ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 23, nº 4 (1973), p. 195-212
<http://www.numdam.org/item?id=AIF_1973__23_4_195_0>

Ann. Inst. Fourier, Grenoble
23, 4 (1973), 195-212.

BOUNDARY BEHAVIOUR OF HARMONIC FUNCTIONS IN A HALF-SPACE AND BROWNIAN MOTION [1]

by D. L. BURKHOLDER and R. F. GUNDY

The behaviour of harmonic functions in the half-space $\mathbf{R}_+^{n+1}$ has been discussed from two points of view: geometrical and probabilistic. In this paper, we compare these two view points with respect to (1) local convergence at the boundary and (2) the H^p-spaces. The results are as follows: (1) The existence of a nontangential limit for almost all points in a set E of positive Lebesgue measure in $\mathbf{R}^n (= \partial \mathbf{R}_+^{n+1})$ is *more* restrictive than the existence of a « fine » or probability limit almost everywhere in E when $n \geqslant 2$. When $n = 1$, the existence of a nontangential limit almost everywhere in E implies the existence of a « fine » limit almost everywhere in E and conversely. (2) For *all* $n \geqslant 1$, the nontangential maximal function of u belongs to $L^p (0 < p < \infty)$ if and only if the Brownian motion maximal function belongs to L^p. That is, in light of the results of Fefferman and Stein [10], we may say that the class H^p, defined probabilistically concides with H^p defined geometrically. This is proved in [3] for the half-plane $\mathbf{R}_+^2$. However, the arguments for $\mathbf{R}_+^2$ cannot be extended to

(1) This research was partially supported by NSF Grant GP-28154 at the University of Illinois and GP-19222 at Rutgers University. The authors would also like to thank Professor Lennart Carleson for many stimulating conversations on the subject of this paper at the Institut Mittag-Leffler, Djursholm, Sweden.

$\mathbf{R}^3_+$, basically, because of the potential-theoretic distinction between dimensions two and three. That this distinction is exhibited in the local statement (1) but not in the global statement (2) is something of a surprise.

From the geometrical view point, the main results on local convergence are due to Marcinkiewicz and Zygmund [14], Spencer [18], and Privalov [17] for $n = 1$, and to Calderón [4], [5] and Stein [19] for $n > 1$. Theorem A below is a summary statement of these results. First, however, we need some notation. The cone in $\mathbf{R}^{n+1}_+$ with vertex at $x \in \mathbf{R}^n$, height k, and angle a, is denoted by

$$\Gamma(x; a, k) = \{(s, y) : |x - s| < ay, 0 < y < k\}.$$

The nontangential maximal function of a function u defined on $\mathbf{R}^{n+1}_+$ is defined as

$$\mathrm{N}(u; a, k)(x) = \sup_{(s, y) \in \Gamma(x; a, k)} |u(s, y)|$$

and the area function

$$\mathrm{A}(u; a, k)(x) = \left(\iint_{\Gamma(x; a, k)} |\nabla u(s, y)|^2 y^{1-n}\, dx\, dy\right)^{\frac{1}{2}}.$$

Notice that both $\mathrm{N}(u; a, k)$ and $\mathrm{A}(u; a, k)$ are monotone increasing in the parameters a and k.

THEOREM A. — *Let u be harmonic in $\mathbf{R}^{n+1}_+$. The following subsets of $\mathbf{R}^n = \partial\mathbf{R}^{n+1}_+$ are equal almost everywhere :*

$$(1) \qquad \{x : \mathrm{N}(u; a, k)(x) < \infty\};$$

$$(2) \qquad \{x : \mathrm{A}(u; a, k)(x) < \infty\};$$

$$(3) \qquad \{x : \lim_{\substack{(s, y) \to x \\ (s, y) \in \Gamma(x; a, k)}} u(s, y) \text{ exists and is finite}\}.$$

A simplified proof of Theorem A, based on distribution function inequalities between the area function and the nontangential maximal function, is given in [2].

In order to state the probabilistic analogue of Theorem A, we recall the following facts : Let u be an harmonic function defined in $\mathbf{R}^{n+1}_+$ and let $z_t = (x_t, y_t)$, $t \geqslant 0$ be $(n + 1)$-dimensional Brownian motion started from the point

$(x_0, y_0) \in \mathbf{R}_+^{n+1}$, stopped at time $\tau = \inf\{t : y_t = 0\}$. We refer to this process as Brownian motion in $\mathbf{R}_+^{n+1}$. It follows from Ito's change of variables formula (see McKean [15]) that $u(x_t, y_t)$ is a stochastic integral of the form

$$u(x_t, y_t) = u(x_0, y_0) + \int_0^t \langle \nabla u(z_s), dz_s \rangle .$$

We let P_{x_0, y_0} denote the measure on the space of trajectories from (x_0, y_0) to $\mathbf{R}^n$ corresponding to the process (x_t, y_t), $t \geqslant 0$. We may also define the conditional measure P_{x_0, y_0}^x corresponding to a « Brownian » process that starts at (x_0, y_0) and terminates at the point $x \in \mathbf{R}^n$. Explicit formulas for P_{x_0, y_0} and P_{x_0, y_0}^x, as well as a discussion of these processes, is given by Doob [9].

Let the Brownian maximal function of u be defined as

$$u^* = \sup_{t<\tau} |u(x_t, y_t)| .$$

The Brownian analogue of the area function $A(u)$ is given by

$$S(u) = \left[u^2(x_0, y_0) + \int_0^\tau |\nabla u(x_t, y_t)|^2 \, dt \right]^{\frac{1}{2}} .$$

With these definitions, we may state the following theorem.

THEOREM A'. — *Let u be harmonic in $\mathbf{R}_+^{n+1}$. The following subsets of $\mathbf{R}^n = \partial \mathbf{R}_+^{n+1}$ are equal almost everywhere (with respect to Lebesgue measure) for every $(x_0, y_0) \in \mathbf{R}_+^{n+1}$:*

(1') $\{x : P_{x_0, y_0}^x (u^* < \infty) > 0\}$

(2') $\{x : P_{x_0, y_0}^x (S(u) < \infty) > 0\}$

(3') $\{x : P_{x_0, y_0}^x (\lim_{t \to \tau} u(x_t, y_t)$ *exists and is finite*$) > 0\}$.

We omit the details of the proof of Theorem A'; it follows from the fact that the sets $\{u^* < \infty\}$ and $\{S(u) < \infty\}$ are equal P_{x_0, y_0}-almost everywhere. The set (3') can also be characterized as the set where u has a fine boundary limit in the sense of Lelong [13] and Naïm [16]. This fact is due to Doob [8].

One purpose of this paper is to compare the local behaviour of u described in Theorems A and A'. We have the following:

THEOREM 1. — *a) For u harmonic in $\mathbf{R}^2_+$, the sets of Theorem A are equal almost everywhere with respect to Lebesgue measure on $\mathbf{R}^1$ to the sets of Theorem A'. b) For u harmonic in $\mathbf{R}^{n+1}_+$, $n \geqslant 2$, the sets of Theorem A are contained in those of Theorem A', up to sets of measure zero. The converse is not true.*

Part *a*) of Theorem 1 is due to Brelot and Doob [1] and Constantinescu and Cornea [7]. Part *b*) is due in part to Brelot and Doob [1]; our contribution is to show that, without additional hypotheses, the sets of Theorem A' can be strictly larger than those of Theorem A when $n \geqslant 2$. (If, however, one adds the hypothesis that u is positive, or even bounded below in each cone $\Gamma(x; a, k)$ for $x \in \mathrm{E}$ of positive measure — the bound may depend on x — then u has a nontangential limit almost everywhere in E (Carleson [6]), as well as a fine limit almost everywhere in E (Brelot and Doob [1]).)

We now consider the geometric and probabilistic descriptions of the Hardy classes H^p. For $\mathbf{R}^2_+$, it is shown in [3] that H^p, $0 < p < \infty$ may be described as the space of real harmonic functions u such that

$$\sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{N}(u; a, k)|^p \, dx < \infty. \tag{4}$$

Fefferman and Stein [10] extend this result to the H^p spaces introduced by Stein and Weiss [20] for harmonic functions in $\mathbf{R}^{n+1}_+$, $n \geqslant 2$. Therefore, we take (4) as the definition of H^p.

The probabilistic analogue of condition (4) is

$$\sup_{y>0} \int_{\mathbf{R}^n} \mathrm{E}_{x,y}(|u^*|^p) \, dx < \infty$$

where $\mathrm{E}_{x,y}$ is the expectation corresponding to $\mathrm{P}_{x,y}$.

Fefferman and Stein show that the area function and nontangential maximal function are related as follows:

THEOREM B. — *Let u be harmonic in $\mathbf{R}^{n+1}_+$. Then for all p in the interval $0 < p < \infty$,*

$$\sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{A}(u; a, k)(x)|^p \, dx \leqslant c_{p,a} \sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{N}(u; a, k)(x)|^p \, dx.$$

Furthermore, if the left-hand side of this inequality is finite, u *may be normalized to vanish at infinity and with this normalization*

$$\sup_{k>0} \int_{\mathbf{R}} |N(u;\, a,\, k)(x)|^p\, dx \leqslant C_{p,a} \sup_{k>0} \int_{\mathbf{R}^n} |A(u;\, a,\, k)(x)|^p\, dx.$$

The probabilistic version of Theorem B is stated in [3] for $\mathbf{R}^2_+$ ([3], Lemma 4). The proof, however, is valid in any number of dimensions. We restate it here as Theorem B′.

THEOREM B′. — *Let* u *be harmonic in* $\mathbf{R}^{n+1}_+$. *For all* p *in the interval* $0 < p < \infty$,

$$c_p \sup_{y>0} \int_{\mathbf{R}^n} E_{x,y}(|S(u)|^p)\, dx \leqslant \sup_{y>0} \int_{\mathbf{R}^n} E_{x,y}(|u^*|^p)\, dx \leqslant C_p \sup_{y>0} \int_{\mathbf{R}^n} E_{x,y}(|S(u)|^p)\, dx.$$

The second purpose of this paper is to compare Theorems B and B′.

THEOREM 2. — *Let* u *be harmonic in* $\mathbf{R}^{n+1}_+$, $n \geqslant 2$. *Then*

$$c_{p,a} \sup_{y>0} \int_{\mathbf{R}^n} E_{x,y}(|u^*|^p)\, dx \leqslant \sup_{k>0} \int_{\mathbf{R}^n} |N(u;\, a,\, k)|^p\, dx \leqslant C_{p,a} \sup_{y>0} \int_{\mathbf{R}^n} E_{x,y}(|u^*|^p)\, dx.$$

Thus, while the probabilistic and nontangential local convergence criteria are different in $\mathbf{R}^{n+1}_+$ for $n \geqslant 2$, the H^p spaces, defined probabilistically or geometrically, coincide in all dimensions. It then follows from Theorems B and B′ that the Brownian and nontangential area functions have equivalent L^p-norms for $0 < p < \infty$.

Proof of Theorem 1. — Since the first two statements of Theorem 1 may be found in Brelot and Doob [1], we prove only the last by constructing an example : There is a function u that is harmonic in $\mathbf{R}^{n+1}_+$ such that *a*) $\lim_{t \to \tau} u(x_t,\, y_t)$ exists and is finite with $P^x_{x_0, y_0}$-probability one for almost all $x \in \mathbf{R}^n$; *b*) nontangential convergence of u holds for no $x \in Q$, the unit cube in $\mathbf{R}^n$. That is, the set (1′) is strictly larger than the set (1).

For simplicity, we carry out the details for $\mathbf{R}^3_+$. Roughly speaking, we construct a bed with an infinite number of vertical spines of varying height on the unit square. The function u defined on $\mathbf{R}^3_+$ is to be large and of varying sign at the end of each spine, but small nearly everywhere else. The set where u is largest — the tips of the spines — has small capacity, so the Brownian paths from (x_0, y_0) miss these points with high probability. On the other hand, any cone $\Gamma(x)$, $x \in Q$ is punctured by infinitely many of the spines, so that the oscillation of u over $\Gamma(x)$ is infinite for every $x \in Q$.

Let

$$D_n = \left\{ \left(\frac{2j-1}{2^n}, \frac{2k-1}{2^n}, \frac{a^{-1}}{2^{n-1}} \right) : j = 1, \ldots, 2^{n-1}, k = 1, \ldots, 2^{n-1} \right\}$$

so that $\Gamma(x; a, k)$ contains at least one point of D_n for each $n \geqslant n(a, k)$. The function u to be constructed satisfies

$$u(x, y) \geqslant n, \qquad (x, y) \in D_n$$

for n odd, and

$$u(x, y) \leqslant -n, \qquad (x, y) \in D_n$$

for n even. Therefore, the oscillation of u over the cone $\Gamma(x; a, k)$, $x \in Q$ is infinite, so that u has a nontangential limit nowhere in the set Q. For simplicity, we may assume that $a = 1$, $k = 2$, and denote the corresponding cone by $\Gamma(x)$.

The function u to be constructed is of the form

$$u = \sum_{j=1}^{\infty} u_j$$

where each u_j is harmonic in all of $\mathbf{R}^3$ and the series is uniformly convergent on compact subsets of $\mathbf{R}^3_+$. Therefore,

$$\lim_{t \to \tau} u_j(x_t, y_t) = u_j(x_\tau, 0)$$

almost everywhere with respect to P_{x_0, y_0}. Also, we show that with P_{x_0, y_0}-probability one,

$$\sum_{j=1}^{\infty} u_j^* < \infty$$

so that by the Lebesgue dominated convergence theorem,

$$\lim_{t\to\tau} u(x_t, y_t) = \sum_{j=1}^{\infty} \lim_{t\to\tau} u_j(x_t, y_t) = \sum_{j=1}^{\infty} u_j(x_\tau, 0)$$

almost everywhere P_{x_0, y_0}. By definition of the conditional measures $P^x_{x_0, y_0}$, we have

$$P^x_{x_0, y_0}\left(\lim_{t\to\tau} u(x_t, y_t) \quad \text{exists and is finite}\right) = 1$$

for almost every $x \in Q$, with respect to Lebesgue measure. In other words, u has a fine limit for almost every $x \in Q$, but a nontangential limit nowhere in Q.

The basic device in the construction is Runge's theorem for harmonic functions in $\mathbf{R}^n$. (Walsh [21]; also see Lelong's review [12], for other references.)

Runge's Theorem for $\mathbf{R}^{n+1}$. — *Let* K *be a compact set in* $\mathbf{R}^{n+1}$ *such that* $\mathbf{R}^{n+1} - K$ *is connected. Suppose that* u *is harmonic on an open set containing* K. *Then* u *can be uniformly approximated by harmonic polynomials on* K.

We now proceed with the construction. For convenience, assume that the initial point (x_0, y_0) for the Brownian motion satisfies $y_0 \geqslant 2$. Let $0 < \varepsilon_n < \frac{1}{2^{n+1}}$, $b_n > y_0 + n$ be chosen so that

$$(5) \quad P_{x_0, y_0}((x_t, y_t) \in Q_n - T_n \quad \text{for all} \quad 0 \leqslant t \leqslant \tau) \geqslant 1 - \frac{1}{2^n}$$

where

$$Q_n = [-b_n, b_n] \times [-b_n, b_n] \times [0, 2b_n]$$

and

$$T_n = \left\{(s, y) : \ |x - s| < \varepsilon_n, \ 0 \leqslant y < \frac{1}{2^{n-1}} + \varepsilon_n, \right.$$
$$\left. \text{for some point} \left(x, \frac{1}{2^{n-1}}\right) \in D_n\right\}.$$

Notice that T_n is the union of 2^{2n-2} disjoint cylinders or « spines » each of which contains a point of D_n in its interior. Notice also that, because of the transience of Brownian motion in $\mathbf{R}^3$, the choice of ε_n, b_n in (5) is possible in $\mathbf{R}^3$ but not in $\mathbf{R}^2$. The set $K_n = (Q_n - T_n) \cup D_n$ is compact and

$\mathbf{R}^3 - K_n$ is connected, so that the hypotheses of Runge's theorem apply. Let U and V be disjoint open sets such that

$$Q_n - T_n \subset U$$

and

$$D_n \subset V.$$

Let $w(x, y)$ be defined on $U \cup V$, equal to zero on U, λ_n on V, where λ_n is a constant to be chosen later. Then w is harmonic on $U \cup V$ and by Runge's theorem, there is a harmonic polynomial u_n such that

$$|u_n(x, y) - w(x, y)| < \frac{1}{2^n} \qquad \text{on} \qquad K_n.$$

Therefore,

$$|u_n(x, y)| < \frac{1}{2^n} \qquad \text{for} \qquad (x, y) \in Q_n - T_n \tag{6}$$

and

$$|u_n(x, y) - \lambda_n| < \frac{1}{2^n} \qquad \text{for} \qquad (x, y) \in D_n.$$

The first claim is that the series $\sum_{n=1}^{\infty} |u_n(x, y)|$ converges uniformly on compact subsets of $\mathbf{R}^3_+$: Any compact subset of $\mathbf{R}^3_+$ is a subset of $Q_n - T_n$ for all large n, so uniform convergence follows from (6). It follows that

$$u = \sum_{j=1}^{\infty} u_j$$

is harmonic in $\mathbf{R}^3_+$.

Finally, we must choose the constants λ_n. Let $\lambda_1 = 2$ and note that the point $\left(\frac{1}{2}, \frac{1}{2}, 1\right) \in D_1$ but

$$\left(\frac{1}{2}, \frac{1}{2}, 1\right) \in Q_n - T_n$$

for all $n > 1$. Therefore

$$\begin{aligned} u\left(\frac{1}{2}, \frac{1}{2}, 1\right) &= \sum_{j=1}^{\infty} u_j\left(\frac{1}{2}, \frac{1}{2}, 1\right) \\ &> 2 - \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1. \end{aligned}$$

Suppose $\lambda_1, \ldots, \lambda_{n-1}$ have been chosen so that $u(x, y) \geqslant k$ for $(x, y) \in D_k$, k odd, and $u(x, y) \leqslant -k$ for $(x, y) \in D_k$, k even. Simply choose λ_n so that

$$\inf_{(x, y) \in D_n} \sum_{k=1}^{n-1} u_k(x, y) + \lambda_n > n + 1$$

if n is odd. Then

$$\begin{aligned} u(x, y) &> n + 1 - \frac{1}{2^n} - \frac{1}{2^{n+1}} - \cdots \\ &\geqslant n \end{aligned}$$

for $(x, y) \in D_n$ since this also implies $(x, y) \in Q_m - T_m$ for $m > n$. If n is even, then choose λ_n so that

$$\sup_{(x, y) \in D_n} \sum_{k=1}^{n-1} u(x, y) + \lambda_n < -(n + 1);$$

then $u(x, y) \leqslant -n$ for $(x, y) \in D_n$ in the same way. Finally, by (5) and (6),

$$P_{x_0, y_0}\left(u_n^* > \frac{1}{2^n}\right) \leqslant \frac{1}{2^n}$$

so that $\sum_{n=1}^{\infty} u_n^* < \infty$ almost everywhere (P_{x_0, y_0}). This completes the construction.

Proof of Theorem 2. — We begin with a series of lemmas.

LEMMA 1. — *For* $b > a > 0$, *and* $\lambda > 0$,

$$m(N(u;\, b, k) > \lambda) \leqslant C m(N(u;\, a, k) > \lambda)$$

The choice of C *depends only on the dimension* n *and the ratio* a/b. *In particular,*

$$\|N(u;\, b, k)\|_p^p \leqslant C \|N(u;\, a, k)\|_p^p.$$

This lemma corresponds to Lemma 2 of [2], stated for $N(u;\, a)$ and $N(u;\, b)$. The proof, however, is valid for any measurable function u defined on $\mathbf{R}_+^{n+1}$. Therefore, we may simply apply that argument to

$$\begin{aligned} u_k(x, y) &= u(x, y) \quad \text{if} \quad y \leqslant k \\ &= 0 \ \text{otherwise.} \end{aligned}$$

The second assertion of the lemma follows from the integration formula

$$\|N(u;\,a,\,k)\|_p^p = p\int_0^\infty \lambda^{p-1} m(N(u;\,a,\,k) > \lambda)\,d\lambda.$$

The next lemma is due to Hardy and Littlewood [11] for the case $p < 1$. They state it without proof; a full proof is given by Fefferman and Stein (Lemma 2 in [10]).

LEMMA 2. — *Let* B_R *be a ball in* $\mathbf{R}^{n+1}$ *with center at* (x_0, y_0), *radius* $R > 0$, *and* $B_r \subset B_R$ *be another ball with the same center but with radius* $r < R$. *Then for* $0 < p < \infty$,

$$\sup_{(s,\,t)\in B_r} |u(s,\,t)|^p \leqslant C_{p,\,r/R}\,\frac{1}{m(B_R)}\int_{B_R} |u(x,\,y)|^p\,dx\,dy.$$

LEMMA 3. — *Let*

$$D(u;\,a,\,k) = \sup_{(s,\,y)\in\Gamma(x;\,a,\,k)} y|\nabla u(s,\,y)|\,;$$

then

$$D(u;\,a,\,k) \leqslant CN(u;\,b,\,2k)$$

for $b > a$, *with* C *depending only on the dimension* n *and the ratio* a/b.

This lemma is taken from Stein [19] (see Lemma 4). We omit the proof.

LEMMA 4. — *Let* u *be harmonic in* $\mathbf{R}_+^{n+1}$ *and satisfy the condition*

$$\sup_{y>0}\int_{\mathbf{R}^n} |u(x,\,y)|^p\,dx < \infty$$

for some p *in the interval* $0 < p < \infty$. *Then*

$$\|N(u_\alpha;\,a,\,k)\|_p < \infty \tag{7}$$

for all $a > 0, k > 0$, *where* $u_\alpha(x,\,y) = u(x,\,y+\alpha)$ *for* $\alpha > 0$. *Furthermore, there exists a* $k_0 > 0$ *such that for all* $k \geqslant k_0$ *we have*

$$\|N(u_\alpha;\,2a,\,2k)\|_p \leqslant C\|N(u_\alpha;\,a,\,k)\|_p. \tag{8}$$

The constant k_0 *depends on* u, *but* C *depends only on* p *and the dimension* n.

Proof. — If $\lim_{k\to\infty} \|N(u_\alpha; a, k)\|_p < \infty$ for some $a > 0$, then the same is true for $2a$ by Lemma 1. Also, (8) holds for $k > 0$ sufficiently large.

We now assume that $\lim_{k\to\infty} \|N(u_\alpha; a, k)\|_p = \infty$ for $a \leqslant \frac{1}{2}$. Consider the ball $B(x, \alpha/2)$ with center at $(x, \alpha/2)$, radius $3\alpha/2$. Then $B(x, \alpha/2)$ contains the cone $\Gamma(x; a, \alpha)$ and all points of $\Gamma(x; a, \alpha)$ lie at a distance of more than $(3/2 - 1/\sqrt{2})\alpha$ from the boundary of the ball $B(x, \alpha/2)$. Therefore, by Lemma 2,

$$|N(u_\alpha; a, \alpha)(x)|^p \leqslant C_p \frac{1}{m(B(x, \alpha/2))} \iint_{B(x, \alpha/2)} |u_\alpha(s, y)|^p \, ds \, dy.$$

If we integrate both sides of the above inequality with respect to x, and use Fubini's theorem, we obtain

$$(9) \quad \int_{\mathbf{R}^n} |N(u_\alpha; a, \alpha)|^p \leqslant C_p \sup_{0<y<3\alpha} \int_{\mathbf{R}^n} |u(x, y)|^p \, dx \\ \leqslant C_p \sup_{y>0} \int_{\mathbf{R}^n} |u(x, y)|^p \, dx < \infty.$$

That is, we have shown that $\|N(u_\alpha; a, k)\|_p < \infty$ for $k = \alpha$ provided $a \leqslant 1/2$. The same kind of argument shows that if $\|N(u_\alpha; a, k)\|_p < \infty$ for some $k < \infty$, then

$$(10) \quad \|N(u_\alpha; a, 2k)\|_p^p \leqslant \|N(u_\alpha; a, k)\|_p^p \\ + C_p \sup_{y>0} \int_{\mathbf{R}^n} |u(x, y)|^p \, dx.$$

In fact, if

$$(11) \quad M(u_\alpha; a, 2k)(x) \\ = \sup \{|u_\alpha(s, y)| : (s, y) \in \Gamma(x; a, 2k) - \Gamma(x; a, k)\}$$

then

$$(12) \quad |N(u_\alpha; a, 2k)|^p \leqslant |N(u_\alpha; a, k)|^p + |M(u_\alpha; a, 2k)|^p.$$

If $B(x, 3k/2)$ is the ball centered at $(x, 3k/2)$, then the « top half » of the cone $\Gamma(x; a, 2k)$, that is, the set $\Gamma(x; a, 2k) - \Gamma(x; a, k)$, is contained in the ball $B(x, 3k/2)$, and lies at a distance of $\left(\frac{3-\sqrt{5}}{2}\right)k$ from the boundary of the ball. Therefore, again by Lemma 1 and the argument leading to (9), we find that

$$\|\mathrm{M}(u_\alpha;\, a,\, 2k)\|_p^p \leqslant \mathrm{C}_p \sup_{y>0} \int_{\mathbf{R}^n} |u(x,\, y)|^p\, dx.$$

Therefore, (10) follows from this and inequality (12).

The argument to this point shows that $\|\mathrm{N}(u_\alpha;\, a,\, k)\|_p < \infty$ for all $k > 0$ since this statement is true for $k = \alpha,\, 2\alpha,\, \ldots$ A slight amplification of the argument shows that $\|\mathrm{N}(u_\alpha;\, a,\, k)\|_p$ is a continuous, increasing function of k with range equal to the interval $[0,\, \infty)$. Therefore, for some $k_0 > 0$, we have

$$\|\mathrm{N}(u_\alpha;\, a,\, k_0)\|_p^p = \sup_{y>0} \int_{\mathbf{R}^n} |u(x,\, y)|^p\, dx.$$

For any $k \geqslant k_0$, from (10) we have

$$\begin{aligned}\|\mathrm{N}(u_\alpha;\, a,\, 2k)\|_p^p &\leqslant \|\mathrm{N}(u_\alpha;\, a,\, k)\|_p^p + \mathrm{C}_p \sup_{y>0} \int_{\mathbf{R}^n} |u(x,\, y)|^p\, dx \\ &\leqslant (1 + \mathrm{C}_p)\|\mathrm{N}(u_\alpha;\, a,\, k)\|_p^p.\end{aligned}$$

Finally, by Lemma 1, we may replace a by $2a$ and obtain

$$\|\mathrm{N}(u_\alpha;\, 2a,\, 2k)\|_p^p \leqslant \mathrm{C}_p'\|\mathrm{N}(u_\alpha;\, a,\, k)\|_p^p.$$

The lemma is proved.

LEMMA 5. — *Given* D > 0 *and* $0 < p < \infty$, *let* $f_i,\, i = 1, 2$, *be a pair of functions that satisfy the inequality*

$$\int |f_2|^p \leqslant \mathrm{D} \int |f_1|^p < \infty. \tag{13}$$

Then

$$\int |f_1|^p \geqslant 2 \int_{\{|f_1| > (2\mathrm{D})^{-1/p}|f_2|\}} |f_1|^p$$

Proof. Since $\|f_1\|_p < \infty$, either the conclusion of the lemma holds, or, with strict inequality, we have

$$\int |f_1|^p < 2 \int_{\{|f_1| \leqslant (2\mathrm{D})^{-1/p}|f_2|\}} |f_1|^p \leqslant \frac{1}{\mathrm{D}} \int |f_2|^p \leqslant \int |f_1|^p,$$

which is a contradiction.

Given any point $(s, y) \in \mathbf{R}_+^{n+1}$, recall that $P_{s,y}^x$ is the measure associated with conditional Brownian motion with initial point (s, y) and terminal point $x \in \mathbf{R}^n$.

LEMMA 6. — *Let* $B(x', y')$ *be the ball in* $\mathbf{R}_+^{n+1}$ *with center at* (x', y'), *radius* $\theta y'$, $0 < \theta < 1$, *and with* $|x' - x| \leqslant ay'$. *If* $|s - x| \leqslant ay$, $y \geqslant 2y'$, *then*

$$P_{s,y}^x((x_t, y_t) \quad \textit{hits} \quad B(x', y')) \geqslant C > 0$$

where C *depends only on* θ *and* a.

Proof. — Let $\tau = \inf\{t : y_t = y'\}$. The conditional measure associated with the random vector (x_τ, y_τ) is given by

$$h_x(s, y) P_{s,y}^x((x_\tau, y_\tau) \in A) = \int h_x(x_\tau, y_\tau) P_{s,y}(dx_\tau, dy_\tau) \quad \{(x_\tau, y_\tau) \in A\}$$

where h_x is the Poisson kernel for $\mathbf{R}_+^{n+1}$ with pole at $x \in \mathbf{R}^n$. This formula may be obtained by a standard stopping time argument. The probability $P_{s,y}^x((x_\tau, y_\tau) \in A)$ has a density with respect to Lebesgue measure on the hyperplane $y = y'$ in $\mathbf{R}_+^{n+1}$ given by

$$q^x(w; (s, y), y') = C_n \frac{y - y'}{(|w - s|^2 + |y - y'|^2)^{\frac{(n+1)}{2}}} \frac{h_x(w, y')}{h_x(s, y)}.$$

It follows that

$$\int_{S(x', y')} q^x(w; (s, y), y')\, dw \geqslant C > 0$$

where $S(x', y')$ is the projection of $B(x', y')$ on the hyperplane $y = y'$. (The constant C depends only on θ and a.) The integral represents the probability that the n-dimensional sphere $S(x', y')$ is hit by a conditional path from (s, y) to x before the path hits the complement of $S(x', y')$ on the hyperplane $y = y'$. Since this probability is smaller than the one we wish to estimate, the lemma is proved.

LEMMA 7. — *Let* u *be a continuous function in* $\mathbf{R}_+^{n+1}$. *Then*

$$\sup_{y>0} \int_{\mathbf{R}^n} P_{x,y}(u^* > \lambda)\, dx \leqslant C \sup_{k>0} m(N(u; a, k) > \lambda).$$

Remarks. — This inequality is stated as part of Theorem 3 of [3] for the case $n = 1$. The proof may be extended without difficulty for $n > 1$, so we omit the details. It should be noted, however, that in Theorem 3 of [3], we assume that u is harmonic. As is clear from the proof, this assumption is used to obtain the converse inequality only.

We are now in a position to prove Theorem 2. First, we note that

$$\sup_{y>0} \int_{\mathbf{R}^n} \mathrm{E}_{xy}(|u^*|^p)\, dx \leqslant \mathrm{C} \sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{N}(u;\, a,\, k)|^p\, dx$$

where C depends only on a. This follows immediately by integrating the stronger estimate given in Lemma 7.

Now, we prove that

$$\sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{N}(u;\, a,\, k)|^p\, dx \leqslant \mathrm{C} \sup_{y>0} \int_{\mathbf{R}^n} \mathrm{E}_{x,y}(|u^*|^p)\, dx$$

where C depends only on a and p. We may assume that the right hand side is finite, so that, in particular,

$$\sup_{y>0} \int_{\mathbf{R}^n} |u(x,\, y)|^p\, dx < \infty.$$

The hypothesis of Lemma 4 is satisfied, and therefore, we have

$$\|\mathrm{N}(u_\alpha;\, 2a,\, 2k)\|_p^p \leqslant \mathrm{C}\|\mathrm{N}(u_\alpha;\, a,\, k)\|_p^p \tag{14}$$

with the right hand side finite for $\alpha > 0$. We now apply Lemma 5 with $f_1 = \mathrm{N}(u_\alpha;\, a,\, k)$ and $f_2 = \mathrm{N}(u_\alpha;\, 2a,\, 2k)$. The hypothesis of Lemma 5 is satisfied with the constant $\mathrm{D} = \mathrm{C}$ where C is given in Lemma 1, independent of a, α, m, and k. Let

$$\mathrm{G} = \{x : \mathrm{N}(u_\alpha;\, a,\, k)(x) \geqslant (2\mathrm{C})^{-1/p}\mathrm{N}(u_\alpha;\, 2a,\, 2k)(x)\}.$$

From this and Lemma 3, we may conclude that

$$\mathrm{G} \subseteq \left\{x : \mathrm{D}\left(u_\alpha;\, \frac{3a}{2},\, \frac{3}{2}\, k\right)(x) \leqslant \mathrm{CN}(u_\alpha;\, a,\, k)(x)\right\} \tag{15}$$

for another constant C independent of a, α, m, or k. Fix a point $x \in \mathrm{G}$ and consider the cone $\Gamma(x;\, a,\, k) = \Gamma$. We may select a point $(x', y') \in \Gamma$ and a ball $B(x', y')$ with center (x', y'), radius $\theta y'$ such that $|u_\alpha(s, t)| \geqslant \frac{1}{4} N(u_\alpha; a, k)(x)$ for every point $(s, t) \in B(x', y')$. This may be done as follows: Choose $(x', y') \in \Gamma$ so that $|u_\alpha^*(x', y')| > \frac{1}{2} N(u_\alpha; a, k)(x)$. Since $x \in G$, we know (15) holds so that

$$t|\nabla u_\alpha(s, t)| \leqslant CN(u_\alpha; a, k)(x)$$

for all points (s, t) in a ball of radius $\theta y'$, centered at (x', y'), and contained in the cone $\Gamma\left(x; \frac{3a}{2}, \frac{3}{2} k\right)(x)$. The constant θ depends only on the angle a, at this point; we may assume $\theta < \frac{1}{2}$. By the mean value theorem,

$$|u_\alpha(s, t) - u_\alpha(x', y')| \leqslant 2CN(u_\alpha; a, k)(x) \frac{(|x' - s)^2 + (y' - t)^2)^{\frac{1}{2}}}{y'}$$

for all points $(s', t') \in B(x', y')$. Now choose θ so that $8\theta C < 1$; it follows from the above inequalities that

$$|u_\alpha(s, t)| \geqslant |u_\alpha(x', y')| - |u_\alpha(s, t) - u_\alpha(x', y')| \geqslant \frac{1}{4} N(u_\alpha; a, k)(x)$$

for all $(s', t') \in B(x', y')$ with radius $\theta y'$.

If we now apply Lemma 6 to $B(x', y')$, we obtain

$$(16) \quad P_{s,y}^x\left(u_\alpha^* \geqslant \frac{1}{4} N(u_\alpha; a, k)(x)\right) \geqslant P_{s,y}^x((x_t, y_t) \text{ hits } B(x', y')) \geqslant C > 0$$

for $x \in G$, $|s - x| \leqslant ay$, $y \geqslant 2y'$. Notice that $y' \leqslant k$, so points (s, y) such that $|s - x| \leqslant ay$, $y \geqslant 2k$ satisfy the above requirements. The last restriction, $y \geqslant 2k$, prevents us from making a direct estimation of integrals from the probabilities (16). To overcome this obstacle, we cut into G in the following way: Let R be chosen large enough so that $G_R = G \cap \{|x| \leqslant R\}$ satisfies

$$(17) \quad \int_{\mathbf{R}^n} |N(u_\alpha; a, k)(x)|^p \, dx \leqslant 2C \int_{G_R} |N(u_\alpha; a, k)(x)|^p \, dx$$

where $C = C(14)$. With this choice of R, let y_0 be chosen so that $y_0 \geqslant 2k$ and such that each point (s, y_0) in the n-dimensional ball $|s| \leqslant \frac{a}{2} y_0$ is contained in the cone $\Gamma(x;\, a,\, y_0)$ for every x such that $|x| \leqslant R$. In particular, since $y_0 \geqslant 2k$, the points in the ball $\left\{(s, y_0) : |s| \leqslant \frac{a}{2} y_0\right\}$ satisfy the requirements of inequality (16) for every $x \in G_R$. Let E^x_{s,y_0} be the conditional expectation corresponding to P^x_{s,y_0} and $\chi_{G_R}(\ )$ the indicator function of the set G_R; we have

$$\begin{aligned} E_{s,y_0}(|u^*_\alpha|^p) &= E_{s,y_0}(E^x_{s,y_0}(|u^*_\alpha|^p)) \\ &\geqslant E_{s,y_0}(E^x_{s,y_0}(|u^*_\alpha|^p)\chi_{G_R}(x)) \\ &\geqslant C E_{s,y_0}(|N(u_\alpha;\, a,\, k)(x)|^p \chi_{G_R}(x)) \\ &= C \int_{R^n} \chi_{G_R}(x)|N(u_\alpha;\, a,\, k)(x)|^p \frac{y_0}{(|s-x|^2 + y_0^2)^{(n+1)/2}}\, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{y>0} \int_{R^n} E_{s,y}(|u^*_\alpha|^p)\, ds \geqslant \int_{|s| \leqslant \frac{a}{2} y_0} E_{s,y_0}(|u^*_\alpha|^p)\, ds \\ &\geqslant C \int_{|s| \leqslant \frac{a}{2} y_0} \int_{R^n} \chi_{G_R}(x)|N(u_\alpha;\, a,\, k)(x)|^p \frac{y_0}{(|s-x|^2 + y_0^2)^{\frac{(n+1)}{2}}}\, dx\, ds \\ &\geqslant C \int_{R^n} \chi_{G_R}(x)|N(u_\alpha;\, a,\, k)(x)|^p\, dx \\ &\geqslant C \int_{R^n} |N(u_\alpha;\, a,\, k)(x)|^p\, dx. \end{aligned}$$

Here we have used Fubini's theorem, inequality (17), and the fact that $y_0 \geqslant 2R$ and $|s| \leqslant \frac{a}{2} y_0$ implies

$$\frac{y_0}{(|s-x|^2 + y_0^2)^{\frac{(n+1)}{2}}} \simeq \frac{1}{y_0^n}.$$

In summary, we have shown that

$$\sup_{y>0} \int_{R^n} E_{s,y}(|u^*_\alpha|^p)\, ds \geqslant C \int_{R^n} |N(u_\alpha;\, a,\, k)(x)|^p\, dx$$

for $k \geqslant k_0$ and all $\alpha > 0$. By successive applications of the monotone convergence theorem, we finally conclude that

$$\sup_{y>0} \int_{\mathbf{R}^n} \mathrm{E}_{s,y}(|u^*|^p)\, dx \geqslant \mathrm{C} \sup_{k>0} \int_{\mathbf{R}^n} |\mathrm{N}(u;\, a,\, k)(x)|^p\, dx.$$

Theorem 2 is proved.

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Manuscrit reçu le 8 décembre 1972.
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