ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 31, nº 2 (1981), p. 61-86
<http://www.numdam.org/item?id=AIF_1981__31_2_61_0>

Ann. Inst. Fourier, Grenoble
31, 2 (1981), 61-86.

CHARACTERISTIC CLASSES OF SUBFOLIATIONS

L. A. CORDERO and X. MASA

1. Introduction.

A flag of foliations of codimensions $q_1, q_2, \ldots, q_k$ $(q_1 \leqslant q_2 \leqslant \cdots \leqslant q_k)$ on a manifold M is defined by Feigin [4] as a chain of foliations F_1, $F_2, \ldots, F_k$ on M, codim $F_i = q_i$, such that for $i < j$ the leaves of F_i contain those of F_j. In his paper, Feigin proposes two constructions for the characteristic classes of flags of foliations, in an attempt to answer the following

FEIGIN'S QUESTION. – Let F be a q-codimensional foliation on a manifold M, and let $0 < p < q$. Do there exist foliations F′, G on M of codimensions q, p respectively such that F′ is integrably homotopic to F and its leaves are contained in those of G ?

In the present paper, we consider those flags of foliations with only two foliations F_1, F_2 and call the couple (F_1, F_2) a (q_1, q_2)-codimensional subfoliation. Then, the main aim of this work is to give a characteristic homomorphism for subfoliations through a differential-geometric construction, generalizing Bott's construction [1] of the characteristic homomorphism of a foliation.

Previous to a detailed discussion of the contents of this paper, it must be pointed out that Cordero-Gadea [3], Moussu [12] and Suzuki [13] also give some partial answers to Feigin's Question.

The paper is structured as follows. First, § 2 is devoted to describe the subfoliation categories; for a subfoliation (F_1, F_2), its normal bundle is defined as $\nu(F_1,F_2) = \nu F_{21} \oplus \nu F_1$, where νF_{21} is the quotient bundle F_1/F_2 and νF_1 is the usual normal bundle of F_1. So, a meaningful exact sequence of vector bundles

$$(1) \qquad 0 \longrightarrow \nu F_{21} \xrightarrow{i} \nu F_2 \xrightarrow{\pi} \nu F_1 \longrightarrow 0$$

appears in a canonical way. This section ends with the study of subfoliation maps and homotopic subfoliations.

§ 3 is devoted to define basic connections on the normal bundle $\nu(F_1,F_2)$; the existence of such basic connections is shown, and Theorem 3.3 states the existence of triples $(\nabla^1, \nabla, \nabla^2)$ of basic connections adapted to the subfoliation, that is, ∇^1, ∇, ∇^2 are basic connections on νF_{21}, νF_2 and νF_1 respectively and compatible with the homomorphisms i and π in (1). The partial flatness of any basic connection on $\nu(F_1,F_2)$ with respect to F_2 leads directly to the Bott's Obstruction Theorem for Subfoliations (Theorem 3.9), firstly stated by Feigin [4]. At the same time, it is deduced that (1) is, in fact, an exact sequence of vector bundles all of them foliated with respect to F_2 and the homomorphisms i and π are compatible with these structures. Nevertheless, this exact sequence does not generally admit a foliated splitting and, therefore, νF_2 and $\nu(F_1,F_2)$ are not, in general, isomorphic as foliated bundles.

In § 4, the characteristic homomorphism of a subfoliation (F_1,F_2) is introducted

$$\lambda^*_{(F_1,F_2)} : H^*(WO_I) \to H^*_{DR}(M)$$

(WO_I, d) being an appropiate graded differential algebra and $H^*(WO_I)$ the associated cohomology groups. The construction of $\lambda^*_{(F_1,F_2)}$ is done following Bott's technique of comparison between a basic and a metric connection on $\nu(F_1,F_2)$. Of course, $\lambda^*_{(F_1,F_2)}$ is natural with respect to subfoliation maps and homotopy invariant.

In § 5, it is first shown that (WO_I, d) is a graded differential subalgebra of a convenable truncated Weil algebra $W_I(g(N))$ of a Lie group and Theorem 5.1 states that $H^*(WO_I)$ and $H^*(W_I(g(N)))$ are isomorphic, which generalizes the well known fact of foliation theory ([5], [6]). Secondly, Theorem 5.2 shows the relation between the characteristic homomorphism of a subfoliation (F_1, F_2) and those of each foliation F_i, $i = 1, 2$.

Finally, in § 6, two applications of the results obtained in § 5 are developped; the first one is the following : Theorem 5.2 gives a necessary condition so that Feigin's Question can have an affirmative answer (Theorem 6.1), and this is used to show that any 2-codimensional foliation in the Yamato's examples [15] cannot be homotopic to F_2 in a (1,2)-codimensional subfoliation (F_1, F_2). The second application is obtained from the techniques used to prove Theorem 5.2, and it is stated as follows : let F be a q-codimensional foliation admitting d everywhere independent transverse infinitesimal transformations $Y_1, \ldots, Y_d$ and such that F and $Y_1, \ldots, Y_d$ generate a new $(q\text{-}d)$-codimensional foliation, then the characteristic homomorphism λ_F^* of F vanishes on the kernel of the canonical homomorphism $\mu^* : H^*(WO_q) \to H^*(WO_{q-d})$. This result gives a generalization of Lazarov-Shulman's results ([9], [10]).

Through this paper all manifolds are differentiable C^∞-manifolds and all maps are smooth C^∞-maps.

2. Subfoliation categories.

To begin with, some basic concepts associated with subfoliations are introduced.

Let M be an n-dimensional manifold, TM its tangent bundle. A *(q_1,q_2)-codimensional subfoliation* on M is a couple (F_1,F_2) of integrable subbundles F_k of TM of dimension $n - q_k$, $k = 1,2$, and F_2 being at the same time a subbundle of F_1.

Therefore, for each $k = 1, 2$, F_k defines a q_k-codimensional foliation on M, $d = q_2 - q_1 \geqslant 0$, and, moreover, the leaves of F_1 contain those of F_2.

Let us remark that a q-codimensional foliation F on M can be considered as a subfoliation on M in three different ways :

$$(C_1) : F_1 = F_2 = F ; \qquad (C_2) : F_1 = TM,\ F_2 = F ;$$
$$(C_3) : F_1 = F,\ F_2 = 0.$$

Let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on M, $\nu F_k = TM/F_k$ the normal bundle of F_k, and let us consider the quotient bundle $\nu F_{21} = F_1/F_2$. Then, the following commutative diagram of short exact sequences of vector bundles is canonically obtained :

$$
\begin{tikzcd}
 & 0 \arrow[ddr] & & & & & & 0 & \\
0 \arrow[drr] & & & & & & & & 0 \\
 & & F_2 \arrow[drr, "i_2"] \arrow[ddr, "i_0"'] & & & & \nu F_1 \arrow[uur] \arrow[urr] & & \\
 & & & & TM \arrow[urr, "\pi_1"] \arrow[dr, "\pi_2"] & & & & \\
 & & & F_1 \arrow[ur, "i_1"] \arrow[ddr, "\pi_0"'] & & \nu F_2 \arrow[uur, "\pi"'] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & & & 0 & & \\
 & & & & \nu F_{21} \arrow[uur, "i"'] \arrow[dr] & & & & \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}
$$

where the i's are the canonical inclusions and the $\pi's$ are the canonical projections.

DEFINITION 2.1. — *The vector bundle*

$$\nu(F_1, F_2) = \nu F_{21} \oplus \nu F_1$$

will be called the normal bundle of (F_1, F_2).

Let be $f : N \to M$ a differentiable map, (F_1, F_2) a (q_1, q_2)-codimensional subfoliation on M; if f is transverse to F_2, the couple $f^{-1}(F_1, F_2) = (f^{-1}(F_1), f^{-1}(F_2))$ defines a (q_1, q_2)-codimensional subfoliation on N which will be called the inverse image of (F_1, F_2) and, then, f is said to be *transverse* to (F_1, F_2). Moreover, we have

$\nu(f^{-1}(F_1,F_2)) = f^*(\nu(F_1,F_2))$, where $f^*(\ )$ denotes the pull-back of the corresponding vector bundle.

DEFINITION 2.2. – *Let* (G_1,G_2) *and* (F_1,F_2) *be* (q_1,q_2)*-codimensional subfoliations on* N *and* M, *respectively. A subfoliation map from* (G_1,G_2) *to* (F_1,F_2) *is a differentiable map* $f : N \to M$ *transverse to* (F_1,F_2) *and such that* $(G_1,G_2) = f^{-1}(F_1,F_2)$.

Now, the notion of homotopy between subfoliations can be defined as follows : let (F_1,F_2), (F'_1,F'_2) be (q_1,q_2)-codimensional subfoliations on M; they are said to be *homotopic subfoliations* if there exists a (q_1,q_2)-codimensional subfoliation $(\bar{F}_1,\bar{F}_2)$ on $M \times \mathbf{R}$ such that :

1) the face maps $j_0, j_1 : M \to M \times \mathbf{R}$ both are transverse to $(\bar{F}_1,\bar{F}_2)$.

2) $j_0^{-1}(\bar{F}_1,\bar{F}_2) = (F_1,F_2)$, $j_1^{-1}(\bar{F}_1,\bar{F}_2) = (F'_1,F'_2)$.

Of course,, the normal bundles of homotopic subfoliations are isomorphic.

A subfoliation (F_1,F_2) will be said *with trivialized normal bundle* if the vector bundles νF_1, νF_2 and $\nu(F_1,F_2)$ are all trivial vector bundles and if there have been chosen trivializations compatible with the projection map $\pi : \nu F_2 \to \nu F_1$ and with the Whitney sum structure of $\nu(F_1,F_2)$.

3. Basic connections and Bott's theorem for subfoliations.

We refer to Bott [1] for the well known definitions and properties of the usual theory of foliations.

Let (F_1,F_2) be a (q_1,q_2)-codimensional subfoliation on M, $d = q_2 - q_1$. Let us consider the exact sequence

$$0 \longrightarrow F_2 \xrightarrow{i_0} F_1 \xrightarrow{\pi_0} \nu F_{21} \longrightarrow 0.$$

DÉFINITION 3.1. – *A connection* ∇ *on* νF_{21} *is said to be basic if*

$$\nabla_X Z = \pi_0[X,\tilde{Z}]$$

for any vector field $X \in \Gamma(F_2)$, $\tilde{Z}$ *being a vector field in* F_1 *such that* $\pi_0(\tilde{Z}) = Z$.

A device similar to that of Bott in [1] permits to show the existence of basic connections in νF_{21}.

Now, consider the exact sequence

$$0 \longrightarrow \nu F_{21} \xrightarrow{i} \nu F_2 \xrightarrow{\pi} \nu F_1 \longrightarrow 0.$$

Then, the following proposition is easily verified.

PROPOSITION 3.2. — *For any* ∇^1, ∇ *and* ∇^2 *basic connections on* νF_{21}, νF_2 *and* νF_1 *respectively, and for any* $X \in \Gamma(F_2)$, *we have*

$$\begin{aligned} i(\nabla^1_X Z) &= \nabla_X i(Z), && \textit{for any} \quad Z \in \Gamma(\nu F_{21}) \\ \pi(\nabla_X Z_2) &= \nabla^2_X \pi(Z_2), && \textit{for any} \quad Z_2 \in \Gamma(\nu F_2). \end{aligned}$$

In fact, we can state the following.

THEOREM 3.3. — *There exist* ∇^1, ∇ *and* ∇^2 *basic connections on* νF_{21}, νF_2 *and* νF_1 *such that, for any vector field* $X \in \Gamma(TM)$,

$$\begin{aligned} i(\nabla^1_X Z) &= \nabla_X i(Z), && \textit{for any} \quad Z \in \Gamma(\nu F_{21}) \\ \pi(\nabla_X Z_2) &= \nabla^2_X \pi(Z_2), && \textit{for any} \quad Z_2 \in (\nu F_2). \end{aligned}$$

Such a triple $(\nabla^1, \nabla, \nabla^2)$ *of basic connections will be said adapted to the subfoliation.*

Proof. — Let us begin by considering a Riemannian metric on M which is compatible with the subfoliated structure, that is (see [14]) : with respect to this metric, νF_2 (respect. νF_1) is isomorphic to the orthogonal complement bundle to F_2 (respect. to F_1) in TM, and νF_{21} is isomorphic to the orthogonal complement bundle to F_2 in F_1; that is, by the choice of such Riemannian metric we obtain isomorphisms

$$\begin{aligned} TM &\cong F_k \oplus \nu F_k, \qquad k = 1,2 \\ F_1 &\cong \nu F_{21} \oplus F_2 \\ \nu F_2 &\cong \nu F_{21} \oplus \nu F_1. \end{aligned}$$

Now, we construct ∇^1 and ∇^2 basic connections on νF_{21} and νF_1, respectively, as follows : for any

$$X \in \Gamma(TM) = \Gamma(F_2) \oplus \Gamma(\nu F_2) = \Gamma(F_1) \oplus \Gamma(\nu F_1),$$

we write $X = X_2 + X'_2 = X_1 + X'_1$, and if $\bar{\nabla}^1$ (respect. $\bar{\nabla}^2$) is an arbitrary connection on νF_{21} (respect. on νF_1), we define

$$\nabla^1_X Z = \pi_0[X_2, \tilde{Z}] + \bar{\nabla}^1_{X'_2} Z, \qquad \text{for any} \quad Z \in \Gamma(\nu F_{21})$$
$$\nabla^2_X Z_1 = \pi_1[X_1, \tilde{Z}_1] + \bar{\nabla}^2_{X'_1} Z_1, \qquad \text{for any} \quad Z_1 \in \Gamma(\nu F_1)$$

where $\tilde{Z} \in \Gamma(F_1)$ and $\tilde{Z}_1 \in \Gamma(TM)$ are such that $\pi_0(\tilde{Z}) = Z$, $\pi_1(\tilde{Z}_1) = Z_1$.

The basic connection ∇ on νF_2 is constructed as follows : for any $Z_2 \in \Gamma(\nu F_2) = \Gamma(\nu F_{21}) \oplus \Gamma(\nu F_1)$, we write $Z_2 = Z'_2 + Z''_2$, and define a connection $\bar{\nabla}$ on νF_2 by

$$\bar{\nabla}_X Z_2 = \nabla^1_X Z'_2 + \nabla^2_X Z''_2$$

for any $X \in \Gamma(TM)$; then, ∇ is given by

$$\dot{\nabla}_X Z_2 = \pi_2[X_2, \tilde{Z}_2] + \bar{\nabla}_{X'_2} Z_2$$

where $\tilde{Z}_2 \in \Gamma(TM)$ verifies $\pi_2(\tilde{Z}_2) = Z_2$.

Now, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(\nu F_{21})$, we have

$$i(\nabla^1_X Z) = i(\pi_0[X_2, \tilde{Z}] + \bar{\nabla}^1_{X'_2} Z) = \pi_2[X_2, \tilde{Z}] + i(\bar{\nabla}^1_{X'_2} Z)$$

$\tilde{Z} \in \Gamma(F_1)$ being such that $\pi_0(\tilde{Z}) = Z$. On the other hand,

$$\nabla_X i(Z) = \pi_2[X_2, \tilde{Z}] + \bar{\nabla}_{X'_2} i(Z)$$

but $i(Z) \in \operatorname{Ker} \pi$, then $\bar{\nabla}_{X'_2} i(Z) = \nabla^1_{X'_2} i(Z)$, and, since $X'_2 \in \Gamma(\nu F_2)$, the result follows immediately.

Analogously, for any $X \in \Gamma(TM)$ and $Z_2 \in \Gamma(\nu F_2)$ we have

$$\pi(\nabla_X Z_2) = \pi(\pi_2[X_2, \tilde{Z}_2] + \bar{\nabla}_{X'_2} Z_2) = \pi_1[X_2, \tilde{Z}_2] + \pi(\bar{\nabla}_{X'_2} Z_2)$$

$\tilde{Z}_2 \in \Gamma(TM)$ being such that $\pi_2(\tilde{Z}_2) = Z_2$. Since $Z_2 = Z'_2 + Z''_2$ and $X'_2 \in \Gamma(\nu F_2)$, and then $X'_2 = (X'_2)' + (X'_2)'' \in \Gamma(\nu F_{21}) \oplus \Gamma(\nu F_1)$, we have

$$\pi(\bar{\nabla}_{X'_2} Z_2) = \nabla^2_{X'_2} Z''_2 = \nabla^2_{X'_2} \pi(Z_2) = \pi_1[(X'_2)', \tilde{Z}_2] + \bar{\nabla}^2_{(X'_2)''} \pi(Z_2)$$

and, since $X_2 + (X'_2)' = X_1$ and $(X'_2)'' = X'_1$, the second part of the theorem is proved.

Q.E.D.

For the later use, we shall explain the relation between the local connection forms of an adapted triple $(\nabla^1, \nabla, \nabla^2)$ of basic connections.

Let $U \subset M$ be an open set of local triviality for νF_{21}, νF_2 and νF_1. A local basis $\{Z_i, i=1,2,\ldots,q_2\}$ of sections for νF_2 will be said *adapted* if $\{\pi(Z_u), u=d+1,\ldots,q_2\}$ is a local basis of sections of νF_1 and $\{Z_a, a=1,2,\ldots,d\}$ is a local basis of sections of νF_{21}.

Now, given an adapted local basis $\{Z_i\}$ of sections, let $\{\tilde{Z}_i, i=1,2,\ldots,q_2\}$ be local vector fields on M such that $\pi_2(\tilde{Z}_i) = Z_i$, $1 \leqslant i \leqslant q_2$, and let $\{\omega_i, i=1,2,\ldots,q_2\}$ be the local 1-forms on M dual to the local vector fields $\{\tilde{Z}_i\}$. Then, the 1-forms $\{\omega_i, 1 \leqslant i \leqslant q_2\}$ are annihilated by F_2 and the 1-forms $\{\omega_u, d+1 \leqslant u \leqslant q_2\}$ are annihilated by F_1. Therefore, by the integrability conditions, there exist local 1-forms τ_{ba}, τ_{ua}, τ_{vu}, $1 \leqslant a, b \leqslant d$, $d+1 \leqslant u, v \leqslant q_2$, on M such that

$$d\omega_a = \sum_{b=1}^{d} \omega_b \wedge \tau_{ba} + \sum_{u=d+1}^{q_2} \omega_u \wedge \tau_{ua}$$

$$d\omega_u = \sum_{v=d+1}^{q_2} \omega_v \wedge \tau_{vu}.$$

Let $(\nabla^1, \nabla, \nabla^2)$ be an adapted triple of basic connections, and suppose that (θ^1_{ab}), (θ_{ij}), (θ^2_{uv}) are respectively their local connection forms with respect to an adapted local basis of sections over an open set $U \subset M$. Then, for any local vector field X on M

$$\nabla^1_X Z_a = \sum_{b=1}^{d} \theta^1_{ab}(X) Z_b$$

$$\nabla_X Z_i = \sum_{j=1}^{q_2} \theta_{ij}(X) Z_j$$

$$\nabla^2_X \pi(Z_u) = \sum_{v=d+1}^{q_2} \theta^2_{uv}(X) \pi(Z_v).$$

Then, a direct computation leads to

$$\begin{aligned} \theta^1_{ab} &= \theta_{ab}, && 1 \leqslant a,b \leqslant d \\ \theta_{au} &= 0, && 1 \leqslant a \leqslant d, \quad d+1 \leqslant u \leqslant q_2 \\ \theta_{uv} &= \theta^2_{uv}, && d+1 \leqslant u, v \leqslant q_2, \end{aligned}$$

and we can state the following :

LEMMA 3.4. — *With respect to an adapted local basis of sections, the local connection forms of an adapted triple* $(\nabla^1, \nabla, \nabla^2)$ *of basic connections are, respectively*

$$\nabla^1 : (\theta_{ab}), \qquad \nabla : \begin{bmatrix} \theta_{ab} & 0 \\ \theta_{ub} & \theta_{uv} \end{bmatrix}, \qquad \nabla^2 : (\theta_{uv}),$$
$$1 \leqslant a, b \leqslant d, \qquad d+1 \leqslant u, v \leqslant q_2 .$$

Moreover, by a similar device to that used in the foliation theory, we get

LEMMA 3.5. — *Let* ∇^1, ∇ *and* ∇^2 *be any basic connections on* νF_{21}, νF_2 *and* νF_1, *respectively. Let us consider an adapted local basis of sections over an open set* $U \subset M$, *and suppose that* (θ^1_{ab}), (θ_{ij}) *and* (θ^2_{uv}) *are the respective local connection forms. Then, we have :*

1. $\theta^1_{ab}(X) = \tau_{ab}(X)$, *for any local vector field* $X \in \Gamma(F_2)$, $1 \leqslant a, b \leqslant d$.

2. $\theta_{ab}(X) = \tau_{ab}(X)$, $\theta_{au}(X) = 0$, $\theta_{ua}(X) = \tau_{ua}(X)$ *and* $\theta_{uv}(X) = \tau_{uv}(X)$ *for any local vector field* $X \in \Gamma(F_2)$, $1 \leqslant a$, $b \leqslant d$, $d+1 \leqslant u$, $v \leqslant q_2$.

3. $\theta^2_{uv}(X) = \tau_{uv}(X)$, *for any local vector field* $X \in \Gamma(F_1)$, $d+1 \leqslant u$, $v \leqslant q_2$.

Now, let us remark once more what happens for the particular case of a foliation considered as a subfoliation :

(C_1) : νF_{21} is the zero vector bundle and the only possible connection is the zero one, and that is trivialy a basic connection.

(C_2) : $\nu F_{21} = \nu F$, the normal bundle of F, and Definition 3.1 is the usual definition of basic connections.

(C_3) : $\nu F_{21} = F$ and, since $F_2 = 0$, any connection on νF_{21} is basic.

From Definition 3.1, we get through a straightforward computation

LEMMA 3.6. — *Let* ∇^1 *be a basic connection on* νF_{21}, K^1 *the curvature of* ∇^1. *Then* $K^1(X,Y) \equiv 0$ *for any vector fields* X, Y *in* F_2.

Now, we shall adopt the following

DEFINITION 3.7. — *A connection* $\nabla = \nabla^1 \oplus \nabla^2$ *on* $\nu(F_1, F_2) = \nu F_{21} \oplus \nu F_1$ *is said to be basic if and only if* ∇^1 *is a basic connection on* νF_{21} *and* ∇^2 *is a basic connection on* νF_1.

It is a well known fact that the curvature K^2 of a basic connection ∇^2 on νF_1 verifies $K^2(X,Y) \equiv 0$ for any X, Y vector fields in F_1. Therefore, the following is immediate :

COROLLARY 3.8. – *Let* ∇ *be a basic connection on* $\nu(F_1, F_2)$, K *the curvature of* ∇. *Then* $K(X,Y) \equiv 0$ *for any* X, Y *vector fields in* F_2.

Remarks. – 1) Lemma 3.6 implies that νF_{21} has a foliated vector bundle structure with respect to F_2 defined by considering the horizontal lift of F_2 with respect to a basic connection ∇^1 on νF_{21}. Lemma 3.5 points out that the foliated structure does not depend on the choice of ∇^1.

2) Corollary 3.8 implies that $\nu(F_1, F_2)$ has also a well-defined foliated bundle structure with respect to F_2 adapted to the Whitney sum structure. Nevertheless, although both νF_2 and $\nu(F_1, F_2)$ have foliated structures with respect to F_2 and are isomorphic as vector bundles, they are not, in general, isomorphic as foliated vector bundles.

3) The result in Proposition 3.2 implies that homomorphisms i and π in the exact sequence of vector bundles

$$0 \longrightarrow \nu F_{21} \xrightarrow{i} \nu F_2 \xrightarrow{\pi} \nu F_1 \longrightarrow 0$$

are, in fact, foliated homomorphisms (i.e. compatible with the respective foliated structures with respect to F_2).

Now, let $U \subset M$ be a simultaneously trivializing neighborhood for νF_1, νF_2 and νF_{21} ; over U, F_1 can be described as the set of tangent vectors on which certain local 1-forms $\omega_{d+1}, \ldots, \omega_{q_2}$ vanish, these 1-forms being linearly independent at each point of U. Analogously, F_2 can be described over U, and, since $F_2 \subset F_1$ we can suppose that the family of local 1-forms which annihilate F_2 is obtained by adding local 1-forms $\omega_1, \ldots, \omega_d$ to the family above, being also linearly independent at each point of U. Let be

I^1_U = ideal in $\Lambda^*(U)$ generated by $\omega_{d+1}, \ldots, \omega_{q_2}$
I^2_U = ideal in $\Lambda^*(U)$ generated by $\omega_1, \ldots, \omega_d, \omega_{d+1}, \ldots, \omega_{q_2}$.

Obviously,

$$I^1_U \subset I^2_U, \qquad (I^1_U)^{q_1+1} = 0, \qquad (I^2_U)^{q_2+1} = 0.$$

Now, if $\nabla = \nabla^1 \oplus \nabla^2$ is a basic connection on $\nu(F_1, F_2)$, then the curvature matrix K_U of ∇ over U will be

$$K_U = \begin{bmatrix} (K_U^1)_{ab} & 0 \\ 0 & (K_U^2)_{uv} \end{bmatrix}$$

with respect to a local basis of sections of $\nu(F_1, F_2)$, dual to the local 1-forms ω_i, $1 \leqslant i \leqslant q_2$. Here, $((K_U^1)_{ab})$ (respect. $((K_U^2)_{uv})$) denotes the curvature matrix of ∇^1 (respect. of ∇^2) over U. Taking into account earlier results, we get

$$\begin{aligned} &(K_U^1)_{ab} \in I_U^2, && 1 \leqslant a, b \leqslant d \\ &(K_U^2)_{uv} \in I_U^1, && d+1 \leqslant u, v \leqslant q_2. \end{aligned}$$

Hence, the following *Bott's Obstruction Theorem for Subfoliations* is obtained

THEOREM 3.9. – *Let* P_1 *and* P_2 *be homogeneous polynomials in the real Pontryagin classes of* νF_{21} *and* νF_1 *respectively, of degree* l_k, $k = 1,2$. *If at least one of the inequalities* $l_2 > 2q_1$, $l_1 + l_2 > 2q_2$ *is satisfied, then* $P_1 P_2 = 0$.

4. The characteristic homomorphism for subfoliations.

In this section we define a characteristic homomorphism for subfoliations, which generalizes the usual characteristic homomorphism for foliations. For this purpose, the technique used by Bott in [1] shall be adopted here.

Let gl_n denote the Lie algebra of $Gl_n = Gl(n, \mathbf{R})$,

$I(gl_n) = \mathbf{R}[c_1, \ldots, c_n]$ the ring of symmetric invariant polynomials on gl_n, $c_1, \ldots, c_n$ being the Chern polynomials given by

$$\det\left(I + \frac{t}{2\pi} A\right) = \sum_{j=0}^{n} c_j(A) t^j$$

where $c_j(A) = \left(\frac{1}{2\pi}\right)^j$ trace $\Lambda^j A$, for any $A \in gl_n$.

In addition, if $E \to M$ is a vector bundle of dimension n over M and

∇ is a connection on E of curvature K, denote

$$\lambda(\nabla) : \mathrm{I}(gl_n) \to \Lambda^*(\mathrm{M})$$

the ring homomorphism defined by

$$\lambda(\nabla)(c_j) = c_j(\mathrm{K}).$$

Moreover, if $\nabla^0, \nabla^1, \ldots, \nabla^m$ are connections on E, define

$$\lambda(\nabla^0,\nabla^1,\ldots,\nabla^m)(c_j) = \pi_*[c_j(\mathrm{K}^{0,1,\ldots,m})|_{\mathrm{M}\times\Delta^m}]$$

where Δ^m is the standard m-simplex, $\mathrm{K}^{0,1,\ldots,m}$ is the curvature of the connection

$$\nabla^{0,1,\ldots,m} = (1-t_1- \cdots -t_m)\nabla^0 + t_1\nabla^1 + \cdots + t_m\nabla^m$$

on the vector bundle $\mathrm{E} \times \mathbf{R}^m \to \mathrm{M} \times \mathbf{R}^m$ and

$$\pi_* : \Lambda^p(\mathrm{M}\times\Delta^m) \to \Lambda^{p-m}(\mathrm{M})$$

denotes the integration along Δ^m.

The following useful properties are verified :

1. $d(\lambda(\nabla^0,\nabla^1,\ldots,\nabla^m)(c_j)) = \sum_{i=0}^{m} (-1)^i\lambda(\nabla^0,\ldots,\hat{\nabla}^i,\ldots,\nabla^m)(c_j).$
2. If $f : \mathrm{N} \to \mathrm{M}$ is a differentiable map, then

$$f^*(\lambda(\nabla^0,\nabla^1,\ldots,\nabla^m)(c_j)) = \lambda(f^*(\nabla^0),f^*(\nabla^1),\ldots,f^*(\nabla^m))(c_j).$$

Now, in order to construct an appropiate cochain complex $(\mathrm{WO}_\mathrm{I},d)$, let us consider $q_1, q_2 \in \mathrm{N}$, with $q_2 \geqslant q_1$ and $d = q_2 - q_1$; denote by

$$\mathrm{I}(gl_d) = \mathbf{R}[c'_1,\ldots,c'_d], \qquad \mathrm{I}(gl_{q_1}) = \mathbf{R}[c''_1,\ldots,c''_{q_1}]$$

the rings of symmetric invariant polynomials, c'_i and c''_i being the corresponding Chern polynomials. Let $\mathrm{I}(gl_d) \otimes \mathrm{I}(gl_{q_1})$ be the tensor product and denote by I the homogeneous ideal generated by binomials $\varphi' \otimes \varphi'' \in \mathrm{I}^{s'}(gl_d) \otimes \mathrm{I}^{s''}(gl_{q_1})$ whose dimensions verify at least one of the inequalities $s'' > q_1$, $s' + s'' > q_2$.

On the other hand, consider the exterior algebras over $\mathbf{R}$

$$\Lambda(h'_1,h'_3,\ldots,h'_{l'}), \qquad \Lambda(h''_1,h''_3,\ldots,h''_{l''})$$

generated by the elements h'_i, h''_i respectively and where

$$l' = 2\left[\frac{d+1}{2}\right] - 1, \qquad l'' = 2\left[\frac{q_1+1}{2}\right] - 1.$$

Now we build a graded differential algebra WO_I as follows :

$$WO_I = \Lambda(h'_1, h'_3, \ldots, h'_{l'}) \otimes \Lambda(h''_1, h''_3, \ldots, h''_{l''}) \otimes \frac{I(gl_d) \otimes I(gl_{q_1})}{I}$$

where

$$\text{degree } (h'_i) = \text{degree } (h''_i) = 2i - 1$$
$$\text{degree } (c'_i) = \text{degree } (c''_i) = 2i,$$

and the unique antiderivation of degree 1, $d : WO_I^{\bullet} \to WO_I^{\bullet+1}$, is defined by

$$d(h'_i) = c'_i, \qquad d(h''_i) = c''_i, \qquad d(c'_i) = d(c''_i) = 0.$$

We shall denote $H^*(WO_I)$ the cohomology of the cochain complex (WO_I, d).

Let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on the manifold M, $\nu(F_1, F_2) = \nu F_{21} \otimes \nu F_1$ its normal bundle. Let $\nabla^0 = {}^1\nabla^0 \oplus {}^2\nabla^0$, $\nabla^1 = {}^1\nabla^1 \oplus {}^2\nabla^1$ be connections on $\nu(F_1, F_2)$, where ${}^1\nabla^0$ (respect. ${}^2\nabla^0$) is a Riemannian connection on νF_{21} (respect. on νF_1) and ∇^1 is a basic connection. Then, a graded algebra homomorphism

$$\lambda_{(F_1,F_2)} : WO_I \to \Lambda^*(M)$$

is defined by

$$\lambda_{(F_1,F_2)}(c'_i) = \lambda({}^1\nabla^1)(c'_i), \qquad \lambda_{(F_1,F_2)}(c''_i) = \lambda({}^2\nabla^1)(c''_i)$$
$$\lambda_{(F_1,F_2)}(h'_i) = \lambda({}^1\nabla^0, {}^1\nabla^1)(c'_i), \qquad \lambda_{(F_1,F_2)}(h''_i) = \lambda({}^2\nabla^0, {}^2\nabla^1)(c''_i).$$

The Obstruction Theorem 3.9 implies that $\lambda_{(F_1,F_2)}$ is well defined and, in fact, it is a cochain complex homomorphism; therefore, it induces a homomorphism of graded algebras on cohomology :

$$\lambda^*_{(F_1,F_2)} : H^*(WO_I) \to H^*_{DR}(M).$$

While the cochain homomorphism $\lambda_{(F_1,F_2)}$ depends on the choice of ∇^0 and ∇^1, the induced homomorphism $\lambda^*_{(F_1,F_2)}$ does not, as one can easily show by means of standard techniques.

DEFINITION 4.1. — $\lambda^*_{(F_1,F_2)}$ *is called the characteristic homomorphism of* (F_1,F_2) *and the classes in the image of* $\lambda^*_{(F_1,F_2)}$ *are said the secondary subfoliation classes of* (F_1,F_2).

The characteristic homomorphism $\lambda^*_{(F_1,F_2)}$ has a certain naturality property which can be expressed as follows :

PROPOSITION 4.2. — *Let* (G_1,G_2) *and* (F_1,F_2) *be subfoliations on* N *and* M *respectively,* $f : N \to M$ *a subfoliation map from* (G_1,G_2) to (F_1,F_2). *Then, the diagram*

$$\begin{tikzcd} H^*(WO_I) \arrow[r, "\lambda^*_{(F_1,F_2)}"] \arrow[dr, "\lambda^*_{(G_1,G_2)}"'] & H^*_{DR}(M) \arrow[d, "f^*"] \\ & H^*_{DR}(N) \end{tikzcd}$$

commutes.

Proof. — This result is an immediate consequence of the following fact : if ∇^0 and ∇^1 are, respectively, a Riemannian and a basic connection on $\nu(F_1,F_2)$, then $f^*(\nabla^0)$ and $f^*(\nabla^1)$ are connections of the same type on $\nu(G_1,G_2) = f^*(\nu(F_1,F_2))$; therefore, from the definition of $\lambda_{(F_1,F_2)}$, the commutativity of the above diagram follows immediately at the cochain complex level. Q.E.D.

Then, from the naturality and through purely homotopy theoretic reasons, we obtain

COROLLARY 4.3. — $\lambda^*_{(F_1,F_2)}$ *only depends on the homotopy class of* (F_1,F_2).

To construct the secondary subfoliation classes of a subfoliation (F_1,F_2) with a trivialized normal bundle, one proceeds exactly as before except that Riemannian connections $\nabla^0 = {}^1\nabla^0 \oplus {}^2\nabla^0$ are replaced by flat connections (Whitney sum of flat connections) and the cochain complex (WO_I, d) is replaced by the cochain complex (W_I, d), where

$$W_I = \Lambda(h'_1, h'_2, \ldots, h'_d) \otimes \Lambda(h''_1, h''_2, \ldots, h''_{q_1}) \otimes \frac{I(gl_d) \otimes I(gl_{q_1})}{I}$$

and with gradation and differential d defined in the same form.

5. Relation between the characteristic homomorphisms of (F_1, F_2) and F_k, $k = 1, 2$.

In order to justify our later constructions, let us recall some well known facts of the theory of characteristic classes of foliations and their relation with the cohomology of truncated relative Weil algebras.

Let $W(gl_n)$ denote the Weil algebra of gl_n; $W(gl_n)$ is a differential graded algebra $W(gl_n) = \bigoplus_{r \geqslant 0} W^r(gl_n)$, where

$$W^r(gl_n) = \bigoplus_{i+2j=r} \{\Lambda^i(gl_n) \otimes S^j(gl_n)\}.$$

In fact, $W(gl_n)$ is a gl_n-algebra and, if $I(gl_n)$ denotes the subalgebra of gl_n-basic elements of $W(gl_n)$, then $I(gl_n)$ admits the Chern polynomials $c_1, c_2, \ldots, c_n$ as a system of generators, which are cocycles of degree $2i$. Moreover, since the complex $(W(gl_n), d)$ is acyclic, there exist $h_i \in W^{2i-1}(gl_n)$, $1 \leqslant i \leqslant n$, such that $dh_i = c_i$. Also, if $W(gl_n, O_n)$ denotes the differential subalgebra of the O_n-basic elements of $W(gl_n)$ ($O_n = O(n,\mathbf{R})$ being the orthogonal group), the generators c_i of $I(gl_n)$ can be chosen in such way that the h_i be O_n-basic for each odd i.

Now, let J_n be the homogeneous ideal of $W(gl_n)$ generated by the elements of $S(gl_n)$ of degree greater than $2n$; denote $W_n(gl_n) = W(gl_n)/J_n$ the quotient algebra and $W_n(gl_n, O_n)$ the subalgebra of O_n-basic elements of $W_n(gl_n)$. Then, one has the following well-known theorem (see [6] or [5]) :

THEOREM. — *$W_n(gl_n)$ (respect. $W_n(gl_n, O_n)$) has the same cohomology that its subalgebra*

$$W_n = \Lambda(h_1, h_2, \ldots, h_n) \otimes \frac{I(gl_n)}{J_n}$$

$$\left(\textit{respect.}\ \ WO_n = \Lambda(h_1, h_3, \ldots,) \otimes \frac{I(gl_n)}{J_n} \right)$$

where the h_i are, for odd i, the O_n-basic elements such that $dh_i = c_i$.

This theorem plays a fundamental role in the theory of characteristic classes of foliations because the cohomology ring $H^*(WO_n)$ (respect.

$H^*(W_n))$ is the domain of the characteristic homomorphism of n-codimensional foliations (respect. with trivialized normal bundle).

Similarly, the domain of the characteristic homomorphism for subfoliations, as defined in the earlier section, can be seen having as domain the cohomology ring of a convenable relative truncated Weil algebra. To explain that, let us consider the Lie algebra $g(\mathrm{N}) = gl_{n_1} \times gl_{n_2}$, its Weil algebra $W(g(\mathrm{N}))$ and the homogeneous ideal I of $W(g(\mathrm{N}))$ generated by the subspaces $S^{i_1}(gl_{n_1}) \otimes S^{i_2}(gl_{n_2})$, where the integers i_1, i_2 satisfy at least one of the inequalities $i_2 > n_2$, $i_1 + i_2 > n_1 + n_2$. Clearly, I is a graded subcomplex of $W(g(\mathrm{N}))$, so the quotient $W_{\mathrm{I}}(g(\mathrm{N})) = W(g(\mathrm{N}))/\mathrm{I}$ is a multiplicative graded complex; moreover, if $O_{\mathrm{N}} = O_{n_1} \times O_{n_2}$ is the product Lie group, we denote $W_{\mathrm{I}}(g(\mathrm{N}), O_{\mathrm{N}})$ the graded subcomplex of $W_{\mathrm{I}}(g(\mathrm{N}))$ of O_{N}-basic elements.

Now, through the canonical isomorphism

$$W(g(\mathrm{N})) \cong W(gl_{n_1}) \otimes W(gl_{n_2})$$

let us consider the graded differential subalgebras W_{I}, WO_{I} of $W_{\mathrm{I}}(g(\mathrm{N}))$ and $W_{\mathrm{I}}(g(\mathrm{N}), O_{\mathrm{N}})$ respectively, given by

$$W_{\mathrm{I}} = \Lambda(h'_1, h'_2, \ldots, h'_{n_1}) \otimes \Lambda(h''_1, h''_2, \ldots, h''_{n_2}) \otimes \frac{\mathrm{I}(gl_{n_1}) \otimes \mathrm{I}(gl_{n_2})}{\mathrm{I}}$$

$$WO_{\mathrm{I}} = \Lambda(h'_1, h'_3, \ldots, \quad) \otimes \Lambda(h''_1, h''_3, \ldots, \quad) \otimes \frac{\mathrm{I}(gl_{n_1}) \otimes \mathrm{I}(gl_{n_2})}{\mathrm{I}},$$

the h'_i (respect. h''_i) being such that $dh'_i = c'_i$ (respect. $dh''_i = c''_i$) and c'_i, $c'_2, \ldots, c'_{n_1}$ (respect. c''_1, $c''_2, \ldots, c''_{n_2}$) the Chern polynomials which generate $\mathrm{I}(gl_{n_1})$ (respect. $\mathrm{I}(gl_{n_2})$), and for odd i, h'_i (respect. h''_i) is supposed to be O_{n_1}-basic (respect. O_{n_2}-basic).

Then, we have

THEOREM 5.1. — *The canonical injections*

$$W_{\mathrm{I}} \longrightarrow W_{\mathrm{I}}(g(\mathrm{N}))$$
$$WO_{\mathrm{I}} \longrightarrow W_{\mathrm{I}}(g(\mathrm{N}), O_{\mathrm{N}})$$

induce isomorphisms on cohomology.

Proof. — The result follows from a device similar to that used in the ordinary case.

We start by defining an even filtration of W_I and WO_I as follows :

$$F^{2p}(W_I) = \Lambda(h'_1, h'_2, \ldots) \otimes \Lambda(h''_1, h''_2, \ldots) \otimes \left\{ \bigoplus_{i_1+i_2 \geqslant p} \frac{I^{i_1}(gl_{n_1}) \otimes I^{i_2}(gl_{n_2})}{I} \right\}$$

$$F^{2p}(WO_I) = \Lambda(h'_1, h'_3, \ldots) \otimes \Lambda(h''_1, h''_3, \ldots) \otimes \left\{ \bigoplus_{i_1+i_2 \geqslant p} \frac{I^{i_1}(gl_{n_1}) \otimes I^{i_2}(gl_{n_2})}{I} \right\}$$

and the second term E_2 of the spectral sequence associated to W_I (respect. WO_I) can be canonically identified with W_I (respect. with WO_I).

On the other hand, we define an even filtration of $W_I(g(N))$ by

$$F^{2p}(W_I(g(N)) = \Lambda^\bullet(g(N)) \otimes \left\{ \bigoplus_{i_1+i_2 \geqslant p} (S^{i_1}(gl_{n_1}) \otimes S^{i_2}(gl_{n_2})) \right\}.$$

Since this filtration is by O_N-invariant ideals, it induces a new and similar filtration of $W_1(g(N), O_N)$. In both cases, the associated spectral sequence is of the Hochschild-Serre type [7], hence its second term is given by

$$E_2^{2p,q} = H^q(g(N), \mathbf{R}) \otimes \left\{ \bigoplus_{i_1, i_2} ([S^{i_1}(gl_{n_1})]^{gl_{n_1}} \otimes [S^{i_2}(gl_{n_2})]^{gl_{n_2}}) \right\}$$

for that associated to $W_I(g(N))$, and by

$$E_2^{2p,q} = H^q(g(N), O_N) \otimes \left\{ \bigoplus_{i_1, i_2} ([S^{i_1}(gl_{n_1})]^{gl_{n_1}} \otimes [S^{i_2} gl_{n_2})]^{gl_{n_2}}) \right\}$$

for that associated to $W_I(g(N), O_N)$; here, in both cases, the summation on the right extends over all couples i_1, i_2 such that $i_1 + i_2 = p$, $i_2 \leqslant n_2$ and $i_1 + i_2 \leqslant n_1 + n_2$, and the symbol $[\ \ ]^{gl_{n_i}}$ denotes the set of gl_{n_i}-invariants.

Now, the canonical projection of $\Lambda(h'_1, h'_2, \ldots) \otimes \Lambda(h''_1, h''_2, \ldots)$ (respect. $\Lambda(h'_1, h'_3, \ldots) \otimes \Lambda(h''_1, h''_3, \ldots))$ over $\Lambda^*(g(N))$ (respect. over $[\Lambda^*(g(N))]_{O_N})$ induces an isomorphism on cohomology and the result follows by applying a comparison theorem.

Q.E.D.

Thus, Theorem 5.1 shows that the graded differential algebras W_I and WO_I, which were introduced in the earlier section, play in the context of subfoliation theory the same role as that of W_n and WO_n in the context of the foliation theory.

Now, let us consider the canonical projection homomorphism of Lie algebras

$$g(N) = gl_{n_1} \times gl_{n_2} \longrightarrow gl_{n_2}.$$

This homomorphism induces an homomorphism of graded differential algebras

$$\mu_1 : W(gl_{n_2}) \longrightarrow W(g(N))$$

which is compatible with the truncation by I_{n_2} and I, and since $O_N = O_{n_1} \times O_{n_2}$ applies onto O_{n_2}, we obtain a graded differential algebra homomorphism

$$\mu_1 : W_{n_2}(gl_{n_2}, O_{n_2}) \longrightarrow W_I(g(N), O_N)$$

and this homomorphism μ_1 acts on the generators as follows :

$$\mu_1(h_i) = h_i'' , \qquad \mu_1(c_i) = c_i'' .$$

Analogously, the canonical injective homomorphism of Lie algebras

$$gl(N) = gl_{n_1} \times gl_{n_2} \longrightarrow gl_{n_1+n_2}$$

induces a graded differential algebra homomorphism

$$\mu_2 : W_{n_1+n_2}(gl_{n_1+n_2}, O_{n_1+n_2}) \longrightarrow W_I(g(N), O_N)$$

and this homomorphism μ_2 acts on the generators as follows :

$$\mu_2(h_i) = h_i' + h_i'' , \qquad \mu_2(c_i) = c_i' + c_i'' .$$

Consequently, and by restricting to the corresponding subalgebras, we obtain two graded differential algebra homomorphisms

$$\mu_1 : WO_{n_2} \longrightarrow WO_I, \qquad \mu_2 : WO_{n_1+n_2} \longrightarrow WO_I .$$

Hereafter, we shall denote μ_k^*, $k = 1, 2$, the induced homomorphism at cohomology level.

Let (F_1, F_2) be a (q_1, q_2)-codimensional subfoliation on a manifold M and put $n_1 = d = q_2 - q_1$, $n_2 = q_1$. Also, denote by

$$\lambda^*_{F_k} : H^*(WO_{q_k}) \longrightarrow H^*_{DR}(M)$$

the characteristic homomorphism of F_k, $k = 1,2$, as defined by Bott [1]. Then, we have

THEOREM 5.2. — *For each* $k = 1,2$, *the diagram*

$$\begin{tikzcd} H^*(WO_{q_k}) \arrow[r, "\mu_k^*"] \arrow[dr, "\lambda^*_{F_k}"'] & H^*(WO_I) \arrow[d, "\lambda^*_{(F_1,F_2)}"] \\ & H^*_{DR}(M) \end{tikzcd}$$

commutes.

Proof. — We shall proceed separately for each value of k.

1. $k = 1$.

Let $\nabla^0 = {}^1\nabla^0 \oplus {}^2\nabla^0$ (respect. $\nabla^1 = {}^1\nabla^1 \otimes {}^2\nabla^1$) be a Riemannian connection (respect. a basic connection) on $\nu(F_1,F_2) = \nu F_{21} \oplus \nu F_1$. Then, ${}^2\nabla^0$ (respect. ${}^2\nabla^1$) is a Riemannian connection (respect. a basic connection) on νF_1. Therefore, from the definitions of $\lambda_{(F_1,F_2)}$, λ_{F_1} and μ_1, we have $\lambda_{F_1} = \lambda_{(F_1,F_2)} \circ \mu_1$, the commutativity of the diagram at the cochain level.

2. $k = 2$.

Let us consider νF_{21}, νF_2 and νF_1 as vector subbundles of TM in such way that $\nu F_2 \cong F_{21} \oplus \nu F_1 = \nu(F_1,F_2)$; that can be done, for example, by using a Riemannian metric to split the exact sequence

$$0 \longrightarrow \nu F_{21} \longrightarrow \nu F_2 \longrightarrow \nu F_1 \longrightarrow 0.$$

Then, let $\{Z_i, i=1,2,\ldots,q_2\}$ be a local basis of sections for νF_2 over an open set $U \subset M$, adapted to such splitting; that is, being $\{Z_a, 1 \leqslant a \leqslant d\}$ (respect. $\{Z_u, d+1 \leqslant u \leqslant q_2\}$) a local basis of sections for νF_{21} (respect. for νF_1); then the change of such local trivializations will be given by a matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A (respect. B) is the matrix of change of local trivialization for νF_{21} (respect. for νF_1).

Now, let us consider an adapted triple $(\nabla^1, \nabla, \nabla^2)$ of basic connections; with respect to the local basis of sections $\{Z_i, 1 \leqslant i \leqslant q_2\}$ above, the matrices of local 1-forms which define these connections are, respectively,

$$\theta^1 = [\theta^1_{ab}], \qquad \theta = \begin{bmatrix} \theta^1_{ab} & 0 \\ \theta_{ub} & \theta^2_{uv} \end{bmatrix}, \qquad \theta^2 = [\theta^2_{uv}]$$

where (θ_{ub}) is non zero in general.

On the other hand, we can consider on νF_2 the connection sum $\bar\nabla = \nabla^1 \oplus \nabla^2$, which is locally defined by the matrix of local 1-forms

$$\bar\theta = \begin{bmatrix} \theta^1_{ab} & 0 \\ 0 & \theta^2_{uv} \end{bmatrix}.$$

Let us remark that $\bar\nabla$ is not a basic connection on νF_2, in general, but it is differentiably $J(> q_2)$-homotopic to the basic connection ∇, in the sense of Lehmann [11], as we shall state in Lemma 5.3 below; therefore, $\bar\nabla$ can be used in order to compute $\lambda^*_{F_2}$.

Keeping all that in mind, the commutativity of the diagram follows directly from the definitions of $\lambda^*_{(F_1,F_2)}$, μ^*_2 and $\lambda^*_{F_2}$, through a straighforward calculation, taking $\bar\nabla = \nabla^1 \oplus \nabla^2$ in the place of a basic connection on νF_2, the same $\bar\nabla$ as a basic connection on $\nu(F_1, F_2)$ and $\nabla^0 = {}^0\nabla^1 \oplus {}^0\nabla^2$ as Riemannian connection in both fiber bundles, ${}^0\nabla^1$ (respect. ${}^0\nabla^2$) being a Riemannian connection on νF_{21} (respect. νF_1).

Q.E.D.

LEMMA 5.3. – *Let* $(\nabla^1, \nabla, \nabla^2)$ *be an adapted triple of basic connections,* $\bar\nabla = \nabla^1 \oplus \nabla^2$ *the connection on* νF_2 *defined as in Theorem 5.2 above. Then,* ∇ *and* $\bar\nabla$ *are differentiably* $J(>q_2)$*-homotopic connections on* νF_2 *(in the sense of Lehmann* [11]*).*

Proof. – First of all, let us remark that both ∇ and $\bar\nabla$ are $J(>q_2)$-connections, because their curvature tensors vanish over F_2.

Now, we define the connection $\nabla' = t\nabla + (1-t)\bar\nabla$ on the vector bundle $\nu F_2 \times I \to M \times I$, $I = [0,1]$. With respect to a local basis of sections $\{Z_i, 1 \leqslant i \leqslant q_2\}$ for this bundle, as above, ∇' is locally defined by the matrix of local 1-forms

$$\begin{bmatrix} \theta^1_{ab} & 0 \\ t\theta_{ub} & \theta^2_{uv} \end{bmatrix}.$$

Therefore, ∇' is also a $J(>q_2)$-connection, because if K' is the curvature form of ∇' and $f \in I^r(gl_{q_2})$, $f(K') = 0$ for $r > q_2$; in fact, locally $f(K')$ is given by

$$f(K') = dt \wedge rf(\theta - \bar{\theta}, K_t) + f(K_t)$$

where K_t denotes the curvature form of the connection for each fixed t. Since $f(\theta - \bar{\theta}, K_t) = 0$ for every r, because

$$\theta - \bar{\theta} = \begin{bmatrix} 0 & 0 \\ t\theta_{ub} & 0 \end{bmatrix}$$

and $f(K_t) = 0$ for $r > q_2$, the result follows immediately.

Q.E.D.

Analogously, we can state :

THEOREM 5.4. – *Let* (F_1,F_2) *be a* (q_1,q_2)*-codimensional subfoliation on* M *with a trivialized normal bundle. Then, for each* $k = 1,2$ *the diagram*

$$\begin{array}{ccc} H^*(W_{q_k}) & \xrightarrow{\mu_k^*} & H^*(W_I) \\ & \searrow^{\lambda^*_{F_k}} & \downarrow{\scriptstyle \lambda^*_{(F_1,F_2)}} \\ & & H^*_{DR}(M) \end{array}$$

commutes.

6. Applications.

In this section, we shall explain two applications of the results and techniques developped in the earlier section.

1. – First, let us remark that Theorems 5.2 and 5.4 allow to approach an answer to Feigin's Question. Obviously, from those Theorems we deduce the following

THEOREM 6.1. — *A necessary condition so that Feigin's Question can have an affirmative answer is the vanishing of those exotic classes of* F *which are obtained from the elements in* $\operatorname{Ker} \mu_2^*$.

In fact, the spirit of the final note in Moussu's article [12] is that of this theorem. On the other hand, Feigin [4] constructs a 2-codimensional foliation with a trivialized normal bundle for which the answer to his Question is negative. Now, by using the results of Yamato [15] and Theorem 5.2, we shall give another new example for which the answer is also negative.

For this purpose, let us remark that the groups $H^r(WO_I)$ have, for $q_1 = 1$, $q_2 = 2$ and $d = 1$, the following dimensions :

$$\dim H^r(WO_I) = \begin{cases} 2 \text{ for } r = 5,6 \\ 1 \text{ for } r = 0,3 \\ 0 \text{ for the remaining } r \end{cases}$$

and, in fact, for $r = 5$, the cohomology classes of $h'_1 \otimes (c'_1)^2$ and $h'_1 \otimes c'_1 \otimes c''_1$ (or its cohomologous $h''_1 \otimes (c'_1)^2$) are generators of $H^5(WO_I)$.

Now, let us recall Yamato's Theorem :

THEOREM [15]. — *For any integer* $q \geqslant 1$, *there exists a q-codimensional foliation* F *on a closed* $(2q+1)$*-manifold* M *such that all the exotic characteristic classes of* F *which correspond to the canonical generators* $[h_j \otimes \varphi]$ *of* $H^{2q+1}(WO_q)$ *are non zero, where* $\varphi \in \mathbf{R}[c_1, \ldots, c_q]$ *is a monomial of degree* $2(q-j+1)$.

Hence, for $q = 2$, the canonical generators of $H^5(WO_2)$ are the cohomology classes of $h_1 \otimes c_1^2$ and $h_1 \otimes c_2$. Hence, Yamato's theorem implies that $\lambda_F^*([h_1 \otimes c_2])$ is not zero, while the class $[h_1 \otimes c_2]$ belongs to $\operatorname{Ker} \mu_2^*$; therefore, *on such a manifold* M *does not exist any* (1,2)*-codimensional subfoliation* (F_1, F_2) *such that* F_2 *be homotopic to the Yamato's 2-codimensional foliation on* M.

2. — Let F_2 be a q_2-codimensional foliation on M and suppose there exist vector fields $Y_1, \ldots, Y_d$ on M such that :

(i) $Y_1, \ldots, Y_d$ are everywhere independent,

(ii) $Y_1, \ldots, Y_d$ are infinitesimal transformations of F_2 and everywhere transverse to F_2,

(iii) F_2 and $Y_1, \ldots, Y_d$ define a new q_1-codimensional foliation F_1

on M, $q_1 = q_2 - d$, and therefore $F_2 \subset F_1$; for this subfoliation (F_1, F_2), the vector bundle νF_{21} is trivial, being its trivialization defined by $Y_1, \ldots, Y_d$.

The techniques used in the proof of Theorem 5.2 and Lemma 5.3., lead us to the following :

THEOREM 6.2. — *Under the hypothesis above, the diagram*

$$\begin{tikzcd} H^*(WO_{q_2}) \arrow[rr, "\lambda^*_{F_2}"] \arrow[dr, "\mu^*"'] & & H^*_{DR}(M) \\ & H^*(WO_{q_1}) \arrow[ur, "\lambda^*_{F_1}"'] & \end{tikzcd}$$

commutes.

Here, μ^* is the homomorphism induced from the canonical injection

$$gl_{q_1} \longrightarrow gl_d \times gl_{q_1} \longrightarrow gl_{q_2}.$$

COROLLARY 6.3. — *The existence of d everywhere independent and transverse infinitesimal transformations of a q-codimensional foliation* F, *satisfying* (iii) *above, implies the vanishing of λ^*_F on the Kernel of*

$$\mu^* : H^*(WO_q) \longrightarrow H^*(WO_{q-d}).$$

Proof of Theorem 6.2. — With the help of a convenable Riemannian metric on M, we may consider the normal vector bundles νF_1, νF_2 and $\nu(F_1, F_2)$ as vector subbundles of TM in such way that $\nu F_2 = \nu F_{21} \oplus \nu F_1$, and νF_{21} being still trivialized by $Y_1, \ldots, Y_d$. In fact, with this identification, the vector fields $Y_1, \ldots, Y_d$ on M are considered as foliated sections for νF_2 and, consequently, the flat connection defined on νF_{21} by this trivialization is a basic connection in the sense of Definition 3.1.

Now, if ${}^0\nabla^2$ and ${}^1\nabla^2$ are, respectively, a metric and a basic connection on νF_1, by using the flat connection on νF_{21}. we get, as in the proof of Theorem 5.2 and in order to compute $\lambda^*_{F_2}$, that the curvature forms of the corresponding connections on νF_2 are expressed, with respect to an adapted basis of sections, by

$${}^0K = \begin{bmatrix} 0 & 0 \\ 0 & {}^0K^2 \end{bmatrix}, \quad {}^1K = \begin{bmatrix} 0 & 0 \\ 0 & {}^1K^2 \end{bmatrix}$$

where ${}^0K^2$ (respect. ${}^1K^2$) is the curvature form of ${}^0\nabla^2$ (respect. of ${}^1\nabla^2$).

The result follows now immediately.

Q.E.D.

The following are two examples where Theorem 6.2 and Corollary 6.3 are applied.

Example 1. – Let G be a d-dimensional Lie group acting locally and freely transverse to a foliation F on M and mapping leaves of F into leaves of F. Then, the Lie algebra of G gives rise to d infinitesimal transformations of F, $Y_1, \ldots, Y_d$, satisfying the hypothesis above. Moreover,

$$[Y_a, Y_b] = \sum_{c=1}^{d} C^c_{ab} Y_c, \quad 1 \leqslant a, b \leqslant d$$

where $C^c_{ab} \in \mathbf{R}$.

Let us remark that this particular case has been firstly considered by Lazarov-Shulman [9], their results being weaker than that of Corollary 6.3. In fact, Lazarov-Shulman announce the result of Corollary 6.3 for the most particular case where $C^c_{ab} = 0$, $1 \leqslant a, b, c \leqslant d$ ([10]).

Example 2. – Let $\pi : P \to M$ be a foliated principal bundle, F being the q-codimensional foliation on M and $\tilde{F}$ the foliation on P. Let us consider the canonical subfoliation $(\pi^{-1}F, \tilde{F})$ on P; then, the following diagram commutes

$$\begin{tikzcd}
H^*(WO_{q'}) \arrow[r, "\lambda^*_{\tilde{F}}"] \arrow[d, "\mu^*"'] \arrow[dr, phantom, "\textcircled{1}", near start] & H^*_{DR}(P) \\
H^*(WO_q) \arrow[r, "\lambda^*_F"'] \arrow[ur, "\lambda^*_{\pi^{-1}F}"'] & H^*_{DR}(M) \arrow[u, "\pi^*"'] \arrow[ul, phantom, "\textcircled{2}", near start]
\end{tikzcd}$$

The commutativity of ① is consequence of Theorem 6.2 and that of ② is given by the naturality of the exotic homomorphism of a foliation. That means

$$\lambda^*_{\tilde{F}} = \pi^* \circ \lambda^*_F \circ \mu^*.$$

If, moreover, νF is a trivialized vector bundle, we have a similar commutative diagram

$$\begin{tikzcd}
H^*(W_{q'}) \arrow[r, "\lambda^*_{\tilde{F}}"] \arrow[d, "\mu^*"'] & H^*_{DR}(P) \\
H^*(W_q) \arrow[r, "\lambda^*_F"] & H^*_{DR}(M) \arrow[u, "\pi^*"']
\end{tikzcd}$$

Now, let us remark that, in both cases, if π^* is injective, the vanishing of an exotic class of $\tilde{F}$ implies the vanishing of the corresponding one of F. A situation where this is applied is the following: suppose νF trivialized and P being the principal bundle of transverse references of F; then π^* is injective because P is topologically a product bundle.

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Manuscrit reçu le 24 avril 1980
révisé le 13 novembre 1980.

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