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BAHMAN SAFFARI
R. C. VAUGHAN

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# ON THE FRACTIONAL PARTS OF $x/n$ AND RELATED SEQUENCES. I

by B. SAFFARI and R. C. VAUGHAN

## 1. Introduction.

**1.** Throughout this paper $\{x\} = x - [x]$ denotes the fractional part of the real number $x$. We write $\|x\| = \min_{k \in \mathbf{Z}} |x - k|$ and $e(x) = e^{2\pi i x}$.

Also, the implied constants in the O symbol of Landau and the $\gg$ and $\ll$ symbols of Vinogradov are absolute.

Finally, by a distribution function we always mean a distribution function in the sense of probability theory, defined on the real line.

**2.** Let $(x_n)$ be a sequence of real numbers. The usual study of the distribution modulo 1 of $(x_n)$ is essentially that of the distribution of the sequence $(e(x_n))$ on the circle $\mathbf{T}$. The main problems are those of investigating

(i) the existence of the asymptotic (or limit) distribution measure

$$\mu = \lim_{k \to \infty} \mu_k \tag{1.1}$$

where

$$\mu_k = \frac{1}{k} \sum_{n=1}^{k} \delta_{e(x_n)} \tag{1.2}$$

with $\delta_v$ denoting the Dirac measure at $v \in \mathbf{T}$, and

(ii) the size of the discrepancy

$$\sup_{\omega} |\mu_k(\omega) - \mu(\omega)| \tag{1.3}$$

where $\omega$ runs through those arcs of $\mathbf{T}$ whose end points have $\mu$-measure zero.

It is classical that the existence of $\mu$ *together* with the assumption that the point $1 \in \mathbf{T}$ has $\mu$-measure zero is equivalent to the existence of a distribution function F such that

$$\text{(1.4)} \qquad F(0+) = 0, \qquad F(1-) = 1$$

and

$$\text{(1.5)} \qquad F(\alpha) = \lim_{k \to \infty} \frac{1}{k} A([0, \alpha), k, (x_n))$$

at every $\alpha$ at which F is continuous, the counting function

$$\text{(1.6)} \quad A([\alpha, \beta), k, (x_n)) = \text{Card}\,\{n : 1 \leqslant n \leqslant k, \alpha \leqslant \{x_n\} < \beta\}$$

being here defined for all real numbers $\alpha$ and $\beta$. The conditions (1.4) mean that F is continuous at 0 and 1, and imply that F is constant on the intervals $(-\infty, 0]$ and $[1, \infty)$. In this case F is called the asymptotic (or limit) distribution function modulo 1 of the sequence $(x_n)$, and the discrepancy (1.3) is equal to

$$\text{(1.7)} \qquad \sup_{0 \leqslant \alpha < \beta \leqslant 1} \left| \frac{1}{k} A([\alpha, \beta), k, (x_n)) - (F(\beta) - F(\alpha)) \right|$$

where $\alpha$ and $\beta$ run through the continuity points of F.

In some situations it may be more appropriate to consider the existence of the A-asymptotic distribution function modulo 1, namely the existence (outside a countable set), *and* the continuity at $\alpha = 0$ and $\alpha = 1$, of

$$\text{(1.8)} \qquad \lim_{k \to \infty} \sum_{n=1}^{k} a_{k,n} c_\alpha(x_n)$$

where

$$\text{(1.9)} \qquad c_\alpha(u) = \begin{cases} 1 & 0 \leqslant \{u\} < \alpha \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function modulo 1 of $[0, \alpha)$, and $A = (a_{k,n})$ is a positive Toeplitz matrix. Here by a positive Toeplitz matrix we mean that

$$a_{k,n} \geqslant 0, \sum_{n=1}^{\infty} a_{k,n} < \infty \qquad \text{and} \qquad \lim_{k \to \infty} \sum_{n=1}^{\infty} a_{k,n} = 1.$$

**3.** The sequence $(x_n)$ is, of course, independant of $k$. Our object is to investigate the distribution modulo 1 of $xh(n)$ with $x$ a large real number, $h(n)$ an arithmetical function, and the integer $n$ belonging to $S \cap [1, k]$ where $S \subset \mathbf{N}$ and $k$ depends on $x$. For our purposes it is somewhat more convenient to replace $k$ by a real parameter $y$. We call $\mathscr{A} = (a_n(y) : y \in [1, \infty),\ n = 1, 2, \ldots)$ a positive Toeplitz transformation if $a_n(y) \geqslant 0$ for all $n$ and $y$, $\sum_{n=1}^{\infty} a_n(y) < \infty$ for every $y$, and $\lim_{y\to\infty} \sum_{n=1}^{\infty} a_n(y) = 1$. We are particularly interested in the special case where the $a_n(y)$ are the simple Riesz means $(R, \lambda_n)$ given by

$$(1.10) \qquad \lambda_n \geqslant 0 \ (n = 1, 2, \ldots), \qquad \lambda_1 > 0$$

and

$$(1.11) \qquad a_n(y) = \begin{cases} \lambda_n \Big/ \sum_{m \leqslant y} \lambda_m & (m \leqslant y) \\ 0 & (m > y) \end{cases}$$

which we assume henceforward, although several of our proofs go through in the general case (see Appendix). Let

$$(1.12) \qquad \Phi_{x,y}(\alpha, h) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(xh(n)).$$

A good deal of our attention will be taken up with $h(n) = 1/n$ and we write

$$(1.13) \qquad \Phi_{x,y}(\alpha) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(x/n).$$

The problems arising from the study of $\Phi_{x,y}(\alpha)$ as $x$ and $y = y(x)$ tend together to infinity are closely related to the Dirichlet divisor problem.

If there exists a distribution function $\Phi_h$ such that

$$(1.14) \qquad \Phi_h(0+) = 0, \qquad \Phi_h(1-) = 1$$

and

$$(1.15) \qquad \Phi_h(\alpha) = \lim_{x\to\infty} \Phi_{x,y(x)}(\alpha, h)$$

at every $\alpha$ at which $\Phi_h$ is continuous, then we call $\Phi_h$ the

$\mathscr{A}$-asymptotic distribution function modulo 1. This situation is equivalent to the existence on the circle **T** of the $\mathscr{A}$-limit (or $\mathscr{A}$-asymptotic) distribution measure

$$(1.16) \qquad \nu = \lim_{x \to \infty} \sum_{n=1}^{\infty} a_n(y)\, \delta_{e(xh(n))}$$

*together* with the fact that the point $1 \in \mathbf{T}$ has $\nu$-measure zero. However, if there exists no distribution function $\Phi_h$ satisfying both (1.14) and (1.15), then it is more appropriate to investigate the distribution modulo 1 of $xh(n)$ via (1.16).

**4**. Our interest in this problem arose from investigating the asymptotic behaviour of

$$\sum_{n \leqslant y} c_\alpha(x/n).$$

During our investigation it became obvious that there were methods which could be applied in a much more general situation. In this paper we present these methods, deferring to the sequel the study of special methods.

As an example of the application of Theorem 2, consider a subset A of $\mathbf{N}^*$ such that the counting function

$$\mathrm{A}(x) = \sum_{\substack{a \leqslant x \\ a \in \mathrm{A}}} 1$$

satisfies

$$\mathrm{A}(x) = x^\sigma \mathrm{L}(x)$$

where $\sigma$ is a constant with $0 < \sigma \leqslant 1$ and L is a slowly varying function, that is

$$\lim_{x \to \infty} \frac{\mathrm{L}(cx)}{\mathrm{L}(x)} = 1$$

for any positive constant $c$. Then

$$(1.17) \qquad \lim_{x \to \infty} \frac{1}{\mathrm{A}(x)} \sum_{\substack{a \leqslant x \\ a \in \mathrm{A}}} c_\alpha(x/a) = \sum_{n=1}^{\infty} (n^{-\sigma} - (n + \alpha)^{-\sigma}).$$

Moreover, there exists a function $y_0(x)$ such that if $y > y_0(x)$

and $y = o(x)$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{1}{A(y)} \sum_{\substack{a \leqslant y \\ a \in A}} c_\alpha(x/a) = \alpha. \tag{1.18}$$

Relation (1.18) means that the fractional parts $\{x/a\}$, where $a$ runs over $[0, y] \cap A$, are asymptotically uniformly distributed, whereas (1.17) means that if $a$ runs over the whole of $[0, x] \cap A$, then the $\{x/a\}$ have the asymptotic distribution function

$$\sum_{n=1}^{\infty} (n^{-\sigma} - (n + \alpha)^{-\sigma}).$$

## 2. Theorems and proofs.

**1.** We first of all state a theorem which gives a sufficient condition for the $(R, \lambda_n)$-asymptotic distribution to be uniform. This is essentially due to Erdös and Turan [1], [2] and is a finite form of Weyl's criterion. It is also possible, of course, to give a necessary condition corresponding to Weyl's criterion, and to give results when the asymptotic distribution is non-uniform but continuous, but we have no applications in mind for these.

Theorem 1 is somewhat divorced from the following theorems. However, it clearly applies to the general situation. As an application we have in mind the case

$$h(n) = \log n. \tag{2.1}$$

THEOREM 1. — *Let the discrepancy* $D_{x,y}(h)$ *be defined by*

$$D_{x,y}(h) = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |\Phi_{x,y}(\beta, h) - \Phi_{x,y}(\alpha, h) - (\beta - \alpha)|. \tag{2.2}$$

*Then, for any positive integer* $m$,

$$D_{x,y}(h) < \frac{6}{m+1} + \frac{4}{\pi} \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{m+1}\right) \left|\sum_{n=1}^{\infty} a_n(y) e(kxh(n))\right|. \tag{2.3}$$

Theorem 1 is a generalization of Theorem 2.2.5 of Kuipers and Niederreiter [3], and can be proved in exactly the same way.

2. The following theorem (together with the observations made in Lemmas 2, 3, 4) shows that the $(\mathrm{R}, \lambda_n)$ asymptotic distribution function modulo 1 of $x/n$ can exist under very general conditions provided that $y$ is not too small compared with $x$.

Whenever $\xi \geqslant 1$ and $\sigma \geqslant 0$ define

$$(2.4)\qquad \mathrm{F}(\alpha;\xi,\sigma) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha;\xi)(1-\xi^\sigma([\xi]+\alpha)^{-\sigma}) & \\ \qquad + \xi^\sigma \sum_{k>\xi} (k^{-\sigma}-(k+\alpha)^{-\sigma}) & (0<\alpha<1, \sigma>0) \\ \alpha & (0<\alpha<1,\ \sigma=0) \end{cases}$$

where

$$(2.5)\qquad \theta(\alpha;\xi) = \begin{cases} 1 & \text{if } (\xi-\alpha, \xi] \cap \mathbf{N} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. — *Suppose that for every real number* $t$ *with* $0 < t < 1$ *the limit*

$$(2.6)\qquad \lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y)$$

*exists and for at least one value of* $t$ *is non-zero. Then there is a non-negative real number* $\sigma$ *such that for every real number* $\varepsilon$ *with* $0 < \varepsilon < \frac{1}{2}$ *there is a real number* $y_0(\varepsilon, \sigma) \geqslant 1$ *so that whenever* $y_0(\varepsilon,\sigma) \leqslant y \leqslant x$ *we have*

$$(2.7)\qquad \Phi_{x,y}(\alpha) = \mathrm{F}(\alpha; x/y, \sigma) + \mathrm{O}(\varepsilon^{1+\sigma}xy^{-1}) + \mathrm{O}(2^\sigma\varepsilon^\sigma).$$

Lemma 1 below will show that the limit (2.6) is $t^\sigma$, which defines $\sigma$. We observe that when $\sigma = 0$ Theorem 2 fails to give non-trivial information. Very likely $\Phi_{x,y}(\alpha) \to \alpha$ still holds in this case, at least when $\sum_{n\leqslant y} \lambda_n \to \infty$, but even when $\lambda_n = 1/n$ this is a deep result.

Before proceeding with the proof of Theorem 2 we state a corollary concerning the case when the integer $n$ is allowed only to run through a shorter interval $[y, z]$.

COROLLARY 2.1. — *With the assumptions of Theorem* 2, *if*

$$y_0(\varepsilon, \sigma) \leqslant y < z \leqslant x/2, \ (y/z)^\sigma < 1 - \varepsilon^{2+\sigma}, \ \varepsilon^z \leqslant y,$$

*and* $\sum_{y<n\leqslant z} \lambda_n > 0$, *then*

$$(2.8) \quad \frac{\sum_{y<n\leqslant z} \lambda_n c_\alpha(x/n)}{\sum_{y<n\leqslant z} \lambda_n} - \alpha \leqslant (\sigma 2^\sigma z x^{-1} + \varepsilon^{1+\sigma} x y^{-1} + 2^\sigma \varepsilon^\sigma)(1 - y^\sigma z^{-\sigma} - \varepsilon^{2+\sigma})^{-1}.$$

We remark that, in this case, the asymptotic distribution is always the uniform one, at least when $\sigma > 0$.

**3**. The proof of Theorem 2 requires the following lemma.

LEMMA 1. — *On the hypothesis of Theorem* 2 *there is a non-negative real number* $\sigma$ *such that for every real number* $\varepsilon$ *with* $0 < \varepsilon < 1/2$ *there is a real number* $y_0(\varepsilon, \sigma) \geqslant 1$ *so that whenever* $y \geqslant y_0(\varepsilon, \sigma)$ *we have, for every* $t$ *with* $\varepsilon \leqslant t \leqslant 1$,

$$(2.9) \qquad \Big| t^\sigma - \sum_{n\leqslant ty} a_n(y) \Big| < \varepsilon^{2+\sigma}.$$

*Proof.* — The existence of (2.6) for every real number $t$ with $0 < t < 1$ together with the assumption that for some $t$ in this range the limit is non-zero imply that there is a non-negative real number $\sigma$ such that for every $t$ with $0 < t \leqslant 1$ we have

$$\lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y) = t^\sigma.$$

Let

$$N = [2e^{\varepsilon^{-2-\varepsilon}} \max(1, \sigma)] + 1$$

and choose $y_0(\varepsilon, \sigma) \geqslant 1$ so that if $y \geqslant y_0(\varepsilon, \sigma)$, then for every integer $r$ with $1 \leqslant r \leqslant N$ we have

$$(2.10) \qquad \left| \left(\frac{r}{N}\right)^\sigma - \sum_{n\leqslant ry/N} a_n(y) \right| < \frac{1}{2}\varepsilon^{2+\sigma}.$$

Now choose an integer $q$ such that

$$(2.11) \qquad \frac{1}{N} \leqslant \frac{q}{N} < t \leqslant \frac{q+1}{N} \leqslant 1,$$

which is always possible if $\varepsilon \leqslant t \leqslant 1$. Note that

$$\left(\frac{q+1}{\mathrm{N}}\right)^{\sigma} - \left(\frac{q}{\mathrm{N}}\right)^{\sigma} = \int_{q/\mathrm{N}}^{(q+1)/\mathrm{N}} \sigma u^{\sigma-1}\, du$$

$$\leqslant \frac{\sigma}{\mathrm{N}} \max\left(\left(\frac{q+1}{\mathrm{N}}\right)^{\sigma-1}, \left(\frac{q}{\mathrm{N}}\right)^{\sigma-1}\right)$$

$$\leqslant \sigma \max(\mathrm{N}^{-1}, \mathrm{N}^{-\sigma}) \leqslant \max(\sigma \mathrm{N}^{-1}, (e \log \mathrm{N})^{-1})$$

$$< \frac{1}{2}\varepsilon^{2+\sigma}.$$

Thus, by (2.10) and (2.11),

$$\sum_{n \leqslant ty} a_n(y) \leqslant \sum_{n \leqslant (q+1)y/\mathrm{N}} a_n(y) < \left(\frac{q+1}{\mathrm{N}}\right)^{\sigma} + \frac{1}{2}\varepsilon^{2+\sigma}$$

$$< \left(\frac{q}{\mathrm{N}}\right)^{\sigma} + \varepsilon^{2+\sigma} \leqslant t^{\sigma} + \varepsilon^{2+\sigma}$$

and

$$\sum_{n \leqslant ty} a_n(y) \geqslant \sum_{n \leqslant qy/\mathrm{N}} a_n(y) > \left(\frac{q}{\mathrm{N}}\right)^{\sigma} - \frac{1}{2}\varepsilon^{2+\sigma}$$

$$> \left(\frac{q+1}{\mathrm{N}}\right) - \varepsilon^{2+\sigma} \geqslant t^{\sigma} - \varepsilon^{2+\sigma}.$$

These last two inequalities give (2.9) as required.

**4.** *Proof of Theorem* 2. — Since (2.7) is trivially true when $\alpha \leqslant 0$ or $\alpha \geqslant 1$, we may assume $0 < \alpha < 1$. Let

$$\mathrm{K} = \left[\frac{x}{\varepsilon y} - \alpha\right].$$

Then, by (1.13), (1.11), (1.9), Lemma 1 and (2.5),

$$\Phi_{x,y}(\alpha) = \sum_{\frac{x}{\mathrm{K}+\alpha} < n \leqslant y} a_n(y) c_\alpha(x/n) + \mathrm{O}\Big(\sum_{n \leqslant 2\varepsilon y} a_n(y)\Big)$$

$$= \sum_{k=1}^{\mathrm{K}} \sum_{\substack{n \leqslant y \\ x/(k+\alpha) < n \leqslant x/k}} a_2(y) + \mathrm{O}(2^{\sigma}\varepsilon^{\sigma})$$

$$= \theta(\alpha\,; x/y)\Big(\sum_{n \leqslant y} a_n(y) - \sum_{n \leqslant x/([x/y]+\alpha)} a_n(y)\Big)$$

$$+ \sum_{x/y \leqslant k \leqslant \mathrm{K}} \Big(\sum_{n \leqslant x/k} a_n(y) - \sum_{n \leqslant x/(k+\alpha)} a_n(y)\Big) + \mathrm{O}(2^{\sigma}\varepsilon^{\sigma}).$$

Hence, by Lemma 1 and (2.4),

$$\Phi_{x,y}(\alpha) = F(\alpha; x/y, \sigma) + O(\varepsilon^{2+\sigma}K) + O(2^\sigma\varepsilon^\sigma) + O\left(\sum_{k>K}\left(\frac{x}{y}\right)^\sigma(k^{-\sigma} - (k+\alpha)^{-\sigma}\right).$$

The proof of (2.7) is completed by observing that $\varepsilon K \leqslant x/y$ and

$$\sum_{k>K}(k^{-\sigma} - (k+\alpha)^{-\sigma}) = \sum_{k>K}\int_k^{k+\alpha}\sigma u^{-\sigma-1}\,du \leqslant \sum_{k>K}\int_k^{k+1}\sigma u^{-\sigma-1}\,du = (K+1)^{-\sigma} < (2\varepsilon y/x)^\sigma.$$

5. *Proof of Corollary* 2.1. — We use (2.7) and Lemma 1. The condition that $(y/z)^\sigma < 1 - \varepsilon^{2+\sigma}$ means that we can assume that $\sigma > 0$. Suppose that $\xi > 1$. Then, by (2.4),

$$\begin{aligned} F(\alpha; \xi, \sigma) &\leqslant \xi^\sigma\int_{[\xi]}^{[\xi]+\alpha} u^{-\sigma}\,du + O\left(\theta(\alpha;\xi)\int_{\xi/([\xi]+\alpha)}^1 \sigma u^{\sigma-1}\,du\right) \\ &\leqslant \alpha(\xi/[\xi])^\sigma \\ &\quad + O\left(\theta(\alpha;\xi)\sigma(1 - \xi/([\xi]+\alpha))\max\left(1, \left(\frac{\xi}{[\xi]+\alpha}\right)^{\sigma-1}\right)\right) \\ &= \alpha + O(\sigma 2^\sigma \xi^{-1}). \end{aligned}$$

Similarly

$$F(\alpha; \xi, \sigma) \geqslant \xi^\sigma\int_{\xi+1}^{\xi+1+\alpha} u^{-\sigma}\,du \geqslant \alpha\left(1 + \frac{1+\alpha}{\xi}\right)^{-\sigma} \geqslant \alpha - \frac{\sigma\alpha(1+\alpha)}{\xi}.$$

Hence, if $y_0(\varepsilon, \sigma) \leqslant y \leqslant x/2$, then by (1.11), (1.13) and (2.7),

$$\sum_{n\leqslant y}\lambda_n c_\alpha(x/n) = (\alpha + O(\sigma 2^\sigma y x^{-1} + x\varepsilon^{1+\sigma}y^{-1} + 2^\sigma\varepsilon^\sigma))\sum_{n\leqslant y}\lambda_n.$$

Thus, if $y_0(\varepsilon, \sigma) \leqslant y < z \leqslant \frac{1}{2}$, then

$$\sum_{y<n\leqslant z}\lambda_n c_\alpha(x/n) = \alpha\sum_{y<n\leqslant z}\lambda_n + O\left((\sigma 2^\sigma y x^{-1} + x_\varepsilon{}^{1+\sigma}y^{-1} + 2^\sigma\varepsilon^\sigma)\sum_{n\leqslant y}\lambda_n\right).$$

We complete the proof of (2.8) by observing that by (1.11) and Lemma 1,

$$\Big(\sum_{n\leqslant z}\lambda_n\Big)\Big/\sum_{y<n\leqslant z}\lambda_n = \Big(1-\Big(\sum_{n\leqslant y}\lambda_n\Big)\Big/\sum_{n\leqslant z}\lambda_n\Big)^{-1} < (1-(y/z)^{\sigma}-\varepsilon^{2+\sigma})^{-1}.$$

**6.** In this section we make some observations concerning the nature of $F(\alpha;\xi,0)$.

Lemma 2. — *Suppose that* $0\leqslant\alpha\leqslant 1$ *and* $\xi\geqslant 1$. *Then*

$$F(\alpha;\xi,\sigma) = \alpha + O(\sigma 2^{\sigma}\xi^{-1}) \quad (\sigma>0), \tag{2.12}$$

$$\lim_{\sigma\to 0^+} F(\alpha;\xi,\sigma) = \alpha = F(\alpha;\xi,0) \tag{2.13}$$

*and*

$$F(\alpha;1,\sigma) = \sum_{k=1}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma}) \quad (\sigma>0). \tag{2.14}$$

By (2.14) with $\sigma=1$, $F(\alpha;1,1)=\Gamma'(\alpha)/\Gamma(\alpha)+\gamma+1/\alpha$ where $\Gamma$ is the gamma function and $\gamma$ is Euler's constant.

*Proof.* — The asymptotic formula (2.12) was established in the proof of (2.8), (2.13) then follows trivially, and (2.14) is immediate from (2.4).

Lemma 3. — *For each* $\xi\geqslant 1$ *and* $\sigma>0$ *the function* $F(\alpha;\xi,0)$ *is a continuous function of* $\alpha$ *and is analytic on* $\mathbf{R}\setminus\{0,\{\xi\},1\}$ *with*

(2.15)

$$F'(\alpha) = \begin{cases} 0 & (\alpha<0,\ \alpha>1)\\ \sigma\xi^{\sigma}\sum_{k>\xi}(k+\alpha)^{-\sigma-1} & (0<\alpha<\{\xi\})\\ \sigma\xi^{\sigma}([\xi]+\alpha)^{-\sigma-1}+\sigma\xi^{\sigma}\sum_{k>\xi}(k+\alpha)^{-\sigma-1} & (\{\xi\}<\alpha<1).\end{cases}$$

*The points* $0$, $\{\xi\}$ *and* $1$ *are angular points of* $F$.

Lemma 4. — *Suppose that* $0<\alpha<1$ *and* $\sigma>0$. *Then considered as a function of* $\xi$, $F(\alpha;\xi,\sigma)$ *is continuous on* $[1,\infty)\setminus\{2,3,4,\ldots\}$ *and for each integer* $n\geqslant 2$,

$$\lim_{\xi\to n-} F(\alpha;\xi,\sigma) = n^{\sigma}\sum_{k=n+1}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma}) \tag{2.16}$$

*and*

$$(2.17)\quad \lim_{\xi \to n+} F(\alpha; \xi, \sigma) = n^\sigma \sum_{k=n}^{\infty} (k^{-\sigma} - (k+\alpha)^{-\sigma}) = F(\alpha; n, \sigma).$$

**7.** We now establish upper and lower bounds for the mean square of $\Phi_{x,y}(\alpha) - \alpha$ which in turn imply respectively

(i) that if $y$ is small compared with $x$ then the only possible $(R, \lambda_n)$ asymptotic distribution modulo 1 is the uniform one, and

(ii) that the discrepancy cannot be too small.

THEOREM 3. — *Suppose that $x_0$ and $x$ are non-negative real numbers, $y \geqslant 1$ and $0 < \alpha < 1$. Then*

$$(2.18)\quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 \, du \leqslant \min (I_1, I_2)$$

*where*

$$(2.19)\quad I_1 = \frac{1}{3}(x + y^2) \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2$$

*and*

$$(2.20)\quad I_2 = \sum_{n=1}^{\infty} \left( \frac{1}{3} x + \frac{1}{2} yn \right) \left( \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2.$$

This theorem can be thought of in a rather loose way as a law of the iterated logarithm. This will be discussed further in a later paper. (See [5]).

THEOREM 4. — *On the hypothesis of Theorem 3,*

$$(2.21)\quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 \, du \geqslant \max (J_1, J_2)$$

*where*

$$(2.22)\quad J_1 = \frac{1}{2} \pi^{-2}(x - y^2) \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y)(1 - e(\alpha m)) \right|^2$$

*and*

$$(2.23)\quad J_2 = ((2\pi)^{-2} \sum_{n=1}^{\infty} (2x - 3yn) \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y)(1 - e(\alpha m)) \right|^2.$$

By taking the real part of the innermost sum in (2.22) and

(2.23) and then discarding all the terms with $m > 1$ one obtains in (2.21) the particularly simple lower bound $\max(L_1, L_2)$, where

$$L_1 = 2\pi^{-2}(\sin \pi\alpha)^4(x - y^2) \sum_{n=1}^{\infty} a_n^2(y)$$

and

$$L_2 = \pi^{-2}(\sin \pi\alpha)^4 \sum_{n=1}^{\infty} (2x - 3yn)a_n^2(y).$$

However, in certain circumstances this loses a factor as large as loglog $y$.

COROLLARY 4.1. — *Let the discrepancy* $D_{x,y}$ *be given by*

$$(2.24) \quad D_{x,y} = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |\Phi_{x,y}(\beta) - \Phi_{x,y}(\alpha) - (\beta - \alpha)|.$$

*Then*

$$(2.25) \qquad \int_{x_0}^{x+x_0} D_{u,y}^2 \, du \geqslant \sup_{\alpha \in [0,1]} \max(J_1, J_2).$$

By analogous methods it is possible to obtain corresponding inequalities for

$$\sum_{n=M+1}^{M+N} |\Phi_{n,y}(\alpha) - \alpha|^2$$

but the bounds obtained are more complicated and not so illuminating.

**8.** To prove Theorems 3 and 4 we require the following lemma which is Theorem 2 of Montgomery and Vaughan [4].

LEMMA 5. — *Suppose that* $x_1, x_2, \ldots, x_R$ *are* R *distinct real numbers, and that* $v_1, v_2, \ldots, v_R$ *are* R *complex numbers. Also, let*

$$(2.26) \quad \delta = \min_{\substack{r,s \\ r \neq s}} |x_r - x_s| \quad \text{and} \quad \delta_r = \min_{\substack{s \\ s \neq r}} |x_r - x_s|.$$

*Then*

$$(2.27) \qquad \left| \sum_{r=1}^{R} \sum_{\substack{s=1 \\ r \neq s}}^{R} \frac{v_r \bar{v}_s}{x_r - x_s} \right| \leqslant \pi \min(K_1, K_2)$$

*where*

$$K_1 = \delta^{-1} \sum_{r=1}^{R} |\varphi_r|^2 \tag{2.28}$$

*and*

$$K_2 = \frac{3}{2} \sum_{r=1}^{R} |\varphi_r|^2 \delta_r^{-1}. \tag{2.29}$$

**9.** *Proofs of Theorems* 3 *and* 4. — Let K be a positive integer. Then it is easily seen that the function $c_\alpha(u)$ given by (1.9) can be written in the form.

$$c_\alpha(u) = \alpha + \sum_{0<|k|\leqslant K} \frac{1 - e(-\alpha k)}{2\pi i k} e(uk) + O\left(\min\left(1, \frac{1}{K\|u\|}\right)\right) + O\left(\min\left(1, \frac{1}{K\|u-\alpha\|}\right)\right). \tag{2.30}$$

Clearly

$$\int_{x_0}^{x_0+x} \min\left(1, \frac{1}{K\left\|\frac{u}{n} - \beta\right\|}\right) du \ll (x+n)\frac{\log K}{K} \quad (0 \leqslant \beta \leqslant 1). \tag{2.31}$$

Hence, by (1.9) and (2.30),

$$\int_{x_0}^{x_0+x} \left|\sum_{n=1}^{\infty} a_n(y) c_\alpha(u/n) - \alpha\right|^2 du = I + O\left((x+y)\frac{\log K}{K}\right) \tag{2.32}$$

where

$$I = \int_{x_0}^{x_0+x} \left|\sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} \left(\sum_{m \leqslant K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha k m))}{2\pi i k m}\right) e\left(\frac{uk}{n}\right)\right|^2 du. \tag{2.33}$$

Clearly, if $n_j \leqslant y$, $0 < |k_j| \leqslant K$, $(n_j, k_j) = 1$ for $j = 1, 2$ and $k_1/n_1 \neq k_2/n_2$, then $|k_1/n_1 - k_2/n_2| \geqslant 1/(yn_1) \geqslant y^{-2}$.

Therefore, by (2.33) and Lemma 5,

$$I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} (x + \theta_1 y^2) \left|\sum_{m \leqslant K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha k m))}{2\pi i k m}\right|^2 \tag{2.34}$$

and

(2.35)

$$I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} \left(x + \frac{3}{2}\theta_2 ny\right) \left| \sum_{m\leqslant K/|k|} \frac{a_{nm}(y)(1-e(-\alpha km))}{2\pi ikm} \right|^2$$

where $|\theta_1| \leqslant 1$, $|\theta_2| \leqslant 1$. Theorem 3 now follows from (2.32) on letting $K \to \infty$. Theorem 4 follows in the same way on discarding all the terms with $|k| \neq 1$.

Sometimes, when the simple Riesz means $(R, \lambda_n)$ are specified, it may be more appropriate to use (2.34) and (2.35) rather than appeal to Theorems 3 and 4.

**10.** By (2.7), (2.8) and (2.13) we see that if $y$ is small compared with $x$ but not too small, then under very general conditions

$$\lim_{x\to\infty} \Phi_{x,y(x)}(\alpha) = \alpha. \tag{2.36}$$

We now show, as a consequence of Theorem 3, and again under very general conditions, that even if $y$ is very small compared with $x$, then (2.36) still holds.

THEOREM 5. — *Suppose that* $0 < \theta < 1$, $0 < \alpha < 1$,

(2.37)

$$\lim_{y\to\infty} \left( \left(1 + y^{\frac{3\theta-1}{2\theta}}\right) \left( \sum_{n\leqslant y - y^{(3\theta-1)/2\theta}} \lambda_n \right)^{-2} \sum_{n\leqslant y} \left( \sum_{m\leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2 \right) = 0$$

*and*

$$\lim_{x\to\infty} \Phi_{x,x^\theta}(\alpha) \tag{2.38}$$

*exists, Then*

$$\lim_{x\to\infty} \Phi_{x,x^\theta}(\alpha) = \alpha. \tag{2.39}$$

We remark that (2.37) is rather a weak condition. For instance, if $\lambda_n = 1$ for every $n$, then it holds for every $\theta$ with $0 < \theta < 1$.

*Proof.* — Let $y$ be large and define $z = y - y^{(3\theta-1)/2\theta}$. Then by Theorem 3, (1.13) and (1.11),

$$(2.40)\quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n \leqslant z} \lambda_n \left( c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll (y^2 + y^{1/\theta} - z^{1/\theta}) \sum_{n \leqslant y} \left( \sum_{m \leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

Furthermore, by Cauchy's inequality (inégalité de Schwarz en français !),

$$\int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{z < n \leqslant u^\theta} \lambda_n \left( c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll (y^{1/\theta} - z^{1/\theta})(1 + y - z) \sum_{n \leqslant y} \lambda_n^2.$$

Hence, by (2.40),

$$(2.41)\quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n \leqslant u^\theta} \lambda_n \left( c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll \left( y^2 + (y^{1/\theta} - z^{1/\theta}) \left(1 + y^{\frac{3\theta-1}{2\theta}}\right) \right) \sum_{n \leqslant y} \left( \sum_{m \leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

It is easily verified that

$$y^2 \ll (y^{1/\theta} - z^{1/\theta}) y^{(3\theta-1)/2\theta}.$$

Thus, by (2.41) and (2.37),

$$\inf_{z^{1/\theta} \leqslant u \leqslant y^{1/\theta}} |\Phi_{u, u^\theta}(\alpha) - \alpha| \to 0 \quad \text{as} \quad y \to \infty.$$

This gives the desired result.

## 3. Appendix.

**1.** Theorem 1 does not require that the $a_n(y)$ be the simple Riesz means $(R, \lambda_n)$. It is valid provided that

$$\sum_{n=1}^{\infty} a_n(y) = 1.$$

**2.** Theorem 2 can be generalized in the following way. We say that the positive Toeplitz transformation $\mathscr{A} = (a_n(y))$ has asymptotic (or limit) distribution function $\varphi$ with respect to the ordinary Cesaro method $(C, 1)$ if there exists a distribution function $\varphi$ such that

$$(3.1)\quad \lim_{y \to \infty} \sum_{n \leqslant ty} a_n(y) = \varphi(t)$$

at every $t$ at which $\varphi$ is continuous. For example, if the $a_n(y)$ are the simple Riesz means $(R, \lambda_n)$ and if $\varphi$ exists, then by Lemma 1 it is either a continuous function given by

$$(3.2) \qquad \varphi(t) = \begin{cases} 0 & (t \leqslant 0) \\ t^\sigma & (0 < t < 1) \quad (\text{with } \sigma > 0), \\ 1 & (t \geqslant 1) \end{cases}$$

or is one of the « Heaviside » functions $Y_0$ and $Y_1$, where $Y_a(t) = 0$ if $t < a$, $Y_a(t) = 1$ if $t \geqslant a$. (In the general case, necessarily $\varphi(t) = 0$ for $t < 0$). On examining the proof of Theorem 2, one sees that provided $\varphi$ exists, is continuous and satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, then it is possible to replace Theorem 2 by a similar but more general statement. In particular $F(\alpha; \xi, \sigma)$ is to be replaced by

$$(3.3) \quad G(\alpha; \xi, \varphi) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha, \xi)\left(1 - \varphi\left(\frac{\xi}{[\xi]+\alpha}\right)\right) + \sum_{k>\xi}\left(\varphi\left(\frac{\xi}{k}\right) - \varphi\left(\frac{\xi}{k+\alpha}\right)\right) \\ \quad (\text{when } 0 < \alpha < 1), \end{cases}$$

but some care is needed with the error terms. Besides the above example where $\varphi$ is given by (3.2), there are other interesting instances in which $\varphi$ exists.

**3.** Theorems 3 and 4 do not require the $a_n(y)$ to be the simple Riesz means $(R, \lambda_n)$. They remain valid without modification provided that $a_n(y) = 0$ for $n > y$. Otherwise, there are extra error-terms involving $\sum_{n>y} a_n(y)$. Thus one can still obtain meaningful information in case $\lim_{y\to\infty} \sum_{n>y} a_n(y) = 0$.

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Proposé par J.-P. Kahane.

B. SAFFARI,
Université de Paris-Sud
Orsay.

R. C. VAUGHAN,
University of Michigan
Ann Arbor.
and Imperial College
London.