Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 26, n° 2 (1976), p. 25-28

<http://www.numdam.org/item?id=AIF_1976__26_2_25_0>

Ann. Inst. Fourier, Grenoble
26, 2 (1976), 25-28.

A CHARACTERIZATION OF WEAKLY SEQUENTIALLY COMPLETE BANACH LATTICES

by A. W. Wickstead

Meyer-Nieberg ([5], Korollar I.8) has given a number of properties of a Banach lattice, E, that are equivalent to weak sequential completeness of the underlying Banach space. Among these is that E is a band in E^{**}; and from [4*b*], Theorem 39.1 this is equivalent to $E = (E^*)^{\times}$, the space of order bounded order continuous linear functionals on E^*, the (ordered) Banach dual of E (we follow [5] for terminology). We give a further equivalence that was first proved for $L^1(\mu)$ (μ a σ-finite measure) by J. P. R. Christensen ([2], Theorem 4). Our tools include a representation theorem for a class of vector lattice due to Fremlin ([3], Theorem 6) and the following theorem of Christensen ([2], Theorem 2).

THEOREM 1. — *Let* N *be the natural numbers and* $K = \{0, 1\}^N$ *with the product topology. If* φ *is a real valued finitely additive set function on the subsets of* N *it may be regarded in an obvious way as a function on* K. *If* φ *is measurable as such a function then* φ *is countably additive as a set function.*

THEOREM 2. — *A Banach lattice* E *is weakly sequentially complete if and only if every* $\sigma(E^*, E)$*-Borel measurable linear functional on* E^* *is* $\sigma(E^*, E)$*-continuous.*

As the sequential $\sigma(E^{**}, E^*)$-closure of E in E^{**} consists of $\sigma(E^*, E)$-Borel measurable linear functionals « if » is obvious.

Conversely suppose E is weakly sequentially complete, and L is a $\sigma(E^*, E)$-Borel measurable linear functional on E^*. We must show that L is induced by an element of E. As E is weakly sequentially complete it follows from [5], Korollar I.8 and the remark above that this is equivalent to showing that L is order bounded and order continuous, i.e. if the net (f_γ) in E^* is directed downward to 0 then $L(f_\gamma) \to 0$.

L is certainly norm measurable and hence ([1], Theorem 2) norm bounded. As E^* is a Banach lattice, L is certainly order bounded, so we must show L is order continuous.

Without loss of generality we may suppose $f_0 \geqslant f_\gamma \geqslant 0$ for all γ and restrict our attention to the band, B, in E^* generated by f_0 which is $\sigma(E^*, E)$-closed, as E has an order continuous norm (this is equivalent to E being an ideal in E^{**} using [4*b*], Theorem 39.1, and this is certainly true as E is a band in E^{**}) by [4*a*], Theorem 36.2. The topology $\sigma(E^*, E)$ on B is the same as $\sigma(B, E/B^0)$ where B^0 is the annihilator of B in E, so we may limit our attention to the Banach lattice E/B^0 and its Banach dual B; i.e. we limit our attention to the case that E^* has a weak order unit.

Using [3], Theorem 6, we may find a locally compact Hausdorff space Σ and a Radon measure μ on Σ such that E^* is vector lattice isomorphic to a lattice ideal in $M(\mu)$, the space of all equivalence classes of μ-measurable extended real valued functions on Σ. We identify E^* with this ideal. Also by [3], Theorem 7, $E = E^{*\times}$ may be identified with the ideal $\{x \in M(\mu) : \int_\Sigma fx\,d\mu < \infty \text{ for all } f \in E^*\}$. Further as E^* has a weak order unit we may suppose $1_\Sigma \in E^*$, and hence $\chi_A \in E^*$ for all Borel sets $A \subset \Sigma$.

Fix $\alpha_i \in \mathbf{R}_+$ and A_i Borel subsets of $\Sigma (i = 1, 2, \ldots)$, such that $\sum_{i=1}^{\infty} \alpha_i \chi_{A_i} \in E^*$. We claim

$$L(\Sigma \alpha_i \chi_{A_i}) = \Sigma L(\alpha_i \chi_{A_i}).$$

Define φ on subsets M of N by $\varphi(M) = L\left(\sum_{i \in M} \alpha_i \chi_{A_i}\right)$, which is defined, as E^* is an ideal in $M(\mu)$. Clearly φ is finitely additive as L is linear. The map

$$\theta : M \longmapsto \sum_{i \in M} \alpha_i \chi_{A_i}$$

is continuous for the $\sigma(E^*, E)$ topology on E^* and the product topology on K. This is because if $x \in E$ then

$$\theta(N)(|x|) = \int_\Sigma \left(\sum_N \alpha_i \chi_{A_i}\right) |x| \, d\mu$$

is finite, so given $\varepsilon > 0$ we can find a finite set $F \subset N$ with $\left|\int_\Sigma \left(\sum_{N \setminus F} \alpha_i \chi_{A_i}\right) x \, d\mu\right| < \varepsilon$. If $M_\gamma, M \subset N$ and $M_\gamma \to M$ for the product topology we can find γ_0 such that $\gamma \geqslant \gamma_0$ implies $M_\gamma \cap F = M \cap F$. Thus $\gamma \geqslant \gamma_0$ implies

$$|\theta(M_\gamma)(x) - \theta(M)(x)| < \varepsilon;$$

i.e. $\theta(M_\gamma) \to \theta(M)$ for $\sigma(E^*, E)$. Hence $\varphi = L \circ \theta$ is measurable as a real valued function on K, so is countably additive as a set function on N, by Theorem 1, which proves the claim.

Define ν on the Borel sets in Σ by $\nu(A) = L(\chi_A)$, which is meaningful as $\chi_A \in E^*$. If A_i are disjoint Borel sets then $\chi_{\cup A_i} = \Sigma \chi_{A_i}$, and the above claim (with $\alpha_i = 1$) shows that ν is countably additive. If $\mu(A) = 0$ then $\chi_A = 0$ (as an element of E^*) so $\nu(A) = L(\chi_A) = 0$. We may thus apply the Radon-Nikodym theorem to find $y \in L^1(\mu)$ with $\nu(A) = \int_A y \, d\mu$ for all Borel subsets A of Σ (y is integrable as $f_1 = \chi_{\{\sigma \in \Sigma : \varphi(\sigma) > 0\}} \in E^*$ and

$$L(f_1) = \int_\Sigma y^+ \, d\mu < \infty \Big).$$

We must next show that $L(f) = \int_\Sigma fy \, d\mu$ for all $f \in E^*$. This will show that $y \in E^{*\times}$, and hence that L is order continuous. If $f \in E^*_+$ (it is no loss of generality to assume this) and $\varepsilon > 0$ we may find Borel sets A_i and $\alpha_i \geqslant 0$ with $\Sigma \alpha_i \chi_{A_i} \leqslant f \leqslant \Sigma \alpha_i \chi_{A_i} + \varepsilon 1_\Sigma$, and hence (as E^* is a Banach lattice) $\|\Sigma \alpha_i \chi_{A_i} - f\| \leqslant \varepsilon \|1_\Sigma\|$. We have

$$L(\Sigma \alpha_i \chi_{A_i}) = \Sigma \alpha_i L(\chi_{A_i}) = \Sigma \alpha_i \int_\Sigma \chi_{A_i} \, d\nu$$
$$= \Sigma \alpha_i \int_\Sigma y \chi_{A_i} \, d\mu = \int_\Sigma (\Sigma \alpha_i \chi_{A_i}) y \, d\mu$$

(this last equality follows from Lebesgues' dominated convergence theorem). As we have seen, L is bounded, so

$$\left|\int_\Sigma fy\, d\mu - L(f)\right| \leqslant \left|\int_\Sigma fy\, d\mu - \int_\Sigma (\Sigma\alpha_i\chi_{A_i})y\, d\mu\right| + |L(\Sigma\alpha_i\chi_{A_i}) - L(f)| \leqslant \varepsilon\|y\|_1 + \varepsilon\|L\|\,\|1_\Sigma\|.$$

Thus $L(f) = \int_\Sigma fy\, d\mu$ for all $f \in E^*_+$, completing the proof.

BIBLIOGRAPHY

[1] J. P. R. Christensen, Borel structures in groups and semi-groups, *Math. Scand.*, 28 (1971), 124-128.

[2] J. P. R. Christensen, Borel structures and a topological zero-one law, *Math. Scand.*, 29 (1971), 245-255.

[3] D. H. Fremlin, Abstract Kothe spaces II, *Proc. Cam. Phil. Soc.*, 63 (1967), 951-956.

[4] W. A. J. Luxemburg and A. C. Zaanen, Notes on Banach function spaces, *Nederl. Akad. Wetensch. Proc. Ser. A.*, 67 (1964) (*a*) 507-518, (*b*) 519-529.

[5] P. Meyer-Nieberg, Zur schwachen Kompaktheit in Banachverbanden, *Math. Z.*, 134 (1973), 303-315.

Manuscrit reçu le 14 mai 1975
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