Representations of the Fundamental Groups of Triangulated 3-Manifolds

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Contents

Disclaimer

During 2014, an Honours thesis was completed by the author which contained some of the topics which are contained in this thesis. In particular, much of the material in Chapter 1 of this thesis, but with less detail in many places and no examples, appeared in that Honours thesis. This material in Chapter 1 has been retained as it comprises a necessary background for the remainder of this thesis.

Notation

- $[m]$ denotes the set $\{1, \ldots, m\}$
- Given sets $X, X_1, \ldots, X_n, Y, Y_1, \ldots, Y_n$ with $X_i \subseteq X$ and $Y_i \subseteq Y$, a map $f : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_n)$ is a map $f : X \to Y$ such that $f(X_i) \subseteq Y_i$
- $\mathrm{dom}(\varphi)$ and $\mathrm{codom}(\varphi)$ denote the domain and codomain, respectively, of a map φ
- Q_8 denotes the quaternionic group $\{\pm 1, \pm i, \pm j, \pm k\}$
- $\mathrm{Sym}(n)$ denotes the symmetric group on $\{0, 1, \ldots, n-1\}$
- V_4 denotes the Klein-4 subgroup $\{1, (01)(23), (02)(13), (03)(12)\}$ inside the symmetric group $\mathrm{Sym}(4)$

Introduction

In this thesis we study representations of the fundamental groups of triangulated 3-manifolds. In [**28**], when the triangulations are even, via labellings of the 0-skeleton, Rubinstein and Tillmann show how to construct representations of the fundamental groups of triangulated n-manifolds into $\mathrm{Sym}(n+1)$. In particular, in the case of evenly triangulated 3-manifolds, we have representations into $\mathrm{Sym}(4)$ via labellings of the vertices. In another context, a hyperbolic structure of finite volume on a 3-manifold M gives rise to a developing map $\mathrm{dev} : \widetilde{M} \to \mathbb{H}^3$, where $\widetilde{M}$ is the universal cover of M, and an embedding $\mathrm{hol} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ which is the holonomy representation associated to the geometric structure and chosen developing map. For details, see, for example, [**26**, Chapter 3] or [**21**, Chapter 8]. In the case that M is triangulated, the decomposition of M into simplices lifts to one of $\widetilde{M}$ which gives rise to a labelling of the 0-skeleton by elements of $\partial\mathbb{H}^3 = \mathbb{C}P^1$ and this labelling encodes all the information necessary to construct the holonomy representation; for the case of torus cusps, see [**27**] and for the closed case, see [**15**]. In [**14**], Luo generalises these labellings to labellings over $\mathbb{P}^1(R)$, the projective line over an arbitrary commutative ring with identity, R, and as such constructs representations into $\mathrm{PGL}_2(R)$.

In Chapter 2, we provide a general framework unifying these generalised holonomy representations and the symmetric representations of Rubinstein and Tillmann. In particular, in Section 2.1, where this framework is defined in greatest generality, we prove some results connecting the combinatorics of the triangulation to the topology of the manifold, the latter in the form of its fundamental group. Moreover, we term the labellings involved in Luo's construction "Thurston labellings" and also prove some results, in Section 3.2, regarding the existence of these Thurston labellings, including one, Proposition 3.33, which connects the existence of these labellings to the symmetric representation of Rubinstein-Tillmann.

As a result of his generalisation, in [**14**], Luo makes the following conjecture: for every connected, compact 3-manifold M and a non-trivial element of its fundamental group, there is a homomorphism $\pi_1(M) \to \mathrm{PSL}_2(R)$, for a finite, commutative (and unital) R, whose kernel does not contain the given element. In Chapter 4, we investigate this conjecture, providing equivalent formulations. Using these formulations, we show that the $(4, 1)$-Dehn filling of the figure-8 knot complement, using the knot theoretic framing, is a counterexample to Luo's conjecture. However, we show that the conjecture does hold true for such classes of spaces as orientable hyperbolic 3-manifolds and S^1-bundles over orientable, connected, compact surfaces.

Chapter 1

Triangulated spaces

In this first chapter we define the notion of triangulation for 3-dimensional spaces and prove some basic results. The two most important of these results is a method to compute the fundamental group of a triangulated space and the existence of a lift of a triangulation to the universal cover of a triangulated space.

1.1. Triangulations

Definition 1.1. An *oriented standard n-simplex* comprises the following:

- a standard n-simplex, that is, the affine span of the standard basis vectors in Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$; this amounts to the following subset of $\mathbb{E}^{n+1}$
$$\left\{(t_1, \ldots, t_{n+1}) \mid t_i \in [0,1], \sum_i t_i = 1\right\}$$
- an ordering of the vertices (the standard basis vectors) of the standard n-simplex above up to even permutations.

Henceforth, unless stated otherwise, when we say n-simplex we shall mean a standard n-simplex. Because, for $n \geq 1$, $\mathrm{Alt}(n+1)$ has index two in $\mathrm{Sym}(n+1)$, there are precisely two orientations on a given n-simplex except the 0-simplex which has just the one orientation. In the $n \geq 1$ case, the two possible orientations are said to be *opposite* or *reverse* to the other. We specify the orientation of a simplex by listing its vertices separated by arrows, for example $v_0 \to v_1 \to v_2$ for a 2-simplex, and we specify a face by enclosing the relevant vertices inside square parentheses, for example $[v_0, v_2, v_3]$ is a codimension-1 face of the 3-simplex $[v_0, v_1, v_2, v_3]$.

Example 1.2. In the $n = 3$ case, say for the 3-simplex $[v_0, v_1, v_2, v_3]$, an orientation is equivalent to a choice of clockwise or counter-clockwise for each codimension-1 face in a consistent manner in the sense that upon fixing the choice of an inner or outer vantage point, the same choice of cyclic direction is made for each face.

Definition 1.3. Suppose that we have an oriented standard n-simplex with vertices v_i for $i = 0, \ldots, n$ and orientation $v_0 \to \cdots \to v_n$. Upon the codimension-1 face $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$, where the cared indicates omission of v_i, the *induced orientation* is given by
$$(-1)^i (v_0 \to \cdots \to \widehat{v_i} \to \cdots \to v_n)$$
where the caret once again indicates omission of v_i; that is, if i is even, the induced orientation is given by $v_0 \to \cdots \to \widehat{v_i} \to \cdots \to v_n$ and if i is odd, the induced orientation is the opposite of this orientation.

It needs to be verified that the induced orientation is well-defined; that is to say, if, using the notation in the definition, $v_0, \ldots, v_n$ are permuted by an even permutation σ, then the induced orientation on $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is not altered. It suffices to show that if

σ is instead a transposition, the induced orientation is reversed and in fact, it suffices to show this only for transpositions of adjacent vertices v_j and v_{j+1} as these generate the symmetric group. Let σ be such a transposition; if neither of the two adjacent vertices is the removed vertex v_i, this is clear. Otherwise, if v_i is tranposed with $v_{i\pm 1}$ the induced orientation is

$$(-1)^{i\pm 1}(v_0 \to \cdots \to \widehat{v_i} \to \cdots \to v_n) = -(-1)^i(v_0 \to \cdots \to \widehat{v_i} \to \cdots \to v_n).$$

Given an oriented standard 3-simplex, we shall be interested in the pairs of opposite edges; we shall also be interested in *ideal simplices* which are simplices with their vertices removed, $[v_0, \ldots, v_n] - \{v_0, \ldots, v_n\}$. In working with pairs of opposite edges and ideal simplices, it is convenient to make the following definitions. Given a 3-simplex, we associate to it three normal quadrilateral types and four normal triangle types, often abbreviated to normal quads and normal triangles respectively; these are depicted below and defined more abstractly in Definition 1.4.

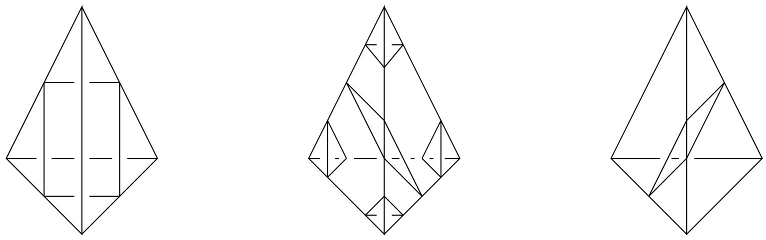


Figure 1. Normal triangle types and normal quadrilateral types

Definition 1.4. Given a 3-simplex $[v_0, v_1, v_2, v_3]$ or ideal 3-simplex, $[v_0, \ldots, v_n] - \{v_0, \ldots, v_n\}$, for each pairwise distinct $i, j, k, l \in \{0, 1, 2, 3\}$, define the partition $\{\{v_i\}, \{v_j, v_k, v_l\}\}$ to be a *normal triangle type* and the partition $\{\{v_i, v_j\}, \{v_k, v_l\}\}$ to be a *normal quadrilateral type*.

The normal quads are defined because they correspond bijectively with the pairs of opposite edges. Similarly, normal triangles are defined because they correspond bijectively with the vertices, which is useful in the case of ideal triangulations, where the vertices have been deleted, as all that can be said via the vertices of a simplex may also be said via its normal triangles. For example, an orientation on it may be specified by an ordering up to even permutations of its normal triangles.

Now, given an oriented 3-simplex, we associate to the orientation a cyclic ordering on the three normal quad types.

Definition 1.5. Given an oriented 3-simplex $[v_0, v_1, v_2, v_3]$ with orientation $v_i \to v_{i+1}$, the cyclic ordering $a \to c \to b \to a$ where $a = \{\{v_0, v_1\}, \{v_2, v_3\}\}$, $b = \{\{v_0, v_2\}, \{v_1, v_3\}\}$ and $c = \{\{v_0, v_3\}, \{v_1, v_2\}\}$ is the *induced cyclic ordering on quads*.

It needs to be verified that the induced cyclic ordering is well-defined; that is to say, if, using the notation in the definition, v_0, v_1, v_2, v_3 are permuted by an even permutation σ, then the induced orientation is not altered. It suffices to show that if σ is instead a transposition of adjacent vertices v_j and v_{j+1}, the induced cyclic orientation is reversed and this is clear as the induced orientation is then found by transposing the two quads which do not separate $[v_j, v_{j+1}]$ from its opposite edge and leaving the remaining quad invariant.

As with orientations on simplices, we will indicate that a quad b follows another quad a in the cyclic ordering on the quads of a given 3-simplex via arrows, viz. $a \to b$. Figure 2

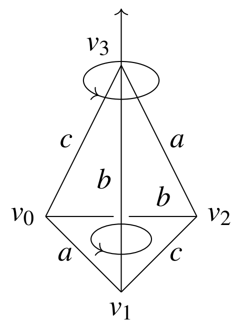


FIGURE 2. The cyclic ordering on normal quads

depicts the situation pictorially. Applying the right-hand rule at each vertex, we order the quads by cyclically ordering the incident edges a, c, b; or alternatively, the edges of the opposite face.

Now we define triangulations.

Definition 1.6. A *triangulation* $\mathcal{T}$ comprises a finite disjoint union $\Sigma = \sqcup_{i=1}^{n} \sigma_i$ of standard n-simplices together with a collection Φ of affine bijections between pairs of distinct codimension-1 faces (possibly within the same n-simplex) in Σ, termed *face-pairings*, such that each face occurs as the domain of precisely one face-pairing and that given any two distinct codimension-1 faces f and g, there either exists no face-pairing $f \to g$ and no face-pairing $g \to f$ or there exists one and only one face-pairing $f \to g$ and one and only one from $g \to f$, denoted $\varphi_{f,g}$ and $\varphi_{g,f}$ respectively, and these are such that $\varphi_{g,f} = \varphi_{f,g}^{-1}$. An *ideal triangulation* comprises the same data but where we use ideal standard n-simplices. Further, *oriented triangulations* and *oriented ideal triangulations* are triangulations and ideal triangulations, respectively, where the simplices are oriented and the face-pairings, orientation-reversing.

In this thesis, though not part of our official definition, we always assume an additional property of a triangulation $\mathcal{T}$: that each n-simplex is connected via a sequence of face-pairings to any other n-simplex. More precisely, we assume that given any n-simplices σ, σ', there exist n-simplices $\sigma_{i_1}, \ldots, \sigma_{i_k}$ for some k and face-pairings $\varphi_1, \ldots, \varphi_{k-1}$ such that $\sigma_{i_1} = \sigma$, $\sigma_{i_k} = \sigma'$, $\mathrm{dom}(\varphi_j) \subseteq \sigma_{i_j}$, $\mathrm{codom}(\varphi_j) \subseteq \sigma_{i_{j+1}}$. It can be shown that this condition on $\mathcal{T}$ is equivalent to that the underlying topological space M of the triangulated space $(M, \mathcal{T})$, as defined below, is path-connected.

Definition 1.7. An *n-dimensional triangulated space* is a pair $(M, \mathcal{T})$ of a topological space M together with a triangulation $\mathcal{T}$ where each σ_i is an n-simplex and where $M = \Sigma/\Phi$. A similar definition applies for *(oriented) (ideally) triangulated spaces*. Triangulated spaces of dimension n are also termed *pseudo n-manifolds*.

The quotient space modulo Φ here is the quotient space modulo $\sim$ where $\sim$ is the equivalence relation generated by the identifications $x \sim \varphi(x)$ for $\varphi \in \Phi$.

Remark 1.8. In [**19**], Moise showed that any topological 3-manifold has an essentially unique piecewise-linear structure and smooth structure. In particular, every closed 3-manifold admits a finite triangulation in that it is a pseudo 3-manifold as defined in Definition 1.7.

Given a (oriented) (ideally) triangulated space, the canonical quotient map $\Sigma \to M$ is denoted π. Denote the collection of i-dimensional simplices in $\mathcal{T}$ (among the n-simplices σ_i together with their faces) by $\mathcal{T}^{(i)}$, termed the *i-skeleton* of $\mathcal{T}$. Skeleta are defined in the same way for stand alone simplices; an ideal simplex is then one which is deprived of its 0-skeleton, denoted $\sigma - \sigma^{(0)}$. If σ itself denotes an ideal simplex and $\mathcal{T}$ an ideal triangulation, $\sigma^{(0)}$ and $\mathcal{T}^{(0)}$ are used to denote the normal triangle types of σ and $\mathcal{T}$ respectively. Finally, simplices *in M* are defined to be images under π of simplices in $\mathcal{T}$ and skeleta $M^{(i)}$ of M are defined similarly.

Proposition 1.9. *Let* $(M, \mathcal{T})$ *be an oriented pseudo* 3*-manifold. Given an edge* $e \in \mathcal{T}^{(1)}$*, there exist simplices* $\sigma_{i_1}, \dots, \sigma_{i_k}$ *where* σ_{i_j} *may coincide with* $\sigma_{i_{j'}}$*, faces* $f_1, g_1, \dots, f_k, g_k$ *where* $f_j, g_j \subset \sigma_{i_j}$ *and edges* $e_j = f_j \cap g_j = [v_j, w_j]$ *such that:*

- *there exist face-pairings* $(g_j, e_j, v_j, w_j) \to (f_{j+1}, e_{j+1}, v_{j+1}, w_{j+1})$ *for* $j \in [k]$*, where the subscripts are taken modulo k*
- *the* e_j *comprise precisely all edges identified to e under* Φ
- *this sequence of simplices, faces and edges is unique up to cyclic permutations and order reversal.*

Note: the notation $(g_j, e_j, v_j, w_j) \to (f_{j+1}, e_{j+1}, v_{j+1}, w_{j+1})$ for a map (the face-pairing) is explained in the second point in the Notation section at the beginning of this thesis.

Proof. It is clear that there exists a uniquely determined collection of say k edges in $\mathcal{T}^{(1)}$ (including e itself) such that these edges and only these edges are identified to e under Φ. First suppose that $k = 1$. Let σ be the simplex containing e, let e have vertices v_0, v_1 and denote by v_2, v_3 the remaining vertices of σ labelled such that the orientation of σ is $v_i \to v_{i+1}$. As $k = 1$, we must have that $[v_0, v_1, v_2]$ is identified to $[v_0, v_1, v_3]$ and such that $e = [v_0, v_1]$ is fixed (though not necessarily pointwise as of yet). But because face-pairings must be orientation reversing the only possibility is that we have the identification where $v_0, v_1, v_2 \mapsto v_0, v_1, v_3$. In this case then we have an obvious and unique sequence of the required form consisting of just the one simplex σ.

Suppose then that $k > 1$. Let σ_{i_1} be the simplex containing e, let f_1, g_1 be the two faces containing e and set $e_1 = e$. Next, let $f_2 = \varphi_{g_1,*}(g_1)$ where $\varphi_{g_1,*}$ is the face-pairing with domain g_1, let $e_2 = \varphi_{g_1,*}(e_1)$, let g_2 be the face $\neq f_2$ which contains e_2 and let σ_{i_2} be the simplex containing e_2. Now repeat this process, which may be carried out indefinitely. For any fixed j, g_j contains e_j and $\varphi_{g_j,*}$ cannot map e_j to e_j for then we would find that e_j is identified to no other edge, contradicting our assumption that $k > 1$. Thus for any j, $e_j \neq e_{j+1}$. Since $|\Sigma^{(1)}| < \infty$, a repetition must occur in the sequence $e_1, e_2, e_3, \dots$. Let the first such instance occur with $e_p = e_q = e'$ at the p^{th} and q^{th} steps with $p \leq q - 2$ since, as just argued, for all j, $e_j \neq e_{j+1}$. Then $\{f_p, g_p\} = \{f_q, g_q\}$. Suppose first that $f_q = g_p$ (so that we re-enter σ_{i_p} from where we took off). Then by definition $e_{q-1} = \varphi_{f_q,*}(e') = \varphi_{g_p,*}(e') = e_{p+1}$ which is a contradiction, as then a repetition occurs prior to that at the q^{th} step. So we must have $f_q = f_p$ and $g_q = g_p$. We claim that $p = 1$ and $q = k + 1$. From $f_q = f_p$ and $g_q = g_p$ we conclude that if $p \neq 1$, $e_{q-1} = \varphi_{f_q,*}(e') = \varphi_{f_p,*}(e') = e_{p-1}$ which is a contradiction to the minimality of the repetition at the q^{th} step. Thus $p = 1$ and then our sequence reads $e_1, e_2, \ldots, e_{q-1}, e_1, e_2, \ldots, e_{q-1}, \ldots$ where there are no repetitions in any consecutive $q-1$ edges and from this we conclude that that $q = k+1$.

We now have a sequence of simplices $\sigma_{i_1}, \ldots, \sigma_{i_k}$, faces $f_1, g_1, \ldots, f_k, g_k$ where $f_j, g_j \subset \sigma_{i_j}$ and edges $e_j = f_j \cap g_j$ such that g_1 is identified to f_2, g_2 to $f_3, \ldots, g_k$ to f_1 where in each identification e_j is identified to e_{j+1} where the indices are interpreted modulo k. We need to verify that no e_j has its two vertices identified. Let e_1 have vertices v_0^1, v_1^1 and let the remaining two vertices of σ_{i_1} be v_2^1, v_3^1 such that $f_1 = [v_0^1, v_1^1, v_2^1]$ and $g_1 = [v_0^1, v_1^1, v_3^1]$. Interchange the labels of e_1 if necessary so that the orientation of σ_{i_1} is given by $v_i^1 \to v_{i+1}^1$. Let the vertices of e_2 which correspond to those of e_1 under $\varphi_{g_1,*}$ be v_0^2 and v_1^2 respectively. Denote the remaining vertex of f_2 by v_2^2 and then the remaining vertex of σ_{i_2} by v_3^2. Then because face-pairings must be orientation reversing, the orientation of f_2 must be $-(v_0^2 \to v_1^2 \to v_2^2)$ so that the orientation of σ_{i_2} is given by $v_i^2 \to v_{i+1}^2$. Further $g_2 = [v_0^2, v_1^2, v_3^2]$ and then has orientation $v_0^2 \to v_1^2 \to v_3^2$. Thus this process may be repeated up until σ_{i_k} which has its vertices denoted $v_0^k, v_1^k, v_2^k, v_3^k$ and has orientation $v_i^k \to v_{i+1}^k$. We also have $e_k = [v_0^k, v_1^k]$, $f_k = [v_0^k, v_1^k, v_2^k]$ and $g_k = [v_0^k, v_1^k, v_3^k]$. Finally g_k is identified to f_1 such that $[v_0^k, v_1^k] \mapsto [v_0^1, v_1^1]$, though not necessarily pointwise, as of yet. However, as face-pairings are orientation reversing, as in the $k = 1$ case above, we find that, in this identification, $v_0^k \mapsto v_0^1$, $v_1^k \mapsto v_1^1$.

The sequence now satisfies all that is required except possibly uniqueness. Uniqueness, as stated, follows because after constructing one sequence, any initial simplex in another sequence for the same edge must lie in this sequence and the remaining properties of the sequences determine the new sequence to be the original sequence, up to cyclic permutations and order reversal. □

Remark 1.10. An analogous result holds in the case of oriented ideally triangulated 3-dimensional spaces, with vertices replaced by normal triangles. If instead $(M, \mathcal{T})$ is only a pseudo 3-manifold (that is, the triangulation is not oriented), the same result holds except that $(g_j, e_j, v_j, w_j) \to (f_{j+1}, e_{j+1}, v_{j+1}, w_{j+1})$ needs to be replaced by $(g_j, e_j) \to (f_{j+1}, e_{j+1})$; that is, a v_j may now be identified to a $w_{j'}$.

Note that the above proof gives an algorithm to find, given an edge $e \in \mathcal{T}^{(1)}$, all other edges in $\mathcal{T}^{(1)}$ which are identified to e via Φ.

Definition 1.11. Given a (oriented) (ideally) triangulated 3-dimensional space $(M, \mathcal{T})$ and $e \in \mathcal{T}^{(1)}$, we term the sequence provided by Proposition 1.9 the *edge cycle* about e.

Example 1.12. The following is an oriented ideal triangulation of the figure-eight knot complement from Regina, [**5**]; the orientations on the simplices here are $t_i \to t_{i+1}$ and $t'_i \to t'_{i+1}$
The face-pairings, specified via the normal triangles t_i, t'_i, here are

$$\varphi_1 : t_0, t_1, t_2 \mapsto t'_2, t'_0, t'_3 \qquad \varphi_2 : t_0, t_1, t_3 \mapsto t'_1, t'_0, t'_3$$
$$\varphi_3 : t_0, t_2, t_3 \mapsto t'_1, t'_0, t'_2 \qquad \varphi_4 : t_1, t_2, t_3 \mapsto t'_1, t'_3, t'_2$$

and there are two edge cycles, indicated in red and blue. If we let $\sigma_1 = [t_0, t_1, t_2, t_3]$ and $\sigma_2 = [t'_0, t'_1, t'_2, t'_3]$, then, if we set $e_1 = [t_0, t_3]$ and $g_1 = [t_0, t_2, t_3]$, the sequence of simplices in the blue edge cycle reads $\sigma_0, \sigma_1, \sigma_0, \sigma_1, \sigma_0, \sigma_1$ and has the following sequence of connecting face-pairings $\varphi_3, \varphi_4^{-1}, \varphi_2, \varphi_1^{-1}, \varphi_4, \varphi_2^{-1}$.

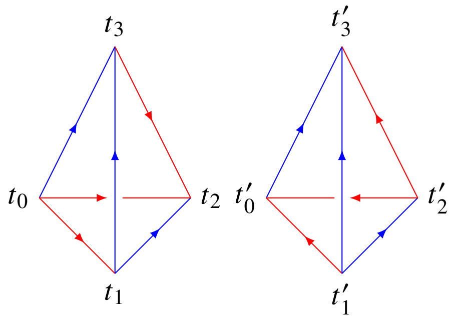


FIGURE 3. Triangulation of the figure-eight knot complement from Regina – “m004: #1” in “Cusped Hyperbolic Census (Orientable)”

Definition 1.13. Given a (oriented) triangulated 3-dimensional space $(M, \mathcal{T})$, we say that $\mathcal{T}$ is *non-singular* if each 3-simplex is imbedded in M and *almost non-singular* if no two edges of the same simplex are identified. If the triangulation is ideal, it is said to be *non-singular* when the associated non-ideal triangulation where the 0-skeleton is included is non-singular.

1.2. Computing the fundamental groups of triangulated spaces

We first focus on graphs, for which we take the following definition.

Definition 1.14. A *graph* is a pair $(G, \mathcal{E})$ where $\mathcal{E}$ comprises a disjoint union of 1-simplices $E = \sqcup_i e_i$ and a partition, given by the equivalence relation $\sim$, of the collection $v(\mathcal{E})$ of vertices of these simplices and where $G = E/\!\sim$.

The canonical map $E \to G$ is denoted π; vertices and edges *in* G are the images under π of those in $\mathcal{E}$. Note that this definition of graph allows both finite and infinite graphs and also multiple edges between given vertices as well as edges which are loops. We always consider only connected graphs.

Definition 1.15. Suppose given a graph $(G, \mathcal{E})$ and a base vertex $v \in v(\mathcal{E})$. Consider the collection, $\ell(\mathcal{E}, v)$, of *combinatorial loops* in G, $\ell = (v_1^1, v_2^1), \dots, (v_1^k, v_2^k)$, where $[v_1^i, v_2^i]$ is an edge of E for each i and $v_1^1 = v$, $\pi(v_2^i) = \pi(v_1^{i+1})$ for $i = 1, \dots, k$, indices taken modulo k. These loops form a monoid under concatenation, which is written left to right. A *spur* is defined to be a trivial path of the form $(v_1, v_2), (v_2, v_1)$. The relation $\sim$ on $\ell(E, v)$ where $\ell \sim \ell'$ if and only if ℓ can be achieved from ℓ' by addition or removal of spurs is an equivalence relation; if $\ell \sim \ell'$, they are said to be *combinatorially path homotopic*. The $\sim$-classes of loops in $\ell(E, v)$ form a group under concatenation, denoted $\pi_1^{\mathrm{comb}}(G, v)$ and termed the *combinatorial fundamental group of* G based at v.

Remark 1.16. Equivalence via combinatorial path homotopies also gives an equivalence relation on combinatorial paths which are not loops; two equivalent paths must necessarily begin and end at the same vertices.

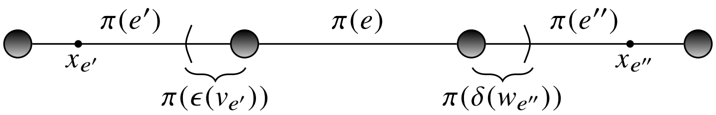
Given $e = [v_1, v_2]$ in E, let $p_{(e,v_1)} : I \to E \xrightarrow{\pi} G$ denote the affine path satisfying $0 \mapsto v_1 \mapsto \pi(v_1)$, $1 \mapsto v_2 \mapsto \pi(v_2)$. That is, if $\{(t_1, t_2) \mid t_i \in [0, 1], t_1 + t_2 = 1\}$ is the 1-simplex e, v_1, v_2 the points $(0, 1), (1, 0)$ respectively, f the map $I \to E : t \mapsto (t, 1 - t)$, then $p_{(e,v_1)}$ denotes the composition $\pi \circ f$. Consider a combinatorial path $p = (v_1^1, v_2^1), \dots, (v_1^k, v_2^k)$ where $\pi(v_2^i) = \pi(v_1^{i+1})$ for $i = 1, \dots, k - 1$. Let $e_i = [v_1^i, v_2^i]$.

We have that $p_{(e_i,v_1^i)}(1) = p_{(e_{i+1},v_1^{i+1})}(0)$ for $i = 1, \ldots, k-1$ so that we may define $p' = p_{(e_1,v_1^i)} \cdots p_{(e_{k+1},v_1^{k+1})}$ (see [**9**, Chapter 1] for the definition of the product of paths). As such, given any combinatorial path p, we associate to it a continuous path, which we denote by p'.

Proposition 1.17. *If G is finite and $\varphi : I \to G$ is any loop based at a vertex v of G, φ is path-homotopic to ℓ' for some combinatorial loop ℓ in G, where ℓ' is the continuous loop corresponding to the combinatorial loop ℓ as defined above.*

Proof. We show this more generally for paths; let $\varphi : I \to G$ be a path. For each edge e in $\mathcal{E}^{(1)}$ such that $\pi(e^\circ) \cap \varphi(I) \neq \emptyset$, fix $x_e \in \pi(e^\circ) \cap \varphi(I)$. For each such edge $e = [v_e, w_e]$, define also n_e as follows:

- if there exists an edge $e' = [v_{e'}, w_{e'}]$ such that $\pi(v_{e'}) = \pi(v_e)$ and an edge $e'' = [v_{e''}, w_{e''}]$ such that $\pi(w_{e''}) = \pi(w_e)$, choose one of each e' and e'', let $\epsilon(v_{e'})$ be a half-open interval in e' containing $v_{e'}$ but not $x_{e'}$ if $\pi((e')^\circ) \cap \varphi(I) \neq \emptyset$, similarly let $\delta(w_{e''})$ be a half-open interval in e'' containing $w_{e''}$ but not $x_{e''}$ if $\pi((e'')^\circ) \cap \varphi(I) \neq \emptyset$ and then finally set $n_e = \pi(\epsilon(v_{e'})) \cup \pi(e) \cup \pi(\delta(w_{e''}))$



- if one or both of e' and e'' as above don't exist, n_e is defined analogously but where the corresponding factor(s) in the union $\pi(\epsilon(v_{e'})) \cup \pi(e) \cup \pi(\delta(w_{e''}))$ is omitted.

Then $\cup_{\pi(e^\circ)\cap\varphi(I)\neq\emptyset} n_e$ is an open cover of $\varphi(I)$ which has no finite subcover if $\varphi(I)$ intersects the interiors of infinitely many e; by compactness of I, this cannot be the case so that $|A| < \infty$ where $A = \{e \in \mathcal{E}^{(1)} \mid \pi(e^\circ) \cap \varphi(I) \neq \emptyset\}$. Given an $e = [v_e, w_e]$ in A, recall that $n_e = \pi(\epsilon(v_{e'})) \cup \pi(e) \cup \pi(\delta(w_{e''}))$ where $e' = [v_{e'}, w_{e'}]$ is such that $\pi(v_{e'}) = \pi(v_e)$ and $e'' = [v_{e''}, w_{e''}]$ is such that $\pi(w_{e''}) = \pi(w_e)$; if no such e' or e'' exists, the corresponding factor in the union is omitted. For this fixed e, let $n_{v_e} = \pi(\epsilon(v_{e'})) \cup \pi(\epsilon'(v_e))$ and $n_{w_e} = \pi(\delta'(w_e)) \cup \pi(\delta(w_{e''}))$ where $\epsilon'(v_e)$ and $\delta'(w_e)$ are half-open intervals containing v_e and w_e respectively. By continuity, for each $t \in I$, there is an open interval a_t about t such that this interval is mapped entirely into an n_e for $e \in A^{(1)}$ or an n_{v_e} for $v_e \in A^{(0)}$. Replace each a_t with a closed interval c_t about t contained in a_t. The open intervals $c_t \setminus \{\text{endpoints} - \{0, 1\}\}$ cover I and so by compactness finitely many c_t cover I. The endpoints of the c_t provide a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for each i, $\varphi([t_{i-1}, t_i])$ is entirely contained in an n_e for $e \in A^{(1)}$ or entirely contained in an n_{v_e} for $v_e \in A^{(0)}$. For each i, let φ_i be the path $\varphi|_{[t_{i-1},t_i]}$ reparametrised affinely to have domain I. Then $\varphi \simeq_p \varphi_1 \cdots \varphi_k$. For each i, by simple connectedness of each n_e and n_{v_e}, φ_i is path-homotopic to the unique affine path φ_i' from $\varphi_i(0)$ to $\varphi_i(1)$. Then we have $\varphi \simeq_p \varphi_1 \cdots \varphi_k \simeq_p \varphi_1' \cdots \varphi_k'$ and this is path-homotopic to the combinatorial path of vertices and edges which it traverses. $\square$

As a result, we have the following.

Proposition 1.18. *Given the graph $(G, \mathcal{E})$ and a vertex $v \in v(\mathcal{E})$, the map $\Psi : \pi_1^{comb}(G, v) \to \pi_1(G, \pi(v))$ given by $[\ell] \mapsto [\ell']$, where ℓ' is the continuous loop corresponding to the combinatorial loop ℓ, as define above, is well-defined and an isomorphism.*

Proof. The map Ψ is well-defined because additions and removals of spurs can be performed via continuous path-homotopies, is a homomorphism because $[(\ell_1 \cdot \ell_2)']$ and $[\ell_1'] \cdot [\ell_2']$ are reparametrisations of each other, surjective by Proposition 1.17 and can also be checked to be injective by showing that continuous path-homotopies can be performed via additions and removals of spurs. $\square$

When G is finite, a finite presentation for $\pi_1(G)$ can be found as follows; the methods can also be extended to deal with the infinite case, see [**25**, Chapter 2]. A *tree* in G is a subgraph such that every combinatorial loop contains a spur; for any finite G, a tree T such that $v(T) = v(G)$, called a *maximal tree*, may be constructed algorithmically. Say that a combinatorial path in G is *reduced* if it contains no spurs.

Proposition 1.19. *Given a maximal tree T for the finite graph $(G, \mathcal{E})$ and $v, w \in v(G)$, there is a unique reduced combinatorial path which joins v to w and is contained in T.*

Proof. ([**25**, Chapter 2]) Suppose that p and p' are two such paths; we may assume that there exist no v', w' such that there exist distinct q, q' contained in T and joining v' and w' such that one of q and q' has length shorter than p. Now $p \cdot (p')^{-1}$ is a cycle in T which then must contain a spur and because each of p and p' is reduced, this spur occurs either with the initial edges of p and p' or with their final edges; in either case, removing the spur contradicts the minimality property of the length of p. $\square$

Proposition 1.20. *Suppose given a finite graph $(G, \mathcal{E})$ and $v \in v(G)$, let T be a maximal tree for $(G, \mathcal{E})$ and for each $v' \in v(G)$, let $p_{v'}$ be the unique reduced path in T from v to v'. For each edge $e = \pi([w, w'])$ of G not in $T^{(1)}$, arbitrarily give e the orientation $w \to w'$ and let $a_e = p_{\pi(w)} \cdot e \cdot p_{\pi(w')}^{-1}$; then $\pi_1^{comb}(G, v) \cong \langle \{[a_e] \mid e \notin T^{(1)}\} \rangle$.*

Proof. To prove that the $[a_e]$, for $e \notin T^{(1)}$, generate $\pi_1^{\mathrm{comb}}(G, v)$, we first also arbitrarily give orientations to $e \in T^{(1)}$ and define a_e for $e \in T^{(1)}$ in exactly the same manner as for $e \notin T^{(1)}$. Then, given a combinatorial loop ℓ in G based at v with sequence of edges $e_1, \ldots, e_k$, it can be checked that $[\ell] = [a_{e_1}^{\epsilon_1} \cdots a_{e_k}^{\epsilon_k}] = [a_{e_1}]^{\epsilon_1} \cdots [a_{e_k}]^{\epsilon_k}$ where ϵ_i is 1 if 1 if e_i is traversed in the direction of the orientation assigned to it and -1 otherwise. Next, since T is a tree, we see that if $e_i \in T^{(1)}$, $[a_{e_i}] = 1$ and so the corresponding factors can be removed from the product $[a_{e_1}]^{\epsilon_1} \cdots [a_{e_k}]^{\epsilon_k}$; thus the $[a_e]$, for $e \notin T^{(1)}$, generate $\pi_1^{\mathrm{comb}}(G, v)$. We claim now that if a product $[\ell_1]^{b_1} \cdots [\ell_k]^{b_k}$ of reduced combinatorial paths in G which admit concatenation is the identity, then there exists some "algebraic spur" $\ell_i^{\pm 1} \ell_{i+1}^{\mp 1}$ in the word $\ell_1^{b_1} \cdots \ell_k^{b_k}$. This is done by induction on the length l of the path $\ell_1^{b_1} \cdots \ell_k^{b_k}$. If $l = 2$, we must have $k = 2$, $b_1 = b_2 = 1$ and ℓ_1, ℓ_2 must be edges in G. If $\ell_2 \neq \ell_1^{-1}$, then the only moves on $\ell_1 \ell_2$ that can be made are additions of spurs and removals of previously added spurs so that $\ell_1 \ell_2 \neq 1$, a contradiction. Now, given arbitrary reduced $\ell_1, \ldots, \ell_k$ and that $[\ell_1]^{b_1} \cdots [\ell_k]^{b_k} = 1$, we have that $\ell_1^{b_1} \cdots \ell_k^{b_k}$ must contain a spur, for otherwise the only moves one can make on this loop are additions of spurs and removals of previously added spurs. Because each a_{e_i} is reduced, there must exist a j such that this spur occurs as the final edge of a_{e_j} and the initial edge of $a_{e_{j+1}}$. Removing this spur gives a product of reduced paths which is the identity and the total length of the product path has decreased. This completes the proof and shows that the $[a_e]$, for $e \notin T^{(1)}$, are free generators. $\square$

Remark 1.21. An intuitive view of the presentation in Proposition 1.20 is that we may collapse the tree T without altering the fundamental group and the result upon collapsing T is a bouquet of circles with an oriented circle for each oriented $e \in \mathcal{E}^{(1)}$ not in T.

Now we turn to (oriented) (ideally) triangulated 3-dimensional spaces. First, we quote a well-known result, a proof of which can be found in [**26**, Chapter 3].

Theorem 1.22. *Given a pseudo* 3*-manifold M, M is a manifold if and only if the link of each vertex in M is, topologically, S^2.*

Ideally triangulated spaces, of any dimension, are always manifolds.

Proposition 1.23. *Given a pseudo* 3*-manifold M which is a manifold, if we denote by M' the ideally triangulated space obtained by deleting the vertices in M and let x be the barycentre of a* 3*-simplex, the inclusion $(M', x) \hookrightarrow (M, x)$ induces an isomorphism $\pi_1(M', x) \xrightarrow{\sim} \pi_1(M, x)$.*

Proof. We need to show that if $\varphi : I \to M$ is a loop based at x, then there is a $\psi : I \to M$ such that $\psi(I) \subseteq M'$ and $\varphi \simeq_p \psi$. Let $M^{(0)}$ be the collection of vertices in M. For each vertex v in M, let B_v be the open 3-ball enclosed by the link of v and let $B = \cup_{v \in M^{(0)}} B_v$. By continuity, for each $t \in I$, there is an open interval n_t about t such that this interval is mapped entirely into B or entirely into $M - M^{(0)} = M'$. Replace each n_t with a closed interval c_t about t contained in n_t. The open intervals $c_t \setminus \{\text{endpoints} - \{0, 1\}\}$ cover I and so by compactness finitely many c_t cover I. The endpoints of the c_t provide a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for each i, $\varphi([t_{i-1}, t_i])$ is entirely contained in M' or entirely contained in B. For each i, let φ_i be the path $\varphi|_{[t_{i-1}, t_i]}$ reparametrised affinely to have domain I. Then $\varphi \simeq_p \varphi_1 \cdots \varphi_k$. Given any i for which $\varphi([t_{i-1}, t_i]) \subseteq B$, by connectedness, there is some unique v_0 such that $\varphi([t_{i-1}, t_i]) \subseteq B_{v_0}$. As B_{v_0} is simply connected, we can choose a path φ'_i such that $\varphi'_i(I) \subseteq B_{v_0} - \{v_0\} \subseteq M'$ and $\varphi'_i \simeq_p \varphi_i$. The latter condition implies $\varphi \simeq_p \varphi_1 \cdots \varphi_k \simeq_p \varphi_1 \cdots \varphi_{i-1} \cdot \varphi'_i \cdot \varphi_{i+1} \cdots \varphi_k$. Repeating this procedure, we have the result. □

Definition 1.24. Given a 3-simplex σ, the *spine* of this simplex is the subspace depicted in Figure 4, denoted $\mathrm{sp}(\sigma)$. Given a (oriented) (ideally) triangulated 3-dimensional space $(M, \mathcal{T})$, because affine maps preserve barycentres and line segments, restrictions of face-pairings in Φ give pairings Φ_s for the spines of $\sigma \in \mathcal{T}^{(3)}$; the space $(\sqcup_i \mathrm{sp}(\sigma_i))/\Phi_s$ is termed the *spine of* M, denoted $\mathrm{sp}(M)$.

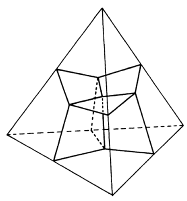


FIGURE 4. The spine of a 3-simplex

The spine is constructed from precisely six quadrilaterals, the *wings* of the spine. Each edge of any wing joins a barycentre of an edge, face or 3-simplex to a barycentre of another.

Proposition 1.25. *If $(M, \mathcal{T})$ is a triangulated 3-manifold, $M \simeq sp(M)$.*

Proof. By Proposition 1.23, we can pass to the corresponding ideally triangulated 3-manifold $(M', \mathcal{T}')$. It is clear that any ideal simplex σ^o deformation retracts to its spine and in fact, because the 0-skeleton of $\mathcal{T}$ has been removed, this may be done simultaneously on all simplices to produce a deformation retraction of M onto $\mathrm{sp}(M)$. □

Definition 1.26. Given a pseudo 3-manifold $(M, \mathcal{T})$, the *dual 1-skeleton* of $\mathcal{T}$ is the graph $(\mathcal{G}, \mathcal{E}_\mathcal{T})$ where $E = \sqcup_{f=g \bmod \Phi} e_{f,g}$ where $e_{f,g} = [\sigma_f, \sigma_g]$, $f \subset \sigma_f$ and $g \subset \sigma_g$, "$f = g \bmod \Phi$" denotes that there exists a face-pairing between f and g, and the partition of endpoints is given by equality. The canonical map $E \to \mathcal{G}$ is denoted $\pi_\mathcal{G}$.

Note that the dual 1-skeleton is 4-valent and because we assume the same of M, connected. Note also that this skeleton can be defined in the case of infinite triangulations as well.

Given a (oriented) (ideally) triangulated 3-dimensional space, for pairs (σ, f) where $\sigma \in \mathcal{T}^{(3)}$ and f is a face of σ, let $q_{(\sigma,f)} : I \to \Sigma \to M$ be the unique affine path where $q_{(\sigma,f)}(0)$ is the barycentre of σ and $q_{(\sigma,f)}(1)$ is the barycentre of f.

Proposition 1.27. *Given a triangulated 3-manifold $(M, \mathcal{T})$, let $(\mathcal{G}, \mathcal{E}_\mathcal{T})$ be the dual 1-skeleton and for each edge cycle in $\mathcal{T}$, attach a 2-cell along the corresponding loop in $\mathcal{G}$. Denoting this 2-dimensional CW-complex by C, we have $sp(M) \cong C$.*

Proof. For any $i \in [n]$, let Γ_{σ_i} be the graph consisting of the segments joining the barycentre of σ_i to the barycentres of its faces; then $(\sqcup_{\sigma \in \mathcal{T}^{(3)}} \Gamma_\sigma)/\Phi_\Gamma \cong \mathcal{G}$ is a subspace of $\mathrm{sp}(M)$ where Φ_Γ comprises the restrictions of face-pairings $f \to g$ in Φ to pairings $f_x \to g_x$ where f_x, g_x are the barycenres of f, g respectively. Note that the wings of a spine of a 3-simplex correspond bijectively to the edges of that simplex and these correspond bijectively to unordered pairs of faces. Given a wing w, let e_w denote the corresponding edge and f_w, g_w the corresponding faces, so that $f_w \cap g_w = e_w$. Conversely, given an $e \in \mathcal{T}^{(1)}$, let w_e denote the corresponding wing. Consider some fixed wing w and let the edge cycle about e_w be given by $f_1, g_1, \ldots, f_k, g_k$, $e_j = f_j \cap g_j$ and $\sigma_{i_j} \supset f_j, g_j$ where g_i is identified to f_{i+1} for $i \in [k]$, interpreting the indices modulo k. Then restrictions of the face-pairings $\varphi_{g_i, f_{i+1}}$ identify the wings w_{e_i} producing a disk with boundary the image of the loop $q_{(\sigma_{i_1}, g_1)} \cdot q^{-1}_{(\sigma_{i_2}, f_2)} \cdot q_{(\sigma_{i_2}, g_2)} \cdot q^{-1}_{(\sigma_{i_3}, f_3)} \cdot q_{(\sigma_{i_3}, g_3)} \cdots q^{-1}_{(\sigma_{i_1}, f_1)}$ in $(\sqcup_{\sigma \in \mathcal{T}^{(3)}} \Gamma_\sigma)/\Phi_\Gamma$, which is the loop in $\mathcal{G}$ corresponding to the edge cycle of e_w. Moreover, no further identifications are made to this disk and each wing occurs as part of such a disk. This completes the proof. □

Proposition 1.28. *Given a triangulated 3-manifold $(M, \mathcal{T})$ with dual 1-skeleton $(\mathcal{G}, \mathcal{E}_\mathcal{T})$, the map $\Psi : \pi_1^{comb}(\mathcal{G}, \sigma) \to \pi_1(M, x)$, where x is the barycentre of σ given by*

$$\begin{aligned}(\sigma_{i_1}, g_1) \to (\sigma_{i_2}, f_2), (\sigma_{i_2}, g_2) \to \cdots \to (\sigma_{i_k}, f_k), (\sigma_{i_k}, g_k) \to (\sigma_{i_1}, f_1) \\ \mapsto q_{(\sigma_{i_1}, g_1)} \cdot q^{-1}_{(\sigma_{i_2}, f_2)} \cdot q_{(\sigma_{i_2}, g_2)} \cdot q^{-1}_{(\sigma_{i_3}, f_3)} \cdot q_{(\sigma_{i_3}, g_3)} \cdots q^{-1}_{(\sigma_{i_1}, f_1)}\end{aligned}$$

is well-defined and surjective. For each edge cycle c, let p be a path in $\mathcal{G}$ from σ to a simplex in c; then $\ker(\Psi)$ is the normal closure of $\{[pcp^{-1}] \mid$ edge cycles $c\}$ in $\pi_1^{comb}(\mathcal{G}, \sigma)$.

Proof. To see that Ψ is surjective, post-compose any loop $I \to M$ with the equivalence $M \to \mathrm{sp}(M)$ in Proposition 1.25, use the simple connectedness of disks to homotope the loop away from the attached disks (here the techniques involved are the same as those in the proofs of Propositions 1.17 and 1.23) and then apply Proposition 1.17. Next, that the normal closure of $\{[pcp^{-1}] \mid \text{edge cycles } c\}$ in $\pi_1^{\mathrm{comb}}(\mathcal{G}, \sigma)$ lies in $\ker(\Psi)$ is a consequence of the result contained within the proof of Proposition 1.27 that edge cycles bound disks. The full result that the kernel is precisely this normal closure, follows from the Seifer-van Kampen theorem; see [**9**, Chapter 1]. □

Thus we have $\pi_1(M) \cong \pi_1^{\text{comb}}(\mathcal{G},\sigma)/\ker(\Psi)$. Analogous to the situation for finite graphs, we can compute a finite presentation for $\pi_1(M)$ as follows.

Proposition 1.29. *Given a triangulated 3-manifold $(M,\mathcal{T})$, let $(\mathcal{G},\mathcal{E}_{\mathcal{T}})$ be the dual 1-skeleton and T a maximal tree for this skeleton. Let $\sigma \in v(\mathcal{G})$ and for $\sigma' \in v(\mathcal{G})$, let $p_{\sigma'}$ be the unique reduced path in $\mathcal{G}$ from σ to σ'. For each edge $e = \pi_{\mathcal{G}}([\sigma_1,\sigma_2])$ of $\mathcal{G}$ not in $T^{(1)}$, arbitrarily give e the orientation $\sigma_1 \to \sigma_2$ and let $a_e = p_{\pi_{\mathcal{G}}(\sigma_1)} \cdot e \cdot p^{-1}_{\pi_{\mathcal{G}}(\sigma_2)}$. For each edge e of $\mathcal{G}$ not in $T^{(1)}$, if the loop in $\mathcal{G}$ corresponding to the edge cycle about e is be $e_{g_1,f_2},\dots,e_{g_k,f_1}$, let also*

$$\delta_i^e = \begin{cases} 1 & \text{if } e_{g_i,f_{i+1}} \in T^{(1)} \\ a^{\pm 1}_{e_{g_i,f_{i+1}}} & \text{otherwise} \end{cases}$$

where the $+$ sign is chosen when $e_{g_i,f_{i+1}}$ is traversed in direction assigned to it; then if x is the barycentre of σ

$$\pi_1(M,x) \cong \langle \{[a_e] \mid e \notin T^{(1)}\} \mid \{[\delta_1^e]\cdots[\delta_k^e] \mid e \in \mathcal{T}^{(1)}\}\rangle.$$

Proof. We know as in Proposition 1.20 that the $[a_e]$, for $e \notin T^{(1)}$, freely generate $\pi_1^{\text{comb}}(G,\sigma)$. In the notation of Proposition 1.28, consider the expression, provided in the proof of Proposition 1.20, of $[pcp^{-1}]$, where c is the edge cycle about say $e' \in \mathcal{T}^{(1)}$, as a product of the $[a_e]$. Because p lies in T, the expression is precisely the product $\delta_1^{e'}\cdots\delta_k^{e'}$ attached to the edge cycle c. Thus the isomorphism in Proposition 1.28 gives this presentation for $\pi_1(M,x)$. □

Remark 1.30. Recall that edge cycles are unique only up to cyclic permutations and order reversals. This may seem to induce an ambiguity in the above presentation in that the same holds for the corresponding introduced relations. This is not so however as order reversals produce inverse elements and cyclically permuting elements in a product produces conjugate elements as $\delta_2\cdots\delta_k\delta_1 = \delta_1^{-1}(\delta_1\cdots\delta_k)\delta_1$ so that the subgroup normally generated by the relations is uniquely determined.

Combining the above, we see how to find the fundamental group of a triangulated 3-manifold (as well as ideally triangulated n-manifolds).

Corollary 1.31. *Given a (ideally) triangulated 3-manifold $(M,\mathcal{T})$, $\pi_1 M$ is finitely-generated.*

Remark 1.32. There exist other combinatorial methods of computing the fundamental group of triangulated spaces. For example, in [**1**, Chapter 6], a standard method is outlined in which the usual, non-dual 1-skeleton carries the generators and the 2-skeleton carries the relaters.

Example 1.33. The following is an oriented triangulation of quaternionic space S^3/Q_8 from Regina, [**5**]; the orientations on the simplices here are $v_i \to v_{i+1}$ and $v_i' \to v_{i+1}'$. Here the action of Q_8 on S^3 is the natural action after identifying S^3 with $\{(z,w) \in \mathbb{C}^2 \mid |z|^2+|w|^2 = 1\}$ and Q_8 with a subgroup of $\mathrm{SL}_2(\mathbb{C})$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

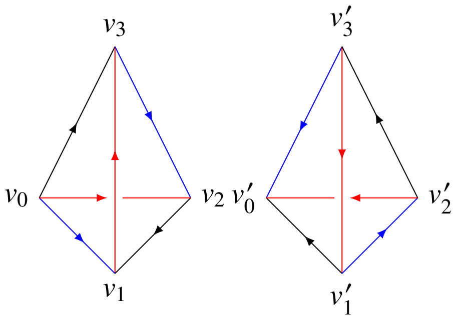


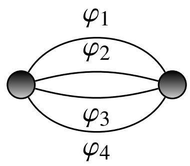
FIGURE 5. Triangulation of quaternionic space from Regina – "SFS [S2: (2,1) (2,1) (2,-1)]: #1" in "Closed Census (Orientable)"

The face-pairings, specified via the vertices v_i, v_i', here are

$$\varphi_1 : v_0, v_1, v_2 \mapsto v_3', v_0', v_1' \qquad \varphi_2 : v_0, v_1, v_3 \mapsto v_1', v_2', v_0'$$

$$\varphi_3 : v_0, v_2, v_3 \mapsto v_2', v_0', v_3' \qquad \varphi_4 : v_1, v_2, v_3 \mapsto v_3', v_2', v_1'$$

giving us the following dual 1-skeleton:



Let $\sigma_1 = [v_0, v_1, v_2, v_3]$ and $\sigma_2 = [v_0', v_1', v_2', v_3']$, choose the φ_1 edge as a maximal tree for this graph and orient the remaining edges $\sigma_0 \to \sigma_1$; each of these three oriented edges gives a generator, say a, b, c for $\varphi_2, \varphi_3, \varphi_4$ respectively. The sequence of face-pairings corresponding to the black edge is $\varphi_3, \varphi_4^{-1}, \varphi_1, \varphi_2^{-1}$, to the red edge is $\varphi_1, \varphi_4^{-1}, \varphi_2, \varphi_3^{-1}$ and corresponding to the blue edge is $\varphi_2, \varphi_4^{-1}, \varphi_3, \varphi_1^{-1}$. These then give relaters $bc^{-1}a^{-1}$, $c^{-1}ab^{-1}$ and $ac^{-1}b$ respectively. Thus, by Proposition 1.29,

$$\pi_1(S^3/Q_8) \cong \langle a, b, c \mid bc^{-1}a^{-1} = 1, c^{-1}ab^{-1} = 1, ac^{-1}b = 1 \rangle.$$

By eliminating c via the relation $c = ab^{-1}$, we have

$$\pi_1(S^3/Q_8) \cong \langle a, b \mid a^2 = b^2, aba^{-1}b = 1 \rangle$$

and this is a well-known presentation for Q_8.

Remark 1.34. The presentation $\langle a, b \mid a^2 = b^2, aba^{-1}b = 1 \rangle$ for Q_8 is sometimes stated with an additional relation, namely $a^4 = 1$. This however, is a consequence of the other two relations as, if those two hold, $a^4 = a^2a^2 = a^2b^2 = a(ab)b = a(b^{-1}a)b = abb^{-2}ab = aba^{-2}ab = aba^{-1}b = 1$.

The result $\pi_1(S^3/Q_8) \cong Q_8$ could have been anticipated via results on the fundamental groups of orbit spaces; see, for example, [**2**].

1.3. Covers and lifting triangulations

Given an oriented triangulated 3-manifold $(M, \mathcal{T})$, as M is a manifold, it possesses a universal cover, let $\widetilde{M}$ denote a fixed such cover with covering map $p : \widetilde{M} \to M$

Proposition 1.35. *Given an oriented triangulated 3-manifold* $(M, \mathcal{T})$*, let* $\sigma_i = [v_0^i, v_1^i, v_2^i, v_3^i]$ *and have orientation* $v_j^i \to v_{j+1}^i$*. There exists a disjoint union of oriented 3-simplices,* $\widetilde{\Sigma} = \sqcup \widetilde{\sigma}_{(i,\gamma)}$ *indexed by* $[n] \times Aut(\widetilde{M})$*, where* $\widetilde{\sigma}_{(i,\gamma)} = [v_0^{(i,\gamma)}, v_1^{(i,\gamma)}, v_2^{(i,\gamma)}, v_3^{(i,\gamma)}]$ *and has orientation* $v_j^{(i,\gamma)} \to v_{j+1}^{(i,\gamma)}$*, and a collection of corresponding orientation-reversing face-pairings* $\widetilde{\Phi}$ *such that:*

- $\widetilde{\Sigma}/\widetilde{\Phi}$ *is simply connected*
- *given a face-pairing* $v_j^{(i,\gamma)}, v_k^{(i,\gamma)}, v_l^{(i,\gamma)} \mapsto v_{j'}^{(i',\gamma')}, v_{k'}^{(i',\gamma')}, v_{l'}^{(i',\gamma')}$ *in* $\widetilde{\Phi}$*, the face-pairing* $v_j^i, v_k^i, v_l^i \mapsto v_{j'}^{i'}, v_{k'}^{i'}, v_{l'}^{i'}$ *is in* Φ*; conversely, if* $v_j^i, v_k^i, v_l^i \mapsto v_{j'}^{i'}, v_{k'}^{i'}, v_{l'}^{i'}$ *is in* Φ *and* $\gamma \in Aut(\widetilde{M})$*, there is a unique* $\gamma' \in Aut(\widetilde{M})$ *such that* $v_j^{(i,\gamma)}, v_k^{(i,\gamma)}, v_l^{(i,\gamma)} \mapsto v_{j'}^{(i',\gamma')}, v_{k'}^{(i',\gamma')}, v_{l'}^{(i',\gamma')}$ *is in* $\widetilde{\Phi}$
- *if* $v_j^{(i,\gamma)}, v_k^{(i,\gamma)}, v_l^{(i,\gamma)} \mapsto v_{j'}^{(i',\gamma')}, v_{k'}^{(i',\gamma')}, v_{l'}^{(i',\gamma')}$ *is in* $\widetilde{\Phi}$*, then, for each* $\gamma'' \in Aut(\widetilde{M})$*,* $v_j^{(i,\gamma''\gamma)}, v_k^{(i,\gamma''\gamma)}, v_l^{(i,\gamma''\gamma)} \mapsto v_{j'}^{(i',\gamma''\gamma')}, v_{k'}^{(i',\gamma''\gamma')}, v_{l'}^{(i',\gamma''\gamma')}$ *is also in* $\widetilde{\Phi}$*.*

For any fixed $i = 1, \ldots, n$ take a fixed lift, the existence of which is guaranteed by the simple connectedness of simplices, of the map

$$\pi|_{\sigma_i} : \sigma_i \hookrightarrow \Sigma \to M$$

and denote it by $\widetilde{\pi|_{\sigma_i}}$; the lifts map σ_i into $\widetilde{M}$. Now for each element of $[n] \times \mathrm{Aut}(\widetilde{M})$, take an oriented standard 3-simplex to form $\widetilde{\Sigma} = \sqcup_{[n]\times\mathrm{Aut}(\widetilde{M})} \widetilde{\sigma}_{(i,\gamma)}$; for each (i, γ), let $\widetilde{\sigma}_{(i,\gamma)} = [v_0^{(i,\gamma)}, v_1^{(i,\gamma)}, v_2^{(i,\gamma)}, v_3^{(i,\gamma)}]$ and have orientation $v_j^{(i,\gamma)} \to v_{j+1}^{(i,\gamma)}$. For each (i, γ), let $\alpha_{(i,\gamma)} : \widetilde{\sigma}_{(i,\gamma)} \to \sigma_i$ be the affine map determined by $v_j^{(i,\gamma)} \mapsto v_j^i$ and then define $\lambda_{(i,\gamma)} = \gamma \circ \widetilde{\pi|_{\sigma_i}} \circ \alpha_{(i,\gamma)}$. Piece together the $\alpha_{(i,\gamma)}$ and $\lambda_{(i,\gamma)}$ to construct the continuous maps

$$\Lambda = \sqcup \lambda_{(i,\gamma)} : \widetilde{\Sigma} \to \widetilde{M} \qquad \Pi = \sqcup \alpha_{(i,\gamma)} : \widetilde{\Sigma} \to \Sigma.$$

Lemma 1.36. *We have the following regarding the behaviour of* Λ*:*
(i) *given* $\widetilde{x} \in \widetilde{\Sigma}$*, suppose that* $\widetilde{x}$ *lies in the interior of a simplex; then given* $\widetilde{y} \in \widetilde{\Sigma}$ *such that* $\widetilde{y} \neq \widetilde{x}$*,* $\Lambda(\widetilde{y}) \neq \Lambda(\widetilde{x})$
(ii) *suppose given* (i, γ) *and let* $\widetilde{f}$ *be a face of* $\widetilde{\sigma}_{(i,\gamma)}$*; then*

- *there is a unique pair* (i', γ') *and unique face* $\widetilde{g}$ *of* $\widetilde{\sigma}_{(i',\gamma')}$ *such that* $\Lambda(\widetilde{f}^\circ) \cap \Lambda(\widetilde{g}) \neq \emptyset$
- *in this case* $\Lambda(\widetilde{f}^\circ) \cap \Lambda(\widetilde{g} - \widetilde{g}^\circ) = \emptyset$ *while there exists* $\varphi_{\Pi(\widetilde{f}),\Pi(\widetilde{g})} \in \Phi$ *and* $\Lambda(\widetilde{f}^\circ)$ *and* $\Lambda(\widetilde{g}^\circ)$ *coincide in such a way that*

$$\Lambda|_{\widetilde{f}^\circ} = \Lambda_{\widetilde{g}^\circ} \circ \left(\alpha^{-1}_{(i',\gamma')}|_{\Pi(\widetilde{g})^\circ} \circ \varphi_{\Pi(\widetilde{f})^\circ,\Pi(\widetilde{g})^\circ} \circ \alpha_{(i,\gamma)}|_{\widetilde{f}^\circ}\right).$$

As a result of (ii), we may define

$$\widetilde{\Phi} = \{\alpha^{-1}_{(i',\gamma')}|_{\Pi(\widetilde{g})} \circ \varphi_{\Pi(\widetilde{f}),\Pi(\widetilde{g})} \circ \alpha_{(i,\gamma)}|_{\widetilde{f}} \mid (\widetilde{f}, (i, \gamma)), (\widetilde{g}, (i', \gamma')) \text{ and } \varphi_{\Pi(\widetilde{f}),\Pi(\widetilde{g})} \text{ are related as in (ii)}\}$$

and letting the equivalence relation generated by identifications via $\widetilde{\Phi}$ *be* $\sim$ *we have:*
(iii) *given* $\widetilde{x} \in \widetilde{e}_1^\circ$*,* $\widetilde{y} \in \widetilde{e}_2^\circ$ *where* $\widetilde{e}_1, \widetilde{e}_2$ *are edges of some* $\widetilde{\sigma}_{(i,\gamma)}, \widetilde{\sigma}_{(i',\gamma')}$*,* $\Lambda(\widetilde{x}) = \Lambda(\widetilde{y}) \Leftrightarrow \widetilde{x} \sim \widetilde{y}$
(iv) *given vertices* $\widetilde{x}$*,* $\widetilde{y}$ *of some* $\widetilde{\sigma}_{(i,\gamma)}, \widetilde{\sigma}_{(i',\gamma')}$*,* $\Lambda(\widetilde{x}) = \Lambda(\widetilde{y}) \Leftrightarrow \widetilde{x} \sim \widetilde{y}$*.*

Proof. (i) Let $\widetilde{x} \in \widetilde{\sigma}^{\circ}_{(i,\gamma)}$ and $\widetilde{y} \in \widetilde{\sigma}^{\circ}_{(i',\gamma')}$ and suppose first that $\Pi(\widetilde{x})$ and $\Pi(\widetilde{y})$ have distinct images under π, that is, they are not identified by Φ. Then, since

$$p \circ (\gamma \circ \widetilde{\pi|_{\sigma_i}}) = \pi|_{\sigma_i} \qquad \text{and} \qquad p \circ (\gamma' \circ \widetilde{\pi|_{\sigma_{i'}}}) = \pi|_{\sigma_{i'}}$$

we have $\Lambda(\widetilde{x}) = (\gamma \circ \widetilde{\pi|_{\sigma_i}})(\Pi(\widetilde{x})) \neq (\gamma' \circ \widetilde{\pi|_{\sigma_{i'}}})(\Pi(\widetilde{y})) = \Lambda(\widetilde{y})$. If instead $\Pi(\widetilde{x})$ and $\Pi(\widetilde{y})$ have the same image under π, because $\Pi(\widetilde{x})$ is in the interior of σ_i, it must be that $i' = i$ and $\Pi(\widetilde{x}) = \Pi(\widetilde{y})$, say both equal to z. Now, $\Lambda(\widetilde{x}) = \gamma(\widetilde{\pi|_{\sigma_i}}(z))$ and $\Lambda(\widetilde{y}) = \gamma'(\widetilde{\pi|_{\sigma_i}}(z))$. If $\Lambda(\widetilde{x}) = \Lambda(\widetilde{y})$, then $\gamma^{-1}\gamma'$ has a fixed point and so, since the action of the deck group on universal covers is free, is the identity; this however implies that $\widetilde{x} = \widetilde{y}$.

(ii) There is a unique face-pairing $\varphi_{\Pi(\widetilde{f}),*} \in \Phi$ with domain $\Pi(\widetilde{f})$; let $\mathrm{codom}(\varphi_{\Pi(\widetilde{f}),*}) \subset \sigma_{i'}$. Suppose that $\Lambda(\widetilde{f}^{\circ}) \cap \Lambda(\widetilde{g}) \neq \emptyset$ where $\widetilde{g} \subset \widetilde{\sigma}_{(i'',\gamma')}$, say $\Lambda(\widetilde{x}) = \Lambda(\widetilde{y})$ for $\widetilde{x} \in \widetilde{f}^{\circ}$, $\widetilde{y} \in \widetilde{g}$. If $i'' \neq i'$, $\Pi(\widetilde{x})$ and $\Pi(\widetilde{y})$ have distinct images under π and so, by the same argument for this case as in (i), $\Lambda(\widetilde{x}) \neq \Lambda(\widetilde{y})$. Thus we must have $i'' = i'$. Note also that the same argument can be applied if $\widetilde{y}$ is not in the interior of $\widetilde{g}$, so that $\widetilde{y} \in \widetilde{g}^{\circ}$ and then using the same argument once more we see that $\widetilde{g}$ is must satisfy $\Pi(\widetilde{g}) = \mathrm{codom}(\varphi_{\Pi(\widetilde{f}),*})$. Let Δ be a model standard 2-simplex and let $\eta_{(i,\gamma)} : \Delta^{\circ} \hookrightarrow \widetilde{\sigma}_{(i,\gamma)}$ and $\eta_{(i',\gamma')} : \Delta^{\circ} \hookrightarrow \sigma_{(i',\gamma')}$ be imbeddings which identify Δ° with $\widetilde{f}^{\circ}$ in $\sigma_{(i,\gamma)}$ and $\widetilde{g}^{\circ}$ in $\sigma_{(i',\gamma)}$ and in such a way that

$$(\star) \qquad \eta_{(i',\gamma')} = \left(\alpha^{-1}_{(i',\gamma')}|_{\Pi(\widetilde{g})^{\circ}} \circ \varphi_{\Pi(\widetilde{f})^{\circ},\Pi(\widetilde{g})^{\circ}} \circ \alpha_{(i,\gamma)}|_{\widetilde{f}^{\circ}}\right) \circ \eta_{(i,\gamma)}.$$

Then $\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)}$ and $\lambda_{(i',\gamma')} \circ \eta_{(i',\gamma')}$ are lifts of $\pi|_{\sigma_i} \circ \alpha_{(i,\gamma)} \circ \eta_{(i,\gamma)}$ and $\pi|_{\sigma_{i'}} \circ \alpha_{(i',\gamma')} \circ \eta_{(i',\gamma')}$ respectively and the latter two are in fact the same map. Now as $\Lambda(\widetilde{f}^{\circ}) \cap \Lambda(\widetilde{g}^{\circ}) \neq \emptyset$, for some $x \in \Delta^{\circ}$, $(\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)})(x) = (\lambda_{(i',\gamma')} \circ \eta_{(i',\gamma')})(x)$ as if $(\widetilde{x}, \widetilde{y}) \in \widetilde{f}^{\circ} \times \widetilde{g}^{\circ}$ is not of the form $(\eta_{(i,\gamma)}(x), \eta_{(i',\gamma')}(x))$, $\Pi(\widetilde{x})$ and $\Pi(\widetilde{y})$ have distinct images under π and, as we have seen, this implies that $\Lambda(\widetilde{x}) \neq \Lambda(\widetilde{y})$. By uniqueness of lifts, the two lifts must coincide everywhere. Suppose now that $\widetilde{\sigma}_{(i',\gamma'')}$ also contains a face, say $\widetilde{g}'$, which satisfies $\Lambda(\widetilde{f}^{\circ}) \cap \Lambda(\widetilde{g'}^{\circ}) \neq \emptyset$; as noted above, this $\widetilde{g}'$ necessarily satisfies $\Pi(\widetilde{g}') = \mathrm{codom}(\varphi_{\Pi(\widetilde{f}),*})$. Defining $\eta_{(i',\gamma'')} = \left(\alpha^{-1}_{(i',\gamma'')}|_{\Pi(\widetilde{g}')^{\circ}} \circ \varphi_{\Pi(\widetilde{f})^{\circ},\Pi(\widetilde{g}')^{\circ}} \circ \alpha_{(i,\gamma)}|_{\widetilde{f}^{\circ}}\right) \circ \eta_{(i,\gamma)}$, we have as before that $\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)}$ and $\lambda_{(i',\gamma'')} \circ \eta_{(i',\gamma'')}$ coincide. In particular, if we let z be the barycentre of Δ, $(\lambda_{(i',\gamma')} \circ \eta_{(i',\gamma')})(z) = (\lambda_{(i',\gamma'')} \circ \eta_{(i',\gamma'')})(z)$. As $(\eta_{(i',\gamma')})(z)$ and $(\eta_{(i',\gamma'')})(z)$ are the barycentres of $\widetilde{g}$ and $\widetilde{g}'$ respectively, they have the same image under $\alpha_{(i',\gamma')}$ and $\alpha_{(i',\gamma'')}$ respectively. and so $(\lambda_{(i',\gamma')} \circ \eta_{(i',\gamma')})(z) = \gamma'(\gamma'')^{-1} \cdot (\lambda_{(i',\gamma'')} \circ \eta_{(i',\gamma'')})(z)$ so that $\gamma'(\gamma'')^{-1}$ has a fixed point and so is the identity. This completes the proof of the uniqueness.

Given an arbitrary γ', let γ'' be the unique deck transformation such that $(\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)})(z) = \gamma'' \cdot (\lambda_{(i',\gamma')} \circ \eta_{(i'),\gamma'})(z)$; then $(\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)})(z) = (\lambda_{(i',\gamma''\gamma')} \circ \eta_{(i',\gamma''\gamma')})(z)$ where $\eta_{(i',\gamma''\gamma')}$ is defined analogously to the other η maps. This proves the required existence and then for the unique γ', as the lifts coincide everywhere, $\lambda_{(i,\gamma)} \circ \eta_{(i,\gamma)} = \lambda_{(i',\gamma')} \circ \eta_{(i',\gamma')}$ and so by $(\star)$, $\lambda_{(i,\gamma)}|_{f^{\circ}} = \lambda_{(i',\gamma')} \circ \left(\alpha^{-1}_{(i',\gamma')}|_{\Pi(\widetilde{g})^{\circ}} \circ \varphi_{\Pi(\widetilde{f})^{\circ},\Pi(\widetilde{g})^{\circ}} \circ \alpha_{(i,\gamma)}|_{\widetilde{f}^{\circ}}\right)$.

The forward implications in (iii) and (iv) follow from the definition of $\widetilde{\Phi}$ and the observation, as in the proof of (i), that if $\Lambda(\widetilde{x}) = \Lambda(\widetilde{y})$, then $\Pi(\widetilde{x})$ and $\Pi(\widetilde{y})$ have the same image under π. The backward implications follow from the observation that in the proof of (ii), we could have use Δ, as opposed to its interior, as the domain of the η maps. ◻

Proof of Proposition 1.35. Using the definition from Lemma 1.36, note that the face-pairings in $\widetilde{\Phi}$ are orientation-reversing because those in Φ are orientation-reversing. It can also be verified, using the corresponding properties of Φ, that every face in $\widetilde{\Sigma}$ occurs as the domain of precisely one face-pairing in $\widetilde{\Phi}$ and that given faces $\widetilde{f}$ and $\widetilde{g}$, there either exists no face-pairing $\widetilde{f} \to \widetilde{g}$ and no face-pairing $\widetilde{g} \to \widetilde{f}$ or there exists one and only one face-pairing $\widetilde{f} \to \widetilde{g}$ and one and only one from $\widetilde{g} \to \widetilde{f}$, denoted $\varphi_{\widetilde{f},\widetilde{g}}$ and $\varphi_{\widetilde{g},\widetilde{f}}$ respectively, and these are such that $\varphi_{\widetilde{g},\widetilde{f}} = \varphi^{-1}_{\widetilde{f},\widetilde{g}}$.

It can be verified by direction construction of open 3-balls that Λ is a quotient map. Lemma 1.36 shows that the equivalence relation generated by $\widetilde{\Phi}$ is given precisely by $x \sim y \Leftrightarrow \Lambda(x) = \Lambda(y)$ and thus Λ descends to a homeomorphism $\widetilde{\Sigma}/\widetilde{\Phi} \xrightarrow{\sim} \widetilde{M}$. In particular, $\widetilde{\Sigma}/\widetilde{\Phi}$ is simply connected.

Finally, the remaining properties of $\widetilde{\Phi}$ and its relation to Φ are immediate consequences of the definition of $\widetilde{\Phi}$. ◻

Remark 1.37. In the case of a non-oriented $(M, \mathcal{T})$, the same holds bar the orientations on the simplices. In the case of ideal triangulations, the same holds with vertices replaced by normal triangle types.

Definition 1.38. Given an oriented triangulated 3-manifold $(M, \mathcal{T})$, any triangulation satisfying the conditions in Proposition 1.35 is said to be a *lift* of $\mathcal{T}$, denoted $\widetilde{\mathcal{T}}$.

Whenever we speak of a lifted triangulation, it is assumed that the vertices in $\mathcal{T}^{(0)}$ and $\widetilde{\mathcal{T}}^{(0)}$ have been labelled in the manner shown to be possible by Proposition 1.35. Also, we denote the canonical surjection $\widetilde{\Sigma} \to \widetilde{\Sigma}/\widetilde{\Phi}$ by $\widetilde{\pi}$.

Note that $\pi \circ \Pi$ descends to a well-defined map $\widetilde{\Sigma}/\widetilde{\Phi} \to M$ and it can be verified that this is a covering map so that, as already seen via Λ,by uniqueness of universal covers, $\widetilde{\Sigma}/\widetilde{\Phi} \cong \widetilde{M}$. As such, in the context of covers and lifted triangulations, we set the following.

Definition 1.39. The symbols $\widetilde{M}$ and p are re-defined to be $\widetilde{\Sigma}/\widetilde{\Phi}$ and the map $\widetilde{\Sigma}/\widetilde{\Phi} \to M$ mentioned above, respectively.

The following proposition shows that though a lifted triangulation may be infinite, it has finite edge cycles.

Proposition 1.40. *Suppose given an oriented triangulated 3-manifold $(M, \mathcal{T})$ and lift $\mathcal{T}$ to a triangulation $\widetilde{\mathcal{T}}$. Given an edge $\widetilde{e} \in \widetilde{\mathcal{T}}^{(1)}$, there exist finitely many simplices $\widetilde{\sigma}_{(i_1,\gamma_1)}, \ldots, \widetilde{\sigma}_{(i_k,\gamma_k)}$ in $\widetilde{\mathcal{T}}^{(3)}$ where $\widetilde{\sigma}_{(i_j,\gamma_j)}$ may coincide with $\widetilde{\sigma}_{(i_{j'},\gamma_{j'})}$, faces $\widetilde{f}_1, \widetilde{g}_1, \ldots, \widetilde{f}_k, \widetilde{g}_k$ where $\widetilde{f}_j, \widetilde{g}_j \subset \widetilde{\sigma}_{(i_j,\gamma_j)}$, edges $\widetilde{e}_j = \widetilde{f}_j \cap \widetilde{g}_j = [\widetilde{v}_j, \widetilde{w}_j]$ where we can set $\widetilde{e}$ to be any $\widetilde{e}_j$ such that:*

- *in $\widetilde{\Phi}$, there exist face-pairings $(\widetilde{g}_j, \widetilde{e}_j, \widetilde{v}_j, \widetilde{w}_j) \to (\widetilde{f}_{j+1}, \widetilde{e}_{j+1}, \widetilde{v}_{j+1}, \widetilde{w}_{j+1})$ for $j \in [k]$, where the subscripts are taken modulo k*
- *the $\widetilde{e}_j$ comprise precisely all edges identified with $\widetilde{e}$ under $\widetilde{\pi}$*
- *this sequence of simplices, faces and edges is unique up to cyclic permutations and order reversal.*

Proof. Let $\widetilde{e} = [v_j^{(i_0,\gamma_0)}, v_k^{(i_0,\gamma_0)}]$ and then let $e = [v_j^{i_0}, v_k^{i_0}] \in \mathcal{T}^{(1)}$. For this e, we have an edge cycle with simplices $\sigma_{i_1}, \ldots, \sigma_{i_k}$, faces $f_j, g_j \subset \sigma_{i_j}$, edges $e_j = f_j \cap g_j = [v_j, w_j]$ where $e_1 = e$ and so $i_1 = i_0$. We know from the proof of the Proposition 1.9 that we can label the vertices of σ_{i_j} as $u_0^j, u_1^j, u_2^j, u_3^j$ such that σ_{i_j} has orientation $u_i^j \to u_{i+1}^j$, $f_j = [u_0^j, u_1^j, u_2^j]$, $g_j = [u_0^j, u_1^j, u_3^j]$, $e_j = [u_0^j, u_1^j]$ and where each identification $g_j \to f_{j+1}$ maps u_0^j, u_1^j, u_3^j to $u_0^{j+1}, u_1^{j+1}, u_2^{j+1}$ respectively, taking the superscripts modulo k.

Let $u_i^j = v_{i'}^{i_j}$; in particular, $u_i^1 = v_{i'}^{i_1} = v_{i'}^{i_0}$. Set $\gamma_1 = \gamma_0$ and then let $\widetilde{\sigma}_{(i_1,\gamma_1)} = \widetilde{\sigma}_{(i_0,\gamma_0)}$, $\widetilde{f}_1 = [v_{0'}^{(i_1,\gamma_1)}, v_{1'}^{(i_1,\gamma_1)}, v_{2'}^{(i_1,\gamma_1)}]$, $\widetilde{g}_1 = [v_{0'}^{(i_1,\gamma_1)}, v_{1'}^{(i_1,\gamma_1)}, v_{3'}^{(i_1,\gamma_1)}]$ and $\widetilde{e}_1 = [v_{0'}^{(i_1,\gamma_1)}, v_{1'}^{(i_1,\gamma_1)}]$ which is $\widetilde{e}$. Now, corresponding to the face-pairing $g_1 \to f_2$ which takes u_0^1, u_1^1, u_3^1 to u_0^2, u_1^2, u_2^2 respectively, that is, $v_{0'}^{i_1}, v_{1'}^{i_1}, v_{3'}^{i_1}$ to $v_{0'}^{i_2}, v_{1'}^{i_2}, v_{2'}^{i_2}$, we have a unique $\gamma_2 \in \mathrm{Aut}(\widetilde{M})$ such that $v_{0'}^{(i_1,\gamma_1)}, v_{1'}^{(i_1,\gamma_1)}, v_{3'}^{(i_1,\gamma_1)} \mapsto v_{0'}^{(i_2,\gamma_2)}, v_{1'}^{(i_2,\gamma_2)}, v_{2'}^{(i_2,\gamma_2)}$ is in $\widetilde{\Phi}$. We have then defined $\widetilde{\sigma}_{(i_2,\gamma_2)}$ and also define $\widetilde{f}_2 = [v_{0'}^{(i_2,\gamma_2)}, v_{1'}^{(i_2,\gamma_2)}, v_{2'}^{(i_2,\gamma_2)}]$, $\widetilde{g}_1 = [v_{0'}^{(i_2,\gamma_2)}, v_{1'}^{(i_2,\gamma_2)}, v_{3'}^{(i_2,\gamma_2)}]$ and $\widetilde{e}_2 = [v_{0'}^{(i_2,\gamma_2)}, v_{1'}^{(i_2,\gamma_2)}]$; here the function $i \to i'$ is that which is involved in $w_i^2 = v_{i'}^{i_2}$. Now we proceed inductively and construct a sequence with simplices $\widetilde{\sigma}_{(i_1,\gamma_1)}, \ldots, \widetilde{\sigma}_{(i_k,\gamma_k)}$, faces $\widetilde{f}_j, \widetilde{g}_j \subset \sigma_{(i_j,\gamma_j)}$, edges $\widetilde{e}_j = \widetilde{f}_j \cap \widetilde{g}_j = [\widetilde{v}_j, \widetilde{w}_j]$ where $\sigma_{(i_j,\gamma_j)} = [v_{0'}^{(i_j,\gamma_j)}, v_{1'}^{(i_j,\gamma_j)}, v_{2'}^{(i_j,\gamma_j)}, v_{3'}^{(i_j,\gamma_j)}]$ such that $\sigma_{(i_j,\gamma_j)}$ has orientation $v_{l'}^{(i_j,\gamma_j)} \to v_{(l+1)'}^{(i_j,\gamma_j)}$, $\widetilde{f}_j = [v_{0'}^{(i_j,\gamma_j)}, v_{1'}^{(i_j,\gamma_j)}, v_{2'}^{(i_j,\gamma_j)}]$, $\widetilde{g}_j = [v_{0'}^{(i_j,\gamma_j)}, v_{1'}^{(i_j,\gamma_j)}, v_{3'}^{(i_j,\gamma_j)}]$, $\widetilde{e}_j = [v_{0'}^{(i_j,\gamma_j)}, v_{1'}^{(i_j,\gamma_j)}]$ and where, for each $j \in [k]$, we have an identification $\widetilde{g}_j \to \widetilde{f}_{j+1}$ mapping $v_{0'}^{(i_j,\gamma_j)}, v_{1'}^{(i_j,\gamma_j)}, v_{3'}^{(i_j,\gamma_j)}$ to $v_{0'}^{(i_{j+1},\gamma_{j+1})}, v_{1'}^{(i_{j+1},\gamma_{j+1})}, v_{2'}^{(i_{j+1},\gamma_{j+1})}$, respectively, taking the indices modulo k. Note that a different function $i \to i'$ is used for each simplex $\widetilde{\sigma}_{(i_j,\gamma_j)}$.

The above constructs a sequence of the required form and the required properties of this sequence automatically imply that the $\widetilde{e}_j$ comprise precisely all edges identified to $\widetilde{e}$ under $\widetilde{\pi}$. It remains to verify uniqueness and that we can set $\widetilde{e}$ to be any $\widetilde{e}_j$. Uniqueness, as stated, follows because any two sequences of the required form will, upon application of Π, give two edge cycles about the same edge for the triangulation $\mathcal{T}$ and then the uniqueness statement here follows from that for edge cycles in $\mathcal{T}$. Further, applying cyclic permutations and/or an order reversal, we see that we can set $\widetilde{e}$ to be any $\widetilde{e}_j$. □

Given an oriented triangulated 3-manifold $(M, \mathcal{T})$ and the associated oriented triangulated universal cover $(\widetilde{M}, \widetilde{\mathcal{T}})$, let $(\widetilde{\mathcal{G}}, \mathcal{E}_{\widetilde{\mathcal{T}}})$ be the dual 1-skeleton of $\widetilde{\mathcal{T}}$. The next proposition describes loops in $\widetilde{\mathcal{G}}$.

Proposition 1.41. *Given an oriented triangulated 3-manifold $(M, \mathcal{T})$, $\widetilde{\sigma} \in \widetilde{\mathcal{T}}^{(3)}$ and a combinatorial loop $\widetilde{\ell}$ in $\widetilde{\mathcal{G}}$ based at $\widetilde{\sigma}$, $[\widetilde{\ell}]$ is a product of conjugates of loop classes of the form $[q_{\widetilde{c}}\widetilde{c}q_{\widetilde{c}}^{-1}]$ where $\widetilde{c}$ is an edge cycle and $q_{\widetilde{c}}$ is a combinatorial path from $\widetilde{\sigma}$ to a 3-simplex in $\widetilde{c}$.*

Proof. Let $\widetilde{\ell} = \widetilde{g}_1 \xrightarrow{\widetilde{\varphi}_1} \widetilde{f}_2, \widetilde{g}_2 \xrightarrow{\widetilde{\varphi}_2} \cdots \xrightarrow{\widetilde{\varphi}_{k-1}} \widetilde{f}_k, \widetilde{g}_k \xrightarrow{\widetilde{\varphi}_k} \widetilde{f}_1$, where $\widetilde{f}_j, \widetilde{g}_j \subset \widetilde{\sigma}_{(i_j,\gamma_j)}$, be a combinatorial loop in $\widetilde{\mathcal{G}}$ where $\widetilde{\sigma}_{(i_1,\gamma_1)} = \widetilde{\sigma}$. Let $\widetilde{f}_j = [\widetilde{v}_0^j, \widetilde{v}_1^j, \widetilde{v}_2^j]$ and $\widetilde{g}_j = [\widetilde{w}_0^j, \widetilde{w}_1^j, \widetilde{w}_2^j]$ such that $\widetilde{\varphi}_j : \widetilde{w}_l^j \mapsto \widetilde{v}_l^{j+1}$, superscripts taken modulo k. Let

$$\widetilde{q} = q_{(\sigma_{(i_1,\gamma_1)},\widetilde{g}_1)} \cdot q^{-1}_{(\sigma_{(i_2,\gamma_2)},\widetilde{f}_2)} \cdot q_{(\sigma_{(i_2,\gamma_2)},\widetilde{g}_2)} \cdots q^{-1}_{(\sigma_{(i_k,\gamma_k)},\widetilde{f}_k)} \cdot q_{(\sigma_{(i_k,\gamma_k)},\widetilde{g}_k)} \cdot q^{-1}_{(\sigma_{(i_1,\gamma_1)},\widetilde{f}_1)}$$

and then let

$$q = p \circ \widetilde{q} = (p \circ q_{(\sigma_{(i_1,\gamma_1)},\widetilde{g}_1)}) \cdot (p \circ q^{-1}_{(\sigma_{(i_2,\gamma_2)},\widetilde{f}_2)}) \cdot (p \circ q_{(\sigma_{(i_2,\gamma_2)},\widetilde{g}_2)})$$
$$\cdots (p \circ q^{-1}_{(\sigma_{(i_k,\gamma_k)},\widetilde{f}_k)}) \cdot (p \circ q_{(\sigma_{(i_k,\gamma_k)},\widetilde{g}_k)}) \cdot (p \circ q^{-1}_{(\sigma_{(i_1,\gamma_1)},\widetilde{f}_1)})$$
$$= q_{(\sigma_{i_1},g_1)} \cdot q^{-1}_{(\sigma_{i_2},f_2)} \cdot q_{(\sigma_{i_2},g_2)} \cdots q^{-1}_{(\sigma_{i_k},f_k)} \cdot q_{(\sigma_{i_k},g_k)} \cdot q^{-1}_{(\sigma_{i_1},f_1)}$$

where, $f_j = [v^j_0, v^j_1, v^j_2]$, $g_j = [w^j_0, w^j_1, w^j_2]$ where $v^j_l = \Pi(\widetilde{v}^j_l)$ and $w^j_l = \Pi(\widetilde{w}^j_l)$. Now set

$$\ell = g_1 \xrightarrow{\varphi_1} f_2, g_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} f_k, g_k \xrightarrow{\varphi_k} f_1$$

where φ_j, which we might denote $\Pi(\widetilde{\varphi}_j)$, is defined by $w^j_l \mapsto v^{j+1}_l$. Because $\widetilde{M}$ is simply connected, $\pi_1(\widetilde{M}, \widetilde{x})$ is trivial where $\widetilde{x}$ is the barycentre of $\widetilde{\sigma}_{(i_1,\gamma_1)}$ and so, letting x denote the barycentre of σ_{i_1}, $[q] = [p \circ \widetilde{q}] = p_*[q]$, where $p_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, x)$ is induced homomoprhism, is the identity element of $\pi_1(M, x)$. Thus ℓ is in $\ker(\Psi)$ where Ψ is the homomorphism in Proposition 1.28 and so $[\ell] \in \pi^{\mathrm{comb}}_1(\mathcal{G}, \sigma_{i_1})$ is a product of conjugates of the loop classes of the form $[q_c c q^{-1}_c]$ where c is an edge cycle in $\mathcal{G}$ and q_c is a combinatorial path from σ_{i_1} to a simplex in c. As such, ℓ is a concatenation of spurs and conjugates of the loops $q_c c q^{-1}_c$. Now, in the associated factorisation of q, it is easy to verify that each a spur factor in this factorisation lifts to a spur in $\widetilde{\mathcal{G}}$ and that a factor which is a conjugate of a loop of the form $q_c c q^{-1}_c$ lifts to a conjugate of a loop of the form $q_{\widetilde{c}} \widetilde{c} q^{-1}_{\widetilde{c}}$; it is also clear that the product of these lifts of factors, say $\widetilde{q}'$, is a lift of q. By uniqueness of lifts, $\widetilde{q}' = \widetilde{q}$ and so $\widetilde{\ell}$ is a concatenation of spurs and conjugates of loops of the form $q_{\widetilde{c}} \widetilde{c} q^{-1}_{\widetilde{c}}$; thus $[\widetilde{\ell}]$ has the required form. □

CHAPTER 2

Representations

In this chapter, we begin to study some methods to construct representations of 3-manifold groups. Given a collection of 3-simplices Σ and of face-pairings Φ, every (Σ, Φ) gives rise to an ideally triangulated 3-manifold while only some, those satisfying the condition in Theorem 1.22, lead to 3-manifolds. Because of this and Proposition 1.23, we focus on representing the fundamental groups of ideally triangulated spaces.

2.1. Via labellings of face-pairings

Given an ideally triangulated 3-manifold $(M, \mathcal{T})$, let $(\mathcal{G}, \mathcal{E}_{\mathcal{T}})$ be the dual 1-skeleton. Let $(\mathcal{E}_{\mathcal{T}}^{(1)})^{\pm} = \{\varphi^{\pm} \mid \varphi \in \mathcal{E}_{\mathcal{T}}^{(1)}\}$ where φ^{+} and φ^{-} are oppositely oriented variants of the edge $\varphi \in \mathcal{E}_{\mathcal{T}}^{(1)}$. Let G be a group and let $f : (\mathcal{E}_{\mathcal{T}})^{\pm} \to G$ be any function such that

- we have $f(\varphi^{-}) = f(\varphi^{+})^{-1}$ for all $\varphi \in \mathcal{E}_{\mathcal{T}}^{(1)}$
- given any edge cycle, say given by face-pairings $\varphi_1, \ldots, \varphi_k$, we have $g_1 \cdots g_k = 1$ where $g_i = f(\varphi_i^{\epsilon_i})$; here $\epsilon_i = \pm$ is determined by the direction in which φ_i is traversed in the edge cycle.

Proposition 2.1. *Suppose given an ideally triangulated 3-manifold $(M, \mathcal{T})$, $(\mathcal{G}, \mathcal{E}_{\mathcal{T}})$ and $f : (\mathcal{E}_{\mathcal{T}})^{\pm} \to G$ as above and let $\sigma \in \mathcal{T}^{(3)}$ and Ψ be the map in Proposition 1.28. The map $\rho : \pi_1^{comb}(\mathcal{G}, \sigma)/\ker(\Psi) \to G$ which maps a sequence of face-pairings $[\varphi_1, \ldots, \varphi_k]$ to $f(\varphi_1^{\varepsilon_1}) \cdots f(\varphi_k^{\varepsilon_k})$ is well-defined and a homomorphism.*

Proof. Recall that $\ell(\mathcal{E}_{\mathcal{T}}, \sigma)$ denotes the collection of combinatorial loops in $\mathcal{G}$ based at σ. The map $\ell(\mathcal{E}_{\mathcal{T}}, \sigma) \to G : \varphi_1, \ldots, \varphi_k \mapsto f(\varphi_1^{\varepsilon_1}) \cdots f(\varphi_k^{\varepsilon_k})$ is a morphism of monoids. Because $f(\varphi^{-}) = f(\varphi^{+})^{-1}$ for all $\varphi \in \mathcal{E}_{\mathcal{T}}^{(1)}$, additions and removals of spurs do not alter the image of a loop under this map and so this map descends to a map $\pi_1^{\mathrm{comb}}(\mathcal{G}, \sigma) \to G$ and this new map is a group homomorphism. Consider the products $g_1 \cdots g_k$ associated to edge cycles $\varphi_1, \ldots, \varphi_k$ where $g_i = f(\varphi_i^{\epsilon_i})$. It can be seen that each element of the image of $\ker(\Psi)$ under our map is a product of conjugates of such products and as such, the second condition imposed on f is equivalent to that the kernel of our map contain $\ker(\Psi)$ and so our map descends to a representation $\pi_1^{\mathrm{comb}}(\mathcal{G}, \sigma)/\ker(\Psi) \to G$ which is precisely ρ. □

Remark 2.2. Given the edge cycle $\varphi_1, \ldots, \varphi_k$ and $g_i = f(\varphi_i^{\epsilon_i})$, $g_i \cdots g_i g_1 \cdots g_{i-1}$ is conjugate to $g_1 \cdots g_k$ and $\varphi_k \cdots \varphi_1$ has image $g_k^{-1} \cdots g_1^{-1} = (g_1 \cdots g_k)^{-1}$. As such, for each edge cycle, we need only check that one of the many possible associated products vanishes.

Definition 2.3. Given a (oriented) (ideally) triangulated 3-dimensional space $(M, \mathcal{T})$, $\mathcal{T}$ is said to be *even* if each edge cycle has even length in the dual 1-skeleton.

For example, the triangulations of the figure-eight knot complement and quaternionic space in Examples 1.12 and 1.33 are both even.

Example 2.4. Given an ideally triangulated 3-manifold $(M, \mathcal{T})$ and dual 1-skeleton $(\mathcal{G}, \mathcal{E}_{\mathcal{T}})$, take $G = \mathbb{Z}_2 = \langle x \rangle$ and define f by $f : \varphi^{\pm} \to x$ for all φ. Then we see that edge cycles are annihilated if and only if $\mathcal{T}$ is even. Thus

$$\text{even triangulation gives } \rho : \pi_1(M) \to \mathbb{Z}_2.$$

It can be seen that

$$\rho \text{ is non-trivial} \Leftrightarrow \mathcal{G} \text{ contains a loop of odd length.}$$

It is a known result that any graph (under our definition of graph as in Definition 1.14), say with vertex set V and edge set E, is bipartite if and only if it contains no loops of odd length. The 'only if' direction is easy to see as the loop must alternate between the two partitioning subsets of the vertex set. To see the converse, given vertices $u, v \in V$, let $d(u, v)$ be the length of the shortest path from u to v. Fix some vertex v_0 in our graph and let $X = \{u \in V \mid d(u, v_0) \text{ is even}\}$, which contains v_0, and let $Y = \{u \in V \mid d(u, v_0) \text{ is odd}\}$. Suppose that there exists an edge $e \in E$ which joins vertices u, v which both lie in X or both in Y. Then $d(u, v_0)$ and $d(v, v_0)$ have the same parity; say p_1 and p_2 are paths from u to v_0, v to v_0 respectively which achieve these minima. Then $p_1^{-1} \cdot e \cdot p_2$ is a loop of odd length, a contradiction. Thus the partition $V = X \sqcup Y$ shows that our graph is bipartite. Thus we have

$$\rho \text{ is non-trivial} \Leftrightarrow \mathcal{G} \text{ is not bipartite.}$$

Note that if the dual 1-skeleton $\mathcal{G}$ is bipartite, then $\mathcal{T}$ is necessarily even. Thus if $(M, \mathcal{T})$ is to have a non-trivial representation into $\mathbb{Z}_2$ as defined above, $\mathcal{T}$ must be even but not because $\mathcal{G}$ is bipartite.

The above example can be used to prove non-triviality results regarding the fundamental groups of ideally triangulated 3-manifolds.

Proposition 2.5. *Given an ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ is even and there exists $\varphi \in \Phi$ such that dom(φ), codom(φ) are contained in the same 3-simplex, then $\pi_1(M) \neq 1$.*

Proof. Because $\mathcal{T}$ is even, we have a representation $\rho : \pi_1(M) \to \mathbb{Z}_2$ as in Example 2.4. The given $\varphi \in \Phi$ gives an edge in $\mathcal{G}$ which is a loop and so $\mathcal{G}$ cannot be bipartite. □

Remark 2.6. Note that in the above proposition, the given φ cannot be a trivial "rotation" across an edge, depicted in Figure 1, as this would contradict the evenness of $\mathcal{T}$.

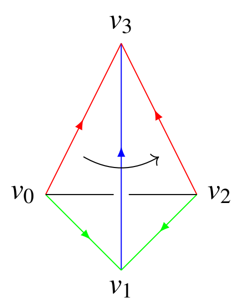
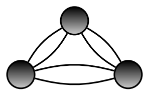


FIGURE 1. A face-pairing, $v_0, v_1, v_3 \mapsto v_2, v_1, v_3$, which is a rotation across an edge

It can be checked that if $\mathcal{T}$ has three 3-simplices and $\mathcal{G}$ has no edges which are loops, because dual 1-skeletons are 4-valent, $\mathcal{G}$ is necessarily



which is not bipartite. Thus we see that if $|\mathcal{T}^{(3)}| = 3$ and $\mathcal{T}$ is even, $\pi_(M) \neq 1$. More generally, we have the following.

Proposition 2.7. *Given an ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ is even and contains an odd number of 3-simplices, then $\pi_1(M) \neq 1$.*

Proof. If $\mathcal{G}$ contains an edge which is a loop, we can apply Proposition 2.5. Assume then that $\mathcal{G}$ does not contain such an edge and that $\mathcal{G}$ is bipartite, say with partition $\mathcal{G}^{(0)} = X \sqcup Y$. Now, any $e \in \mathcal{G}^{(1)}$ contributes exactly 1 to both the sums $\sum_{v \in X} \deg(v)$ and $\sum_{v \in Y} \deg(v)$. As such, these sums are equal. However, as $\mathcal{G}$ is 4-valent, we have that these sums are $4|X|$ and $4|Y|$ respectively. Thus $|X| = |Y|$ which implies that $|\mathcal{G}^{(0)}|$ is even, a contradiction. □

Example 2.8. The Lens space $L(4, 1)$ has the following even triangulation, with degree sequence 2,4, from Regina, [**5**]. It has one 3-simplex, and so, by Proposition 2.7, $L(4, 1)$ must have a non-trivial fundamental group.

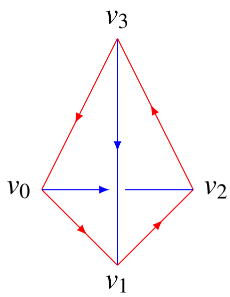


FIGURE 2. Triangulation of $L(4, 1)$ from Regina – "L(4,1): #1" in "Closed Census (Orientable)"

Remark 2.9. As $\mathbb{Z}_2$ is abelian, the representation here into $\mathbb{Z}_2$ can only detect non-triviality in the case that the abelianisation of π_1, that is, first homology, is non-trivial.

2.2. Via labellings of vertices

Suppose that $(M, \mathcal{T})$ is an oriented ideally triangulated 3-manifold and that G is a group which acts simply 3-transitively on a set X where $|X| \geq 3$. Suppose further that:

(A1): there is a map
$$P : \{\text{injections } f^{(0)} \hookrightarrow X\} \to \{\text{injections } \sigma^{(0)} \hookrightarrow X\}$$
which, given an injection $\iota : f^{(0)} \hookrightarrow X$, where $f \subset \sigma$, assigns an extension $\kappa : \sigma^{(0)} \hookrightarrow X$ which is also an injection

(A2): given an injection $\kappa : \sigma^{(0)} \hookrightarrow X$, if $\kappa = P(\kappa|_f)$ for some $f \subset \sigma$, $\kappa = P(\iota|_f)$ for all $f \subset \sigma$ (alternatively, given $\iota : f^{(0)} \to X$, $f \subset \sigma$, $P(P(\iota)|_g) = P(\iota)$ for all $g \subset \sigma$); here $\kappa|_f$ denotes $\kappa|_{f^{(0)}}$

(A3): P commutes with $\psi \in G$ in that for all $\iota : f^{(0)} \hookrightarrow X$, $P(\psi \circ \iota) = \psi \circ P(\iota)$; this can be described as the commutativity of the following diagram

$$\begin{array}{ccc}\{\text{injections } f^{(0)} \hookrightarrow X\} & \xrightarrow{\psi} & \{\text{injections } f^{(0)} \hookrightarrow X\} \\ P\big\downarrow & & \big\downarrow P \\ \{\text{injections } \sigma^{(0)} \hookrightarrow X\} & \xrightarrow[\psi]{} & \{\text{injections } \sigma^{(0)} \hookrightarrow X\}\end{array}$$

or in words that P is G-equivariant

(A4): given an edge cycle with face-pairings $g_1 \to f_2, g_2 \to f_3, \ldots, g_k \to f_1$, $f_i = [v_0^i, v_1^i, v_2^i]$, $g_i = [v_0^i, v_1^i, v_3^i]$ and $\iota_1 : f_1^{(0)} \hookrightarrow X$; if we set $\iota_2 : f_2^{(0)} \hookrightarrow X : v_0^2, v_1^2, v_2^2 \mapsto P(\iota_1)(v_0^1), P(\iota_1)(v_1^1), P(\iota_1)(v_3^1)$, then inductively construct $\iota_1, \ldots, \iota_k, \iota_{k+1}$ where $\mathrm{dom}((\iota_{k+1}) = f_1^{(0)}$, then we have $\iota_{k+1} = \iota_1$.

Definition 2.10. Given G, X and P as above satisfying (A1)-(A4), we call the triple (G, X, P) a *G-equivariant transport* for $(M, \mathcal{T})$.

The basic idea which we are trying to capture with (A1)-(A4) is that there is a procedure which allows one, when given a labelling of the vertices of a base 3-simplex, to continue this labelling onto vertices of other 3-simplices along combinatorial paths in the dual 1-skeleton (and which also satisfies some nice properties). This procedure is made precise in the following definition.

Definition 2.11. Let $(M, \mathcal{T})$ be an oriented ideally triangulated 3-manifold and (G, X, P) a G-equivariant transport for $(M, \mathcal{T})$. Let σ be some base simplex in $\mathcal{T}^{(3)}$, $f \subset \sigma$ and $\iota : f^{(0)} \hookrightarrow X$ an injection. Consider a combinatorial path α in the dual 1-skeleton $\sigma_1, \ldots, \sigma_k$, where $\sigma_1 = \sigma$, with connecting face-pairings $\varphi_1 : g_1 \to f_2, \varphi_2 : g_2 \to f_3, \ldots, \varphi_{k-1} : g_{k-1} \to f_k$. Let $f_i = [v_0^i, v_1^i, v_2^i]$ and $g_i = [w_0^i, w_1^i, w_2^i]$ where $\varphi_i : w_j^i \mapsto v_j^{i+1}$. Set $\iota_1 = \iota$, $\kappa_1 = P(\iota_1)$ and then inductively define $\kappa_j = P(\iota_j)$ where $\iota_j : f_j^{(0)} \hookrightarrow X : v_l^i \mapsto \kappa_{j-1}(w_l^{i-1})$; κ_k is then termed the *transported labelling along* α and denoted $\kappa(\iota, \alpha)$.

Proposition 2.12. *Let $(M, \mathcal{T})$ be an oriented ideally triangulated 3-manifold and (G, X, P) a G-equivariant transport for $(M, \mathcal{T})$. Let σ be some base simplex in $\mathcal{T}^{(3)}$, $f \subset \sigma$ and $\iota : f^{(0)} \hookrightarrow X$ an injection, then:*
(i) *for each combinatorial path α in the dual 1-skeleton with initial simplex σ and each face $g \subset \sigma$, $\kappa(\iota, \alpha) = \kappa(P(\iota)|_g, \alpha)$*
(ii) *if α and β are combinatorially path homotopic, then $\kappa(\iota, \alpha) = \kappa(\iota, \beta)$*
(iii) *if α and β can be concatenated, say at σ', $\kappa(\iota, \alpha\beta) = \kappa(\kappa(\iota, \alpha)|_g, \beta)$ for any $g \subset \sigma'$.*

Proof. (i) By (A2), P applied to $P(\iota)|_g$ returns $P(\iota)$.

(ii) We need to show that transports of labellings along spurs are trivial. Let $\sigma_1^{(0)}$ have the labelling κ where we may assume that $P(\kappa|_{g'}) = \kappa$ for some, and so all, $g' \subset \sigma_1$ and consider the spur σ_1, σ_2 connected via the face-pairings $\varphi_1 : g \to f$, $\varphi_2 : f \to g$, $g = [v_0, v_1, v_2], f = [w_0, w_1, w_2]$ and $\varphi_1 : v_i \mapsto w_i$, $\varphi_2 : w_i \mapsto v_i$. We set $\kappa_1 = \kappa$, $\kappa_2 = P(\iota_2)$ where $\iota_2 : f^{(0)} \hookrightarrow X : w_i \mapsto \kappa(v_i)$ and then $\kappa_3 = P(\iota_3)$ where $\iota_3 : g^{(0)} \hookrightarrow X : v_i \mapsto \kappa_2(w_i)$. It is clear that $\iota_3 = \kappa|_g$ and so using (A2), $\kappa_3 = P(\kappa|_g) = \kappa_1$.

(iii) This follows using (A2) once more to conclude that P applied to $\kappa(\iota,\alpha)|_g$ returns $\kappa(\iota,\alpha)$. □

Given an oriented ideally triangulated 3-manifold $(M,\mathcal{T})$ with dual 1-skeleton $(\mathcal{G},\mathcal{E}_{\mathcal{T}})$, $f\subset\sigma$ and $\iota : f^{(0)}\to X$, if we transport a labelling from a base simplex $\sigma\in\mathcal{T}^{(3)}$ along a combinatorial loop $g_1 \xrightarrow{\varphi_1} f_2, g_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} f_k, g_k \xrightarrow{\varphi_k} f_1$ in $\mathcal{G}$, we have two labellings κ_1 and κ_{k+1} of $\sigma^{(0)}$. Choosing any $g\subset\sigma$, by simple 3-transitivity of the action of G on X, we have a unique $\psi\in G$ such that $\psi\circ\kappa_1|_g = \kappa_{k+1}|_g$. Using (A2), $P(\kappa_1|_g) = \kappa_1$ as $P(\kappa_1|_f) = \kappa_1$ by definition and $P(\kappa_{k+1}|_g) = \kappa_{k+1}$ as $P(\kappa_{k+1}|_{f_1}) = \kappa_{k+1}$ by construction and so the stronger equality $\psi\circ\kappa_1 = \psi\circ P(\kappa_1|_g) = P(\psi\circ\kappa_1|_g) = P(\kappa_{k+1}|_g) = \kappa_{k+1}$ holds due to (A3) so that the extended labellings of all four vertices of σ are related via ψ; as a result, ψ is independent of g. As a result of Proposition 2.12 (ii), transports may be defined along elements of $\pi_1^{\mathrm{comb}}(\mathcal{G},\sigma)$ so that we have a map $\pi_1^{\mathrm{comb}}(\mathcal{G},\sigma)\to G$.

Proposition 2.13. *Suppose given an oriented ideally triangulated 3-manifold $(M,\mathcal{T})$, a G-equivariant transport (G,X,P), a base simplex $\sigma\in\mathcal{T}^{(3)}$, $f\subset\sigma$ and $\iota : f^{(0)}\to X$. Let Ψ be the map in Proposition 1.28. The map $\pi_1^{comb}(\mathcal{G},\sigma)\to G$ defined above is a homomorphism and descends to a representation $\rho : \pi_1^{comb}(\mathcal{G},\sigma)/\ker(\Psi)\to G$.*

Proof. Let α and β be two combinatorial loops based at σ, say $\sigma_1,\dots,\sigma_k$, where $\sigma_1=\sigma$, with connecting face-pairings $g_1 \xrightarrow{\varphi_1} f_2, g_2 \xrightarrow{\varphi_2} f_3, g_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{k-1}} f_k, g_k \xrightarrow{\varphi_k} f_1$ and $\sigma'_1,\dots,\sigma'_l$, where $\sigma'_1=\sigma$, with connecting face-pairings $g'_1 \xrightarrow{\varphi'_1} f'_2, g'_2 \xrightarrow{\varphi'_2} f'_3, g'_3 \xrightarrow{\varphi'_3} \cdots \xrightarrow{\varphi'_{k-1}} f'_l, g'_l \xrightarrow{\varphi'_k} f'_1$, respectively. Let $[\alpha]$ and $[\beta]$ have images $\psi_1,\psi_2\in G$ respectively. By Proposition 2.12 (iii), $\kappa(\iota,\alpha\beta) = \kappa(\kappa(\iota,\alpha)|_f,\beta)$. By definition, $\kappa(\iota,\alpha) = \psi_1\circ P(\iota)$ so that $\kappa(\iota,\alpha)|_f = \psi_1\circ\iota$. In computing $\kappa(\iota,\beta)$ define $\kappa_1,\dots,\kappa_{l+1},\iota_1,\dots,\iota_{l+1}$, where $\kappa_j = P(\iota_j)$, as in Definition 2.11 and similarly, in computing $\kappa(\psi_1\circ\iota,\beta)$, we have labellings $\kappa'_1,\dots,\kappa'_{k+1},\iota'_1,\dots,\iota'_{k+1}$, where $\kappa'_j = P(\iota'_j)$. By definition, $\iota'_1 = \psi_1\circ\iota_1$ so that $\kappa'_1 = P(\iota'_1) = P(\psi_1\circ\iota_1) = \psi_1\circ P(\iota_1) = \psi_1\circ\kappa_1$. Then inductively $\iota'_j = \psi_1\circ\iota_j$ and $\kappa'_j = \psi_1\circ\kappa_j$ for all j and in particular, $\kappa_{(l+1)'} = \psi_1\circ\kappa_{l+1} = \psi_1\circ(\psi_2\circ P(\iota)) = (\psi_1\circ\psi_2)\circ P(i)$; thus $[\alpha][\beta]$ has image $\psi_1\circ\psi_2$ and our map is a group homomorphism.

Finally combining Proposition 2.12 and (A4), which says precisely that transports along edge cycles are trivial, we see that our map kills $\ker(\Psi)$ and so descends to a representation $\rho : \pi_1^{\mathrm{comb}}(\mathcal{G},\sigma)/\ker(\Psi)\to G$. □

Proposition 2.14. *Given an oriented ideally triangulated 3-manifold $(M,\mathcal{T})$, a G-equivariant transport (G,X,P), a base simplex $\sigma\in\mathcal{T}^{(3)}$ and an injection $\iota : f^{(0)}\to X$ for some $f\subset\sigma$, let the associated representation be ρ. If ρ' is the representation constructed by replacing ι with ι', then $\rho'(\cdot) = \psi\rho(\cdot)\psi^{-1}$ where ψ is defined by $\psi\circ\iota = \iota'$.*

Proof. Let α be a combinatorial loop based at σ, say $\sigma_1,\dots,\sigma_k$, where $\sigma_1=\sigma$, with connecting face-pairings $g_1 \xrightarrow{\varphi_1} f_2, g_2 \xrightarrow{\varphi_2} f_3, g_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{k-1}} f_k, g_k \xrightarrow{\varphi_k} f_1$. Let $\rho([\alpha]) = \psi_1$ and $\rho'([\alpha]) = \psi_2$. In computing $\kappa(\iota,\alpha)$ define $\kappa_1,\dots,\kappa_{k+1},\iota_1,\dots,\iota_{k+1}$, where $\kappa_j = P(\iota_j)$, as in Definition 2.11 and similarly, in computing $\kappa(\psi\circ\iota,\alpha)$, remembering $\psi\circ\iota=\iota'$, we have labellings $\kappa'_1,\dots,\kappa'_{l+1},\iota'_1,\dots,\iota'_{l+1}$, where $\kappa'_j = P(\iota'_j)$. By definition, $\iota'_1 = \psi\circ\iota_1$ so that $\kappa'_1 = P(\iota'_1) = P(\psi\circ\iota_1) = \psi\circ P(\iota_1) = \psi\circ\kappa_1$. Then inductively $\iota'_j = \psi\circ\iota_j$ and $\kappa'_j = \psi\circ\kappa_j$ for all j and in particular, $\kappa_{(k+1)'} = \psi\circ\kappa_{k+1} = \psi\circ(\psi_1\circ P(\iota)) = (\psi\psi_1\psi^{-1})\circ(\psi\circ P(i)) = (\psi\psi_1\psi^{-1})\circ(P(\psi\circ\iota)) = (\psi\psi_1\psi^{-1})\circ P(\iota')$; thus $\psi_2 = \psi\psi_1\psi^{-1}$. □

As such, each (G, X, P) gives a conjugacy class of representations $\pi_1(M, x) \to G$ for a fixed x, the barycentre of the base simplex σ.

In practice, these representations can be computed, using the presentation in Proposition 1.29, as follows.

Proposition 2.15. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, a G-equivariant transport (G, X, P), a base simplex $\sigma \in \mathcal{T}^{(3)}$ and an injection $\iota : f^{(0)} \to X$ for some $f \subset \sigma$, let the associated representation be ρ; let also T be a maximal tree for the dual 1-skeleton $(\mathcal{G}, \mathcal{E}_{\mathcal{T}})$. Given any $\sigma' \in \mathcal{T}^{(3)}$, let $p_{\sigma'}$ be the unique reduced path in $\mathcal{G}$ from σ to σ'; this allows us to define the labelling $I = \sqcup_{\sigma'} \kappa(\iota, p_{\sigma'}) : \mathcal{T}^{(0)} \hookrightarrow X$. For each edge $e \notin T^{(1)}$, there is a face-pairing $\varphi_e : v_0, v_1, v_2 \mapsto w_0, w_1, w_2$ to which there corresponds $\psi_e \in G$ defined by $\psi_e : I(v_i) \mapsto I(w_i)$; the image of ρ is generated by these ψ_e.*

Proof. For each $e \notin T^{(1)}$, corresponding to the face-pairing φ_e between say σ_1, σ_2, let $a_e = p_{\sigma_1} \cdot e \cdot p_{\sigma_2}^{-1}$; we know from Proposition 1.20 that the $[a_e]$ generate $\pi_1^{\text{comb}}(\mathcal{G}, \sigma)$. We claim that $\rho : [a_e] \mapsto \psi_e^{-1}$. By Proposition 2.12, $\kappa(\iota, a_e) = \kappa(\kappa(\iota, p_{\sigma_1})|_f, e \cdot p_{\sigma_2}^{-1}) = \kappa(I|_{g_1}, e \cdot p_{\sigma_2}^{-1})$ where $g_1 = \text{dom}(\varphi_e)$. Let $f_2 = \text{codom}(\varphi_e)$ and $p_{\sigma_2}^{-1}$ be given by $g_2 \xrightarrow{\varphi_2} f_3, g_3 \xrightarrow{\varphi_3} f_4, g_4 \xrightarrow{\varphi_4} \cdots \xrightarrow{\varphi_{k-1}} f_k$ where $f_j, g_j \subset \sigma_j$, $\sigma_k = \sigma$. In computing $\kappa(I|_f, e \cdot p_{\sigma_2}^{-1})$ define $\kappa_1, \ldots, \kappa_k, \iota_1, \ldots, \iota_k$, where $\kappa_j = P(\iota_j)$, as in Definition 2.11. By definition, $\iota_1 = I|_{g_1}$, $\kappa_1 = P(I|_{g_1}) = I|_{\sigma_1}$, $\iota_2 = \psi_e^{-1} \circ I|_{f_2}$, $\kappa_2 = P(\psi_e^{-1} \circ I|_{f_2}) = \psi_e^{-1} \circ P(I|_{f_2}) = \psi_e^{-1} \circ I|_{\sigma_2}$ and then inductively, $\iota_j = \psi_e^{-1} \circ I|_{f_j}$ and $\kappa_j = \psi_e^{-1} \circ I|_{\sigma_j}$ for $j > 1$. In particular, $\kappa_k = \psi_e^{-1} \circ I|_\sigma$ and this completes the proof. □

Remark 2.16. Proposition 2.15 can also be used to show that the representations here constructed via transports are special cases of those constructed via labellings of face-pairings in the previous section. To see this, in the notation of the proposition, if one labels those face-pairings in T, in both directions, with the identity and those $e \notin T^{(1)}$ with the element ψ_e^{-1} in the direction traversed by a_e, we see that the representation constructed in Proposition 2.15 is precisely that constructed in Proposition 2.1.

Proposition 2.17. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ and a G-equivariant transport (G, X, P), if $\mathcal{T}$ is singular, then any associated representation ρ is non-trivial.*

Note that, because altering the initial data conjugates ρ, the condition that ρ be non-trivial is independent of the initial data.

Proof. Denote by $\mathcal{T}'$ the non-ideal triangulation associated to $\mathcal{T}$ where the 0-skeleton is included. Recall that $\mathcal{T}$ is said to be singular exactly when $\mathcal{T}'$ is singular. We blur the distinction between $\mathcal{T}$ and $\mathcal{T}'$ by calling normal triangles in $\mathcal{T}$ vertices, saying that two normal triangles are identified when the corresponding vertices in $\mathcal{T}'$ are identified in the corresponding pseudo 3-manifold and saying that a 3-simplex of $\mathcal{T}$ is not imbedded in M if either it is not imbedded in the usual sense or if two of its normal triangles are identified. Thus, as $\mathcal{T}$ is singular, there is some 3-simplex, say σ_{i_1}, which is not imbedded in M. Since points (including the normal triangles, which we are thinking of as vertices) are identified if and only if they can be connected via a sequence of face-pairings, there must be a combinatorial loop, say

$$\alpha : (\sigma_{i_1}, g_1) \xrightarrow{\varphi_1} (\sigma_{i_2}, f_2), (\sigma_{i_2}, g_2) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{k-1}} (\sigma_{i_k}, f_k), (\sigma_{i_k}, g_k) \xrightarrow{\varphi_k} (\sigma_{i_1}, f_1)$$

based at σ_{i_1} and a point $x \in g_1$ which is mapped to a distinct point in the boundary of σ_{i_1} under this sequence of face-pairings; that is, $\varphi_k(\varphi_{k-1}(\cdots\varphi_1(x))\cdots)$ is defined and not equal to x. It cannot be that x is in the interior of g_1 for then $\varphi_1(x)$ would lie in the interior of f_2 and our sequence of face-pairings would then be of the form $\varphi_1, \varphi_1^{-1}, \varphi_1, \varphi_1^{-1}, \ldots$. Thus x lies in some edge $e = [v, w]$ of σ_{i_1}. If x is one of v, w, it will be a vertex of σ_{i_1} which is identified to another vertex of this same 3-simplex. On the other hand, if x lies in the interior of e, the composition $\varphi_k \circ \cdots \circ \varphi_1$ would be defined on e and could not fix v and w as otherwise it would have to be the identity; thus we see that in this case, v and w are identified. As such, together with our sequence of face-pairings above, there must exist a sequence of vertices $v_j \in \sigma_{i_j}^{(0)}$ such that $v_1 \overset{\varphi_1}{\mapsto} \cdots \overset{\varphi_{k-1}}{\mapsto} v_k \overset{\varphi_k}{\mapsto} v_1'$ where $v_1' \in \sigma_{i_1}^{(0)}$ and $v_1' \neq v_1$. Let ρ be constructed with an initial labelling $\iota : f_1^{(0)} \to X$. Then by definition, $\kappa(\iota, \alpha)(v_1') = P(\iota)(v_1)$ so that $\rho([\alpha]) : P(\iota)(v_1') \mapsto P(\iota)(v_1)$ and so $\rho([\alpha]) \neq 1$. □

2.3. Symmetric representation

In this section, we introduce our first example of a G-equivariant transport, which one might term the *symmetric transport*.

Let $(M, \mathcal{T})$ be an oriented ideally triangulated 3-manifold such that $\mathcal{T}$ is even. Let $G = \mathrm{Sym}(4)$, $X = \{0, 1, 2, 3\}$ with the natural action of G on X and given an injection $\iota : f^{(0)} \to X$ for an $f \subset \sigma$, let $P(\iota)(v)$, where v is the unique element of $\sigma^{(0)} - f^{(0)}$, be the unique element of $X - \mathrm{im}(\iota)$. We have then satisfied (A1) and (A2) is easily verified. Further, (A3) follows from that if $\psi \in \mathrm{Sym}(4)$ and $\mathrm{im}(\iota) = \{i, j, k\}$, then the unique element of $X - \{\psi(i), \psi(j), \psi(k)\}$ is precisely the image under ψ of the unique element of $X - \{i, j, k\}$. Finally (A4) follows from the observation that, in the notation of the statement of (A4), if $X = \{i, j, k, l\}$, $\iota_1 : v_0^1, v_1^1, v_2^1 \mapsto i, j, k$, then $\iota_2 : v_0^2, v_1^2, v_2^2 \mapsto i, j, l$, $\iota_3 : v_0^3, v_1^3, v_2^3 \mapsto i, j, k$, and so on; see the figure below, which depicts the situation in the non-singular case.

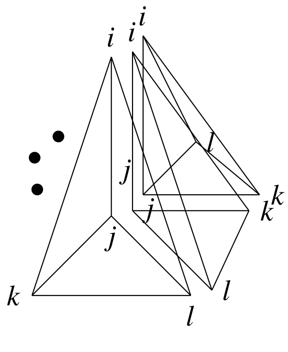


Figure 3. Transporting across a non-singular edge cycle

As such we may construct representations $\rho_{\mathrm{sym}} : \pi_1(M) \to \mathrm{Sym}(4)$. Appropriating Proposition 2.17 to this special case, we have:

Corollary 2.18. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ such that $\mathcal{T}$ is even, if $\mathcal{T}$ is singular, then any symmetric representation ρ_{sym} is non-trivial; in particular, $\pi_1(M) \neq 1$.*

Remark 2.19. While the existence of a face-pairing whose domain and codomain are contained in a single 3-simplex implies that the triangulation in question is singular, singularity of a triangulation does not imply the existence of such a self-identification (or more generally, of any loop of odd length in the dual 1-skeleton); as examples, we have our triangulations of the figure-eight knot complement and quaternionic space in Examples 1.12 and 1.33. Thus, Corollary 2.18 is a strictly stronger result than Proposition 2.5.

Example 2.20. Consider the triangulation of the figure-eight knot complement in Example 1.12. Label the normal triangles t_i with i, choose the face-pairing φ_1 as a maximal tree in the dual 1-skeleton (the dual 1-skeleton is the same as that for the triangulation of S^3/Q_8 in Example 1.33) and then construct a labelling of all normal triangles as in Proposition 2.15. See Figure 4.

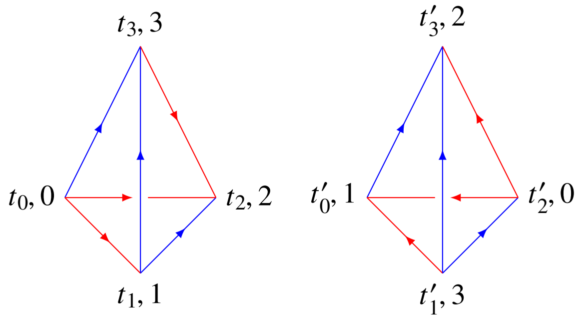


FIGURE 4. Computing the symmetric representation for the figure-eight knot complement

Now we find that the image of the symmetric representation for this triangulation of the figure-eight knot complement is generated by the following permutations:

$$\begin{aligned}
\varphi_1 &: t_0, t_1, t_2 \mapsto t'_2, t'_0, t'_3 \leftrightarrow 0,1,2 \mapsto 0,1,2 \rightsquigarrow 1\\
\varphi_2 &: t_0, t_1, t_3 \mapsto t'_1, t'_0, t'_3 \leftrightarrow 0,1,3 \mapsto 3,1,2 \rightsquigarrow (0\,3\,2)\\
\varphi_3 &: t_0, t_2, t_3 \mapsto t'_1, t'_0, t'_2 \leftrightarrow 0,2,3 \mapsto 3,1,0 \rightsquigarrow (0\,3)(1\,2)\\
\varphi_4 &: t_1, t_2, t_3 \mapsto t'_1, t'_3, t'_2 \leftrightarrow 1,2,3 \mapsto 3,2,0 \rightsquigarrow (0\,1\,3).
\end{aligned}$$

Thus the image lies in Alt(4). It can be checked that $(0\,3)(1\,2)$ and $(0\,3\,1) = (0\,1\,3)^2$ generate this alternating group so that the image is precisely Alt(4).

Example 2.21. Consider the triangulation of quaternionic space in Example 1.33. Label the vertices v_i with i, choose the face-pairing φ_1 as a maximal tree in the dual 1-skeleton and then construct a labelling of all vertices as in Proposition 2.15. See Figure 5.
Now we find that the image of the symmetric representation for this triangulation of the figure-eight knot complement is generated by the following permutations:

$$\begin{aligned}
\varphi_1 &: v_0, v_1, v_2 \mapsto v'_3, v'_0, v'_1 \leftrightarrow 0,1,2 \mapsto 0,1,2 \rightsquigarrow 1\\
\varphi_2 &: v_0, v_1, v_3 \mapsto v'_1, v'_2, v'_0 \leftrightarrow 0,1,3 \mapsto 2,3,1 \rightsquigarrow (0\,2)(1\,3)\\
\varphi_3 &: v_0, v_2, v_3 \mapsto v'_2, v'_0, v'_3 \leftrightarrow 0,2,3 \mapsto 3,1,0 \rightsquigarrow (0\,3)(1\,2)\\
\varphi_4 &: v_1, v_2, v_3 \mapsto v'_3, v'_2, v'_1 \leftrightarrow 1,2,3 \mapsto 0,3,2 \rightsquigarrow (0\,1)(2\,3).
\end{aligned}$$

Thus, the image in this case is the Klein-4 group V_4.

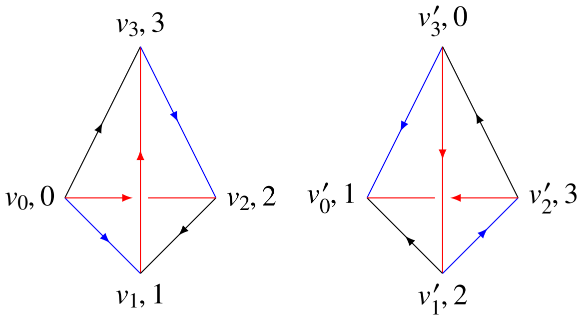


FIGURE 5. Computing the symmetric representation for S^3/Q_8

One might be tempted now to define other similar constructions. One possibility is that, given an ideally triangulated 3-manifold $(M, \mathcal{T})$, one labels the edges of a fixed simplex $\sigma \in \mathcal{T}^{(3)}$ such that opposite edges receive the same label and then transports this labelling via the same "reflection" mechanism as in the definition of the symmetric transport, see Figure 6.

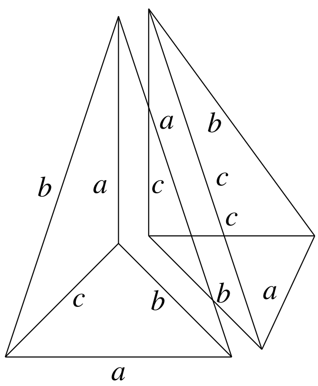


FIGURE 6. Transporting edge labels

It is clear again that, in the case of (and only in the case of) even $\mathcal{T}$, this defines a transport which is equivariant with respect to the obvious (identifying a, b, c with $0, 1, 2$ respectively) action of Sym(3) on $\{a, b, c\}$. As such we get (conjugacy classes of) representations into Sym(3), denoted by ρ_{edges}.

These representations however are simply post-compositions of the previous symmetric representations with the map $\alpha : \mathrm{Sym}(4) \twoheadrightarrow \mathrm{Sym}(4)/V_4 \xrightarrow{\sim} \mathrm{Sym}(3)$ where the latter map takes a coset ψV_4 to the element-wise action of ψ on the partitions $\{\{i, j\}, \{k, l\}\}$. To see this, let $a = \{\{0, 1\}, \{2, 3\}\}$, $b = \{\{0, 2\}, \{l, 3\}\}$, $c = \{\{0, 3\}, \{1, 2\}\}$, label the vertices of the initial simplex used in constructing ρ_{edges} with $0, 1, 2, 3$ such that opposite edges give the partitions indicated by their labels. Then if, under ρ_{sym}, the image of some combinatorial loop is ψ, the image under ρ_{edges} will be $\alpha(\psi)$.

2.4. Pseudo-developing maps and holonomy representations

In this section, we give an alternative viewpoint on the representations constructed via transports in Section 2.2.

Proposition 2.22. *Suppose given an oriented ideally triangulated 3-manifold* $(M, \mathcal{T})$ *and a lift* $\widetilde{\mathcal{T}}$ *of* $\mathcal{T}$. *If* $\mathcal{T}$ *supports a* G*-equivariant transport* (G, X, P), *then* $\widetilde{\mathcal{T}}$ *is non-singular.*

Proof. Recall the notation from Section 1.3. Suppose that $v_j^{(i_0,\gamma_0)}$ and $v_k^{(i_0,\gamma_0)}$ are identified under $\widetilde{\pi}$. Consider the edge cycle about $\widetilde{e} = [v_j^{(i_0,\gamma_0)}, v_k^{(i_0,\gamma_0)}]$, say with simplices $\widetilde{\sigma}_{(i_1,\gamma_1)}, \ldots, \widetilde{\sigma}_{(i_k,\gamma_k)}$, where $i_1 = i_0$. From the proof of Proposition 1.40, the vertices of the $\widetilde{\sigma}_{(i_j,\gamma_j)}$ can be labelled $u_l^{(i_j,\gamma_j)}$ so that $\sigma_{(i_j,\gamma_j)} = [u_0^{(i_j,\gamma_j)}, u_1^{(i_j,\gamma_j)}, u_2^{(i_j,\gamma_j)}, u_3^{(i_j,\gamma_j)}]$, $\widetilde{f}_j = [u_0^{(i_j,\gamma_j)}, u_1^{(i_j,\gamma_j)}, u_2^{(i_j,\gamma_j)}]$, $\widetilde{g}_j = [u_0^{(i_j,\gamma_j)}, u_1^{(i_j,\gamma_j)}, u_3^{(i_j,\gamma_j)}]$, $\widetilde{e}_j = [u_0^{(i_j,\gamma_j)}, u_1^{(i_j,\gamma_j)}]$ and each face-pairing $\widetilde{g}_j \to \widetilde{f}_{j+1}$ maps $u_0^{(i_j,\gamma_j)}, u_1^{(i_j,\gamma_j)}, u_3^{(i_j,\gamma_j)}$ to $u_0^{(i_{j+1},\gamma_{j+1})}$, $u_1^{(i_{j+1},\gamma_{j+1})}$, $u_2^{(i_{j+1},\gamma_{j+1})}$ respectively. Let $u_l^{(i_j,\gamma_j)} = v_{l'}^{(i_j,\gamma_j)}$. Now it cannot be that $\widetilde{\sigma}_{(i_j,\gamma_j)} \neq \widetilde{\sigma}_{(i_1,\gamma_1)}$ for $j > 1$ as if this was the case, $v_j^{(i_0,\gamma_0)}$ and $v_k^{(i_0,\gamma_0)}$ would not be identified under $\widetilde{\pi}$. Let then $1 < j_0 < k$ be such that $\widetilde{\sigma}_{(i_{j_0},\gamma_{j_0})} = \widetilde{\sigma}_{(i_1,\gamma_1)}$; we must have $\widetilde{e}_{j_0} \neq \widetilde{e}_1$. Consider

$$q = q_{(\widetilde{\sigma}_{(i_1,\gamma_1)},\widetilde{g}_1)} \cdot q^{-1}_{(\widetilde{\sigma}_{(i_2,\gamma_2)},\widetilde{f}_2)} \cdot q_{(\widetilde{\sigma}_{(i_2,\gamma_2)},\widetilde{g}_2)} \cdot q^{-1}_{(\widetilde{\sigma}_{(i_3,\gamma_3)},\widetilde{f}_3)} \cdot q_{(\widetilde{\sigma}_{(i_3,\gamma_3)},\widetilde{g}_3)} \cdots q^{-1}_{(\widetilde{\sigma}_{(i_{j_0},\gamma_{j_0})},\widetilde{f}_{j_0})}$$

and also the corresponding loop $p \circ q : I \to M$ which we know is homotopically trivial as $[p \circ q] = p_*([q])$ where $p_* : \pi_1(\widetilde{M}, \widetilde{x}) \to \pi_1(M, x)$ is the induced map; here $\widetilde{x}$ is the barycentre of $\widetilde{\sigma}_{(i_1,\gamma_1)}$ and x is the barycentre of σ_{i_1}. Now,

$$\begin{aligned} p \circ q &= (p \circ q_{(\widetilde{\sigma}_{(i_1,\gamma_1)},\widetilde{g}_1)}) \cdot (p \circ q^{-1}_{(\widetilde{\sigma}_{(i_2,\gamma_2)},\widetilde{f}_2)}) \cdot (p \circ q_{(\widetilde{\sigma}_{(i_2,\gamma_2)},\widetilde{g}_2)}) \\ &\quad \cdot (p \circ q^{-1}_{(\widetilde{\sigma}_{(i_3,\gamma_3)},\widetilde{f}_3)}) \cdot (p \circ q_{(\widetilde{\sigma}_{(i_3,\gamma_3)},\widetilde{g}_3)}) \cdots (p \circ q^{-1}_{(\widetilde{\sigma}_{(i_{j_0},\gamma_{j_0})},\widetilde{f}_{j_0})}) \\ &= q_{(\sigma_{i_1},g_1)} \cdot q^{-1}_{(\sigma_{i_2},f_2)} \cdot q_{(\sigma_{i_2},g_2)} \cdot q^{-1}_{(\sigma_{i_3},f_3)} \cdot q_{(\sigma_{i_3},g_3)} \cdots q^{-1}_{(\sigma_{i_{j_0}},f_{j_0})} \end{aligned}$$

where $\sigma_{i_j} = [v_{0'}^{i_j}, v_{1'}^{i_j}, v_{2'}^{i_j}, v_{3'}^{i_j}]$, $f_j = [v_{0'}^{i_j}, v_{1'}^{i_j}, v_{2'}^{i_j}]$, $g_j = [v_{0'}^{i_j}, v_{1'}^{i_j}, v_{3'}^{i_j}]$ and we also set $e_j = [v_{0'}^{i_j}, v_{1'}^{i_j}]$; here the function $i \to i'$ changes with j and is that involved in $u_l^{(i_j,\gamma_j)} = v_{l'}^{(i_j,\gamma_j)}$. Note that, for $1 \leq j < j_0$, there are face-pairings $g_j \to f_{j+1}$ in Φ mapping $v_{0'}^{i_j}, v_{1'}^{i_j}, v_{3'}^{i_j}$ to $v_{0'}^{i_{j+1}}, v_{1'}^{i_{j+1}}, v_{2'}^{i_{j+1}}$ respectively. Note that $\sigma_{i_{j_0}} = \sigma_{i_1}$, so that we have a loop in the dual 1-skeleton but $e_{j_0} \neq e_1$. We claim that this loop has non-trivial image, denoted by ψ, in G. Let $\iota : f_1^{(0)} \hookrightarrow X : v_{0'}^{i_1}, v_{1'}^{i_1}, v_{2'}^{i_1} \mapsto x_0, x_1, x_2$ be some initial labelling. Transporting this labelling along the combinatorial loop corresponding to $p \circ q$ up to $\sigma_{i_{j_0-1}}$, we see that $v_{0'}^{i_{j_0-1}}, v_{1'}^{i_{j_0-1}}, v_{3'}^{i_{j_0-1}}$ receive the labels x_0, x_1 and x_2', to be determined, respectively. If $v_{0'}^{i_{j_0}}$ is any of $v_{1'}^{i_1}, v_{2'}^{i_1}$ or $v_{3'}^{i_1}$, $\psi \neq 1$ as then $\psi^{-1}(x_0) \neq x_0$; so assume that $v_{0'}^{i_{j_0}} = v_{0'}^{i_1}$. It cannot then be that $v_{1'}^{i_{j_0}} = v_{1'}^{i_1}$ since $e_{j_0} \neq e_1$ so that $v_{1'}^{i_{j_0}}$ is one of $v_{2'}^{i_1}, v_{3'}^{i_1}$ and so once again $\psi \neq 1$ as $\psi^{-1}(x_1) \neq x_1$. □

Recall the map Π from Section 1.3.

Definition 2.23. Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ and a G-equivariant transport (G, X, P) for $\mathcal{T}$, we can lift this to a G-equivariant transport $(G, X, \widetilde{P})$ for $\widetilde{\mathcal{T}}$. The definition of $\widetilde{P} : \{\text{injections } \widetilde{f}^{(0)} \hookrightarrow X\} \to \{\text{injections } \widetilde{\sigma}^{(0)} \hookrightarrow X\}$ is as follows. Given $\widetilde{f}^{(0)} = [\widetilde{u}, \widetilde{v}, \widetilde{w}]$, and $\iota : \widetilde{f}^{(0)} \hookrightarrow X$, define $\Pi(\iota) : \Pi(\widetilde{f})^{(0)} \hookrightarrow X$ by $\Pi(\widetilde{u}), \Pi(\widetilde{v}), \Pi(\widetilde{w}) \mapsto \iota(\widetilde{u}), \iota(\widetilde{v}), \iota(\widetilde{w})$ and then set $\widetilde{P}(\iota) = P(\Pi(\iota)) \circ \Pi$.

The ideal behind the definition of $\widetilde{\Pi}$ is that at any one simplex in the cover, one applies the same procedure as they would at the corresponding simplex in the original manifold. It can be checked that $(G, X, \widetilde{P})$ satisfies (A1)-(A4).

Definition 2.24. Suppose given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ and a G-equivariant transport (G, X, P). A *pseudo-developing map* is a map $D : \widetilde{\mathcal{T}}^{(0)} \to X$ such that $D|_{\widetilde{\sigma}}$ is an injection for each $\widetilde{\sigma} \in \widetilde{\mathcal{T}}^{(3)}$, $D(v) = D(w)$ when $\widetilde{\pi}(v) = \widetilde{\pi}(w)$ and for each $\widetilde{f} \subset \widetilde{\sigma}$, $\widetilde{P}(D|_{\widetilde{f}}) = D|_{\widetilde{\sigma}}$.

Proposition 2.25. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ and (G, X, P), there exists a pseudo-developing map. If D and D' are both pseudo-developing maps, $D' = \psi \circ D$ for a unique $\psi \in G$.*

Proof. Let $\widetilde{\sigma} \in \widetilde{\mathcal{T}}^{(3)}$ be a base 3-simplex, $\widetilde{f} \subset \widetilde{\sigma}$ and ι be an injection $\widetilde{f}^{(0)} \hookrightarrow X$. Given any other simplex $\widetilde{\sigma}' \in \widetilde{\mathcal{T}}^{(3)}$, there exists a combinatorial path α from $\widetilde{\sigma}$ to $\widetilde{\sigma}'$ and we may define $D|_{\widetilde{\sigma}'} = \kappa(\iota, \alpha)$. We claim that if α is a loop, $\kappa(\iota, \alpha) = \widetilde{P}(\iota)$; that is, the transported labelling on $\widetilde{\sigma}^{(0)}$ coincides with the initial labelling $\widetilde{P}(\iota)$. Due to Proposition 1.41, this follows from Proposition 2.12 and (A4). Now, if α and β are combinatorial paths from $\widetilde{\sigma}$ to $\widetilde{\sigma}'$, then $\kappa(\iota, \alpha) = \kappa(\iota, \alpha\beta^{-1}\beta)$ as $[\alpha] = [\alpha\beta^{-1}\beta]$ and $\kappa(\iota, \alpha\beta^{-1}\beta) = \kappa(\kappa(\iota, \alpha\beta^{-1})|_{\widetilde{f}}, \beta) = \kappa(\widetilde{P}(\iota)|_{\widetilde{f}}, \beta) = \kappa(\iota, \beta)$. Thus $D|_{\widetilde{\sigma}'}$ is uniquely determined by σ' and as such we have defined a map $D : \widetilde{\mathcal{T}}^{(0)} \to X$. That each $D|_{\widetilde{\sigma}}$ is an injection is clear by construction, that $D(v) = D(w)$ when $\widetilde{\pi}(v) = \widetilde{\pi}(w)$ is clear when $w = \widetilde{\varphi}(v)$ for some $\varphi \in \widetilde{\Phi}$ and this implies the general result and finally that $\widetilde{P}(D|_{\widetilde{f}}) = D|_{\widetilde{\sigma}}$ for each $\widetilde{f} \subset \widetilde{\sigma}$ also follows immediately from the construction.

Suppose now that D and D' are both pseudo-developing maps. Given any $\widetilde{\sigma} \in \widetilde{\mathcal{T}}^{(3)}$ (not necessarily the base 3-simplex above), and any face $\widetilde{f} \subset \widetilde{\sigma}$, there exists a unique $\psi \in G$ such that $\psi \circ D|_{\widetilde{f}} = D'|_{\widetilde{f}}$. Further, for this ψ, we have $\psi \circ D|_{\widetilde{\sigma}} = \psi \circ \widetilde{P}(D|_{\widetilde{f}}) = \widetilde{P}(\psi \circ D|_{\widetilde{f}}) = \widetilde{P}(D'|_{\widetilde{f}}) = D'|_{\widetilde{\sigma}}$ so that ψ relates the extended labellings on $\widetilde{\sigma}^{(0)}$ and so is independent of $\widetilde{f}$. The same construction associates a $\psi' \in G$ to any other $\widetilde{\sigma}' \in \widetilde{\mathcal{T}}^{(3)}$. We claim that $\psi = \psi'$. Because the dual 1-skeleton $(\widetilde{\mathcal{G}}, \mathcal{E}_{\widetilde{\mathcal{T}}})$ is connected, it suffices to prove this in the case that there exists a face-pairing between $\widetilde{\sigma}$ and $\widetilde{\sigma}'$, say $\widetilde{\varphi} : \widetilde{f} \to \widetilde{g}$ where $\widetilde{f} = [\widetilde{v}_0, \widetilde{v}_1, \widetilde{v}_2]$, $g = [\widetilde{w}_0, \widetilde{w}_1, \widetilde{w}_2]$ and $\widetilde{\varphi}(\widetilde{v}_i) = \widetilde{w}_i$. Then $D(\widetilde{v}_i) = D(\widetilde{w}_i)$ and $D'(\widetilde{v}_i) = D'(\widetilde{w}_i)$ so that by simple 3-transitivity, $\psi = \psi'$. This completes the proof. $\square$

The representations arising from transports may now be given an alternative characterisation, showing that they are generalisations of the holonomy representations associated to geometric structures.

Proposition 2.26. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, a G-equivariant transport (G, X, P) and $\gamma \in Aut(\widetilde{M})$, if D is a pseudo-developing map, so is $D \circ \gamma_{comb}$ where $\gamma_{comb} : \widetilde{\mathcal{T}}^{(0)} \to \widetilde{\mathcal{T}}^{(0)} : v_j^{(i,\gamma')} \mapsto v_j^{(i,\gamma\gamma')}$. Thus there is a unique $\psi \in G$ such that $D \circ \gamma_{comb} = \psi \circ D$ and this map $\rho_D : Aut(\widetilde{M}) \to G : \gamma \mapsto \psi$ is a homomorphism. If $D' = \psi' \circ D$ is another pseudo-developing map, then $\rho_{D'}(\cdot) = \psi' \rho_D(\cdot)(\psi')^{-1}$.*

Proof. It is easy to check that, for each $\gamma \in \mathrm{Aut}(\widetilde{M})$, γ_{comb} is a bijection and from this it follows that $(D \circ \gamma_{\mathrm{comb}})|_{\widetilde{\sigma}}$ is an injection for any $\widetilde{\sigma} \in \widetilde{\mathcal{T}}^{(3)}$. Given $\widetilde{v}, \widetilde{w} \in \widetilde{\mathcal{T}}^{(0)}$, if there exists $\widetilde{\varphi} \in \widetilde{\Phi}$ such that $\widetilde{w} = \widetilde{\varphi}(\widetilde{v})$, then, from the third property of $\widetilde{\mathcal{T}}$ in

Proposition 1.35, there also exists $\widetilde{\varphi}' \in \widetilde{\Phi}$ such that $\gamma_{\text{comb}}\widetilde{w} = \widetilde{\varphi}(\gamma_{\text{comb}}\widetilde{v})$ so that $(D \circ \gamma_{\text{comb}})(v) = D(\gamma_{\text{comb}}\widetilde{v}) = D(\gamma_{\text{comb}}\widetilde{w}) = (D \circ \gamma_{\text{comb}})(w)$. Finally, given $\widetilde{f} \subset \widetilde{\sigma}$, $\widetilde{P}((D \circ \gamma_{\text{comb}})|_{\widetilde{f}}) = \widetilde{P}(D|_{\gamma_{\text{comb}}\widetilde{f}}) = D|_{\gamma_{\text{comb}}\widetilde{\sigma}} = (D \circ \gamma_{\text{comb}})|_{\widetilde{\sigma}}$.

Now, suppose that $\rho_D : \gamma_1, \gamma_2 \mapsto \psi_1, \psi_2$. Then $D \circ (\gamma_1 \circ \gamma_2)_{\text{comb}} = (D \circ (\gamma_1)_{\text{comb}}) \circ (\gamma_2)_{\text{comb}} = (\psi_1 \circ D) \circ (\gamma_2)_{\text{comb}} = \psi_1 \circ (D \circ (\gamma_2)_{\text{comb}}) = \psi_1 \circ (\psi_2 \circ D) = (\psi_1 \circ \psi_2) \circ D$ so that $\rho_D(\gamma_1 \circ \gamma_2) = \rho_D(\gamma_1) \circ \rho_D(\gamma_2)$ and ρ_D is a homomorphism.

Finally, given that $\rho_D : \gamma \mapsto \psi$, we have that $D \circ \gamma_{\text{comb}} = \psi \circ D$ so that $D' \circ \gamma_{\text{comb}} = \psi' \circ (D \circ \gamma_{\text{comb}}) = (\psi'\psi(\psi')^{-1}) \circ (\psi' \circ D) = (\psi'\psi(\psi')^{-1}) \circ D'$ so that $\rho_{D'}(\gamma) = \psi'\rho_D(\gamma)(\psi')^{-1}$. □

Thus we see that (G, X, P) gives a conjugacy class of representations $\text{Aut}(\widetilde{M}) \to G$.

The following describes the connection with the earlier construction of representations via transports.

Proposition 2.27. *Suppose given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, a G-equivariant transport (G, X, P), let Ψ be the map in Proposition 1.28 and let Θ be the equivalence $\pi_1^{comb}(\mathcal{G}, \sigma)/\ker(\Psi) \to \pi_1(M, x) \to Aut(\widetilde{M})$ where x is the barycentre of $\sigma_j \in \mathcal{T}^{(3)}$ and the latter map is the standard isomorphism with the barycentre, say $\widetilde{x}$, of any fixed $\widetilde{\sigma}_{(j,\gamma)}$ as the chosen lift of x. Let ρ be the representation constructed as in Proposition 2.13 with initial labelling $\iota : f^{(0)} \to X$ where $f \subset \sigma_j$ and let ρ_D be the representation constructed in Proposition 2.26 where D is constructed from the initial labelling $\widetilde{\iota} = \iota \circ \Pi|_{\widetilde{f}} : \widetilde{f}^{(0)} \to X$ where $\widetilde{f} \subset \widetilde{\sigma}_{(j,\gamma)}$ and $\Pi(\widetilde{f}) = f$; then $\rho = \rho_D \circ \Theta$.*

Proof. Let α be a combinatorial loop class representative based at σ_j, say $\sigma_{i_1}, \dots, \sigma_{i_k}$, where $i_1 = j$, with connecting face-pairings $g_1 \xrightarrow{\varphi_1} f_2, g_2 \xrightarrow{\varphi_2} f_3, g_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{k-1}} f_k, g_k \xrightarrow{\varphi_k} f_1$. This loop lifts to a combinatorial loop, $\widetilde{\alpha}$, in the dual 1-skeleton of the lifted triangulation, $(\widetilde{\mathcal{G}}, \mathcal{E}_{\widetilde{\mathcal{T}}})$, between simplices $\widetilde{\sigma}_{(i_1,\gamma_1)}, \dots, \widetilde{\sigma}_{(i_{k+1},\gamma_{k+1})}$, where $\gamma_1 = \gamma, i_{k+1} = i_1 = j$, and connecting face-pairings $\widetilde{g}_1 \xrightarrow{\widetilde{\varphi}_1} \widetilde{f}_2, \widetilde{g}_2 \xrightarrow{\widetilde{\varphi}_2} \widetilde{f}_3, \widetilde{g}_3 \xrightarrow{\widetilde{\varphi}_3} \cdots \xrightarrow{\widetilde{\varphi}_{k-1}} \widetilde{f}_k, \widetilde{g}_k \xrightarrow{\widetilde{\varphi}_k} \widetilde{f}_{k+1}$ where $\Pi(\widetilde{f}_j) = f_j$, $\Pi(\widetilde{f}_{k+1}) = f_1$ and $\Pi(\widetilde{g}_j) = g_j$. Then Θ maps $[\alpha]$ (taken modulo $\ker(\Psi)$) to $\gamma^{-1}\gamma_{k+1}$ and $\rho_D(\gamma^{-1}\gamma_{k+1})$ is defined by $\rho_D(\gamma^{-1}\gamma_{k+1}) : P(\widetilde{\iota})(v_l^{(i_1,\gamma_1)}) \mapsto \kappa(\widetilde{\iota}, \widetilde{\alpha})(v_l^{(i_1,\gamma_{k+1})})$. In computing $\kappa(\iota, \alpha)$ define $\kappa_1, \dots, \kappa_{k+1}, \iota_1, \dots, \iota_{k+1}$, where $\kappa_j = P(\iota_j)$, as in Definition 2.11 and similarly, in computing $\kappa(\widetilde{\iota}, \widetilde{\alpha})$, we have labellings $\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_{k+1}, \widetilde{\iota}_1, \dots, \widetilde{\iota}_{k+1}$, where $\widetilde{\kappa}_j = \widetilde{P}(\widetilde{\iota}_j)$. By definition, $\widetilde{\iota}_1 = \widetilde{\iota} = \iota \circ \Pi|_{\widetilde{f}} = \iota_1 \circ \Pi|_{\widetilde{f}}$, $\widetilde{\kappa}_1 = \widetilde{P}(\iota_1 \circ \Pi|_{\widetilde{f}}) = P(\iota_1) \circ \Pi|_{\widetilde{\sigma}_{(i_1,\gamma_1)}} = \kappa_1 \circ \Pi|_{\widetilde{\sigma}_{(i_1,\gamma_1)}}$ where the second to last equality comes from the definition of $\widetilde{P}$. The, inductively, for $j > 1$, $\widetilde{\iota}_j = \iota_j \circ \Pi|_{\widetilde{f}_j}$ and $\widetilde{\kappa}_j = \kappa_j \circ \Pi|_{\widetilde{\sigma}_{(i_j,\gamma_j)}}$. In particular, $\kappa(\widetilde{\iota}, \widetilde{\alpha}) = \widetilde{\kappa}_{k+1} = \kappa_{k+1} \circ \Pi|_{\widetilde{\sigma}_{(i_k,\gamma_k)}} = \kappa(\iota, \alpha) \circ \Pi|_{\widetilde{\sigma}_{(i_k,\gamma_k)}}$. Further, by definition of $\widetilde{P}$, $P(\widetilde{\iota}) = P(\iota) \circ \Pi|_{\widetilde{\sigma}_{(i_1,\gamma_1)}}$. As such, $\kappa(\widetilde{\iota}, \widetilde{\alpha})(v_l^{(i_k,\gamma_k)}) = \kappa(\iota, \alpha)(v_l^{i_k})$ and $P(\widetilde{\iota})(v_l^{(i_k,\gamma_k)}) = P(\iota)(v_l^{i_k})$. Thus $\rho_D(\gamma^{-1}\gamma_{k+1}) \circ P(\iota) = \kappa(\iota, \alpha)$ and so $\rho_D(\gamma^{-1}\gamma_{k+1}) = \rho([\alpha])$. □

CHAPTER 3

Hyperbolic gluing equations over commutative rings

In this chapter we introduce the hyperbolic gluing equations and associated Thurston labellings which help us to construct a second example of equivariant transports. In this and the next chapter, every ring R is assumed to be non-zero, unital and commutative unless specified otherwise. Further, for each such R, $R^\times$ is used to denote the group of units of R.

3.1. Representations into $\mathbf{PGL}_2(R)$

Definition 3.1. Given a ring R, define the *projective line over R* to be

$$\mathbb{P}^1(R) = \frac{\{A \in R^2 \mid \text{if } A = (a,b)^t, Ra + Rb = R\}}{\sim}$$

where $A \sim B$ if and only if $B = \lambda A$ for some $\lambda \in R^\times$.

Remark 3.2. Equivalence up to multiplication by a unit is also an equivalence relation on the entire R^2 and also on R, giving equivalence up to associates in the latter case. In all three cases, the equivalence relation will be denoted by $\sim$ and equivalence classes will be indicated by enclosure inside square parentheses. Note that $\mathbb{P}^1(R) \subseteq R^2/\sim$.

Now, let $G = \mathrm{PGL}_2(R)$ for some unspecified ring R and let $X = \mathbb{P}^1(R)$. Here G does not necessarily act simply 3-transitively, but nevertheless G is transitive on a restricted subset of X^3 which we will describe below.

Definition 3.3. Given a ring R, on $R^2 \times R^2$, define the skew-symmetric bilinear form $\langle , \rangle$ by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = ad - bc.$$

Proposition 3.4. *Given $A, B \in \mathbb{P}^1(R)$, $\langle A, B\rangle$ is well-defined as an element of $R/\sim$, denoted $[A, B]$.*

Proof. Fix representatives for A and B, denoted by these same symbols and further take two other representatives, λA and μB for some $\lambda, \mu \in R^\times$. Then $\langle \lambda A, \mu B\rangle = \lambda\mu\langle A, B\rangle$ using bilinearity of $\langle , \rangle$. □

Example 3.5. If R is a field and $A, B \in \mathbb{P}^1(R)$, $[A, B] = [0] \in R/\sim$ if and only if $A = B$.

Definition 3.6. Given a collection of points $A_1, \ldots, A_n \in R^2$, say that this collection is *admissible* if and only if $\langle A_i, A_j\rangle \in R^\times$ for $i \neq j$; similarly if $A_1, \ldots, A_n \in \mathbb{P}^1(R)$, say that these form an *admissible* collection if and only if $[A_i, A_j] = [1]$ for all $i \neq j$.

Proposition 3.7. *We have the following:*
(i) *if R is a field, $PGL_2(R)$ is simply 3-transitive on $\mathbb{P}^1(R)$; if R is not a field, we retain existence and uniqueness in the weaker form below*
(ii) *given $A_i, B_i \in \mathbb{P}^1(R)$ for $i = 0, 1, 2$ such that $\{A_i\}$ and $\{B_i\}$ are admissible, there exists a unique $X \in PGL_2(R)$ such that $X : A_i \mapsto B_i$*

Proof. (i) This is implied by (ii) and Example 3.5.

(ii) Let $A_i, B_i \in \mathbb{P}^1(R)$ for $i = 0, 1, 2$. Suppose there exists an $X \in \mathrm{PGL}_2(R)$ such that $X : A_i \mapsto B_i$. Say $(a, b)^t, (c, d)^t, (e, f)^t, (g, h)^t$ are representatives for A_0, A_1, B_0, B_1 respectively and set

$$Y = \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad \text{and} \quad Z = \frac{1}{eh - fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix}$$

which are elements of $\mathrm{GL}_2(R)$ with determinants $\frac{1}{ad-bc}$ and $\frac{1}{eh-fg}$ respectively. Then $Y(a, b)^t = Z(e, f)^t = (1, 0)^t$, $Y(c, d)^t = Z(g, h)^t = (0, 1)^t$. Choose some representatives for A_3, B_3 and then with these representaties let $Y(\text{rep. of } A_3) = (s, t)^t$, $Z(\text{rep. of } B_3) = (u, v)^t$. Further, let

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

be a representative for X. Then

$$Z\begin{pmatrix} m & n \\ p & q \end{pmatrix}Y^{-1}$$

will take $Y(a, b)^t = (1, 0)^t$ to a unit multiple of $Z(e, f)^t = (1, 0)^t$ and $Y(c, d)^t = (0, 1)^t$ to a unit multiple of $Z(g, h)^t = (0, 1)^t$. This implies that

$$Z\begin{pmatrix} m & n \\ p & q \end{pmatrix}Y^{-1} = \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix}$$

for some units i and j of R. Further this matrix takes $(s, t)^t$ to a unit multiple of $(u, v)^t$. That is, $(is, jt)^t = (\mu u, \mu v)^t$ for some $\mu \in R^\times$. For our representatives $\langle A_i, A_j\rangle$ is a unit for $i \neq j$ (these i and j not being related to the matrix entries above) and further, using the previous proposition, for the same representatives (so as to be able to multiply by Y), $\langle YA_i, YA_j\rangle$ are units for $i \neq j$. Then upon computation, using the representatives chosen, we find that s, t are units. Thus, we must have $i = \mu s^{-1}u$ and $j = \mu t^{-1}v$ and X is determined up to multiplication by units. As such, if such an X exists, it is unique. Conversely, retaining all notation, define X to be the image of $Z^{-1}\mathrm{diag}(s^{-1}u, t^{-1}v)Y$ in $\mathrm{PGL}_2(R)$. Using that $\{B_i\}$ is admissible, we find that u, v are units and so that $X : A_i \mapsto B_i$ for $i = 0, 1$. Further chasing the definitions and directly calculating reveals that this holds also for $i = 2$. □

Next, to define P, we need to introduce Thurston labellings and cross-ratios.

Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, denote by $\square$ the collection of all normal quads of all 3-simplices in $\mathcal{T}^{(3)}$ and by $\triangle$ the collection of all normal triangles of all 3-simplices in $\mathcal{T}^{(3)}$. Further, for each simplex of $\mathcal{T}^{(3)}$, order its normal quads as outlined in § 1.1.

Definition 3.8. Let $(M, \mathcal{T})$ be an oriented ideally triangulated 3-manifold. A *Thurston labelling* of $\mathcal{T}$ is a function $x : \square \to R$ for some ring R, such that the following holds:

- given $\sigma \in \mathcal{T}^{(3)}$, if the normal quads of σ are q, q', q'', $q \to q' \to q''$ and $x(q) = r$, $x(q') = r'$, $x(q'') = r''$, then

  $$r'(1 - r) = r''(1 - r') = r(1 - r'') = 1$$

  noting that these equations are invariant under cyclic permutations of r, r', r''

- given any edge cycle comprising edges $e_1, \ldots, e_k \in \mathcal{T}^{(1)}$, if q_j corresponds to e_j and $x(q_j) = r_j$, then

$$r_1 \cdots r_k = 1. \tag{$\star\star$}$$

Remark 3.9. If we did not use the notion of normal quad types, we could say that Thurston labellings are labellings of edges in $\mathcal{T}^{(1)}$ such that opposite edges within each 3-simplex receive the same label and the above two conditions hold for the resulting three labels within each 3-simplex.

Definition 3.10. The equations $r'(1-r) = r''(1-r') = r(1-r'') = 1$ are termed the *parameter relations* and the equations $r_1 \cdots r_k = 1$ the *gluing consistency equations*.

Note that the parameter relations enforce that the labels must come from $R \setminus \{0, 1\}$ and that for all labels r, r and $1 - r$ must be units.

Remark 3.11. The motivation for the definition of Thurston labellings is as follows. Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, to place a hyperbolic structure on M one can try to use ideal hyperbolic tetrahedra for the ideal 3-simplices. In [**27**], Thurston gave a parametrisation of oriented ideal hyperbolic tetrahedra up to oriented congruence which labelled the edges of ideal hyperbolic tetrahedra with elements $z \in \mathbb{C}$, called the *shape parameters*, such that opposite edges received the same label and the parameter equations above are satisfied. Using ideal hyperbolic tetrahedra together with the gluings in $\mathcal{T}$ gives a hyperbolic structure on $M - M^{(1)}$. Again in [**27**], the oriented ideal hyperbolic tetrahedra with arbitrary shape parameters are then shown to "fit" around edges in M, extending the hyperbolic structure to the 1-skeleton of M, if and only if the gluing consistency equations above are satisfied.

Example 3.12. Consider the triangulation of S^3/Q_8 in Example 1.33. We search for Thurston labellings by labelling edges by arbitrary elements of some ring R as depicted in Figure 1.

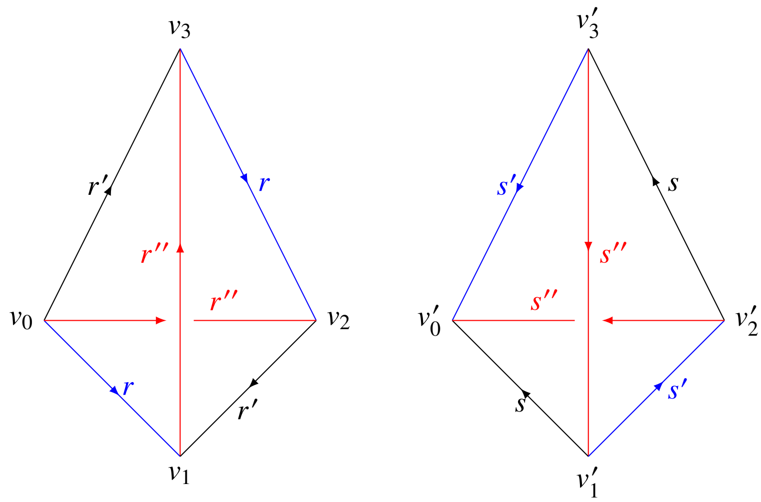


FIGURE 1. Computing Thurston labellings for S^3/Q_8

The gluing equations are then

$$r^2(s')^2 = 1 \qquad (r')^2 s^2 = 1 \qquad (r'')^2(s'')^2 = 1.$$

Combining these with the parameter relations, one finds that the parameter relations together with

$$r^2(s')^2 = 1 \qquad 2(rs' - 1) = 0$$

are necessary and sufficient conditions on the labels (to see this, re-write the gluing equations in terms of r and s' alone). Thus in the case of an R in which 2 is not a zero-divisor, for example $\mathbb{C}$, we have that $s' = r^{-1}$ and

$$(r, r', r'', s, s', s'') = \left(r, \frac{1}{1-r}, \frac{r-1}{r}, 1-r, \frac{1}{r}, \frac{r}{r-1}\right)$$

constitutes all possible Thurston labellings. There are however, other possible labellings. For example, it can be checked that setting $R = \mathbb{F}_4[x]/(x^2)$, where $\mathbb{F}_4 = \{0, 1, a, b\}$ is the field with four elements, and $r = a$, $s' = b + x$ gives a Thurston labelling, namely

$$(r, r', r'', s, s', s'') = (a, a, a, b + bx, b + x, b + ax).$$

Definition 3.13. Suppose given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$. A *homogeneous Thurston labelling* of $\mathcal{T}$ is a function $x : \square \to R$ for some ring R, such that the following holds:

- given $\sigma \in \mathcal{T}^{(3)}$, the sum of the labels of the quads of σ is zero
- given any edge cycle comprising edges $e_1, \ldots, e_k \in \mathcal{T}^{(1)}$, if q_j corresponds to e_j, $q_j \to q'_j$, $x(q_j) = r_j$ and $x(q'_j) = r'_j$, then

$$(\star\star\star) \qquad \prod_{i=1}^{k} r_i = \prod_{i=1}^{k} (-r'_i).$$

Remark 3.14. The justification of the adjective "homogeneous" is that, if one has a homogeneous Thurston labelling, within any given 3-simplex one can multiply each of the labels by some constant element of R, even allowing a different choice of element for different 3-simplices, and still satisfy the conditions required of a homogeneous Thurston labelling.

Proposition 3.15. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, we have the following relationship between Thurston and homogeneous Thurston labellings:*
(i) *suppose that $x : \square \to R$ is a homogeneous Thurston labelling such that im$(x) \subseteq R^\times$, then $y : \square \to R : q \mapsto -\frac{x(q)}{x(q')}$ where $q \to q'$ is a Thurston labelling*
(ii) *given a Thurston labelling y, there is a homogeneous Thurston labelling x such that im$(x) \subseteq R^\times$ and for each $q \in \square$, $y(q) = -\frac{x(q)}{x(q')}$ where again $q \to q'$.*

Proof. (i) The gluing consistency equations ($\star\star$) follow by division from the corresponding equations ($\star\star\star$) for homogeneous Thurston labellings. Let $q \in \square$ and q', q'' the remaining two quads of the same 3-simplex such that $q \to q' \to q''$. Then

$$y(q')(1 - y(q)) = -\frac{r(q')}{r(q'')}\left(1 + \frac{r(q)}{r(q')}\right) = -\frac{r(q')}{r(q'')}\left(\frac{r(q) + r(q')}{r(q')}\right) = 1$$

using the zero sum condition required of homogeneous Thurston labellings.

(ii) Define $x : \square \to R$ as follows. Given any $\sigma \in \mathcal{T}^{(3)}$, choose some quad q_0 in σ, let the remaining two quads be q'_0 and q''_0 such that $q_0 \to q'_0 \to q''_0$ and then define $x(q_0) = y(q_0), x(q'_0) = -1, x(q''_0) = 1 - y(q_0)$. Note then that $-\frac{x(q_0)}{x(q'_0)} = y(q_0)$, $-\frac{x(q'_0)}{x(q''_0)} = (1 - y(q_0))^{-1} = y(q'_0)$ and $-\frac{x(q''_0)}{x(q_0)} = 1 - y(q_0)^{-1} = 1 - (1 - y(q''_0)) = y(q''_0)$.

As such, $\mathrm{im}(x) \subseteq R^\times$, the sum of the labels of the three quads of any given $\sigma \in \mathcal{T}^{(3)}$ is zero and for all $q \in \square$, $y(q) = -\frac{x(q)}{x(q')}$ where $q \to q'$. Finally, $(\star\star\star)$ follows from $(\star\star)$ and the relation $y(q) = -\frac{x(q)}{x(q')}$. $\square$

Remark 3.16. The method of inverting the induction of a Thurston labelling above is not unique; we could have altered the choice within one or more $\sigma \in \mathcal{T}^{(3)}$ as to which quad we call q_0.

Definition 3.17. Given $A_0, A_1, A_2, A_3 \in R^2$, define their *cross-ratio* to be

$$(A_0, A_1; A_2, A_3) = \begin{pmatrix} \langle A_0, A_3\rangle\langle A_1, A_2\rangle \\ \langle A_0, A_2\rangle\langle A_1, A_3\rangle \end{pmatrix}.$$

Example 3.18.

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} -ad \\ -bc \end{pmatrix}.$$

Proposition 3.19. *Given $A_0, A_1, A_2, A_3 \in \mathbb{P}^1(R)$, $(A_1, A_2; A_3, A_4)$ is well-defined as an element of $R^2/{\sim}$, denoted $[A_0, A_1; A_2, A_3]$.*

Proof. Fix representatives for the A_i, denoted by these same symbols and further take four other representatives, $\lambda_i A_i$ for $\lambda_i \in R^\times$. Then we find that

$$\begin{aligned}(\lambda_0 A_0, \lambda_1 A_1; \lambda_2 A_2, \lambda_3 A_3) &= \begin{pmatrix} \langle \lambda_0 A_0, \lambda_3 A_3\rangle\langle \lambda_1 A_1, \lambda_2 A_2\rangle \\ \langle \lambda_0 A_0, \lambda_2 A_2\rangle\langle \lambda_1 A_1, \lambda_3 A_3\rangle \end{pmatrix} \\ &= \lambda_0\lambda_1\lambda_2\lambda_3 \begin{pmatrix} \langle A_0, A_3\rangle\langle A_1, A_2\rangle \\ \langle A_0, A_2\rangle\langle A_1, A_3\rangle \end{pmatrix}.\end{aligned}$$

$\square$

Proposition 3.20. *Suppose that $A_0, A_1, A_2 \in \mathbb{P}^1(R)$ form an admissible collection and $C = [c, d]^t \in \mathbb{P}^1(R)$, then:*
(i) *there is a unique $A_3 \in \mathbb{P}^1(R)$ such that $[A_0, A_1; A_2, A_3] = C$; with this unique A_3, we have $[A_0, A_2; A_3, A_1] = [c - d, c]^t$ and $[A_0, A_3; A_1, A_2] = [d, d - c]^t$*
(ii) *the augmented collection A_0, A_1, A_2, A_3 is admissible if and only if $c, d, c - d \in R^\times$.*

Proof. (i) Assume such an A_3 exists and fix a representative for it and A_0, A_1, A_2. Define X in the same way that Y and Z were defined in the proof of Proposition 3.7 (ii) so that $XA_0 = (1, 0)^t$, $XA_1 = (0, 1)^t$ and note that also, by Proposition 3.19, $(A_0, A_1; A_2, A_3)$ and $(XA_0, XA_1; XA_2, XA_3)$ are equivalent up to multiplication by units. Let $XA_2 = (w, x)^t$ and $XA_3 = (y, z)^t$. Because $\{A_0, A_1, A_2\}$ is admissible, by a previous proposition, (XA_i, XA_j) are units for $0 \le i < j \le 2$, so that $w, x \in R^\times$. Now, equality of the cross-ratio with C implies that

$$\begin{pmatrix} -wz \\ -xy \end{pmatrix} = \lambda \begin{pmatrix} c \\ d \end{pmatrix}$$

for some $\lambda \in R^\times$, so that

$$XA_3 = \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} -x^{-1}d \\ -w^{-1}c \end{pmatrix}.$$

Applying X^{-1} shows that A_3 is determined up to multiplication by units. Further, as $C \in \mathbb{P}^1(R)$, there exists some $D = (e, f)^t \in R^2$ such that $\langle (d, c)^t, (e, f)^t\rangle \in R^\times$. Let $Y = \mathrm{diag}(-x^{-1}, -w^{-1}) \in \mathrm{GL}_2(R)$, then $\langle X^{-1}Y \cdot (d, c)^t, X^{-1}Y \cdot (e, f)^t\rangle \in R^\times$, showing that A_3 is uniquely determined as an element of $\mathbb{P}^1(R)$ using a previous proposition (alternatively, rather than defining Y, we could have just stated that $(-xf)(-x^{-1}d) + (-we)(w^{-1}c) = fd - ec \in R^\times$).

Conversely, if after having fixed representatives for A_0, A_1, A_2 and subsequently defining X and w, x as above, we define $A_3 = [X^{-1} \cdot (-x^{-1}d, -w^{-1}c)]^t$ then we find that with these representatives and $X^{-1} \cdot (-x^{-1}d, -w^{-1}c)$ for A_3,

$$(A_0, A_1; A_2, A_3) = \det(X)^{-1}((1,0)^t, (0,1)^t; (w,x)^t, (-x^{-1}d, -w^{-1}c)^t)$$

which is a unit multiple of (c, d). Finally, with $A_3 = [X^{-1} \cdot (-x^{-1}d, -w^{-1}c)]^t$, direct computation, using the calculation in an earlier example, reveals that $[XA_0, XA_2; XA_3, XA_1] = [d-c, -c]^t$ and $[XA_0, XA_3; XA_1, XA_2] = [-d, c-d]^t$, completing the proof of (i).

(ii) We see from a previous proposition that if we fix any representatives for A_1, A_2, A_3, A_4 and define X as in (i) above, then $\{A_i\}$ forms an admissible collection if and only if $\{XA_i\}$ forms an admissible collection. The latter statement is true if and only if $\langle XA_i, XA_4\rangle \in R^\times$ for $i = 1, 2, 3$, which upon explicit calculation using the constructions in (i) above gives the required result. □

Now we may define P. Let $(M, \mathcal{T})$ be an oriented ideally triangulated 3-manifold, suppose that $\mathcal{T}$ supports a Thurston labelling $y : \square \to R$ and let x be a homogeneous Thurston labelling, provided by Proposition 3.15 (ii), such that, for all $q \in \square$, $x(q) \in R^\times$ and $y(q) = -\frac{x(q)}{x(q')}$ where $q \to q'$. Given some $f \subset \sigma$ and $\iota : f^{(0)} \hookrightarrow \mathbb{P}^1(R)$, say $\sigma = [v_0, v_1, v_2, v_3]$ with orientation $v_i \to v_{i+1}$, $f = [v_0, v_1, v_2]$; $P(\iota)(v_3)$ is then defined to be the unique element of $\mathbb{P}^1(R)$, provided by Proposition 3.20 (recall that $\mathrm{im}(x) \subseteq R^\times$), such that $[\iota(v_0), \iota(v_1); \iota(v_2), P(\iota)(v_3)] = [x(q), -x(q')]^t$ where $q = \{\{v_0, v_1\}, \{v_2, v_3\}\}$. We need to show that P is well-defined.

Proposition 3.21. *We have the following:*
(i) $(A_0, A_1; A_2, A_3) = (A_2, A_3; A_0, A_1) = (A_1, A_0; A_3, A_2)$ *for all* $A_i \in R^2$
(ii) *if* $(A_0, A_1; A_2, A_3) = (a, b)^t$*, then* $(A_1, A_0; A_2, A_3) = (A_0, A_1; A_3, A_2) = (b, a)^t$.

Proof. Both of these are immediate from the definition of cross-ratios. □

Remark 3.22. The statement in Proposition 3.21(i) may be stated alternatively as that if $\psi \in V_4$, where $V_4 = \{1, (01)(23), (02)(13), (03)(12)\} \leq \mathrm{Sym}(4)$, then for all $A_i \in R^2$, $(A_0, A_1; A_2, A_3) = (A_{\psi(0)}, A_{\psi(1)}; A_{\psi(2)}, A_{\psi(3)})$. Further, combining (i) and (ii) in Proposition 3.21, there are then at most $|\mathrm{Sym}(4)|/|V_4| = 6$ different cross-ratios of a given quadruple of points in the generic scenario.

Now, given some $f \subset \sigma$ and $\iota : f^{(0)} \hookrightarrow \mathbb{P}^1(R)$, say $\sigma = [v_0, v_1, v_2, v_3]$ with orientation $v_i \to v_{i+1}$, $f = [v_0, v_1, v_2]$, let $\iota(v_i) = A_i$ and let A_3 be the unique element of $\mathbb{P}^1(R)$ such that $[A_0, A_1; A_2, A_3] = [x(q), -x(q')]^t$ where $q = \{\{v_0, v_1\}, \{v_2, v_3\}\}$. If then $\sigma = [w_0, w_1, w_2, w_3]$, has orientation $w_i \to w_{i+1}$ and $f = [w_0, w_1, w_2]$, it must be that $w_3 = v_3$ and (w_0, w_1, w_2) is one of (v_0, v_1, v_2), (v_1, v_2, v_0) and (v_2, v_0, v_1). Using Propositions 3.20 and 3.21, $[A_1, A_2; A_0, A_3] = [A_0, A_3; A_1, A_2] = [-x(q'), -x(q') - x(q)]^t = [x(q'), -x(q'')]^t$ and $[A_2, A_0; A_1, A_3] = [A_0, A_2; A_3, A_1] = [x(q) + x(q'), x(q)]^t = [x(q''), -x(q)]^t$. Noting that $q' = \{\{v_0, v_3\}, \{v_1, v_2\}\}$ and $q'' = \{\{v_0, v_2\}, \{v_1, v_3\}\}$, P is seen to be well-defined and this gives us (A1).

To see (A2), let $\sigma = [v_0, v_1, v_2, v_3]$, have orientation $v_i \to v_{i+1}$ and $\kappa : \sigma^{(0)} \to X : v_i \mapsto A_i$ such that $[A_0, A_1; A_2, A_3] = [x(q), -x(q')]^t$ where $q = \{\{v_0, v_1\}, \{v_2, v_3\}\}$. Using Propositions 3.20 and 3.21 again, we verify that $\kappa = P(\kappa|_f)$ successively for $f = [v_0, v_1, \hat{v}_2, v_3]$, $f = [v_0, \hat{v}_1, v_2, v_3]$, $f = [\hat{v}_0, v_1, v_2, v_3]$ via the calculations $[A_0, A_3; A_1, A_2] = [-x(q'), -x(q') - x(q)]^t = [x(q'), -x(q'')]^t$, $[A_0, A_2; A_3, A_1] = [x(q) + x(q'), x(q)]^t = [x(q''), -x(q)]^t$ and $[A_1, A_3; A_2, A_0] = [A_2, A_0; A_1, A_3] = [A_0, A_2; A_3, A_1] = [x(q''), -x(q)]^t$.

Next, (A3) is a simple consequence of the following proposition.

Proposition 3.23. *Given $A_0, A_1, A_2, A_3 \in \mathbb{P}^1(R)$ and an $X \in PGL_2(R)$ we have $[XA_0, XA_1; XA_2, XA_3] = [A_0, A_1; A_2, A_3]$.*

Proof. After choosing representatives for the A_i and X we have

$$
\begin{aligned}
(XA_0, XA_1; XA_2, XA_3) &= \begin{pmatrix} \langle XA_0, XA_3\rangle\langle XA_1, XA_2\rangle \\ \langle XA_0, XA_2\rangle\langle XA_1, XA_3\rangle \end{pmatrix} \\
&= (\det X)^2 \begin{pmatrix} \langle A_0, A_3\rangle\langle A_1, A_2\rangle \\ \langle A_0, A_2\rangle\langle A_1, A_3\rangle \end{pmatrix}.
\end{aligned}
$$

□

Finally, we need to verify (A4). Consider an edge cycle with simplices $\sigma_{i_1}, \ldots, \sigma_{i_k}$, faces $f_j, g_j \subset \sigma_{i_j}$, edges $e_j = f_j \cap g_j = [v_j, w_j]$. We know from the proof of Proposition 1.9 that we can label the vertices of σ_{i_j} as $v_0^j, v_1^j, v_2^j, v_3^j$ such that σ_{i_j} has orientation $v_i^j \to v_{i+1}^j$, $f_j = [v_0^j, v_1^j, v_2^j]$, $g_j = [v_0^j, v_1^j, v_3^j]$, $e_j = [v_0^j, v_1^j]$ and where each identification $g_j \to f_{j+1}$ maps v_0^j, v_1^j, v_3^j to $v_0^{j+1}, v_1^{j+1}, v_2^{j+1}$ respectively, taking the superscripts modulo k. Label the vertices of σ_{i_1} so that $v_i^1 \mapsto A_i$ such that $[A_i, A_j; A_k, A_l] = [x(q), -x(q')]^t$ where $q = \{\{v_i^1, v_j^1\}, \{v_k^1, v_l^1\}\}$. We can suppose that $A_0 = [0, 1]^t$, $A_1 = [1, 0]^t$, $A_2 = [1, 1]^t$ (for the more general case, we can apply an element of $\mathrm{PGL}_2(R)$ to reduce most of the proof to this case). Let $[x(q_j), -x(q_j')]^t = [c_j, d_j]^t$ where $q_j = \{\{v_0^j, v_1^j\}, \{v_2^j, v_3^j\}\}$. Using Example 3.18, we have that if we let $A_3 = [x, y]^t$, $[-x, -y]^t = [c_1, d_1]^t$ so that the uniquely determined definition of A_3 which we are after is $A_3 = [c_1, d_1]^t$. Similarly, if we now let the desired label of v_3^2 be $[x, y]^t$, we have $[-d_1 x, -c_1 y]^t = [c_2, d_2]^t$ so that the uniquely determined label which we are after is $[c_2 d_1^{-1}, d_2 c_1^{-1}]^t$. Inductively, we find that, upon transporting this labelling, $v_i^k \mapsto B_i$ where $B_0 = A_0$, $B_1 = A_1$ and

$$
B_3 = \begin{bmatrix} c_k d_{k-1}^{-1} c_{k-2} d_{k-3}^{-1} \cdots \\ d_k c_{k-1}^{-1} d_{k-2} c_{k-3}^{-1} \cdots \end{bmatrix}.
$$

As such, we find that the labels on the vertices of σ_{i_1} upon completion of the loop are $v_i^1 \mapsto A_i'$ where $A_0' = A_0$, $A_1' = A_1$, $A_2' = B_3$ and A_3' is uniquely determined from these three labels via $[A_0', A_1'; A_2', A_3'] = [x(q), -x(q')]^t$ where $q = \{\{v_0^1, v_1^1\}, \{v_2^1, v_3^1\}\}$. If $A_2' = A_2$, we have $A_3' = A_3$. We know that there exist $\lambda_j \in R^\times$ such that $c_j = \lambda_j x(q_j)$, $d_j = \lambda_j(-x(q_j'))$ and because we have a homogeneous Thurston labelling, we have

$$
\prod_{j=1}^{k} x(q_j) = \prod_{j=1}^{k} (-x(q_j')) \quad \text{which gives} \quad \prod_{j=1}^{k} c_j = \prod_{j=1}^{k} d_j
$$

and the latter equation implies that $c_k d_{k-1}^{-1} c_{k-2} d_{k-3}^{-1} \cdots = d_k c_{k-1}^{-1} d_{k-2} c_{k-3}^{-1} \cdots$. Thus $B_3 = [1, 1]^t = A_3$. This verifies (A4).

As such, we have constructed a representation $\pi_1(M) \to \mathrm{PGL}_2(R)$. Appropriating Proposition 2.17 to this special case, we have:

Corollary 3.24. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ supports a Thurston labelling over some R and is singular, then the associated representations $\rho : \pi_1(M) \to PGL_2(R)$ are non-trivial; in particular, $\pi_1(M) \neq 1$.*

As each Thurston labelling provides a conjugacy class of representations into $\mathrm{PGL}_2(R)$, the solution set of the equations defining such labellings parametrises such conjugacy classes of representations.

3.2. Existence of Thurston labellings

We consider now the question of what combinatorial conditions are required on triangulations to support Thurston labellings. In this line of thought, in [**14**], Feng Luo made the following conjecture.

Conjecture. If $M \neq S^3$ is a closed oriented 3-manifold, then there exists a 1-vertex triangulation $\mathcal{T}$ of M and a commutative ring R with identity so that $\mathcal{T}$ supports a Thurston labelling over R.

The original appearance of the Thurston labellings comes from $\mathbb{H}^3$. There the labels happened to be cross-ratios of points in $\mathbb{P}^1(\mathbb{C})$. As a first step, we wish to emulate some results from that situation in the case of a general commutative ring with identity and see how far we can go with cross-ratios alone in this general scenario.

Proposition 3.25. *Given $A_0, A_1, A_2, A_3 \in R^2$, we have $(A_0, A_1; A_2, A_3) + (A_0, A_2; A_3, A_1) + (A_0, A_3; A_1, A_2) = 0$.*

Proof. We have

$$\begin{aligned}(A_0, A_1; A_2, A_3) &+ (A_0, A_2; A_3, A_1) + (A_0, A_3; A_1, A_2)\\ &= (\langle A_0, A_3\rangle\langle A_1, A_2\rangle + \langle A_0, A_1\rangle\langle A_2, A_3\rangle + \langle A_0, A_2\rangle\langle A_3, A_1\rangle,\\ &\qquad \langle A_0, A_2\rangle\langle A_1, A_3\rangle + \langle A_0, A_3\rangle\langle A_2, A_1\rangle + \langle A_0, A_1\rangle\langle A_3, A_2\rangle)^t = (a, -a)^t\end{aligned}$$

where $a = \langle A_0, A_3\rangle\langle A_1, A_2\rangle + \langle A_0, A_1\rangle\langle A_2, A_3\rangle + \langle A_0, A_2\rangle\langle A_3, A_1\rangle = \langle\langle A_1, A_2\rangle A_0, A_3\rangle + \langle\langle A_0, A_1\rangle A_2, A_3\rangle + \langle\langle A_2, A_0\rangle A_1, A_3\rangle = \langle b, A_3\rangle$ where $b = \langle A_1, A_2\rangle A_0 + \langle A_0, A_1\rangle A_2 + \langle A_2, A_0\rangle A_1$ and this quantity, b, may be directly computed to be zero. $\square$

Proposition 3.26. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, let $f : \triangle \to R^2$ be a function such that $f(t) = f(t')$ when t, t' are paired by a face-pairing. Define $x : \square \to R$ as follows. Given $q \in \square$, say $q \subset \sigma$, let $q = \{\{t_0, t_1\}, \{t_2, t_3\}\}$ such that σ has orientation $t_i \to t_{i+1}$, let $(f(t_0), f(t_1); f(t_2), f(t_3)) = (a, b)^t$ and then set $x(q) = a$. Then x is well-defined and a homogeneous Thurston labelling.*

Proof. To see that x is well-defined, note that if the q in the proposition is also equal to $\{\{t'_0, t'_1\}, \{t'_2, t'_3\}\}$, where σ has orientation $t'_i \to t'_{i+1}$, then there must exist $\psi \in V_4$ such that $t'_i = t_{\psi(i)}$ for each i and then the result follows from Proposition 3.21 and Remark 3.22.

Now, let $\sigma \in \mathcal{T}^{(3)}$ have normal triangles t_0, t_1, t_2, t_3 and orientation $t_i \to t_{i+1}$. Then the orientation of σ is also given by $t_0 \to t_3 \to t_1 \to t_2$ and $t_0 \to t_2 \to t_3 \to t_1$. Thus if we set $q = \{\{t_0, t_1\}, \{t_2, t_3\}\}$, $q' = \{\{t_0, t_3\}, \{t_1, t_2\}\}$, $q'' = \{\{t_0, t_2\}, \{t_3, t_1\}\}$, we have that $x(q) + x(q') + x(q'')$ is the first coordinate of $(f(t_0), f(t_1); f(t_2), f(t_3)) + (f(t_0), f(t_2); f(t_3), f(t_1)) + f(t_0), f(t_3); f(t_1), f(t_2))$, which we know to be zero by Proposition 3.25.

Finally, let $e \in \mathcal{T}^{(1)}$ and let its edge cycle be given by the 3-simplices $\sigma_{i_1}, \ldots, \sigma_{i_k}$, faces $f_1, g_1, \ldots, f_k, g_k$ where $f_j, g_j \subset \sigma_{i_j}$, edges $e_j = f_j \cap g_j = [v_j, w_j]$. We know from the proof of Proposition 1.9 that we can label the normal triangles of σ_{i_j} as $t_0^j, t_1^j, t_2^j, t_3^j$ such that σ_{i_j} has orientation $t_i^j \to t_{i+1}^j$, $f_j = [t_0^j, t_1^j, t_2^j]$, $g_j = [t_0^j, t_1^j, t_3^j]$, $e_j = [t_0^j, t_1^j]$ and where each face-pairing $\varphi_j : g_j \to f_{j+1}$ maps t_0^j, t_1^j, t_3^j to $t_0^{j+1}, t_1^{j+1}, t_2^{j+1}$ respectively, taking the superscripts modulo k. We have that $f(t_0^j)$ and $f(t_1^j)$ are constant for varying j and so we may denote $A = f(t_a^j)$ and $B = f(t_b^j)$. Let $C_j = f(v_2^j)$. Then as $\varphi_j : t_3^j \mapsto t_2^{j+1}$, $f(v_3^j) = C_{j+1}$ where C_{k+1} is set to be C_1. Then if q_j is the quad corresponding to e_j and $q_j \to q_j'$, $x(q_j)$ is the first coordinate of $(A, B; C_j, C_{j+1})$, $x(q_j')$ is the first coordinate of $(A, C_{j+1}; B, C_j)$ and it is immediate that

$$\prod_{j=1}^{k} x(q_j) = \prod_{j=1}^{k} \langle A, C_{j+1}\rangle\langle B, C_j\rangle = \prod_{j=1}^{k} -\langle A, C_j\rangle\langle C_{j+1}, B\rangle = \prod_{j=1}^{k} -x(q_j').$$

□

As such, we see that homogeneous Thurston labellings may always be constructed via cross-ratios. However the construction in Proposition 3.26 will usually not provide Thurston labellings via Proposition 3.15 as if $\mathcal{T}$ is singular, the condition imposed on f in Proposition 3.26 will necessitate the existence of quads with the label zero. For example, in the case of a 1-vertex triangulation, this construction yields only the trivial homogeneous Thurston labelling where every quad is labelled with the zero element.

We now consider more general constructions.

Proposition 3.27. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, let $\square = \{q_1, q_1', q_1'', \ldots, q_n, q_n', q_n''\}$, $R = \mathbb{C}[q_1, q_1', q_1'', \ldots, q_n, q_n', q_n'']$ and*

$$I = \langle \{\textit{parameter relations}\} \cup \{\textit{gluing equations}\}\rangle \trianglelefteq R.$$

If $I \neq R$, $\mathcal{T}$ supports a Thurston labelling over some ring, namely R/I. Conversely, if $\mathcal{T}$ supports a Thurston labelling over some ring, $I \neq R$.

Proof. If $I \neq R$, $x : \square \to R/I : q_i, q_i', q_i'' \mapsto \overline{q_i}, \overline{q_i'}, \overline{q_i''}$ gives a Thurston labelling. Conversely, if x is a Thurston labelling over the ring S,

$$1, q_1, q_1', q_1'', \ldots, q_1, q_1', q_1'' \mapsto 1, x(q_1), x(q_1'), x(q_1''), \ldots, x(q_1), x(q_1'), x(q_1'')$$

defines a homomorphism $R \to S$ which kills I but not R. □

In the notation of Proposition 3.27, Corollary 3.24 may now instead be stated as follows.

Corollary 3.28. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ is singular and the associated ideal $I = \langle \{\textit{parameter relations}\} \cup \{\textit{gluing equations}\}\rangle$ in $\mathbb{C}[q_1, q_1', q_1'', \ldots, q_n, q_n', q_n'']$ is not the unit ideal, then $\pi_1(M) \neq 1$.*

Proposition 3.29. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ supports a Thurston labelling over some R, it supports a Thurston labelling over $\mathbb{C}$.*

Proof. By Proposition 3.27, we know that the ideal $I = \langle\{\text{parameter relations}\} \cup \{\text{gluing equations}\}\rangle$ in $\mathbb{C}[q_1, q_1', q_1'', \ldots, q_n, q_n', q_n'']$ is not the unit ideal. Thus we may find a maximal ideal $(q_1 - a_1, q_1' - a_1', q_1'' - a_1'', \ldots, q_n - a_n, q_n' - a_n', q_n'' - a_n'')$ containing I and then note that $x : \square \to \mathbb{C} : q_1, q_1', q_1'', \ldots, q_n, q_n', q_n'' \mapsto a_1, a_1', a_1'', \ldots, a_n, a_n', a_n''$ is a Thurston labelling over $\mathbb{C}$. □

Remark 3.30. Note that the construction, in particular the ring R and ideal I, analogous to that in Proposition 3.27 for homogeneous Thurston labellings is not useful as $(0, \ldots, 0)$ is always a homogeneous Thurston labelling. However, as what we really need are homogeneous Thurston labellings where each coordinate of each label is a unit, we could add in extra generators as inverse elements to ensure a more functional, but much larger, construction.

We consider now the simplest Thurston labellings, the constant labellings. The following proposition classifies all constant Thurston labellings. Note that $x(q) = r$ for all q gives a constant Thurston labelling if and only if $r(1 - r) = 1 \Leftrightarrow r^2 - r + 1 = 0$ and $r^\ell = 1$ for every edge cycle length ℓ.

Proposition 3.31. *Suppose given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ with edge cycle lengths $\ell_1, \ldots, \ell_m$ and let $d = \gcd(\ell_1, \ldots, \ell_m)$. Let also e be the (principal) residue of d modulo 6. The following table describes all possible constant Thurston labellings, with constant label $r \in R$.*

e	*Constant labellings*
0	*Any R, r satisfies* $r^2 - r + 1 = 0$
1	*None*
2	*R has characteristic 3,* $r = -1$
3	*R has characteristic 2, r satisfies* $r^2 - r + 1 = 0$
4	*R has characteristic 3,* $r = -1$
5	*None*

Proof. Given $R \neq 0$ and $r \in R$, labelling each and every quad with r gives a Thurston labelling if and only if $r^2 - r + 1 = 0$ and $r^d = 1$. As $(r+1)(r^2 - r + 1) = r^3 + 1$, this will be satisfied if and only if $r^2 - r + 1 = 0$ and $r^e = 1$ (as the former of these gives $r^3 = -1$ which gives $r^6 = 1$). If $e = 0$, then setting R to be any non-zero ring which contains an element r such that $r^2 - r + 1 = 0$ we have a Thurston labelling; for example $R = \mathbb{C}$ and $r = \omega$ where ω is one of the two non-trivial cube roots of -1 or $R = \mathbb{Z}_3$ and $r = 2$. If $e = 1$, we require $r = 1$ which contradicts that $R \neq 0$. In the case $e = 2$, note that $r^2 - r + 1 = 0, r^2 = 1$ imply that $r = 2$ and then $2^2 = 1$ implies that R has characteristic 3, and then that for such an R, $r = 2$ does indeed satisfy $r^2 - r + 1 = 0, r^2 = 1$. Similarly, the only possibilities in the case $e = 3$ are characteristic 2 rings R containing an element r satisfying $r^2 - r + 1 = 0$, for example either of the two elements $\neq 0, 1$ in $\mathbb{F}_{2^2}$, and in the case $e = 4$ are characteristic 3 rings and $r = 2$. Finally, similarly to the $e = 1$ case, there are no constant Thurston labellings in the case $e = 5$. □

In particular, for all even $\mathcal{T}$, that is, when $e = 0, 2, 4$, $\mathcal{T}$ supports the constant labelling $r = -1$ over $\mathbb{Z}_3$; this was noted in [**14**]. As a result of the computations in the proof of Proposition 3.31 above and Proposition 3.29, we have the following.

Corollary 3.32. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$ with edge cycle lenghts $\ell_1, \ldots, \ell_m$ and $d = gcd(\ell_1, \ldots, \ell_m)$, if $d \not\equiv 1, 5 \ (mod\, 6)$, which is true in particular for even $\mathcal{T}$, then $\mathcal{T}$ supports a Thurston labelling over $\mathbb{C}$.*

The next proposition shows how, under a certain condition, one can explicitly construct a non-constant Thurston labelling over $\mathbb{C}$ in the case of an even triangulation. Recall that $V_4 \leq \mathrm{Sym}(4)$ denotes the Klein-4 group $\{1, (01)(23), (02)(13), (03)(12)\}$ inside $\mathrm{Sym}(4)$.

Proposition 3.33. *Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ is even and has a symmetric representation ρ_{sym} such that $im(\rho_{sym}) \leq V_4$, then $\mathcal{T}$ supports a non-constant Thurston labelling over $\mathbb{C}$.*

Note that since V_4 is normal in $\mathrm{Sym}(4)$ and altering initial data conjugates ρ_{sym}, the condition $\mathrm{im}(\rho_{\mathrm{sym}}) \leq V_4$ is independent of initial data. Note also that this condition may be rephrased as that the related representations ρ_{edges} into $\mathrm{Sym}(3)$ constructed in Section 2.5 are trivial.

Proof. Let D be a pseudo-developing map associated with the given symmetric representation. Define $D' : \widetilde{\mathcal{T}}^{(0)} \to \mathbb{C}^2$ by $v_j^{(i,\gamma)} \mapsto A_{D(v_j^{(i,\gamma)})}$ where A_0, A_1, A_2, A_3 are an arbitrary but admissible (which amounts to being pairwise distinct) collection in $\mathbb{P}^1(\mathbb{C})$. Given any $\sigma_i = [v_0^i, v_1^i, v_2^i, v_3^i]$ in $\mathcal{T}^{(3)}$ with orientation $v_j^i \to v_{j+1}^i$, let $\widetilde{\sigma}_{(i,\gamma)} = [v_0^{(i,\gamma)}, v_1^{(i,\gamma)}, v_2^{(i,\gamma)}, v_3^{(i,\gamma)}]$ and $\widetilde{\sigma}_{(i,\gamma')} = [v_0^{(i,\gamma')}, v_1^{(i,\gamma')}, v_2^{(i,\gamma')}, v_3^{(i,\gamma')}]$ be two lifts in $\widetilde{\mathcal{T}}^{(3)}$ of σ_i. Let $D'(v_j^{(i,\gamma)}) = B_j$ and $D'(v_j^{(i,\gamma')}) = C_j$. Let $\psi = \rho_{\mathrm{sym}}(\gamma'\gamma^{-1})$; then $C_j = B_{\psi(j)}$ and so, by Remark 3.22, $(B_p, B_q; B_r, B_s) = (C_p, C_q; C_r, C_s)$. As such, we may define $x : \square \to \mathbb{C}$ as follows. Given a $q \subset \sigma_i$, say $\{\{v_p^i, v_q^i\}, \{v_r^i, v_s^i\}\}$ such that σ_i has orientation $v_p^i \to v_q^i \to v_r^i \to v_s^i$, let $\widetilde{\sigma}_{(i,\gamma)} = [v_p^{(i,\gamma)}, v_q^{(i,\gamma)}, v_r^{(i,\gamma)}, v_s^{(i,\gamma)}]$ be any lift of σ_i in $\widetilde{\mathcal{T}}^{(3)}$ and then define $x(q) = a$ where $\left(A_{D(v_p^{(i,\gamma)})}, A_{D(v_q^{(i,\gamma)})}; A_{D(v_r^{(i,\gamma)})}, A_{D(v_s^{(i,\gamma)})}\right) = (a, b)^t$. By reasoning very similar to that in the proof of Proposition 3.26, x is a homogeneous Thurston labelling. Because the A_j are admissible, $x(q)$ is a unit for all q and so $y(q) = -x(q)x(q')^{-1}$ gives a Thurston labelling as in Proposition 3.15. $\square$

Example 3.34. We found in Example 2.21 that the image of the symmetric representation for our triangulation of S^3/Q_8 is precisely V_4. In the notation of Proposition 3.33, set A_0, A_1, A_2, A_3 to be $(1, 0)^t, (0, 1)^t, (1, 1)^t, (z, 1)^t$ for some $z \neq 0, 1$. We take lifts of our simplices so as to form the fundamental domain given by the maximal tree in the dual 1-skeleton which we chose in Example 2.21. There is then also a lift of the face-pairing φ_1 between these two lifts and these two lifts, upon replacing i with A_i, have the following labellings.

Now, the quads $q_1 = \{\{v_0, v_1\}, \{v_2, v_3\}\}$, $q_1' = \{\{v_0, v_3\}, \{v_1, v_2\}\}$ and $q_1'' = \{\{v_0, v_2\}, \{v_1, v_3\}\}$ are ordered $q_1 \to q_1' \to q_1''$. We have

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -z \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} z \\ z-1 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} z \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1-z \\ 1 \end{pmatrix}$$

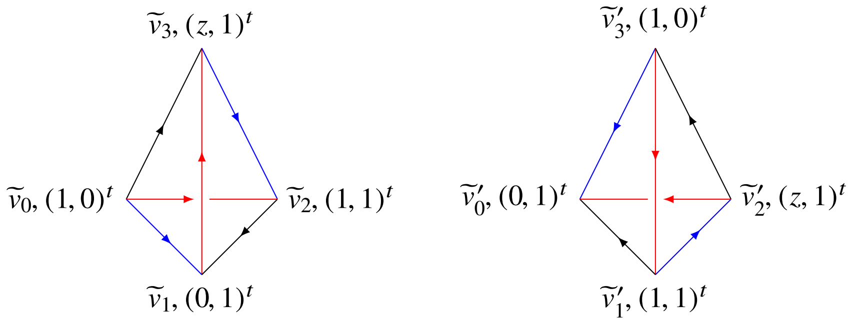


FIGURE 2. Computing a Thurston labelling for S^3/Q_8

so that, under the homogeneous Thurston labelling of Proposition 3.33, q_1, q_1', q_1'' receive the labels $-1, z, 1-z$, respectively. Further, the quads $q_2 = \{\{v_0', v_1'\}, \{v_2', v_3'\}\}$, $q_2' = \{\{v_0', v_3'\}, \{v_1', v_2'\}\}$ and $q_2'' = \{\{v_0', v_2'\}, \{v_1', v_3'\}\}$ are ordered $q_2 \to q_2' \to q_2''$ and we have

$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} z \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} z-1 \\ z \end{pmatrix}$$
$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -z \\ -1 \end{pmatrix}$$
$$\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1-z \end{pmatrix}.$$

Thus, under the homogeneous Thurston labelling of Proposition 3.33, q_2, q_2', q_2'' receive the labels $z-1, -z, 1$, respectively. We now have a homogeneous Thurston labelling and may apply Proposition 3.15 (i) to find a Thurston labelling. This labelling is given by

$$q_1, q_1', q_1'', q_2, q_2', q_2'' \mapsto \frac{1}{z}, \frac{z}{z-1}, z-1, \frac{z-1}{z}, z, \frac{1}{1-z}.$$

Substituting $z = \frac{1}{r}$, we see that we have found all those labellings over $\mathbb{C}$ which we found in Example 3.12.

Remark 3.35. Another method by which one can search for non-constant Thurston labellings under the condition of Proposition 3.33 is via the labelling of edges in Section 2.5, resulting in a representation ρ_{edges}. As mentioned above, the condition that $\operatorname{im}(\rho_{\text{sym}}) \leq V_4$ is equivalent to that ρ_{edges} is trivial. One can then label the edges of a base 3-simplex with arbitrary elements of a ring R such that opposite edges receive the same label, resulting in three labels r, r', r'', and such that the parameter relations are satisfied and then transport this labelling via a combinatorial path in the dual 1-skeleton to any other 3-simplex – the triviality of ρ_{edges} guarantees that this is independent of the chosen path. What remains to be satisfied then are equations of the form $r^{e_1} = 1$, $(r')^{e_2} = 1$ and $(r'')^{e_3} = 1$.

3.3. Some computations of holonomy representations

Example 3.36. Recall from the previous section that given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, if $\mathcal{T}$ is even, that is, in the notation of Proposition 3.31, $e = 0, 2, 4$, then it supports the constant Thurston labelling $x(q) = -1 \in \mathbb{Z}_3$ for all q. We consider the holonomy representations arising from this labelling. Following Proposition 3.15, in constructing an associated homogeneous Thurston labelling, the three quad labels within any 3-simplex are $-1, -1, 1-(-1) = 2 = -1$. As such, in transporting labels, the required cross-ratio is always $[1, -1]^t$. Consider now a base 3-simplex σ with normal triangles t_0, t_1, t_2, t_3 and orientation $t_i \to t_{i+1}$ and suppose that t_0, t_1, t_2 have been labelled with $A_i \in \mathrm{PGL}_2(\mathbb{Z}_3)$ where $A_0 = [1, 0]^t, A_1 = [0, 1]^t, A_2 = [1, 1]^t$. The unique such A_3 then such that $[A_0, A_1; A_2, A_3] = [1, -1]^t$ is $[1, -1]^t$ and we thus label t_3 by this A_3. Suppose that σ' is another 3-simplex, with normal triangles t'_i and orientation $t'_i \to t'_{i+1}$, which contains a face which is identified to a face of σ, say via the face-pairing $t_p, t_q, t_r \mapsto t'_p, t'_q, t'_r$. See Figure 3.

Upon transporting labels, the normal triangles t'_p, t'_q, t'_r receive the labels A_p, A_q, A_r and we denote the remaining label, that of t'_s, by B. Now, we can assume that $t_p \to t_q \to t_r \to t_s$ gives the orientation of σ and then, because our face-pairing must be orientation-reversing, the orientation of σ' is given by $t'_p \to t'_q \to t'_s \to t'_r$ and so then B is defined by $[A_p, A_q; B, A_r] = [1, -1]^t$. Now, by construction, $[A_p, A_q; A_r, A_s] = [1, -1]^t$ and then by Proposition 3.21 (ii), we have $[A_p, A_q; A_s, A_r] = [-1, 1]^t = [1, -1]^t$. Thus $B = A_s$.

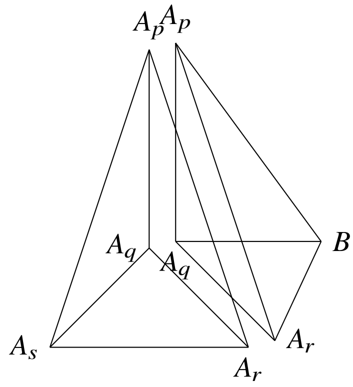


Figure 3. Transporting in the case of a constant Thurston labelling

As such, we see that the transport is essentially the symmetric transport and the symmetric representations constructed for evenly triangulated spaces can be recovered as holonomy representations associated to this particular Thurston labelling. There is a well-known equivalence between $\mathrm{PGL}_2(\mathbb{Z}_3)$ and $\mathrm{Sym}(4)$ which arises from the faithful natural action of the former on $\mathbb{P}^1(\mathbb{Z}_3)$, which is precisely $\{[1, 0]^t, [0, 1]^t, [1, 1]^t, [1, -1]^t\}$ (in general, if k is a finite field of size q, $|\mathbb{P}^1(k)| = q + 1$). This equivalence is precisely the relation between the symmetric transports construction in § 2.5 and those which can be constructed with the constant Thurston labelling over $\mathbb{Z}_3$ here.

Example 3.37. Given an oriented ideally triangulated 3-manifold $(M, \mathcal{T})$, in the case that $\mathcal{T}$ is even and, in the notation of Proposition 3.31, $d \equiv 0 \pmod 6$, we saw in the proof of that same proposition that other constant labellings exist. For example, the constant labelling $x(q) = \omega \in \mathbb{C}$ for all q, where ω is a non-trivial cube root of -1. Following Proposition 3.15, in constructing an associated homogeneous Thurston labelling, the three quad labels within any 3-simplex in this case are $\omega, -1, 1-\omega$ and as such, we cannot expect the transport in this case to be the symmetric transport as, in transporting labels, the required cross-ratio may be one of $[\omega, 1]^t$, $[-1, \omega-1]^t$ and $[1-\omega, -\omega]^t$.

Consider, for example, our ideal triangulation of the figure-eight knot complement, for which $d = 6$; whereas for our triangulation of quaternionic space, $d = 4$.

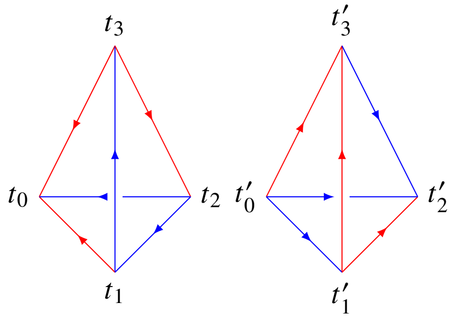


FIGURE 4. An even ideal triangulation, with edge degree sequence 6,6, of the figure-8 knot complement

Recall that the face-pairings are

$$\varphi_1 : t_0, t_1, t_2 \mapsto t'_2, t'_0, t'_3 \qquad \varphi_2 : t_0, t_1, t_3 \mapsto t'_1, t'_0, t'_3$$
$$\varphi_3 : t_0, t_2, t_3 \mapsto t'_1, t'_0, t'_2 \qquad \varphi_4 : t_1, t_2, t_3 \mapsto t'_1, t'_3, t'_2.$$

We define our homogeneous Thurston labelling by setting that, in the notation of the proof of Proposition 3.15 (ii), in $\sigma = [t_0, t_1, t_2, t_3]$, $q_0 = \{\{t_0, t_1\}, \{t_2, t_3\}\}$ and in $\sigma' = [t'_0, t'_1, t'_2, t'_3]$, $q_0 = \{\{t'_0, t'_1\}, \{t'_2, t'_3\}\}$. We then label t_0, t_1, t_2 by $[1, 0]^t, [0, 1]^t, [1, 1]^t$ and then note that the label of t_3, say A, is defined by $[[1, 0]^t, [0, 1]^t; [1, 1]^t, A] = [\omega, 1]^t$ and as such, using Example 3.18, we see that A is $[1, \omega]^t$. We now choose the φ_1 edge as a maximal tree for the dual 1-skeleton and so label t'_2, t'_0, t'_3 by $[1, 0]^t, [0, 1]^t, [1, 1]^t$ respectively. The remaining label, that of t'_1, say A', is defined by $[[0, 1]^t, A'; [1, 0]^t, [1, 1]^t] = [\omega, 1]^t$ and is then given by $A' = [\omega, 1]^t$. We may now apply Proposition 2.15 to conclude that the image, in $\mathrm{PGL}_2(\mathbb{C})$, of the associated holonomy representation is generated by M, N, P where

$$M : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix} \mapsto \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow M = \begin{bmatrix} \omega & 0 \\ 1 & \omega \end{bmatrix}$$
$$N : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix} \mapsto \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} \omega & -\omega \\ 1 & \omega^2 \end{bmatrix}$$
$$P : \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix} \mapsto \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} \omega + 1 & -1 \\ 1 & \omega^2 \end{bmatrix}.$$

Example 3.38. Consider the holonomy representations arising from the labellings defined in the Proposition 3.33. In the notation of the proof of that proposition, we may define the map $\alpha : \mathrm{Sym}(4) \to \mathrm{PGL}_2(\mathbb{C})$ which takes ψ to the unique element M of $\mathrm{PGL}_2(\mathbb{C})$ satisfying $M : A_i \mapsto A_{\psi(i)}$. This map is a group embedding and it is clear that the image of the holonomy representation is just the image under α of the image of the given symmetric representation.

Example 3.39. Consider the Thurston labellings which we found for our triangulation of S^3/Q_8 in Example 3.12. First, consider the labellings over $\mathbb{C}$, namely, in the notation of Example 3.12,

$$(r, r', r'', s, s', s'') = \left(z, \frac{1}{1-z}, \frac{z-1}{z}, 1-z, \frac{1}{z}, \frac{z}{z-1}\right)$$

for $z \neq 0, 1$. We define our homogeneous Thurston labelling by setting that, in the notation of the proof of Proposition 3.15 (ii), in $\sigma = [v_0, v_1, v_2, v_3]$, $q_0 = \{\{v_0, v_1\}, \{v_2, v_3\}\}$ and in $\sigma' = [v_0', v_1', v_2', v_3']$, $q_0 = \{\{v_0', v_1'\}, \{v_2', v_3'\}\}$. We then label v_0, v_1, v_2 by $[1, 0]^t$, $[0, 1]^t$, $[1, 1]^t$ and then note that the label of v_3, say A, is defined by $[[1, 0]^t, [0, 1]^t; [1, 1]^t, A] = [z, 1]^t$ and as such, using Example 3.18, we see that A is $[1, z]^t$. We now choose the φ_1 edge as a maximal tree for the dual 1-skeleton and so label v_3', v_0', v_1' by $[1, 0]^t$, $[0, 1]^t$, $[1, 1]^t$ respectively. The remaining label, that of v_2', say A', is defined by $[[0, 1]^t, [1, 1]^t; A', [1, 0]^t] = [1 - z, 1]^t$ and is then given by $A' = [1, z]^t$. We may now apply Proposition 2.15 to conclude that the image, in $\mathrm{PGL}_2(\mathbb{C})$, of the associated holonomy representation is generated by M, N, P where

$$M : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow M = \begin{bmatrix} z & -1 \\ z & -z \end{bmatrix}$$

$$N : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} 1 & -1 \\ z & -1 \end{bmatrix}$$

$$P : \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ z \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}.$$

It is clear that M, N, P are pairwise distinct and it can be checked that any two of these (in either order) multiply to give the third. Thus, for any z, the image of the holonomy representation is the Klein-4 group.

Consider now the Thurston labelling over $\mathbb{F}_4[x]/(x^2)$ given by

$$(r, r', r'', s, s', s'') = (a, a, a, b + bx, b + x, b + ax).$$

We will see that we can achieve a larger image by not working over $\mathbb{C}$ and using this labelling. We again define our homogeneous Thurston labelling by setting that, in the notation of the proof of Proposition 3.15 (ii), in $\sigma = [v_0, v_1, v_2, v_3]$, $q_0 = \{\{v_0, v_1\}, \{v_2, v_3\}\}$ and in $\sigma' = [v_0', v_1', v_2', v_3']$, $q_0 = \{\{v_0', v_1'\}, \{v_2', v_3'\}\}$ and again label v_0, v_1, v_2 by $[1, 0]^t, [0, 1]^t, [1, 1]^t$. Then the label of v_3, denoted A, is defined by $[[1, 0]^t, [0, 1]^t; [1, 1]^t, A] = [a, 1]^t$ and as such, using Example 3.18, we see that A is $[1, a]^t$. Choosing, as before, the φ_1 edge as a maximal tree for the dual 1-skeleton, we label v_3', v_0', v_1' by $[1, 0]^t, [0, 1]^t, [1, 1]^t$ respectively. The remaining label, that of v_2', denoted A', is defined by $[[0, 1]^t, [1, 1]^t; A', [1, 0]^t] = [b + bx, 1]^t$ and is then given by $A' = [1, a + bx]^t$. We may now apply Proposition 2.15 to conclude that the image, in $\mathrm{PGL}_2(\mathbb{C})$, of the associated holonomy representation is generated by M, N, P where

$$M : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a + bx \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow M = \begin{bmatrix} a & 1 \\ a & a + bx \end{bmatrix}$$

$$N : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ a + bx \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} 1 & 1 \\ a + bx & 1 + ax \end{bmatrix}$$

$$P : \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ a + bx \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} x & 1 + x \\ a + bx & 0 \end{bmatrix}.$$

It can now be checked that

$$M^2 = N^2 = P^2 = \begin{bmatrix} 1 & bx \\ x & 1 \end{bmatrix}$$

and if we denote this common square J, that

$$J^2 = 1 \qquad MNP = J$$

so that this holonomy representation is faithful with image isomorphic to Q_8 and an explicit isomorphism is given by $J, M, N, P \mapsto -1, i, j, k$.

CHAPTER 4

Work on a conjecture of Feng Luo

In this chapter, we present work on a conjecture of Feng Luo, proving it to hold in certain cases but providing a counterexample for the general case.

4.1. The conjecture

As a result of his generalisation of the hyperbolic gluing equations to the context of commutative rings, Luo made the following conjecture in [**14**].

Conjecture. If M is a compact 3-manifold and $\gamma \in \pi_1(M) - \{1\}$, there exists a finite commutative ring R with identity and a homomorphism $\pi_1(M) \to \mathrm{PSL}_2(R)$ whose kernel does not contain γ.

Definition 4.1. Given a group G, say that G is *residually PSL*$_2$ if and only if for any $g \in G - \{1\}$, there exists a finite commutative ring R with identity and a homomorphism $G \to \mathrm{PSL}_2(R)$ whose kernel does not contain g. Define, in an analogous manner, *residually PGL*$_2$, *residually SL*$_2$ and *residually GL*$_2$.

Luo's conjecture then says that every compact 3-manifold group is residually PSL_2. As a first observation, we have the following result, proven by Hempel, see [**10**], in the case of Haken manifolds and extendible to the general case via geometrisation, see [**11**].

Theorem 4.2. *Every compact* 3*-manifold group is residually finite.*

Remark 4.3. In [**12**], Luo mentions that one motivation for his conjecture is that it's verification would provide a list of specific finite groups which detect non-triviality.

Due to Theorem 4.2, as finite groups embed into symmetric groups, if we had that $\mathrm{Sym}(n)$ is residually PSL_2 for all n, we would have verified Luo's conjecture. However, we have the following result, which is proven in [**18**] using the characterisation of the property of being residually PSL_2 provided by Proposition 4.11 in the next section.

Theorem 4.4. *$\mathrm{Sym}(n)$ is residually PSL*$_2$ *if and only if $n < 5$.*

We can, however, use Theorem 4.2 to show that a weakened version of Luo's conjecture holds.

Proposition 4.5. *$\mathrm{Sym}(n)$ embeds into $\mathrm{GL}_n(R)$ for any non-zero commutative ring with identity R.*

Proof. This is via permutation matrices. Given a $\sigma \in \mathrm{Sym}(n)$, associate to it the matrix E_σ which, in the i^{th} column, has 1 in the $\sigma(i)^{\text{th}}$ place and 0's elsewhere. It can be checked that, for $\sigma, \sigma' \in \mathrm{Sym}(n)$, $E_{\sigma\sigma'} = E_\sigma E_{\sigma'}$. The permutation matrices E_σ are invertible as $\det(E_\sigma) = \mathrm{sgn}(\sigma)$; this is clear as, if τ is a transposition, E_τ is an elementary matrix which determinant -1. The map $\mathrm{Sym}(n) \to \mathrm{GL}_n(R) : \sigma \mapsto E_\sigma$ is then a group embedding. □

Define *residually PGL$_n$* in a manner similar to that in Definition 4.1.

Proposition 4.6. *Sym(n) is residually PGL$_n$.*

Proof. This follows from Proposition 4.5 upon composition with the canonical surjection $\mathrm{GL}_n(R) \to \mathrm{PGL}_n(R)$ which is injective on the copy of $\mathrm{Sym}(n)$ in $\mathrm{GL}_n(R)$. □

The combination of Theorem 4.2 and Proposition 4.6 shows that Luo's conjecture holds if we weaken it to allow arbitrary dimension of matrices.

4.2. Alternative characterisations

The following result, which is proven in [**3**], is crucial to our work.

Theorem 4.7. *If a commutative ring R is finitely-generated as a $\mathbb{Z}$-algebra, then R is residually finite.*

Let K be one of the symbols $\mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PSL}_2, \mathrm{PGL}_2$. The proofs of the Propositions 4.8, 4.9 and 4.11 are generalisations of the proofs contained in [**7**] of the case $K = \mathrm{SL}_2$.

Proposition 4.8. *If G is finitely-generated, then G is residually K if and only if it admits a faithful representation $G \to K(R)$ for some, not necessarily finite, R.*

Proof. We first focus on the "only if" direction; suppose first that K is one of $\mathrm{SL}_2, \mathrm{GL}_2$. Since G is finitely-generated, it is countable. Let $G = \{g_0, g_1, \dots\}$ where $g_0 = 1$. For each $i > 0$, we have a representation $\rho_i : G \to K(R_i)$ where R_i is finite and $\rho_i(g_i) \neq 1$. Let $R = \prod_{i>0} R_i$ and if given a matrix $A_i = (a^i_{kl})_{kl} \in K(R_i)$ for each $i > 0$, define

$$\prod_{i>0} A_i = \left(\prod_{i>0} a^i_{kl}\right)_{kl} ;$$

that is to say, the k, l entry of $\prod_{i>0} A_i$ is the "product" of the k, l entries of the A_i. As everything here is defined component-wise, so that, for example, given $a_i \in R_i$, $\prod_{i>0} a_i = 1 \in R$ if and only if $a_i = 1 \in R_i$ for each i, we see that if $A_i \in \mathrm{SL}_2(R_i)$ for each i, $\prod_{i>0} A_i \in \mathrm{SL}_2(R)$. Now define $\rho : G \to K(R) : g \mapsto \prod_{i>0} \rho_i(g)$ which we can see to be a homomorphism via

$$\begin{aligned}
\rho(gh) = \prod_{i>0} \rho_i(gh) = \prod_{i>0} \rho_i(g)\rho_i(h) &= \prod_{i>0} (a^i_{kl})_{kl}(b^i_{kl})_{kl} = \prod_{i>0} (a^i_{k1}b^i_{1l} + a^i_{k2}b^i_{2l})_{kl} \\
&= \left(\prod_{i>0} (a^i_{k1}b^i_{1l} + a^i_{k2}b^i_{2l})\right)_{kl} = \left(\prod_{i>0} a^i_{k1} \prod_{i>0} b^i_{1l} + \prod_{i>0} a^i_{k2} \prod_{i>0} b^i_{2l}\right)_{kl} \\
&= \left(\prod_{i>0} a^i_{kl}\right)_{kl} \left(\prod_{i>0} b^i_{kl}\right)_{kl} = \prod_{i>0} (a^i_{kl})_{kl} \prod_{i>0} (b^i_{kl})_{kl} = \rho(g)\rho(h).
\end{aligned}$$

We can also see that ρ is faithful. For suppose that $\rho(g) = 1$, then if we set $\rho_i(g) = (a^i_{kl})_{kl}$, we have $a^i_{12} = a^i_{21} = 0, a^i_{11} = a^i_{22} = 1$ for all i. This shows that $\rho_i(g) = 1$ for all i so that $g = g_0 = 1$.

Now suppose that K is one of $\mathrm{PSL}_2, \mathrm{PGL}_2$ and again let $G = \{g_0, g_1, \dots\}$, where $g_0 = 1$. For each $i > 0$, we take a representation $\rho_i : G \to K(R_i)$ where R_i is finite and $\rho_i(g_i) \neq 1$. As earlier, let $R = \prod_{i>0} R_i$ and if given a class of matrices $A_i = [a^i_{kl}]_{kl} \in K(R_i)$ for each $i > 0$, we define $\prod_{i>0} A_i = \left[\prod_{i>0} a^i_{kl}\right]_{kl}$. In this case, this product needs to be checked to be well-defined which is seen via the observation that if $(\lambda_i a^i_{kl})_{kl}$ are other representatives of the A_i, then for each k, l, $\prod_{i>0} \lambda_i a^i_{kl} = \prod_{i>0} \lambda_i \prod_{i>0} a^i_{kl}$ and $(\prod_{i>0} \lambda_i)^2 = (1, 1, \dots)$ if $\lambda_i^2 = 1$ for each i. As before, we then define $\rho : G \to K(R) : g \mapsto \prod_{i>0} \rho_i(g)$. Then ρ is a homomorphism and we can also see that ρ is faithful. For suppose that $\rho(g) = 1$, then there exist representatives for the $\rho_i(g)$, say $(a^i_{kl})_{kl}$, such that $a^i_{12} = a^i_{21} = 0$ for all i and there exist $\lambda_i \in R^\times$ such that $a^i_{11} = a^i_{22} = \lambda_i$. This shows that $\rho_i(g) = 1$ for all i so that $g = g_0 = 1$.

Now we prove the "if" direction; suppose first again that K is one of SL_2, GL_2. Let $G = \langle g_1, \dots, g_k \rangle$ and let $\rho : G \to K(R)$ be faithful. For $i = 1, \dots, k$, let $A_i = \rho(g_i)$ and let R' be the ring generated by the entries of $A_1, \dots, A_k$ as well as the elements $(\det A_1)^{-1}, \dots, (\det A_k)^{-1}$; in the case that $K = \mathrm{SL}_2$, the inclusion of the determinants is superfluous. Note that R' contains the entries of $A_1^{-1}, \cdots, A_k^{-1}$. As such, we can restrict ρ to attain a faithful representation $\rho' : G \to \mathrm{SL}_2(R')$ and because R' is a finitely generated $\mathbb{Z}$-algebra, it is residually finite. Now consider an arbitrary non-identity $g \in G$. If $\rho'(g)$ has a non-zero off-diagonal entry, say a, we let $\phi : R' \to R''$ be such that $\phi(a) \neq 0$ and R'' is finite, then the image of g under the map $G \xrightarrow{\rho'} \mathrm{SL}_2(R') \xrightarrow{\phi_*} \mathrm{SL}_2(R'')$ is non-trivial. Now suppose that $\rho'(g)$ is diagonal, say $\mathrm{diag}(a, b)$. If $a = b$, say with both equal to $c \neq 1$, let $\phi : R' \to R''$ be such that $\phi(c - 1) \neq 0$ and R'' is finite, then the image of g under the map $G \xrightarrow{\rho'} \mathrm{SL}_2(R') \xrightarrow{\phi_*} \mathrm{SL}_2(R'')$ is non-trivial. Finally, if $a - b \neq 0$ and we can choose $\phi : R' \to R''$ be such that $\phi(a) - \phi(b) \neq 0$ and R'' is finite; then the image of g under the map $G \xrightarrow{\rho'} \mathrm{SL}_2(R') \xrightarrow{\phi_*} \mathrm{SL}_2(R'')$ is non-trivial.

Now suppose that K is one of PSL_2, PGL_2 and let again $G = \langle g_1, \dots, g_k \rangle$ and $\rho : G \to K(R)$ a faithful representation. Let $A_1, \dots, A_k$ be representatives of $\rho(g_1), \dots, \rho(g_k)$ respectively and let R' be the ring generated by the entries of $A_1, \dots, A_k$; note that R' contains the entries of represenatives for $\rho(g_1^{-1}), \cdots, \rho(g_k^{-1})$. Let $\iota : K(R') \to K(R)$ denote the obvious injection and note that $\mathrm{im}(\rho) \subseteq \mathrm{im}(\iota)$ so that by a composition we can attain a faithful representation $\rho' : G \to K(R')$. Because R' is a finitely generated $\mathbb{Z}$-algebra, it is residually finite. The remainder of the proof is analogous to the proof in the case that K is one of SL_2, GL_2 except that we observe that a representative for $\rho'(g)$ cannot be scalar for $g \neq 1$. □

Proposition 4.9. *If G is finitely-presentable, then there exists a commutative ring S_K, an ideal $I_K \trianglelefteq S_K$ and a map $\varphi_K : G \to K(S_K/I_K)$ such that any representation $G \to K(R)$ factors through φ_K; that is, for each $\rho : G \to K(R)$, there exists a mediating map $\psi : K(S_K/I_K) \to K(R)$ such that the following diagram commutes.*

$$\begin{tikzcd} G \arrow[r, "\varphi_K"] \arrow[dr, "\rho"'] & K(S_K/I_K) \arrow[d, "\psi"] \\ & K(R) \end{tikzcd}$$

Proof. Let $G = \langle g_1, \dots, g_n \mid r_1 = s_1, \dots, r_m = s_m \rangle$ and suppose first that $K = \mathrm{SL}_2$. Let

$$S_{\mathrm{SL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \dots, x_{na}, x_{nb}, x_{nc}, x_{nd}]$$

and then define

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix}.$$

Define also $p(r_i)$ and $p(s_i)$ by setting that p be multiplicative and then set

$$I_{\mathrm{SL}_2} = \langle \{\det p(g_i) - 1\}_i \cup \{(p(r_i) - p(s_i))_{k,l}\}_{i,k,l}\rangle .$$

Then we set

$$\varphi_{\mathrm{SL}_2} : G \to \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2}) : g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix}$$

which can be checked to be well-defined. Now, suppose that $\rho : G \to \mathrm{SL}_2(R)$ is given. Let $\rho(g_i) = (a^i_{kl})_{kl}$ and define $q : S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2} \to R$ by

$$1, x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd} \mapsto 1, a^1_{11}, a^1_{12}, a^1_{21}, a^1_{22}, \ldots, a^n_{11}, a^n_{12}, a^n_{21}, a^n_{22}.$$

The map q is well-defined because $a^i_{11}a^i_{22} - a^i_{12}a^i_{21} - 1 = 0$ for each i and because computation of $\rho(r_j)$ and $\rho(s_j)$ will give the required remaining equations defining I. This map q induces a map

$$\psi : \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2}) \xrightarrow{q_*} \mathrm{SL}_2(R)$$

by applying q to each entry and one can then verify that $\psi \circ \varphi_{\mathrm{SL}_2} = \rho$ holds.

If $K = \mathrm{GL}_2$, we alter the definitions as follows:

$$S_{\mathrm{GL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd}, y_1, \ldots, y_n],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, p(g_i^{-1}) = y_i \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix},$$

$p(r_i)$ and $p(s_i)$ are defined by setting that p be multiplicative,

$$I_{\mathrm{GL}_2} = \langle \{(\det p(g_i))y_i - 1\}_i \cup \{(p(r_i) - p(s_i))_{k,l}\}_{i,k,l}\rangle,$$

$$\varphi_{\mathrm{GL}_2} : G \to \mathrm{GL}_2(S_{\mathrm{GL}_2}/I_{\mathrm{GL}_2}) : g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix}$$

and finally given $\rho : G \to \mathrm{GL}_2(R)$ and $\rho(g_i) = (a^i_{kl})_{kl}$, $q : S_{\mathrm{GL}_2}/I_{\mathrm{GL}_2} \to R$: $1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, (a^i_{11}a^i_{22} - a^i_{12}a^i_{21})^{-1}$ and $\psi = q_*$.

If $K = \mathrm{PSL}_2$, we alter the definitions as follows:

$$S_{\mathrm{PSL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd}, \lambda_1, \ldots, \lambda_m],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix},$$

$p(r_i)$ and $p(s_i)$ are defined by setting that p be multiplicative,

$$I_{\mathrm{PSL}_2} = \left\langle \{\det p(g_i) - 1\}_i \cup \{\lambda_i^2 - 1\}_i \cup \{(p(r_i) - \lambda_i p(s_i))_{k,l}\}_{i,k,l}\right\rangle,$$

$$\varphi_{\mathrm{PSL}_2} : G \to \mathrm{PSL}_2(S_{\mathrm{PSL}_2}/I_{\mathrm{PSL}_2}) : g_i \mapsto \begin{bmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{bmatrix}$$

and finally given $\rho : G \to \mathrm{PSL}_2(R)$, $\rho(g_i) = [a^i_{kl}]_{kl}$ and that the corresponding representative for $p(r_i)$ is equal to μ_i multiplied by the corresponding representative for $\rho(s_i)$, $q : S_{\mathrm{PSL}_2}/I_{\mathrm{PSL}_2} \to R : 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, \mu_i$ and $\psi = q_*$.

If $K = \mathrm{PGL}_2$, we alter the definitions as follows:

$$S_{\mathrm{PGL}_2} = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd}, y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_m],$$

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix},$$

$p(r_i)$ and $p(s_i)$ are defined by setting that p be multiplicative,

$$I_{\mathrm{PGL}_2} = \left\langle \{(\det\, p(g_i))y_i - 1\}_i \cup \{\lambda_i^2 - 1\}_i \cup \{(p(r_i) - \lambda_i p(s_i))_{k,l}\}_{i,k,l} \right\rangle,$$

$$\varphi_{\mathrm{PGL}_2} : G \to \mathrm{PGL}_2(S_{\mathrm{PGL}_2}/I_{\mathrm{PGL}_2}) : g_i \mapsto \begin{bmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{bmatrix}$$

and finally given $\rho : G \to \mathrm{PGL}_2(R)$, $\rho(g_i) = [a^i_{kl}]_{kl}$ and that the corresponding representative for $p(r_i)$ is equal to μ_i multiplied by the corresponding representative for $\rho(s_i)$, $q : S_{\mathrm{PGL}_2}/I_{\mathrm{PGL}_2} \to R : 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, y_i, \lambda_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, (a^i_{11}a^i_{22} - a^i_{12}a^i_{21})^{-1}, \mu_i$ and $\psi = q_*$. □

Remark 4.10. We could use any other characteristic zero ring instead of $\mathbb{Z}$ for the coefficients in S_K.

Proposition 4.11. *Given a finitely-presentable group G, it is residually K if and only if the map $\varphi_K : G \to K(S_K/I_K)$ above is an injection.*

Proof. By Proposition 4.8, if G it is residually K, there exists a faithful $\rho : G \to K(R)$ for some R so that, as ρ factors through φ_K, φ_K too is an injection. Conversely, if φ_K is faithful, we apply Proposition 4.8 again with φ_K as the injection to conclude that G is residually K. □

Proposition 4.12. *We have the following implications:*

$$\begin{tikzcd}[column sep=huge, row sep=large]
\textit{residually } SL_2 \arrow[r, Rightarrow, "\forall G"] \arrow[d, Rightarrow, shift left, "\textit{f.g. 2-t.f. } G"] & \textit{residually } GL_2 \arrow[d, Rightarrow, shift right, "\textit{f.g. c.l. } G"'] \\
\textit{residually } PSL_2 \arrow[u, Rightarrow, shift left, "\textit{f.p. } G"] \arrow[r, Rightarrow, shift left, "\forall G"] & \textit{residually } PGL_2 \arrow[l, Rightarrow, shift left, "\textit{f.g. } G"] \arrow[u, Rightarrow, shift right, "\textit{f.p. } G"']
\end{tikzcd}$$

Here f.g., f.p., 2-t.f. and c.l mean, respectively, finitely-generated, finitely-presentable, 2-torsion-free and centreless.

Note that in passing across these implications, it may be necessary to alter the ring over which the relevant matrix group is considered. For example, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ embeds into $\mathrm{PSL}_2(\mathbb{C})$ but this embedding cannot be lifted to one into $\mathrm{SL}_2(\mathbb{C})$ as there is no such embedding for $\mathrm{SL}_2(\mathbb{C})$ contains only one element of order two.

Note also that all compact 3-manifold groups are finitely-generated; for a proof, see [**13**].

Proof. That residually SL_2 and residually PSL_2 imply, respectively, residually GL_2 and residually PGL_2 for all G is clear. To see that, if G is finitely-generated and 2-torsion-free, residually SL_2 implies residually PSL_2, via Proposition 4.8 the former gives us a faithful representation into $\mathrm{SL}_2(R)$ for some R and 2-torsion-freeness implies that the image of this representation cannot contain non-identity scalar matrices (we could have instead imposed the condition that the centre of G be 2-torsion-free). A similar proof shows that if G is finitely-generated and centreless, residually GL_2 implies residually PGL_2.

Next, we show that, for finitely-generated G, residually PGL_2 implies residually PSL_2. Let $G = \langle g_1, \ldots, g_k \rangle$; via Proposition 4.8, we have a faithful $\rho : G \to \mathrm{PGL}_2(R)$ for some R. Choose representatives of the generators $\rho(g_1), \ldots, \rho(g_k)$ of $\rho(G)$, let

$a_i = \det(\rho(g_i))$ and let $R' = R[x_1, \ldots, x_k]/I$ where $I = (x_1^2 - a_1^{-1}, \ldots, x_k^2 - a_k^{-1})$. It can be checked that $I \cap R = \{0\}$, and so as a result the map $\iota : \mathrm{PGL}_2(R) \to \mathrm{PGL}_2(R')$ which applies the inclusion $R \hookrightarrow R'$ to each entry is an embedding which then gives us the faithful representation $\iota \circ \rho : G \to \mathrm{PGL}_2(R')$. For each i, choosing the same representatives of the $\rho(g_i)$ as earlier we note that the representative $x_i(\iota \circ \rho)(g_i)$ has unit determinant. Thus the image of $\iota \circ \rho$ lies in the copy of $\mathrm{PSL}_2(R')$ inside $\mathrm{PGL}_2(R')$.

Finally we will show that, for finitely-presentable G, residually PSL_2 implies residually SL_2; this will, using the other implications proven so far, show also that, under the same conditions, residually PGL_2 implies residually GL_2. To show this, we first show that, given a finitely-presentable group G and a representation $\rho : G \to \mathrm{PSL}_2(R)$, there exists an R' and a map $\varphi : G \to \mathrm{SL}_2(R')$ through which ρ factors. The construction involved is the same as that for the $K = \mathrm{PSL}_2$ case in the proof of Proposition 4.9. Let $G = \langle g_1, \ldots, g_n \mid r_1 = s_1, \ldots, r_m = s_m \rangle$, let

$$S = \mathbb{Z}[x_{1a}, x_{1b}, x_{1c}, x_{1d}, \ldots, x_{na}, x_{nb}, x_{nc}, x_{nd}, \lambda_1, \ldots, \lambda_m]$$

and then define

$$p(g_i) = \begin{pmatrix} x_{ia} & x_{ib} \\ x_{ic} & x_{id} \end{pmatrix}, p(g_i^{-1}) = \begin{pmatrix} x_{id} & -x_{ib} \\ -x_{ic} & x_{ia} \end{pmatrix}.$$

Define also $p(r_i)$ and $p(s_i)$ by setting that p be multiplicative and then define

$$I = \left\langle \{\det p(g_i) - 1\}_i \cup \{\lambda_i^2 - 1\}_i \cup \{(p(r_i) - \lambda_i p(s_i))_{k,l}\}_{i,k,l} \right\rangle.$$

Now set $R' = S/I$ and

$$\varphi : G \to \mathrm{SL}_2(R') : g_i \mapsto \begin{pmatrix} \overline{x_{ia}} & \overline{x_{ib}} \\ \overline{x_{ic}} & \overline{x_{id}} \end{pmatrix}$$

which can be checked to be well-defined. Now, given $\rho : G \to \mathrm{PSL}_2(R)$, let $(a^i_{kl})_{kl}$ be representatives for $\rho(g_i)$ and let $\mu_i \in R^\times$ be such that $\mu_i^2 = 1$ for each i and the corresponding representative for $\rho(r_i)$ is equal to μ_i multiplied by the corresponding representative for $\rho(s_i)$. Define $q : R' \to R : 1, x_{ia}, x_{ib}, x_{ic}, x_{id}, \lambda_i \mapsto 1, a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, \mu_i$ and set $\psi = q_*$. Then $\rho = \psi \circ \phi$.

Now, if G is finitely-presentable and residually PSL_2, it admits a faithful $\rho : G \to \mathrm{PSL}_2(R)$. This ρ factors through a representation $\rho' : G \to \mathrm{SL}_2(R')$ which is then also faithful and so G is residually SL_2. Alternatively, in case it sheds any light, we can prove the contrapositive. Say $g \in G$ is such that given any $\rho : G \to \mathrm{SL}_2(R)$ for a finite R, $\rho(g) = 1$. Then, using the same argument as in proof of the 'if' direction of Proposition 4.8, given any $\rho : G \to \mathrm{SL}_2(R)$ for any commutative R, $\rho(g) = 1$. Now, given some $\rho : G \to \mathrm{PSL}_2(R)$ for an even not necessarily finite R, it factors through a representation $\rho' : G \to \mathrm{SL}_2(R')$. We must have $\rho'(g) = 1$ so that $\rho(g) = 1$. Thus G is not residually PSL_2. □

4.3. A counterexample

Let M be the $(4, 1)$-Dehn filling, using the knot theoretic framing, of the figure-8 knot complement. We will show that M is a counterexample to Luo's conjecture. In SnapPy, [**6**], one can construct a triangulation of M and this triangulation can then be imported into Regina, [**5**]. See A.1 in the Appendix. Regina then gives the following presentation for $\Gamma = \pi_1(M)$:

$$\Gamma = \langle a, b \mid a^{-1}b^2a^{-3}b^2 = 1, ba^{-2}ba^{-2}b^3a^{-2} = 1\rangle.$$

We re-write this presentation by making the substitutions $a \rightsquigarrow b^{-1}$, $b \rightsquigarrow a^{-1}$ and set $c = b^2a^{-2}$; this leads to the following presentation:

$$\Gamma = \langle a, b, c \mid ca^2 = b^2, c^{-1}b = bc, ac^{-1}a^{-1} = cac\rangle.$$

Remark 4.13. Note that the above presentation can be re-written as

$$G = \langle a, b, c \mid c = b^2a^{-2}, \underbrace{1 = bcb^{-1}c}_{\text{Klein bottle}}, \overbrace{a^2 = (ac)^3}^{\text{trefoil complement}} \rangle$$

which highlights the presence of a trefoil knot complement and a Klein bottle; in fact, M can be constructed as the identification of a trefoil knot complement and a twisted I-bundle over a Klein bottle. Specifically, letting $\Gamma_1 = \langle u, v \mid u^3 = v^2\rangle$ and $\Gamma_2 = \langle j, k \mid jkj^{-1}k = 1\rangle$, at the level of the fundamental group, this identification is given by gluing $\langle v^{-1}u, u^3\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$ and $\langle k, j^2\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$ via the identifications $v^{-1}u \leftrightarrow k$, $u^3 \leftrightarrow k^{-1}j^2$.

Now, we return to the second presentation for Γ above and construct the universal representation $\varphi_{\mathrm{SL}_2} : G \to \mathrm{SL}_2(S_{\mathrm{SL}_2}/I_{\mathrm{SL}_2})$ as in Proposition 4.9 where $S_{\mathrm{SL}_2} = \mathbb{Z}[i, j, k, l, p, q, r, s, w, x, y, z]$,

$$a \mapsto \begin{pmatrix} i & j \\ k & l \end{pmatrix} \qquad b \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix} \qquad c \mapsto \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and $il - kj - 1, ps - rq - 1, wz - yx - 1$ as well as 12 equations arising from the relations generate I_{SL_2}. Now, we have

$$\begin{aligned} &\begin{pmatrix} p & q \\ r & s \end{pmatrix}^4 \\ &= \begin{pmatrix} (p^2 + qr)^2 + qr(p + s)(p + s) & q(p^2 + qr)(p + s) + q(p + s)(qr + s^2) \\ r(p + s)(p^2 + qr) + r(qr + s^2)(p + s) & qr(p + s)(p + s) + (qr + s^2)^2 \end{pmatrix} \\ &= \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \end{aligned}$$

and it can be verified via SageMath, [**24**], that $f_1 - 1, f_2, f_3, f_4 - 1 \in I_{\mathrm{SL}_2}$ so that $\varphi_{\mathrm{SL}_2}(b^4) = 1$ and so φ_{SL_2} is not injective. See A.2 in the Appendix. Thus b^4 is killed in any representation $G \to \mathrm{SL}_2(R)$, G is not residually SL_2 and so, using Proposition 4.12, is also not residually PSL_2.

Remark 4.14. Our manifold M is a closed graph manifold. These manifolds have played a special role in many results in recent years. For example, in [**8**], it is shown that if N is an irreducible 3-manifold with empty or toroidal boundary, then, if N is not a closed graph manifold, $\pi_1(N)$ is residually a torsion-free and elementary amenable group. Thus one can ask the question whether Luo's conjecture holds for irreducible manifolds with empty or toroidal boundary whose fundamental groups are, residually, torsion-free and elementary amenable.

4.4. Geometric manifolds

Proposition 4.15. *Given an orientable hyperbolic 3-manifold M, $\pi_1(M)$ embeds into $PSL_2(\mathbb{C})$.*

Proof. The embedding is the holonomy representation, see [**21**, Chapter 8] for details. □

Corollary 4.16. *Given an orientable hyperbolic 3-manifold M, $\pi_1(M)$ is residually PSL_2.*

Proof. This follows from Propositions 4.8 and 4.15. □

Remark 4.17. It is a well-known result due to Thurston that we can lift the holonomy representation into $\mathrm{PSL}_2(\mathbb{C})$ to one into $\mathrm{SL}_2(\mathbb{C})$; see [**23**]. As such, any orientable hyperbolic 3-manifold group is also residually SL_2; this can also be seen via Proposition 4.12

In [**7**], there is work towards Luo's conjecture for other classes of geometric manifolds. In particular, the following is shown.

Proposition 4.18. *Given an orientable compact M modelled on $\mathbb{H}^2 \times \mathbb{E}^1$, $\pi_1(M)$ is residually GL_2.*

Proof. Using the notation $\mathrm{Isom}^+(\cdot)$ for the groups of orientation-preserving isometries, we have $\mathrm{Isom}^+(\mathbb{H}^2 \times \mathbb{E}^1) = \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{Isom}^+(\mathbb{E}^1) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. Now we post-compose this map by the projections of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ onto either factor and then use Proposition 4.7 as it was used in the proof of Proposition 4.8 to reduce the ring of entries to a finite ring. □

We also have:

Proposition 4.19. *Given an orientable compact M modelled on $\mathbb{S}^3$, $\pi_1(M)$ is residually SL_2.*

Proof. The holonomy representation arising from the geometric structure gives an embedding of $\pi_1(M)$ into $\mathrm{SO}_4(\mathbb{R})$. It can be shown that this embedding may be lifted to one into $\mathrm{SU}_2 \times \mathrm{SU}_2$, see [**7**] or [**4**, Section 9.2], and then we again use Proposition 4.7 as it was used in the proof of Proposition 4.8 to reduce the ring of entries to a finite ring. □

4.5. S^1-bundles over surfaces

In this section we prove Luo's conjecture in the case of S^1-bundles over orientable, compact and connected surfaces, denoted $S_{g,b}$, or S_g when $b = 0$, where g denotes the genus and b the number of boundary components.

Proposition 4.20. *For all g, b, $\pi_1(S_{g,b})$ is residually PSL_2.*

Proof. If $b > 0$, $\pi_1(S_{g,b})$ is a free group on $2g + b - 1$ generators. It is well-known that a free group of at most countable rank embeds into the free group on two generators which is known to embed into $\mathrm{SL}_2(\mathbb{Z})$ as the subgroup generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Because free groups are torsion-free, upon application of Propositions 4.8 and 4.12, we have the result.

Now suppose that $b = 0$; that is, $S_{g,b}$ is closed. If $g = 0$, we have the 2-sphere which has trivial fundamental group and there is nothing to show. If $g = 1$, we have the 2-torus which as $\mathbb{Z}\times\mathbb{Z}$ as its fundamental group and, using the projections onto each factor, that this group is residually PSL_2 follows from the same property just shown to be possessed by $\mathbb{Z}$. Finally, it is shown in [**20**] that, for $g \geq 2$, $\pi_1(S_g)$ embeds into $\pi_1(S_2)$. It is shown in [**16**] that $\pi_1(S_2)$ embeds into $\mathrm{SL}_2(\mathbb{C})$ and using torsion-freeness of closed surface groups and Propositions 4.8 and 4.12, we have the result. □

Remark 4.21. The proof of Proposition 4.20 shows that free groups of at most countable rank are residually PGL_2. Further, as it is clear that $\mathbb{Z}$ is residually PGL_2, we see that Luo's conjecture holds for $S_{g,b}\times S^1$. Proposition 4.22 below generalises this observation.

Proposition 4.22. *The fundamental group of an S^1-bundle over an orientable, compact and connected surface is residually PSL_2.*

The part of the following proof up to the reduction to the case of $\langle c\rangle \leq H$ is work from [**17**].

Proof. Let N be the S^1-bundle, over say the surface F, and first consider the case that N is a trivial S^1-bundle, that is, $N \cong S^1\times F$. Then $\pi_1(N) \cong \mathbb{Z}\times\pi_1(F)$ and so, using the projections onto each factor and that $\mathbb{Z}$ and $\pi_1(F)$ are residually PSL_2, $\pi_(N)$ is residually PSL_2. Now assume that N is a non-trivial S^1-bundle over F. If N has boundary, then F also has boundary and we obtain $H^2(F;\mathbb{Z}) = 0$, so the Euler class of the S^1-bundle $N \to F$ is trivial and so N would be a trivial S^1-bundle. Hence F is a closed surface. If $F \cong S^2$, then the long exact sequence in homotopy theory shows that $\pi_1(N)$ is cyclic, hence residually PSL_2. If $F \not\cong S^2$, then it follows again from the long exact sequence in homotopy theory that the subgroup $\langle t\rangle$ of $\pi_1(N)$ generated by a fibre is normal and infinite cyclic, and that we have a short exact sequence

$$1 \to \langle t\rangle \to \pi_1(N) \to \pi_1(F) \to 1.$$

Let $e \in H^2(F;Z) \cong \mathbb{Z}$ be the Euler class of F. By the discussion in [**22**], p.435, a presentation for $\pi_1(N)$ is given by

$$\pi_1(N) \cong \left\langle a_1, b_1, \dots, a_n, b_n, t \mid \prod_{i=1}^{n}[a_i, b_i] = t^e, t \text{ central}\right\rangle.$$

As $\pi_1(F)$ is residually PSL_2, we see that we need only show that for any $\gamma \in \langle t\rangle$, there exists a representation $\pi_1(N) \to \mathrm{PSL}_2(R)$ for a finite R whose kernel does not contain γ. First we pass through the homomorphism $\pi_1(N) \to \langle a, b, c \mid [a, b] = c^e, c \text{ central}\rangle$ via the homomorphism which kills $a_2, b_2, \dots, a_n, b_n$ and maps a_1, b_1, t to a, b, c; note that this homomorphism is injective on $\langle t\rangle$ so that we have not lost any generality. Let $H = \langle a, b, c \mid [a, b] = c^e, c \text{ central}\rangle$ and note that we may assume $e \neq 0$ for otherwise $H = \mathbb{Z}^3$ which may be seen to be residually PSL_2 using the projections onto each factor. Let $R = \mathbb{C}[x, y, z]/(x(1 - y^2)^2, yz - 1)$ and define $\rho : H \to \mathrm{SL}_2(R)$ as follows

$$a \mapsto \begin{pmatrix} 1 & \overline{x} \\ 0 & 1 \end{pmatrix} \quad b \mapsto \begin{pmatrix} \overline{y} & 0 \\ 0 & \overline{z} \end{pmatrix} \quad c \mapsto \begin{pmatrix} 1 & \overline{e^{-1}x(1-y^2)} \\ 0 & 1 \end{pmatrix}.$$

It can be checked that the defining relations of H are indeed satisfied and also that $x(1 - y^2) \notin I$ (to see a verification via SageMath, see A.3 in the Appendix) so that

$\rho(c) \neq 1$. Similarly, as

$$\begin{pmatrix} 1 & e^{-1}x(1-y^2) \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & ne^{-1}x(1-y^2) \\ 0 & 1 \end{pmatrix}$$

we see that $\rho(c^n) \neq 1$ for all $n \neq 0$. Because H is finitely-generated, we may use Proposition 4.7 as it was used in the proof of Proposition 4.8 to reduce the ring of entries to a finite ring and then we may also projectivise without killing $\rho(c^n)$ because an off-diagonal entry is non-zero. □

4.6. Seifert fibred spaces

We have seen that Luo's conjecture holds for S^1-bundles over surfaces. These are examples of Seifert fibred spaces. The following is a well-known result, proven in [**17**], on such spaces.

Proposition 4.23. *Every Seifert fibred 3-manifold is finitely covered by an S^1-bundle over an orientable, compact, connected surface.*

Corollary 4.24. *Seifert fibred 3-manifold groups are virtually residually PSL_2.*

Proof. This follows from Propositions 4.22 and 4.23. □

References

[1] Armstrong, M.A., *Basic Topology*, Springer, 1997

[2] Armstrong, M.A., *Calculating the Fundamental Group of an Orbit Space*, Proceedings of the American Mathematical Society, Vol. 84, No. 2 (Feb., 1982), pp. 267-271

[3] Baumslag, G., *Lecture notes on nilpotent groups*, Regional Conference Series in Mathematics, No. 2 American Mathematical Society, Providence, R.I. 1971

[4] Boileau, M., Leeb, B., Porti, J., *Geometrization of 3-dimensional orbifolds*, Annals of Mathematics, 162 (2005), 195-290

[5] Burton, B.A. *Regina: Normal surface and 3–manifold topology software*, `http://regina.sourceforge.net/`, 1999–2009.

[6] Culler, M., Dunfield, N. M., Weeks, J.R., *SnapPy*, a computer program for studying the geometry and topology of 3-manifolds, `http://snappy.computop.org`

[7] Friedl, S., *A Conjecture of Feng Luo*, unpublished manuscript

[8] Friedl, S., Schreve, K., Tillmann, S., *Thurston norm via Fox calculus*, arXiv:1507.05660 [math.GT]

[9] Hatcher, A. *Algebraic Topology*, Cambridge University Press, 2001

[10] Hempel, *Residual Finiteness for 3-Manifolds*, 1987

[11] http://mathoverflow.net/questions/61661/residual-finiteness-of-fundamental-group-of-compact-3-manifold

[12] http://www.math.rutgers.edu/ fluo/problemlist1.pdf

[13] Kirby, Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull AMS 75 (1969)

[14] Luo, F., *Solving Thurston Equation in a Commutative Ring*, arXiv:1201.2228

[15] Luo, F., Tillmann, S., Yang, T., *Thurston's Spinning Construction and Solutions to the Hyperbolic Gluing Equations for Closed Hyperbolic 3-Manifolds*, Proceedings of the American Mathematical Society 141 (2013), no. 1, 335-350

[16] Magnus, W. *Noneuclidean tesselations and their groups*, Academic Press, New York, 1974.

[17] Matthias Aschenbrenner, Stefan Friedl, Henry Wilton, *3-manifold groups*, arXiv:1205.0202

[18] Matura, Philip, *Linearität von 3-Mannigfaltigkeitsgruppen*, Bachelorarbeit at Universität zu Köln

[19] Moise, E.E., *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung*, Annals of Mathematics. Second Series 56: 96-114, 1952

[20] Newman, M., *A Note on Fuchsian Groups*, Illinois J. Math. 29 (1985), 682-686

[21] Ratcliffe, J.G., *Foundations of Hyperbolic Manifolds*, Springer, 2006

[22] Scott, P., *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), no. 5, 401-487.

[23] Shalen, P. *Representations of 3-manifold groups*, Handbook of Geometric Topology, pp. 955-1044 (2002)

[24] Stein, W.A. et al. *Sage Mathematics Software* (Version 6.4.1), The Sage Development Team, 2013, http://www.sagemath.org.

[25] Stillwell, J., *Classical Topology and Combinatorial Group Theory*, Springer, 1995

[26] Thurston, W., *Three-Dimensional Geometry and Topology*, Princeton University Press (January 17, 1997)

[27] Thurston, W., *The geometry and topology of 3-manifolds*, Princeton Univ. Math. Dept. (1978). Available from `http://msri.org/publications/books/gt3m/`.

[28] Rubinstein, J. Hyam, Tillmann, S., *Even triangulations of n-dimensional pseudo-manifolds*, arXiv:1406.1002 [math.GT]

Appendix

A.1

The following is SnapPy, [**6**], code which will produce a triangulation of the (4, 1)-Dehn filling, using the knot theoretic framing, of the figure-8 knot complement and save it to a file entitled "fig-eight-4-1.tri" which can be imported into Regina.

```
In [ 1 ]: M = Manifold('4_1')

In [ 2 ]: M.dehn_fill( (4,1) )

In [ 3 ]: N = M.filled_triangulation()

In [ 4 ]: N.save('fig-eight-4-1.tri')
```

To import this triangulation into Regina, [**5**], follow: File -> Import -> SnapPea triangulation.

A.2

The following is SageMath, [**24**], code verifying the claim about the element b^4 in Section 4.3. Here the polynomials $f1, f2, f3$ arise as determinants, $f4, f5, f6, f7$ from the relation $ca^2 = b^2$, $f8, f9, f10, f11$ from the relation $c^{-1}b = bc$ and $f12, f13, f14, f15$ from the relation $ac^{-1}a^{-1} = cac$.

```
R = PolynomialRing(ZZ,12,'ijklpqrsxywz')

R

   Multivariate Polynomial Ring in i,j,k,l,p,q,r,s,w,x,y,z over Integer
      Ring

i,j,k,l,p,q,r,s,w,x,y,z = R.gens()

f1 = i*l-k*j-1

f2 = p*s-r*q-1

f3 = w*z-y*x-1

f4 = w*i^2+w*j*k+x*i*k+x*l*k-p^2-q*r

f5 = w*i*j+w*l*j+x*j*k+x*l^2-p*q-s*q
```

```
f6 = y*i^2+y*j*k+z*i*k+z*l*k-r*p-r*s

f7 = y*i*j+y*l*j+z*j*k+z*l^2-q*r-s^2

f8 = p*w+q*y-p*z+x*r

f9 = p*x+q*z-q*z+x*s

f10 = r*w+s*y-w*r+p*y

f11 = r*x+s*z-w*s+y*q

f12 = w^2*i+w*j*y+x*k*w+x*l*y-i*z*l-i*x*k+j*y*l+j*w*k

f13 = w*i*x+w*j*z+k*x^2+x*l*z-y*j^2-j*w*i+i*z*j+x*i^2

f14 = y*i*w+j*y^2+z*k*w+z*l*y-k*z*l-x*k^2+y*l^2+l*w*k

f15 = y*i*x+y*j*z+z*k*x+l*z^2-l*y*j-l*w*i+k*z*j+k*x*i

I = (f1,f2,f3,f4,f5,f6,f7,f8,f9,f10,f11,f12,f13,f14,f15)*R

I

  Ideal (i*l-k*j-1, p*s-r*q-1, w*z-y*x-1,
     w*i^2+w*j*k+x*i*k+x*l*k-p^2-q*r, w*i*j+w*l*j+x*j*k+x*l^2-p*q-s*q,
     y*i^2+y*j*k+z*i*k+z*l*k-r*p-r*s, y*i*j+y*l*j+z*j*k+z*l^2-q*r-s^2,
     p*w+q*y-p*z+x*r, p*x+q*z-q*z+x*s, r*w+s*y-w*r+p*y,
     r*x+s*z-w*s+y*q, w^2*i+w*j*y+x*k*w+x*l*y-i*z*l-i*x*k+j*y*l+j*w*k,
     w*i*x+w*j*z+k*x^2+x*l*z-y*j^2-j*w*i+i*z*j+x*i^2,
     y*i*w+j*y^2+z*k*w+z*l*y-k*z*l-x*k^2+y*l^2+l*w*k,
     y*i*x+y*j*z+z*k*x+l*z^2-l*y*j-l*w*i+k*z*j+k*x*i) of Multivariate
     Polynomial Ring in i,j,k,l,p,q,r,s,w,x,y,z over Integer Ring

(p^2+qr)^2+qr(p+s)(p+s)-1 in I

  True

q(p^2+qr)(p+s)+q(p+s)(qr+s^2) in I

  True

r(p+s)(p^2+qr)+r(qr+s^2)(p+s) in I

  True

qr(p+s)(p+s)+(qr+s^2)^2-1 in I

  True
```

A.3

The following is SageMath, [**24**], code verifying the claim in the proof of Proposition 4.22 that $x(1-y^2) \notin I$ where I is the ideal $(x(1-y^2)^2, yz-1)$ in $\mathbb{C}[x, y, z]$.

```
R = PolynomialRing(CC,3,'xyz')

R

   Multivariate Polynomial Ring in x,y,z over Complex Field with 53
      bits of precision

x,y,z = R.gens()

I = (x(1-y^2)^2,yz-1)*R

I

   Ideal (x*y^4 + (-2.00000000000000)*x*y^2+x,y*z-1.00000000000000) of
      Multivariate Polynomial Ring in x,y,z over Complex Field with 53
      bits of precision

x(1-y^2) in I

   False
```