Invariant integration on classical and quantum Lie supergroups

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Invariant integrals on Hopf superalgebras, in particular, the classical and quantum Lie supergroups, are studied. The uniqueness (up to scalar multiples) of a left integral is proved, and a $\mathbb{Z}_2$-graded version of Maschke's theorem is discussed. A construction of left integrals is developed for classical and quantum Lie supergroups. Applied to several classes of examples the construction yields the left integrals in explicit form.
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I. INTRODUCTION

This article studies invariant integrals on Hopf superalgebras. We shall focus on the Hopf superalgebras of functions on classical Lie supergroups and their quantum counterparts, developing aspects of the general theory of integrals on them, and also establishing an explicit construction of such integrals.

An important feature of classical and quantum Lie superalgebras is that their finite-dimensional representations are not completely reducible. This imposes severe restrictions on the possible integrals on the corresponding classical and quantum Lie supergroups. We shall extensively investigate this fact, arriving at a result which may be regarded as a $\mathbb{Z}_2$-graded version of Maschke's theorem in an infinite-dimensional setting.

Recall that if the dual of a given finite-dimensional Hopf algebra is semisimple, then a generalization of Maschke's theorem (see Refs. 1 and 2) applies, and the invariant integral on the Hopf algebra can be obtained by considering a Peter–Weyl type basis of the Hopf algebra. Such a construction of integrals fails badly in the supersymmetric setting [except for $\mathrm{OSP}(1|2n)$ and $\mathrm{OSP}_q(1|2n)$]. Here we develop an explicit construction of integrals, which can be implemented on classical Lie supergroups and also on type I quantum supergroups. The construction can also be adapted to produce integrals on quantum groups at roots of unity.

The study of this article is motivated by the great importance of the Haar measure in the theory of locally compact Lie groups. The first place we know of where integrals in the sense of Hopf algebra theory have shown up is Hochschild's proof of Tannaka's duality theorem for compact groups.[3] Later on they played an important role in the structure theory of finite-dimensional Hopf algebras (for example, see Refs. 4–7).

With the appearance of quantum groups and quantum algebras, it became obvious that integrals have to play an important role there, too. In fact, the quantum Haar functional is a basic tool in the C^*-algebra approach to quantum groups,[8] and it can also be used to introduce topologies on Hopf algebras which originally are defined by purely algebraic means.[9] Correspondingly, there are various attempts to define integration on quantum groups, quantum spaces and their braided generalizations (see Ref. 10 and the references therein).

In principle, the braided case includes Hopf superalgebras as a special example, but it seems worthwhile to investigate the super case separately. Needless to say, there is a huge literature dealing with the integration on supermanifolds and supergroups, but a theory of integrals on Hopf superalgebras seems to be missing. This will be the topic of the present work. We hope that integrals will also prove to be useful in the further investigation of the structure and representations of quantum supergroups, and that our results will shed some new light on the integration over classical, i.e., undeformed Lie supergroups.

At present, we know of only one related work.[11] In that reference, integrals on quantum supergroups of the special linear type are constructed by means of the R-matrix formalism. However, even for the $\mathrm{SL}_q(m|n)$ quantum supergroups the techniques used and the results derived in that paper are totally different from those to be presented here [even though, because of the uniqueness theorem to be proved in Sec. II, the integrals on $\mathrm{SL}_q(m|n)$ constructed here and in Ref. 11 must be proportional].

The organization of the article is as follows. In Sec. II we develop some general theory of integrals on Hopf superalgebras and establish results generalizing Maschke's theorem. In Sec. III we study classical Lie supergroups. A general construction of integrals is developed, and applied to the type I Lie supergroups, and also the type II Lie supergroups $\mathrm{OSP}(1|2n)$ and $\mathrm{OSP}(3|2)$. In Sec. IV we extend the results to the quantum setting, obtaining a method for constructing integrals on quantum supergroups. As examples, the type I quantum supergroups are studied in detail. Section V contains a brief discussion of our results. Finally, in the Appendix we have collected some information on the finite dual of $U(\mathfrak{gl}(1))$.

We close this introduction by recalling some conventions related to $\mathbb{Z}_2$-graded algebraic structures. The two elements of $\mathbb{Z}_2$ are denoted by $\overline{0}$ and $\overline{1}$. Unless stated otherwise, all gradations considered in this work will be $\mathbb{Z}_2$-gradations. For any superspace, i.e., $\mathbb{Z}_2$-graded vector space $V=V_{\overline{0}}\oplus V_{\overline{1}}$, we define the gradation index $[\ ]: V_{\overline{0}}\cup V_{\overline{1}}\to\mathbb{Z}_2$ by $[x]=\alpha$ if $x\in V_\alpha$, where $\alpha\in\mathbb{Z}_2$. All algebraic notions and constructions are to be understood in the super sense, i.e., they are assumed to be consistent with the $\mathbb{Z}_2$-gradations and to include the appropriate sign factors.

II. INTEGRALS ON HOPF SUPERALGEBRAS

Let $\mathcal{A}$ be a Hopf superalgebra with comultiplication Δ, counit ε, and antipode S. A left integral $\int^l$ on $\mathcal{A}$ is an element of $\mathcal{A}^*$, such that

$$\left(\mathrm{id}_{\mathcal{A}}\otimes\int^l\right)\Delta=\mathbb{1}_{\mathcal{A}}\int^l. \tag{1}$$

Equivalently, this means that

$$a^*\cdot\int^l=a^*(\mathbb{1}_{\mathcal{A}})\int^l,\quad \forall a^*\in\mathcal{A}^*, \tag{2}$$

where the dot denotes the multiplication in $\mathcal{A}^*$ deduced from Δ. A right integral $\int^r\in\mathcal{A}^*$ on $\mathcal{A}$ is defined by a similar requirement

$$\left(\int^r\otimes\ \mathrm{id}_{\mathcal{A}}\right)\Delta=\mathbb{1}_{\mathcal{A}}\int^r. \tag{3}$$

Let $\mathcal{A}^{\mathrm{aop,cop}}$ be the Hopf superalgebra opposite to $\mathcal{A}$ both regarded as an algebra and a coalgebra. Then a linear form $\int\in\mathcal{A}^*$ is a left/right integral on $\mathcal{A}$ if and only if it is a right/left integral on $\mathcal{A}^{\mathrm{aop,cop}}$. In particular, if the antipode S of $\mathcal{A}$ (and hence of $\mathcal{A}^{\mathrm{aop,cop}}$) is invertible, then $S^{\pm1}$ are isomorphisms of $\mathcal{A}$ onto $\mathcal{A}^{\mathrm{aop,cop}}$, and hence $\int$ is a left integral on $\mathcal{A}$ if and only if $\int S^{\pm1}$ are right integrals on $\mathcal{A}$. Thus we only need to consider left integrals (or right integrals).

We have the following result:

Theorem 1: *The dimension of the space of left integrals on $\mathcal{A}$ is not greater than 1. In particular, any integral on $\mathcal{A}$ is even or odd.*

Proof: The proof is carried out by reducing the problem to the classical nongraded case. In principle, this can be viewed as an application of Majid's bosonization,[12] but for the present simple case the technique has been known for quite some time.

For notational convenience (and for reasons that will become obvious at the end of this proof) we define a map

$$\tau:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{C}$$

by

$$\tau(\alpha,\beta)=(-1)^{\alpha\beta},\quad \forall\alpha,\beta\in\mathbb{Z}_2.$$

Let $\mathbb{C}\mathbb{Z}_2$ be the group Hopf algebra of $\mathbb{Z}_2$. The canonical basis elements will be denoted by g_α, $\alpha\in\mathbb{Z}_2$. In particular, we have

$$g_\alpha g_\beta=g_{\alpha+\beta},\quad \forall\alpha,\beta\in\mathbb{Z}_2.$$

Then

$$\overline{\mathcal{A}}=\mathcal{A}\otimes\mathbb{C}\mathbb{Z}_2$$

is made into a usual Hopf algebra by means of the following definitions (where $a,b\in\mathcal{A}$ and $\alpha,\beta\in\mathbb{Z}_2$):
product:

$$(a\otimes g_\alpha)(b\otimes g_\beta)=\tau(\alpha,[b])\,ab\otimes g_{\alpha+\beta},$$

coproduct (in Sweedler's notation):

$$\overline{\Delta}(a\otimes g_\alpha)=\sum_{(a)}\,(a_{(1)}\otimes g_{[a_{(2)}]+\alpha})\otimes(a_{(2)}\otimes g_\alpha),$$

counit:

$$\overline{\varepsilon}(a\otimes g_\alpha)=\varepsilon(a),$$

antipode:

$$\overline{S}(a\otimes g_\alpha)=\tau([a],\alpha+[a])\,S(a)\otimes g_{-\alpha-[a]}.$$

Now let s be a left integral on $\mathcal{A}$, i.e., a linear form $s\in\mathcal{A}^*$ such that

$$(\mathrm{id}_{\mathcal{A}}\otimes s)\Delta=\mathbb{1}_{\mathcal{A}}s,$$

and let us assume that s is homogeneous of degree σ. Recall that $\otimes$ denotes the tensor product in the *graded* sense. Nevertheless, it is easy to see that the equation above is equivalent to

$$\sum_{(a)}\,a_{(1)}s(a_{(2)})=s(a)\,\mathbb{1}_{\mathcal{A}},\quad \forall a\in\mathcal{A},$$

i.e., it takes the same form as in the nongraded case.

Define the linear form t_σ on $\mathbb{C}\mathbb{Z}_2$ by

$$t_\sigma(g_\alpha)=\delta_{\sigma,\,\alpha},\quad \forall\alpha\in\mathbb{Z}_2.$$

Then

$$\overline{s} = s \otimes t_\sigma$$

(nongraded tensor product) is a left integral on $\overline{\mathcal{A}}$. We prove this by showing that $\overline{s}$ satisfies the equation analogous to that given above for s: For all $a \in \mathcal{A}$ and $\alpha \in \mathbb{Z}_2$, we have

$$\begin{aligned}(\mathrm{id}_{\overline{\mathcal{A}}} \otimes \overline{s})(\overline{\Delta}(a \otimes g_\alpha)) &= (\mathrm{id}_{\overline{\mathcal{A}}} \otimes \overline{s}) \sum_{(a)} (a_{(1)} \otimes g_{[a_{(2)}]+\alpha}) \otimes (a_{(2)} \otimes g_\alpha) \\ &= \sum_{(a)} (a_{(1)} \otimes g_{[a_{(2)}]+\alpha})\, s(a_{(2)})\, t_\sigma(g_\alpha) \\ &= \sum_{(a)} (a_{(1)} \otimes g_{-\sigma+\alpha})\, s(a_{(2)})\, t_\sigma(g_\alpha) \\ &= \left(\sum_{(a)} a_{(1)} s(a_{(2)}) \right) \otimes g_0\, t_\sigma(g_\alpha) \\ &= s(a) t_\sigma(g_\alpha)\, \mathbb{1}_{\mathcal{A}} \otimes g_0 \\ &= \overline{s}(a \otimes g_\alpha)\, \mathbb{1}_{\mathcal{A}} \otimes g_0,\end{aligned}$$

as required.

Now let us suppose that $s \neq 0$ and that s' is a second nonzero integral on $\mathcal{A}$ which is homogeneous of degree σ'. Then $\overline{s} = s \otimes t_\sigma$ and $\overline{s'} = s' \otimes t_{\sigma'}$ are nonzero integrals on $\overline{\mathcal{A}}$. According to Sullivan's theorem on the uniqueness of integrals on ordinary (nongraded) Hopf algebras (see Refs. 13 and 2) these integrals must be proportional. This implies that $\sigma = \sigma'$ (otherwise, t_σ and $t_{\sigma'}$ would be linearly independent) and hence that s and s' are proportional.

Finally, let $s \in \mathcal{A}^*$ be an arbitrary linear form on $\mathcal{A}$, and let $s = \Sigma_{\sigma \in \mathbb{Z}_2} s_\sigma$, with $s_\sigma \in (\mathcal{A}^*)_\sigma$, be its decomposition into homogeneous components. Obviously, s is a left integral on $\mathcal{A}$ if and only if all of the s_σ are. Applying the foregoing result to the s_σ, we conclude that, for a left integral s, at most one of the s_σ can be different from zero, i.e., that s is homogeneous. This proves the theorem.

The reader will notice that the same proof applies to arbitrary color Hopf algebras (and this was the other reason to introduce the map τ).

The uniqueness result of the theorem enables us to investigate how a left integral behaves under "right translations." Thus, let $\int$ be a nontrivial left integral on $\mathcal{A}$. We know that the linear form $\int$ is homogeneous, let γ be its degree. We consider the linear map

$$g: \mathcal{A} \to \mathcal{A}, \quad g = \left(\int \otimes \mathrm{id} \right) \Delta.$$

Obviously, it is homogeneous of degree γ and not equal to zero (otherwise, $\varepsilon\, g = \int$ would be equal to zero). Using the coassociativity of the coproduct, it is easy to check that

$$(g \otimes \mathrm{id})\Delta = \Delta\, g, \tag{4}$$

$$(\mathrm{id} \otimes g)\Delta = j\, g, \tag{5}$$

where

$$j: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad j(a) = \mathbb{1}_{\mathcal{A}} \otimes a$$

is the right canonical injection of $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$.

Now let $a^* \in \mathcal{A}^*$ be an arbitrary homogeneous linear form on $\mathcal{A}$. Equation (5) implies that $a^*\, g$ is a left integral on $\mathcal{A}$ and hence proportional to $\int$. In particular, $a^*\, g$ vanishes on the kernel of $\int$. Since this is true for all homogeneous elements $a^* \in \mathcal{A}^*$, it follows that g itself vanishes on the kernel of $\int$. Consequently, there exists a unique element $a_0 \in \mathcal{A}$ such that

$$g(a) = \left\langle \int , a \right\rangle a_0, \quad \forall a \in \mathcal{A},$$

and a_0 is even. Equation (4) now means that

$$\Delta(a_0) = a_0 \otimes a_0 .$$

Since a_0 is nonzero (because g is nonzero), we see that a_0 is a grouplike element of $\mathcal{A}$. Thus we have proved the following proposition.

Proposition 1: Let $\int$ *be a nontrivial left integral on a Hopf superalgebra* $\mathcal{A}$. *Then there exists a unique even grouplike element* $a_0 \in \mathcal{A}$ *such that*

$$\left(\int \otimes \mathrm{id} \right) \Delta = a_0 \int .$$

In particular, $\int$ *is also a right integral if and only if* $a_0 = \mathbb{1}_{\mathcal{A}}$.

Let V be a finite-dimensional $\mathbb{Z}_2$-graded right $\mathcal{A}$-comodule, and let

$$\omega: V \to V \otimes \mathcal{A}$$

be its structure map (which, according to our general conventions, is supposed to be even). The antipode of $\mathcal{A}$ enables one to introduce a right $\mathcal{A}$-comodule structure on the dual space V^* of V, with the structure map

$$\overline{\omega}: V^* \to V^* \otimes \mathcal{A}$$

uniquely defined by

$$\langle v^*, w \rangle \mathbb{1}_{\mathcal{A}} = (\langle , \rangle \otimes M)(\mathrm{id}_{V^*} \otimes T \otimes \mathrm{id}_{\mathcal{A}}) \overline{\omega}(v^*) \otimes \omega(w), \quad \forall v^* \in V^*, w \in V,$$

where T is the flipping map, M denotes the multiplication in $\mathcal{A}$ and $\langle , \rangle$ is the dual space pairing. It follows that $\mathrm{End}(V) = V \otimes V^*$ has a natural right $\mathcal{A}$-comodule structure

$$\delta: \mathrm{End}(V) \to \mathrm{End}(V) \otimes \mathcal{A}.$$

For later use we note that a map $g \in \mathrm{End}(V)$ is a comodule endomorphism of V if and only if it is even and coinvariant, i.e., it satisfies

$$\delta(g) = g \otimes \mathbb{1}_{\mathcal{A}} .$$

If $\int$ is a left integral on $\mathcal{A}$, we define the linear map

$$\Phi = \left(\mathrm{id} \otimes \int \right) \delta : \mathrm{End}(V) \to \mathrm{End}(V).$$

Consider $\Phi(m) \in \mathrm{End}(V)$ for any $m \in \mathrm{End}(V)$. Left invariance of $\int$ immediately leads to

$$\delta(\Phi(m)) = \Phi(m) \otimes \mathbb{1}_{\mathcal{A}};$$

that is, we have the following.

Lemma 1: $\mathrm{Im}\,\Phi$ *is contained in the subspace of coinvariant elements of* $\mathrm{End}(V)$.

Now we consider the case when V contains a sub-comodule V_1. Let $P \in \mathrm{End}(V)$ be a projection onto V_1, i.e., $\mathrm{Im}\,P = V_1$ and $P^2 = P$. It can be easily shown that $\Phi(P)$ satisfies

$$\Phi(P)V\subset V_1 \quad \text{and} \quad \Phi(P)v_1 = v_1\int \mathbb{1}_{\mathcal{A}}, \quad \forall v_1\in V_1.$$

Suppose now that $\int \mathbb{1}_{\mathcal{A}}\neq 0$. This implies that $\int$ is even. Thus $\Phi(P)$ is even as well, and hence it is a comodule endomorphism of V. It follows that $\operatorname{Ker}\Phi(P)$ is a comodule complement of V_1 in V. Since this holds for any finite-dimensional right $\mathcal{A}$-comodule V and any of its sub-comodules, we conclude that all finite-dimensional right $\mathcal{A}$-comodules are completely reducible. Using the basic fact that all finitely generated comodules are finite-dimensional, it follows by means of standard arguments (known, for example, from the general theory of semisimple modules over rings) that all (not necessarily finite-dimensional) right $\mathcal{A}$-comodules are completely reducible.

Conversely, let $\mathcal{A}$ be a Hopf superalgebra such that all right $\mathcal{A}$-comodules are completely reducible. In particular, $\mathcal{A}$ regarded as a right $\mathcal{A}$-comodule with structure map Δ is completely reducible. Let $\mathcal{A}_0$ be a comodule complement of $\mathbb{C}\mathbb{1}_{\mathcal{A}}$ in $\mathcal{A}$. Then any linear form $\int^r$ on $\mathcal{A}$ with kernel $\mathcal{A}_0$ is a right(!) integral on $\mathcal{A}$ such that $\int^r \mathbb{1}_{\mathcal{A}}\neq 0$. Applying the foregoing to $\mathcal{A}^{\mathrm{aop,cop}}$ and $\int^r$ (which is a left integral on $\mathcal{A}^{\mathrm{aop,cop}}$) we conclude that all left $\mathcal{A}$-comodules are completely reducible and that $\mathcal{A}$ also has a left integral $\int^l$ such that $\int^l \mathbb{1}_{\mathcal{A}}\neq 0$. It should be noted that according to Larson[6] analogous results hold for Hopf algebras, comodules and integrals living in an arbitrary tensor category.

Actually, much more can be said. Let $\{V(\lambda)|\lambda\in\Lambda\}$ be a complete representative set of all finite-dimensional right $\mathcal{A}$-comodules, where Λ is some index set. Among these, there is a one-dimensional comodule, $V(0)$, say, such that under the coaction, $v\mapsto v\otimes \mathbb{1}_{\mathcal{A}}$. We call $V(0)$ the trivial $\mathcal{A}$-comodule. For each $V(\lambda)$, we choose a basis $\{v_a^{(\lambda)}|a=1,2,\ldots,\dim V(\lambda)\}$. Then under the coaction of $\mathcal{A}$, we have

$$\omega(v_a^{(\lambda)}) = \sum_b v_b^{(\lambda)}\otimes t_{ba}^{(\lambda)},$$

and the $t_{ab}^{(\lambda)}$ form a Peter-Weyl type of basis for $\mathcal{A}$. If $\int$ denotes the linear form on $\mathcal{A}$ defined by

$$\int \mathbb{1}_{\mathcal{A}}=1, \quad \int t_{ab}^{(\lambda)}=0, \quad \forall \lambda\neq 0,$$

then $\int$ is both a left and right integral on $\mathcal{A}$ and, obviously, it is even.

Summarizing part of our results, we have proved the following generalization of the well-known Maschke's theorem to the case of Hopf superalgebras (see Refs. 1, 6, and 2).

Proposition 2: The Hopf superalgebra $\mathcal{A}$ admits a left integral $\int$ with $\int \mathbb{1}_{\mathcal{A}}\neq 0$ if and only if all right $\mathcal{A}$-comodules are completely reducible.

In the present work we are mainly interested in the case where $\mathcal{A}$ is a sub-Hopf-superalgebra of the finite dual $\mathcal{U}^\circ$ of a Hopf superalgebra $\mathcal{U}$. The comultiplication, counit, and antipode of $\mathcal{U}$ will also be denoted by Δ, ε, and S, respectively. In this case, if V is a right $\mathcal{A}$-comodule, then V also has a natural left $\mathcal{U}$-module structure defined by

$$x\,v = (-1)^{[x][v]}(\omega(v))(x), \quad \forall x\in\mathcal{U}, v\in V.$$

We denote by $\mathcal{U}-\mathrm{Mod}_r$ the collection of all the left $\mathcal{U}$-modules obtained from finite-dimensional right $\mathcal{A}$-comodules, which forms a monoidal category. The above proposition is equivalent to the following statement: The category $\mathcal{U}-\mathrm{Mod}_r$ is semisimple if and only if $\mathcal{A}$ admits a left integral which does not vanish on the identity.

Let us close this section by the following simple remark. As above, let $\mathcal{A}$ be a sub-Hopf-superalgebra of $\mathcal{U}^\circ$. The even grouplike elements of $\mathcal{U}^\circ$ are exactly the characters of $\mathcal{U}$, i.e., the superalgebra homomorphisms of $\mathcal{U}$ into $\mathbb{C}$. By convention, $\mathcal{A}$ always contains the unit element of $\mathcal{U}^\circ$, i.e., the counit $\varepsilon_{\mathcal{U}}$ of $\mathcal{U}$. This is the so-called trivial character of $\mathcal{U}$. Now Proposition 1 implies the following lemma.

Lemma 2: Suppose that $\mathcal{A}$ does not contain any non-trivial character of $\mathcal{U}$. Then every left integral on $\mathcal{A}$ is also a right integral.

III. INTEGRALS ON CLASSICAL SUPERGROUPS

Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra,[14,15] where $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ are the even and odd subspaces respectively. We take $\mathcal{U}$ to be the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. $U(\mathfrak{g})$ contains the enveloping algebra $U(\mathfrak{g}_{\bar{0}})$ of the Lie subalgebra $\mathfrak{g}_{\bar{0}}$ as a subalgebra. We denote $U(\mathfrak{g}_{\bar{0}})$ by $\mathcal{U}_e$, and let

$$\mathcal{I}: \mathcal{U}_e \to \mathcal{U}$$

be the embedding, which is a Hopf superalgebra map. It is well-known that the dual $\mathcal{I}^*$ of $\mathcal{I}$ induces a Hopf superalgebra map

$$\mathcal{P}:\ \mathcal{U}^\circ \to \mathcal{U}_e^\circ,$$

which is given by

$$\langle \mathcal{P}(a), u\rangle = \langle a, \mathcal{I}(u)\rangle, \quad \forall a \in \mathcal{U}^\circ, \quad u \in \mathcal{U}_e .$$

In the present work, a Lie supergroup will be defined in terms of its Hopf superalgebra of functions, i.e., we proceed as in the usual definition of quantum groups[16] or quantum supergroups[17] (for a related treatment of supergroups, see Refs. 18 and 19). More precisely, if $\mathfrak{g}$ is a Lie superalgebra, the superalgebra of functions on a Lie supergroup associated to $\mathfrak{g}$ will be a sub-Hopf-superalgebra $\mathcal{A}$ of $\mathcal{U}^\circ = U(\mathfrak{g})^\circ$, subject to the condition that $\mathcal{A}$ be dense in $U(\mathfrak{g})^*$. Actually, in our discussion of integrals, this latter property will not be used.

Thus, let $\mathcal{A}$ be a sub-Hopf-superalgebra of $\mathcal{U}^\circ$. We set

$$\mathcal{P}(\mathcal{A}) = \mathcal{A}_e ,$$

which is a Hopf subalgebra of $\mathcal{U}_e^\circ$. Then there exist the following natural Hopf superalgebra maps (which are injective if $\mathcal{A}$ is dense in $\mathcal{U}^*$ and, consequently, $\mathcal{A}_e$ is dense in $\mathcal{U}_e^*$):

$$\nu: U(\mathfrak{g}) \to \mathcal{A}^\circ,$$

$$x \mapsto \nu(x), \ \ \langle \nu(x), a\rangle = (-1)^{[x][a]}\langle a, x\rangle, \quad \forall a \in \mathcal{A}; \tag{6}$$

$$\nu_e : \mathcal{U}_e \to \mathcal{A}_e^\circ,$$

$$u \mapsto \nu_e(u) = \tilde{u}, \ \ \langle \tilde{u}, a_0\rangle = \langle a_0, u\rangle, \quad \forall a_0 \in \mathcal{A}_e ;$$

$$\hat{\mathcal{I}} = \nu\mathcal{I}: \mathcal{U}_e \to \mathcal{A}^\circ,$$

$$u \mapsto \hat{u}, \ \ \langle \hat{u}, a\rangle = \langle \tilde{u}, \mathcal{P}(a)\rangle = \langle \mathcal{P}(a), u\rangle = \langle a, \mathcal{I}(u)\rangle, \quad \forall a \in \mathcal{A}. \tag{7}$$

Let

$$\int_0 : \mathcal{A}_e \to \mathbb{C}$$

be a left integral on $\mathcal{A}_e$ with $\int_0 \mathbb{1}_{\mathcal{A}_e} = 1$. The existence of $\int_0$ depends on properties of $\mathfrak{g}_{\bar{0}}$ and $\mathcal{A}_e$. In the case when $\mathfrak{g}_{\bar{0}}$ is semisimple or reductive as a Lie algebra, such an $\int_0$ is known to exist and is right invariant as well. (However, see the Appendix about the reductive case.)

Lemma 3: The linear form $\int_0 \mathcal{P}$: $\mathcal{A} \to \mathbb{C}$ is left invariant with respect to $\mathcal{U}_e$ in the sense that

$$\hat{\mathcal{I}}(u)\cdot\left(\int_0 \mathcal{P}\right)=\varepsilon(u)\int_0 \mathcal{P}, \quad \forall u\in\mathcal{U}_e .$$

Proof: Lemma 3 can be confirmed by a direct calculation. For any $u\in\mathcal{U}_e$ and $a\in\mathcal{A}$, we have

$$\begin{aligned}\left\langle \hat{u}\cdot\left(\int_0 \mathcal{P}\right), a\right\rangle &= \sum_{(a)} \langle \hat{u}, a_{(1)}\rangle \int_0 \mathcal{P}(a_{(2)})\\ &= \sum_{(a)} \langle \tilde{u}, \mathcal{P}(a_{(1)})\rangle \int_0 \mathcal{P}(a_{(2)})\\ &= \left\langle \tilde{u}\otimes \int_0, \Delta\mathcal{P}(a)\right\rangle\\ &= \left\langle \tilde{u}\cdot \int_0, \mathcal{P}(a)\right\rangle = \varepsilon(u)\int_0 \mathcal{P}(a).\end{aligned}$$

Let $J=\mathcal{U}\mathfrak{g}_{\bar{0}}^{-}$. By using the Poincaré–Birkhoff–Witt theorem for Lie superalgebras,[15] one immediately sees the following.

Lemma 4: The subspace J is a left ideal of $\mathcal{U}$ with finite codimension.

Consequently, the quotient space $\mathcal{U}/J$ is a left $\mathcal{U}$-module in the standard fashion:

$$x(y+J)=xy+J, \quad \forall x\in\mathcal{U},\ y+J\in\mathcal{U}/J.$$

Note that this module is isomorphic to the $\mathcal{U}$-module induced from the trivial $\mathcal{U}_e$-module. According to the usual definition, an element $z+J\in\mathcal{U}/J$, with $z\in\mathcal{U}$, is said to be invariant (under the action of $\mathcal{U}$) if

$$x(z+J)=\varepsilon(x)z+J, \quad \forall x\in\mathcal{U}.$$

Let $z+J$ be any invariant of this type, and let $\nu(z)$ be the image of z in $\mathcal{A}^{\circ}$ under the natural Hopf superalgebra map (6). Then we have the following theorem.

Theorem 2: *The linear form $\int=\nu(z)\cdot\int_0\mathcal{P}$ is a left integral on $\mathcal{A}$ and does not depend on the choice of the representative for $z+J$. If $z\notin J$ and if the matrix elements of the $\mathcal{U}$-module $\mathcal{U}/J$ belong to $\mathcal{A}$, the integral $\int$ is not equal to zero.*

Remark: The definition of $\int$ involves implicitly the comultiplication of $\mathcal{A}$. For any $a\in\mathcal{A}$,

$$\int a=\left\langle \nu(z)\otimes\int_0\mathcal{P},\ \Delta(a)\right\rangle.$$

Proof of Theorem 2: It follows from Lemma 3 that for any $X_0\in\mathfrak{g}_{\bar{0}}^{-}$, $\nu(X_0)\cdot\int_0\mathcal{P}=0$. As ν is an algebra homomorphism, $\nu(y)\cdot\int_0\mathcal{P}=0$ for all $y\in J$. This proves the second part of the theorem. Now the invariance property of $z+J$ leads to

$$\nu(x)\cdot\int=\varepsilon(x)\int, \quad \forall x\in\mathcal{U}.$$

This implies Eq. (1). [Indeed, since $\mathcal{A}$ is contained in $\mathcal{U}^*$, it is sufficient to check Eq. (2) for all $a^*=\nu(x)$, $x\in\mathcal{U}$.]

To prove the last part of the theorem, we choose a homogeneous basis $(v_i)_{1\leqslant i\leqslant r}$ of $\mathcal{U}/J$ such that $v_1=\mathbb{1}_{\mathcal{U}}+J$. Let π be the representation of $\mathcal{U}$ in $\mathcal{U}/J$, and let $\pi_{i,j}$ be the matrix elements of π with respect to the basis (v_i), i.e.,

$$\pi(x)v_j=\sum_{i=1}^{r}\pi_{i,j}(x)v_i \quad \text{if } x\in\mathcal{U},\ 1\leqslant j\leqslant r.$$

Since v_1 is $\mathcal{U}_e$-invariant, we have

$$\pi_{i,1}(x)=\varepsilon_{\mathcal{U}_e}(x)\,\delta_{i,1} \quad \text{if } x\in\mathcal{U}_e,\ 1\leqslant i\leqslant r.$$

This implies that

$$\int\pi_{i,1}=(-1)^{[v_i]}\pi_{i,1}(z), \quad 1\leqslant i\leqslant r$$

(recall that we are assuming that $\int_0 \mathbb{1}_{\mathcal{A}_e}=1$). Since $z\notin J$ and since

$$z+J=\pi(z)v_1=\sum_{i=1}^{r}\pi_{i,1}(z)v_i,$$

at least one of the matrix elements $\pi_{i,1}(z)$ must be different from zero. This proves the theorem.

We notice that

$$\int\mathbb{1}_{\mathcal{A}}=\varepsilon(z)\int_0\mathbb{1}_{\mathcal{A}_e}.$$

Taking for granted that $\int_0\mathbb{1}_{\mathcal{A}_e}$ is different from zero, we see that $\int\mathbb{1}_{\mathcal{A}}\neq 0$ if and only if $\varepsilon(z)\neq 0$.

Remark: Suppose that the Lie algebra $\mathfrak{g}_{\bar 0}$ is reductive, and that the adjoint representation of the center of $\mathfrak{g}_{\bar 0}$ in $\mathfrak{g}_{\bar 1}$ is diagonalizable. Then the subspace of $\mathcal{U}$-invariant elements of $\mathcal{U}/J$ is at most one-dimensional. This follows at once from Theorems 1 and 2, applied to a suitable sub-Hopf-superalgebra $\mathcal{A}$ of $\mathcal{U}^\circ$ (see the Appendix).

Let us now consider examples.

Example 1: The Berezin integral

Consider the purely odd Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar 1}$ with the basis $\{\xi_i,\ i=1,2,\ldots,n\}$ and with the super bracket

$$[\xi_i,\xi_j]=0, \quad \forall i,j.$$

Obviously, $\mathcal{U}=U(\mathfrak{g})$ is the Grassmann algebra on the n generators ξ_i, and $\mathcal{U}_e=\mathbb{C}\mathbb{1}_{\mathcal{U}}$. It is well-known that $\mathcal{U}$ has the basis

$$\Xi_{j_1\cdots j_l}=\xi_{j_1}\cdots\xi_{j_l}, \quad 1\leqslant j_1<\cdots<j_l\leqslant n,$$

where the $l=0$ element is understood to be the unity. The Hopf structure of $\mathcal{U}$ is the standard one for enveloping algebras of Lie superalgebras.

Introduce a basis $\{\Theta_{i_1\cdots i_k},\ 1\leqslant i_1<\cdots<i_k\leqslant n,\ 0\leqslant k\leqslant n\}$ for $\mathcal{U}^*$ (the $k=0$ case corresponds to the unit element) such that

$$\langle\Theta_{i_1\cdots i_k},\Xi_{j_1\cdots j_l}\rangle=(-1)^{(1/2)k(k-1)}\delta_{kl}\,\delta_{i_1j_1}\cdots\delta_{i_kj_k},$$

and set

$$\theta_i=\Theta_i, \quad i=1,2,\ldots,n.$$

The cocommutativity of $\mathcal{U}$ implies that

$$\theta_i\theta_j+\theta_j\theta_i=0, \quad \forall i,j.$$

It is also easy to show that

$$\Theta_{i_1 \cdots i_k} = \theta_{i_1} \cdots \theta_{i_k}, \quad i_1 < \cdots < i_k .$$

As a Hopf superalgebra, $\mathcal{U}^*$ has the unique comultiplication such that

$$\Delta(\theta_i) = \theta_i \otimes \mathbb{1} + \mathbb{1} \otimes \theta_i ,$$

the counit is fixed by

$$\varepsilon(\theta_i) = 0,$$

and the antipode is specified by

$$S(\theta_i) = -\theta_i .$$

Since $(\mathcal{U}^*)^* \cong \mathcal{U}$ in this case, we make the identification. It is obvious that $\int_0 \mathcal{P} = \mathbb{1}_{\mathcal{U}}$. Moreover, the $\mathcal{U}$-invariant elements of $\mathcal{U}$ are the scalar multiples of $\Xi_{1\,2\,\ldots\,n}$. Thus upon choosing an appropriate normalization we obtain the unique integral

$$\int = (-1)^{(1/2)\, n(n-1)} \xi_1 \xi_2 \cdots \xi_n ,$$

which yields the standard Berezin integral on the Grassmanian algebra $\mathcal{U}^*$:

$$\int \theta_{i_1} \theta_{i_2} \cdots \theta_{i_k} = 0, \quad \text{if } \; k < n,$$

$$\int \theta_1 \theta_2 \cdots \theta_n = 1.$$

To explain the left (and right) invariance of $\int$ in more familiar terms, note that if $P(\theta)$ is any polynomial in the θ_i's, then

$$\Delta P(\theta) = P(\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta).$$

Left invariance of the integral means

$$\left(\mathrm{id} \otimes \int \right) \Delta(P(\theta)) = \int P(\theta).$$

One may write $\mathbb{1} \otimes \theta_i$ as θ_i, and denote $\theta_i \otimes \mathbb{1}$ by λ_i, which is regarded as an independent Grassmann number. The above equation states that

$$\int_\theta P(\theta + \lambda) = \int P(\theta),$$

where the subscript θ on the left hand side indicates the fact that the "integration" is carried out over the θ's. The last equation is nothing but the translational invariance of the Berezin integral.

Example 2: The Lie supergroup $\mathrm{SL}(m|n)$

Let $\mathfrak{g}$ denote the Lie superalgebra $\mathfrak{sl}(m|n)$, which we shall regard as a subalgebra of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. Let $\{E_{ab} \,|\, a,b = 1,2,\ldots,m+n\}$ be the standard homogeneous basis of $\mathfrak{gl}(m|n)$, which satisfies the commmutation relations

$$[E_{ab}, E_{cd}] = \delta_{bc} E_{ad} - (-1)^{[E_{ab}][E_{cd}]} \delta_{da} E_{cb} ,$$

where $[\cdot,\cdot]$ should be understood as the graded brackets, namely, it is symmetric when both arguments are odd, and antisymmetric otherwise.

The standard basis for $\mathfrak{g}$ then is given by

$$E_{ab}, \quad a\neq b; \;\; h_a = E_{aa} - (-1)^{\delta_{am}} E_{a+1,a+1}, \;\; a<m+n.$$

The maximal even subalgebra of $\mathfrak{g}$ is $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(m)\oplus\mathfrak{sl}(n)\oplus\mathfrak{gl}(1)$. Let $\mathfrak{g}_{\bar{1}+}$ be the odd subalgebra spanned by $E_{i\mu}$, $i\leqslant m$, $\mu>m$, and $\mathfrak{g}_{\bar{1}-}$ be that spanned by $E_{\mu i}$. Then $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_{\bar{1}-}\oplus\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}+}$ (as vector spaces). Under the Lie superbracket,

$$\begin{gathered}[\mathfrak{g}_{\bar{1}+},\mathfrak{g}_{\bar{1}+}] = \{0\}, \quad [\mathfrak{g}_{\bar{1}-},\mathfrak{g}_{\bar{1}-}] = \{0\}, \\ [\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{1}\pm}]\subset\mathfrak{g}_{\bar{1}\pm}, \quad [\mathfrak{g}_{\bar{1}+},\mathfrak{g}_{\bar{1}-}]\subset\mathfrak{g}_{\bar{0}}.\end{gathered} \tag{8}$$

Next, we observe that $U(\mathfrak{g}_{\bar{1}+})$ and $U(\mathfrak{g}_{\bar{1}-})$ are both isomorphic to the Grassmann algebra on mn generators. The subspaces of the highest Grassmann degree in $U(\mathfrak{g}_{\bar{1}+})$ and $U(\mathfrak{g}_{\bar{1}-})$ are both one-dimensional. We choose the following bases for them, respectively,

$$\mathbb{E} = \mathbb{E}_m\mathbb{E}_{m-1}\cdots\mathbb{E}_1, \quad \mathbb{F} = \mathbb{F}_1\mathbb{F}_2\cdots\mathbb{F}_m,$$

where

$$\mathbb{E}_i = E_{i,m+1}E_{i,m+2}\cdots E_{i,m+n},$$

$$\mathbb{F}_i = E_{m+n,i}E_{m+n-1,i}\cdots E_{m+1,i}.$$

Then we have

$$[X,\mathbb{E}] = [X,\mathbb{F}] = 0, \;\; \forall X\in\mathfrak{g}_{\bar{0}},$$

$$\xi_+\mathbb{E} = 0, \;\; \forall\xi_+\in\mathfrak{g}_{\bar{1}+},$$

$$\xi_-\mathbb{F} = 0, \;\; \forall\xi_-\in\mathfrak{g}_{\bar{1}-},$$

$$\xi_-\mathbb{E} - (-1)^{mn}\mathbb{E}\,\xi_-\in U(\mathfrak{g})\mathfrak{g}_{\bar{0}}, \;\; \forall\xi_-\in\mathfrak{g}_{\bar{1}-}.$$

Defining

$$\Gamma = \mathbb{E}\,\mathbb{F},$$

it follows that

$$X\Gamma\in U(\mathfrak{g})\mathfrak{g}_{\bar{0}}, \;\; \forall X\in\mathfrak{g}.$$

Let t be the defining representation of $\mathfrak{sl}(m|n)$, with

$$t(E_{ab}) = e_{ab}, \quad a\neq b,$$

$$t(h_a) = e_{aa} - (-1)^{\delta_{am}} e_{a+1,a+1},$$

where the e_{ab}'s are the matrix units, and let t_{ab}, $a,b=1,2,\ldots,m+n$, be the elements of $\mathcal{U}^\circ = U(\mathfrak{g})^\circ$ defined by

$$(t_{ab}(x))_{a,\,b=1}^{m+n} = t(x), \quad \forall x \in U(\mathfrak{g}).$$

Moreover, let $\overline{t}$ be the dual representation of t, and let us similarly introduce the matrix elements $\overline{t}_{ab} \in U(\mathfrak{g})^\circ$ of $\overline{t}$. We note that

$$\sum_c \overline{t}_{ca} t_{cb} (-1)^{([a]+[c])([b]+\overline{1})} = \delta_{ab}.$$

The standard comultiplication on $U(\mathfrak{g})$ is super cocommutative. Therefore the finite dual $U(\mathfrak{g})^\circ$ is a super commutative Hopf superalgebra. The matrix elements t_{ab} and $\overline{t}_{ab}$ of the vector representation and the dual vector representation generate a sub-Hopf-superalgebra $\mathcal{A}$ of $U(\mathfrak{g})^\circ$, with the comultiplication

$$\Delta(t_{ab}) = \sum_c t_{ac} \otimes t_{cb} (-1)^{([a]+[c])([c]+[b])},$$

$$\Delta(\overline{t}_{ab}) = \sum_c \overline{t}_{ac} \otimes \overline{t}_{cb} (-1)^{([a]+[c])([c]+[b])},$$

the counit $\varepsilon(t_{ab}) = \varepsilon(\overline{t}_{ab}) = \delta_{ab}$, and the involutary antipode $S(t_{ab}) = (-1)^{[a]([a]+[b])} \overline{t}_{ba}$, where

$$[a] = \begin{cases} \overline{0}, & a \leqslant m, \\ \overline{1}, & a > m. \end{cases}$$

An important fact is the following.

Proposition 3: The subspace $\mathcal{A}$ is dense in $U(\mathfrak{sl}(m|n))^*$.

Proof: This follows from a slight strengthening of a theorem which in the nongraded case is due to Harish-Chandra. Let V be a finite-dimensional graded vector space and let $\mathfrak{g}$ be a graded subalgebra of the Lie superalgebra $\mathfrak{sl}(V)$. We regard V as a $\mathfrak{g}$-module. Arguing as in the nongraded case (see the proof of Theorem 2.5.7 in Ref. 20) one can easily prove that for any nonzero element $x \in U(\mathfrak{g})$ there exists an integer $r \geqslant 0$ such that x acts nontrivially on $V^{\otimes r}$. Actually, there is a minor complication: Dixmier's proof only applies if $\dim V_{\overline{0}} \neq \dim V_{\overline{1}}$. But if $\dim V_{\overline{0}} = \dim V_{\overline{1}}$, we can embed V into $W = V \oplus \mathbb{C}$, where $\mathbb{C}$ is regarded as a trivial $\mathfrak{g}$-module. Then his arguments apply to W, and the tensorial powers of W are isomorphic to direct sums of tensorial powers of V. This proves the proposition.

As at the beginning of this section, let $\mathcal{P}$ be the dual of the embedding of $U(\mathfrak{g}_{\overline{0}})$ in $U(\mathfrak{g})$. We have

$$\mathcal{P}(t_{i\mu}) = \mathcal{P}(t_{\mu i}) = \mathcal{P}(\overline{t}_{i\mu}) = \mathcal{P}(\overline{t}_{\mu i}) = 0, \quad 1 \leqslant i \leqslant m, \quad m < \mu \leqslant m+n.$$

Set

$$\mathcal{A}_e = \mathcal{P}(\mathcal{A}).$$

Then $\mathcal{A}_e$ has a Peter-Weyl type basis in terms of the matrix elements of irreducible finite-dimensional representations of $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{gl}(1)$. Thus it follows from the discussion of the last section that there exists a unique normalized left integral

$$\int_0 : \ \mathcal{A}_e \to \mathbb{C},$$

which also turns out to be right invariant (see the Appendix). Denote by $\nu(\Gamma) \in \mathcal{A}^\circ$ the image of Γ under the natural embedding $U(\mathfrak{g}) \to \mathcal{A}^\circ$. Recalling Lemma 2, we have the following theorem.

Theorem 3: *The linear form $\int = \nu(\Gamma) \cdot \int_0 \mathcal{P}$ is a nontrivial left and right integral on $\mathcal{A}$.*

To see that $\int$ is indeed nontrivial, we consider $\int \Theta \overline{\Theta}$, where

$$\Theta_i = t_{i,m+n} t_{i,\,m+n-1} \cdots t_{i,m+1},$$

$$\overline{\Theta}_i = \overline{t}_{i,m+n} \overline{t}_{i,m+n-1} \cdots \overline{t}_{i,m+1}, \quad i = 1, 2, \ldots, m,$$

$$\Theta = \Theta_m \Theta_{m-1} \cdots \Theta_1,$$

$$\overline{\Theta} = \overline{\Theta}_m \overline{\Theta}_{m-1} \cdots \overline{\Theta}_1.$$

We have

$$\int \Theta \overline{\Theta} = \langle \Theta \overline{\Theta}, \mathbb{E}\mathbb{F} \rangle \int_0 \mathcal{P}(\det(t_{\mu\nu}) \det(\overline{t}_{\mu\nu}))^m.$$

As

$$\det(t_{\mu\nu}) \det(\overline{t}_{\mu\nu})(u) = \varepsilon(u), \quad \forall u \in U(\mathfrak{g}_{\bar{0}}),$$

we immediately obtain

$$\int_0 \mathcal{P}(\det(t_{\mu\nu}) \det(\overline{t}_{\mu\nu}))^m = 1.$$

By induction we can show that

$$\langle \Theta \overline{\Theta}, \mathbb{E}\mathbb{F} \rangle = (-1)^{mn(mn+1)/2},$$

hence

$$\int \Theta \overline{\Theta} = (-1)^{mn(mn+1)/2}.$$

Example 3: The Lie supergroup $\mathrm{OSP}(2|2n)$

The Lie superalgebras $\mathfrak{g} = \mathfrak{osp}(2|2n)$ form the other series of type I (basic classical) Lie superalgebras besides $\mathfrak{sl}(m|n)$. They share many properties with the latter. In particular, the odd subspace of $\mathfrak{osp}(2|2n)$ is a direct sum of $\mathfrak{g}_{\bar{1}+}$ and $\mathfrak{g}_{\bar{1}-}$. Both $U(\mathfrak{g}_{\bar{1}+})$ and $U(\mathfrak{g}_{\bar{1}-})$ are isomorphic to the Grassmann algebra on $2n$ generators. The maximal even subalgebra of $\mathfrak{osp}(2|2n)$ is $\mathfrak{sp}(2n) \oplus \mathfrak{gl}(1)$, and $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}\pm}$ satisfy relations of the same form as (8).

The subspaces of $U(\mathfrak{g}_{\bar{1}\pm})$ of the highest Grassmann degree are both one-dimensional. We choose bases $\mathbb{E}$ and $\mathbb{F}$ for them, respectively, and set $\Gamma = \mathbb{E}\mathbb{F}$. Then

$$X\Gamma \in U(\mathfrak{g})\mathfrak{g}_{\bar{0}}, \quad \forall X \in \mathfrak{g}.$$

Let t be the defining representation of $\mathfrak{osp}(2|2n)$. It is known that t is self-dual. Introduce the matrix elements of t,

$$t_{ab} \in U(\mathfrak{g})^\circ, \quad a,b = 1,2,\ldots,2n+2,$$

with t_{ij} and $t_{\mu\nu}$ being even, and $t_{i\mu}$ and $t_{\mu i}$ odd, where $i,j=1,2;\ \mu,\nu=3,4,\ldots,2n+2$.

Proposition 4: The elements t_{ab} *generate a sub-Hopf-superalgebra* $\mathcal{A}$ *of* $U(\mathfrak{osp}(2|2n))^\circ$, *and* $\mathcal{A}$ *is dense in* $U(\mathfrak{osp}(2|2n))^*$.

Proof: This follows from the proof of Proposition 3.

Set $\mathcal{A}_e=\mathcal{P}(\mathcal{A})$. Then $\mathcal{A}_e$ admits a unique (up to scalar multiples) left integral $\int_0$. Denoting by $\nu(\Gamma)$ the canonical image of Γ in $\mathcal{A}^\circ$, we have

Theorem 4: *The linear form* $\int=\nu(\Gamma)\cdot\int_0\mathcal{P}$ *is a nontrivial left and right integral on* $\mathcal{A}$.

The $t_{i\mu}$ and $t_{\mu i}$ generate a Grassmann algebra contained in $\mathcal{A}$. We take Θ to be a nonzero element of the highest degree in this Grassmann algebra. Then direct computations can show that

$$\int\Theta\neq 0.$$

Example 4: The Lie supergroup OSP$(1|2n)$

Let us start with the simplest case, $n=1$. The Dynkin diagram of $\mathfrak{osp}(1|2)$ is just ●, and the simple Chevalley generators are $\{e,f,h\}$, where e and f are odd while h is even, with the commutation relations

$$[h,e]=e,\ \ [h,f]=-f,\ \ [e,f]=h.$$

It is important to observe that $[e,e]=E$, $[f,f]=F$ and h span an $\mathfrak{sl}(2)$ subalgebra, which is the maximal even subalgebra $\mathfrak{osp}(1|2)_{\bar{0}}$. This is a general feature of any type II superalgebra, where some simple generators of the maximal even subalgebra are generated by odd elements. We denote $\mathfrak{g}=\mathfrak{osp}(1|2)$, $\mathfrak{g}_{\bar{0}}=\mathfrak{sl}(2)\subset\mathfrak{osp}(1|2)$, $\mathcal{U}=U(\mathfrak{g})$ and $\mathcal{U}_e=U(\mathfrak{g}_{\bar{0}})$.

Now

$$\mathbb{1}+ef+\mathcal{U}\mathfrak{g}_{\bar{0}}$$

is an invariant of the left $\mathcal{U}$-module $\mathcal{U}/\mathcal{U}\mathfrak{g}_{\bar{0}}$, and we have the left and right integral

$$\int\ =\nu(\mathbb{1}+ef)\cdot\int_0\mathcal{P}:\mathcal{U}^\circ\rightarrow\mathbb{C},$$

where $\int_0:\mathcal{U}_e^\circ\rightarrow\mathbb{C}$ is the standard Haar functional on $\mathcal{U}_e^\circ$. Consider $\int\mathbb{1}_{\mathcal{U}^\circ}$. We have

$$\int\mathbb{1}_{\mathcal{U}^\circ}=\langle\mathbb{1}_{\mathcal{U}^\circ},\mathbb{1}+ef\rangle\int_0\mathcal{P}(\mathbb{1}_{\mathcal{U}^\circ})=\int_0\mathbb{1}_{\mathcal{U}_e^\circ}\neq 0.$$

That is, the integral does not vanish on the identity element of $\mathcal{U}^\circ$. It follows from the discussion of Sec. II that all finite-dimensional representations of $\mathfrak{osp}(1|2)$ are completely reducible, which, of course, is a well-known fact.

The general case can be treated similarly. We do not go into details but only mention that an even element $u_0\in\mathcal{U}=U(\mathfrak{osp}(1|2n))$ such that $\varepsilon(u_0)\neq 0$ and such that $u_0+\mathcal{U}\mathfrak{g}_{\bar{0}}$ is invariant in $\mathcal{U}/\mathcal{U}\mathfrak{g}_{\bar{0}}$ has been constructed by Djoković and Hochschild in Ref. 21. Moreover, they have proved the following theorem:

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra over a field of characteristic zero. Then all finite-dimensional representations of $\mathfrak{g}$ are completely reducible if and only if the following two conditions are satisfied.

(1) The Lie algebra $\mathfrak{g}_{\bar{0}}$ is semisimple.
(2) There is an element u_0 in $U(\mathfrak{g})$ such that $u_0+U(\mathfrak{g})\mathfrak{g}_{\bar{0}}$ is an invariant element of $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_{\bar{0}}$ and satisfies $\varepsilon(u_0)\neq 0$.

Visibly, in the cited reference the element u_0 has been a decisive tool in the proof that all finite-dimensional representations of $\mathfrak{osp}(1|2n)$ are completely reducible. It is remarkable that in the present work it serves to construct a left integral on $\mathcal{U}^\circ$ which does not vanish on the unit element, a result which, in turn, implies the complete reducibility of the $\mathcal{U}^\circ$-comodules and hence of the finite-dimensional $\mathcal{U}$-modules.

Example 5: The Lie supergroup $\mathrm{OSP}(3|2)$

Let $\mathfrak{g}$ denote the Lie superalgebra $\mathfrak{osp}(3|2)$. It is the simplest of those orthosymplectic Lie superalgebras which are not of type I and not one of the special algebras $\mathfrak{osp}(1|2n)$. Its maximal even subalgebra is $\mathfrak{g}_{\bar{0}}=\mathfrak{so}(3)\oplus\mathfrak{sp}(2)$. The $\mathfrak{g}$-module $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_{\bar{0}}$ will be denoted by W. We shall also need the quadratic Casimir element $C\in U(\mathfrak{g})$ and the corresponding Casimir operator C_W acting on W.

In the subsequent investigation of the $\mathfrak{g}$-module W we are going to use the classification of finite-dimensional irreducible $\mathfrak{g}$-modules obtained by Van der Jeugt in Ref. 22. Both the $\mathfrak{g}$-modules and the $\mathfrak{g}_{\bar{0}}$-modules are characterized by a pair of numbers $p,q\in\{0,\frac{1}{2},1,\frac{3}{2},\ldots\}$. By a slight abuse of notation, we denote the corresponding $\mathfrak{g}$-module by $[p,q]$, and the corresponding $\mathfrak{g}_{\bar{0}}$-module by (p,q). [We remark that p is associated in the obvious way to $\mathfrak{so}(3)$ and q to $\mathfrak{sp}(2)$.]

A version of the Poincaré–Birkhoff–Witt theorem implies that W, regarded as a $\mathfrak{g}_{\bar{0}}$-module, is isomorphic to the Grassmann algebra constructed over $\mathfrak{g}_{\bar{1}}$. Using the representation theory of $\mathfrak{sl}(2)$, we conclude that the $\mathfrak{g}_{\bar{0}}$-module W decomposes into the direct sum of the modules contained in the following list, where the first line gives the Grassmann degree to which the modules underneath belong.

0	1	2	3	4	5	6
$(0,0)$	$(1,\frac{1}{2})$	$(1,1)$	$(2,\frac{1}{2})$	$(1,1)$	$(1,\frac{1}{2})$	$(0,0)$
		$(2,0)$	$(1,\frac{1}{2})$	$(2,0)$		
		$(0,0)$	$(0,\frac{3}{2})$	$(0,0)$		

Comparison with the lower-dimensional irreducible $\mathfrak{g}$-modules then shows that for a Jordan-Hölder sequence of the $\mathfrak{g}$-module W the irreducible quotients must be isomorphic to the following modules:

$$[0,\tfrac{3}{2}],\quad [1,1],\quad [1,\tfrac{1}{2}],\quad [0,0],\quad [0,0].$$

For the convenience of the reader and for later use, we also note how these modules decompose into irreducible $\mathfrak{g}_{\bar{0}}$-submodules, moreover, in the first column we give the eigenvalue of the quadratic Casimir operator (normalized as in Ref. 22) in these modules:

$$\begin{array}{rl}
-6 & [0,\tfrac{3}{2}]\cong(0,\tfrac{3}{2})\oplus(1,1)\oplus(1,\tfrac{1}{2})\oplus(0,0)\\
0 & [1,1]\cong(1,1)\oplus(1,\tfrac{1}{2})\oplus(2,\tfrac{1}{2})\oplus(2,0)\\
2 & [1,\tfrac{1}{2}]\cong(1,\tfrac{1}{2})\oplus(2,0)\oplus(0,0)\\
0 & [0,0]\cong(0,0)
\end{array}$$

Note that at this point it is obvious that the $\mathfrak{g}$-module W is not completely reducible: It is generated, as a $\mathfrak{g}$-module, by a $\mathfrak{g}_{\bar{0}}$-invariant element; the multiplicity of $(0,0)$ in the $\mathfrak{g}_{\bar{0}}$-module W is equal to 4, but the length of the $\mathfrak{g}$-module W (i.e., the number of irreducible quotients of a Jordan-Hölder sequence) is equal to 5.

The eigenvalues given previously imply that the primary decomposition of W with respect to C_W takes the following form:

$$W=W_{-6}\oplus W_2\oplus W_0, \tag{9}$$

where W_r, $r\in\{-6,2,0\}$, is the primary subspace of W corresponding to the eigenvalue r of C_W. Of course, the W_r's are $\mathfrak{g}$-submodules of W. Regarded as $\mathfrak{g}$-modules, we have

$$W_{-6}\cong[0,\tfrac{3}{2}],\quad W_2\cong[1,\tfrac{1}{2}],$$

whereas W_0 has a Jordan-Hölder sequence of the form

$$W_0\supset W_0'\supset W_0''\supset\{0\},$$

where one of the three modules W_0/W_0', W_0'/W_0'', W_0'' is isomorphic to $[1,1]$, while the other two are trivial one-dimensional. In any case we have

$$C_W(W_0)\subset W_0', \tag{10}$$

$$C_W(W_0')\subset W_0'', \tag{11}$$

$$C_W(W_0'')=\{0\}. \tag{12}$$

We stress that while W_{-6} and W_2 are eigenspaces of C_W, this is not so for W_0. In fact, we shall see that the restriction of C_W to W_0 is not equal to zero but only nilpotent.

Lemma 5: The subspace $C_W(W_0)$ of W_0 is either a trivial one-dimensional $\mathfrak{g}$-submodule of W_0 or else it is equal to $\{0\}$.

Proof: In the subsequent discussion, it is important to keep the following fact in mind: (∗) The $\mathfrak{g}$-module $[1,1]$ does not contain a trivial $\mathfrak{g}_{\bar{0}}$-submodule.

There are three cases to consider.

(a) The module W_0/W_0' is isomorphic to $[1,1]$. This case is not possible since W_0, like W_2, is generated by a $\mathfrak{g}_{\bar{0}}$-invariant element which, under the present assumption and because of (∗), would have to belong to W_0'.

(b) The module W_0'/W_0'' is isomorphic to $[1,1]$. In this case, W_0'' consists of $\mathfrak{g}$-invariant elements, hence the existence of nonzero $\mathfrak{g}$-invariant elements in W_0 is obvious. However, we want to find an explicit expression for them, and a first step towards this end is the lemma. According to Eq. (11) we have $C_W(W_0')\subset W_0''$. Using Eq. (12) and recalling (∗), we can even conclude that $C_W(W_0')=\{0\}$. Thus C_W induces a $\mathfrak{g}$-module map $W_0/W_0'\to W_0$, and this implies our claim. Actually, it is easy to see that $C_W(W_0)\subset W_0''$.

(c) The module W_0'' is isomorphic to $[1,1]$. Equation (12) says that $C_W(W_0'')=\{0\}$, hence C_W induces a $\mathfrak{g}$-module map $W_0'/W_0''\to W_0'$ which, according to Eq. (11), is even a map into W_0''. Invoking (∗) we conclude that $C_W(W_0')=\{0\}$, and our claim follows as in part (b). This proves the lemma.

Let us now recall the decomposition (9) of W and also the fact that the $\mathfrak{g}$-module W is generated by the element $\mathbb{1}_{U(\mathfrak{g})}+U(\mathfrak{g})\mathfrak{g}_{\bar{0}}^-$. Then the lemma above can be rephrased as follows: Either the element

$$z=C(C-2)(C+6)\in U(\mathfrak{g})$$

belongs to $U(\mathfrak{g})\mathfrak{g}_{\bar{0}}^-$, or else $z+U(\mathfrak{g})\mathfrak{g}_{\bar{0}}^-$ is a nontrivial invariant element of $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_{\bar{0}}^-$.

Thus all that remains to be shown is that z does not belong to $U(\mathfrak{g})\mathfrak{g}_{\bar{0}}^-$. This is an easy consequence of the Poincaré–Birkhoff–Witt (PBW) theorem, which allows us to construct a suitable basis of W. Actually, the task can be simplified, as follows. The Casimir element C can be decomposed (in various ways) into the sum of two pieces,

$$C=C_o+C_e,$$

where C_o is quadratic in the elements of $\mathfrak{g}_{\bar{1}}$, and where C_e belongs to $U(\mathfrak{g}_{\bar{0}})$. Since C commutes with all elements of $U(\mathfrak{g})$, it follows that

$$z \in C_o(C_o-2)(C_o+6)+U(\mathfrak{g})\mathfrak{g}_{\bar{0}},$$

and hence we can replace z by

$$z_o = C_o(C_o-2)(C_o+6).$$

Applying Theorem 2 to z or z_o and recalling Lemma 2 we obtain a nonzero left and right integral on $U(\mathfrak{osp}(3|2))^\circ$.

IV. INTEGRALS ON QUANTUM SUPERGROUPS

We shall extend the construction of integrals on classical supergroups to quantum supergroups at generic q. Recall that the Drinfeld-Jimbo quantum superalgebra $U_q(\mathfrak{g})$ associated with a simple basic classical Lie superalgebra $\mathfrak{g}$ is usually defined with respect to the distinguished simple root system of $\mathfrak{g}$ where only one odd simple root exists. By removing the odd simple generators (but retaining the corresponding Cartan generator), one obtains a graded quantum subalgebra $U_q(\mathfrak{g}_0) \subset U_q(\mathfrak{g})$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is an even subalgebra of $\mathfrak{g}$, which is a reductive Lie algebra. We stress that while for the basic classical Lie superalgebras of type I we have $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, this is *not* the case for type II.

An important fact is that $U_q(\mathfrak{g}_0)$ forms a Hopf subalgebra of $U_q(\mathfrak{g})$, with its structure inherited from the latter. We have the following Hopf superalgebra maps:

$$\mathcal{I}\colon \;\; U_q(\mathfrak{g}_0) \to U_q(\mathfrak{g}),$$

$$\mathcal{P}\colon \;\; U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{g}_0)^\circ,$$

where $\mathcal{I}$ is the natural embedding and $\mathcal{P}$ is induced from its dual $\mathcal{I}^*$.

A quantum supergroup associated with $U_q(\mathfrak{g})$ is defined by specifying its superalgebra of functions $\mathcal{A}$, where $\mathcal{A}$ should meet two basic requirements, namely, it forms a sub-Hopf-superalgebra of $U_q(\mathfrak{g})^\circ$, and it is dense in $U_q(\mathfrak{g})^*$. In general, $\mathcal{A}$ is generated by the matrix elements of some finite-dimensional irreducible representations of $U_q(\mathfrak{g})$. The structure of $\mathcal{A}$ associated with a type I quantum superalgebra has been extensively studied. The fact that $\mathcal{A}$ is dense in $U_q(\mathfrak{g})^*$ implies that the natural Hopf superalgebra maps

$$\nu\colon \;\; U_q(\mathfrak{g}) \to \mathcal{A}^\circ,$$

$$\hat{\mathcal{I}} = \nu\mathcal{I}\colon \;\; U_q(\mathfrak{g}_0) \to \mathcal{A}^\circ,$$

are embeddings.

Denote $\mathcal{A}_e = \mathcal{P}(\mathcal{A})$. Then $\mathcal{A}_e$ separates points of $U_q(\mathfrak{g}_0)$, i.e., it is dense in $U_q(\mathfrak{g}_0)^*$. Furthermore, $\mathcal{A}_e$ admits a Peter-Weyl type basis in terms of the matrix elements of finite-dimensional irreducible representations of $U_q(\mathfrak{g}_0)$, and there exists a unique (up to scalar multiples) left integral

$$\int_0\colon \;\; \mathcal{A}_e \to \mathbb{C},$$

which also turns out to be right invariant, and it is nonvanishing on $1_{\mathcal{A}_e}$.

Similar to the classical case, we consider

$$\int_0 \mathcal{P}\colon \;\; \mathcal{A} \to \mathbb{C},$$

which is clearly left invariant with respect to $U_q(\mathfrak{g}_0)$, i.e.,

$$\hat{\mathcal{I}}(u)\cdot\int_0 \mathcal{P}=\varepsilon(u)\int_0 \mathcal{P}, \quad \forall u\in U_q(\mathfrak{g}_0).$$

Let K denote the ideal of $U_q(\mathfrak{g}_0)$ defined by

$$K=\{u\in U_q(\mathfrak{g}_0)|\,\varepsilon(u)=0\},$$

where ε is the counit of $U_q(\mathfrak{g})$. Then

$$J=U_q(\mathfrak{g})K \tag{13}$$

is a left ideal of $U_q(\mathfrak{g})$.

Lemma 6: If $\mathfrak{g}$ *is one of the Lie superalgebras* $\mathfrak{sl}(m|n)$ *or* $\mathfrak{osp}(2|2n)$ (*i.e., if* $\mathfrak{g}$ *is basic classical of type I*), *the left ideal* J *has finite codimension in* $U_q(\mathfrak{g})$.

Proof: This follows immediately from the PBW theorems for these quantum superalgebras established in Refs. 23 and 24.

Clearly $U_q(\mathfrak{g})/J$ forms a left $U_q(\mathfrak{g})$-module under the natural action

$$x(y+J)=xy+J, \quad \forall x,y\in U_q(\mathfrak{g}).$$

Let $z+J$ be an invariant of $U_q(\mathfrak{g})/J$, i.e., $x(z+J)=\varepsilon(x)z+J$, $\forall x\in U_q(\mathfrak{g})$. Nontrivial invariants of this kind exist for type I quantum superalgebras, as we will see later. However, we doubt that the type II quantum superalgebras admit such invariants, as in this case J is expected to have infinite codimension.

Theorem 5: *Let* $\int=\nu(z)\cdot\int_0\mathcal{P}$. *Then* $\int$ *is a left integral on* $\mathcal{A}$, *that does not depend on the representative of* $z+J$. *As before,* $\nu(z)$ *is the image of* z *under the natural embedding* $U_q(\mathfrak{g})\to\mathcal{A}^\circ$.

Proof: The proof goes in the same way as in the classical case.

Example 6: *The quantum supergroup* $\mathrm{SL}_q(m|n)$

We study the quantum supergroup $\mathrm{SL}_q(m|n)$. The quantum superalgebra $U_q(\mathfrak{sl}(m|n))$ is generated by the simple and the Cartan generators

$$E_{a,\,a+1}, \quad E_{a+1,a}, \quad k_a^{\pm 1}, \quad a=1,2,\ldots,m+n-1,$$

subject to the standard relations. (Here $k_a=K_aK_{a+1}^{-1}$ in the notation of Ref. 23.) The generators $E_{m,\,m+1}$ and $E_{m+1,\,m}$ are odd, while all the others are even. Define recursively

$$E_{ab}=E_{ac}E_{cb}-q_c^{-1}E_{cb}E_{ac},$$

$$E_{ba}=E_{bc}E_{ca}-q_cE_{ca}E_{bc}, \quad a<c<b,$$

where $q_c=q^{(-1)^{[c]}}$. The vector representation t of $U_q(\mathfrak{sl}(m|n))$ is given by

$$t(E_{a,\,a\pm 1})=e_{a,\,a\pm 1},$$

$$t(k_a)=q_a^{e_{aa}}q_{a+1}^{-e_{a+1,\,a+1}}=1+(q_a-1)e_{aa}+(q_{a+1}^{-1}-1)e_{a+1,\,a+1}.$$

We shall denote the dual vector representation by $\overline{t}$, and let

$$t_{ab},\,\overline{t}_{ab}\in U_q(\mathfrak{sl}(m|n))^\circ, \quad a,b=1,2,\ldots,m+n,$$

be the matrix elements of t and $\overline{t}$, respectively. Then the superalgebra $\mathcal{A}$ of functions on the quantum supergroup $\mathrm{SL}_q(m|n)$ is defined to be the subalgebra of $U_q(\mathfrak{sl}(m|n))^\circ$ generated by the t_{ab}, $\overline{t}_{ab}$. In Ref. 25 the following was shown.

Proposition 5: The algebra $\mathcal{A}$ *is a sub-Hopf-superalgebra of* $U_q(\mathfrak{sl}(m|n))^\circ$ *and is dense in* $U_q(\mathfrak{sl}(m|n))^*$.

The quantum even subalgebra $U_q(\mathfrak{g}_0)$ is $U_q(\mathfrak{sl}(m)\oplus\mathfrak{gl}(1)\oplus\mathfrak{sl}(n))$ with generators

$$k_a^{\pm 1},\ E_{b,b+1},\ E_{b+1,b},\quad a,b=1,2,\ldots,m+n-1,\quad b\neq m.$$

The images of t and $\overline{t}$ under $\mathcal{P}$ give rise to representations of $U_q(\mathfrak{g}_0)$, with

$$\mathcal{P}(t)=\begin{pmatrix}\mathcal{P}(t_{ij}) & 0\\ 0 & \mathcal{P}(t_{\mu\nu})\end{pmatrix},\quad \mathcal{P}(\overline{t})=\begin{pmatrix}\mathcal{P}(\overline{t}_{ij}) & 0\\ 0 & \mathcal{P}(\overline{t}_{\mu\nu})\end{pmatrix}.$$

The matrix elements of these representations generate $\mathcal{A}_e$, which forms a Hopf subalgebra of $U_q(\mathfrak{g}_0)^\circ$. On $\mathcal{A}_e$ there exists a unique left integral $\int_0$ which annihilates the matrix elements of all nontrivial irreducible representations and satisfies

$$\int_0 \mathbb{1}_{\mathcal{A}_e}=1.$$

Introduce

$$\mathbb{E}_i=E_{i,m+1}\ E_{i,m+2}\cdots E_{i,m+n},$$

$$\mathbb{F}_i=E_{m+n,i}\ E_{m+n-1,i}\cdots E_{m+1,i},$$

$$\mathbb{E}=\mathbb{E}_m\,\mathbb{E}_{m-1}\cdots\mathbb{E}_1,$$

$$\mathbb{F}=\mathbb{F}_1\,\mathbb{F}_2\cdots\mathbb{F}_m,$$

$$\Gamma=\mathbb{E}\,\mathbb{F}.$$

Lemma 7: Let J *be defined as in (13). Then the image of* Γ *under the canonical map* $U_q(\mathfrak{sl}(m|n))\rightarrow U_q(\mathfrak{sl}(m|n))/J$ *is an invariant.*

Proof: In Ref. 23 it was shown that

$$k_a\Gamma=\Gamma k_a,\quad \forall a,$$

$$[E_{c,c+1},\mathbb{E}]=[E_{c,c+1},\mathbb{F}]=0,\quad c\neq m,$$

$$[E_{c+1,c},\mathbb{E}]=[E_{c+1,c},\mathbb{F}]=0,\quad c\neq m.$$

It is also clear that

$$E_{m,m+1}\Gamma=0.$$

This immediately leads to

$$E_{i,m+1}\Gamma=0,\quad \forall i\leqslant m.$$

What remains to be shown is that

$$E_{m+1,m}\Gamma\in J. \tag{14}$$

By using the fact that $E_{m+1,m}$ q-anticommutes with all $E_{\mu,i}$, $\mu \geqslant m+1$, $i \leqslant m$, and $(E_{m+1,m})^2 = 0$, we have

$$E_{m+1,m}\,\mathbb{F} = 0.$$

Thus

$$E_{m+1,m}\Gamma = [E_{m+1,m}, \mathbb{E}]\,\mathbb{F}.$$

To determine the right hand side, we need the following commutation relations:

$$[E_{m+1,m}, \mathbb{E}_i] = q^{m+n-2} E_{i,m+2} E_{i,m+3} \cdots E_{i,m+n} k_m E_{i,m}, \quad i < m,$$

$$[E_{i,m}, \mathbb{E}_j] = 0, \quad i > j.$$

Now

$$[E_{m+1,m}, \Gamma] = [E_{m+1,m}, \mathbb{E}_m]\,\mathbb{E}_{m-1} \cdots \mathbb{E}_1\,\mathbb{F},$$

where $[E_{m+1,m}, \mathbb{E}_m]$ can be easily calculated to yield

$$\begin{aligned}[E_{m+1,m}, \mathbb{E}_m] = {} & \frac{k_m - k_m^{-1}}{q - q^{-1}} E_{m,m+2} \cdots E_{m,m+n} \\ & + \sum_{\alpha=2}^{n} (-1)^{\alpha} q^{-(n-\alpha)} E_{m,m+1} \cdots \hat{E}_{m,m+\alpha} \cdots E_{m,m+n} E_{m+1,m+\alpha} k_{\alpha}^{-1},\end{aligned}$$

with $\hat{E}_{m,m+\alpha}$ indicating that $E_{m,m+\alpha}$ is removed from the second term. By using

$$E_{m+1,m+\alpha}\,\mathbb{E}_i - q^{-2}\,\mathbb{E}_i\,E_{m+1,m+\alpha} = 0, \quad i = 1,2,\ldots,m, \ \ \alpha = 2,3,\ldots,n,$$

we immediately see that (14) indeed holds.

Let $\nu: U_q(\mathfrak{g}) \to \mathcal{A}^{\circ}$ be the natural embedding.

Theorem 6: *There exists the following nontrivial left integral on $\mathcal{A}$:*

$$\int = \nu(\Gamma) \cdot \int_0 \mathcal{P}.$$

Example 7: The quantum supergroup $\mathrm{OSP}_q(2|2n)$.

We denote by $\mathfrak{g}$ the Lie superalgebra $\mathfrak{osp}(2|2n)$ and recall that in this case $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ is the maximal even subalgebra $\mathfrak{sp}(2n) \oplus \mathfrak{gl}(1)$ of $\mathfrak{g}$. Introduce the $(n+1)$-dimensional Minkowski space $\mathfrak{h}^*$ with a basis $\{\delta_i \mid i = 0,1,2,\ldots,n\}$ and the bilinear form $(,):\mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ defined by

$$(\delta_i, \delta_j) = -(-1)^{\delta_{0,i}}\,\delta_{i,j}, \quad \forall i,j.$$

Then the simple roots can be expressed as $\alpha_i = \delta_i - \delta_{i+1}$, $0 \leqslant i < n$, $\alpha_n = 2\delta_n$, with α_0 being the unique odd simple root. A convenient version of the Cartan matrix $A = (a_{ij})_{i,j=0}^{n}$ is $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, $\forall i > 0$, $a_{0,j} = (\alpha_0, \alpha_j)$. The quantum superalgebra $U_q(\mathfrak{g})$ is the universal complex superalgebra with generators $\{k_i^{\pm 1}, e_i, f_i, i \in \mathbb{N}_n\}$, $\mathbb{N}_n = \{0,1,2,\ldots,n\}$, where e_0 and f_0 are odd and the rest are even. The defining relations are

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = \mathbb{1},$$

$$k_i e_j k_i^{-1} = q_i^{a_{ij}/2} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}/2} f_j,$$

$$[e_i, f_j] = \delta_{ij}(k_i^2 - k_i^{-2})/(q_i - q_i^{-1}), \quad i,j \in \mathbb{N}_n,$$

$$(e_0)^2 = (f_0)^2 = 0,$$

$$\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_{q_i} e_i^{1-a_{ij}-\mu} e_j e_i^\mu = 0, \quad i \neq 0,$$

$$\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_{q_i} f_i^{1-a_{ij}-\mu} f_j f_i^\mu = 0, \quad i \neq 0,$$

where $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q$ is a q-binomial coefficient. As is well-known, the quantum superalgebra $U_q(\mathfrak{g})$ has the structure of a Hopf superalgebra. Note that $\{e_i, f_i, k_i^{\pm 1} \mid i = 1,2,\ldots,n\}$ generate a Hopf subalgebra $U_q(\mathfrak{sp}(2n)) \subset U_q(\mathfrak{g})$. Together with $\{k_0^{\pm 1}\}$, they generate $U_q(\mathfrak{g}_0) = U_q(\mathfrak{sp}(2n) \oplus \mathfrak{gl}(1))$.

Define the odd elements

$$\psi_1 = e_0,$$

$$\psi_{i+1} = \psi_i e_i - q e_i \psi_i, \quad 1 \leq i < n,$$

$$\psi_{-n} = \psi_n e_n - q^2 e_n \psi_n,$$

$$\psi_{-i} = \psi_{-i-1} e_i - q e_i \psi_{-i-1}, \quad 1 \leq i < n;$$

$$\phi_0 = f_0,$$

$$\phi_{i+1} = f_i \phi_i - q^{-1} \phi_i f_i, \quad 1 \leq i < n,$$

$$\phi_{-n} = f_n \phi_n - q^{-2} \phi_n f_n,$$

$$\phi_{-i} = f_i \phi_{-i-1} - q^{-1} \phi_{-i-1} f_i, \quad 1 \leq i < n,$$

which satisfy the following relations

$$\psi_{\pm i} \psi_{\pm j} + q^{\pm 1} \psi_{\pm j} \psi_{\pm i} = 0, \quad i \leq j,$$

$$\psi_i \psi_{-j} + q \psi_{-j} \psi_i = 0, \quad \forall i \neq j,$$

$$\psi_n \psi_{-n} + q^2 \psi_{-n} \psi_n = 0,$$

$$\psi_{-i-1} \psi_{i+1} + \psi_{i+1} \psi_{-i-1} + q \psi_{-i} \psi_i + q^{-1} \psi_i \psi_{-i} = 0, \quad i < n;$$

$$\psi_j e_i - q^{(\alpha_i, \delta_0 - \delta_j)} e_i \psi_j = \delta_{ij} \psi_{i+1}, \quad \forall i,j,$$

$$\psi_{-j} e_i - q^{(\alpha_i, \delta_0 + \delta_j)} e_i \psi_{-j} = \delta_{i+1,j} \psi_{-i+1}, \quad i > 1,$$

and also similar relations for $\phi_{\pm i}$, where ψ_{n+1} and ϕ_{n+1} are understood as ψ_{-n} and ϕ_{-n}, respectively. Let

$$E_{1,2} = e_1,$$

$$E_{1,i+1} = E_{1,i} e_i - q e_i E_{1,i}, \quad 1 < i < n,$$

$$E_{1,\bar{n}} = E_{1,n} e_n - q^2 e_n E_{1,n},$$

$$E_{1,\overline{i}} = E_{1,\overline{i+1}} e_i - q e_i E_{1,\overline{i+1}}, \quad 1 < i < n,$$

$$E_{1,\overline{1}} = E_{1,\overline{2}} e_1 q^{-1} - q e_1 E_{1,\overline{2}},$$

where we have introduced the notation $\overline{i} = -i$. Then

$$\{\psi_i, f_0\} = E_{1,i} k_0^{-2}, \quad \{\psi_{-i}, f_0\} = E_{1,\overline{i}} k_0^{-2}.$$

Define

$$\mathbb{E} = \psi_1 \psi_2 \cdots \psi_n \psi_{-n} \psi_{-n+1} \cdots \psi_{-1},$$

$$\mathbb{F} = \phi_{-1} \phi_{-2} \cdots \phi_{-n} \phi_n \phi_{n-1} \cdots \phi_1,$$

$$\Gamma = \mathbb{E}\mathbb{F}.$$

We have the following lemma.

Lemma 8: Let J be defined as in (13). Then

(i) $[v, \mathbb{E}] = [v, \mathbb{F}] = 0$, $\forall v \in U_q(\mathfrak{sp}(2n)) \subset U_q(\mathfrak{g}_0)$,
(ii) $[u, \Gamma] = 0$, $\forall u \in U_q(\mathfrak{g}_0)$,
(iii) $x\Gamma \in \varepsilon(x)\Gamma + J$, $\forall x \in U_q(\mathfrak{g})$.

Of particular importance for us is the vector representation t of $U_q(\mathfrak{g})$. Introduce the index $a = i$ or $\overline{i}$, with $i = 0,1,\ldots,n$, $\overline{i} = \overline{0}, \overline{1}, \ldots, \overline{n}$. We have

$$t(e_0) = e_{0,1} + e_{\overline{1},\overline{0}}, \quad t(f_0) = e_{1,0} - e_{\overline{0},\overline{1}},$$

$$t(e_i) = e_{i,i+1} - e_{\overline{i+1},\overline{i}}, \quad t(f_i) = e_{i+1,i} - e_{\overline{i},\overline{i+1}}, \quad 1 \leqslant i < n,$$

$$t(e_n) = e_{n,\overline{n}}, \quad t(f_n) = e_{\overline{n},n},$$

$$t(k_i) = q_i^{H_i/2}, \quad 0 \leqslant i \leqslant n,$$

where

$$H_0 = \delta_0^* + \delta_1^*,$$

$$H_i = \delta_i^* - \delta_{i+1}^*, \quad 0 < i < n,$$

$$H_n = \delta_n^*;$$

$$\delta_i^* = e_{i,i} - e_{\overline{i},\overline{i}}, \quad 0 \leqslant i \leqslant n.$$

Let $t_{ab} \in U_q(\mathfrak{g})^\circ$, $a,b = 0,1,\ldots,n,\overline{0},\overline{1},\ldots,\overline{n}$, be the matrix elements of the vector representation t,

$$\langle t_{ab}, x \rangle = t(x)_{ab}, \quad \forall x \in U_q(\mathfrak{g}).$$

We will take the algebra $\mathcal{A}$ of functions on $\mathrm{OSP}_q(2|2n)$ to be the subalgebra of $U_q(\mathfrak{g})^\circ$ generated by the elements t_{ab}. In Ref. 26 we have shown the following.

Proposition 6: The algebra $\mathcal{A}$ is a sub-Hopf-superalgebra of $\mathcal{U}_q(\mathfrak{osp}(2|2n))^\circ$ and is dense in $\mathcal{U}_q(\mathfrak{osp}(2|2n))^*$.

As usual, let $\mathcal{P}:U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{g}_0)^\circ$ be the map induced by the dual of the embedding $\mathcal{I}:U_q(\mathfrak{g}_0)\to U_q(\mathfrak{g})$, let $\nu:U_q(\mathfrak{g})\to \mathcal{A}^\circ$ be the canonical map, and let $\hat{\mathcal{I}}=\nu\,\mathcal{I}$. Set $\mathcal{A}_e=\mathcal{P}(\mathcal{A})$. Then $\mathcal{A}_e$ admits a left integral $\int_0$, which we normalize by setting $\int_0 \mathbb{1}_{\mathcal{A}_e}=1$. Now

$$\int_0 \mathcal{P}:\mathcal{A}\to\mathbb{C}$$

is a well-defined linear map, which is left invariant with respect to $U_q(\mathfrak{g}_0)\subset U_q(\mathfrak{g})$:

$$\hat{\mathcal{I}}(u)\cdot\int_0 \mathcal{P}=\varepsilon(u)\int_0 \mathcal{P},\quad \forall u\in U_q(\mathfrak{g}_0).$$

We define

$$\int \;=\nu(\Gamma)\cdot\int_0 \mathcal{P}.$$

Theorem 7: *The linear form $\int:\mathcal{A}\to\mathbb{C}$ is a left integral on $\mathcal{A}$.*

Consider $\int\Lambda$, where

$$\Lambda=t_{\overline{1}\,\overline{0}}\cdots t_{\overline{n}\,\overline{0}}\,t_{n\,\overline{0}}\cdots t_{1\,\overline{0}}\,t_{\overline{1}\,0}\cdots t_{\overline{n}\,0}\,t_{n\,0}\cdots t_{1\,0}.$$

Using the following property of the Hopf superalgebra homomorphism $\mathcal{P}$,

$$\mathcal{P}(t_{a\,0})=\mathcal{P}(t_{a\,\overline{0}})=0,\quad \forall a\neq 0,\overline{0},$$

$$\mathcal{P}(t_{0\,\overline{0}})=\mathcal{P}(t_{\overline{0}\,0})=0,$$

we have

$$\int\Lambda=\langle\Lambda,\Gamma\rangle\int_0 \mathcal{P}((t_{\overline{0}\,\overline{0}})^{2n}(t_{0\,0})^{2n}).$$

Now

$$\mathcal{P}(t_{\overline{0}\,\overline{0}}\,t_{0\,0})=\mathcal{P}(t_{0\,0}\,t_{\overline{0}\,\overline{0}})=\mathbb{1}_{U_q(\mathfrak{g}_0)^\circ},$$

thus

$$\int\Lambda=\langle\Lambda,\Gamma\rangle,$$

which does not vanish if its $q\to 1$ limit is nonzero. A brute force calculation shows

$$|\langle\Lambda,\Gamma\rangle|\to 1,\ \text{as}\ \ q\to 1.$$

V. DISCUSSION

In the present work we have introduced and investigated the integrals on Hopf superalgebras, with special emphasis on the classical and quantum supergroups. In the undeformed case, there is obviously one problem that we have not solved completely, namely, to prove the existence of nonzero integrals for all of the basic classical Lie supergroups. However, in the meantime we have shown that nonzero integrals exist for a large class of Lie supergroups, including the classical simple ones. For further details, we refer the reader to Ref. 27.

In the quantum case, we have only been able to treat the type I supergroups. In particular, we could not say anything about most of the orthosymplectic quantum supergroups. There are clear indications that our method will not work (or, at least, has to be modified) in this case. However, one should remember that, at present, only very little is known about the orthosymplectic quantum supergroups anyway.

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APPENDIX: DESCRIPTION OF $U(\mathfrak{gl}(1))^\circ$

In Example 2 of Sec. III we need to choose a (left) integral on

$$\mathcal{A}_e \subset \mathcal{U}_e^\circ,$$

where

$$\mathcal{U}_e = U(\mathfrak{g}_{\bar{0}}) = U(\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{g}(1)) \cong U(\mathfrak{sl}(m)) \otimes U(\mathfrak{sl}(n)) \otimes U(\mathfrak{gl}(1)),$$

and hence

$$\mathcal{U}_e^\circ \cong U(\mathfrak{sl}(m))^\circ \otimes U(\mathfrak{sl}(n))^\circ \otimes U(\mathfrak{gl}(1))^\circ$$

(the isomorphisms are to be interpreted in the Hopf algebra sense). According to the discussion in Sec. II, the Hopf algebras $U(\mathfrak{sl}(n))^\circ$ are sufficiently well understood. In particular, there is a unique (up to scalar multiples) left integral on $U(\mathfrak{sl}(n))^\circ$, which turns out to be right invariant as well. For $\mathfrak{osp}(2|2n)$ and for the quantum counterparts the situation is similar. Correspondingly, in the present appendix we would like to comment on $U(\mathfrak{gl}(1))^\circ$. Needless to say, the results to be presented are well-known,[28,29] and we summarize them here in order to clarify some slightly subtle issues.

The Lie algebra $\mathfrak{gl}(1)$ is one-dimensional, hence $U(\mathfrak{gl}(1))$ is isomorphic (as a Hopf algebra) to the polynomial algebra $\mathbb{C}[X]$ in one indeterminate X. The Hopf algebra structure is the one known from enveloping algebras: The structure maps are uniquely fixed by the equations

$$\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X,$$

$$\varepsilon(X) = 0,$$

$$S(X) = -X.$$

It follows that

$$\Delta(X^r) = \sum_{s=0}^{r} \binom{r}{s} X^s \otimes X^{r-s},$$

for all integers $r \geqslant 0$.

The finite dual $\mathbb{C}[X]^\circ$ of $\mathbb{C}[X]$ can be described as follows. Define, for any element $a \in \mathbb{C}$ and any integer $r \geqslant 0$, the linear form u_a^r on $\mathbb{C}[X]$ by

$$\langle u_a^r, P\rangle = \left.\frac{d^r P}{dX^r}\right|_{X=a}, \quad \forall P \in \mathbb{C}[X].$$

Using some elementary algebra, it is not difficult to prove that these linear forms, with a and r as described above, form a basis of the vector space $\mathbb{C}[X]^\circ$. The multiplication in $\mathbb{C}[X]^\circ$ is given by

$$u_a^r u_b^s = u_{a+b}^{r+s},$$

in particular, the unit element is equal to u_0^0 (which is the counit of $\mathbb{C}[X]$), the coproduct is given by

$$\Delta(u_a^r) = \sum_{s=0}^{r} \binom{r}{s} u_a^s \otimes u_a^{r-s},$$

the counit by

$$\varepsilon(u_a^r) = \delta_{r,0},$$

and the antipode by

$$S(u_a^r) = (-1)^r u_{-a}^r,$$

where, in all cases, $a,b \in \mathbb{C}$ and $r,s \geqslant 0$ are integers.

Let us next recall that the dual $\mathbb{C}[X]^*$ of the vector space $\mathbb{C}[X]$ can be identified (in various ways) with the space of formal power series $\mathbb{C}[[Y]]$ in one indeterminate Y. If the dual pairing

$$\langle\ ,\ \rangle : \mathbb{C}[[Y]] \times \mathbb{C}[X] \to \mathbb{C}$$

is chosen such that

$$\left\langle \sum_{n\geqslant 0} c_n Y^n, X^r \right\rangle = r!\, c_r, \quad \forall r,$$

then the coalgebra structure of $\mathbb{C}[X]$ induces just the usual algebra structure on $\mathbb{C}[[Y]]$. Using this identification, the corresponding injection

$$\mathbb{C}[X]^\circ \to \mathbb{C}[[Y]]$$

is given by

$$u_a^r \to Y^r \exp(aY),$$

which immediately gives the product rule for the u_a^r's. Similarly, we find

$$\Delta(Y^r \exp(aY)) = (Y\otimes \mathbb{1} + \mathbb{1}\otimes Y)^r (\exp(aY)\otimes \exp(aY)).$$

Under the canonical embedding of $\mathbb{C}[[Y]]\otimes\mathbb{C}[[Y]]$ into $\mathbb{C}[[Y\otimes\mathbb{1}, \mathbb{1}\otimes Y]]$, the algebra of formal power series in $Y\otimes\mathbb{1}$ and $\mathbb{1}\otimes Y$, the right hand side of this equation can be written in the form

$$(Y\otimes\mathbb{1} + \mathbb{1}\otimes Y)^r \exp(a(Y\otimes\mathbb{1} + \mathbb{1}\otimes Y)).$$

In this sense, the coproduct in $\mathbb{C}[X]^\circ$ is fixed by the simple rule

$$\Delta(Y) = Y\otimes\mathbb{1} + \mathbb{1}\otimes Y,$$

just as for $\mathbb{C}[X]$.

Let us now turn to the object of our main concern, the integrals. It is easy to see that on $\mathbb{C}[X]^\circ$ a nontrivial integral does not exist. However, there is a way out. Obviously, the elements u_a^0, $a \in \mathbb{C}$, span a Hopf subalgebra $\mathcal{K}$ of $\mathbb{C}[X]^\circ$, and the linear form $\int$ on $\mathcal{K}$, defined by

$$\int u_a^0 = \delta_{a,0}, \quad \forall a \in \mathbb{C},$$

is a left and right integral on $\mathcal{K}$. Note that the u_a^0's are exactly the characters of the algebra $\mathbb{C}[X]$, i.e., the grouplike elements of $\mathbb{C}[X]^\circ$, and that $\mathcal{K}$ is isomorphic to the group Hopf algebra of the additive group $\mathbb{C}$.

Now we recall that, for an arbitrary algebra A (associative, with unit element), the finite dual A° consists exactly of the matrix elements (regarded as linear forms on A) of the representations of A. (Here and in the following, all representations are assumed to be finite-dimensional.) It is easy to see that the matrix elements of the completely reducible representations of $\mathbb{C}[X]$ (i.e., the representations for which the image of X is diagonalizable) belong to $\mathcal{K}$, whereas the other elements of $\mathbb{C}[X]^\circ$ stem from those representations which are not completely reducible. Note that, once again, the close relationship between complete reducibility and the existence of nontrivial integrals shows up.

Returning to the situation at the beginning of this appendix, we have to assume that

$$\mathcal{A}_e \subset U(\mathfrak{sl}(m))^\circ \otimes U(\mathfrak{sl}(n))^\circ \otimes \mathcal{K}.$$

According to the foregoing discussion, this corresponds to the requirement to consider only those representations of $\mathfrak{g} = \mathfrak{sl}(m|n)$ for which the one-dimensional center of $\mathfrak{g}_{\bar{0}}$ is represented by diagonalizable operators, which is usually assumed anyway.

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