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Vector bundles on manifolds without divisors and a theorem on deformations

VECTOR BUNDLES
ON MANIFOLDS WITHOUT DIVISORS
AND A THEOREM ON DEFORMATIONS

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Introduction.

The motivation for this paper was to gather some information on holomorphic vector bundles on some non-algebraic compact complex manifolds, especially manifolds without divisors. As a first step, we treat the case of 2-bundles. Examples of such 2-bundles are given by extensions

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0 \qquad (*)$$

where L and M are line bundles and $\mathscr{I}_Z$ is the ideal sheaf of a 2-codimensional locally complete intersection. On a projective algebraic manifold every 2-bundle is of this form, however L, M and Z are not uniquely determined by E. In sharp contrast to this, on a manifold without divisors, the « devissage » $(*)$ is uniquely determined for an indecomposable bundle E (cf. Theorem 2.2). On the other hand, on such highly non-algebraic manifolds there might exist 2-bundles without any such devissage; we call them non-filtrable. More precisely, E admits a devissage if and only if there exists a line bundle L such that $E \otimes L^*$ has non-trivial sections.

In order to prove the existence of non-filtrable bundles on 2-tori with Picard number zero, we prove (in § 3) some general theorems on the deformation of vector bundles and projective bundles which might be of independant interest. Roughly speaking, any deformation of a vector bundle on a compact complex space is composed of a deformation of det (E) and a deformation of the associated projective bundle $P(E)$; for a precise formulation see Theorem 3.4. As a corollary we get : If

dim $H^2(X, \text{End } E) = \dim H^2(X, \mathcal{O}_X)$, then the basis of the versal deformation of E is smooth. We use this last fact to deform a certain filtrable bundle on a 2-torus into a non-filtrable one (Proposition 4.9).

In an Appendix we gather some facts on algebraic dimension and Picard number of 2-tori.

Notations. — By a vector bundle on a complex space X we always mean a holomorphic vector bundle which we consider as a locally free $\mathcal{O}_X$-module of constant (finite) rank. The dimensions of cohomology groups are denoted by $h^q(X, \mathcal{F}) : = \dim H^q(X, \mathcal{F})$.

1. Filtration of bundles.

In this section we collect some more or less well known facts about vector bundles which are extensions of the form

$$0 \to L \to E \to M \otimes \mathcal{I}_Z \to 0,$$

where L and M are line bundles and Z is a 2-codimensional locally complete intersection.

1.1. If E is a vector bundle of rank r on a smooth curve, then there exists a (not uniquely determined) filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E,$$

where E_k is a subbundle of rank k (cf. Atiyah [1]). On a complex manifold of dimension > 1 this is no longer true. Instead of subbundles one has to consider coherent subsheaves $\mathcal{F} \subset E$. Such subsheaves are always torsion-free. The following facts are well known :

a) Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X. Then the set

$$\text{Sing}(\mathcal{F}) = \{x \in X : \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\}$$

is analytic of codimension ≥ 1.

If $\mathcal{F}$ is torsion-free, $\text{Sing}(\mathcal{F})$ is of codim ≥ 2. If $\mathcal{F}$ is reflexive, i.e. $\mathcal{F} = \mathcal{F}^{**}$, then codim $\text{Sing}(\mathcal{F}) \geq 3$. If $\mathcal{F}$ is reflexive and has rank 1, it is locally free, i.e. a line bundle.

b) Let E be a vector bundle on a complex manifold X and $\mathscr{F} \subset E$ a coherent subsheaf. Then the set $\text{Sing}(E/\mathscr{F})$ is equal to the set

$$S = \{x \in X : \mathscr{F}_x \text{ is not a direct summand of } E_x\}$$

and $\mathscr{F}|X\backslash S$ is a subbundle of $E|X\backslash S$.

c) For every $\mathscr{F} \subset E$ we denote by $\hat{\mathscr{F}}$ the following coherent subsheaf of E: Let $p : E \to E/\mathscr{F}$ be the canonical projection and $\text{Tors}(E/\mathscr{F})$ the torsion submodule of $E/\mathscr{F}$. Define

$$\hat{\mathscr{F}} : = p^{-1}(\text{Tors}(E/\mathscr{F})).$$

Then $\mathscr{F} \subset \hat{\mathscr{F}} \subset E$ and $\hat{\mathscr{F}}$ coincides with $\mathscr{F}$ outside an analytic set of codimension $\geqslant 1$. The quotient $E/\hat{\mathscr{F}}$ is torsion-free, hence $\hat{\mathscr{F}}$ is a subbundle of E outside an analytic set of codimension $\geqslant 2$.

1.2. DEFINITION. — *A vector bundle* E *of rank* r *on a complex manifold* X *is called filtrable if there exists a filtration*

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_r = E$$

where $\mathscr{F}_k$ *is a coherent subsheaf of rank* k.

Of course every vector bundle on a compact algebraic manifold is filtrable, but we will prove the existence of bundles on certain non-algebraic manifolds which are not filtrable.

Remark. — According to 1.1.*c*) we may assume all quotients $E/\mathscr{F}_k$ to be torsion-free. In that case the $\mathscr{F}_k/\mathscr{F}_{k-1}$ are torsion-free of rank 1 and $L_k : = (\mathscr{F}_k/\mathscr{F}_{k-1})^{**}$ are line bundles. Moreover,

$$\det E \cong L_1 \otimes \ldots \otimes L_r.$$

This last formula comes from the fact that $0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_r = E$ is a filtration of subbundles outside a set of codimension $\geqslant 2$.

1.3. LEMMA. — *Let* $\mathscr{F} \subset E$ *be a coherent subsheaf of rank* 1 *of a vector bundle on a complex manifold. If* $E/\mathscr{F}$ *is torsion-free, then* $\mathscr{F}$ *is locally free.*

Proof. — It suffices to show that $\mathscr{F}$ is reflexive. Let $\mathscr{F}^{**} \to E$ denote the bidual of the inclusion morphism $\mathscr{F} \to E$ and consider the sheaf $\hat{\mathscr{F}} : = \text{Im}(\mathscr{F}^{**} \to E)$. Then $\hat{\mathscr{F}}/\mathscr{F} \subset E/\mathscr{F}$ is a torsion sheaf, hence $\hat{\mathscr{F}}/\mathscr{F} = 0$. Since $\mathscr{F}^{**} \to E$ is a monomorphism, $\mathscr{F}^{**} \cong \hat{\mathscr{F}} \cong \mathscr{F}$, q.e.d.

1.4. COROLLARY. — *A vector bundle of rank* 2 *on a connected complex manifold* X *is filtrable if and only if there exists a line bundle* L *on* X *such that* $\Gamma(X, L^* \otimes E) \neq 0$.

1.5. COROLLARY. — *On a complex manifold* X *let* E *be a vector bundle,* L *a line bundle and* $\alpha : L \to E$ *a sheaf monomorphism. Then*

$$\mathrm{Supp}\ (\mathrm{Tors}\ (E/\mathrm{Im}\ (L \overset{\alpha}{\to} E)))$$

is (empty or) of pure codimension 1.

Proof. — Set $\mathscr{F} := \mathrm{Im}\ (L \to E)$ and define $\hat{\mathscr{F}} \subset E$ as in 1.1.c). $\mathscr{F}$ is isomorphic to L and $\hat{\mathscr{F}}$ is locally free by Lemma 1.3. The inclusion map $\mathscr{F} \hookrightarrow \hat{\mathscr{F}}$ may be considered as a section of the line bundle $\mathscr{F}^* \otimes \hat{\mathscr{F}}$, hence $\mathrm{Supp}\ (\hat{\mathscr{F}}/\mathscr{F})$ has pure codimension 1. But

$$\mathrm{Tors}\ (E/\mathrm{Im}\ (L \to E)) \cong \hat{\mathscr{F}}/\mathscr{F}.$$

1.6. PROPOSITION. — *For every filtrable 2-bundle* E *on a complex manifold* X *there exist line bundles* L, M *on* X *and a 2-codimensional (possibly empty) analytic subspace* $Z \subset X$ *such that* E *fits into an exact sequence*

$$0 \to L \overset{\alpha}{\to} E \overset{\beta}{\to} M \otimes \mathscr{I}_Z \to 0.$$

Proof. — Let $0 \subset \mathscr{F} \subset E$ be a filtration such that $E/\mathscr{F}$ is torsion free. By Lemma 1.3 the sheaf $L := \mathscr{F}$ is locally free of rank 1. Let $\alpha : L \to E$ be the inclusion map. Set $M := (E/\mathscr{F})^{**}$. The image of the natural inclusion map

$$E/\mathscr{F} \to (E/\mathscr{F})^{**} = M$$

is of the form $M \otimes \mathscr{I}_Z$, where $\mathscr{I}_Z$ is the ideal sheaf of a subspace $Z \subset X$ of codimension ≥ 2. But Z may also be defined by the vanishing of $\alpha \in \Gamma(X, L^* \otimes E)$, hence is a locally complete intersection of codimension $= 2$ (or empty). The morphism β is the quotient map $E \to E/\mathscr{F}$ composed with the isomorphism $E/\mathscr{F} \overset{\sim}{\to} M \otimes \mathscr{I}_Z$.

1.7. *Notation.* — We call an exact sequence

$$(*)\qquad\qquad 0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

as in Proposition 1.6 a *devissage* of E. We have

$$\det(E) \cong L \otimes M,$$

in particular $c_1(E) = c_1(L) + c_1(M)$. The bundle $L^* \otimes E$ has a section vanishing on the subspace Z. Hence

$$c_2(L^* \otimes E) = \text{dual class of } [Z].$$

Since $E^* \cong E \otimes \det E^*$, we can tensor $(*)$ by $L^* \otimes M^*$ to get the dual devissage

$$0 \rightarrow M^* \rightarrow E^* \rightarrow L^* \otimes \mathcal{I}_Z \rightarrow 0.$$

1.8. Recall that a vector bundle E on a compact complex connected manifold is *simple* if $\text{End}(E) = \mathbf{C}$. This is equivalent to the fact that every non-zero endomorphism is invertible. If rank $E = 2$ and E is not simple, then E is filtrable. In fact, if $\sigma : E \rightarrow E$ is a non-zero, non-invertible endomorphism, then $\text{Ker } \sigma \subset E$ is a subsheaf of rank 1.

1.9. LEMMA. — *Let E be an indecomposable 2-bundle on a compact connected complex manifold X and $\sigma \in \text{End } E$ a non-invertible endomorphism. Then $\sigma^2 = 0$.*

Proof. — Consider the eigenvalues λ_1, λ_2 of σ. (Since X is compact connected, the eigenvalues of σ in all fibres of E are the same.) Necessarily $\lambda_1 = \lambda_2$, otherwise the eigenspaces would define a decomposition of E. Since $\det(\sigma) = 0$, we have $\lambda_1 = \lambda_2 = 0$, which implies $\sigma^2 = 0$.

1.10. In general, the devissage of a 2-bundle is not uniquely determined. However we shall discuss conditions which guarantee uniqueness.

Let X be a compact connected complex manifold and L, L' line bundles on X. Following Atiyah [1] we shall write $L' \leqslant L$ if there exists a non-zero morphism $L' \rightarrow L$.

We call a devissage $L \rightarrowtail E \twoheadrightarrow M \otimes \mathcal{I}_Z$ of a 2-bundle E *maximal*, if for every other devissage $L' \rightarrowtail E \twoheadrightarrow M' \otimes \mathcal{I}_Z$, we have $L' \leqslant L$.

1.11. PROPOSITION. — *Let E be a non-simple, indecomposable 2-bundle on a compact connected complex manifold X. Then E admits a uniquely determined maximal devissage*

$$0 \rightarrow L \rightarrow E \rightarrow M \otimes \mathcal{I}_Z \rightarrow 0.$$

This maximal devissage is characterized by the fact that $M \leqslant L$.

Proof. — Let $\sigma : E \to E$ be a non-zero, non-invertible endomorphism. Let $L := \mathrm{Ker}\,\sigma$. Since $E/\mathrm{Ker}\,\sigma \cong \mathrm{Im}\,\sigma$ is torsion-free, L is a line bundle by Lemma 1.3. We may write $\mathrm{Im}\,\sigma \cong M \otimes \mathscr{I}_Z$, where M is a line bundle and $Z \subset X$ a subspace of codimension 2. By Lemma 1.9 we have $\mathrm{Im}\,\sigma \subset \mathrm{Ker}\,\sigma$, hence there exists a monomorphism $M \otimes \mathscr{I}_Z \to L$, which extends to a monomorphism $M \to L$. So we get a devissage

$$0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} M \otimes \mathscr{I}_Z \to 0$$

with $M \leqslant L$.

We will now show that a devissage with $M \leqslant L$ is the uniquely determined maximal devissage.

i) Maximality. Let $f : L' \to E$ be any non-zero morphism. If $\beta \circ f : L' \to M \otimes \mathscr{I}_Z$ is non-zero, then $L' \leqslant M \leqslant L$. If however $\beta \circ f = 0$, we have $L' \cong \mathrm{Im}\,f \subset \mathrm{Im}\,\alpha \cong L$, i.e. $L' \leqslant L$ in every case.

ii) Uniqueness. Let

$$0 \to L' \xrightarrow{f} E \to M' \otimes \mathscr{I}_{Z'} \to 0$$

be a second maximal devissage. Then $L' \leqslant L \leqslant L'$, hence $L' \cong L$. If $\beta \circ f : L' \to M \otimes \mathscr{I}_Z$ is non-zero, the composite map

$$L' \xrightarrow{\beta \circ f} M \otimes \mathscr{I}_Z \to M \to L$$

is non-zero, hence an isomorphism. This implies in particular $Z = \varnothing$ and $\beta \circ f : L' \to M$ is an isomorphism. But then $E = L \oplus L'$, which was excluded. So necessarily $\beta \circ f = 0$ and we get a factorization

Since $L' \cong L$, g is an isomorphism. This implies that the two devissages are isomorphic, q.e.d.

2. Vector bundles of rank 2
on manifolds without divisors.

2.1. Let L and L' be two line bundles on a complex connected manifold and $f : L' \to L$ a non-zero morphism. Let D be the zero divisor of f. Then $L \cong L' \otimes [D]$, where $[D]$ denotes the line bundle associated to D. Therefore, if X is a complex connected manifold without divisors, the relation $L' \leqslant L$ implies $L' \cong L$.

Recall that a compact connected complex manifold without divisors has algebraic dimension zero, i.e. the only meromorphic functions are constant. The converse is not true (think of blow-ups !), however a torus has algebraic dimension zero if and only if it admits no divisors.

We will now give a rough classification of 2-bundles on manifolds without divisors.

2.2. THEOREM. — *Let* X *be a compact connected complex manifold without divisors. Then we have the following classification of vector bundles of rank* 2 *on* X :

I. *Filtrable bundles.*

1) *Indecomposable bundles.*

A filtrable 2-bundle is indecomposable if and only if its devissage is uniquely determined.

i) *Simple bundles. They have a devissage*

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

with $L \not\cong M$ *and endomorphism ring* $\mathrm{End}\, E \cong \mathbf{C}$.

ii) *Non-simple bundles. Their devissage is*

$$0 \to L \to E \to L \otimes \mathscr{I}_Z \to 0$$

and $\mathrm{End}\, E \cong \mathbf{C}[\varepsilon]$, $\varepsilon^2 = 0$, *is the ring of dual numbers.*

2) *Decomposable bundles*

i) *Bundles of the form* $E \cong L \oplus M$ *with* $L \not\cong M$.

In this case $\text{End } E \cong C \oplus C$ *with componentwise multiplication.*

ii) *Bundles of the form* $E \cong L \oplus L$.

In this case $\text{End } E \cong M_2(C)$ *is the full matrix ring.*

II. *Non-filtrable bundles.*

These bundles are all simple, i.e. $\text{End } E = C$.

Proof. — *a*) Let E be an indecomposable 2-bundle on X with a devissage

$$0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} M \otimes \mathscr{I}_Z \to 0.$$

We will show that the devissage is uniquely determined and that the assertions in 1i), ii) hold.

i) Suppose $M \not\cong L$. Let $f : L' \to E$ be any monomorphism of a line bundle L' in E. We claim that

$$\beta \circ f : L' \to M \otimes \mathscr{I}_Z$$

is zero. Otherwise Z would be empty (since X has no divisors) and $\beta \circ f : L' \to M$ an isomorphism. But this would imply $E \cong L \oplus M$, contradicting the indecomposability of E. Therefore f factorizes as follows

$$
\begin{array}{ccc}
L & \xrightarrow{\alpha} & E \\
 & \underset{g}{\nwarrow} & \uparrow{\scriptstyle f} \\
 & & L'
\end{array}
$$

and g is necessarily an isomorphism. This implies the uniqueness of the devissage.

Tensoring the dual devissage $M^* \rightarrowtail E^* \twoheadrightarrow L^* \otimes \mathscr{I}_Z$ by E, we get an exact sequence

$$0 \to M^* \otimes E \to E^* \otimes E \to L^* \otimes E \otimes \mathscr{I}_Z \to 0,$$

which implies

$$\dim \text{End } E \leqslant \dim \Gamma(X, M^* \otimes E) + \dim \Gamma(X, L^* \otimes E \otimes \mathscr{I}_Z).$$

The uniqueness of the devissage of E implies

$$\Gamma(X, M^* \otimes E) = 0 \quad \text{and} \quad \Gamma(X, L^* \otimes E \otimes \mathscr{I}_Z) \subset \Gamma(X, L^* \otimes E) \cong \mathbb{C},$$

hence dim End $E = 1$, i.e. E is simple.

ii) If $M \cong L$, denote by ε the composed morphism

$$E \to M \otimes \mathscr{I}_Z \hookrightarrow M \overset{\sim}{\to} L \to E.$$

Obviously $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. In particular E is non-simple and the uniqueness of the devissage follows from Proposition 1.10. Since $\mathbb{C}[\varepsilon] \subset$ End E, it remains to be shown that dim End $E \leqslant 2$. To see this, we use the same inequality as above

$$\dim \text{End } E \leqslant \dim \Gamma(X, M^* \otimes E) + \dim \Gamma(X, L^* \otimes E \otimes \mathscr{I}_Z).$$

Since $M \cong L$, the uniqueness of the devissage implies $\dim \Gamma(X, M^* \otimes E) = 1$ and $\dim \Gamma(X, L^* \otimes E \otimes \mathscr{I}_Z) \leqslant 1$, hence dim End $E \leqslant 2$. Therefore End $E \cong \mathbb{C}[\varepsilon]$.

b) It is clear that the devissage of a decomposable bundle $E \cong L \oplus M$ is not uniquely determined. Furthermore

$$\text{End } (E) \cong \text{End } (L) \oplus \text{End } (M) \oplus \text{Hom } (L, M) \oplus \text{Hom } (M, L),$$

which gives the endomorphism rings as asserted in 2i), ii).

c) That non-filtrable 2-bundles are simple follows from 1.7. This completes the proof of Theorem 2.2. We now look at a relative situation.

2.3. THEOREM. — *Let X be a compact complex manifold without divisors, S a Stein manifold with $H^2(S, \mathbb{Z}) = 0$ and E a vector bundle of rank 2 on $X \times S$. For $s \in S$ denote by i_s the inclusion map*

$$i_s : X \overset{\sim}{\to} X \times \{s\} \hookrightarrow X \times S$$

and $E_s := i_s^* E$. Suppose that E_s is filtrable and indecomposable for all $s \in S$ (i.e. belongs to class I.1 in the classification of Theorem 2.2). Then there exist line bundles $L \to X \times S$, $M \to X \times S$ and a subspace $Z \subset X \times S$ of codimension 2 which is flat over S, such that E fits into

an exact sequence

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

whose restriction to every fibre $X \times \{s\}$ *is the uniquely determined devissage of* E_s.

Proof. — Let $L \to X \times \mathrm{Pic}(X)$ be the universal line bundle. Consider the bundle

$$L^* \boxtimes E \to X \times (\mathrm{Pic}(X) \times S).$$

Let $p : X \times (\mathrm{Pic}(X) \times S) \to \mathrm{Pic}(X) \times S$ be the projection. By the semi-continuity theorem the set

$$S' := \{(\xi,s) \in \mathrm{Pic}(X) \times S : H^0(p^{-1}(\xi,s), L_\xi^* \otimes E_s) \neq 0\}$$

is analytic. Since the devissage of every bundle E_s is uniquely determined, the projection $q : S' \to S$ is bijective, hence biholomorphic if we provide S' with the structure of a reduced subspace of $\mathrm{Pic}(X) \times S$. Let $\varphi : S \to S' \subset \mathrm{Pic}(X) \times S$ be the inverse map of q and define the line bundle $L \to X \times S$ by

$$L := (\mathrm{id}_X \times \varphi)^* L.$$

For every $s \in S$, the vector space $\mathrm{Hom}(L_s, E_s)$ is one-dimensional, hence the direct image sheaf

$$\pi_* \, \mathrm{Hom}(L,E),$$

where $\pi : X \times S \to S$ is the projection, is locally free of rank 1 on S. The hypothesis $H^2(S,\mathbb{Z}) = 0$ implies $\pi_* \, \mathrm{Hom}(L,E) \cong \mathcal{O}_S$. Let $\alpha : L \to E$ be the morphism corresponding to a global non-vanishing section of $\pi_* \, \mathrm{Hom}(L,E)$. The restriction $\alpha_s : L_s \to E_s$ of α to any fibre $\pi^{-1}(s)$ is up to a constant factor the unique monomorphism of a line bundle into E_s. The image $\alpha(L)$ is a direct summand of E outside a set of codimension 2. Corollary 1.5 implies that $E/\alpha(L)$ is torsion free. Define the line bundle $M \to X \times S$ by

$$M := (E/\alpha(L))^{**}.$$

Then $E/\alpha(L) \cong M \otimes \mathscr{I}_Z$ for a certain 2-codimensional subspace $Z \subset X \times S$. Since Z is locally a complete intersection whose intersection

with every fibre $\pi^{-1}(s)$ is 2-codimensional, Z is flat over S. The morphism $\alpha : L \to E$ together with the quotient map $E \to E/\alpha(L) \cong M \otimes \mathscr{I}_Z$ gives the desired exact sequence

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0.$$

2.4. Theorem 2.3 implies the following : Let $E \to X \times S$ be a vector bundle as in Theorem 2.3 and

$$0 \to L_s \to E_s \to M_s \otimes \mathscr{I}_{Z_s} \to 0$$

the unique devissage of E_s. Then

$$s \mapsto [L_s] \quad \text{and} \quad s \mapsto [M_s]$$

define holomorphic maps $S \to \text{Pic}(X)$. Moreover there is a holomorphic map

$$S \to D(X), \quad s \mapsto Z_s,$$

where $D(X)$ denotes the Douady space of all compact analytic subspaces of X, cf. [4].

3. Deformations of vector bundles and projective bundles.

3.1. Holomorphic fibre bundles with fibre $\mathbf{P}_{r-1}$ and structure group $\text{PGL}(r,\mathbf{C})$ on a complex space X (we will call them briefly projective $(r-1)$-bundles or $\mathbf{P}_{r-1}$-bundles) are classified by $H^1(X,\text{PGL}(r,\mathcal{O}))$. Every vector bundle E of rank r on X gives rise to a projective $(r-1)$-bundle $\mathbf{P}(E)$. The relevant exact sequence is

$$0 \to \mathcal{O}^* \to \text{GL}(r,\mathcal{O}) \to \text{PGL}(r,\mathcal{O}) \to 0,$$

to which is associated the exact cohomology sequence

$$H^1(X,\text{GL}(r,\mathcal{O})) \to H^1(X,\text{PGL}(r,\mathcal{O})) \to H^2(X,\mathcal{O}^*).$$

Thus if $H^2(X,\mathcal{O}^*) = 0$ (in particular if X is a curve or $\mathbf{P}_n$) every projective bundle is of the form $\mathbf{P}(E)$ where E is a vector bundle (cf. Atiyah [1]).

For general X this is no longer true. However we will show that if P_0 is a projective bundle associated to a vector bundle, then any small deformation of P_0 also comes from a vector bundle.

3.2. THEOREM. — *Let E_0 be a vector bundle of rank r on a compact complex space X. Let $P \to X \times S$ be a deformation of $P(E_0)$ over the germ $(S,0)$. Then there exists a deformation $E \to X \times S$ of the vector bundle E_0 such that $P \cong P(E)$. Moreover one can choose E such that $\det E$ is a trivial deformation of $\det E_0$. With this supplementary condition E is uniquely determined.*

Proof. — The deformation P is given by a cocycle

$$\xi \in H^1(X \times S, PGL(r,\mathcal{O}))$$

which can be represented by a cochain

$$(g_{ij}) \in C^1(\mathcal{U} \times S, GL(r,\mathcal{O})),$$

where $\mathcal{U} = (U_i)_{i \in I}$ is a suitable open covering of X. We may assume all intersections $U_i \cap U_j$ to be simply connected. We may further assume that

$$(g_{ij}(0)) \in C^1(\mathcal{U}, GL(r,\mathcal{O}))$$

is a cocycle defining the vector bundle E_0. Therefore there exists a cochain

$$(c_{ijk}) \in C^2(\mathcal{U} \times S, \mathcal{O}^*)$$

with

$$c_{ijk}(0) = 1$$

and

$$g_{ij}g_{jk} = c_{ijk}g_{ik} \quad \text{on} \quad (U_i \cap U_j \cap U_k) \times S.$$

Since the $U_i \cap U_j$ are simply connected, there exist functions

$$\gamma_{ij} \in \mathcal{O}^*((U_i \cap U_j) \times S)$$

with

$$\det g_{ij} = \gamma_{ij}^r.$$

We define

$$\tilde{g}_{ij} := g_{ij} \frac{\gamma_{ij}(0)}{\gamma_{ij}} \in \mathrm{GL}(r, \mathcal{O}((\mathrm{U}_i \cap \mathrm{U}_j) \times \mathrm{S})).$$

We have then

$$\tilde{g}_{ij}(0) = g_{ij}(0)$$

and

$$\det \tilde{g}_{ij}(s) = \gamma_{ij}(0)^r \quad \text{for all} \quad s \in \mathrm{S}.$$

We will show that $(\tilde{g}_{ij})$ is a cocycle, i.e.

$(*)$ $$\tilde{g}_{ij}\tilde{g}_{jk} = \tilde{g}_{ik}.$$

Indeed, we have $\tilde{g}_{ij}\tilde{g}_{jk} = \tilde{c}_{ijk}\tilde{g}_{ik}$ with a cochain

$$\tilde{c}_{ijk} \in \mathrm{C}^2(\mathcal{U} \times \mathrm{S}, \mathcal{O}^*), \qquad \tilde{c}_{ijk}(0) = 1.$$

Then taking determinants we get

$$\gamma_{ij}(0)^r \gamma_{jk}(0)^r = (\tilde{c}_{ijk})^r \gamma_{ik}(0)^r.$$

On the other hand $\gamma_{ij}(0)^r \gamma_{jk}(0)^r = \gamma_{ik}(0)^r$, hence

$$(\tilde{c}_{ijk})^r = 1.$$

Since $\tilde{c}_{ijk}(0) = 1$, this implies $\tilde{c}_{ijk} = 1$ as an element of $\mathcal{O}^*((\mathrm{U}_i \cap \mathrm{U}_j \cap \mathrm{U}_k) \times \mathrm{S})$. Thus we have proved the cocycle relation $(*)$. The cocycle

$$(\tilde{g}_{ij}) \in \mathrm{Z}^1(\mathcal{U} \times \mathrm{S}, \mathrm{GL}(r, \mathcal{O}))$$

defines the desired deformation E of E_0 for which $\mathrm{P(E)} \cong \mathrm{P}$ and det E is the trivial deformation of $\det \mathrm{E}_0$.

Uniqueness. Let $\mathrm{E}' \to \mathrm{X} \times \mathrm{S}$ be another deformation of E_0 with $\mathrm{P(E')} \cong \mathrm{P}$. Then $\mathrm{E}' \cong \mathrm{E} \otimes \mathrm{L}$, where $\mathrm{L} \to \mathrm{X} \times \mathrm{S}$ is a deformation of the trivial line bundle. If both det E and $\det \mathrm{E}' = (\det \mathrm{E}) \otimes \mathrm{L}^r$ are trivial deformations of $\det \mathrm{E}_0$, it follows that L^r is trivial. Since L_0 is trivial, L must be trivial itself.

3.3. Given a vector bundle E of rank r on a complex space X, we have a canonical injection

$$\mathcal{O}_X \to End\ E,\ f \mapsto f.\mathrm{id}_E.$$

This injections splits by the map

$$\varphi \mapsto \frac{1}{r}\,\mathrm{trace}\,(\varphi)$$

and we get a direct sum decomposition

$$End\ E \cong \mathcal{O}_X \oplus End_0\ E,$$

where $End_0\ E$ is the sheaf of endomorphisms of trace zero. In particular, we have for any $q \in \mathbf{N}$

$$H^q(X, End\ E) \cong H^q(X, \mathcal{O}_X) \otimes H^q(X, End_0\ E).$$

Consider the projective bundle $\mathbf{P}(E)$ associated to E. If X is compact, the versal deformation of $\mathbf{P}(E)$ exists and the tangent space of the basis of the versal deformation is $H^1(X, End_0\ E)$.

3.4. THEOREM. — *Let E_0 be a vector bundle on the compact complex space X. Let $E' \to X \times \Sigma$ be a deformation of E_0 such that $\mathbf{P}(E') \to X \times \Sigma$ is the versal deformation of $\mathbf{P}(E_0)$. Let $L \to X \times \Pi$ be the versal deformation of the trivial line bundle on X. Then the exterior tensor product*

$$L \boxtimes E' \to X \times (\Pi \times \Sigma)$$

is the versal deformation of E_0.

Remarks. — *a*) The deformation $E' \to X \times \Sigma$ exists by Theorem 3.2.

b) The versal deformation $L \to X \times \Pi$ of the trivial line bundle can be obtained as follows: Choose cocycles $(h_{ij}^\mu) \in Z^1(\mathcal{U}, \mathcal{O}_X)$, $\mu = 1, \ldots, m$ whose cohomology classes form a basis of $H^1(X, \mathcal{O}_X)$. Then $\Pi = (\mathbf{C}^m, 0)$ and

$$g_{ij} := \exp\left(\sum_{\mu=1}^m t_\mu h_{ij}^\mu\right),$$

where $t_1, \ldots, t_m$ are the coordinates in $\mathbf{C}^m$, is the cocycle defining L.

Proof of Theorem 3.4. — Let

$$E \to X \times S$$

be the versal deformation of E_0. Then $P(E) \to X \times S$ is a deformation of $P(E_0)$, hence there exists a map $\alpha : S \to \Sigma$ such that

$$P(E) \cong \alpha^*P(E') = P(\alpha^*E').$$

Then there exists a deformation $M \to X \times S$ of the trivial line bundle such that

$$E \cong M \otimes \alpha^*E'.$$

By the versal property of $L \to X \times \Pi$, there exists a map $\beta : S \to \Pi$ such that $M \cong \beta^*L$. Thus, letting

$$f := (\beta, \alpha) : S \to \Pi \times \Sigma,$$

we have

$$E \cong f^*(L \otimes E').$$

On the other hand, by the versal property of $E \to X \times S$, there exists a map $g : \Pi \times \Sigma \to S$ such that

$$L \boxtimes E' \cong g^*E.$$

Therefore $E \cong (g \circ f)^*E$, which implies

$$(dg)_0 \circ (df)_0 = d(g \circ f)_0 = id_{T_0S}.$$

Consider the diagram

$$T_0S \xrightarrow{(df)_0} T_{(0,0)}(\Pi \times \Sigma) \xrightarrow{(dg)_0} T_0S.$$

Since $T_0\Pi = H^1(X, \mathcal{O}_X)$, $T_0\Sigma = H^1(X, End_0 E)$ and $T_0S = H^1(X, End E)$, we have

$$\dim T_0S = \dim T_{(0,0)}(\Pi \times \Sigma),$$

hence $(df)_0$ and $(dg)_0$ are isomorphisms. This implies that $f : S \to \Pi \times \Sigma$ is an isomorphism of germs and

$L \boxtimes E' \to X \times (\Pi \times \Sigma)$ isomorphic to the versal deformation $E \to X \times S$, q.e.d.

3.5. COROLLARY. — *Let* E *be a vector bundle on a compact complex space* X *such that*

$$\dim H^2(X, \text{End } E) = \dim H^2(X, \mathcal{O}_X).$$

Then the basis S *of the versal deformation of* E *is smooth.*

Proof. — The hypothesis implies $H^2(X, \text{End}_0 E) = 0$. Therefore the basis Σ of the versal deformation of $P(E)$ is smooth, so $S = \Pi \times \Sigma$ is also smooth.

3.6. COROLLARY. — *Let* X *be a smooth compact complex surface with trivial canonical bundle (for example a torus or a K3-surface) and* E *be a simple vector bundle on* X. *Then the basis of the versal deformation of* E *is smooth.*

Proof. — By Serre duality

$$H^2(X, \text{End } E) \cong H^0(X, \text{End } E)^* \cong \mathbf{C}$$

and

$$H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)^* \cong \mathbf{C}.$$

Therefore we can apply Corollary 3.5.

4. Vector bundles on tori with trivial Néron-Severi group.

4.1. Recall the theorem of Riemann-Roch for a (smooth, compact complex) surface X. If E is a vector bundle of rank r on X, we have

$$\chi(X, E) = r\chi(X, \mathcal{O}_X) + \frac{1}{2}(c_1(X)c_1(E) + c_1(E)^2) - c_2(E),$$

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

In particular we can apply Riemann-Roch to the endomorphism bundle *End* E. Since

$$c_1 \, (\text{End E}) = 0,$$
$$c_2 \, (\text{End E}) = r^2 c_2 \, (\text{E}) - (r-1) c_1 \, (\text{E})^2,$$

we get

$$\chi(\text{X}, \text{End E}) = r^2 \chi(\text{X}, \mathcal{O}_\text{X}) + (r-1) c_1 (\text{E})^2 - r^2 c_2 (\text{E}).$$

4.2. The Néron-Severi group of a surface X is defined by

$$\text{NS(X)} : = \text{Im} \, (\text{H}^1 (\text{X}, \mathcal{O}_\text{X}^*) \xrightarrow{c_1} \text{H}^2 (\text{X}, \text{Z})).$$

In the following we shall deal with surfaces X (especially tori) having NS(X) = 0. If in addition X is Kähler, then X has no divisors, in particular its algebraic dimension is zero. (Hopf surfaces always have NS(X) = 0, whereas their algebraic dimension may be zero or one.)

4.3. PROPOSITION. — *Let* X *be a Kähler surface with* NS(X) = 0. *Then for any vector bundle* E *of rank* 2 *on* X *we have* $c_2(\text{E}) \geqslant 0$.

Proof. — Since NS(X) = 0, we have $c_1(\text{X}) = c_1(\text{E}) = 0$, hence

$$\chi(\text{X}, \text{E}) \ = \frac{1}{6} c_2(\text{X}) - c_2(\text{E}),$$

$$\chi(\text{X}, \mathcal{O}_\text{X}) = \frac{1}{12} c_2(\text{X}).$$

Since for a surface with algebraic dimension zero we have $\chi(\text{X}, \mathcal{O}_\text{X}) \geqslant 0$ (cf. [3], part 6, Prop. 1.5), it follows $c_2(\text{X}) \geqslant 0$. We consider first the case that E is not filtrable. Then

$$\text{H}^0(\text{X}, \text{E}) = 0 \quad \text{and} \quad \text{H}^2(\text{X}, \text{E})^* \cong \text{H}^0(\text{X}, \text{E}^* \otimes \text{K}_\text{X}) = 0,$$

hence

$$0 \leqslant h^1(\text{X}, \text{E}) = -\chi(\text{X}, \text{E}) = c_2(\text{E}) - \frac{1}{6} c_2(\text{X}) \leqslant c_2(\text{E}).$$

If E is filtrable we have a devissage

$$0 \to \text{L} \to \text{E} \to \text{M} \otimes \mathscr{I}_\text{Z} \to 0.$$

Since $c_1(L) = c_1(M) = 0$, $c_2(E)$ is the dual class of Z, hence non-negative.

4.4. We consider now bundles on a two-dimensional torus X. Since the tangent bundle of X is trivial, we have

$$\chi(X, \mathcal{O}_X) = 0.$$

Serre duality gives

$$h^2(X, E) = h^0(X, E^*)$$

for every vector bundle E on X.

4.5. PROPOSITION. — *Let E be a simple vector bundle of rank r on a two-dimensional torus X with*

$$c_1(E) = c_2(E) = 0.$$

Then E is homogeneous, i.e. invariant under translations.

Proof. — Since $h^0(X, \text{End } E) = h^2(X, \text{End } E) = 1$, we have by Riemann-Roch $h^1(X, \text{End } E) = 2$, hence

$$h^1(X, \text{End}_0 E) = h^1(X, \text{End } E) - h^1(X, \mathcal{O}_X) = 0.$$

By Theorem 3.4 the versal deformation of E is given by

$$E \boxtimes L \to X \times \Pi,$$

where $L \to X \times \Pi$ is the versal deformation of the trivial line bundle.

We now construct a family $F \to X \times X$ in the following way : Let

$$a : X \times X \to X$$

be the addition map $a(x,y) := x + y$ and define

$$F := a^*E.$$

By versality, we get a map

$$\varphi : (X, 0) \to \Pi$$

of space germs such that

$$F|X \times (X,0) \cong \varphi^*(E \boxtimes L).$$

Let $\tau_x : X \to X$ be the translation $y \mapsto x + y$. Then for x in a sufficiently small neighborhood of $0 \in X$ we have

$$\tau_x^* E \cong E \otimes L_{\varphi(x)}.$$

Taking determinants, we get

$$\tau_x^*(\det E) \cong (\det E) \otimes L_{\varphi(x)}^2.$$

Since $\det E$ is a topologically trivial line bundle, it is homogeneous, which shows that $L_{\varphi(x)}^2$ is the trivial line bundle. Since $L_{\varphi(0)}$ is trivial, $L_{\varphi(x)}$ itself is trivial. Hence $\tau_x^* E \cong E$ for all sufficiently small x. Since every neighborhood of zero generates X, the bundle E is homogeneous.

4.6. COROLLARY — *Every 2-bundle E on a 2-dimensional torus with $c_1(E) = c_2(E) = 0$ is filtrable.*

Proof. — By (1.8) we may assume that E is simple. Then E is homogeneous by Proposition 4.5. By a theorem of Matsushima ([5], Prop. 3.2) E is filtrable.

4.7. PROPOSITION. — *Let X be a two-dimensional torus with $NS(X) = 0$. A two-bundle E on X is induced by a representation*

$$\sigma : \pi_1(X) \to GL(2,\mathbb{C})$$

if and only if $c_2(E) = 0$.

Remark. — If one drops the hypothesis $NS(X) = 0$, the result does not necessarily hold. Oda [7] has constructed a 2-bundle E on an algebraic 2-dimensional torus with $c_1(E) = c_2(E) = 0$ which does not admit a connection, hence is not induced by a representation.

Proof of Proposition 4.7. — A bundle induced by a representation of $\pi_1(X)$ possesses an integrable connection, hence all its Chern classes are zero (cf. Atiyah [2]).

Conversely suppose $c_2(E) = 0$. Then by Corollary 4.6, E is filtrable (since automatically $c_1(E) = 0$). We now distinguish two cases.

i) If E is decomposable, it is a sum of two topologically trivial line bundles, hence induced by a representation (Appell-Humbert).

ii) If E is indecomposable, we have a devissage

$$0 \to L \to E \to M \otimes \mathscr{I}_Z \to 0$$

with $L, M \in \mathrm{Pic}_0(X)$. Since $c_2(E) = 0$, Z must be empty. We have (by Riemann-Roch)

$$\dim H^1(X, M^* \otimes L) = \begin{cases} 2 & \text{if} \quad L \cong M, \\ 0 & \text{if} \quad L \cong M. \end{cases}$$

Since E is indecomposable, the second possibility is excluded and we have an exact sequence

$$0 \to L \to E \to L \to 0.$$

The extensions of L by L are classified by

$$H^1(X, \mathrm{Hom}(L,L)) = H^1(X, \mathcal{O}).$$

Now the translations operate trivially on $H^1(X, \mathcal{O})$, which shows that E is homogeneous, hence induced by a representation [5].

4.8. *Example of a non-filtrable bundle.* — Let X be a two dimensional torus with $\mathrm{NS}(X) = 0$. Let $L, M \in \mathrm{Pic}_0(X) = \mathrm{Pic}(X)$ be two line bundles on X with $L \ncong M$ and $Z \subset X$ a subspace consisting of two simple points. Consider a 2-bundle E_0 on X which is an extension

$$0 \to L \to E_0 \to M \otimes \mathscr{I}_Z \to 0.$$

We will show that in the versal deformation of E_0 there occur non-filtrable bundles.

Let us first convince ourselves that there is such a bundle E_0. The extensions of $M \otimes \mathscr{I}_Z$ by L are classified by the group $\mathrm{Ext}^1(M \otimes \mathscr{I}_Z, L)$. There is an exact sequence

$$0 \to H^1(X, \mathit{Hom}(M \otimes \mathscr{I}_Z, L)) \to \mathrm{Ext}^1(M \otimes \mathscr{I}_Z, L) \to$$
$$\to \Gamma(X, \mathit{Ext}^1(M \otimes \mathscr{I}_Z, L)) \to H^2(X, \mathit{Hom}(M \otimes \mathscr{I}_Z, L)).$$

Since Z has codimension 2, we have

$$\mathit{Hom}(M \otimes \mathscr{I}_Z, L) \cong M^* \otimes L.$$

By Serre duality $H^2(X,M^*\otimes L) \cong H^0(X,M\otimes L^*)^* = 0$, hence by Riemann-Roch $H^1(X,M^*\otimes L) = 0$. On the other hand, since Z is a locally complete intersection consisting of discrete points.

$$\mathcal{E}xt^1 (M\otimes\mathcal{I}_Z,L) \cong \mathcal{O}_Z,$$

which proves

$$\mathrm{Ext}^1 (M\otimes\mathcal{I}_Z,L) \cong \Gamma(X,\mathcal{O}_Z) \cong \mathbf{C} \oplus \mathbf{C}.$$

By Serre [8], the sheaf corresponding to an extension $\xi \in \mathrm{Ext}^1 (M\otimes\mathcal{I}_Z,L)$ is locally free if and only if its image in $\mathbf{C} \oplus \mathbf{C}$ under the above isomorphism has both coordinates different from zero. Extensions ξ_1, ξ_2 which differ only by a constant factor $\lambda \in \mathbf{C}^*$ give rise to isomorphic sheaves.

4.9. PROPOSITION. — *On a two-torus* X *with* $NS(X) = 0$ *there exist non-filtrable vector bundles* E *of rank* 2 *with* $c_2(E) = 2$.

Proof. — Let E_0 be a 2-bundle with devissage

$$0 \to L \to E_0 \to M \otimes \mathcal{I}_Z \to 0$$

as in 4.8. By Theorem 2.2 this bundle is simple, hence the basis $(V,0)$ of its versal deformation $E \to X \times V$ is smooth (Corollary 3.6). The dimension of V equals $h^1(X, \mathrm{End}\, E_0)$ and can be calculated by Riemann-Roch : We have $\chi(X,\mathcal{O}_X) = 0$ and $c_1(E_0) = 0$, hence

$$h^1(X, \mathrm{End}\, E_0) = h^0(X, \mathrm{End}\, E_0) + h^2(X, \mathrm{End}\, E_0)$$
$$+ 4c_2(E) = 2 + 8 = 10.$$

Since small deformations of simple bundles are simple and have the same Chern classes, this dimension is invariant under small deformations. This implies that the versal deformation of E_0 is also versal in neighboring points.

Suppose now that all bundles $E_s, s \in V$, are filtrable. Then they belong all (for s sufficiently close to 0) to class I.1.i) of the classification of Theorem 2.2. By Theorem 2.3 there exist deformations $\mathcal{L} \to X \times V$ and $\mathcal{M} \to X \times V$ of L resp. M and a two-codimensional subspace $\mathcal{Z} \subset X \times V$, flat over V, such that E fits into an exact sequence

$$0 \to \mathcal{L} \to E \to M \otimes \mathcal{I}_{\mathcal{Z}} \to 0.$$

Since $Z = \mathscr{Z}_0$ consists of two simple points, also $\mathscr{Z}_s$ consists of two simple points for s sufficiently near 0. We can define a holomorphic map

$$\varphi : V \to \mathrm{Pic}_0(X) \times \mathrm{Pic}_0(X) \times S^2 X$$

by

$$s \mapsto (\mathscr{L}_s, \mathscr{M}_s, \mathscr{Z}_s).$$

Since $\dim (\mathrm{Pic}_0(X) \times \mathrm{Pic}_0(X) \times S^2 X) = 8$,

$$S := \varphi^{-1}(L, M, Z)$$

is a subgerm of V of dimension $\geqslant 2$ and we get a family

$$0 \to q^* L \to E|X \times S \to q^* M \otimes \mathscr{I}_{Z \times S} \to 0,$$

where $q : X \times S \to X$ is the projection. This family of extensions defines a holomorphic map

$$\psi : S \to \mathrm{Ext}^1 (M \otimes \mathscr{I}_Z, L) \cong \mathbf{C}^2.$$

Since $0 \notin \psi(S)$, we have an associated map

$$\bar{\psi} : S \to \mathbf{P} (\mathrm{Ext}^1 (M \otimes \mathscr{I}_Z, L)) \cong \mathbf{P}_1.$$

If $\bar{\psi}(s) = \bar{\psi}(s')$, then $E_s \cong E_{s'}$. Since $\dim S \geqslant 2$, the fibres of $\bar{\psi}$ have dimension $\geqslant 1$. Thus there exists a 1-dimensional subgerm $C \subset S$, such that $E|X \times C$ is a trivial deformation of E_0. But this is a contradiction to the versality of the deformation $E \to X \times V$. Hence there must exist non-filtrable bundles E_s in this deformation, q.e.d.

Appendix
Picard number and algebraic dimension of tori.

1. *Generalities.* Let X be a compact complex connected manifold of dimension n. Its algebraic dimension $a(X)$ is defined as the transcendence degree of its field of meromorphic functions. As is well known, $a(X) \leqslant n$. We denote by $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$ the group of isomorphism classes of holomorphic line bundles on X, and by

$$\mathrm{Pic}_0(X) = \mathrm{Ker} (H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbf{Z}))$$

the subgroup of line bundles with vanishing first Chern class. The Néron-Severi group $NS(X)$ is defined by the exact sequence

$$0 \to \text{Pic}_0(X) \to \text{Pic}(X) \to NS(X) \to 0.$$

Hence we can write

$$NS(X) = \text{Im}\,(H^1(X,\mathcal{O}^*) \xrightarrow{\;c_1\;} H^2(X,Z)).$$

The rank of $NS(X)$ is called the Picard number of X and is denoted by $\rho(X)$:

$$\rho(X) = \text{rank}_Z\,NS(X).$$

Assume now that X is a Kähler manifold and consider the Hodge decomposition

$$H^2(X,C) = H^{20}(X,C) \oplus H^{11}(X,C) \oplus H^{02}(X,C).$$

Denote by $j \colon H^2(X,Z) \to H^2(X,C)$ the map induced by the inclusion $Z \hookrightarrow C$. Then the famous Lefschetz Theorem on $(1,1)$-classes reads

$$NS(X) = j^{-1}(H^{11}(X,C)).$$

So, denoting as usual $\dim_C H^{11}(X,C)$ by $h^{11}(X)$, we have

(i) $\qquad\qquad\qquad \rho(X) \leqslant h^{11}(X).$

Equality does not necessarily hold, however we have

(ii) $\rho(X) = h^{11}(X) \Rightarrow X$ projective algebraic

(iii) $\rho(X) = 0 \Rightarrow a(X) = 0.$

2. *The case of tori.* Suppose now X is a torus,

$$X = V/\Gamma,$$

where V is a vector space of dimension n over C and $\Gamma \subset V$ a lattice of rank $2n$. One has a natural isomorphism

$$H^2(X,Z) \cong \text{Alt}_Z^2(\Gamma,Z)$$

of $H^2(X,Z)$ with the space of alternating integer-valued 2-forms on Γ. Let

$$H(V,\Gamma) = \{H : H \text{ hermitian form on } V \text{ with } \operatorname{Im} H(\Gamma \times \Gamma) \subset Z\}.$$

Since the imaginary part $\operatorname{Im} H$ of a hermitian form H is an alternating 2-form which determines completely H, we may consider $H(V,\Gamma)$ as a subgroup of $\operatorname{Alt}_Z^2(\Gamma,Z) \cong H^2(X,Z)$. With this identification one has by the theorem of Appell-Humbert (cf. Mumford [6])

$$NS(X) = H(V,\Gamma).$$

Following Weil [9], let us call Riemann form of X any hermitian form $H \in H(V,\Gamma)$ which is positive semi-definite. Then the algebraic dimension of X is given by

$$a(X) = \max \{\operatorname{rank} H : H \text{ Riemann form of } X\}.$$

In order to be able to make explicit calculations, we introduce coordinates. Let $V = C^n$ and let Γ be the lattice generated by the vectors $\gamma_1, \ldots, \gamma_{2n} \in C^n$, which we consider as column vectors. Define the $n \times 2n$ period matrix

$$\Pi : = (\gamma_1, \ldots, \gamma_{2n}).$$

Then $H(V,\Gamma)$ is identified with the space of all hermitian $n \times n$ matrices A for which

$$(*) \qquad\qquad \operatorname{Im} ({}^t\Pi A \bar{\Pi}) \in Z^{2n \times 2n}.$$

3. *Examples.* In this section we consider two-dimensional tori. We want to give examples for all possible pairs $(a(X), \rho(X))$. For these examples we consider tori determined by period matrices of the form

$$\Pi = \begin{pmatrix} 1 & 0 & ip & ir \\ 0 & 1 & iq & is \end{pmatrix} = (I, iP); \qquad P = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in R^{2 \times 2}.$$

An hermitian 2×2 matrix can be written as

$$A = \begin{pmatrix} x & u + iv \\ u - iv & y \end{pmatrix}, \qquad x, y, u, v \in R.$$

The condition ($*$) above becomes

   (i) $v \in \mathbf{Z}$, $(ps - qr)v \in \mathbf{Z}$.

   (ii) $px + qu \in \mathbf{Z}$, $pu + qy \in \mathbf{Z}$,
      $rx + su \in \mathbf{Z}$, $ru + sy \in \mathbf{Z}$.

Obviously the conditions (i) are independent of (ii) and yield a contribution of 1 or 0 to the Picard number of X, according as $ps - qr$ is rational or not. Since $ps - qr \neq 0$, the system

   (iii) $px + qu = n_1$, $pu + gy = n_3$,
      $rx + su = n_2$, $ru + sy = n_4$,

has at most one solution for fixed $(n_1, n_2, n_3, n_4) \in \mathbf{Z}^4$. Hence the group of triples (x, y, u) satisfying (ii) is isomorphic to the group of those $(n_1, n_2, n_3, n_4) \in \mathbf{Z}^4$ for which (iii) has a solution. But this system has a solution if and only if the value of u deduced from the first pair of equations is the same as that deduced from the second pair, that is if and only if

   (iv) $n_1 r - n_2 p + n_3 s - n_4 q = 0$.

The subgroup of $\mathbf{Z}^4$ defined by this equation has rank equal to

$$4 - \mathrm{rank}_{\mathbf{Q}}(p, q, r, s).$$

Summing up, we have proved

   PROPOSITION. $-$ *Let Γ be the lattice in $\mathbf{C}^2$ spanned by the columns of the matrix*

$$\begin{pmatrix} 1 & 0 & ip & ir \\ 0 & 1 & iq & is \end{pmatrix}, \qquad p, q, r, s \in \mathbf{R}.$$

Then the Picard number of the torus $X = \mathbf{C}^2/\Gamma$ is given by the formula

$$\rho(X) = 4 - \mathrm{rank}_{\mathbf{Q}}(p, q, r, s) + \begin{cases} 1 & \text{if } ps - qr \in \mathbf{Q}, \\ 0 & \text{if } ps - qr \notin \mathbf{Q}. \end{cases}$$

Since for a two-torus X we have $h^{11}(X) = 4$, from (App. 1), (i) - (iii) follow the following restrictions for the Picard number :

$$\begin{aligned} 0 \leqslant \rho(X) \leqslant 3, & \quad \text{if} \quad a(X) = 0, \\ 1 \leqslant \rho(X) \leqslant 3, & \quad \text{if} \quad a(X) = 1, \\ 1 \leqslant \rho(X) \leqslant 4, & \quad \text{if} \quad a(X) = 2. \end{aligned}$$

Besides these there are no other restrictions as is shown by the following examples. In the table we give the matrix P determining the period matrix $\Pi = (I, iP)$ of the required torus.

	$a = 0$	$a = 1$	$a = 2$
$\rho = 0$	$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$	impossible	impossible
$\rho = 1$	$\dfrac{1}{\sqrt{6} - \sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$	$\begin{pmatrix} 1 & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$	$\begin{pmatrix} -\sqrt{2} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$
$\rho = 2$	$\begin{pmatrix} 1 & -3\sqrt{2} \\ 3\sqrt{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 3\sqrt{2} & 1 \\ 0 & 3\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$
$\rho = 3$	$\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix}$
$\rho = 4$	impossible	impossible	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The values of $\rho(X)$ follow from the proposition. We leave it as an exercise to the reader to verify the values of $a(X)$ by determining the maximal rank of a Riemann form.

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