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THE MARKOV PROPERTY FOR GENERALIZED GAUSSIAN RANDOM FIELDS

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1. Introduction.

In this paper we consider Gaussian generalized processes or random fields in R^n which are Markovian in the sense first made clear by Paul Lévy for ordinary processes. Our study was largely motivated by the papers on Lévy's n-parameter Brownian motion of H. P. McKean, Jr., G. M. Molchan and P. Cartier ([5], [6], [3]; see also P. Assouad's note [2]) and by the more recent paper of L. Pitt which undertakes a general investigation of the Markov property for Gaussian stochastic processes, [9].

Since our aim has been to pursue the Hilbert space approach initiated by Cartier and Pitt we have devoted a good part of the paper to exploring the interrelationships among the various concepts connected with the Markov property. These are given in a series of lemmas (particularly in Sections 2 and 4) from which is extracted our main result (Theorem 1). It gives verifiable necessary and sufficient conditions in order that a Gaussian generalized process have the Markov property relative to a given family of open sets. This theorem applies also to (ordinary) Gaussian stochastic processes (see the lemma in Section 5).

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A natural (and, in our opinion, illuminating) concept related to the Markov property is the notion of a *dual* generalized process which we discuss in some detail in Section 3. Most of the lemmas as well as Theorem 1 are stated for generalized random fields which have a dual. The idea of the dual process occurs in a recent note by Molchan [7] who has announced a number of conditions equivalent to the Markov property relative to the class of all subsets of an open subset T of R^n. It seems to us that our approach closely parallels that of Molchan in [7]. A more precise comparison of the results is not possible (his seemingly more general than ours) since, unfortunately, no proofs are given in his note.

In discussing applications of Theorem 1 we have contented ourselves with three. A direct and straightforward deduction from Theorem 1 yields the not surprising result that every Gaussian white noise is Markov. In Section 6 we give a simple derivation of the Markov property of Lévy's Brownian motion in R^n (n odd). An application to ordinary (i.e. not generalized) Gaussian processes is made in Theorem 2 of Section 5 which contains a result of Pitt (Theorem 5.2 of [9]). It might be mentioned here in passing that Pitt's condition (2.5) or (3.4) forms part of his definition of the Markov property but not of ours. The verification of (2.5) in pratice appears not to be an easy matter. Sufficient conditions can be given but we do not pursue the question in this paper.

The lemmas of Sections 2 and 4 can also be used to obtain results on the Markovian character of another generalization to many parameters of the single parameter Wiener process, viz., the Cameron-Yeh process. These results will be reported in a later paper.

2. Definitions and Basic Lemmas.

Let T be either the n-dimensional Euclidean space R^n or a domain contained in it. We consider a generalized Gaussian random field (GRF) $\xi(\varphi)$ defined over some probability space $(\Omega, \mathscr{F}, P)$ and where φ ranges over $C_0^\infty(T)$ the space of all indefinitely differentiable real functions each with compact support. We assume $E\xi(\varphi) = 0$ for each φ

and $R(\varphi, \psi) = E[\xi(\varphi)\xi(\psi)]$. The work of P. Cartier has shown that for Gaussian processes (or generalized processes) the Markov property can be characterized in terms of certain Hilbert spaces [3]. For the purposes of this paper it is convenient to take this characterization as our definition of the Markov property. For more information on this point the reader is referred, in addition to Cartier's work already cited, to the papers of H. P. McKean, Jr. [5] and of L. Pitt [9]. The relevant Hilbert spaces are the following: $H(\xi)$, the linear space of the GRF ξ is defined as the closed linear subspace of $L^2(\Omega, \mathscr{F}, P)$ spanned by $\{\xi(\varphi), \varphi \in C_0^\infty(T)\}$. $\mathscr{K}(\xi)$ denotes the reproducing kernel Hilbert space (RKHS) determined by the continuous covariance (bilinear) functional R and its elements are continuous linear functionals on $C_0^\infty(T)$. Write $\langle , \rangle_{\mathscr{K}(\xi)}$ for the inner product in $\mathscr{K}(\xi)$. (When there is no possibility of confusion we write $\mathscr{K}$ for $\mathscr{K}(\xi)$.) Then we use the following well known facts about $\mathscr{K}(\xi)$: For each φ in $C_0^\infty(T)$, the linear functional $R(.,\ \varphi) \in \mathscr{K}(\xi)$; if $F \in \mathscr{K}(\xi)$ is any element, $F(\varphi) = \langle F,\ R(.,\ \varphi)\rangle_{\mathscr{K}(\xi)}$ for each φ. The Hilbert space $\mathscr{K}(\xi)$ is spanned by the set

$$\{R(.,\varphi), \varphi \in C_0^\infty(T)\}.$$

The spaces $H(\xi)$ and $\mathscr{K}(\xi)$ are linked by a congruence, denoted by J, which sends $\xi(\varphi)$ in $H(\xi)$ to $R(.,\varphi)$ in $\mathscr{K}(\xi)$. By congruence we mean an isometric isomorphism of $H(\xi)$ onto $\mathscr{K}(\xi)$. Let D_- be an open subset of T whose boundary is Γ. Write $D_+ = T \setminus (D_- \cup \Gamma)$. For any subset A of T, $\overline{A}$ is the closure of A in T. We need the following subspaces of $\mathscr{K}(\xi)$:

$$(2.1) \qquad \begin{aligned} \mathfrak{M}(D_-) &= V[F : F \in \mathscr{K}(\xi), \text{supp } F \subset D_-]; \\ \mathscr{K}(\xi; D_-) &= V[R(.,\varphi) : \text{supp } \varphi \subset D_-]. \end{aligned}$$

(Supp F stands for the support of F in the sense of support of a distribution. Supp φ is the support of the function φ in the usual sense.) If C is any closed subset of T,

$$\mathscr{K}(\xi; C) = \bigcap_{0 \supset C} \mathscr{K}(\xi; 0)$$

where the intersection is over all open sets 0 containing C.

The closed linear subspaces of $H(\xi)$ which correspond to the above subspaces under J will be denoted by $M(D_-)$, $H(\xi;\ D_-)$ and $H(\xi;\ C)$.

Definition of Markov property with respect to D_- :

A generalized GRF ξ has the Markov property relative to an open set D_- if the following two properties are satisfied: Let $\overline{P}_\mp$ denote the orthoprojector with domain $H(\xi)$ and range $H(\xi;\overline{D}_\mp)$. Then

$$\overline{P}_- \text{ and } \overline{P}_+ \text{ commute;} \tag{2.2}$$

$$H(\xi;\overline{D}_-) \cap H(\xi;\overline{D}_+) = H(\xi;\Gamma). \tag{2.3}$$

Let $\mathcal{O}$ be a family of open subsets of T. The generalized GRF ξ is said to have the Markov property relative to the class $\mathcal{O}$ if (2.2) and (2.3) hold for every D_- in $\mathcal{O}$ (D_+ being defined as $\overline{D}_-^c$). The most interesting choice for $\mathcal{O}$ seems to be (i) $\mathcal{O}$ = the class of all open subsets of T and (ii) $\mathcal{O}$ = the class of all relatively compact open subsets of T.

The Markov property for ordinary Gaussian stochastic processes form an important special case in our work. If $X(t)$ $(t \in T)$ is a zero-mean Gaussian process with continuous covariance function $R(t, s)$ we associate with it the generalized GRF

$$\xi(\varphi) = \int_R X(t)\varphi(t)\,dt \quad (\varphi \in C_0^\infty(T)). \tag{2.4}$$

For convenience (since in any event this assumption is fulfilled in the most important examples) we shall assume $X(t)$ to be sample continuous so that the right side of (2.4) is the ordinary (Lebesgue) integral of the sample function $X(t)$. Furthermore $R(t, s)$ being continuous all elements of $\mathcal{H}(X)$, the RKHS of X, are continuous functions on T.

If $H(X)$ denotes the linear space of the process $X(t)$ it is easily verified that

$$H(X) = H(\xi). \tag{2.5}$$

The well known congruence between $H(X)$ and the RKHS $\mathcal{H}(X)$ is given by J' (say) which sends $X(t)$ into $R(.\,, t)$. Now let $F \in \mathcal{H}(\xi)$ and $z \in H(\xi)$ with $Jz = F$. Since

$z \in H(X)$ we have $J'z \in \mathscr{H}(X)$. Setting $f = J'z$ it is easy to see that the following relations hold.

$$F(\varphi) = \int f(t)\varphi(t)\,dt, \tag{2.6}$$

$$\text{supp}\ F = \text{supp}\ f. \tag{2.7}$$

Here supp F is the support as defined for a distribution while supp f is the usual support of a function. We get (2.7) from (2.6) because f is continuous. Furthermore if F_i, z_i and f_i are as above $(i = 1, 2)$ then we have

$$\langle F_1, F_2 \rangle_{\mathscr{H}(\xi)} = \langle f_1, f_2 \rangle_{\mathscr{H}(X)}. \tag{2.8}$$

Facts (2.5)-(2.8) will be used later on in the paper.

The Markov property for $X(t)$ is defined exactly as for a generalized GRF, by means of the conditions (2.2) and (2.3). Now, however, the spaces $H(\xi; \overline{D}_\mp)$ are to be replaced by $H(X; \overline{D}_\mp)$ where, for any closed set C we define

$$H(X; C) = \bigcap_{0 \supset C} H(X; 0),$$

0 being open and $H(X; 0) = V[X(t),\ t \in 0]$. The definition of $\mathfrak{M}(D_-)$ is analogous:

$$\mathfrak{M}(D_-) = V[f : f \in \mathscr{H}(X),\ \text{supp}\ f \subset D_-].$$

Later in the paper (in Section 5) we shall show that the Gaussian process $X(t)$, $(t \in T)$ has the Markov property relative to $\mathcal{O}$ provided the generalized GRF ξ given by (2.5) has it. This fact enables us to work only with generalized GRF's even when interested in deriving the Markov property for ordinary processes (e.g. the Lévy Brownian motion in Section 6).

First, we shall prove two useful results giving a decomposition of the RKHS of ξ.

Lemma 1. — *Let* $D_- \subset T$ *be any open set. Then*

$$\mathscr{H}(\xi) = \mathfrak{M}(D_-) \oplus \mathscr{H}(\xi; \overline{D}_+). \tag{2.9}$$

Proof. — Let $F \in \mathfrak{M}(D_-)$. Since $\text{supp}\ F \subset D_-$, supp F and $\overline{D}_+$ are disjoint closed sets so that there exists an open set $0 \supset \overline{D}_+$ and disjoint with supp F. Hence if φ is any C_0^∞-function with supp $\varphi \subset 0$, we have $F(\varphi) = 0$, i.e. $\langle F, R(.,\varphi)\rangle_{\mathcal{K}(\xi)} = 0$. This proves that $F \perp \mathcal{H}(\xi; 0)$. Hence $\mathfrak{M}(D_-) \perp \mathcal{H}(\xi; \overline{D}_+)$ or, equivalently,

$$(2.10) \qquad \mathfrak{M}(D_-) \subset \mathcal{H}(\xi) \ominus \mathcal{H}(\xi; \overline{D}_+).$$

Next let 0 be any open set containing $\overline{D}_+$ and F any element of $\mathcal{H}(\xi) \perp \mathcal{H}(\xi; 0)$. $\overline{D}_+$ and 0^c being disjoint closed sets we can find an open set V such that $0 \supset V \supset \overline{D}_+$. Then $F \perp \mathcal{H}(\xi; V)$ which implies that

$$F(\varphi) = \langle F, R(.,\varphi)\rangle_{\mathcal{K}(\xi)} = 0$$

for all C_0^∞-functions φ with support contained in V. Since $V \subset (\text{supp } F)^c$ we have $\text{supp } F \subset V^c \subset \overline{D}_+^c = D_-$, i.e. $F \in \mathfrak{M}(D_-)$. Thus we have shown that for every open set $0 \supset \overline{D}_+$, $[\mathcal{H}(\xi; 0)]^\perp \subset \mathfrak{M}(D_-)$ which proves that

$$(2.11) \qquad \mathfrak{M}(D_-) \supset \mathcal{H}(\xi) \ominus \mathcal{H}(\xi; \overline{D}_+).$$

(2.10) and (2.11) establish the lemma.

Remark. — We may interchange D_- and D_+ in (2.9) to get a second relation

$$(2.12) \qquad \mathcal{H}(\xi) = \mathfrak{M}(D_+) \oplus \mathcal{H}(\xi; \overline{D}_-).$$

Lemma 2. — *Let* D_- *be any open set and* $D_+ = \overline{D}_-^c$. *Then*

$$(2.13) \qquad \mathcal{H}(\xi) = \mathcal{H}(\xi; D_-) \vee \mathcal{H}(\xi; \overline{D}_+).$$

Proof. — Let $F \in \mathcal{H}(\xi)$ such that $F \perp \mathcal{H}(\xi; D_-)$ and $F \perp \mathcal{H}(\xi; \overline{D}_+)$. The latter fact and (2.9) of Lemma 1 together imply that $F \in \mathfrak{M}(D_-)$ so that $\text{supp } F \subset D_-$. Since supp F and $\overline{D}_+$ are disjoint there exists an open set 0 containing $\overline{D}_+$ and disjoint with supp F. For every C_0^∞-function φ with $\text{supp } \varphi \subset 0 \subset (\text{supp } F)^c$ we have $F(\varphi) = 0$, i.e., $\langle F, R(.,\varphi)\rangle_{\mathcal{K}} = 0$. Thus $F \perp \mathcal{H}(\xi; 0)$. Now let $\{\alpha_1, \alpha_2\}$ be a partition of unity corresponding to the covering $\{D_-, 0\}$ of T([4], p. 45). The functions α_1 and α_2 are in C^∞ with

$\operatorname{supp} \alpha_1 \subset D_-$, $\operatorname{supp} \alpha_2 \subset 0$ and $\alpha_1 + \alpha_2 = 1$. Hence for every $\varphi \in C_0^\infty$ we have $\varphi = \alpha_1\varphi + \alpha_2\varphi$ where $\alpha_i\varphi \in C_0^\infty (i = 1, 2)$, $\operatorname{supp} \alpha_1\varphi \subset D_-$ and $\operatorname{supp} \alpha_2\varphi \subset 0$. The orthogonality of F to $\mathcal{H}(\xi; D_-)$ and $\mathcal{H}(\xi; 0)$ yields $\langle F, R(., \alpha_i\varphi)\rangle_{\mathfrak{K}} = 0$ $(i = 1, 2)$. So $\langle F, R(., \varphi)\rangle_{\mathfrak{K}} = 0$ for all $\varphi \in C_0^\infty$. Hence $F = 0$ and (2.13) is proved.

Lemmas 1 and 2 yield the following simple decomposition for a Markov generalized GRF ξ.

LEMMA 3. — *If ξ has the Markov property with respect to an open set D_- then*

$$\mathfrak{M}(D_\mp) = \mathcal{H}(\xi; \overline{D}_\mp) \theta \mathcal{H}(\xi; \Gamma) \tag{2.14}$$

where $\Gamma = \partial D_-$ *and* $D_+ = (D_- \cup \Gamma)^c$.

Proof. — From (2.2) and (2.3) defining the Markov property and the relation

$$\mathcal{H}(\xi; \overline{D}_-) \vee \mathcal{H}(\xi; \overline{D}_+) = \mathcal{H}(\xi) \tag{2.13'}$$

(which is implied by (2.13)) we obtain

$$\mathcal{H}(\xi) = \mathcal{H}(\xi; \overline{D}_\mp) \oplus [\mathcal{H}(\xi; \overline{D}_\pm) \theta \mathcal{H}(\xi; \Gamma)]. \tag{2.15}$$

Comparison of (2.15) with (2.9) and (2.12) completes the proof.

Our next condition pertains to a structural property of $\mathcal{H}(\xi)$.

$$(A_1). \qquad \langle F, G\rangle_{\mathfrak{K}(\xi)} = 0$$

for all $F, G \in \mathcal{H}(\xi)$ with disjoint supports.

LEMMA 4. — *Let ξ have the Markov property* w.r.t. *all open sets. Then $\mathcal{H}(\xi)$ satisfies condition* (A_1).

Proof. — Let D_- be any open set and $D_+ = \overline{D}_-^c$. We first show that $F \in \mathfrak{M}(D_-)$ and $G \in \mathfrak{M}(D_+)$ implies

$$\langle F, G\rangle_{\mathfrak{K}(\xi)} = 0.$$

From (2.14) we get $F \perp \mathcal{H}(\xi; \Gamma)$, i.e., $\overline{P}_+\overline{P}_-F = 0$, where we use the same symbols $\overline{P}_\mp$ also to denote orthoprojectors onto $\mathcal{H}(\xi; \overline{D}_\mp)$ (See (2.2)). Hence noting that $F = \overline{P}_-F$ and $G = \overline{P}_+G$ it follows that $\langle F, G\rangle = \langle \overline{P}_+\overline{P}_-F, G\rangle = 0$.

Now suppose that $F, G \in \mathscr{K}(\xi)$, $\operatorname{supp} F \cap \operatorname{supp} G = \varnothing$. Let U, V be disjoint open sets such that $U \supset \operatorname{supp} F$ and $V \supset \operatorname{supp} G$. If we denote U by D_-, then

$$V \subset U^c = D_-^c = \overline{D}_+.$$

Since $F \in \mathfrak{M}(D)_-$ and $G \in \mathfrak{M}(D_+)$ we get $\langle F, G\rangle_{\mathcal{K}(\xi)} = 0$ from the first part.

Lemma 5. — *Let* ξ *be a generalized* GRF *such that*

$$\text{(2.16)} \qquad H^\perp(\xi; D_-) \subset H(\xi; \overline{D}_+).$$

Then the orthoprojectors $\overline{P}_+$ *and* $\overline{P}_-$ *commute.*

Proof. — An elementary argument shows that it is enough to show that

$$\text{(2.17)} \qquad \overline{P}_+ H(\xi; \overline{D}_-) \subset H(\xi; \overline{D}_-).$$

Let $u \in H(\xi; \overline{D}_-)$ and write $u = u_1 + u_2$ where $u_1 = \overline{P}_+ u$ and $u_2 = (I - \overline{P}_+)u$.

From condition (2.16) of the lemma,

$$u_2 \in H(\xi; D_-) \subset H(\xi; \overline{D}_-).$$

Hence $\overline{P}_+ u = u - u_2 \in H(\xi; \overline{D}_-)$, which proves (2.17). It follows that $\overline{P}_- \overline{P}_+$ is the orthoprojector with range

$$H(\xi; \overline{D}_-) \cap H(\xi; \overline{D}_+).$$

Lemma 6. — *If* ξ *has the Markov property with respect to all open sets then the inverse* $J^{-1} : \mathscr{K}(\xi) \to H(\xi)$ *given by* $J^{-1}R(., \varphi) = \xi(\varphi)$ *is a local map,* i.e., $F \in \mathscr{K}(\xi)$ *with* $\operatorname{supp} F \subset C$ *implies* $J^{-1}F \in H(\xi; C)$, *where* C *is any closed subset of* T.

The proof is immediate. For any open set 0 containing C let V be an open set such that $C \subset \overline{V} \subset 0$. Since $\operatorname{supp} F \subset C$, $F \in \mathfrak{M}(V)$ which implies $F \in \mathscr{K}(\xi; \overline{V})$ from Lemma 3. Thus $J^{-1}F \in H(\xi; 0)$ and the assertion follows.

3. The Concept of a Dual Process.

In many cases the Markov property of a generalized GRF ξ can be described in terms of another generalized GRF $\hat{\xi}$ which stands in a dual relation to ξ and which we shall designate the dual process (or the dual of ξ). This concept is naturally suggested by the following observation:

$$\text{Let } \xi(\varphi) = \int X(t)\varphi(t)\, dt \ \ (\varphi \in C_0^\infty),\ X(t)$$

being a Gaussian process as in Section 2. Suppose further that $\mathscr{K}(X)$ contains $C_0^\infty(T)$ as a dense subset. Denote by $\hat{\xi}(\psi)$ the element in $H(X)$ which corresponds to ψ in $\mathscr{K}(X)$ $(\psi \in C_0^\infty(T))$. Then $E[\hat{\xi}(\psi)X(t)] = \psi(t)$, $(t \in T)$ and hence $E\xi(\varphi)\hat{\xi}(\psi) = (\varphi, \psi)_0$ (4). It also follows that the linear space $H(\hat{\xi})$ coincides with $H(X)$ or $H(\xi)$. We shall consider later a class of Gaussian processes which possess a dual in our sense. An example of a generalized GRF $\xi(\varphi)$ not determined by an ordinary process $X(t)$ as in (2.5) and for which the dual exists is the Gaussian white noise, also to be discussed below.

Definition of $\hat{\xi}$. — *Let ξ be a generalized* GRF. *A generalized process $\hat{\xi}$, defined on the same probability space as ξ, is called the dual of ξ if the following two conditions are satisfied: For all φ, $\psi \in C_0^\infty$.*

$$E[\xi(\varphi)\hat{\xi}(\psi)] = (\varphi, \psi)_0; \tag{3.1}$$
$$H(\hat{\xi}) = H(\xi). \tag{3.2}$$

When $\hat{\xi}$ exists it is easy to see that it is unique in the sense that if $\hat{\xi}_1$ is another generalized GRF satisfying (3.1) and (3.2) then $\hat{\xi}(\psi) = \hat{\xi}_1(\psi)$ as elements of $H(\xi)$. The proof of the following result is omitted. Let $\mathscr{D}'$ denote the space of Schwarz distributions over $C_0^\infty(T)$ and let F be the linear map from $C_0^\infty \to \mathscr{D}'$ defined by

$$F_\varphi(\psi) = (\varphi, \psi)_0 (\varphi, \psi \in C_0^\infty).$$

(4) $(\ , \ )_0$ denotes inner product in L_2.

(In other words F_φ is the image of φ under the natural imbedding of C_0^∞ in $\mathscr{D}'$.)

PROPOSITION 1. — *A generalized* GRF ξ *has a dual process* $\hat{\xi}$ *if and only if the following three conditions hold: For all* $\varphi \in C_0^\infty$

(3.3) $$F_\varphi \in \mathscr{K}(\xi);$$
(3.4) $$F : C_0^\infty \to \mathscr{K}(\xi) \text{ is continuous};$$
(3.5) $$\text{The family } \{F_\varphi, \varphi \in C_0^\infty\} \text{ is total in } \mathscr{K}(\xi).$$

Remark. — If $X(t) = 0$ on a closed subset T_0 of T it is often convenient to consider $\xi(\varphi) = \int_G X(t)\varphi(t)\,dt$ where $G = T \cap T_0^c$ and $\varphi \in C_0^\infty(G)$. Hence, in this case, the sufficient conditions (3.3), (3.4) and (3.5) above should be satisfied for

$$\varphi \in C_0^\infty(G)$$

for the existence of a dual $\hat{\xi}$.

Let $\mathscr{K}_R$ be the completion of C_0^∞ with respect to the inner product $R(\varphi, \psi)$. Then the linear map which sends φ into $R(., \varphi)$ in $\mathscr{K}(\xi)$ extends to an isometry of $\mathscr{K}_R$ onto $\mathscr{K}(\xi)$. If f is any element of $\mathscr{K}_R$ (it is not claimed that f is a function) it is convenient and consistent to represent its image under this isometry by $R(., f)$. Furthermore, if $u \in H(\xi)$ is an arbitrary element and $Ju = F$ (J being the isometry of $H(\xi)$ onto $\mathscr{K}(\xi)$ introduced earlier), we extend our notation and denote u by $\xi(f)$ where $F = R(., f)$ $(f \in \mathscr{K}_R)$. Thus every element of $H(\xi)$ is of the form $\xi(f)$ where f is the corresponding unique element in $\mathscr{K}_R$ such that $J[\xi(f)] = R(., f)$. Now for each $\psi \in C_0^\infty$, $\hat{\xi}(\psi) \in H(\xi)$. Hence there is a unique element $f_\psi \in \mathscr{K}_R$ such that

$$\hat{\xi}(\psi) = \xi(f_\psi).$$

Thus we have

(3.6) $$B(\varphi, \psi) = E[\xi(f_\varphi)\hat{\xi}(\psi)] = \hat{f}_\varphi(\psi), \text{ say.}$$

At this stage we recall that a generalized stochastic process $\hat{\xi}$ is said to have independent values at every point if, the random variables $\hat{\xi}(\varphi)$ and $\hat{\xi}(\psi)$ are independent whenever φ and ψ have disjoint supports. Since, in our case, $\hat{\xi}$ is Gaussian, $\hat{\xi}$ has independent values if and only if $B(\varphi, \psi)=0$ whenever φ and ψ have disjoint supports.

The next lemma enables us to characterize $B(\varphi, \psi)$.

LEMMA 7. — *(a) The linear map* $\hat{f} : C_0^\infty \to \mathcal{D}'$ *which sends* φ *into* $\hat{f}_\varphi$ *where* $\hat{f}_\varphi$ *is defined by* (3.6) *is continuous.*

(b) $\hat{\xi}$ *has independent values at every point if and only if for every* $\varphi \in C_0^\infty$

$$\operatorname{supp} \hat{f}_\varphi \subset \operatorname{supp} \varphi. \tag{3.7}$$

Proof. — *(a)* The linearity of the map $\hat{f}$ is obvious. Since by assumption $\hat{\xi}$ is the dual process of ξ, (3.4) of Proposition 1 implies that $\hat{f}_\varphi \in \mathcal{D}'$. Also $\varphi_n \to \varphi$ in C_0^∞ implies $\hat{f}_{\varphi_n} \to \hat{f}_\varphi$ weakly. Hence $\hat{f}_{\varphi_n} \to \hat{f}_\varphi$ in $\mathcal{D}'$ (i.e. strongly) ([4], p. 48).

(b) Suppose $\hat{\xi}$ has independent values at every point. Fix $\varphi \in C_0^\infty$. Then for every $\psi \in C_0^\infty$ whose support is disjoint with $\operatorname{supp} \varphi$ we have

$$B(\varphi, \psi) = 0, \qquad \text{i.e.,}\ \hat{f}_\varphi(\psi) = 0$$

which implies that

$$\operatorname{supp} \psi \subset (\operatorname{supp} \hat{f}_\varphi)^c. \tag{3.8}$$

From (3.8) it follows that

$$(\operatorname{supp} \varphi)^c \subset (\operatorname{supp} \hat{f}_\varphi)^c. \tag{3.9}$$

If (3.9) is not true let t_0 be a point in $(\operatorname{supp} \hat{f}_\varphi) \cap (\operatorname{supp} \varphi)^c$. Since $(\operatorname{supp} \varphi)^c$ is open, it contains the closed ball S_ε with t_0 as center and radius ε where ε is sufficiently small. Consider the function

$$\begin{aligned} \psi_\varepsilon(t) &= \varepsilon^{-n} \exp\left[-\frac{\varepsilon^2}{\varepsilon^2 - |t - t_0|^2}\right] && \text{if} \quad |t - t_0| < \varepsilon, \\ &= 0 && \text{if} \quad |t - t_0| \geqslant \varepsilon. \end{aligned}$$

$$\psi_\varepsilon \in C_0^\infty, \quad \operatorname{supp} \psi_\varepsilon = S_\varepsilon \quad \text{and} \quad t_0 \in \operatorname{supp} \psi_\varepsilon.$$

From (3.8) therefore, $t_0 \in (\operatorname{supp} \hat{f}_\varphi)^c$ which is a contradiction.

Hence (3.9) holds. Using (3.6) for converse implication we get (*b*).

We now prove.

Proposition 2. — *The dual process* $\hat{\xi}$ *has independent values at every point if and only if its covariance functional has the following form*

$$B(\varphi, \psi) = \int \sum_{\alpha, \beta} a_{\alpha\beta}(t) D^{\alpha}\varphi D^{\beta}\psi \, dt \tag{3.10}$$

where the $a_{\alpha\beta}(t)$ are locally in L^2 and where on each compact set all but a finite number of the coefficients $a_{\alpha\beta}$ vanish.

Proof. — If $B(\varphi, \psi)$ is of the form (3.10) it is obvious that $\hat{\xi}$ has independent values at every point. The key results which establish the converse part are Lemma 7 and a result due to J. Peetre [9]. Let $\hat{\xi}$ have independent values at every point. Then by Lemma 7 the map $\hat{f}$ defined by (3.6) is a continuous map of C_0^∞ into $\mathcal{D}'$ and moreover, from (3.7)

$$\text{supp}\, \hat{f}_\varphi \subset \text{supp}\, \varphi$$

for each φ in C_0^∞. Thus from Peetre's theorem it follows that

$$\hat{f}_\varphi = P\varphi = \Sigma c_j D^j \varphi \tag{3.11}$$

where $\{c_j\}$ is a locally finite family of distributions, i.e., on every compact subset of T all but a finite number of c_j's vanish. The operator $P = \Sigma c_j D^j$ we shall call the Peetre operator in this context for convenience of reference. We have shown that

$$B(\varphi, \psi) = (P\varphi)(\psi). \tag{3.12}$$

The expression in (3.12) can be written in the form (3.10) (see [8] or [9], p. 377 for details).

In some respects the form for $B(\varphi, \psi)$ given by (3.12) is more illuminating since the processes ξ and $\hat{\xi}$ are linked to each other by the Peetre operator P. This point is illustrated by the following.

COROLLARY. — *Suppose the operator* P *satisfies the condition*

$$P(C_0^\infty) \subset C_0^\infty. \tag{3.13}$$

Then

$$\hat{\xi}(\varphi) = \xi(P\varphi) \quad (\varphi \in C_0^\infty) \tag{3.14}$$

and

$$H(\hat{\xi}; 0) \subset H(\xi; 0) \tag{3.15}$$

for every open set 0.

Proof. — Condition (3.13) implies that $P\varphi$ is the distribution determined by the C_0^∞-function φ and hence $(P\varphi)(\psi) = (P\varphi, \psi)_0$. (3.14) follows from the relation $(\varphi, \psi \in C_0^\infty)$

$$E[\hat{\xi}(\varphi)\hat{\xi}(\psi)] = B(\varphi, \psi) = (P\varphi, \psi)_0 = E[\xi(P\varphi)\hat{\xi}(\psi)].$$

Next, suppose that $\operatorname{supp}\varphi \subset 0$. Then $\operatorname{supp} P\varphi \subset 0$ so that $\hat{\xi}(\varphi) = \xi(P\varphi) \in H(\xi; 0)$ and (3.15) is proved.

Remark. — Proposition 2 has been derived by L. Pitt using an adaptation of Peetre's result and in connection with the Markov property of (ordinary) Gaussian processes. The proof given here based on Lemma 7 is linked directly to Peetre's result through the crucial property $\operatorname{supp} \hat{f}_\varphi \subset \operatorname{supp}\varphi$.

4. Some Necessary and Sufficient Conditions for the Markov Property for Generalized GRF's.

Our object in this section is to derive necessary and sufficient conditions for the Markov property of ξ in terms of the dual process $\hat{\xi}$. Hence in our next sequence of lemmas the existence of $\hat{\xi}$ will necessarily be a part of our hypotheses even when not explicitly stated. First we need the following conditions which, along with condition (A_1) provide operational criteria for checking the Markov property in examples.

DEFINITION. — ξ (or $\hat{\xi}$) *satisfies Condition* (A_2) *if for every open set* 0, *for all* φ *in* C_0^∞ *with* $\operatorname{supp}\varphi \subset 0$, *we have* $\hat{\xi}(\varphi) \in H(\xi; 0)$.

Obviously, Condition (A_2) is equivalent to saying $H(\hat{\xi}; 0) \subset H(\xi; 0)$ for every open set 0.

For any open set 0 write $\hat{\mathfrak{M}}(0)$ = closed linear subspace of $\mathscr{K}(\xi)$ spanned by $\{F_\varphi, \text{supp}\,\varphi \subset 0\}$. Clearly $\hat{\mathfrak{M}}(0) \subset \mathfrak{M}(0)$.

DEFINITION. — ξ (or $\hat{\xi}$) *satisfies Condition* (A_3) *if for every open set* 0, $\hat{\mathfrak{M}}(0) = \mathfrak{M}(0)$.

We now proceed to the final group of lemmas which leads to our main results.

LEMMA 8. — *Let the dual* $\hat{\xi}$ *exist. Then Condition* (A_1) *implies Condition* (A_2).

Proof. — Let D_- be an open subset of T and let $F \in \mathfrak{M}(D_-)$ such that $F \perp \hat{\mathfrak{M}}(D_-)$. I.e.,

$$\langle F, F_\varphi \rangle_{\mathcal{K}} = 0 \tag{4.1}$$

for all φ in C_0^∞ with $\text{supp}\,\varphi \subset D_-$.

Next, for all φ (we consider only C_0^∞-functions here and below unless otherwise noted) with $\text{supp}\,\varphi \subset \overline{D}_+$ we have $\text{supp}\,F_\varphi \subset \overline{D}_+$, so that $\text{supp}\,F \cap \text{supp}\,F_\varphi = \varnothing$. Hence from (A_1) we get

$$\langle F, F_\varphi \rangle_{\mathcal{K}} = 0 \tag{4.2}$$

for all φ with $\text{supp}\,\varphi \subset \overline{D}_+$.

Let 0 be an open set containing $\overline{D}_+$ and disjoint with $\text{supp}\,F$. Then, introducing a partition of unity corresponding to the covering $\{0, D_-\}$ of T and proceeding as in the proof of Lemma 2 we conclude that

$$\langle F, F_\varphi \rangle_{\mathcal{K}} = 0$$

for all $\varphi \in C_0^\infty$, giving $F = 0$ Hence $\hat{\mathfrak{M}}(D_-) = \mathfrak{M}(D_-)$ and the lemma is proved.

LEMMA 9. — *If the dual* $\hat{\xi}$ *exists then* (A_1) *implies* (A_2).

Proof. — Denote by P_{D_-} the orthoprojector in $\mathscr{K}(\xi)$ with range $\mathscr{K}(\xi;\, D_-)$ (D_- open). Suppose φ has support in D_-. Since $P_{D_-}R(\cdot, \varphi) = R(\cdot, \varphi)$ we have for every

C_0^∞-function ψ,

$$F_\psi(\varphi) - P_{D\_}F_\psi(\varphi) = \langle F_\psi - P_{D\_}F_2, R(.,\varphi)\rangle_{\mathcal{K}} = 0$$

yielding $\operatorname{supp} \varphi \subset [\operatorname{supp}(F_\psi - P_{D\_}F_\psi)]^c$. Arguing as in Lemma 7 we find that

$$\text{(4.4)} \qquad D_- \subset [\operatorname{supp}(F_\psi - P_{D\_}F_\psi)]^c.$$

If $\operatorname{supp} \psi \subset D_-$, (4.4) shows that the distributions F_ψ and $F_\psi - P_{D\_}F_\psi$ have disjoint supports. Let Condition (A_1) hold. Then

$$\text{(4.5)} \qquad \langle F_\psi, F_\psi - P_{D\_}F_\psi\rangle_{\mathcal{K}} = 0.$$

Hence

$$\text{(4.6)} \qquad F_\psi = P_{D\_}F_\psi.$$

Recalling that under the isometry J, F_ψ corresponds to $\hat{\xi}(\psi)$ and $\mathcal{K}(\xi; D_-)$ corresponds to $H(\xi; D_-)$ we conclude from (4.6) that $\hat{\xi}(\psi) \in H(\xi; D_-)$, i.e., Condition (A_2) is satisfied.

LEMMA 10. — *If* $\hat{\xi}$ *exists then the Markov property for* ξ *implies Condition* (A_2).

Proof. — Consider the element $F_\varphi \in \mathcal{K}(\xi)$. From Lemma 6 we know that J^{-1} is a local map. Hence if $\operatorname{supp} \varphi \subset 0$ (0 an open set), $J^{-1}F_\varphi \in H(\xi; 0)$ proving Condition (A_2).

LEMMA 11. — *Condition* (A_2) *implies that* $\hat{\xi}$ *has independent values at every point.*

The proof is obvious.

LEMMA 12. — *For a generalized* GRF ξ *having a dual* $\hat{\xi}$ *Conditions* (A_2) *and* (A_3) *imply that* ξ *has the Markov property with respect to all open sets.*

Proof. — In the definition of (A_3) take 0 successively to be D_-, D_+ and $D_- \cup D_+$. The decomposition of Lemma 1 then yields the following.

$$\text{(4.7)} \qquad \begin{cases} H(\hat{\xi}; D_-) \oplus H(\xi; \overline{D}_+) = H(\xi) \\ H(\hat{\xi}; D_+) \oplus H(\xi; \overline{D}_-) = H(\xi) \\ H(\hat{\xi}; D_- \cup D_+) \oplus H(\xi; \Gamma) = H(\xi). \end{cases}$$

From the first equality in (4.7) $H^{\perp}(\hat{\xi}; D_-) = H(\xi; \overline{D}_+)$ and by Condition (A_2), $H^{\perp}(\xi; D_-) \subseteq H^{\perp}(\hat{\xi}; D_-)$. Hence the hypothesis of Lemma 3 is satisfied and it follows that the orthoprojectors $\overline{P}_+$ and $\overline{P}_-$ commute. Letting $\hat{P}_{\mp}$ be the orthoprojectors with range $H(\hat{\xi}; D_{\mp})$ and noting that from (4.7), $\overline{P}_{\mp} = (I - \hat{P}_{\mp})$ we have

$$(4.8) \quad \overline{P}_+\overline{P}_- = (I - \hat{P}_-)(I - \hat{P}_+) = I - (\hat{P}_- + \hat{P}_+)$$

since $\hat{P}_-\hat{P}_+ = 0$ because $\hat{\xi}$ has independent values at every point. The latter fact is a consequence of Condition (A_2) (Lemma 11). Now $\hat{P}_- + \hat{P}_+$ is the orthoprojector with range $H(\hat{\xi}; D_-) \oplus H(\hat{\xi}; D_+)$ which we now show is $H(\hat{\xi}; D_- \cup D_+)$. This follows at once from the fact that for every C_0^∞-function φ with $\operatorname{supp} \varphi \subset D_- \cup D_+$ we may write $\hat{\xi}(\varphi) = \hat{\xi}(\alpha_1\varphi) + \hat{\xi}(\alpha_2\varphi)$ where $\{\alpha_1, \alpha_2\}$ is a partition of unitry of $D_- \cup D_+$ given by the open sets $\{D_-, D_+\}$. This gives

$$H(\hat{\xi}; D_- \cup D_+) \subset H(\hat{\xi}; D_-) \oplus H(\hat{\xi}; D_+)$$

and the reverse inclusion is obvious. From the third equality in (4.7) it now follows that $I - (\hat{P}_- + \hat{P}_+)$ is the orthoprojector onto $H(\xi; \Gamma)$. Hence (4.8) proves that $\overline{P}_+\overline{P}_-$ is the orthoprojector onto $H(\xi; \Gamma)$. Thus we have verified conditions (2.2) and (2.3) defining the Markov property for ξ, with respect to every open set D_- with boundary Γ and $D_+ = T\setminus(D_- \cup \Gamma)$. This completes the proof.

LEMMA 13. — *For the dual process $\hat{\xi}$ condition (A_3) and independent values at every point implies Condition (A_1).*

Proof. — Let D_- and D_+ be open sets as before $(D_+ = \overline{D}_-^c)$ and let $F \in \mathfrak{M}(D_-)$ and $G \in \mathfrak{M}(D_+)$. Because of Condition (A_3) there exist sequences $\{F_{\varphi_n}\}$ and $\{F_{\psi_n}\}$ where for each n, $\operatorname{supp} \varphi_n \subset D_-$ and $\operatorname{supp} \psi_n \subset D_+$ such that $F_{\varphi_n} \to F$ and $F_{\psi_n} \to G$ in $\mathscr{K}(\xi)$. Since $\hat{\xi}$ has independent values at every point $\langle F_{\varphi_n}, F_{\psi_n}\rangle = 0$ for every n. Hence

$$\langle F, G\rangle_{\mathcal{K}} = \langle F - F_{\varphi_n}, G\rangle_{\mathcal{K}} + \langle F_{\varphi_n}, G - F_{\psi_n}\rangle_{\mathcal{K}}$$

and both terms on the right hand side tend to zero as $n \to \infty$ giving $\langle F, G\rangle_{\mathfrak{K}} = 0$.

If F, G are arbitrary elements in $\mathscr{K}(\xi)$ with disjoint supports we repeat the argument at the end of the proof of Lemma 4. This proves the assertion of the lemma.

From the series of lemmas derived in this Section and Sections 2 and 3 we extract our principal general result.

THEOREM 1. — *Let the generalized* GRF ξ *have a dual generalized* GRF $\hat{\xi}$. *Then the following statements are equivalent:*

(*a*) ξ *has the Markov property relative to all open sets;*

(*b*) *For every pair* F, G *in* $\mathscr{K}(\xi)$ *having disjoint supports,* $\langle F, G\rangle_{\mathfrak{K}(\xi)} = 0$;

(*c*) *For every open subset* D_- *of* T

(i) $H(\hat{\xi}; D_-) \subset H(\xi; D_-)$, *and*

(ii) $\mathfrak{M}(D_-) = \hat{\mathfrak{M}}(D_-)$;

(*d*) $\hat{\xi}$ *has independent values at every point and*

$$\mathfrak{M}(D_-) = \hat{\mathfrak{M}}(D_-)$$

for any open set D_- *in* T.

Proof. — Recall the definitions of conditions (A_1), (A_2) and (A_3) given above. The theorem is established by the following chain of implications.

(*a*) implies (*b*) by Lemma 4. Since (A_1) implies (A_3) by Lemma 8 and (A_2) by Lemma 9 it follows that (*b*) implies (*c*).

(*c*) $\Longrightarrow$ (*d*) since by Lemma 11 (A_2) implies that $\hat{\xi}$ has independent values at every point. By Lemma 13 (*d*) implies (A_1) and hence (A_2) and (A_3) by Lemmas 8 and 9. Finally (A_2) and (A_3) together imply (*a*) by Lemma 12.

Example. — *Gaussian white noise in* R^n. ξ is a generalized GRF with covariance functional $R(\varphi, \psi) = (\varphi, \psi)_0$. It is easy to see that the dual $\hat{\xi}$ is ξ itself and that

$$\mathscr{K}(\xi) = L^2(R^n) \text{ i.e., } F \in \mathscr{K}(\xi)$$

is determined by a (unique) element $F \in L^2(R^n)$,

$$F(\varphi) = (F, \varphi)_0$$

for every φ in C_0^∞ and

$$\langle F, G\rangle_{\mathcal{K}} = (F, G)_0 \tag{4.9}$$

where $G \in \mathcal{K}(\xi)$. (4.9) shows that condition (A_1) is satisfied. Hence from Theorem 1 it follows that the Gaussian white noise ξ is Markov with respect to all open sets.

5. The Markov Property for Gaussian Stochastic Processes.

Suppose $X(t)$ $(t \in T)$ is a Gaussian process as described in Section 2. As in (2.4) let $\xi(\varphi) = \int X(t)\varphi(t)\,dt$. We shall need the following lemma.

LEMMA 14. — *Let the generalized* GRF ξ *have the Markov property relative to a family* $\mathcal{O}$ *of open sets. Then the process* $X(t)$ *also has the Markov property relative to* $\mathcal{O}$.

Proof. — In view of the definitions of the Markov properties relative to $\mathcal{O}$ for X and ξ it suffices to prove that for all $0 \in \mathcal{O}$, $H(X:0) = H(\xi:0)$. Since $\xi(\varphi) = \int X(t)\varphi(t)\,dt$ for $\varphi \in C_0^\infty$ and the integral is in Riemann sense

$$H(\xi:0) \subset H(X:0).$$

To prove the reverse inclusion, let $t_0 \in 0$ and N be an integer such that $\left\{t \,\middle|\, |t - t_0| < \frac{1}{n}\right\} \subset 0$ for all $n \geqslant N$. Choose for $n \geqslant N$, C_0^∞-functions φ_n such that

$$\operatorname{supp}\varphi_n \subset \left\{t : |t - t_0| < \frac{1}{n}\right\} \quad \int \varphi_n(t)\,dt = 1$$

(e.g., put $\varphi_n = \psi_{1/n}$ where ψ_ε for $\varepsilon > 0$ is as in Section 3). Then

$$E|X(t_0) - \xi(\varphi_n)|^2 = E\left|\int X(t_0)\varphi_n(t)\,dt - \int X(t)\varphi_n(t)\,dt\right|^2 \leqslant \sup_{t \in \left\{s : |s - t_0| < \frac{1}{n}\right\}} E|X(t_0) - X(t)|^2.$$

By mean continuity of $\{X(t),\ t \in T\}$ the limit as $n \to \infty$ on the right side is zero giving $E|X(t_0) - \xi(\varphi_n)|^2 \to 0$ for each $t_0 \in 0$. But $\operatorname{supp} \varphi_n \subset 0$ implies $H(X:0) \subset H(\xi:0)$ completing the proof.

We shall now deduce as an application of Theorem 1 a result which bears a close resemblance to Theorem 5.2 of Pitt [9]. Assume $T = R^n$ and write C_0^∞ for $C_0^\infty(R^n)$.

THEOREM 2. — *Suppose the* RKHS $\mathcal{K}(X)$ *of the process* $X(t)$ *has the following properties:*

(5.1) C_0^∞ *is a dense subset of* $\mathcal{K}(X)$.

The inner product in $\mathcal{K}(X)$ has the form

$$(5.2) \qquad \langle \varphi, \psi \rangle = \int \sum_{\substack{|\alpha| \leqslant p \\ |\beta| \leqslant p}} a_{\alpha\beta}(t) D^\alpha \varphi(t) D^\beta \psi(t)\, dt,$$

$(\varphi, \psi \in C_0^\infty)$ where p is a non negative integer and the $a'_{\alpha\beta}s$ are bounded C^∞-functions;

There exists a positive constant c such that for all φ in C_0^∞

$$(5.3) \qquad \|\varphi\|_{\mathcal{K}(X)} \geqslant c \,.\, \|\varphi\|_{0,p}$$

where $\|\,.\,\|_{0,p}$ denotes the Sobolev norm of the Hilbert space $H_0^{2,p}$.

Then the dual $\hat{\xi}$ exists and $X(t)$ has the Markov property relative to all open sets. Moreover, $\mathcal{K}(X)$ and $H_0^{2,p}$ have equivalent norms and

$$(5.4) \qquad \mathcal{K}(X) = H_0^{2,p}.$$

Proof. — First of all observe that from the definition of $H_0^{2,p}$ (see [10], p. 323) if $f \in H_0^{2,p}$, the L^2-derivatives $D^\alpha f$ exist and belong to $L^2(R^n)$ for all α, $|\alpha| \leqslant p$. Hence from (5.2) the $\mathcal{K}(X)$-norm is defined and finite for each such f. In fact, we have

$$(5.5) \qquad \|f\|_{\mathcal{K}(X)} \leqslant c' \|f\|_{0,p'}$$

c' being a suitable constant. From (5.1), (5.5) and the fact that C_0^∞ is dense in $H_0^{2,p}$ it follows that $f \in H_0^{2,p}$ can be approximated in $\mathcal{K}(X)$-norm by a sequence of C_0^∞-functions.

Assumption (5.3) then gives

$$\|f\|_{\mathcal{K}(X)} \geqslant c.\|f\|_{0,p} \tag{5.3)'$$

for all f in $H_0^{2,p}$. We have thus shown that $H_0^{2,p}$ is contained in $\mathcal{H}(X)$ (in the sense that every f in $H_0^{2,p}$ is an element of $\mathcal{H}(X)$) and from (5.5) and (5.3)' that on $H_0^{2,p}$ the $\|\cdot\|_{0,p}$-norm and $\|\cdot\|_{\mathcal{K}(X)}$-norm are equivalent. Furthermore, it follows from (5.3)' that $H_0^{2,p}$ is a closed linear subspace of $\mathcal{H}(X)$. Since C_0^∞ is a subspace of $H_0^{2,p}$ and is dense in $\mathcal{H}(X)$ we have (5.4).

Noting that the existence of $\hat{\xi}$ is assured by (5.1) and Proposition 1 it remains only to show Condition (A_3) is satisfied. The Markov property of $X(t)$ then follows from Theorem 1 and Lemma 14.

Let $F \in \mathcal{H}(\xi)$ such that $F \in \mathfrak{M}(D_-)$ and let f be the element in $H_0^{2,p}$ which determines F by (2.6), i.e.

$$F(\varphi) = \int f(t)\varphi(t)\, dt.$$

Since $\operatorname{supp} F \subset D_-$ by assumption, it is easily seen that

$$D^\alpha f(t) = 0, \tag{5.6}$$

for all α with $|\alpha| \leqslant p$, for a.e. t in $\overline{D}_+$.

We use now the well known fact that the restriction $\bar{f}$ of f to D_- is in $H_0^{2,p}(D_-)$ ([10], p. 328) and hence there exists a sequence (φ_n) in $C_0^\infty(D_-)$ such that

$$\int_{D_-} \sum_{|\alpha|\leqslant p} [D^\alpha \bar{f}(t) - D^\alpha \varphi_n(t)]^2\, dt \to 0 \tag{5.7}$$

as $n \to \infty$. Since the $\varphi_n's$ have support contained in D_- we obtain from (5.6) and (5.7) that

$$\int_{R^n} \sum_{|\alpha|\leqslant p} [D^\alpha f(t) - D^\alpha \varphi_n(t)]^2\, dt \to 0. \tag{5.8}$$

Hence (from (5.5)),

$$\|f - \varphi_n\|_{\mathcal{K}(X)} \to 0. \tag{5.9}$$

Recalling (2.6) and (2.8) we see that (5.9) implies

$$\|F - F_{\varphi_n}\|_{\mathcal{K}(\xi)} \to 0 \tag{5.10}$$

where $\varphi_n \in C_0^\infty$ and $\operatorname{supp} \varphi_n \subset D_-$ for each n. Thus $F \in \hat{\mathfrak{M}}(D_-)$ and (A_3) is proved.

We now give an example of a stochastic process for which $\mathcal{K}(X) = H_0^{2,p}(R^n)$ with the same Hilbertian structure. This example first occurred in the work of Pitt ([9], p. 379). In view of the above theorem this process has Markov property on all open subsets of R^n.

Example. — Let $\{X(t), t \in R^n\}$ be a continuous in quadratic mean stochastic process with $EX(t) = 0$ for all t and covariance

$$\rho(t-s) = \int e^{i(t,x)} e^{-i(s,x)} \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Since $\rho(t, x) = EX(t)\overline{X(s)}$ and each $f \in \mathcal{K}(X)$ has the form $f(t) = EYX(t)$ with $Y \in H(X)$, we get that each $f \in \mathcal{K}(X)$ can be written as

$$(5.11) \quad f(t) = \int e^{i(t,x)} g(x) \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx$$

where

$$g \in L_2\Big(R^n, \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx\Big).$$

We now show that f of the form (5.11) belongs to $H_0^{2,p}(R^n)$. Let $D^\alpha = \dfrac{\partial^\alpha}{\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n}}$ then from (5.11) we have

$$(5.12) \quad D^\alpha f = \int i^{|\alpha|} x_1^{\alpha_1} \dots x_n^{\alpha_n} e^{i(t,x)} g(x) \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx.$$

Hence by Plancherel's Theorem for $|\alpha| \leqslant p$, $D^\alpha f \in L^2(R^n)$. Also

$$\begin{aligned}(5.13) \quad & \sum_{|\alpha| \leqslant p} \int_{R^n} |(D^\alpha f)(t)|^2 \, dt \\ &= \sum_{|\alpha| \leqslant p} \int_{R^n} x_1^{2\alpha_1} \dots x_n^{2\alpha_n} |i^{2\alpha}| \, |g(x)|^2 \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-2} dx. \\ &= \sum_{|\alpha| \leqslant p} \int_{R^n} |g(x)|^2 \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx.\end{aligned}$$

Since $g \in L_2\Big(R^n, \Big(\sum_{|\alpha| \leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx\Big)$, (5.13) implies

$f \in H_0^{2,p}(R^n)$. Thus $\mathscr{H}(X) \subset H_0^{2,p}(R^n)$. In particular,

$$\rho(\,.\,-s) \in H_0^{2,p}(R^n) \qquad \text{for each} \qquad s \in R^n$$

and

$$(5.14)\quad D^\alpha\rho(t-s) = \int e^{i(t,x)-i(s,x)} i^{|\alpha|} x_1^{\alpha_1} \dots x_n^{\alpha_n} \Big(\sum_{|\alpha|\leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx.$$

From (5.12) and (5.14) and Plancherel's Theorem, we get

$$(5.15)\quad \sum_{|\alpha|\leqslant p} \int D^\alpha f(t) D^\alpha \rho(t-s)\, dt = \int e^{i(s,x)} g(x) \Big(\sum_{|\alpha|\leqslant p} x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\Big)^{-1} dx = f(s).$$

Thus we have shown that $\mathscr{H}(X) \subset H_0^{2,p}(R^n)$ and for all $f \in \mathscr{H}(X)$

$$(5.16)\quad (f, \rho(\,.\,-x))_{0,p} = f(s) = \langle f. \rho(\,.\,-s)\rangle_{\mathcal{K}(X)}.$$

Let h be a finite combination $\sum_j c_j \rho(\,.\,-s_j)$ then (5.16) implies

$$(5.17)\qquad \|h\|_{0,p}^2 = \|h\|_{\mathcal{K}(X)}^2.$$

But $\mathscr{H}(X)$ is the completion w.r.t. $\|\ \|_{\mathcal{K}(X)}$ of the linear manifold generated by $\{\rho(\,.\,-s),\ s \in R^n\}$. Hence by (5.17), $\mathscr{H}(X)$ is the $\|\ \|_{0,p}$-completion of the linear manifold generated by $\{\rho(\,.\,-s),\ s \in R^n\}$. That is

(5.18) $\mathscr{H}(X)$ is the closed linear subspace of $H_0^{2,p}(R^n)$ generated by $\{\rho(\,.\,-s),\ s \in R^n\}$.

Let us now calculate $(\varphi, \rho(\,.\,-s))_{0,p}$. By definition $(\varphi, \rho(\,.\,-s))_{0,p} = \sum_{|\alpha|\leqslant p} \int_{R^n} D^\alpha\varphi(t) D^\alpha\rho(t-s)\, dt$. By Plancherel's Theorem and (5.14)

$$(\varphi, \rho(\,.\,-s))_{0,p} = \int e^{i(s,x)} \hat{\varphi}(x)\, dx$$

where $\hat{\varphi}(x)$ is the Fourier-Plancherel transform of φ. Hence $(\varphi, \rho(\,.\,-s))_{0,p} = \varphi(s)$ for all $\varphi \in C_0^\infty(R^n)$. Using continuity of $(\cdot, \cdot)_{0,p}$ in $\|\ \|_p$ and denseness of $C_0^\infty(R^n)$ in $H_0^{2,p}(R^n)$ we get that for all $f \in H_0^{2,p}(R^n)$

$$(5.19)\qquad (f, \rho(\,.\,-s))_{0,p} = f(s) \quad \text{a.e.}$$

Suppose that there is an $f \in H_0^{2,p}(R^n)$ such that

$$(f, \rho(\cdot - s))_{0,p} = 0$$

for all s. Then $f = 0$ a.e. by (5.19). That is, $H_0^{2,p}(R^n) = \mathscr{H}(X)$ in view of (5.18).

6. P. Lévy's Brownian Motion in Multiple Parameters.

P. Lévy's Brownian motion in R^n is the Gaussian process $X(t)$ $(t \in R^n)$ such that for each t, $EX(t) = 0$ and

$$R(t, s) = \frac{1}{2}(|t| + |s| + |t - s|)$$

where $|.|$ denotes the Euclidean distance in R^n. Molchan has shown in [6] that if $f \in \mathscr{H}(X)$ then for n an odd integer (and $2p = n + 1$),

(*a*) $D^\alpha f \in L^2(R^n)$ for $|\alpha| = p$,

(*b*) $D^\alpha f$ is locally in $L^2(R^n)$ for $|\alpha| < p$ and

(*c*) $$\langle f, g\rangle_{\mathscr{H}(X)} = c_n(\Delta^k f, \Delta^k g)_0 \qquad \text{for} \quad n = 4k - 1$$
$$= c_n \sum_{i=1}^{n} \left(\frac{\partial}{\partial t_i}\Delta^k f, \frac{\partial}{\partial t_i}\Delta^k g\right)_0 \qquad \text{for} \quad n = 4k + 1.$$

The explanation of the notation in (*a*), (*b*) and (*c*) is as follows.

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}}, \quad (\alpha_i' s \text{ are non negative integers}),$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

$D^\alpha f$ is the L^2-derivative of f, $(D^0 f = f)$. Δ is the Laplacian $\sum_{i=1}^{n} \frac{\partial^2}{\partial t_i^2}$, $(\ ,\ )_0$ is the $L^2(R^n)$ inner product and c_n is a constant depending only on n.

Let $G = R^n \setminus \{0\}$ and let

$$\text{(6.1)} \qquad \xi(\varphi) = \int_G X(t)\varphi(t)\,dt, \qquad \varphi \in C_0^\infty(G).$$

It is not hard to verify that $C_0^\infty(G)$ is contained in $\mathscr{H}(X; G)$, the RKHS of $X(t)$ with t restricted to G. (E.g., one may use Molchan's representation (4) of [6] to do this.) The following argument shows that $C_0^\infty(G)$ is a dense subset of $\mathcal{K}(X; G)$. Let u be an element of the latter space orthogonal to every φ in $C_0^\infty(G)$. Define the distribution

$$F_u(\varphi) = \int_G u(t)\varphi(t)\, dt. \tag{6.2}$$

Since $u \perp \varphi$ from (6.2) we have

$$\Delta^p F_u = 0 \quad \text{in } G \tag{6.3}$$

in the sense of distributions. Since the operator Δ^p is elliptic and hence hypoelliptic it follows that F_u belongs to $C^\infty(G)$ ([10], p. 535). This means that F_u is determined by a C^∞ function, say, v. From (6.2) we obtain $u = v$ since u is continuous. Finally from (6.2) and (6.3) we get

$$\Delta^p u = 0 \quad \text{in } G. \tag{6.4}$$

We shall show that (6.4) implies $u = 0$. Let $\tilde{u} \in \mathcal{K}(X)$ be the unique extension of u. If $n = 4k - 1$ (noting that $2p = n + 1$) the first formula in (c) gives

$$\|\tilde{u}\|^2_{\mathcal{K}(X)} = (-1)^p c_n(\Delta^p \tilde{u}, \tilde{u})_0 = 0$$

from (6.4) because $\tilde{u} = u$ on G and $G \cup \{0\} = R^n$. Thus $\tilde{u} = 0$ which implies $u = 0$ and our assertion is proved. The case $n = 4k + 1$ follows similarly upon using the second expression in (c). What we have shown is that $C_0^\infty(G)$ is dense in $\mathcal{K}(X; G)$. Since $\xi(\varphi)$ is given by (6.1), (2.8) holds (with $T = G$) and we conclude that $\{F_\varphi, \varphi \in C_0^\infty(G)\}$ is dense in $\mathcal{K}(\xi)$. The existence of the dual $\hat{\xi}$ now follows from the remark following Proposition 1 of Section 3. To find $\hat{\xi}$ one can apply the general Proposition 2 but it is simpler in this case to do it directly. From (2.6) and (2.8),

$$B(\varphi, \psi) = \langle F_\varphi, F_\psi \rangle_{\mathcal{K}(\xi)} = \langle \varphi, \psi \rangle_{\mathcal{K}(X;\, G)}.$$

It is now easy to verify that $\hat{\xi}(\varphi) = \xi(L\varphi)$ where L is the Peetre operator introduced in Proposition 2. In this case L turns out to be equal to $c.(-1)^p\Delta^p$, c being a constant. Thus we have.

Proposition 3. — *Let* ξ *be the generalized* GRF *in* (6.1) *given by Lévy's Brownian motion in* R^n *(n, an odd* *integer). Then the dual process* $\hat{\xi}$ *exists and is given by*

$$\hat{\xi}(\varphi) = c\xi((-1)^p \Delta^p \varphi). \tag{6.5}$$

Using sufficient condition (*b*) of Theorem 1, (2.6), (2.7) and Lemma 14, we get the following proposition.

PROPOSITION 4. — P. *Lévy's Brownian motion in* R^n *has Markov property relative to all open subsets of* DR^n such that $0 \notin \overline{D}$.

The above proposition generalizes and includes the work of [5], *and* [6].

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