JOURNAL OF MATHEMATICAL PHYSICS **49**, 042302 (2008)

Noncommutative fields and actions of twisted Poincaré algebra

M. Chaichian,[1] P. P. Kulish,[1,2] A. Tureanu,[1,a)] R. B. Zhang,[3] and Xiao Zhang[4]

[1]*Department of Physics, University of Helsinki and Helsinki Institute of Physics, P.O. Box 64, 00014 Helsinki, Finland*

[2]*St. Petersburg Department of Steklov Institute of Mathematics, Fontanka, 27, St. Petersburg 191023, Russia*

[3]*School of Mathematics and Statistics, University of Sydney, Sydney, New South Wales 2006, Australia*

[4]*Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences 100190, Beijing, People's Republic of China*

(Received 20 December 2007; accepted 14 March 2008; published online 22 April 2008)

Within the context of the twisted Poincaré algebra, there exists no noncommutative analog of the Minkowski space interpreted as the homogeneous space of the Poincaré group quotiented by the Lorentz group. The usual definition of commutative classical fields as sections of associated vector bundles on the homogeneous space does not generalize to the noncommutative setting, and the twisted Poincaré algebra does not act on noncommutative fields in a canonical way. We make a tentative proposal for the definition of noncommutative classical fields of any spin over the Moyal space, which has the desired representation theoretical properties. We also suggest a way to search for noncommutative Minkowski spaces suitable for studying noncommutative field theory with deformed Poincaré symmetries.
 [DOI: 10.1063/1.2907580]

I. INTRODUCTION

There have been intensive research activities in quantum field theory on noncommutative spaces (see, e.g., Refs. 1 and 2 and references therein) in recent years. All aspects of noncommutative quantum field theory on the Moyal space have been studied, which include foundational issues, renormalization, as well as model building for particle physics. We mention, in particular, that noncommutative quantum field theories behave very differently from their commutative counterparts, as can be seen, e.g., from the UV/IR mixing[3] appearing in the noncommutative case.

A major conceptual advance was the recognition[4] that the twisted Poincaré algebra should play the same role in noncommutative quantum field theory on the Moyal space as that played by the Poincaré group in usual relativistic quantum field theory. The merit of the twisted Poincaré symmetry of the noncommutative quantum field theory (NC QFT) is that its particle representations are identical to the ones of the usual Poincaré symmetry since the structure of the twisted Poincaré algebra is identical to the one of the Poincaré algebra and, hence, the Casimir operators are the same. As a result, the particle states of NC QFT are classified according to their mass and spin[4] as ordinary. The study of the consequences of this twisted Poincaré symmetry [1] has increased

[1]Recently several papers[6–8] claimed that twisted Poincaré invariant noncommutative quantum field theory on the Moyal space had the same S-matrix as its commutative counterpart. This is very surprising in view of the drastic differences between the commutative and noncommutative theories.

[a)]Electronic mail: anca.tureanu@helsinki.fi.

the interest in the subject since the publication of Ref. 4. Attempts[5] have also been made to gauge the twisted Poincaré algebra in order to construct a noncommutative theory of general relativity. Other possible noncommutative spacetime symmetries have also been studied in the literature, e.g., the κ-Poincaré algebra.[9]

The twisted Poincaré invariance of noncommutative quantum field theory is an extremely important issue, which should be systematically investigated by starting from first principles. To consider it, one needs to have a representation theoretical interpretation of the fields and also a precise definition of the actions of the twisted Poincaré algebra on them. Unfortunately, neither is well understood, especially for fields with nonzero spin (of course, we cannot give precise meanings to the terms "fields" and "spin" yet). In the literature, there is enough material for one to extract a general definition of a classical noncommutative scalar field on the Moyal space and specify the precise transformation rule for it under the twisted Poincaré algebra (see, e.g., Ref. 10 and also later treatments by other authors). However, there is hardly any discussion on what fields with nonzero spin should be, leaving alone any precise formulation, from first principles, of their transformation rules under the twisted Poincaré algebra. Some researchers are aware of aspects of this problem. For example, there was a lengthy discussion in Ref. 6 on the need of formulating a transformation rule of fields under the twisted Poincaré algebra, but the authors did not directly address the issue, rather they suggested a way to side step it instead. Also, Ref. 11 aimed at addressing similar issues for general Hopf algebras.

For simplicity, we shall consider twisted Poincaré invariance of noncommutative classical field theory. Recall that in the commutative setting, Minkowski space is realized as the quotient of the Poincaré group by the Lorentz group, and a classical field is a section of a vector bundle induced by some representation of the Lorentz group (or its double cover). The space of sections of the bundle, which is the well-known induced module, carries a natural action of the entire Poincaré group. One would expect that the Moyal space and noncommutative fields on it should be understood in such terms as well.

We shall carefully examine the induced module construction in Sec. II, and then investigate the possibility of generalizing it to the twisted Poincaré algebra in Sec. III A. Unfortunately, we find that the natural generalization does not go through, primarily because the universal enveloping algebra of the Lorentz Lie algebra is not a Hopf subalgebra of the twisted Poincaré algebra. We shall explain in detail the obstacle preventing the generalization in the second half of Sec. III A. To further illustrate the problems, we examine in Sec. III C the two noncommutative algebras in the literature, which are closely related to the Moyal space and arise from the representation theory of the twisted Poincaré algebra, and explain why they are not useful for defining classical fields. These rather unexpected difficulties indicate that one cannot use the same canonical definitions in the case of noncommutative fields on the Moyal space to address the representation theoretical properties relative to the twisted Poincaré algebra.

One could, however, approach the problem differently. In Sec. IV, we propose a definition of noncommutative classical fields, which agrees with what noncommutative scalar fields were implicitly taken to be in the literature (see, e.g., Ref. 10) and recovers the usual definition of scalar fields in the commutative case. We hope that the proposal will provide a useful framework for studying twisted Poincaré invariance of quantum field theory on the Moyal space.

A further useful aspect of results in this paper is that they provide a theoretical basis for the search of noncommutative analogs of the Minkowski space, which are suitable for studying noncommutative field theory with deformed Poincaré symmetries. We shall discuss this point in more detail in Sec. V.

Before closing this section, we mention that we shall limit ourselves to the case where the noncommutative fields carry no internal degrees of freedom. This enables us to better focus on properties of noncommutative fields relative to the twisted Poincaré algebra. All results of this paper can be generalized to include internal degrees of freedom in a straightforward manner.

II. INDUCED MODULES OF POINCARÉ GROUP AND CLASSICAL FIELDS

We review the basic definition of Poincaré group actions on commutative classical fields. This material is needed later when we investigate the possibilities/difficulties of generalizing it to the twisted Poincaré algebra.

Choose the metric $\eta = \mathrm{diag}(-1,1,1,1)$ for $\mathbb{R}^{1,3}$. Let G denote the Poincaré group, which is the semidirect product of the Lorentz group and the Abelian group of translations on $\mathbb{R}^{1,3}$, where the Lorentz group is defined with respect to the metric η. Since spinor fields should be included in the framework as well, we consider, instead, the covering group $\widetilde{G} = \mathrm{Spin}(1,3) \ltimes \mathbb{R}^{1,3}$, the semi-direct product of Spin(1, 3) with the group of translations. Now Spin(1, 3) acts on translations via the surjection π from Spin(1, 3) to the Lorentz group. For convenience, we shall denote Spin(1, 3) by L.

Denote the coordinate of $\mathbb{R}^{1,3}$ by $x = (x^0, x^1, x^2, x^3)$. For computational purposes, it is the best to write an element of $\widetilde{G}$ as $\Lambda \exp(iPx)$, where $\Lambda \in L$ and $x \in \mathbb{R}^{1,3}$, with the product of two elements $\Lambda \exp(iPx)$ and $\Lambda' \exp(iPy)$ given by

$$\Lambda' \exp(iPy) \Lambda \exp(iPx) = \Lambda' \Lambda \exp(iP(\Lambda^{-1}(y) + x)).$$

Here, $\Lambda^{-1}(y)^\mu = \pi(\Lambda^{-1})^\mu_\nu y^\nu$.

Let $C^\infty(\widetilde{G})$ be the *set* of smooth functions on $\widetilde{G}$. In the present (untwisted) case, it forms a commutative algebra under the usual pointwise multiplication from calculus.

Remark 2.1: The pointwise multiplication of functions on $\widetilde{G}$ is intimately related to the fact that we give the group algebra the cocommutative comultiplication

$$\Delta_0(g) = g \otimes g,$$

for all elements g in the group. This comultiplication is compatible with the standard cocommutative comultiplication (3.2) for the Poincaré algebra.

There are two natural actions of $\widetilde{G}$ on $C^\infty(\widetilde{G})$, the left and right translations, which we shall denote by $\mathcal{L}$ and $\mathcal{R}$, respectively. For any $\phi \in C^\infty(\widetilde{G})$ and $g \in \widetilde{G}$, $\mathcal{L}_g(\phi)$ and $\mathcal{R}_g(\phi)$ are respectively, defined by

$$\mathcal{L}_g(\phi)(g_1) = \phi(g^{-1} g_1), \quad \mathcal{R}_g(\phi)(g_1) = \phi(g_1 g), \quad \forall\, g_1 \in \widetilde{G}.$$

It is these actions that give rise to actions of the Poincaré group on classical fields. We shall carefully examine this point now with the view of possible generalizations to the noncommutative case.

To discuss properties of classical fields on $\mathbb{R}^{1,3}$ in relation to the Poincaré group, we first note that $\mathbb{R}^{1,3} \cong \widetilde{G}/L$. At this point, we need to make a choice in interpreting this either as a left or right coset space. We shall take $\widetilde{G}/L$ as the right coset space consisting of equivalence classes with the following equivalence relation: $\Lambda \exp(iPx) \sim \Lambda' \exp(iPx)$, for all $\Lambda, \Lambda' \in L$.

Let V be a finite dimensional L-module and denote by ρ the representation of L on V relative to some choice of basis. Then, a classical field of a type characterized by V is a section of the associated C^∞ vector bundle $\widetilde{G} \times_L V \to \widetilde{G}/L$. Denote by $\Gamma(V)$ the space of the smooth sections of this vector bundle, which is a subspace of $C^\infty(\widetilde{G}) \otimes_{\mathbb{C}} V$, where the latter vector space is endowed with an action of L defined for any $\phi \otimes v \in C^\infty(\widetilde{G}) \otimes_{\mathbb{C}} V$ by

$$\Lambda(\phi \otimes v) = (\mathcal{L} \otimes \rho)\Delta_0(\Lambda)(\phi) = \mathcal{L}_\Lambda(\phi) \otimes \rho(\Lambda) v, \quad \forall\, \Lambda \in L.$$

Then, $\Gamma(V)$ is the subspace of invariants of $C^\infty(\widetilde{G}) \otimes_{\mathbb{C}} V$ with respect to this L-action, that is,

$$\Gamma(V) = (C^{\infty}(\tilde{G}) \otimes_{\mathbb{C}} V)^{L}. \tag{2.1}$$

As is well known, the space $\Gamma(V)$ of sections forms a module, the induced module, for the entire Poincaré group $\tilde{G}$ with the group action defined by

$$g(\Phi) = (\mathcal{R}_g \otimes \mathrm{id}_V)\Phi, \quad g \in \tilde{G}, \quad \Phi \in \Gamma(V). \tag{2.2}$$

Remark 2.2: Note that in formulating (2.1), it is of crucial importance that the group algebra of L is a Hopf subalgebra of the group algebra of $\tilde{G}$ under the comultiplication Δ_0.

Both Eqs. (2.1) and (2.2) can be made more explicit. If $\Phi \in \Gamma(V)$, then (2.1) implies that

$$(\mathcal{L}_{\Lambda} \otimes \mathrm{id}_V)\Phi = (\mathrm{id} \otimes \rho(\Lambda^{-1}))\Phi, \quad \forall\, \Lambda \in L, \tag{2.3}$$

where the id on the right side is the identity map on $C^{\infty}(\tilde{G})$. Therefore,

$$\Phi(\Lambda \exp(iPx)) = \rho(\Lambda)\Phi(\exp(iPx)).$$

Denote

$$\phi(x) = \Phi(\exp(iPx)), \quad (g \cdot \phi)(x) = (g(\Phi))(\exp(iPx)), \quad g \in \tilde{G}.$$

Then, by using (2.3), we easily see (with notations as above) that Eq. (2.2) is equivalent to

$$(\Lambda \exp(iPa) \cdot \phi)(x) = \rho(\Lambda)\phi(\Lambda^{-1}x + a), \quad \Lambda \exp(iPa) \in \tilde{G}. \tag{2.4}$$

This is the familiar transformation rule for a classical field on $\mathbb{R}^{1,3}$ under the action of the Poincaré group. The type of a classical field is determined by V. For example, the field is a vector if V is the natural module for the Lorentz group and a spinor if V is a spinor module.

Let V and V' be L-modules. If $f: V \to V'$ is an L-module homomorphism, it induces a morphism between the associated vector bundles,

$$f_*: \Gamma(V) \to \Gamma(V'), \quad \Phi \mapsto (\mathrm{id} \otimes f)\Phi.$$

Also note that the multiplication of $\mathbb{C}^{\infty}(\tilde{G})$ induces a tensor product map for the associated vector bundles,

$$\Gamma(V) \otimes \Gamma(V') \to \Gamma(V \otimes V'), \tag{2.5}$$

defined for $\Phi = \Sigma_i \phi_i \otimes v_i \in \Gamma(V)$ and $\Psi = \Sigma_j \psi_j \otimes v'_j \in \Gamma(V')$ by

$$\Phi \otimes \Psi \mapsto \Phi\Psi = \sum_{i,j} \phi_i \psi_j \otimes v_i \otimes v'_j.$$

By a direct computation, one can show that the right hand side, indeed, belongs to $\Gamma(V \otimes V')$. There is an obvious generalization of the map to more than two bundles.

Given a classical field $\Phi \in \Gamma(V)$, we may consider, say, $\Phi^k \in \Gamma(V^{\otimes k})$. If there exists a module map f from $V^{\otimes k}$ to the one-dimensional trivial L-module $\mathbb{C}$, then $f_*(\Phi^k)$ is a complex valued function on $\mathbb{R}^{1,3}$. Then, for all $\Lambda \exp(iPa) \in \tilde{G}$,

$$\int dx (\Lambda \exp(iPa) \cdot (f_*(\Phi^k)))(x) = \int dx (f_*(\Phi^k))(\Lambda^{-1}x + a) = \int dx (f_*(\Phi^k))(x),$$

that is, the integral $\int dx (f_*(\Phi^k))(x)$ [which means $\int dx (f_*(\Phi^k))(\exp(iPx))$] is Poincaré invariant. The construction of the invariant integral can obviously generalize to the case with more than one classical field, which can be sections of different vector bundles on $\tilde{G}/L$ (derivatives of a section are considered as a section of different vector bundles). This is how one constructs Poincaré invariant Lagrangians in classical field theory.

Remark 2.3: Unitarity of the induced module $\Gamma(V)$ *is required in order to have a sensible field theory.*

III. INDUCED MODULES FOR THE TWISTED POINCARÉ ALGEBRA

A. Generalities on induced modules for the twisted Poincaré algebra

In this section, we shall first discuss induced modules of the twisted Poincaré algebra in general terms, then explain the obstruction preventing the generalization of the constructions of Sec. II to the noncommutative setting.

There are detailed treatments of induced modules for quantum groups in the literature, see, e.g., Refs. 12 and 13. Two equivalent approaches were followed by using the languages of comodules[12] and modules.[13] Our discussion below is presented in terms of modules.

Let $\mathfrak{g}$ be the complexification of the Lie algebra Lie$(\widetilde{G})$ of the Poincaré group. Then, $\mathfrak{g}=\mathfrak{l}+\mathfrak{p}$, where $\mathfrak{l}$ is the complexification of the Lie algebra of the Lorentz group and $\mathfrak{p}$ is the complexification of the Lie algebra of the group of translations on $\mathbb{R}^{1,3}$. A basis for $\mathfrak{g}$ is $\{J_{\mu\nu},P_\mu|\mu,\nu=0,1,2,3\}$ with the following commutation relations:

$$[J_{\mu\nu},J_{\sigma\rho}]=\frac{1}{i}(\eta_{\nu\sigma}J_{\mu\rho}-\eta_{\mu\sigma}J_{\nu\rho}-\eta_{\nu\rho}J_{\mu\sigma}+\eta_{\mu\rho}J_{\nu\sigma}),$$

$$[J_{\mu\nu},P_\sigma]=i\eta_{\mu\sigma}P_\nu-i\eta_{\nu\sigma}P_\mu, \tag{3.1}$$

$$[P_\mu,P_\nu]=0.$$

We denote by $\mathcal{U}$ the universal enveloping algebra of the Poincaré algebra $\mathfrak{g}$. The standard cocommutative comultiplication Δ_0 is given by

$$\Delta_0(X)=X\otimes 1+1\otimes X, \qquad \forall\, X\in\mathfrak{g}. \tag{3.2}$$

The twisted Poincaré algebra is the associative algebra $\mathcal{U}$ equipped with a twisted comultiplication defined in the following way. Let $\theta=(\theta^{\mu\nu})$ be a real 4×4 skew symmetric matrix. Set

$$\mathcal{F}=\exp\left(\sum_{\mu,\nu}\frac{1}{2}i\theta^{\mu\nu}P_\mu\otimes P_\nu\right),$$

which is understood as belonging to some appropriate completion of $\mathcal{U}\otimes\mathcal{U}$. The twisted comultiplication is then defined by

$$\Delta:\mathcal{U}\to\mathcal{U}\otimes\mathcal{U}, \quad u\mapsto\mathcal{F}\Delta_0(u)\mathcal{F}^{-1}, \tag{3.3}$$

which is, indeed, coassociative as can be easily shown. Now, for any $\omega\in\mathfrak{l}$ and $P\in\mathfrak{p}$,

$$\Delta(\omega)=\omega\otimes 1+1\otimes\omega-\frac{1}{2}\sum i\theta^{\mu\nu}([\omega,P_\mu]\otimes P_\nu+P_\mu\otimes[\omega,P_\nu]),$$

$$\Delta(P)=P\otimes 1+1\otimes P. \tag{3.4}$$

If we also define the counit ϵ and antipode S, respectively, by $\epsilon(1)=1$, $\epsilon(X)=0$, and $S(X)=-X$ for all $X\in\mathfrak{g}$, then $\mathcal{U}$ is a Hopf algebra with comultiplication Δ.

It is worth mentioning that the $\mathcal{F}$ used to twist Δ_0 to obtain the new comultiplication Δ is an example of a special type of gauge transformations in the powerful theory of quasi-Hopf algebras[14,15] (see also Ref. 16). We refer to Ref. 17 for more details on twisting comultiplications.

The coalgebra structure of $\mathcal{U}$ induces a natural associative algebra structure on the dual space $\mathcal{U}^*$ of $\mathcal{U}$. Since Δ is clearly noncocommutative, $\mathcal{U}^*$ is noncommutative. Note that $\mathcal{U}^*$ is a huge object, which contains $C^\infty(\widetilde{G})$ as a subspace in some appropriate sense. There exist two left actions $\mathcal{L},\mathcal{R}:\mathcal{U}\otimes\mathcal{U}^*\to\mathcal{U}^*$, respectively, defined for any $f\in\mathcal{U}^*$ and $u\in\mathcal{U}$ by

$$\mathcal{L}_u(f)(w) = f(S^{-1}(u)w), \quad \mathcal{R}_u(f)(w) = f(wu), \quad \forall\, w \in \mathcal{U}. \tag{3.5}$$

Let $\mathcal{A}(\mathfrak{g})$ be either $\mathcal{U}^*$ itself or an appropriate subalgebra of it. In the latter case, we require that for any nonzero $u\in\mathcal{U}$, there exists some $a\in\mathcal{A}(\mathfrak{g})$ such that $a(u)\neq 0$. Also, $\mathcal{A}(\mathfrak{g})$ should be stable under both the left and right translations, that is, $\mathcal{L}_u(\mathcal{A}(\mathfrak{g})),\mathcal{R}_u(\mathcal{A}(\mathfrak{g}))\subset\mathcal{A}(\mathfrak{g})$ for all $u\in\mathcal{U}$. The algebra $\mathcal{A}(\mathfrak{g})$ will be taken as defining some noncommutative space following the general philosophy of noncommutative geometry.[19]

Let $\mathcal{C}$ be a two-sided coideal of $\mathcal{U}$ satisfying $c(1)=0$ for all $c\in\mathcal{C}$, where 1 is the identity element of $\mathcal{U}$. Being a two-sided coideal means that $\Delta(\mathcal{C})\subset\mathcal{C}\otimes\mathcal{U}+\mathcal{U}\otimes\mathcal{C}$. Now, define

$$\mathcal{A}(\mathfrak{g},\mathcal{C}) \coloneqq \{f \in \mathcal{A}(\mathfrak{g}) | \mathcal{L}_c(f) = 0,\ \forall\, c \in \mathcal{C}\}. \tag{3.6}$$

Then, $\mathcal{A}(\mathfrak{g},\mathcal{C})$ is a subalgebra of $\mathcal{A}(\mathfrak{g})$. The proof of this is quite illuminating. If $f,g\in\mathcal{A}(\mathfrak{g},\mathcal{C})$, then for all $c\in\mathcal{C}$ and $u\in\mathcal{U}$, we have

$$\mathcal{L}_c(fg)(u) = (f\otimes g)\Delta(S^{-1}(c)u) = \sum_{(c),(u)} \mathcal{L}_{c_{(2)}}(f)(u_{(1)})\mathcal{L}_{c_{(1)}}(g)(u_{(2)}),$$

where we have used Sweedler's notation[19] for the comultiplications of c and u. Since $\mathcal{C}$ is a two-sided coideal, we have

$$\mathcal{L}_c(fg)(u) = 0, \quad \forall\, u \in \mathcal{U}.$$

The algebra $\mathcal{A}(\mathfrak{g},\mathcal{C})$ is taken as defining a noncommutative analog of some homogeneous space of $\widetilde{G}$.

As far as we are aware, this is the definition of noncommutative homogeneous spaces that requires the weakest conditions on $\mathcal{C}$. If we also want to develop a theory of induced representations similar to that in the setting of Lie groups, we need to impose the stronger condition that $\mathcal{C}$ generates a Hopf subalgebra of $\mathcal{U}$.

Now, we make the assumption that $\mathcal{C}$ generates a Hopf subalgebra $\mathcal{H}$ of $\mathcal{U}$. Then,

$$\mathcal{A}(\mathfrak{g},\mathcal{C}) = \mathcal{A}(\mathfrak{g})^{\mathcal{L}_{\mathcal{H}}},$$

which is the subalgebra of $\mathcal{A}(\mathfrak{g})$ consisting of the $\mathcal{H}$ invariant elements. Let V be a finite dimensional $\mathcal{H}$-module. We define the vector space

$$\Gamma(V) \coloneqq \left\{\zeta \in \mathcal{A}(\mathfrak{g})\otimes_{\mathbb{C}}V \Big| \sum_{(u)} (\mathcal{L}_{u_{(1)}} \otimes u_{(2)})\zeta = \epsilon(u)\zeta,\ \forall\, u \in \mathcal{H}\right\}, \tag{3.7}$$

where we have used Sweedler's notation $\Delta(u)=\Sigma_{(u)}u_{(1)}\otimes u_{(2)}$ for the comultiplication of u. Then, $\Gamma(V)$ is a two-sided $\mathcal{A}(\mathfrak{g},\mathcal{C})$-module under the multiplication in $\mathcal{A}(\mathfrak{g})$: for any $a\in\mathcal{A}(\mathfrak{g},\mathfrak{l})$ and $\zeta=\Sigma\phi_i\otimes v_i\in\Gamma(V)$, both

$$a\zeta = \sum a\phi_i \otimes v_i \quad \text{and} \quad \zeta a = \sum \phi_i a \otimes v_i$$

belong to $\Gamma(V)$.

Remark 3.1: Both the definition of $\Gamma(V)$ *and its* $\mathcal{A}(\mathfrak{g},\mathfrak{l})$*-module structures rely in a crucial way on the Hopf algebra structure of* $\mathcal{H}$.

It is important to observe that $\Gamma(V)$ forms a left $\mathcal{U}$-module under the action

$$\mathcal{U} \otimes \Gamma(V) \to \Gamma(V), \quad u \otimes \zeta \mapsto (\mathcal{R}_u \otimes \mathrm{id}_V)\zeta. \tag{3.8}$$

Furthermore, if V and V' are both $\mathcal{H}$-modules, then there exists a map

$$\Gamma(V) \otimes \Gamma(V') \to \Gamma(V \otimes V') \tag{3.9}$$

defined in exactly the same way as (2.5).

Remark 3.2: By imposing appropriate conditions on the algebra $\mathcal{A}(\mathfrak{g})$, *we can reproduce the results in* Sec. II *this way by using the usual comultiplication* Δ_0 for $\mathcal{U}$.

B. Difficulties with representation theoretical interpretation of noncommutative fields

In order for the twisted Poincaré algebra to play a similar role in noncommutative field theory as that played by the Poincaré group in commutative field theory, it appears to be quite necessary to have a noncommutative analog of the construction of induced representations given in Sec. II. It is that construction that provides the definition of classical fields on $\mathbb{R}^{1,3}$ and also specifies the action of the Poincaré group on them.

If we wish to generalize Sec. II to the noncommutative setting, we have to take such a subspace $\mathcal{C}$ of $\mathcal{U}$ that contains $\mathfrak{l}$ but not any nontrivial subspace of $\mathfrak{p}$. However, in this case, $\mathcal{C}$ *cannot* be a two-sided coideal as one can easily see by inspecting the comultiplication (3.4). For example, the natural choice $\mathcal{C}=\mathfrak{l}$ does not give us a two-sided coideal. It still makes sense to define $\mathcal{A}(\mathfrak{g},\mathfrak{l})=\mathcal{A}(\mathfrak{g},\mathcal{C})$ in this case; however, $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ will not be a subalgebra of $\mathcal{U}^*$ since the universal enveloping of $\mathfrak{l}$ is not a Hopf subalgebra of $\mathcal{U}$.

This means that conceptually, we cannot regard $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ as defining a noncommutative geometry. An immediate practical problem caused by this is the following. If we follow the type of thinking in Sec. II, we would like to interpret elements of $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ as a "scalar field." Since $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ is not an algebra, we do not know how to multiply two "fields" (or a "field" with itself) together.

Now, we consider the induced module construction. If V is merely an $\mathfrak{l}$-module, then the corresponding $\Gamma(V)$ as that in (3.7) cannot be defined. One way out is to take V to be a $\mathcal{U}$-module with a trivial $\mathfrak{p}$ action. Then, at least, we can define a $\Gamma(V)$ by (3.7). Now, the map (3.9) is not defined. Therefore, we cannot simply generalize the classical construction to build Lagrangians from elements of $\Gamma(V)$ and defining Wightman functions in the corresponding quantum theory.

In summary, there does not exist a noncommutative analog of $\mathbb{R}^{1,3}=\widetilde{G}/L$ in terms of the twisted Poincaré algebra, and the induced module construction for the Poincaré group in Sec. II cannot be generalized to the twisted setting.

Therefore, one cannot generalize the canonical definition of actions of the Poincaré group on classical fields to the noncommutative setting. Note that a similar situation is encountered in the case of the κ-Poinincaré algebra,[9] for the same reason that the enveloping algebra of the Lorentz subalgebra is not a Hopf subalgebra.

C. Representation theoretical constructs related to Moyal space

There are two noncommutative algebras in the literature, which arise from the representation theory of the twisted Poincaré algebra and are related to the Moyal space. We discuss difficulties which one encounters when trying to take any of these algebras as the algebra of functions on some noncommutative space and develops field theory on it. There are also various inaccurate statements concerning the relationship between these algebras and the Moyal space in the literature, which we hope to clarify here. We should mention that it is not hard to deduce the material below from appropriate mathematical sources, e.g., Ref. 19.

1. A module algebra

Consider an indecomposable module $V=X\oplus\mathbb{C}1$ for the Poincaré algebra $\mathcal{U}$, where $\mathbb{C}1$ is a one-dimensional submodule and $X=\oplus_{\mu=0}^{3}\mathbb{C}x^{\mu}$ forms the natural module for $\mathfrak{l}$. Explicitly, the $\mathcal{U}$ action on V is given by

$$J_{\mu\nu}(x^\sigma)=\frac{1}{i}(\delta^\sigma_\nu x_\mu-\delta^\sigma_\mu x_\nu),\quad P_\mu(x^\sigma)=\frac{1}{i}\delta^\sigma_\mu\mathbb{1},\quad Y(\mathbb{1})=0,\quad \forall\, Y\in\mathfrak{g}.$$

Let $T(V)$ be the tensor algebra of V. Then, $T(V)=\Sigma_{k=0}^{\infty}T(V)_k$ with $T(V)_k=V^{\otimes k}$ and $T(V)_0=\mathbb{C}$. Now, $T(V)$ has a natural $\mathcal{U}$-module structure with respect to the *twisted* comultiplication Δ.

Let $\omega=\frac{1}{2}\Sigma\,\omega^{\mu\nu}J_{\mu\nu}$ and $P=\Sigma c^\mu P_\mu$, where $\omega^{\mu\nu}$ and c^μ are complex numbers. Set $\omega^\mu_\nu=\Sigma_\sigma\omega^{\mu\sigma}\eta_{\sigma\nu}$. For the following elements of $V\otimes V$:

$$A^{\mu\nu}:=x^\mu\otimes x^\nu-x^\nu\otimes x^\mu-i\theta^{\mu\nu}\mathbb{1}\otimes\mathbb{1},$$

$$V^\mu:=x^\mu\otimes\mathbb{1}-\mathbb{1}\otimes x^\mu,$$

we have

$$\Delta(\omega)A^{\mu\nu}=i\sum_\sigma(\omega^\mu_\sigma A^{\sigma\nu}-i\omega^\nu_\sigma A^{\sigma\mu}),\quad \Delta(P)A^{\mu\nu}=-i(c^\mu V^\nu-c^\nu V^\mu),$$

$$\Delta(\omega)V^\mu=i\sum_\sigma\omega^\mu_\sigma V^\sigma,\quad \Delta(P)V^\mu=0.$$

Also observe that the element $1-\mathbb{1}\in T(V)_0\oplus T(V)_1$ is an invariant. Therefore, the two-sided ideal $\mathcal{I}$ of $T(V)$ generated by all $A^{\mu\nu}$, V^μ, and $1-\mathbb{1}$ is a $\mathcal{U}$-submodule with respect to the twisted comultiplication. Define the unital associative algebra

$$\mathcal{A}:=T(V)/\mathcal{I},\tag{3.10}$$

which admits a natural action of the twisted Poincaré algebra $\mathcal{U}$. The algebra $\mathcal{A}$ frequently appears in the literature. It may be regarded as generated by x^μ $(\mu=0,1,2,3)$ and the identity subject to the relation

$$x^\mu x^\nu-x^\nu x^\mu=i\theta^{\mu\nu}.\tag{3.11}$$

These are the same as the familiar relations satisfied by the coordinate functions of the Moyal space. However, one cannot simply assign numerical values to x^μ to obtain numbers from elements of $\mathcal{A}$. This fact prevents one from constructing field theory by directly using the algebra $\mathcal{A}$.

Let us now investigate the algebra $\mathcal{A}$ a little further. Since the ideal $\mathcal{I}$ is not homogeneous as the generator $1-\mathbb{1}$ is not, the $\mathbb{Z}_+$ grading of $T(V)$ does not descend to $\mathcal{A}$ but induces a filtration

$$\mathcal{A}_0\subset\mathcal{A}_1\subset\mathcal{A}_2\subset\cdots,$$

where $\mathcal{A}_i=T(V)_{\leqslant i}/(\mathcal{I}\cap T(V)_{\leqslant i})$ and $T(V)_{\leqslant i}=\Sigma_{k\leqslant i}T(V)_k$. Every $\mathcal{A}_i$ is obviously a $\mathcal{U}$-submodule; thus,

$$\mathrm{gr}\,\mathcal{A}_i:=\mathcal{A}_i/\mathcal{A}_{i-1}$$

admits a natural $\mathcal{U}$-action. Then, $\mathrm{gr}\,\mathcal{A}=\Sigma_i\mathrm{gr}\,\mathcal{A}_i$ is a graded algebra with the momentum operators P_μ acting on it by zero and the Lorentz generators $J_{\mu\nu}$ acting through the usual untwisted comultiplication Δ_0. Results from classical invariant theory of orthogonal groups state that the subalgebra of $\mathcal{U}$ invariants in $\mathrm{gr}\,\mathcal{A}$ is the polynomial algebra generated by the image $(X^2)_0\in\mathrm{gr}\,\mathcal{A}$ of the element

$$X^2:=\sum_{\mu,\nu}\eta_{\mu\nu}x^\mu x^\nu\in\mathcal{A}.$$

A simple calculation shows that $J_{\mu\nu}(X^2)=0$ for all μ and ν, and this in turn leads to $J_{\mu\nu}(X^2)^k=0$ for all k. Now, let $\mathcal{A}^0$ be the subset of $\mathcal{A}$ consisting of elements annihilated by all $J_{\mu\nu}$, that is,

$$\mathcal{A}^0 = \{\phi \in \mathcal{A} | J_{\mu\nu}(\phi) = 0, \forall \mu, \nu\}.$$

If $\phi \in \mathcal{A}^0$ belongs to $\mathcal{A}_i$ but not to $\mathcal{A}_{i-1}$, then its image in gr $\mathcal{A}$ is a polynomial of degree i in the variable $(X^2)_0$. Then, there exists some complex number c such that $\phi - c(X^2)^k \in \mathcal{A}_{i-1} \cap \mathcal{A}^0$. By induction on i, we can show that $\mathcal{A}^0$, in fact, consists of polynomials in X^2. Therefore, we have the following result: the set $\mathcal{A}^0$ of Lorentz invariant elements of $\mathcal{A}$ consists of polynomials in X^2 and thus forms a subalgebra of $\mathcal{A}$.

Remark 3.3: One may think that the result is intuitively clear but, in fact, this is far from the truth because the first fundamental theorem of invariant theory breaks down in the present situation as the algebra $\mathcal{A}$ is noncommutative. Therefore, the result is quite interesting mathematically from the point of view of invariant theory.

2. Algebra generated by matrix elements of a representation

Let us now consider the subalgebra of $\mathcal{U}^*$ generated by the matrix elements of the representation of $\mathcal{U}$ associated with the module V. We shall denote this algebra by $\mathcal{A}(\mathfrak{g})$.

Order the basis elements of V as $x^0, x^1, x^2, x^3, \mathbb{1}$ and denote $\mathbb{1}$ by x^4. Consider the matrix elements t^a_b $(a,b=0,1,\ldots,4)$ of the representation of $\mathcal{U}$ furnished by the module V relative to this basis. Here, $t^a_b \in \mathcal{U}^*$ such that for any $u \in \mathcal{U}$, $ux^a = \sum_{b=0}^4 t^a_b(u)x^b$. From $ux^4 = \epsilon(u)x^4$, we obtain $t^4_4 = \epsilon$, the counit of $\mathcal{U}$. Also note that $t^4_\mu = 0$ for $\mu = 0,1,2,3$. A further property of the matrix elements is that if $\mu, \nu \leqslant 3$,

$$t^\nu_\mu(uP_\sigma) = t^\nu_\mu(P_\sigma u) = 0, \quad \forall\, u \in \mathcal{U}.$$

Form the 5×5 matrix $t = (t^b_a)$, where a is the row index and b is the column index, and write $t(u) = (t^b_a(u))$ for any $u \in \mathcal{U}$. Then, $t(u)t(u') = t(uu')$ for all $u, u' \in \mathcal{U}$. Let $C^{\mu\nu} := \sum_{\sigma,\rho=1}^3 \eta_{\sigma\rho} t^\sigma_\mu t^\rho_\nu$. Then, $C^{\mu\nu}$ satisfies $C^{\mu\nu}(uP_\mu) = 0$ for all $u \in \mathcal{U}$. Also, $\sum_{\sigma,\rho=1}^3 \eta_{\sigma\rho} x^\sigma \otimes x^\rho$ is invariant under the action of the Lorentz subalgebra; thus, we conclude that

$$\sum_{\sigma,\rho=1}^{3} \eta_{\sigma\rho} t^\sigma_\mu t^\rho_\nu = \eta_{\mu\nu}\epsilon. \tag{3.12}$$

This is the familiar orthogonality relation satisfied by the matrix elements of the natural representation of the orthogonal group.

It is easy to show that the opposite comultiplication Δ' of $\mathcal{U}$ is related to Δ through

$$\mathcal{F}^{-2}\Delta = \Delta' \mathcal{F}^{-2},$$

where $\mathcal{F}^{-2}$ satisfies all the defining properties of a universal R-matrix. Thus, it follows that

$$t^b_a t^d_c - t^d_c t^b_a = -\delta^4_a \delta^4_c \left(i\theta^{bd}\epsilon - \sum_{\mu,\nu} i\theta^{\mu\nu} t^b_\mu t^d_\nu \right), \tag{3.13}$$

where $\theta^{bd} = 0$ if any of the indices is 4. Now, (3.13) is equivalent to the following relations:

$$t^\nu_\mu t^d_c = t^d_c t^\nu_\mu, \quad t^\mu_4 t^\nu_4 = t^\nu_4 t^\mu_4 - i\theta^{\mu\nu}\epsilon - \sum_{\sigma,\rho=0}^{3} i\theta^{\sigma\rho} t^\mu_\sigma t^\nu_\rho, \quad \mu,\nu \leqslant 3. \tag{3.14}$$

It follows from the first relation that the elements t^ν_μ $(\mu,\nu \leqslant 3)$ commute among themselves and also commute with all the other matrix elements. In view of (3.12), the t^ν_μ are nothing else but the matrix elements of the natural module of the orthogonal group.

The second relation in (3.14) is reminiscent of the relation (3.11). We may define $\omega^{\mu\nu} := \theta^{\mu\nu}\epsilon + \sum_{\sigma,\rho=0}^3 \theta^{\sigma\rho} t^\mu_\sigma t^\nu_\rho$. Then, $\omega^{\mu\nu}$ is skew symmetric in the indices μ and ν and commutes with t^ρ_4 for all ρ. Also, the components $\omega^{\mu\nu}$ commute with one another. Denote $\zeta^\mu := t^\mu_4$. Then,

$$\zeta^\mu \zeta^\nu - \zeta^\nu \zeta^\mu = i\omega^{\mu\nu}. \tag{3.15}$$

The elements ζ^{μ} and $\omega^{\mu\nu}$ together generate a subalgebra of $\mathcal{A}(\mathfrak{g})$. We may consider the commutative ring R generated by all the components of $\omega^{\mu\nu}$ and consider this subalgebra over R. Denote the R-algebra by $\mathcal{A}_{\omega}$, then again the relations (3.15) are the same relations as those satisfied by the coordinate functions of the Moyal space but with $\theta^{\mu\nu}$ replaced by $\omega^{\mu\nu}$.

One may be tempted to identify some completion of $\mathcal{A}_{\omega}$ with the Moyal space, which, however, is not possible. Note that $\mathcal{A}_{\omega}$ is not stable under the action of the Lorentz subalgebra corresponding to the left or right translations, e.g., $\mathcal{L}_{J_{\alpha\beta}}(\zeta^{\mu}\zeta^{\nu})$ contains terms of the form $t^{\mu}_{\alpha}\Sigma_{\sigma}\theta^{\sigma}_{\beta}t^{\nu}_{\sigma}$, which do not belong to the subalgebra. This is not surprising since the Lorentz generators do not generate a Hopf subalgebra of $\mathcal{U}$. It is also this fact which causes the induced module construction, which works so well in the classical setting, to fail badly in the context of the twisted Poincaré algebra.

IV. NONCOMMUTATIVE FIELDS ON MOYAL SPACE

As we have already seen in Sec. III A, it is not possible to generalize the induced module construction of Sec. II to the noncommutative setting. This probably means that there is no canonical definition of noncommutative fields in relation to the representation theory of the twisted Poincaré algebra.

However, we shall make a tentative proposal for the definition of noncommutative classical fields with twisted Poincaré algebra actions on the Moyal space. It agrees with what is assumed in the literature for scalar fields on the Moyal space. We hope that this will provide a framework for studying twisted Poincaré invariance of theories involving noncommutative fields with nonzero spin.

A more systematic treatment of the problem addressed in this section will require us to develop a theory of twisted Poincaré algebra equivariant noncommutative vector bundles on the Moyal space. This is well beyond the scope of the present paper. Furthermore, there is no other known way to construct such bundles except for the induced module construction.

A. Special type of commutative classical fields

Hereafter, we denote by $C^{\infty}(\mathbb{R}^{1,3})$ the space of complex valued smooth functions on $\mathbb{R}^{1,3}$. Denote by $x=(x^0,x^1,x^2,x^3)$ the coordinate of $\mathbb{R}^{1,3}$. Then, regardless of what algebraic structure we impose on $C^{\infty}(\mathbb{R}^{1,3})$, we can always assign numerical values to the x^{μ} to obtain numbers from elements of $C^{\infty}(\mathbb{R}^{1,3})$. This is in sharp contrast to the situation of Sec. III C 1.

We return to Sec. II and consider the associated vector bundle $\widetilde{G}\times_L V\to\widetilde{G}/L$ in the special case when V is a finite dimensional module for the twisted Poincaré algebra and $\mathcal{U}$ with *trivial action* of all the generators P_{μ}.

The bundle is trivial; thus, its space of sections $\Gamma(V)$ is a free module over $(C^{\infty}(\widetilde{G}))^{\mathcal{L}_L}$. Note that

$$C^{\infty}(\mathbb{R}^{1,3}) = (C^{\infty}(\widetilde{G}))^{\mathcal{L}_L}, \tag{4.1}$$

and this is an identification of commutative associative algebras if we equip $C^{\infty}(\mathbb{R}^{1,3})$ with the usual commutative multiplication, which shall be denoted by $\cdot$. Therefore, we have the $(C^{\infty}(\mathbb{R}^{1,3}),\cdot)$-module isomorphism

$$\Gamma(V) \cong C^{\infty}(\mathbb{R}^{1,3}) \otimes V.$$

In the special case under consideration, we can easily describe the isomorphism. Now, $C^{\infty}(\widetilde{G})$ contains a subalgebra, which is spanned by the matrix elements of the finite dimensional representations of $\widetilde{G}$ with trivial actions of all P_{μ}. Denote this algebra by $\mathcal{A}(\mathfrak{l})$. Then, $\mathcal{A}(\mathfrak{l})$, in fact, has the structure of a commutative Hopf algebra.

Being a finite dimensional $\widetilde{G}$-module, V forms a right $\mathcal{A}(\mathfrak{l})$ comodule. We denote the comodule map by

$$\delta: V \to V \otimes \mathcal{A}(\mathfrak{l}),$$

and also use Sweedler's notation $\delta(v)=\Sigma_{(v)}v_{(1)}\otimes v_{(2)}$ for any $v\in V$. Then, the isomorphism is given by

$$\psi: C^\infty(\mathbb{R}^{1,3}) \otimes V \to \Gamma(V), \quad a \otimes v \mapsto \sum_{(v)} v_{(2)}a \otimes v_{(1)}.$$

It is a useful exercise to check that the image of ψ is, indeed, contained in $\Gamma(V)$, but we omit the details and refer to Ref. 13 for general ideas. Since $C^\infty(\mathbb{R}^{1,3})$ is a $\tilde{G}$-module under the action $\mathcal{L}$, $C^\infty(\mathbb{R}^{1,3})\otimes V$ admits a natural $\tilde{G}$ action via the usual coproduct. One can easily show that ψ is a $\tilde{G}$-module map when $\Gamma(V)$ is regarded as a $\tilde{G}$-module in the sense of (2.2).

Elements of $\Gamma(V)$ are a special class of classical fields determined by the inducing module V of the spinor group, which is, in fact, the restriction of a module for the entire Poincaré group $\tilde{G}$. The reason for us to consider this special case is that this generalizes to the noncommutative setting.

B. Generalization to noncommutative setting

Let us equip $C^\infty(\mathbb{R}^{1,3})$ with the standard $*$-product defined for any functions f and g by

$$(f * g)(x) = \lim_{y\to x} \exp\left(\frac{i}{2}\sum_{\mu,\nu} \theta^{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right) f(x)g(y), \quad x,y \in \mathbb{R}^{1,3}.$$

Then, $(C^\infty(\mathbb{R}^{1,3}),*)$ is a noncommutative associative algebra. There is the natural action of the twisted Poincaré algebra on $C^\infty(\mathbb{R}^{1,3})$ given by

$$\begin{aligned} P_\mu(f)(x) &= - i\partial_\mu f(x), \\ J_{\mu\nu}(f)(x) &= - ix_\mu\partial_\nu f(x) + ix_\nu\partial_\mu f(x). \end{aligned} \tag{4.2}$$

By modifying this action, we obtain another action $\varpi:\mathcal{U}\otimes C^\infty(\mathbb{R}^{1,3})\to C^\infty(\mathbb{R}^{1,3})$ of the twisted Poincaré algebra on $C^\infty(\mathbb{R}^{1,3})$ given by

$$\varpi(u_1u_2)(f) = S(u_2)(S(u_1)(f)),$$

for all $u_1,u_2\in\mathcal{U}$ and $f\in C^\infty(\mathbb{R}^{1,3})$.

As was first pointed out in Ref. 4 and very well known by now, $C^\infty(\mathbb{R}^{1,3})$ has the structure of a module algebra over the twisted Poincaré algebra as a Hopf algebra with the twisted comultiplication Δ. For any elements f and g of $C^\infty(\mathbb{R}^{1,3})$, and any $u\in\mathcal{U}$,

$$u(f * g) = \sum_{(u)} u_{(1)}(f) * u_{(2)}(g). \tag{4.3}$$

It also follows that $C^\infty(\mathbb{R}^{1,3})$ has the structure of a module algebra over $\mathcal{U}$ under the action ϖ with respect to the opposite twisted comultiplication Δ',

$$\varpi(u)(f * g) = \sum_{(u)} \varpi(u_{(2)})(f) * \varpi(u_{(1)})(g). \tag{4.4}$$

A *noncommutative scalar field* is an element ϕ of $C^\infty(\mathbb{R}^{1,3})$ regarded as a $(\mathcal{U},\varpi)$ module, where ϕ rapidly vanishes at infinity. This definition is in agreement with that implied in the literature on noncommutative field theory and reduces to the usual definition of scalar fields in the commutative setting.

Two observations are important for the proposal of a definition of noncommutative fields with nonzero spin. One is that the space $\mathcal{A}(\mathfrak{l})$ of matrix elements of the finite dimensional representations of $\mathcal{U}$ with trivial P_μ actions for all μ forms a commutative subalgebra of the dual $\mathcal{U}^*$ of the twisted Poincaré algebra and, furthermore, $\mathcal{A}(\mathfrak{l})$ commutes with all elements of $\mathcal{U}^*$. Another observation is that there exists a canonical *vector space* embedding $j: C^\infty(\mathbb{R}^{1,3}) \to C^\infty(\tilde{G})$ given by Eq. (4.1) as a subset of functions on the classical Poincaré group, since the algebraic structure with the $*$-product is imposed afterward. Now, $\mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})$ naturally has an associative algebra structure with the multiplication, which we still denote by $*$, given by

$$(a \otimes f) * (b \otimes g) = ab \otimes f * g,$$

for any $a \otimes f$ and $b \otimes g$ in $\mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})$. Consider the vector space embedding

$$i: \mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3}) \to C^\infty(\tilde{G}),$$

$$i(a \otimes f)(\Lambda \exp(iPx)) = a(\Lambda) f(x), \qquad \forall\, \Lambda \exp(iPx) \in \tilde{G},$$

and denote

$$\mathcal{A} := i(\mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})).$$

We can introduce a noncommutative algebraic structure on $\mathcal{A}$ by setting

$$i(a \otimes f) * i(b \otimes g) = i(ab \otimes f * g).$$

Then, obviously, i is an algebra isomorphism between $\mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})$ and $\mathcal{A}$, and we shall denote the resulting algebra by $(\mathcal{A}, *)$.

The twisted Poincaré algebra $\mathcal{U}$ acts on $\mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})$,

$$\begin{gathered} \mathfrak{R}: \mathcal{U} \otimes \mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3}) \to \mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3}), \\ u \otimes a \otimes f \mapsto \mathfrak{R}_u(a \otimes f) = \sum_{(u)} \mathcal{R}_{u_{(1)}}(a) \otimes \varpi(u_{(2)})(f). \end{gathered} \tag{4.5}$$

This leads to a well defined action on $\mathcal{A}$,

$$\hat{\mathfrak{R}}: \mathcal{U} \otimes \mathcal{A} \to \mathcal{A}, \tag{4.6}$$

given for any $g = i(a \otimes f)$ with $a \otimes f \in \mathcal{A}(\mathfrak{l}) \otimes C^\infty(\mathbb{R}^{1,3})$ by

$$u \otimes i(a \otimes f) \mapsto \hat{\mathfrak{R}}_u i(a \otimes f) = i\left(\sum_{(u)} \mathcal{R}_{u_{(1)}}(a) \otimes \varpi(u_{(2)})(f)\right).$$

This turns $(\mathcal{A}, *)$ into a module algebra for the twisted Poincaré algebra.

Remark 4.1: The $\hat{\mathfrak{R}}$ action on $\mathcal{A}$ can, in fact, be obtained by differentiating the right translation by the Poincaré group.

Any finite dimensional $\mathcal{U}$-module V with trivial actions of all P_μ automatically has an $\mathcal{A}(\mathfrak{l})$ comodule structure, which we still denote by

$$\delta: V \to V \otimes \mathcal{A}(\mathfrak{l}), \qquad v \mapsto \sum_{(v)} v_{(1)} \otimes v_{(2)}.$$

Define the map

$$\psi_\theta: C^\infty(\mathbb{R}^{1,3}) \otimes V \to \mathcal{A} \otimes V \tag{4.7}$$

by $a \otimes v \mapsto \Sigma_{(v)} i(v_{(2)} \otimes a) \otimes v_{(1)}$ and set

$$\Gamma_\theta(V) \coloneqq \psi_\theta(C^\infty(\mathbb{R}^{1,3}) \otimes V), \tag{4.8}$$

where we emphasize again that V is assumed to be a finite dimensional $\mathcal{U}$-module with trivial actions for all P_μ.

Then, $\Gamma_\theta(V)$ forms a $\mathcal{U}$-module with the action defined for any $u \in \mathcal{U}$ and $\zeta \in \Gamma(V)$ by

$$u \cdot \zeta \coloneqq (\hat{\mathfrak{R}}_u \otimes \mathrm{id}_V)\zeta.$$

Regard $C^\infty(\mathbb{R}^{1,3})$ as a $\mathcal{U}$-module with the action ϱ. Then, $C^\infty(\mathbb{R}^{1,3}) \otimes V$ has a natural $\mathcal{U}$-module structure. It can be shown that ψ_θ is $\mathcal{U}$ linear.

For any element $\zeta = \Sigma_i g_i \otimes v_i$ of $\Gamma_\theta(V)$ and the special type of elements $\exp(iPx) \in \widetilde{G}$, we write

$$\zeta(x) \coloneqq \sum_i v_i g_i(\exp(iPx)).$$

Then,

$$(u \cdot \zeta)(x) = \sum (\varpi(u_{(1)})g_i)(x) u_{(2)}(v_i). \tag{4.9}$$

Note that if we rewrite the action (2.4) of the Poincaré group on commutative classical fields in terms of the the universal enveloping algebra of the Poincaré algebra, the resulting formula will have the same form as (4.9). Therefore, elements ζ of $\Gamma_\theta(V)$ may be regarded as *noncommutative classical fields*. (We have excluded internal degrees of freedom throughout the paper.)

We define the spin of the field $\zeta \in \Gamma_\theta(V)$ to be the spin of the irreducible module V of the Lorentz subalgebra determined by the highest weight. For example, $\zeta \in \Gamma_\theta(V)$ will be called a scalar field if V is the one-dimensional trivial module, and a spinor field if V is the spinor module. However, we should note that when all $\theta^{\mu\nu}=0$, this definition of fields reduces to a special case of that in the commutative setting over the usual Minkowski space.

Remark 4.2: The discussion after Eq. (2.5) at the end of Sec. II generalizes to the noncommutative setting for the $\Gamma_\theta(V)$ defined by (4.8).

As a simple example, we consider the case of scalar fields. Since V is a one-dimensional trivial module, the matrix element of the corresponding representation is the identity of the algebra $\mathcal{U}^*$. From (4.8), we obtain $\Gamma_\theta(V) = C^\infty(\mathbb{R}^{1,3})$. Thus, a scalar field $\Phi(x)$ is nothing else but an element of $C^\infty(\mathbb{R}^{1,3})$. Assuming that Φ is real, then the classical action

$$S = \frac{1}{2} \int d^4x \partial^\mu \Phi(x) * \partial_\mu \Phi(x)$$

is twisted Poincaré invariant.[10]

It will be very interesting to construct twisted Poincaré invariant theories involving spinor fields. Work in this direction is in progress.

C. Transformation rules of noncommutative quantum fields

Before closing this section, let us make some remarks on the quantum case. After quantization, all g_i in a field $\zeta = \Sigma g_i \otimes v_i \in \Gamma_\theta(V)$ become operators (field operators) acting on some Hilbert space. Denote the algebra of field operators by $\mathcal{O}$. Then, every g_i belongs to the algebra $\mathcal{O} \otimes C^\infty(\mathbb{R}^{1,3})$ with the natural algebraic structure of the tensor product of two algebras,

$$(A \otimes f)(B \otimes h) = AB \otimes f * h, \quad \forall A,B \in \mathcal{O}, \quad f,h \in C^\infty(\mathbb{R}^{1,3}).$$

The twisted Poincaré algebra is realized in terms of the field operators $\iota : \mathcal{U} \to \mathcal{O}$. In order for the action of the twisted Poincaré algebra on $\mathcal{O}$ to respect the algebraic structure of the latter, one has to define the action of $\mathcal{U}$ on a quantum field ζ by

$$(u \cdot \zeta)(x) := \sum \iota(u_{(1)}) g_i(x) \iota(S(u_{(2)})) \otimes v_i, \quad u \in \mathcal{U}. \tag{4.10}$$

One can show that this, indeed, defines an action of $\mathcal{U}$ on quantum fields by noting that the right hand side involves the well-known adjoint action of a Hopf algebra.

The transformation rule of the quantum field is then given by

$$(u \cdot \zeta)(x) = \sum (\varpi(u_{(1)}) g_i)(x) \otimes u_{(2)}(v_i), \tag{4.11}$$

which is formally of the same form as (4.9) but with the left hand side given by (4.10) and the $\varpi(u_{(1)})$ on the right side acting on the $C^\infty(\mathbb{R}^{1,3})$ component of g_i only.

V. CONCLUSION, COMMENTS, AND OUTLOOK

As we mentioned earlier, some researchers were clearly aware of the necessity of formulating a precise transformation rule for noncommutative fields under twisted Poincaré algebra. For example, this was discussed at length by Fiore and Wess in Ref. 6 (Sec. IV). Lacking such a rule, they suggested[6] to replace it by a condition imposed on the Wightman functions. This condition formally looked the same as that in the commutative case. Even assuming that one would eventually find any justification for this, there is still the need of a general rule to associate a field with a representation of the Lorentz subalgebra of the twisted Poincaré algebra in order to state the condition. So in this sense, the transformation rule for noncommutative fields under twisted Poincaré algebra cannot be entirely avoided. Our proposal for such a transformation rule is self-consistent and should be the correct form. It hopefully provides the necessary framework for studying twisted Poincaré invariance of noncommutative quantum field theories on the Moyal space.

There are many important issues in noncommutative field theory related to the present work, which all deserve independent in depth treatments. Below, we shall briefly discuss some of the issues, which we shall return to in the future.

A. Wigner's construction for twisted Poincaré algebra

Recall that usually (see, e.g., Ref. 20), one starts with a Poincaré invariant classical field theory, then performs quantization (say, canonical quantization) to arrive at quantum fields. The in and out states of the quantum field theory correspond to particles classified by Wigner's theory in terms of unitary representations of the Poincaré group, where single particle states correspond to irreducible representations and multiparticle states to tensor products of representations. The investigations in the present paper follows this general line of thinking.

However, a rather different approach to quantum fields was advocated by Weinberg in Ref. 12 (Sec. V). The starting point now is Wigner's classification of particles. Relativistic quantum fields are directly constructed from particle states through cluster decomposition. It will be very interesting to generalize this approach to the noncommutative setting, especially in view of the difficulties discussed in Sec. III B. We plan to develop this approach in a future publication.

Since Weinberg's approach relied on Wigner's classification of particle states, here we briefly describe the generalization of Wigner's construction of unitary representations of the Poincaré group to the twisted Poincaré algebra. One can take any four momentum $k=(k_0,k_1,k_2,k_3)$, on which the Lorentz subalgebra $\mathfrak{l}$ of the twisted Poincaré algebra $\mathcal{U}$ acts in the standard way. Let

$$\mathfrak{l}_k = \{X \in \mathfrak{l} | X(k) = 0\},$$

which is the subalgebra of the Lorentz algebra that leaves k invariant. This is the Lie algebra of the "little group" of Wigner. Recall that we used $\mathfrak{p}$ to denote the subalgebra of translations in the Poincaré algebra. Now, $\mathfrak{l}_k+\mathfrak{p}$ is again a Lie subalgebra. It is important to note the following fact.

Lemma 5.1: Let $\mathcal{U}_k=\mathrm{U}(\mathfrak{l}_k+\mathfrak{p})$ *be the universal enveloping algebra of* $\mathfrak{l}_k+\mathfrak{p}$. *Then,* $\mathcal{U}_k$ *is a Hopf subalgebra of* $\mathcal{U}$.

Given any irreducible $\mathfrak{l}_k$-module V_k, we can define a $\mathcal{U}_k$-module structure on it by taking any nonzero vector $v \in V_k$ and requiring $p_\mu v = k_\mu v$ for all μ. Then, it follows that for all $w \in V_k$, we have $p_\mu w = k_\mu w$. This way, we arrive at an irreducible $\mathcal{U}_k$-module, which we still denote by V_k.

Remark 5.1: One can extend any given $\mathfrak{l}_k$-module to a $\mathcal{U}_k$-module by requiring p_μ to act by k_μ for all μ.

Because of Lemma 5.1, one can now use (3.7) with $V = V_k$ and $\mathcal{H} = \mathcal{U}_k$ to define an induced module $\Gamma(V_k)$ for the entire twisted Poincaré algebra $\mathcal{U}$. This is the generalization of Wigner's construction to the twisted Poincaré algebra in a nut shell, leaving aside issues on unitarity and irreducibility, which are beyond the scope of this paper.

B. Noncommutative Minkowski spaces

Another useful aspect of results reported here is that they point out a way to look for possible noncommutative Minkowski spaces suitable for developing quantum field theory with space-time symmetries described by Hopf algebras, which are deformations of the Poincaré algebra. In order for fields to naturally emerge within such a framework, one might require the deformed Poincaré algebra to contain the enveloping algebra of the Lorentz algebra or a deformation of it as a Hopf subalgebra.

As an example, we consider the Poincaré algebra twisted by

$$\mathcal{F}_\tau = \exp(-i\tau J_{12} \otimes J_{34}).$$

(We could have antisymmetrized the exponent as in Ref. 22 but have not done so because we want to have simple expressions for the $*$-product.) Now, the universal enveloping algebra of the Lorentz algebra is contained in this twisted Poincaré algebra as a Hopf subalgebra. Thus, there exists a noncommutative analog M_τ of the homogeneous space $\widetilde{G}/L$ (the usual Minkowski space), and noncommutative fields then naturally emerge as sections of noncommutative homogeneous vector bundles on M_τ. The noncommutative Minkowski space M_τ is also easy to describe. Write $\mathcal{F}_\tau^{-1} = \Sigma F_\alpha \otimes G_\alpha$. Define the following noncommutative product $*_\tau$ on the space of functions on $\mathbb{R}^{1,3}$:

$$(f * g)(x) = \sum (F_\alpha f)(x)(G_\alpha g)(x),$$

and denote the resulting algebra by $(C^\infty(\mathbb{R}^{1,3}), *_\tau)$. Then, $(C^\infty(\mathbb{R}^{1,3}), *_\tau)$ is the algebra of functions on the noncommutative Minkowski space. Denote by X_μ, $\mu = 1,2,3,4$, the coordinate functions, that is,

$$X_\mu(x) = x_\mu, \quad x \in \mathbb{R}^{1,3}.$$

Then, $(X_\alpha *_\tau X_\beta)(x) = x_\alpha x_\beta$, if $\alpha \geqslant \beta$, or $\alpha, \beta \in \{1,2\}$, or $\alpha, \beta \in \{3,4\}$, and

$$\begin{aligned}
(X_1 *_\tau X_3)(x) &= x_1 x_3 \cos\tau - i x_2 x_4 \sin\tau, \\
(X_1 *_\tau X_4)(x) &= x_1 x_4 \cos\tau + i x_2 x_3 \sin\tau, \\
(X_2 *_\tau X_3)(x) &= x_2 x_3 \cos\tau + i x_1 x_4 \sin\tau, \\
(X_2 *_\tau X_4)(x) &= x_2 x_4 \cos\tau - i x_1 x_3 \sin\tau.
\end{aligned} \tag{5.1}$$

It will be interesting to construct quantum field theoretical models on such a noncommutative Minkowski space, which are invariant with respect to the twisted Poincaré algebra.

Other possible examples are the quantum Poincaré algebras constructed in Refs. 23 and 24 in the context of the complexified conformal algebra $\mathrm{so}(6,\mathbb{C}) = \mathrm{sl}(4,\mathbb{C}) \subset \mathrm{gl}(4,\mathbb{C})$. These quantum Poincaré algebras are quantized parabolic subalgebras of the enveloping algebra of $\mathrm{gl}(4,\mathbb{C})$ and contain the quantum group $\mathrm{U}_q(\mathrm{sl}_2) \otimes \mathrm{U}_q(\mathrm{sl}_2)$ as a Hopf subalgebra, which is the quantized version of the enveloping algebra of the complexified Lorentz algebra. In these cases, there exist natural quantum homogeneous spaces, which play the role of the Minkowski space. The noncommutativity of the quantum Minkowski spaces is now much more severe than that of the standard Moyal space or the previous example. Nevertheless, by using appropriate analogs of the quantum Haar measure[13] one may be able to construct quantum Poincaré invariant field theory on such quantum Minkowski spaces.

There remains the possibility that the Seiberg–Witten map[1] allows for a realization of space-time symmetry of the twisted Poincaré type. We also mention that quantum group symmetries manifest themselves in conformal field theory as well[25] but in a manner different from space-time symmetries. It will be interesting to understand such quantum group symmetries from the point of view of noncommutative geometry.

ACKNOWLEDGMENTS

We are indebted to Jerzy Lukierski for helpful comments. P.P.K., R.B.Z., and X.Z. wish to thank the Department of Physical Sciences, University of Helsinki, and the Helsinki Institute of Physics for the hospitality extended to them during their visits in September–October 2007 when this work was completed. Partial financial support from the Australian Research Council, National Science Foundation of China (Grant No. 10421001), NKBRPC 2006CB805905), the Chinese Academy of Sciences, the Academy of Finland, and RFBR Grant No. 06-01-00451 is also gratefully acknowledged. A.T. acknowledges Project No. 121720 of the Academy of Finland.

[1] N. Seiberg and E. Witten, J. High Energy Phys. 9909, 032 (1999).
[2] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[3] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 0002, 020 (2000).
[4] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004).
[5] P. Aschieri, M. Dimitrijevic, F. Meyer, and J. Wess, Class. Quantum Grav. **23**, 1883 (2006).
[6] G. Fiore and J. Wess, Phys. Rev. D **75**, 105022 (2007).
[7] E. Joung and J. Mourad, J. High Energy Phys. 0705, 098 (2007).
[8] A. P. Balachandran, A. Pinzul, and B. A. Qureshi, Phys. Rev. D **77**, 025021 (2008).
[9] J. Lukierski, A. Nowicki, and H. Ruegg, Phys. Lett. B **293**, 344 (1992).
[10] M. Chaichian, P. Prešnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005).
[11] A. Demichev, e-print arXiv:hep-th/9701079.
[12] A. Gonzlez-Ruiz and L. A. Ibort, Phys. Lett. B **296**, 104 (1992).
[13] A. R. Gover and R. B. Zhang, Rev. Math. Phys. **11**, 533 (1999).
[14] V. G. Drinfeld, Algebra Anal. **1**, 114 (1989).
[15] V. G. Drinfeld, Algebra Anal. **2**, 149 (1990).
[16] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, 1994).
[17] M. Chaichian and A. Demichev, *Introduction to Quantum Groups* (World Scientific, River Edge, NJ, 1996).
[18] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).
[19] M. E. Sweedler, *Hopf Algebras*, Mathematics Lecture Note Series (Benjamin, New York, 1969).
[20] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, 1961).
[21] S. Weinberg, *The Quantum Theory of Fields: Foundations* (Cambridge University Press, Cambridge, 2005), Vol. I.
[22] J. Lukierski and M. Woronowicz, Phys. Lett. B **633**, 116 (2006).
[23] O. Ogievetsky, W. B. Schmidke, J. Wess, and B. Zumino, Commun. Math. Phys. **150**, 495 (1992).
[24] M. Chaichian and A. Demichev, J. Math. Phys. **36**, 398 (1995).
[25] L. Alvarez-Gaumé, C. Gómez, and G. Sierra, Nucl. Phys. B **319**, 155 (1989).