xi}_{i-}, \, \widetilde{\xi}_{i+1-}).$$

Here, $\widetilde{h}$ is the variational principle associated to L in the sense of §4, so it is a function defined on $\mathbb{R} \times \mathbb{R}$. The minimum is taken over all

$\widetilde{\xi} = (\cdots, \widetilde{\xi}_i, \cdots) \in \mathbb{R}^{\mathbb{Z}}$ such that $\widetilde{\xi}_{i-} \leqslant \widetilde{\xi}_i \leqslant \widetilde{\xi}_{i+}$ and $\widetilde{\xi}_0 = \widetilde{a}$. The condition $\widetilde{\xi}_{i-} \leqslant \widetilde{\xi}_i \leqslant \widetilde{\xi}_{i+}$ guarantees that the sum above is absolutely convergent, since $\sum \widetilde{\xi}_{i+1-} - \widetilde{\xi}_{i-} \leqslant 1$ in the case that ω is irrational. Note that if $\widetilde{\xi}_-$ is replaced by $\widetilde{\xi}_+$ in the formula above for $P_\omega(a)$, it is still valid.

To prove that $B_c(a) = P_\omega(a)$, we consider how the definition of $P_\omega(a)$ may be put in a form more closely resembling that of $B_c(a)$. If we replace $\widetilde{h}$ by the variational principle $\widetilde{h}_c$ associated to $L - \eta_c - \alpha(c)$ in the expression defining $P_\omega(a)$, we get the same quantity. From Aubry's crossing lemma, we then obtain

$$P_\omega(a) = \lim_{k,l \to +\infty} h_c^k\,(\xi_{-k-},\, a) + h_c^l\,(a,\, \xi_{l-}) - h_c^{k+l}\,(\xi_{-k-},\, \xi_{l-}).$$

Note that $h_c^{k+l}(\xi_{-k-}, \xi_{l-}) = h_c^\infty(\xi_{-k-}, \xi_{l-})$, since $(\cdots, \xi_{l-}, \cdots)$ is c-minimal.

Since $\omega = \alpha'(c)$ is irrational, it follows from the well known theory of twist maps that Σ_c^0 is a Denjoy minimal set for the Poincaré map f_c. Thus, every orbit of f_c is dense in Σ_c^0. In particular, we may choose $\xi \in \Sigma_c^0$ and sequences $k_j, l_j \to +\infty$, $j = 1, 2 \cdots$ such that $\xi_{-k_j-} \to \xi$ and $\xi_{l_j-} \to \xi$ as $j \to \infty$. Clearly, $\lim_{j\to\infty} h_c^\infty\,(\xi_{-k_j-}, \xi_{l_j-}) = h_c^\infty(\xi, \xi) = 0$. Moreover, $\lim_{j\to\infty} h_c^{k_j}\,(\xi_{-k_j-}, a) = h_c^\infty(\xi, a)$ and $\lim_{j\to\infty} h_c^{l_j}\,(a, \xi_{l_j}) = h_c^\infty\,(a, \xi)$. Therefore

$$B_c(a) = h_c^\infty\,(a, \xi) + h_c^\infty\,(\xi, a) = P_\omega(a). \qquad \square$$

Proposition 7.2. — *If $\alpha'(c) = p/q$ in lowest terms and $c = \max\{c^*\colon \alpha'(c^*) = p/q\}$ (resp. $c = \min\{c^*\colon \alpha'(c^*) = p/q\}$) then $B_c(a) = P_{p/q+}(a)$ (resp. $P_{p/q-}(a)$).*

The proof is similar to the proof of Proposition 7.1 and we omit it.

Thus, we have related $B_c(a)$ to Peierls's barrier in all cases when $(c, \alpha(c))$ is an extremal point of the epigraph of α. When $(c, \alpha(c))$ is not an extremal point of the epigraph of α, then it lies on a flat part of the graph of α. Let p/q be the slope of this flat part, expressed in lowest terms. Then B_c and $P_{p/q}$ have the same zero set. However, in general they are not equal.

8. Another barrier.

In the next section, we will state versions of Theorems A and B of the introduction (Theorems 4.1 and 4.2 of [Ma5]) in our more general setting. For this we need a variant of the barrier defined in §6. In this section we define this variant and develop some of its properties.

We set

$$B_c^*(m) = \min\left\{h_c^\infty(\xi, m) + h_c^\infty(m, \eta) - h_c^\infty(\xi, \eta)\colon\ \xi, \eta \in \Sigma_c^0\right\}.$$

It is easily seen that $B_c^*(m)$ is a Lipschitz function of m, with a uniform Lipschitz constant for c in a compact set. Note that

$$B_c(m) = \min\left\{h_c^\infty(\xi, m) + h_c^\infty(m, \eta) - h_c^\infty(\xi, \eta)\colon\ \xi, \eta \in \Sigma_c^0,\ d_c(\xi, \eta) = 0\right\}.$$

It follows that $0 \leqslant B_c^* \leqslant B_c$. Clearly, $\{B_c = 0\}$ is the union of the regular c-minimal configurations and $\{B_c^* = 0\}$ is the union of the c-minimal configurations. Moreover, if d_c vanishes on $\Sigma_c^0 \times \Sigma_c^0$, then $B_c^* = B_c$.

In §6, we observed that if $(\cdots, m_i, \cdots)$ is a regular c-minimal configurations, then $d_c(m_i, m_j) = 0$. Thus, if the Poincaré map $f_c\colon \Sigma_c^0 \to \Sigma_c^0$ has a dense orbit, then d_c vanishes identically on $\Sigma_c^0 \times \Sigma_c^0$.

In the case of a twist map, if $\omega = \alpha'(c)$ is irrational, then $f_c\colon \Sigma_c^0 \longrightarrow \Sigma_c^0$ has a dense orbit, so d_c vanishes identically on $\Sigma_c^0 \times \Sigma_c^0$, and $B_c = B_c^*$.

Let d be a metric on M associated to a smooth Riemannian metric. The pseudo-metric d_c satisfies a Hölder condition of exponent 2 with respect to d, viz.,

$$d_c(\xi, \eta) \leqslant C\, d(\xi, \eta)^2, \qquad \xi \in \Sigma_c^{0'}, \eta \in M.$$

Here, C is a constant which depends only on the Lagrangian L and the cohomology class c. Moreover, C may be chosen to be independent of c for c in a compact subset of $H^1(M, \mathbb{R})$.

To prove this, we use the fact that there is a regular c-minimal configuration $(\cdots, m_i, \cdots)$ such that $\xi = m_0$. Let α be an α-limit point and ω an ω-limit point of $(\cdots, m_i, \cdots)$. Then

$$\begin{aligned} d_c(\xi, \eta) &= h_c^\infty(\xi, \eta) + h_c^\infty(\eta, \xi) \\ &= h_c^\infty(\alpha, \eta) + h_c^\infty(\eta, \omega) - h_c^\infty(\alpha, \xi) - h_c^\infty(\xi, \omega) \\ &\leqslant h_c(m_{-1}, \eta) + h_c(\eta, m_1) - h_c(m_{-1}, \xi) - h_c(\xi, m_1) \\ &\leqslant C\, d(\xi, \eta)^2. \end{aligned}$$

Here, the second equation is consequence of the equations $h_c^\infty(\alpha, \xi) + h_c^\infty(\xi, \eta) = h_c^\infty(\alpha, \eta)$ and $h_c^\infty(\eta, \xi) + h_c^\infty(\xi, \omega) = h_c^\infty(\eta, \omega)$, which hold because $d_c(\alpha, \xi) = d_c(\xi, \omega) = 0$. To prove the first inequality, we consider sequences $k_j, l_j \to +\infty$ such that $m_{-k_j} \to \alpha$ and $m_{l_j} \to \omega$. Then $h_c^{k_j}(m_{-k_j}, \xi) = \inf_l h_c^l(m_{-k_j}, \xi) \to h_c^\infty(\alpha, \xi)$ and $h_c^{l_j}(\xi, m_{l_j}) = \inf_l h_c^l(\xi, m_{l_j}) \to h_c^\infty(\xi, \omega)$. The first inequality follows easily. The second inequality is elementary.

Since d_c satisfies a Hölder condition of exponent 2, we have that $d_c(\xi, \eta) = 0$, if $\xi, \eta \in \Sigma_c^0$ and ξ and η can be connected in Σ_c^0 by a rectifiable curve. Thus, it follows that d_c vanishes identically if $\Sigma_c^0 = M$. For example, in the twist map case, we have seen that if p/q is a rational number and $\{c\colon \alpha'(c) = p/q\}$ is reduced to one point, then $\Sigma_c^0 = \mathbb{T}^1$. Thus d_c vanishes identically in this case.

Continuing with the twist map case, we next consider the generic situation, viz. $\{c\colon \alpha'(c) = p/q\}$ is an interval $[c_0, c_1]$, with $c_0 < c_1$. For any $c \in [c_0, c_1]$, Σ_c^0 is the union of the minimal configurations of rotation symbol p/q. If, furthermore, $c \in (c_0, c_1)$, then $\Sigma_c^{0'} = \Sigma_c^0$. On the other hand, if $c = c_0$ (resp. $c = c_1$), then $\Sigma_c^{0'}$ is the union of the minimal configurations of rotation symbol $p/q-$ (resp. $p/q+$) and those of rotation symbol p/q, and properly contains Σ_c^0 (c.f. [Ma2]).

In the cases $c = c_0$ and $c = c_1$, the pseudo-metric d_c vanishes on $\Sigma_c^{0'} \times \Sigma_c^{0'}$. In the case $c_0 < c < c_1$, two points in Σ_c^0 are at positive distance with respect to d_c if and only if their images in the quotient space Σ_c^0/f_c are in distinct connected components. For a generic twist diffeomorphism, Σ_c^0/f_c is one point (in the case $\alpha'(c) \in \mathbb{Q}$), but in exceptional cases, it has several points and d_c does not vanish on $\Sigma_c^0 \times \Sigma_c^0$. In the case that Σ_c^0/f_c is one point, $B_c^* = B_c$, but in the remaining cases, $B_c^* < B_c$, and $\{B_c^* = 0\}$ properly contains $\{B_c = 0\}$.

Continuing with the twist map case with the restriction that $\{c\colon \alpha(c) = p/q\}$ is an interval $[c_0, c_1]$, we now consider the situation when Σ_c^0/f_c $\big(c \in [c_0, c_1]\big)$ has two or more points. (Note that this set is independent of $c \in [c_0, c_1]$, since Σ_c^0 is the union of the minimal configurations of rotation symbol p/q.) Consider $\xi, \eta \in \Sigma_c^0$ which have positive distance with respect to d_c and hence represent different elements in Σ_c^0/f_c, and let

$$\Sigma_c^0(\xi, \eta) = \big\{m \in \mathbb{T}^1\colon\ h_c^\infty(\xi, m) + h_c^\infty(m, \eta) - h_c^\infty(\xi, \eta) = 0\big\}.$$

Since Σ_c^0 inherits a cyclic order from $\mathbb{T}^1$, there is an induced cyclic order on Σ_c^0/f_c, and also on the quotient of the set of complementary intervals of

Σ_c^0 by f_c. Now $\Sigma_c^0(\xi, \eta)$ may be described in the following way. There is a critical value c' with $c_0 < c' < c_1$ such that for $c' < c < c_1$ (resp. $c_0 < c < c'$), $\Sigma_c^0(\xi, \eta)$ is the set of all $m \in \mathbb{T}^1$ such that either m is in the orbit of ξ or η under f_c, or m is in a configuration of rotation symbol $p/q+$ (resp. $p/q-$) and (ξ, m, η) is positively (resp. negatively) oriented with respect to the quotient cyclic order. Moreover $\Sigma_{c'}^0(\xi, \eta)$ is the union of the two sets just described.

Since $\{B_c^* = 0\}$ is, by definition, the union of all the sets $\Sigma_c^0(\xi, \eta)$ just described, this provides a description of $\{B_c^* = 0\}$.

9. Versions of Theorems A and B in more degrees of freedom.

In this section, we state versions of Theorems A and B of the introduction (Theorems 4.1 and 4.2 of [Ma5]) in more degrees of freedom. We will also discuss the extent to which these generalize Theorems A and B.

We let $W_L = \{c \in H^1(M, \mathbb{R})$: there exists an open neighborhood U of $\{B_c^* = 0\}$ in M such that the inclusion map $H_1(U, \mathbb{R}) \longrightarrow H_1(M, \mathbb{R})$ is the zero map$\}$. From the fact that $\{B_c^* = 0\}$ is the union of the c-minimal configurations, it follows that the set function $c \to \{B_c^* = 0\}$ is upper semi-continuous. Consequently, W_L is open in $H^1(M, \mathbb{R})$.

THEOREM 9.1. — *Suppose c_0 and c_1 are in the same connected component of W_L. Then there is a trajectory of the Euler-Lagrange flow whose α-limit set lies in M'_{c_0} and whose ω-limit set lies in M'_{c_1}.*

THEOREM 9.2. — *Consider a bi-infinite sequence $(\cdots, c_i, \cdots)$ of cohomology classes, each of which lies in the same connected component of W_L. Let $(\cdots, \epsilon_i, \cdots)$ be a sequence of positive numbers. Then there is a trajectory of the Euler-Lagrange flow which passes within a distance of ϵ_i of each of the sets M'_{c_i} in turn.*

These are our versions of Theorems A and B in more degrees of freedom.

To see to what extent these generalize Theorems A and B, we examine the relationship between the connected components of W_L and the Birkhoff regions of instability.

Thus, we consider a twist mapping f. According to Moser [Mo], there is a Lagrangian $L\colon T\mathbb{T}\times\mathbb{T}\longrightarrow\mathbb{R}$, satisfying the hypotheses we considered in §1, whose time one map is f. Clearly, a cohomology class c is a member of W_L if and only if $\{B_c^*=0\}$ is properly contained in $\mathbb{T}$. In the case that $\alpha'(c)$ is irrational, this holds if and only if there is no homotopically non-trivial f-invariant curve of rotation number $\alpha'(c)$. However, in the case that $\alpha'(c)$ is rational, the situation is more complicated.

Let p/q be a rational number, expressed in lowest terms. Let $[c_0, c_1] = \{c\in H^1(M,\mathbb{R})\colon \alpha'(c)=p/q\}$. If $[c_0, c_1]$ is reduced to one point c, then there exists a homotopically non-trivial invariant curve of rotation number p/q, consisting entirely of periodic orbits of period q (cf. [Aub], [Ban3], [Ma7]). In this case, $\{B_c^*=0\}=\{B_c=0\}=\mathbb{T}$, so $c\notin W_L$.

Thus, we restrict our attention to the case when $c_0<c_1$. This case divides into several subcases, depending on how many action minimizing orbits of rotation number p/q there are.

In the generic situation there is just one. When there is just one, we see from the description in §8 of $\{B_c^*=0\}$ that $c\in W_L$ if $c\in(c_0, c_1)$. Moreover, in the case $c=c_0$ (resp. $c=c_1$), $c\in W_L$ if and only if there does not exist a homotopically non-trivial f-invariant curve of rotation number p/q consisting entirely of orbits of symbol p/q or $p/q-$ (resp. $p/q+$).

When there are two action minimizing orbits of rotation number p/q, the situation is the same as before for $c=c_0, c_1$, but more complicated for $c_0<c<c_1$. The assumption that there are two minimizing orbits means that Σ_c^0/f_c has two elements. Let $\xi, \eta\in\Sigma_c^0$ represent the two different elements of Σ_c^0/f_c. As discussed in §8, we have $\{B_c^*=0\} = \Sigma_c^0(\xi,\eta)\cup\Sigma_c^0(\eta,\xi)$. After some thought, the reader should be able to see that if there is a homotopically non-trivial f-invariant curve Γ of rotation number p/q, then $\{B_c^*=0\}=\mathbb{T}$ in at least one of the following cases : $c=c_0$, $c=c_1$, $c=c'$, or $c=c''$, where c' is the bifurcation value of $\Sigma_c^0(\xi,\eta)$ (i.e. the unique value of c between c_0 and c_1 where it changes) and c'' is the bifurcation value of $\Sigma_c^0(\eta,\xi)$. More specifically, there are the following possibilities : Γ consists of orbits of rotation symbol $p/q+$ (resp. $p/q-$) and orbits of rotation symbol p/q, in which case $\{B_{c_1}^*=0\}=\mathbb{T}$ (resp. $\{B_{c_0}^*=0\}=\mathbb{T}$); or it contains orbits both of rotation symbol $p/q-$ and of rotation symbol $p/q+$, in which case $\Sigma_{c'}^0(\xi,\eta)=\mathbb{T}$ or $\Sigma_{c''}^0(\eta,\xi)=\mathbb{T}$.

In the case of two action minimizing orbits of rotation symbol p/q, it may happen that $\{B_c^*=0\}=\mathbb{T}$ for some $c_0<c<c_1$ even though there is no homotopically non-trivial f-invariant curve of rotation number p/q. If

this happens at all, it must happen when c is one of the critical values c' or c''.

To see how this can happen, we consider for simplicity the case when $p/q = 0$. Then Σ_c^0 has two points ξ and η. Let I (resp. J) denote the arc in $\mathbb{T}$ consisting of all points θ for which (ξ, θ, η) (resp. (η, θ, ξ)) is positively oriented with respect to the cyclic order in $\mathbb{T}$. We could consider, e.g., a twist mapping f, for which every element of I is part of a configuration of rotation symbol 0+ and every element of J is part of a configuration of rotation symbol 0+ or 0−, but there exist an element in J which is not part of a configuration of rotation symbol 0+ and an element in J which is not part of a configuration of rotation symbol 0−. Such twist maps are easily constructed. We may do this in such a way that $c'' < c'$. For such mappings, there is no homotopically non-trivial invariant curve of rotation number 0, but $\{B_{c'} = 0\} = \mathbb{T}$.

Such examples are very exceptional, but they do show that Theorems 9.1 and 9.2 do not generalize Theorems A and B. Such examples give a kind of extraneous obstruction to finding connecting orbits - extraneous in the sense that the connecting orbits exist in these examples, even though their existence does not follow from Theorems 9.1 and 9.2.

Presumably, it should be possible to improve Theorems 9.1 and 9.2 by weakening the hypothesis, so that there are no such extraneous obstructions. However, we have not done so until now.

10. Proof of Theorem 9.1.

Let c_0, c_1 be in the same connected component of W_L. Since W_L is an open subset of the finite dimensional vector space $H^1(M, \mathbb{R})$, we may choose a simple smooth curve Γ in W_L joining c_0 and c_1. For each $c \in \Gamma$ we choose a smooth closed one form η_c whose de Rham cohomology class is c. We choose η_c so that it depends smoothly on c and the other variables jointly and so that for any $c^* \in \Gamma$, we have a neighborhood U_{c^*} of $\{B_{c^*}^* = 0\}$ in M and a neighborhood J_{c^*} of c^* in Γ such that $\eta_c \big| U_{c^*} = \eta_{c^*} \big| U_{c^*}$ for $c \in J_{c^*}$. The possibility of choosing η_c in this way is a consequence of the assumption that $\{B_{c^*}^* = 0\}$ has a neighborhood V in M such that the inclusion map $H_1(V, \mathbb{R}) \longrightarrow H_1(M, \mathbb{R})$ is the zero map, together with the fact that the set function $c \longrightarrow \{B_c^* = 0\}$ is upper semi-continuous.

Given a sequence $\vec{c} = (c^0, \cdots, c^N)$ of elements of Γ, an increasing sequence $\vec{j} = (j_1 < \cdots < j_N)$ of integers, an M-configuration $(\cdots, m_i, \cdots)$, and integers $a < j_1$ and $b > j_N$, we define

$$h_{\vec{c},\vec{j}}(m_a, \cdots, m_b) = \sum_{i=0}^{N} \sum_{k=j_i}^{j_{i+1}-1} h_{c^i}(m_k, m_{k+1}),$$

where we set $j_0 = a$ and $j_{N+1} = b$. Note that h_{c^i} depends on the choice of η_{c^i}. For what follows, it is essential that η_c be chosen as indicated above.

We will say that the segment $(m_a, \cdots, m_b)$ of an M-configuration is $(\vec{c}, \vec{j})$-*minimal* if for every increasing sequence $\vec{j}' = (j'_1 < \cdots < j'_N)$ of integers satisfying $j'_{i+1} - j'_i \leqslant j_{i+1} - j_i$, $i = 1, \cdots, N$, any integers $c < j'_1$ and $d > j'_N$, and any segment of an M-configuration $(m'_c, \cdots, m'_d)$ satisfying the boundary condition $m_a = m'_c$, $m_b = m'_d$, we have

$$h_{\vec{c},\vec{j}}(m_1, \cdots, m_b) \leqslant h_{\vec{c},\vec{j}}(m'_c, \cdots, m'_d).$$

We will say that an M-configuration is $(\vec{c}, \vec{j})$-*minimal* if every segment of it is $(\vec{c}, \vec{j})$-minimal.

An easy compactness argument shows that for any sequence $\vec{c} = (c^0, \cdots, c^N)$ of elements of Γ, and any increasing sequence $\vec{j} = (j_1 < \cdots < j_N)$ of integers, there exists a $(\vec{c}, \vec{j})$-minimal configuration.

The strategy of proof of Theorem 9.1 is to choose a sequence $\vec{c} = (c^0 = c_0, c^1, c^2, \cdots, c^N = c_1)$ of elements of Γ, with c^{i+1} very close to c^i, an increasing sequence $(j_1 < \cdots < j_N)$ of integers, with $j_{i+1} - j_i$ very large, and a $(\vec{c}, \vec{j})$-minimal configuration $(\cdots, m_i, \cdots)$. Then we construct a curve $\gamma : \mathbb{R} \to M$ by letting $\gamma(t)$, $i \leqslant t \leqslant i + 1$, be a Tonelli minimizer satisfying the boundary condition $\gamma(i) = m_i$, $\gamma(i + 1) = m_{i+1}$.

Our assertion is that if $\vec{c}$ and $\vec{j}$ are chosen appropriately (i.e., if c^{i+1} is sufficiently close to c^i, and $j_{i+1} - j_i$ is sufficiently large), then γ satisfies the required conditions, i.e., it satisfies the Euler-Lagrange equation, and the curve $t \longrightarrow (d\gamma(t), t \bmod 1)$ in $TM \times \mathbb{T}$ has its α-limit set in M'_{c^0} and its ω-limit set in M'_{c^N}.

The assertions about the α- and ω-limit sets are easy consequences of the theory which we have already developed. We leave their verification to the reader.

The crux is the assertion that γ satisfies the Euler-Lagrange equation. It is obvious that γ satisfies the Euler-Lagrange equation except possibly at

$t = j_i$, $i = 1, \cdots, N$. To verify that γ satisfies the Euler-Lagrange equation at $t = j_i$, it is enough to check that $\eta_{c^{i-1}} = \eta_{c^i}$ in a neighborbood of m_{j_i}.

We may first choose $c^{*0}, \cdots, c^{*k}$, ordered along Γ, starting at $c^{*0} = c_0$ and ending at $c^{*k} = c_1$, such that for each i, there is an open set $U_i \subset M$ with the property that if c is in the arc $[c^{*i-1}, c^{*i+1}]$ in Γ, then $\eta_c = \eta_{c^{*i}}$ on U_i and $\{B^*_c = 0\} \subset U_i$. This is a consequence of the way that the η_c were chosen and the upper semi-continuity of the set-function $c \to \{B^*_c = 0\}$.

We think of $c^{*0}, \cdots, c^{*k}$ as defining a partition of Γ. We choose $c^0, \cdots c^N$, ordered along Γ, to define a refinement of this partition. In other words, we choose the latter so that $\{c^{*0}, \cdots, c^{*k}\} \subset \{c^0, \cdots, c^N\}$. It is easy to see that if c^{i-1} and c^i are sufficiently close and $j_i - j_{i-1}$ and $j_{i+1} - j_i$ are sufficiently large, then $m_{j_i} \in U_l$ where l is chosen so that c^{i-1} and c^i are in $[c^{*l-1}, c^{*l}]$. Thus, $\eta_{c^{i-1}} = \eta_{c^i}$ in a neighborhood of m_{j_i}, as required.

11. Proof of Theorem 9.2.

This is very similar to the proof of Theorem 9.1.

Let $(\cdots, c_i, \cdots)$ be a bi-infinite sequence consisting of elements of the same connected component of W_L. This time, we choose a smooth parameterized curve Γ: $\mathbb{R} \longrightarrow W_L$ such that $\Gamma(i) = c_i$ and $\Gamma\big|[i, i+1]$ is an embedding. For each $t \in \mathbb{R}$, we choose a smooth closed one form η_t whose de Rham cohomology class is $\Gamma(t)$. We choose η_t so that it depends smoothly on t and the other variables jointly and so that for any $t_0 \in \mathbb{R}$, we have a neighborhood U_{t_0} of $\{B^*_{\Gamma(t_0)} = 0\}$ in M and a neighborhood J_{t_0} of t_0 in $\mathbb{R}$ such that $\eta\big|U_{t_0} = \eta_{t_0}\big|U_{t_0}$ for $t \in J_{t_0}$. The possibility of choosing η_t in this way is again a consequence of the assumption that $\{B^*_{\Gamma(t_0)} = 0\}$ has a neighborhood V and M such that the inclusion map $H_1(V, \mathbb{R}) \to H_1(M, \mathbb{R})$ is the zero map, together with the fact that the set function $c \to \{B^*_c = 0\}$ is upper semi-continuous.

The strategy of proof of Theorem 9.2 follows the strategy of proof of Theorem 9.1, with appropriate modifications.

Since we may have $\eta_t \neq \eta_{t'}$, even though $\Gamma(t) = \Gamma(t')$, we need to introduce the function h_t: $M \times M \to \mathbb{R}$ defined by

$$h_t(m, m') = \min \int_0^1 (L - \eta_t)\,\big(d\gamma(t), t\big)dt - \alpha\big(\Gamma(t)\big),$$

where the minimum is taken over all curves γ: $[0, 1] \longrightarrow M$ such that $\gamma(0) = m$ and $\gamma(1) = m'$. This is the same as the definition of h_c in §6 (with $c = \Gamma(t)$), but now the dependence on η_c is taken into account explicitly.

Given a bi-infinite sequence $\vec{t} = (\cdots, t_i, \cdots)$, a bi-infinite increasing sequence $\vec{j} = (\cdots < j_i < \cdots)$ of integers, an M-configuration $(\cdots, m_i, \cdots)$ and integers a, b, we define $h_{\vec{t}, \vec{j}}(m_1, \cdots, m_b)$ by an obvious modification of the definition of $h_{\vec{c}, \vec{j}}$ given in the previous section. Likewise, we introduce the notion of a segment $(m_a, \cdots, m_b)$ of an M-configuration being $(\vec{t}, \vec{j})$-minimal by an obvious modification of the notion of a $(\vec{c}, \vec{j})$-minimal configuration introduced in the previous section. Finally, we say that an M-configuration $(\cdots, m_i, \cdots)$ is $(\vec{t}, \vec{j})$-minimal if each finite segment of it is $(\vec{t}, \vec{j})$-minimal.

An easy compactness argument shows that for any sequence $\vec{t} = (\cdots, t_i, \cdots)$ of real numbers, and any bi-infinite increasing sequence $\vec{j} = (\cdots, j_i, \cdots)$ of integers, there exists a $(\vec{t}, \vec{j})$-minimal configuration.

To prove Theorem 9.2, we choose an increasing sequence $\vec{t} = (\cdots < t_i < \cdots)$ of real numbers, with t_{i+1} very close to t_i, an increasing sequence $(\cdots < j_i < \cdots)$ of integers, with $j_{i+1} - j_i$ very large, and a $(\vec{t}, \vec{j})$-minimal configuration $(\cdots, m_i, \cdots)$. Then we construct a curve γ: $\mathbb{R} \longrightarrow M$ by letting $\gamma(t)$, $i \leqslant t \leqslant i + 1$, be a Tonelli minimizer satisfying the boundary condition $\gamma(i) = m_i$, $\gamma(i + 1) = m_{i+1}$.

Our claim is that if $\vec{t}$ and $\vec{j}$ are chosen appropriately (i.e. if t_{i+1} is sufficiently close to t_i, and $j_{i+1} - j_i$ is sufficiently large), then γ satisfies the required conditions. The proof of this is similar to the corresponding argument in the proof of Theorem 9.1, and we omit it.

12. A weaker hypothesis.

Just before the deadline for submitting this paper for the proceedings of the conference in honor of Malgrange's 65^{th} birthday, I noticed that the proofs of Theorems 9.1 and 9.2 work under a weaker hypothesis, which I will explain in this section.

Given $c \in H^1(M, \mathbb{R})$, we define

$$V_c = \bigcap_U \{i_{U*}H_1(U, \mathbb{R})\colon\ U \text{ is a neighborhood of } \{B_c^* = 0\}\}.$$

Here, i_U: $U \longrightarrow M$ denotes the inclusion map. Thus, V_c is a vector subspace of $H_1(M, \mathbb{R})$. Moreover, $V_c = 0$ if and only if $c \in W_L$.

We define $V_c^\perp$ to be the annihilator of V_c. In other words, if $c' \in H^1(M, \mathbb{R})$, then $c' \in V_c^\perp$ if and only if $< c', h >= 0$ for all $h \in V_c$. Clearly,

$$V_c^\perp = \bigcup_U \{ \ker\ i_U^*\colon U \text{ is a neighborhood of } \{B_c^* = 0\}\}.$$

Note that there exists a neighborhood U of $\{B_c^* = 0\}$ in M such that $V_c = i_{U*}\ H_1(U, \mathbb{R})$ and $V_c^\perp = \ker i_U^*$.

We will say that c_0, $c_1 \in H^1(M, \mathbb{R})$ are *C-equivalent* if there exists a continuous curve Γ: $[0, 1] \longrightarrow M$ such that $\Gamma(0) = c_0$ and $\Gamma(1) = c_1$, and for each $t_0 \in [0, 1]$, there exists $\delta > 0$ such that $\Gamma(t) - \Gamma(t_0) \in V_{\Gamma(t_0)}^\perp$ whenever $t \in [0, 1]$ and $|t - t_0| < \delta$.

Theorem 9.1 remains true if the hypothesis that c_0 and c_1 are in the same connected component of W_L is replaced by the hypothesis that c_0 and c_1 are C-equivalent. Likewise, Theorem 9.2 remains true if the hypothesis that the c_i are all in the same connected component of W_L is replaced by the hypothesis that the c_i are all C-equivalent. The proofs are the same.

13. A conjecture.

To demonstrate the usefulness of our theory, we would need to give examples to which it applies. At present we have no real examples beyond twist maps. In this section, we will state a conjecture which we hope to prove at some future date by an extension of the methods of this paper. This conjecture gives an example of what we are aiming for in developing our theory.

Our conjecture concerns generic Lagrangians in the sense of Mañé [Man3]. We consider a smooth Lagrangian L_0 on a smooth compact manifold M, i.e., a C^∞ mapping L_0: $TM \times \mathbb{T} \longrightarrow \mathbb{R}$ satisfying the hypotheses listed in §1, viz., positive definiteness, superlinear growth, and completeness of the Euler-Lagrange flow. We consider the family of Lagrangians of the form $L = L_0 + \psi$, where ψ: $M \times \mathbb{T} \to \mathbb{R}$ is a C^∞ function. Here, we identity ψ with $\psi \circ \pi$, where π: $TM \times \mathbb{T} \to M \times \mathbb{T}$ denotes the projection. We will also assume that for any L of this form, the Euler-Lagrange flow is complete.

We will say that a property of the Euler-Lagrange flow E_L is generic (in the sense of Mañé) if for any L_0, the set of ψ for which it is satisfied is residual, with respect to the C^∞ topology on $C^\infty(M \times \mathbb{T})$. We conjecture that if $\dim M \geqslant 2$, then generically there exists an orbit γ which escapes to infinity, in the sense that for every compact subset K of $TM \times \mathbb{T}$, there exists t_0 such that $\gamma(t) \notin K$, for $t \geqslant t_0$.

Such a result is false when $\dim M = 1$, by KAM theory. When $\dim M \geqslant 2$, the usual KAM tori do not separate phase space, so KAM theory does not tell whether our conjecture is true.

Our conjecture belongs to a class of speculations which go back to Boltzmann's quasi-ergodic hypothesis. Recently, Herman (see [Yoc]) has produced examples of Hamiltonian systems for which Boltzmann's quasi-ergodic hypothesis is false. It is noteworthy that in Herman's examples, variational methods do not apply.

Let us mention also the famous paper of Arnold [Arn2] who gave an example to show that certain results guaranteeing boundedness of orbits in Hamiltonain systems in two degrees of freedom in the autonomous case or one degree of freedom in the non-autonomous case have no analogue in more degrees of freedom. The method of [Arn2] is another method one might try to use to prove our conjecture, but it seems (at least to the author) that variational methods such as described here are more likely to succeed for proving the conjecture we have stated here.

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