JOURNAL OF MATHEMATICAL PHYSICS **52**, 042106 (2011)

# Generalized MICZ-Kepler problems and unitary highest weight modules

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(Received 30 November 2010; accepted 16 March 2011; published online 14 April 2011)

For each integer $n \geq 1$, we demonstrate that a $(2n+1)$-dimensional generalized MICZ-Kepler problem has a Spin$(2, 2n+2)$ dynamical symmetry which extends the manifest Spin$(2n+1)$ symmetry. The Hilbert space of bound states is shown to form a unitary highest weight Spin$(2, 2n+2)$-module with the minimal positive Gelfand–Kirillov dimension. As a byproduct, we obtain a simple geometric realization for such a unitary highest weight Spin$(2, 2n+2)$-module. © *2011 American Institute of Physics*. [doi:10.1063/1.3574886]

## I. INTRODUCTION

The Kepler problem is a physics problem in dimension three about two bodies which attract each other by a force proportional to the inverse square of their distance. As is well known, its exact solution in classical mechanics gives a very satisfactory explanation of the Kepler's laws of planetary motion, and its exact solution in quantum mechanics gives an equally satisfactory explanation of the spectral lines for the hydrogen atom. The MICZ-Kepler problems, discovered in the late 1960s by McIntosh and Cisneros[1] and independently by Zwanziger,[2] are natural cousins of the Kepler problem. Roughly speaking, a MICZ-Kepler problem is the Kepler problem for which the nucleus of a hypothetic hydrogen atom also carries a magnetic charge.

In the early 1990s, Iwai[3] obtained non-Abelian analogs of the MICZ-Kepler problems in dimension five; more recently, the first author constructed and solved[4] analogs of the MICZ-Kepler problems in all dimensions bigger than or equal to three which extends the aforementioned work of McIntosh and Cisneros, Zwanziger, and Iwai. We shall refer to the MICZ-Kepler problems and their higher dimensional analogs as the *generalized MICZ-Kepler problems*.

Recall that the MICZ-Kepler problems all have a large dynamical symmetry group—Spin$(2, 4)$—as shown by Barut and Bornzin.[5] These authors also used the symmetry to provide an elegant solution for the problems in Ref. 5. Similar results were also established in dimension five by Pletyukhov and Tolkachev[6] for the generalized MICZ-Kepler problems of Iwai. The purpose of the present paper is to investigate the dynamical symmetry and explore its representation theory for the generalized MICZ-Kepler problems in all odd dimensions.

We shall show that for each positive integer $n$, a $(2n+1)$-dimensional generalized MICZ-Kepler problem always has a Spin$(2, 2n+2)$[7] dynamical symmetry, i.e., *its Hilbert space of bound states*.[8] In fact, we shall show that the Hilbert space of bound states forms a unitary highest weight module for Spin$(2, 2n+2)$; more precisely, we shall establish the following result:[9]

**Theorem 1:** *Assume $n \geq 1$ is an integer and $\mu$ is an half integer. Let $\mathscr{H}(\mu)$ be the Hilbert space of bound states for the $(2n+1)$-dimensional generalized MICZ-Kepler problem with magnetic charge $\mu$, and $l_\mu := l + |\mu| + n - 1$ for any integer $l \geq 0$.*

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(1) *There is a natural unitary action of* Spin$(2, 2n+2)$ *on* $\mathscr{H}(\mu)$ *which extends the manifest unitary action of* Spin$(2n+1)$*. In fact,* $\mathscr{H}(\mu)$ *is the unitary highest weight module of* Spin$(2, 2n+2)$ *with highest weight* $(-(n+|\mu|), |\mu|, \cdots, |\mu|, \mu)$*.*

(2) *As a representation of subgroup* Spin$(2, 1) \times_{\mathbb{Z}_2}$ Spin$(2n+1)$*,*

$$\mathscr{H}(\mu) = \hat{\bigoplus}_{l=0}^{\infty} \left(\mathcal{D}^{-}_{2l_\mu+2} \otimes D_l\right), \tag{1}$$

*where* $D_l$ *is the irreducible module of* Spin$(2n+1)$ *with highest weight* $(l+|\mu|, |\mu|, \cdots, |\mu|)$ *and* $\mathcal{D}^{-}_{2l_\mu+2}$ *is the anti-holomorphic discrete series representation of* Spin$(2, 1)$ *with highest weight* $-l_\mu - 1$*.*

(3) *As a representation of the maximal compact subgroup* Spin$(2) \times_{\mathbb{Z}_2}$ Spin$(2n+2)$*,*

$$\mathscr{H}(\mu) = \hat{\bigoplus}_{l=0}^{\infty} \left(D(-l_\mu - 1) \otimes D^l\right), \tag{2}$$

*where* $D^l$ *is the irreducible module of* Spin$(2n+2)$ *with highest weight* $(l+|\mu|, |\mu|, \cdots, |\mu|, \mu)$ *and* $D(-l_\mu - 1)$ *is the irreducible module of* Spin$(2)$ *with weight* $-l_\mu - 1$*.*

Readers who wish to have a quick geometric description of the aforementioned unitary highest weight module of Spin$(2, 2n+2)$ may consult the Appendix. Readers who wish to know more details about the classification[10–12] of unitary highest weight modules may start with a fairly readable account from Ref. 12. Note that there is no general classification result for the family of unitary modules of real noncompact simple Lie groups, and the subfamily of unitary *highest weight* modules is special enough so that such a nice classification result can possibly exist.

In Sec. II, we give a quick review of the generalized MICZ-Kepler problems in odd dimensions. For the computational purpose in the subsequent section, we quickly review the gauge potential[13] for the background gauge field (i.e., *connection*) under a particular local gauge (i.e., *bundle trivialization*), and then quote from Ref. 4 some key identities satisfied by the gauge potential. In Sec. III, we introduce the dynamical symmetry operators and show that they satisfy the commutation relations for the generators[14] of $\mathfrak{so}_0(2, 2n+2)$. We also show that these dynamical symmetry operators satisfy a set of quadratic relations.[15] In Sec. IV, we start with a preliminary discussion of the representation problem and point out the need of "twisting." Then we give a review of the (bound) energy eigenspaces (i.e., *eigenspaces of the harmiltonian* viewed as a hermitian operator on the physical Hilbert space) and finally introduce the notion of "twisted" energy eigenspaces. The "twisted" Hilbert space of bound states, defined as the Hilbert space completion of the direct sum of "twisted" energy eigenspaces, turns out to be the space of $L^2$-sections of a canonical hermitian bundle. In Sec. V, we solve the representation problem by proving two propositions from which Theorem 1 follows quickly. In the Appendix, each of the unitary highest weight representations of Spin$(2, 2n+2)$ encountered here is geometrically realized as the space of all $L^2$-sections of a canonical hermitian bundle. Via communications with Professors R. Howe and N. Wallach, we learned that these representations can be imbedded into the kernel of certain canonical differential operators, see Refs. 16 and 17 for the case $\mu = 0$ and Ref. 18 for the general case. Professor Feher informed us of Ref. 19 in which a related interesting model with a conjectured dynamical $O(2, 4)$ symmetry is investigated.

Via private communication with Professor Vogan, the unitary highest weight Reps appeared in the above theorem are precisely those with the minimal positive Gelfand–Kirillov dimension. In other words, a unitary highest weight Reps of Spin$(2, 2n+2)$ has the minimal positive Gelfand–Kirillov dimension if and only if it can be realized as the Hilbert space of bound states of a generalized MICZ-Kepler problem in dimension $2n+1$. This phenomenon has been vastly extended.[20]

The main concern of this paper is to further investigate the generalized MICZ-Kepler problems constructed in Ref. 4. As a byproduct of this investigation, we obtain both the $L^2$-model of all unitary highest weight modules of Spin(2, 2n + 1) with minimal positive Gelfand–Kirillov dimension and their branching laws with respect to certain subgroups. It turns out that these representation theoretical results have been obtained earlier in much more general setting by Kobayashi and Ørsted in a series of papers[17] especially when the magnetic charge $\mu$ is zero (see also Ref. 21 for a more general branching laws). For *nonzero* magnetic charge $\mu$, our $L^2$-model seems to be new. Finally we would like to remark that we have a second model for the unitary highest weight modules concerned here, i.e., the Hilbert space of bound states for the generalized MICZ-Kepler problem; and these two different models are related via the twisting map in Eq. (42).

## II. REVIEW OF GENERALIZED MICZ-KEPLER PROBLEMS

From the physics point of view, a MICZ-Kepler problem is a generalization of the Kepler problem by adding a suitable background magnetic field, while at the same time making an appropriate adjustment to the scalar Coulomb potential so that the problem is still integrable. The configuration space is the punctured 3D Euclidean space, and the background magnetic field is a Dirac monopole. To be more precise, the (dimensionless) hamiltonian of a MICZ-Kepler problem with magnetic charge $\mu$ is

$$H = -\frac{1}{2}\Delta_{\mathcal{A}} + \frac{\mu^2}{2r^2} - \frac{1}{r}. \tag{3}$$

Here $\Delta_{\mathcal{A}}$ is the Laplace operator twisted by the gauge potential $\mathcal{A}$ of a Dirac monopole under a particular gauge, and $\mu$ is the magnetic charge of the Dirac monopole, which must be a half integer.

To extend the MICZ-Kepler problems beyond dimension three, one needs a suitable generalization of the Dirac monopoles. Fortunately this problem was solved in Refs. 4, 22, and 23. We review the work here.

### A. Generalized MICZ-Kepler problems

Let $D \geq 3$ be an integer, $\mathbb{R}^D_*$ be the punctured $D$-space, i.e., $\mathbb{R}^D$ with the origin removed. Let $ds^2$ be the cylindrical metric on $\mathbb{R}^D_*$. Then $(\mathbb{R}^D_*, ds^2)$ is the product of the straight line $\mathbb{R}$ with the round sphere $\mathrm{S}^{D-1}$. Since we are interested in the odd dimensional generalized MICZ-Kepler problems only in this paper, we assume $D$ is odd.

Let $\mathcal{S}_\pm$ be the positive/negative spinor bundle of $(\mathbb{R}^D_*, ds^2)$, then $\mathcal{S}_\pm$ correspond to the fundamental spin representations $\mathbf{s}_\pm$ of $\mathfrak{so}_0(D-1)$ [the Lie algebra of $\mathrm{SO}(D-1)$]. Note that each of the above spinor bundles is endowed with a natural $\mathrm{SO}(D)$ invariant connection—the Levi–Civita spin connection of $(\mathbb{R}^D_*, ds^2)$. As a result, the Young product of $I$ copies of these bundles, denoted by $\mathcal{S}^I_+$, $\mathcal{S}^I_-$ respectively, is also equipped with natural $\mathrm{SO}(D)$ invariant connections.

When $\mu$ is a positive half integer, we write $\mathcal{S}^{2\mu}_+$ as $\mathcal{S}^{2\mu}$ and $\mathcal{S}^{2\mu}_-$ as $\mathcal{S}^{-2\mu}$. We also adopt this convention for $\mu = 0$: to denote by $\mathcal{S}^0$ the product complex line bundle with the product connection. Note that $\mathcal{S}^{2\mu}$ with $\mu$ being a half integer is our analog of the Dirac monopole with magnetic charge $\mu$, and the corresponding representation of $\mathfrak{so}_0(D-1)$ will be denoted by $\mathbf{s}^{2\mu}$.

*Definition 1: Let $n \geq 1$ be an integer, $\mu$ a half integer. The $(2n+1)$-dimensional generalized MICZ-Kepler problem with magnetic charge $\mu$ is defined to be the quantum mechanical system on $\mathbb{R}^{2n+1}_*$ for which the wave-functions are sections of $\mathcal{S}^{2\mu}$, and the hamiltonian is*

$$H = -\frac{1}{2}\Delta_\mu + \frac{(n-1)|\mu| + \mu^2}{2r^2} - \frac{1}{r}, \tag{4}$$

*where $\Delta_\mu$ is the Laplace operator twisted by $\mathcal{S}^{2\mu}$.*

Upon choosing a local gauge, the background gauge field (i.e., the natural connection on $\mathcal{S}^{2\mu}$) can be represented by a gauge potential $\mathcal{A}_\alpha$ in an explicit form; then $\Delta_\mu$ can be represented explicitly by $\sum_\alpha(\partial_\alpha + i\mathcal{A}_\alpha)^2$. Since the gauge potential is of crucial importance, we review some of its properties in Subsection II B.

### B. Basic identities for the gauge potential

We write $\vec{r} = (x_1, x_2, \ldots, x_{D-1}, x_D)$ for a point in $\mathbb{R}^D$ and $r$ for the length of $\vec{r}$. The small Greek letters $\mu$, $\nu$, etc. run from 1 to $D$ and the lower case Latin letters $a$, $b$ etc. run from 1 to $D-1$. We use the Einstein convention that repeated indices are always summed over.

Under a suitable choice of local gauge on $\mathbb{R}^D$ with the negative $D$th axis removed, the gauge field can be represented by the following gauge potential:

$$\mathcal{A}_D = 0, \qquad \mathcal{A}_b = -\frac{1}{r(r+x_D)} x_a \gamma_{ab}, \tag{5}$$

where $\gamma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$ with $\gamma_a$ being the "gamma matrix" for physicists. Note that $\gamma_a = i e_a$ with $e_a$ being the element in the Clifford algebra that corresponds to the $a$th standard coordinate vector of $\mathbb{R}^{D-1}$.

The field strength of $\mathcal{A}_\alpha$ is then given by

$$\begin{aligned} F_{Db} &= \frac{1}{r^3} x_a \gamma_{ab}, \\ F_{ab} &= -\frac{2\gamma_{ab}}{r(r+x_D)} + \frac{1}{r^2(r+x_D)^2} \cdot \\ &\quad \left( (2 + \frac{x_D}{r}) x_c (x_a \gamma_{cb} - x_b \gamma_{ca}) + i x_d x_c [\gamma_{da}, \gamma_{cb}] \right). \end{aligned} \tag{6}$$

Here are some identities from Ref. 4 that our later computations will crucially depend on:

*Lemma 1: Let $\mathcal{A}_\alpha$ be the gauge potential defined by Eq. (5) and let $F_{\alpha\beta}$ be its field strength.*

(1) *The following identities are valid in any representation of $\mathfrak{so}_0(D-1)$:*

$$\begin{aligned} F_{\mu\nu} F^{\mu\nu} = \frac{2}{r^4} c_2, \quad [\nabla_\kappa, F_{\mu\nu}] &= \frac{1}{r^2} \left( x_\mu F_{\nu\kappa} + x_\nu F_{\kappa\mu} - 2 x_\kappa F_{\mu\nu} \right), \\ x_\mu \mathcal{A}_\mu = 0, \qquad x_\mu F_{\mu\nu} &= 0, \qquad [\nabla_\mu, F_{\mu\nu}] = 0, \\ r^2 [F_{\mu\nu}, F_{\alpha\beta}] + i F_{\mu\beta} \delta_{\alpha\nu} - i F_{\nu\beta} \delta_{\alpha\mu} &+ i F_{\alpha\mu} \delta_{\beta\nu} - i F_{\alpha\nu} \delta_{\beta\mu} \\ = \frac{i}{r^2} \left( x_\mu x_\alpha F_{\beta\nu} + x_\mu x_\beta F_{\nu\alpha} \right. &\left. - x_\nu x_\alpha F_{\beta\mu} - x_\nu x_\beta F_{\mu\alpha} \right), \end{aligned} \tag{7}$$

*where $\nabla_\alpha = \partial_\alpha + i \mathcal{A}_\alpha$, and $c_2 = c_2[\mathfrak{so}_0(D-1)] = \frac{1}{2} \gamma_{ab} \gamma_{ab}$ is the (quadratic) Casimir operator of $\mathfrak{so}_0(D-1)$.*

(2) *When $D = 2n+1$, $\mu$ is a half integer, the following identity*

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{c_2}{n} \left( \frac{1}{r^2} \delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{r^4} \right) + i(n-1) F_{\alpha\beta} \tag{8}$$

*holds in the irreducible representation $\mathbf{s}^{2\mu}$ of $\mathfrak{so}_0(2n)$ with highest weight $(|\mu|, \cdots, |\mu|, \mu)$.*

Note that $\frac{c_2}{n} = \mu^2 + (n-1)|\mu|$ in the irreducible representation $\mathbf{s}^{2\mu}$. We remark that $\mathcal{A}_r = \mathcal{A}_\theta = 0$, where $\mathcal{A}_r$ and $\mathcal{A}_\theta$ are the $r$ and $\theta$ components of $\mathcal{A}$ in the polar coordinate system $(r, \theta, \theta_1, \cdots, \theta_{D-3}, \phi)$ for $\mathbb{R}^D_*$ with $\theta$ being the angle between $\vec{r}$ and the positive $D$th axis.

## III. THE DYNAMICAL SYMMETRY

For the remainder of this paper, we only consider a fixed $(2n+1)$-dimensional generalized MICZ-Kepler problem with magnetic charge $\mu$. Recall that the configuration space is $\mathbb{R}^D_*$ where $D = 2n+1$. For our computational purposes, it suffices to work on $\mathbb{R}^D$ with the negative $D$-axis removed. We introduce the notations $\pi_\alpha := -i\nabla_\alpha$, $c := \mu^2 + (n-1)|\mu|$. Then $[\pi_\alpha, \pi_\beta] = -i F_{\alpha\beta}$.

Following Barut and Bornzin,[5] we let

$$\begin{cases} \vec{\Gamma} := r\vec{\pi}, \quad X := r\pi^2 + \frac{c}{r}, \quad Y := r, \\ J_{\alpha\beta} := i[\Gamma_\alpha, \Gamma_\beta], \quad \vec{Z} := i[\vec{\Gamma}, X], \quad \vec{W} := i[\vec{\Gamma}, Y] = \vec{r}; \end{cases} \tag{9}$$

and

$$\begin{cases} \Gamma_{D+1} := \frac{1}{2}(X - Y), & \Gamma_{-1} := \frac{1}{2}(X + Y), \\ \vec{A} := \frac{1}{2}\left(\vec{Z} - \vec{W}\right), \quad \vec{M} := \frac{1}{2}\left(\vec{Z} + \vec{W}\right), & T := i[\Gamma_{D+1}, \Gamma_{-1}]. \end{cases} \tag{10}$$

Some relatively straightforward but lengthy computations yield

$$\begin{cases} J_{\alpha\beta} = x_\alpha \pi_\beta - x_\beta \pi_\alpha + r^2 F_{\alpha\beta}, \\ A_\alpha = \frac{1}{2} x_\alpha \pi^2 - \pi_\alpha(\vec{r}\cdot\vec{\pi}) + r^2 F_{\alpha\beta}\pi_\beta - \frac{c}{2r^2} x_\alpha + \frac{i}{2}(D-3)\pi_\alpha - \frac{1}{2}x_\alpha, \\ M_\alpha = \frac{1}{2} x_\alpha \pi^2 - \pi_\alpha(\vec{r}\cdot\vec{\pi}) + r^2 F_{\alpha\beta}\pi_\beta - \frac{c}{2r^2} x_\alpha + \frac{i}{2}(D-3)\pi_\alpha + \frac{1}{2}x_\alpha, \\ T = \vec{r}\cdot\vec{\pi} - i\frac{D-1}{2}, \\ \Gamma_\alpha = r\pi_\alpha, \\ \Gamma_{-1} = \frac{1}{2}\left(r\pi^2 + r + \frac{c}{r}\right), \\ \Gamma_{D+1} = \frac{1}{2}\left(r\pi^2 - r + \frac{c}{r}\right). \end{cases} \tag{11}$$

Let the capital Latin letters $A$, $B$ run from $-1$ to $D+1$. Introduce $J_{AB}$ as follows:

$$J_{AB} = \begin{cases} J_{\mu\nu} & \text{if } A = \mu,\ B = \nu \\ A_\mu & \text{if } A = \mu,\ B = D+1 \\ M_\mu & \text{if } A = \mu,\ B = -1 \\ \Gamma_\mu & \text{if } A = \mu,\ B = 0 \\ T & \text{if } A = D+1,\ B = -1 \\ \Gamma_{D+1} & \text{if } A = D+1,\ B = 0 \\ \Gamma_{-1} & \text{if } A = -1,\ B = 0 \\ -J_{BA} & \text{if } A > B \\ 0 & \text{if } A = B. \end{cases} \tag{12}$$

**Theorem 2:** *Let $C^\infty(\mathcal{S}^{2\mu})$ be the space of smooth sections of $\mathcal{S}^{2\mu}$. Let $J_{AB}$ be defined by (12).*
(1) *As operators on $C^\infty(\mathcal{S}^{2\mu})$, $J_{AB}$'s satisfy the following commutation relations:*

$$[J_{AB}, J_{A'B'}] = -i\eta_{AA'}J_{BB'} - i\eta_{BB'}J_{AA'} + i\eta_{AB'}J_{BA'} + i\eta_{BA'}J_{AB'}, \tag{13}$$

*where the indefinite metric tensor $\eta$ is* $\mathrm{diag}\{+ + - \cdots -\}$ *relative to the following order: $-1$, 0, 1, ..., $2n+2$ for the indices.*
(2) *As operators on $C^\infty(\mathcal{S}^{2\mu})$,*

$$\{J_{AB}, J^A{}_C\} := J_{AB}J^A{}_C + J^A{}_C J_{AB} = -2a\eta_{BC}, \tag{14}$$

*where $a = n - c$.*

The proof of this theorem is purely algebraic and computational, but quite long. It will be carried out in Subsections III A and III B.

### A. Proof of part (1)

By exploiting the symmetry properties of both sides of Eq. (13), we can see that it suffices to verify the commutation relations in the cases where $(A, B) \neq (A', B')$, $A < B$, $A' < B'$, and $B' \leq B$. The proof crucially depends on Lemma 1.

The following lemma is quite useful.

*Lemma 2:*

$$\begin{cases} [J_{\alpha\beta}, r] = [J_{\alpha\beta}, \frac{1}{r}] = 0, \\ [J_{\alpha\beta}, x_\nu] = -i(x_\alpha\delta_{\beta\nu} - x_\beta\delta_{\alpha\nu}), \\ [J_{\alpha\beta}, \pi_\nu] = -i(\pi_\alpha\delta_{\beta\nu} - \pi_\beta\delta_{\alpha\nu}), \\ [J_{\alpha\beta}, F_{\alpha'\beta'}] = i\delta_{\alpha\alpha'}F_{\beta\beta'} + i\delta_{\beta\beta'}F_{\alpha\alpha'} - i\delta_{\alpha\beta'}F_{\beta\alpha'} - i\delta_{\beta\alpha'}F_{\alpha\beta'}. \end{cases} \tag{15}$$

*Proof:*

$$
\begin{aligned}
[J_{\alpha\beta}, r] &= [x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2 F_{\alpha\beta}, r] = [x_\alpha\pi_\beta - x_\beta\pi_\alpha, r] \\
&= -i(x_\alpha \frac{x_\beta}{r} - x_\beta \frac{x_\alpha}{r}) = 0. \\
[J_{\alpha\beta}, \frac{1}{r}] &= [x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2 F_{\alpha\beta}, \frac{1}{r}] = [x_\alpha\pi_\beta - x_\beta\pi_\alpha, \frac{1}{r}] \\
&= +i(x_\alpha \frac{x_\beta}{r^3} - x_\beta \frac{x_\alpha}{r^3}) = 0.
\end{aligned}
$$

$$
\begin{aligned}
[J_{\alpha\beta}, x_\nu] &= [x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2 F_{\alpha\beta}, x_\nu] = [x_\alpha\pi_\beta - x_\beta\pi_\alpha, x_\nu] \\
&= -i(x_\alpha\delta_{\beta\nu} - x_\beta\delta_{\alpha\nu}).
\end{aligned}
$$

$$
\begin{aligned}
[J_{\alpha\beta}, \pi_\nu] &= [x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2 F_{\alpha\beta}, \pi_\nu] \\
&= -i(\pi_\alpha\delta_{\beta\nu} - \pi_\beta\delta_{\alpha\nu}) - i x_\alpha F_{\beta\nu} + i x_\beta F_{\alpha\nu} \\
&\quad +2i x_\nu F_{\alpha\beta} + i r^2 [\nabla_\nu, F_{\alpha\beta}] \\
&= -i(\pi_\alpha\delta_{\beta\nu} - \pi_\beta\delta_{\alpha\nu}).
\end{aligned}
$$

$$
\begin{aligned}
[J_{\alpha\beta}, F_{\alpha'\beta'}] &= [x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2 F_{\alpha\beta}, F_{\alpha'\beta'}] \\
&= x_\alpha[\pi_\beta, F_{\alpha'\beta'}] - x_\beta[\pi_\alpha, F_{\alpha'\beta'}] + r^2[F_{\alpha\beta}, F_{\alpha'\beta'}] \\
&= i\frac{x_\alpha}{r^2}(2x_\beta F_{\alpha'\beta'} + x_{\alpha'} F_{\beta\beta'} + x_{\beta'} F_{\alpha'\beta}) - i\frac{x_\beta}{r^2}(2x_\alpha F_{\alpha'\beta'} + x_{\alpha'} F_{\alpha\beta'} + x_{\beta'} F_{\alpha'\alpha}) \\
&\quad + r^2[F_{\alpha\beta}, F_{\alpha'\beta'}] \\
&= r^2[F_{\alpha\beta}, F_{\alpha'\beta'}] - \frac{i}{r^2}\left(-x_\alpha x_{\alpha'} F_{\beta\beta'} - x_\alpha x_{\beta'} F_{\alpha'\beta} + x_\beta x_{\alpha'} F_{\alpha\beta'} + x_\beta x_{\beta'} F_{\alpha'\alpha}\right) \\
&= i\delta_{\alpha\alpha'} F_{\beta\beta'} + i\delta_{\beta\beta'} F_{\alpha\alpha'} - i\delta_{\alpha\beta'} F_{\beta\alpha'} - i\delta_{\beta\alpha'} F_{\alpha\beta'}.
\end{aligned}
$$

□

By using Lemma 2 and the definition of $J_{\alpha\beta}$, one can easily check that $J_{\alpha\beta}$'s satisfy the standard commutation relation of $\mathfrak{so}(D)$ Lie algebra. Then Lemma 2 may be paraphrased as follows: under the commutation action of $J_{\alpha\beta}$'s, $r$ and $\frac{1}{r}$ transform as $\mathfrak{so}(D)$ scalars, $x_\alpha$'s and $\pi_\alpha$'s transform as $\mathfrak{so}(D)$ vectors, and $F_{\alpha\beta}$'s transform as a $\mathfrak{so}(D)$ bi-vectors. It is then clear that $T$, $\Gamma_{D+1}$, and $\Gamma_{-1}$ transform as $\mathfrak{so}(D)$ scalars; $\vec{A}$, $\vec{M}$, and $\vec{\Gamma}$ transform as $\mathfrak{so}(D)$ vectors. This completes the proof for Eq. (13) in the case when it involves $J_{\alpha\beta}$.

By using identities $x_\alpha \mathcal{A}_\alpha = 0$ and $x_\alpha F_{\alpha\beta} = 0$, one can check that $[-\vec{r}\cdot\nabla, \vec{r}] = -\vec{r}$, $[-\vec{r}\cdot\nabla, r] = -r$, $[-\vec{r}\cdot\nabla, \frac{1}{r}] = \frac{1}{r}$, $[-\vec{r}\cdot\nabla, \vec{\pi}] = \vec{\pi}$. That is, $-\vec{r}\cdot\nabla$ is the dimension operator in physics. It is then clear that

$$
[\Gamma_{-1}, T] = -i\Gamma_{D+1}, \quad [\Gamma_{D+1}, T] = -i\Gamma_{-1}, \quad [\vec{\Gamma}, T] = \vec{0}. \tag{16}
$$

Consequently,

$$
\begin{aligned}
[M_\alpha, T] &= [i[\Gamma_\alpha, \Gamma_{-1}], T] = i[\Gamma_\alpha, [\Gamma_{-1}, T]] + i[[\Gamma_\alpha, T], \Gamma_{-1}] \\
&= [\Gamma_\alpha, \Gamma_{D+1}] = -iA_\alpha, \\
[A_\alpha, T] &= [i[\Gamma_\alpha, \Gamma_{D+1}], T] = i[\Gamma_\alpha, [\Gamma_{D+1}, T]] + i[[\Gamma_\alpha, T], \Gamma_{D+1}] \\
&= [\Gamma_\alpha, \Gamma_{-1}] = -iM_\alpha.
\end{aligned} \tag{17}
$$

This completes the proof of Eq. (13) in the case when it involves $T$.

The remaining verifications are divided into four cases.

*Case 1:*

$$[\Gamma_\alpha, \Gamma_\beta] = -iJ_{\alpha\beta}, \quad [\Gamma_\alpha, \Gamma_{D+1}] = -iA_\alpha,$$
$$[\Gamma_\alpha, \Gamma_{-1}] = -iM_\alpha, \quad [\Gamma_{D+1}, \Gamma_{-1}] = -iT, \tag{18}$$

which are just the defining relations. So case 1 is done.

*Case 2:*

$$[M_\alpha, \Gamma_\beta] = -i\eta_{\alpha\beta}\Gamma_{-1}, \quad [A_\alpha, \Gamma_\beta] = -i\eta_{\alpha\beta}\Gamma_{D+1}$$

or equivalently

$$[Z_\alpha, \Gamma_\beta] = -i\eta_{\alpha\beta}X, \quad [W_\alpha, \Gamma_\beta] = -i\eta_{\alpha\beta}Y. \tag{19}$$

*Proof:* Since $[W_\alpha, \Gamma_\beta] = [x_\alpha, r\pi_\beta] = ir\delta_{\alpha\beta} = -i\eta_{\alpha\beta}Y$, we just need to verify the first identity. Note that $[Z_\alpha, r] = 2i\Gamma_\alpha$ (see case 3 below), so

$$\begin{aligned}
[Z_\alpha, \Gamma_\beta] &= r[Z_\alpha, \pi_\beta] + 2i\Gamma_\alpha\pi_\beta \\
&= r[x_\alpha\pi^2 - 2\pi_\alpha(\vec{r}\cdot\vec{\pi}) + 2r^2F_{\alpha\beta}\pi_\beta - \frac{c}{r^2}x_\alpha + i(D-3)\pi_\alpha, \pi_\beta] + 2ir\pi_\alpha\pi_\beta \\
&= r\left(i\delta_{\alpha\beta}\pi^2 - 2ix_\alpha F_{\gamma\beta}\pi_\gamma\right) + r(2iF_{\alpha\beta}(\vec{r}\cdot\vec{\pi}) - 2i\pi_\alpha\pi_\beta) + r(-2ir^2F_{\alpha\gamma}F_{\gamma\beta} \\
&\quad + [2r^2F_{\alpha\gamma}, \pi_\beta]\pi_\gamma) + cr[\pi_\beta, \frac{x_\alpha}{r^2}] + (D-3)rF_{\alpha\beta} + 2ir\pi_\alpha\pi_\beta \\
&= i\delta_{\alpha\beta}r\pi^2 - 2irx_\alpha F_{\gamma\beta}\pi_\gamma + 2irF_{\alpha\beta}(\vec{r}\cdot\vec{\pi}) + r(-2ir^2F_{\alpha\gamma}F_{\gamma\beta} + 4ix_\beta F_{\alpha\gamma}\pi_\gamma \\
&\quad - 2r^2[\pi_\beta, F_{\alpha\gamma}]\pi_\gamma) + cr[\pi_\beta, \frac{x_\alpha}{r^2}] + (D-3)rF_{\alpha\beta} \\
&= i\delta_{\alpha\beta}r\pi^2 - 2ir^3F_{\alpha\gamma}F_{\gamma\beta} + 2ir(2x_\beta F_{\alpha\gamma} + x_\alpha F_{\beta\gamma} + x_\gamma F_{\alpha\beta} + r^2[\nabla_\beta, F_{\alpha\gamma}])\pi_\gamma \\
&\quad + cr[\pi_\beta, \frac{x_\alpha}{r^2}] + (D-3)rF_{\alpha\beta} \\
&= i\delta_{\alpha\beta}r\pi^2 + 2ir^3F_{\gamma\alpha}F_{\gamma\beta} - icr\left(\delta_{\alpha\beta} - 2\frac{x_\alpha x_\beta}{r^4}\right) + 2(n-1)rF_{\alpha\beta} \\
&= i\delta_{\alpha\beta}(r\pi^2 + \frac{c}{r^2}) = -i\eta_{\alpha\beta}X.
\end{aligned}$$

□

*Case 3:*

$$[M_\alpha, \Gamma_{-1}] = i\Gamma_\alpha, \quad [M_\alpha, \Gamma_{D+1}] = 0$$
$$[A_\alpha, \Gamma_{-1}] = 0, \quad [A_\alpha, \Gamma_{D+1}] = -i\Gamma_\alpha.$$

or equivalently,

$$[\vec{W}, Y] = [\vec{Z}, X] = 0, \quad [\vec{W}, X] = [\vec{Z}, Y] = 2i\vec{\Gamma}. \tag{20}$$

*Proof:* It is clear that $[\vec{W}, Y] = 0$. Now $[W_\alpha, X] = r[x_\alpha, \pi^2] = ir\{\pi_\beta, \delta_{\alpha\beta}\} = 2i\Gamma_\alpha$. Next, using the identity $F_{\alpha\beta}x_\beta = 0$, we have

$$\begin{aligned}
[Z_\alpha, Y] &= [x_\alpha\pi^2 - 2\pi_\alpha(\vec{r}\cdot\vec{\pi}) + 2r^2F_{\alpha\beta}\pi_\beta - \frac{c}{r^2}x_\alpha + i(D-3)\pi_\alpha, r]\\
&= x_\alpha[\pi^2, r] - 2[\pi_\alpha(\vec{r}\cdot\vec{\pi}), r] - 2irF_{\alpha\beta}x_\beta + (D-3)\frac{x_\alpha}{r}\\
&= -ix_\alpha\{\pi_\beta, \frac{x_\beta}{r}\} - 2\pi_\alpha[(\vec{r}\cdot\vec{\pi}), r] - 2[\pi_\alpha, r](\vec{r}\cdot\vec{\pi}) + (D-3)\frac{x_\alpha}{r}\\
&= -(D-1)\frac{x_\alpha}{r} - 2i\frac{x_\alpha}{r}\vec{r}\cdot\vec{\pi} + 2i\pi_\alpha r + 2i\frac{x_\alpha}{r}(\vec{r}\cdot\vec{\pi}) + (D-3)\frac{x_\alpha}{r}\\
&= -2\frac{x_\alpha}{r} + 2i\pi_\alpha r = 2ir\pi_\alpha = 2i\Gamma_\alpha.
\end{aligned}$$

Finally,

$$\begin{aligned}
[\frac{1}{r}, Z_\alpha] &= [\frac{1}{r}, x_\alpha\pi^2 - 2\pi_\alpha(\vec{r}\cdot\vec{\pi}) + 2r^2F_{\alpha\beta}\pi_\beta - \frac{c}{r^2}x_\alpha + i(D-3)\pi_\alpha]\\
&= x_\alpha[\frac{1}{r}, \pi^2] - 2[\frac{1}{r}, \pi_\alpha(\vec{r}\cdot\vec{\pi})] + 2r^2F_{\alpha\beta}[\frac{1}{r}, \pi_\beta] + i(D-3)[\frac{1}{r}, \pi_\alpha]\\
&= -ix_\alpha\{x_\beta r^3, \pi_\beta\} - 2\pi_\alpha[\frac{1}{r}, (\vec{r}\cdot\vec{\pi})] - 2[\frac{1}{r}, \pi_\alpha](\vec{r}\cdot\vec{\pi}) - 2ir^2F_{\alpha\beta}x_\beta r^3 + (D-3)\frac{x_\alpha}{r^3}\\
&= -ix_\alpha[\pi_\beta, x_\beta r^3] - 2i\frac{x_\alpha}{r^3}\vec{r}\cdot\vec{\pi} + 2i\pi_\alpha\frac{1}{r} + 2i\frac{x_\alpha}{r^3}(\vec{r}\cdot\vec{\pi}) + (D-3)\frac{x_\alpha}{r^3}\\
&= 2i\pi_\alpha\frac{1}{r};
\end{aligned}$$

$$\begin{aligned}
[r\pi^2, Z_\alpha] &= [r, Z_\alpha]\pi^2 + r[\pi^2, Z_\alpha]\\
&= -2i\Gamma_\alpha\pi^2 + r[\pi^2, Z_\alpha]\\
&= -2i\Gamma_\alpha\pi^2 + r[\pi^2, x_\alpha\pi^2 - 2\pi_\alpha(\vec{r}\cdot\vec{\pi}) + 2r^2F_{\alpha\beta}\pi_\beta - \frac{c}{r^2}x_\alpha + i(D-3)\pi_\alpha]\\
&= -2i\Gamma_\alpha\pi^2 + r\left([\pi^2, x_\alpha]\pi^2 - 2[\pi^2, \pi_\alpha(\vec{r}\cdot\vec{\pi})] + 2[\pi^2, r^2F_{\alpha\beta}\pi_\beta]\right)\\
&\quad + r\left(-c[\pi^2, \frac{x_\alpha}{r^2}] + i(D-3)[\pi^2, \pi_\alpha]\right)\\
&= -2i\Gamma_\alpha\pi^2 + r\left(-2i\pi_\alpha\pi^2 - 2[\pi^2, \pi_\alpha](\vec{r}\cdot\vec{\pi}) + 4i\pi_\alpha\pi^2 + 2[\pi^2, r^2F_{\alpha\beta}\pi_\beta]\right)\\
&\quad + r\left(-c[\pi^2, \frac{x_\alpha}{r^2}] + i(D-3)[\pi^2, \pi_\alpha]\right)\\
&= r\left(-2[\pi^2, \pi_\alpha](\vec{r}\cdot\vec{\pi}) + 2[\pi^2, r^2F_{\alpha\beta}]\pi_\beta + 2r^2F_{\alpha\beta}[\pi^2, \pi_\beta] - c[\pi^2, \frac{x_\alpha}{r^2}]\right.\\
&\quad \left. + i(D-3)[\pi^2, \pi_\alpha]\right)\\
&= r\left(-2[\pi^2, \pi_\alpha](\vec{r}\cdot\vec{\pi}) - 2[\pi^2, x_\alpha\pi_\beta - x_\beta\pi_\alpha]\pi_\beta\right)\\
&\quad + r\left(2r^2F_{\alpha\beta}[\pi^2, \pi_\beta] - c[\pi^2, \frac{x_\alpha}{r^2}] + i(D-3)[\pi^2, \pi_\alpha]\right)\\
&= r\left(-2[\pi^2, \pi_\alpha](\vec{r}\cdot\vec{\pi}) - 2[\pi^2, x_\alpha\pi_\beta]\pi_\beta + 2[\pi^2, x_\beta\pi_\alpha]\pi_\beta\right)\\
&\quad + r\left(2r^2F_{\alpha\beta}[\pi^2, \pi_\beta] - c[\pi^2, \frac{x_\alpha}{r^2}] + i(D-3)[\pi^2, \pi_\alpha]\right)\\
&= r\left(-2[\pi^2, \pi_\alpha](\vec{r}\cdot\vec{\pi}) - 2x_\alpha[\pi^2, \pi_\beta]\pi_\beta + 2x_\beta[\pi^2, \pi_\alpha]\pi_\beta + 4F_{\alpha\beta}\pi_\beta\right)\\
&\quad + r\left(2r^2F_{\alpha\beta}[\pi^2, \pi_\beta] - c[\pi^2, \frac{x_\alpha}{r^2}] + i(D-3)[\pi^2, \pi_\alpha]\right).
\end{aligned}$$

To continue we note that $[\pi^2, \pi_\alpha] = 2iF_{\alpha\gamma}\pi_\gamma$, so

$$
\begin{aligned}
[r\pi^2, Z_\alpha] &= r\left(-4iF_{\alpha\gamma}\pi_\gamma(\vec{r}\cdot\vec{\pi}) - 4ix_\alpha F_{\beta\gamma}\pi_\gamma\pi_\beta + 4ix_\beta F_{\alpha\gamma}\pi_\gamma\pi_\beta + 4F_{\alpha\beta}\pi_\beta\right)\\
&\quad + r\left(4ir^2F_{\alpha\beta}F_{\beta\gamma}\pi_\gamma - c[\pi^2, \frac{x_\alpha}{r^2}] - 2(D-3)F_{\alpha\gamma}\pi_\gamma]\right)\\
&= r\left(2x_\alpha F_{\beta\gamma}F_{\beta\gamma} + 4ir^2F_{\alpha\beta}F_{\beta\gamma}\pi_\gamma - c[\pi^2, \frac{x_\alpha}{r^2}] - 2(D-3)F_{\alpha\gamma}\pi_\gamma]\right)\\
&= 4c_2\frac{x_\alpha}{r^3} - cr[\pi^2, \frac{x_\alpha}{r^2}] + 4ir\left(r^2F_{\alpha\beta}F_{\beta\gamma} + i\frac{D-3}{2}F_{\alpha\gamma}\right)\pi_\gamma\\
&= 4c_2\frac{x_\alpha}{r^3} + icr\{\pi_\beta, [\nabla_\beta, \frac{x_\alpha}{r^2}]\} - 4ir\frac{c_2}{n}\left(\frac{\delta_{\alpha\gamma}}{r^2} - \frac{x_\alpha x_\gamma}{r^4}\right)\pi_\gamma\\
&= 4c_2\frac{x_\alpha}{r^3} + icr\{\pi_\beta, \frac{\delta_{\alpha\beta}}{r^2} - 2\frac{x_\alpha x_\beta}{r^4}\} + 4irc\left(-\frac{1}{r^2}\pi_\alpha + \frac{x_\alpha}{r^4}\vec{r}\cdot\vec{\pi}\right)\\
&= 4c_2\frac{x_\alpha}{r^3} + icr[\pi_\beta, \frac{\delta_{\alpha\beta}}{r^2} - 2\frac{x_\alpha x_\beta}{r^4}] - 2ic\frac{1}{r}\pi_\alpha.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[X, Z_\alpha] &= [r\pi^2 + \frac{c}{r}, Z_\alpha] = 4c_2\frac{x_\alpha}{r^3} + icr[\pi_\beta, \frac{\delta_{\alpha\beta}}{r^2} - 2\frac{x_\alpha x_\beta}{r^4}] - 2ic[\frac{1}{r}, \pi_\alpha]\\
&= 4nc\frac{x_\alpha}{r^3} - 2c(D-2)\frac{x_\alpha}{r^3} - 2c\frac{x_\alpha}{r^3} = 0.
\end{aligned}
$$

□

*Case 4:*

$$[M_\alpha, M_\beta] = -iJ_{\alpha\beta}, \quad [A_\alpha, M_\beta] = -i\eta_{\alpha\beta}T, \quad [A_\alpha, A_\beta] = iJ_{\alpha\beta}$$

or equivalently,

$$[Z_\alpha, Z_\beta] = [W_\alpha, W_\beta] = 0, \quad [Z_\alpha, W_\beta] = -2i\left(\eta_{\alpha\beta}T + J_{\alpha\beta}\right). \tag{21}$$

*Proof:* It is clear that $[W_\alpha, W_\beta] = 0$ because $W_\alpha = x_\alpha$. Next,

$$
\begin{aligned}
[Z_\alpha, W_\beta] &= [x_\alpha\pi^2 - 2\pi_\alpha(\vec{r}\cdot\vec{\pi}) + 2r^2F_{\alpha\gamma}\pi_\gamma - \frac{c}{r^2}x_\alpha + i(D-3)\pi_\alpha, x_\beta]\\
&= x_\alpha[\pi^2, x_\beta] - 2[\pi_\alpha(\vec{r}\cdot\vec{\pi}), x_\beta] + 2r^2F_{\alpha\gamma}[\pi_\gamma, x_\beta] + i(D-3)[\pi_\alpha, x_\beta]\\
&= -2ix_\alpha\pi_\beta - 2\pi_\alpha[(\vec{r}\cdot\vec{\pi}), x_\beta] - 2[\pi_\alpha, x_\beta](\vec{r}\cdot\vec{\pi}) - 2ir^2F_{\alpha\beta} + (D-3)\delta_{\alpha\beta}\\
&= -2ix_\alpha\pi_\beta + 2i\pi_\alpha x_\beta + 2i\delta_{\alpha\beta}(\vec{r}\cdot\vec{\pi}) - 2ir^2F_{\alpha\beta} + (D-3)\delta_{\alpha\beta}\\
&= -2i\left(x_\alpha\pi_\beta - x_\beta\pi_\alpha + r^2F_{\alpha\beta}\right) + 2i\delta_{\alpha\beta}\left(\vec{r}\cdot\vec{\pi} - i\frac{D-1}{2}\right)\\
&= -2i\left(\eta_{\alpha\beta}T + J_{\alpha\beta}\right).
\end{aligned}
$$

Finally, using results from case 2 and case 3, we have

$$
\begin{aligned}
-i[Z_\alpha, Z_\beta] &= [[\Gamma_\alpha, X], Z_\beta] = [\Gamma_\alpha X - X\Gamma_\alpha, Z_\beta]\\
&= [\Gamma_\alpha X, Z_\beta] - [X\Gamma_\alpha, Z_\beta] = [\Gamma_\alpha, Z_\beta]X - X[\Gamma_\alpha, Z_\beta]\\
&= [[\Gamma_\alpha, Z_\beta], X] = [i\eta_{\alpha\beta}X, X] = 0.
\end{aligned}
$$

End of the proof of part (1) of Theorem 2. □

### B. Proof of part (2)

We just need to verify equality

$$\sum_{1\le A\le D+1}\{J_{AB}, J_{AC}\} - \sum_{-1\le A\le 0}\{J_{AB}, J_{AC}\} = 2a\eta_{BC} \tag{22}$$

under the condition that $B \le C$, to be more specific, we need to verify the following identities:

$$\begin{cases} \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, J_{\alpha\gamma}\} + \{A_\beta, A_\gamma\} - \{M_\beta, M_\gamma\} - \{\Gamma_\beta, \Gamma_\gamma\} = 2a\eta_{\beta\gamma}, \\ \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, A_\alpha\} - \{M_\beta, T\} - \{\Gamma_\beta, \Gamma_{D+1}\} = 0, \\ \sum_{1\le\alpha\le D} A_\alpha^2 - T^2 - \Gamma_{D+1}^2 = -a, \\ \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, M_\alpha\} - \{A_\beta, T\} - \{\Gamma_\beta, \Gamma_{-1}\} = 0, \\ \sum_{1\le\alpha\le D}\{A_\alpha, M_\alpha\} - \{\Gamma_{D+1}, \Gamma_{-1}\} = 0, \\ \sum_{1\le\alpha\le D} M_\alpha^2 + T^2 - \Gamma_{-1}^2 = a, \\ \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, \Gamma_\alpha\} - \{A_\beta, \Gamma_{D+1}\} + \{M_\beta, \Gamma_{-1}\} = 0, \\ \sum_{1\le\alpha\le D}\{A_\alpha, \Gamma_\alpha\} + \{T, \Gamma_{-1}\} = 0, \\ \sum_{1\le\alpha\le D}\{M_\alpha, \Gamma_\alpha\} + \{\Gamma_{D+1}, T\} = 0, \\ \sum_{1\le\alpha\le D} \Gamma_\alpha^2 + \Gamma_{D+1}^2 - \Gamma_{-1}^2 = a. \end{cases} \tag{23}$$

The checking is then divided into six cases.

*Case 1:*

$$\sum_{1\le\alpha\le D}\Gamma_\alpha^2 + \Gamma_{D+1}^2 - \Gamma_{-1}^2 = a. \tag{24}$$

*Proof:*

$$\begin{aligned}\sum_{1\le\alpha\le D}\Gamma_\alpha^2 + \Gamma_{D+1}^2 - \Gamma_{-1}^2 &= r\pi_\alpha r\pi_\alpha - \frac{1}{2}(XY + YX) \\ &= r^2\pi^2 - i\vec{r}\cdot\vec{\pi} - \frac{1}{2}(r\pi^2 r + c + r^2\pi^2 + c) \\ &= -i\vec{r}\cdot\vec{\pi} - \frac{1}{2}r[\pi^2, r] - c = -i\vec{r}\cdot\vec{\pi} + \frac{i}{2}r\{\pi_\mu, \frac{x_\mu}{r}\} - c \\ &= \frac{i}{2}r[\pi_\mu, \frac{x_\mu}{r}] - c = \frac{D-1}{2} - c = a. \end{aligned}$$

□

*Case 2:*

$$\sum_{1\le\alpha\le D}\{A_\alpha, \Gamma_\alpha\} + \{T, \Gamma_{-1}\} = 0, \qquad \sum_{1\le\alpha\le D}\{M_\alpha, \Gamma_\alpha\} + \{\Gamma_{D+1}, T\} = 0;$$

or equivalently

$$\sum_{1\le\alpha\le D}\{Z_\alpha, \Gamma_\alpha\} + \{X, T\} = 0, \qquad \sum_{1\le\alpha\le D}\{W_\alpha, \Gamma_\alpha\} - \{Y, T\} = 0. \tag{25}$$

*Proof:* We check the second identity first:

$$\begin{aligned}\sum_{1\le\alpha\le D}\{W_\alpha, \Gamma_\alpha\} - \{Y, T\} &= \{x_\alpha, r\pi_\alpha\} - \{r, \vec{r}\cdot\vec{\pi} - \frac{D-1}{2}i\} \\ &= 2r\vec{r}\cdot\vec{\pi} + r[\pi_\alpha, x_\alpha] - 2r\vec{r}\cdot\vec{\pi} - [\vec{r}\cdot\vec{\pi}, r] + i(D-1)r \\ &= 0. \end{aligned}$$

Then we check the first identity:

$$\begin{aligned}
\sum_{1\le\alpha\le D}\{Z_\alpha,\Gamma_\alpha\}+\{X,T\} &= 2\Gamma_\alpha Z_\alpha + 2TX + [Z_\alpha,\Gamma_\alpha]+[X,T]\\
&= 2(r\pi_\alpha x_\alpha \pi^2 - 2r\pi^2(\vec r\cdot\vec\pi) + 2r\pi_\alpha r^2 F_{\alpha\beta}\pi_\beta - r\pi_\alpha \frac{c}{r^2}x_\alpha + i(D-3)r\pi^2)\\
&\quad + 2(\vec r\cdot\vec\pi - \frac{D-1}{2}i)(r\pi^2+\frac{c}{r}) - i\eta_{\alpha\alpha}X - iX\\
&= 2\left(r(\vec r\cdot\pi)\pi^2 - 2r\pi^2(\vec r\cdot\vec\pi) + 2r^3F_{\alpha\beta}\pi_\alpha\pi_\beta - cr\pi_\alpha\frac{x_\alpha}{r^2} - 3ir\pi^2\right)\\
&\quad + 2\vec r\cdot\vec\pi(r\pi^2+\frac{c}{r})\\
&= 2\left(2ir\pi^2 - r\pi^2(\vec r\cdot\vec\pi) - ir^3F_{\alpha\beta}F_{\alpha\beta} - \frac{c}{r}\vec r\cdot\vec\pi - cr[\pi_\alpha,\frac{x_\alpha}{r^2}] - 3ir\pi^2\right)\\
&\quad + 2\vec r\cdot\vec\pi(r\pi^2+\frac{c}{r})\\
&= 2\left([\vec r\cdot\vec\pi, r\pi^2] - ir^3F_{\alpha\beta}F_{\alpha\beta} + [\vec r\cdot\vec\pi,\frac{c}{r}] - cr[\pi_\alpha,\frac{x_\alpha}{r^2}] - ir\pi^2\right)\\
&= 2\left(-ir^3F_{\alpha\beta}F_{\alpha\beta} + i\frac{c}{r} + ic\frac{D-2}{r}\right)\\
&= 2\left(-2i\frac{c_2}{r} + ic\frac{D-1}{r}\right) = 0.
\end{aligned}$$

□

*Case 3:*

$$\sum_{1\le\alpha\le D}\{J_{\alpha\beta},\Gamma_\alpha\} - \{A_\beta,\Gamma_{D+1}\} + \{M_\beta,\Gamma_{-1}\} = 0. \tag{26}$$

*Proof:*

$$\begin{aligned}
&\sum_{1\le\alpha\le D}\{J_{\alpha\beta},\Gamma_\alpha\} - \{A_\beta,\Gamma_{D+1}\} + \{M_\beta,\Gamma_{-1}\}\\
&= 2J_{\alpha\beta}\Gamma_\alpha + [\Gamma_\alpha, J_{\alpha\beta}] + \frac12\left(\{X,W_\beta\}+\{Y,Z_\beta\}\right)\\
&= 2(x_\alpha\pi_\beta r\pi_\alpha - x_\beta\pi_\alpha r\pi_\alpha + r^2F_{\alpha\beta}r\pi_\alpha) - i(D-1)\Gamma_\beta\\
&\quad + XW_\beta + YZ_\beta + 2i\Gamma_\beta\\
&= 2(\pi_\beta r\vec r\cdot\vec\pi + [x_\alpha,\pi_\beta r]\pi_\alpha - x_\beta[\pi_\alpha,r]\pi_\alpha - x_\beta r\pi^2 + r^3F_{\alpha\beta}\pi_\alpha) - i(D-3)\Gamma_\beta\\
&\quad + (r\pi^2+\frac{c}{r})x_\beta + rx_\beta\pi^2 - 2r\pi_\beta(\vec r\cdot\vec\pi) + 2r^3F_{\beta\gamma}\pi_\gamma - \frac{c}{r}x_\beta + i(D-3)r\pi_\beta\\
&= 2(\pi_\beta r\vec r\cdot\vec\pi + ir\pi_\beta + i\frac{x_\beta}{r}\vec r\cdot\pi) - x_\beta r\pi^2\\
&\quad + r\pi^2x_\beta - 2r\pi_\beta(\vec r\cdot\vec\pi)\\
&= 2(r\pi_\beta\vec r\cdot\vec\pi + ir\pi_\beta) + [r\pi^2,x_\beta] - 2r\pi_\beta(\vec r\cdot\vec\pi)\\
&= 2([r\pi_\beta,\vec r\cdot\vec\pi] + ir\pi_\beta) - 2ir\pi_\beta = 0.
\end{aligned}$$

□

*Case 4:*

$$\sum_{1\le\alpha\le D} A_\alpha^2 - T^2 - \Gamma_{D+1}^2 = -a,$$

$$\sum_{1\le\alpha\le D} M_\alpha^2 + T^2 - \Gamma_{-1}^2 = a,$$

$$\sum_{1\le\alpha\le D} \{A_\alpha, M_\alpha\} - \{\Gamma_{D+1}, \Gamma_{-1}\} = 0;$$

or equivalently

$$\sum_{1\le\alpha\le D} Z_\alpha^2 = X^2, \qquad \sum_{1\le\alpha\le D} \{Z_\alpha, W_\alpha\} + 4T^2 - \{X, Y\} = 4a. \tag{27}$$

Here we have used the fact that $\sum_{1\le\alpha\le D} W_\alpha^2 = Y^2$.

*Proof:* To check the first identity, we note that $Z_\alpha = i[\Gamma_\alpha, X]$ and $[Z_\alpha, X] = 0$, so

$$Z_\alpha^2 = \frac{i}{2}[\{Z_\alpha, \Gamma_\alpha\}, X].$$

Then

$$\begin{aligned}\sum_{1\le\alpha\le D} Z_\alpha^2 &= \frac{i}{2}\sum_{1\le\alpha\le D}[\{Z_\alpha, \Gamma_\alpha\}, X] \\ &= -\frac{i}{2}[\{X, T\}, X] \qquad \text{use results from case 2} \\ &= -\frac{i}{2}[T, X^2] = X^2.\end{aligned}$$

To check the second identity, we note that $Z_\alpha = i[\Gamma_\alpha, X]$ and $[W_\alpha, X] = 2i\Gamma_\alpha$, so

$$\{Z_\alpha, W_\alpha\} = i[\{W_\alpha, \Gamma_\alpha\}, X] + 4\Gamma_\alpha^2.$$

Then

$$\begin{aligned}\sum_{1\le\alpha\le D}\{Z_\alpha, W_\alpha\} &= i[\{Y, T\}, X] + 4\sum_{1\le\alpha\le D}\Gamma_\alpha^2 \qquad \text{use results from case 2} \\ &= i\left(\{Y, [T, X]\} + \{[Y, X], T\}\right) + 4\sum_{1\le\alpha\le D}\Gamma_\alpha^2 \\ &= i\left(\{Y, iX\} + \{2iT, T\}\right) + 4\sum_{1\le\alpha\le D}\Gamma_\alpha^2 \\ &= -\{X, Y\} - 4T^2 + 4\sum_{1\le\alpha\le D}\Gamma_\alpha^2.\end{aligned}$$

So

$$\begin{aligned}\sum_{1\le\alpha\le D}\{Z_\alpha, W_\alpha\} + 4T^2 - \{X, Y\} &= 4\sum_{1\le\alpha\le D}\Gamma_\alpha^2 - 2\{X, Y\} \\ &= 4a \qquad \text{use results from case 1.}\end{aligned}$$

□

*Case 5:*

$$\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, A_\alpha\} - \{M_\beta, T\} - \{\Gamma_\beta, \Gamma_{D+1}\} = 0,$$

$$\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, M_\alpha\} - \{A_\beta, T\} - \{\Gamma_\beta, \Gamma_{-1}\} = 0;$$

or equivalently

$$\begin{cases} \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, Z_\alpha\} - \{Z_\beta, T\} - \{\Gamma_\beta, X\} = 0, \\ \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, W_\alpha\} + \{W_\beta, T\} - \{\Gamma_\beta, Y\} = 0. \end{cases} \tag{28}$$

*Proof:* We check the second identity first:

$$\begin{aligned} &\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, W_\alpha\} + \{W_\beta, T\} - \{\Gamma_\beta, Y\} \\ &= 2x_\alpha J_{\alpha\beta} + [J_{\alpha\beta}, x_\alpha] + 2x_\beta \vec{r}\cdot\vec{\pi} + [\vec{r}\cdot\vec{\pi}, x_\beta] - i(D-1)x_\beta - 2r^2\pi_\beta - r[\pi_\beta, r] \\ &= 2r^2\pi_\beta - 2x_\beta\vec{r}\cdot\vec{\pi} + [J_{\alpha\beta}, x_\alpha] + 2x_\beta\vec{r}\cdot\vec{\pi} + [\vec{r}\cdot\vec{\pi}, x_\beta] - i(D-1)x_\beta - 2r^2\pi_\beta - r[\pi_\beta, r] = 0. \end{aligned}$$

To check the first identity, we note that $Z_\alpha = i[\Gamma_\alpha, X]$ and $[J_{\alpha\beta}, X] = 0$, so

$$\begin{aligned} \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, Z_\alpha\} &= i[\{J_{\alpha\beta}, \Gamma_\alpha\}, X] \\ &= -i[XW_\beta + Z_\beta Y, X] \qquad \text{use results from case 3} \\ &= -iX[W_\beta, X] - i[Z_\beta, X]Y - iZ_\beta[Y, X] \\ &= 2X\Gamma_\beta + 2Z_\beta T \qquad \text{use results from commutation relations} \\ &= \{X, \Gamma_\beta\} + \{Z_\beta, T\} + [X, \Gamma_\beta] + [Z_\beta, T] \\ &= \{X, \Gamma_\beta\} + \{Z_\beta, T\} + iZ_\beta - iZ_\beta = \{X, \Gamma_\beta\} + \{Z_\beta, T\}. \end{aligned}$$

□

So the first identity is checked.

*Case 6:*

$$\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, J_{\alpha\gamma}\} + \{A_\beta, A_\gamma\} - \{M_\beta, M_\gamma\} - \{\Gamma_\beta, \Gamma_\gamma\} = 2a\eta_{\beta\gamma}. \tag{29}$$

*Proof:*

$$\begin{aligned} \sum_{1\le\alpha\le D}\{J_{\alpha\beta}, J_{\alpha\gamma}\} &= i\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, [\Gamma_\alpha, \Gamma_\gamma]\} \\ &= i\sum_{1\le\alpha\le D}\big([\{J_{\alpha\beta}, \Gamma_\alpha\}, \Gamma_\gamma] - \{[J_{\alpha\beta}, \Gamma_\gamma], \Gamma_\alpha\}\big) \\ &= -i[XW_\beta + Z_\beta Y, \Gamma_\gamma] - \{\Gamma_\alpha\delta_{\beta\gamma} - \Gamma_\beta\delta_{\alpha\gamma}, \Gamma_\alpha\} \\ &= -iX[W_\beta, \Gamma_\gamma] - iZ_\beta[Y, \Gamma_\gamma] - i[X, \Gamma_\gamma]W_\beta - i[Z_\beta, \Gamma_\gamma]Y \\ &\qquad \text{use results from case 3} \\ &\quad -2\delta_{\beta\gamma}\sum_{1\le\alpha\le D}\Gamma_\alpha^2 + \{\Gamma_\beta, \Gamma_\gamma\} \\ &= -\eta_{\beta\gamma}YX + Z_\beta W_\gamma + Z_\gamma W_\beta - \eta_{\beta\gamma}YX \\ &\quad +2\eta_{\beta\gamma}\sum_{1\le\alpha\le D}\Gamma_\alpha^2 + \{\Gamma_\beta, \Gamma_\gamma\}. \end{aligned}$$

So

$$\begin{aligned}\sum_{1\le\alpha\le D}\{J_{\alpha\beta}, J_{\alpha\gamma}\}-\{\Gamma_\beta,\Gamma_\gamma\} &= \eta_{\beta\gamma}\left(2\sum_{1\le\alpha\le D}\Gamma_\alpha^2-2YX\right)+Z_\beta W_\gamma+Z_\gamma W_\beta\\ &= \eta_{\beta\gamma}\left(2\sum_{1\le\alpha\le D}\Gamma_\alpha^2-2YX\right)+\frac{1}{2}\left(\{Z_\beta, W_\gamma\}+\{Z_\gamma, W_\beta\}\right)\\ &\quad -\frac{1}{2}\left([Z_\beta, W_\gamma]+[Z_\gamma, W_\beta]\right)\\ &= \eta_{\beta\gamma}\left(2\sum_{1\le\alpha\le D}\Gamma_\alpha^2-2YX+2iT\right)+\frac{1}{2}\left(\{Z_\beta, W_\gamma\}+\{Z_\gamma, W_\beta\}\right)\\ &= \eta_{\beta\gamma}\left(2\sum_{1\le\alpha\le D}\Gamma_\alpha^2-\{X,Y\}\right)-\{A_\beta, A_\gamma\}+\{M_\beta, M_\gamma\}.\end{aligned}$$

So the identity is true because in case 1 we have verified that

$$2\sum_{1\le\alpha\le D}\Gamma_\alpha^2-\{X,Y\}=2a. \qquad \square$$

End of the proof of part (2) of Theorem 2.

## IV. REPRESENTATION THEORETICAL ASPECTS—THE PRELIMINARY PART

The main objective in the rest of this paper is to show that the algebraic direct sum $\mathcal{H}$ of the energy eigenspaces of a generalized MICZ-Kepler problem in dimension $(2n+1)$ is a unitary highest weight $(\mathfrak{g}, K)$-module where $\mathfrak{g}=\mathfrak{so}(2n+4)$ and $K=\mathrm{Spin}(2)\times_{\mathbb{Z}_2}\mathrm{Spin}(2n+2)$. Along the way, we prove Theorem 1.

We can label the generators of $\mathfrak{g}_0$ [the Lie algebra of $\mathrm{Spin}(2, 2n+2)$] as follows:

$$M_{AB}=-M_{BA}\quad \text{for } A, B=-1, 0, 1, \ldots, 2n+2,$$

where in the $(2n+4)$-dimensional defining representation, the matrix elements of $M_{AB}$ are given by

$$[M_{AB}]_{JK}=-i(\eta_{AJ}\eta_{BK}-\eta_{BJ}\eta_{AK}),$$

with the indefinite metric tensor $\eta$ being $\mathrm{diag}\{++-\cdots-\}$ relative to the following order: $-1$, 0, 1, ..., $2n+2$ for the indices.

One can easily show that

$$[M_{AB}, M_{A'B'}]=i(\eta_{AA'}M_{BB'}+\eta_{BB'}M_{AA'}-\eta_{AB'}M_{BA'}-\eta_{BA'}M_{AB'}). \tag{30}$$

In view of the sign difference between the right hand sides of Eqs. (13) and (30), we define the representation $(\tilde\pi, C^\infty(\mathcal{S}^{2\mu}))$ of $\mathfrak{g}$ as follows: for $\psi\in C^\infty(\mathcal{S}^{2\mu})$,

$$\boxed{\tilde\pi(M_{AB})(\psi)=-\hat J_{AB}\psi}, \tag{31}$$

where, by definition, $\hat J_{AB}:=\frac{1}{\sqrt r}J_{AB}\sqrt r$.

However, what is really relevant for us is just a subspace of $C^\infty(\mathcal{S}^{2\mu})$, i.e., $\mathcal{H}$. Actually, the story is bit more involved: what is really invariant under $\tilde\pi$ is not $\mathcal{H}$, but a twisted version of $\mathcal{H}$ which is denoted by $\tilde{\mathcal{H}}$ later; and there is a twist linear equivalence

$$\tau:\ \mathcal{H}\to\tilde{\mathcal{H}}$$

which preserves the $L^2$-norm, such that, viewing $\tau$ as an equivalence of representations, we get representation $(\pi, \mathcal{H})$. Because of this intricacy, we shall devote Subsections IV A and IV B to some preparations.

### A. Review of the (bound) energy eigenspaces

The bound eigen-states (i.e., $L^2$ *eigen-sections* of the Hamiltonian) of the generalized MICZ-Kepler problems have been analyzed in section 5.1 of Ref. 4 by using the classical analytic methods with the help of the representation theory for compact Lie groups. Recall that the (bound) energy spectrum is

$$E_I = -\frac{1}{2(I+n+|\mu|)^2}, \tag{32}$$

where $I = 0, 1, 2, \cdots$.

Denote by $\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}}$ the restriction bundle of $\mathcal{S}^{2\mu}$ to the unit sphere $\mathrm{S}^{2n}$. As a hermitian bundle with a hermitian connection, $\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}}$ is just the vector bundle

$$\mathrm{Spin}(2n+1)\times_{\mathrm{Spin}(2n)}\mathbf{s}^{2\mu}\to \mathrm{S}^{2n}$$

with the natural $\mathrm{Spin}(2n+1)$-invariant connection. Note that, as a hermitian bundle with a hermitian connection, $\mathcal{S}^{2\mu}$ is the pullback of $\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}}$ under the natural projection $\mathbb{R}^{2n+1}_* \to \mathrm{S}^{2n}$. Let $L^2(\mathcal{S}^{2\mu})$, $L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}})$ be the $L^2$-sections of $\mathcal{S}^{2\mu}$ and $\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}}$, respectively. It is clear that $\mathrm{Spin}(2n+1)$ acts on both $L^2(\mathcal{S}^{2\mu})$ and $L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}})$ unitarily. In fact, as a representation of $\mathrm{Spin}(2n+1)$, $L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}})$ is the induced representation of $\mathbf{s}^{2\mu}$ from $\mathrm{Spin}(2n)$ to $\mathrm{Spin}(2n+1)$; therefore, by the Frobenius reciprocity plus a branching rule[24] for $(\mathrm{Spin}(2n+1), \mathrm{Spin}(2n))$, one has

$$L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{2n}}) = \hat{\bigoplus}_{l\ge 0}\mathscr{R}_l, \tag{33}$$

where $\mathscr{R}_l$ is the irreducible representation of $\mathrm{Spin}(2n+1)$ with highest weight $(l+|\mu|, |\mu|, \cdots, |\mu|)$. Observe that, if we use $\widetilde{X}$ to denote the horizontal lift of a vector field $X$ on $\mathbb{R}^{2n+1}_*$, then the vector field $[\widetilde{r\partial_\alpha}, \widetilde{r\partial_\beta}]$ can be shown to be just the right invariant vector field on $\mathbb{R}_+\times\mathrm{Spin}(2n+1)$ whose value at $(r, e)$ (where $e$ is the group identity element) is $(0, -i\gamma_{\alpha\beta})$, i.e., $(0, -\frac{1}{4}[e_\alpha, e_\beta])$. Consequently, the infinitesimal action of $\mathrm{Spin}(2n+1)$ on $C^\infty(\mathcal{S}^{2\mu})$ is just the restriction of $\tilde{\pi}$ to $\mathrm{span}_{\mathbb{R}}\{M_{\alpha\beta}\mid 1\le\alpha<\beta\le 2n+1\} = \mathfrak{so}_0(2n+1)$. It is then clear that $\tilde{\pi}(M_{\alpha\beta})$'s act only on the angular part of the wave sections—a consequence which can also be deduced from the fact that $\hat{J}_{\alpha\beta}$'s commute with the multiplication by a smooth function of $r$.

Let $\{Y_{l\mathbf{m}}(\Omega)\}_{\mathbf{m}\in\mathcal{I}(l)}$ be an orthonormal (say Gelfand–Zeltin) basis for $\mathscr{R}_l$, and

$$\boxed{l_\mu = l+|\mu|+n-1}.$$

Then, an orthonormal basis for the energy eigenspace $\mathscr{H}_I$ with energy $E_I$ is

$$\{\psi_{kl\mathbf{m}} := R_{kl_\mu}(r)Y_{l\mathbf{m}}(\Omega)\mid k+l = I+1, k\ge 1, l\ge 0, \mathbf{m}\in\mathcal{I}(l)\}, \tag{34}$$

where $R_{kl_\mu}\in L^2(\mathbb{R}_+, r^{2n}\,dr)$ is a square integrable (with respect to the measure $r^{2n}\,dr$) solution of the radial Schrödinger equation:

$$\left(-\frac{1}{2r^{2n}}\partial_r r^{2n}\partial_r + \frac{l_\mu(l_\mu+1)-n(n-1)}{2r^2} - \frac{1}{r}\right)R_{kl_\mu} = E_{k-1+l}R_{kl_\mu}. \tag{35}$$

Note that $R_{kl_\mu}$ is of the form

$$r^{-n}y_{kl_\mu}(r)\exp\left(-\frac{r}{k+l_\mu}\right)$$

with $y_{kl_\mu}(r)$ satisfying equation

$$\left(\frac{d^2}{dr^2} - \frac{2}{k+l_\mu}\frac{d}{dr} + \left[\frac{2}{r} - \frac{l_\mu(l_\mu+1)}{r^2}\right]\right)y_{kl_\mu}(r) = 0. \tag{36}$$

In term of the generalized Laguerre polynomials,

$$y_{kl_\mu}(r) = c(k,l) r^{l_\mu+1} L_{k-1}^{2l_\mu+1}\left(\frac{2}{k+l_\mu} r\right),$$

where $c(k,l)$ is a constant, which can be uniquely determined by requiring $c(k,l)>0$ and $\int_0^\infty |R_{kl_\mu}(r)|^2 r^{2n} dr = 1$.

We are now ready to state the following remark.

*Remark 1: (1) $\mathscr{H}_I$ is the space of square integrable solutions of Eq. $H\psi = E_I\psi$.*

(2) *As representation of* $\mathfrak{so}(2n+1)$

$$\mathscr{H}_I = \bigoplus_{l=0}^{I} D_l \tag{37}$$

*where* $D_l := \mathrm{span}\{\psi_{(I-l+l)l\boldsymbol{m}} \,|\, \boldsymbol{m} \in \mathcal{I}(l)\}$ *is the highest weight module with highest weight* $(l+|\mu|, |\mu|, \cdots, |\mu|)$.

(3) $\{\mathscr{H}_I \mid I = 0, 1, 2, \ldots\}$ *is the complete set of (bound) energy eigenspaces.*

For the completeness of this review, we state part of Theorem 1 from Ref. 4 below:

**Theorem 3:** *For the* $(2n+1)$*-dimensional generalized MICZ-Kepler problem with magnetic charge* $\mu$*, the following statements are true:*

(1) *The negative energy spectrum is*

$$E_I = -\frac{1/2}{(I+n+|\mu|)^2},$$

*where* $I = 0, 1, 2, \ldots$*;*

(2) *The Hilbert space* $\mathscr{H}(\mu)$ *of negative-energy states admits a linear* $\mathrm{Spin}(2n+2)$*-action under which there is a decomposition*

$$\mathscr{H}(\mu) = \hat{\bigoplus}_{I=0}^{\infty} \mathscr{H}_I,$$

*where* $\mathscr{H}_I$ *is the irreducible* $\mathrm{Spin}(2n+2)$*-module with highest weight* $(I+|\mu|, |\mu|, \cdots, |\mu|, \mu)$*;*

(3) *The linear action in part (2) extends the manifest linear action of* $\mathrm{Spin}(2n+1)$*, and* $\mathscr{H}_I$ *in part (2) is the energy eigenspace with eigenvalue* $E_I$ *in part (1).*

It was shown in Ref. 4 that the bound eigen-states are precisely the ones with negative energy eigenvalues. We would like to remark that, in dimension five, a similar result obtained with a similar method has already appeared in Ref. 25.

## B. Twisting

As we said before, because of the technical intricacy, we need to introduce the notion of twisting. Let us start with the listing of some important spaces used later:

- $\mathscr{H}_I$ — the $I$th bound energy eigenspace;
- $\mathcal{H}$ — the algebraic direct sum of all bound energy eigenspaces;
- $\mathscr{H}$ or $\mathscr{H}(\mu)$ — the completion of $\mathcal{H}$ under the standard $L^2$-norm;
- $\mathcal{H}_{l\mathbf{m}}$ — the subspace of $\mathcal{H}$ spanned by $\{\psi_{kl\mathbf{m}} \,|\, k \geq 1,\ 1, \mathbf{m}\, fixed\}$;
- $\mathscr{H}_{l\mathbf{m}}$ — the completion of $\mathcal{H}_{l\mathbf{m}}$ under the standard $L^2$-norm.

Note that these spaces are all endowed with the unique hermitian inner product which yields the standard $L^2$-norm, i.e.,

$$\boxed{\langle\psi, \phi\rangle := \int_{\mathbb{R}^D_*} (\psi, \phi)\, d^D x}, \tag{38}$$

where $(\psi, \phi)$ is the point-wise hermitian inner product and $d^D x$ is the Lebesgue measure.

It is clear from Subsection IV A that

$$\mathscr{B} := \{\psi_{kl\mathbf{m}} \mid k \geq 1, l \geq 0, \mathbf{m} \in \mathcal{I}(l)\} \tag{39}$$

is an orthonormal basis for both $\mathcal{H}$ and $\mathscr{H}$.

To study the action of $\hat{J}_{AB}$'s, we need to "twist" $\mathscr{B}$, $\mathscr{H}_I$, $\mathcal{H}_{l\mathbf{m}}$, $\mathscr{H}_{l\mathbf{m}}$, $\mathcal{H}$, and $\mathscr{H}$ to get $\tilde{\mathscr{B}}$, $\tilde{\mathscr{H}}_I$, $\tilde{\mathcal{H}}_{l\mathbf{m}}$, $\tilde{\mathscr{H}}_{l\mathbf{m}}$, $\tilde{\mathcal{H}}$, and $\tilde{\mathscr{H}}$, respectively. It suffices to twist the elements of $\mathscr{B}$. Let $\tau\colon \mathscr{B} \to \tilde{\mathscr{B}}$ be defined as follows:

$$\begin{aligned}\tau(\psi_{kl\mathbf{m}})(r,\Omega) &:= (k+l_\mu)\, e^{-i\theta_{k+l_\mu}\hat{T}} \left(\frac{1}{\sqrt{r}}\psi_{kl\mathbf{m}}(r,\Omega)\right)\\ &= (k+l_\mu)^{n+1}\frac{1}{\sqrt{r}}\psi_{kl\mathbf{m}}((k+l_\mu)r,\Omega)\\ &\propto r^{l+|\mu|-\frac{1}{2}}\, L^{2l_\mu+1}_{k-1}(2r)\, e^{-r}\, Y_{l\mathbf{m}}(\Omega),\end{aligned} \tag{40}$$

where $\hat{T} = \frac{1}{\sqrt{r}} T \sqrt{r}$, and $\theta_I = -\ln I$ for any positive number $I$. For simplicity, we write $\tau(\psi_{kl\mathbf{m}})$ as $\tilde{\psi}_{kl\mathbf{m}}$. One can check that

$$\int_{\mathbb{R}^D_*} (\tilde{\psi}_{kl\mathbf{m}}, \tilde{\psi}_{kl\mathbf{m}})\, d^D x = \int_{\mathbb{R}^D_*} (\psi_{kl\mathbf{m}}, \psi_{kl\mathbf{m}})\, d^D x = 1.$$

By using Eq. (40) and the orthogonality identities for the generalized Laguerre polynomials, one can see that $\tilde{\psi}_{kl\mathbf{m}}$ is orthogonal to $\tilde{\psi}_{k'l\mathbf{m}}$ when $k \neq k'$.

It is now clear how to twist all the relevant spaces listed in the beginning of this subsection. For example,

$$\tilde{\mathscr{H}}_I := \left\{ e^{-i\theta_{I_\mu+1}\hat{T}} \left(\frac{1}{\sqrt{r}}\psi\right) \mid \psi \in \mathscr{H}_I \right\}. \tag{41}$$

Since $\mathscr{H}_I$ is spanned by $\{\psi_{kl\mathbf{m}} \mid k + l = I + 1, k \geq 1, l \geq 0, \mathbf{m} \in \mathcal{I}(l)\}$, it follows that $\tilde{\mathscr{H}}_I$ is spanned by

$$\{\tilde{\psi}_{kl\mathbf{m}} \mid k + l = I + 1, k \geq 1, l \geq 0, \mathbf{m} \in \mathcal{I}(l)\}.$$

We shall call $\tilde{\mathscr{H}}(\mu)$ the twisted Hilbert space of the bound states for the $(2n+1)$-dimensional generalized MICZ-Kepler problem with magnetic charge $\mu$. We remark that the twisting map[26]

$$\tau:\ \mathscr{H}(\mu) \to \tilde{\mathscr{H}}(\mu) \tag{42}$$

is the unique linear isometry which sends $\psi_{kl\mathbf{m}}$ to $\tilde{\psi}_{kl\mathbf{m}}$; moreover, $\tau$ maps all relevant subspaces of $\mathscr{H}(\mu)$ isomorphically onto the corresponding relevant twisted subspaces. Note that $\hat{J}_{\alpha\beta} = \frac{1}{\sqrt{r}} J_{\alpha\beta}\sqrt{r} = J_{\alpha\beta}$ obviously acts on $\tilde{\mathscr{H}}_I$ as hermitian operator, so $\mathfrak{r} := \mathrm{span}\{M_{\alpha\beta} \mid 1 \leq \alpha < \beta \leq 2n+1\} = \mathfrak{so}(2n+1)$ acts unitarily on $\tilde{\mathscr{H}}_I$ via $\tilde{\pi}$.

Recall that for non-negative integer $I$, we use $I_\mu$ to denote $I + n + |\mu| - 1$.

*Proposition 1: (1) $\tilde{\psi}_{kl\mathbf{m}}$ is an eigenvector of $\hat{\Gamma}_{-1}$ with eigenvalue $k + l_\mu$.*

*(2) $\tilde{\mathscr{H}}_I$ is the space of square integrable solutions of Eq. $\hat{\Gamma}_{-1}\psi = (I_\mu + 1)\psi$.*

*(3) $\hat{\Gamma}_{-1}$ is a self-adjoint operator on $\tilde{\mathscr{H}}(\mu)$ and $\tilde{\mathscr{H}}_I$ is the eigenspace of $\hat{\Gamma}_{-1}$ with eigenvalue $I_\mu + 1$.*

*(4) As representation of $\mathfrak{r}$,*

$$\tilde{\mathscr{H}}_I = \bigoplus_{l=0}^{I} \tilde{D}_l, \tag{43}$$

*where $\tilde{D}_l := \mathrm{span}\{\tilde{\psi}_{(I-l+1)l\mathbf{m}} \mid \mathbf{m} \in \mathcal{I}(l)\}$ is the highest weight module with highest weight $(l + |\mu|, |\mu|, \cdots, |\mu|)$.*

*(5)* $\tilde{\mathscr{H}}(\mu) = L^2(\mathcal{S}^{2\mu})$.

*Proof:* (1) The proof is based on the ideas from Ref. 5. Since

$$H\psi_{kl\mathbf{m}} = E_{k+l-1}\psi_{kl\mathbf{m}}, \tag{44}$$

we have $\sqrt{r}(H - E_{k+l-1})\psi_{kl\mathbf{m}} = 0$ which can be rewritten as

$$(\frac{1}{2}\hat{X} - 1 - E_{k+l-1}\hat{Y})(\frac{1}{\sqrt{r}}\psi_{kl\mathbf{m}}) = 0,$$

where $X$ and $Y$ are given by Eq. (9). In terms of $\hat{\Gamma}_{-1}$ and $\hat{\Gamma}_{D+1}$, we can recast the above equation as

$$\left((\frac{1}{2} - E_{k+l-1})\hat{\Gamma}_{-1} + (\frac{1}{2} + E_{k+l-1})\hat{\Gamma}_{D+1} - 1\right)(\frac{1}{\sqrt{r}}\psi_{kl\mathbf{m}}) = 0.$$

Plugging $\psi_{kl\mathbf{m}} = \frac{1}{k+l_\mu}\sqrt{r}e^{i\theta_{k+l_\mu}\hat{T}}\left(\tilde{\psi}_{kl\mathbf{m}}\right)$ into the above equation and using identities

$$\begin{cases} e^{-i\theta\hat{T}}\,\hat{\Gamma}_{-1}\,e^{i\theta\hat{T}} = \cosh\theta\,\hat{\Gamma}_{-1} + \sinh\theta\,\hat{\Gamma}_{D+1} \\ e^{-i\theta\hat{T}}\,\hat{\Gamma}_{D+1}\,e^{i\theta\hat{T}} = \sinh\theta\,\hat{\Gamma}_{-1} + \cosh\theta\,\hat{\Gamma}_{D+1}, \end{cases} \tag{45}$$

we arrive at the following equation:

$$\boxed{\hat{\Gamma}_{-1}\tilde{\psi}_{kl\mathbf{m}} = (k + l_\mu)\tilde{\psi}_{kl\mathbf{m}}}. \tag{46}$$

(2) Note that the Barut–Bornzin process going from Eq. (44) to Eq. (46) is completely reversible. Therefore, part (2) is just a consequence of part (1) of Remark 1.

(3) Note that $\hat{\Gamma}_{-1}$ is defined on the dense linear subspace $\tilde{\mathcal{H}}$ of $\tilde{\mathscr{H}}(\mu)$. It is easy to check that

$$\langle\tilde{\psi}_{k'l'\mathbf{m}'}, \hat{\Gamma}_{-1}\tilde{\psi}_{kl\mathbf{m}}\rangle = \langle\hat{\Gamma}_{-1}\tilde{\psi}_{k'l'\mathbf{m}'}, \tilde{\psi}_{kl\mathbf{m}}\rangle$$

for any $\tilde{\psi}_{kl\mathbf{m}}$ and $\tilde{\psi}_{k'l'\mathbf{m}'}$. Therefore, $\hat{\Gamma}_{-1}$ (to be precise, it should be its closure) is a self-adjoint operator on $\tilde{\mathscr{H}}(\mu)$. In view of part (2), $\tilde{\mathscr{H}}_I$ is the eigenspace of $\hat{\Gamma}_{-1}$ with eigenvalue $I_\mu + 1$.

(4) This part is clear due to part (2) of Remark 1.

(5) Recall that $\tilde{\psi}_{kl\mathbf{m}}(r, \Omega) = \tilde{R}_{kl_\mu}(r)\,Y_{l\mathbf{m}}(\Omega)$ where $\tilde{R}_{kl_\mu}(r) \propto r^{l+|\mu|-\frac{1}{2}}\,L_{k-1}^{2l_\mu+1}(2r)\,e^{-r}$. By the well-known property for the generalized Laguerre polynomials, for any $l \geq 0$, $\{\tilde{R}_{kl_\mu}\}_{k=1}^{\infty}$ form an orthonormal basis for $L^2(\mathbb{R}_+, r^{2n}\,dr)$.

By virtue of Theorem II. 10 of Ref. 27 and Eq. (33),

$$\begin{aligned} L^2(\mathcal{S}^{2\mu}) &= L^2(\mathbb{R}_+, r^{2n}\,dr) \otimes L^2(\mathcal{S}^{2\mu}\mid_{\mathrm{S}^{2n}}) \\ &= \hat{\bigoplus}_{l=0}^{\infty}\left(L^2(\mathbb{R}_+, r^{2n}\,dr) \otimes \mathscr{R}_l\right). \end{aligned}$$

Therefore, $\tilde{\mathscr{B}}$ is an orthonormal basis for $L^2(\mathcal{S}^{2\mu})$, consequently $\tilde{\mathscr{H}}(\mu) = L^2(\mathcal{S}^{2\mu})$.

We end this subsection with □

*Remark 2: $\tilde{\mathscr{H}}_I$ is the eigenspace of $\tilde{\pi}(H_0)$ with eigenvalue $-(I_\mu + 1)$. Here $\tilde{\pi}(H_0) = -\hat{\Gamma}_{-1}$ is viewed as an endomorphism of $\tilde{\mathcal{H}}$.*

## V. REPRESENTATION THEORETICAL ASPECTS—THE FINAL PART

We start with some notations:

- $G = \mathrm{Spin}(2, 2n+2)$ — the double cover of $\mathrm{SO}_0(2, 2n+2)$ characterized by the homomorphism $\pi_1(\mathrm{SO}_0(2, 2n+2)) = \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ sending $(a, b)$ to $\bar{a} + b$;
- $\mathfrak{g}_0$ — the Lie algebra of $\mathrm{Spin}(2, 2n+2)$;
- $\mathfrak{g}$ — the complexfication of $\mathfrak{g}_0$, so $\mathfrak{g} = \mathfrak{so}(2n+4)$;
- $H_0$ — defined to be $M_{-1,0}$;

- $H_j$ — defined to be $-M_{2j-1,2j}$ for $1 \le j \le n+1$;
- $K := \mathrm{Spin}(2) \times_{\mathbb{Z}_2} \mathrm{Spin}(2n+2)$ — a maximal compact subgroup of $\mathrm{Spin}(2, 2n+2)$;
- $\mathfrak{k}_0$ — the Lie algebra of $K$;
- $\mathfrak{k}$ — the complexfication of $\mathfrak{k}_0$, so $\mathfrak{k} = \mathfrak{so}(2) \oplus \mathfrak{so}(2n+2)$;
- $\mathfrak{r}$ — the subalgebra of $\mathfrak{g}$ generated by $\{M_{AB} \mid 1 \le A < B \le 2n+1\}$, so $\mathfrak{r}_0 := \mathfrak{g}_0 \cap \mathfrak{r} = \mathfrak{so}_0(2n+1)$;
- $\mathfrak{s}$ — the subalgebra of $\mathfrak{g}$ generated by $\{M_{AB} \mid 1 \le A < B \le 2n+2\}$, so $\mathfrak{s}_0 := \mathfrak{g}_0 \cap \mathfrak{s} = \mathfrak{so}_0(2n+2)$;
- $\mathfrak{sl}(2)$ — the subalgebra of $\mathfrak{g}$ generated by $M_{-1,D+1}$, $M_{0,D+1}$ and $M_{-1,0}$, so $\mathfrak{sl}_0(2) := \mathfrak{g}_0 \cap \mathfrak{sl}(2) = \mathfrak{so}_0(2,1)$;
- $U(\mathfrak{sl}(2))$ — the universal enveloping algebra of $\mathfrak{sl}(2)$.

### A. $\tilde{\mathcal{H}}$ is a unitary highest weight Harish-Chandra module

The goal of this subsection is to show that $(\tilde\pi, \tilde{\mathcal{H}})$ is a unitary highest weight $(\mathfrak{g}, K)$-module.

*Proposition 2: (1) Each $\tilde\pi(M_{AB})$ maps $\tilde{\mathcal{H}}$ into $\tilde{\mathcal{H}}$, so $(\tilde\pi, \tilde{\mathcal{H}})$ is a representation of $\mathfrak{g}$.*
*(2) Each $\tilde\pi(M_{AB})$ is a hermitian operator on $\tilde{\mathcal{H}}$, so $(\tilde\pi, \tilde{\mathcal{H}})$ is a unitary representation of $\mathfrak{g}$.*
*(3) $(\tilde\pi|_{\mathfrak{sl}(2)}, \tilde{\mathcal{H}}_{l\mathbf{m}})$ is the discrete series representation of $\mathfrak{so}(2,1)$ with highest weight $-l_\mu - 1$.*

*Proof:* (1) We follow the convention of Ref. 28 for describing the root space of $\mathfrak{g} = \mathfrak{so}(2n+4)$. Take as a basis of the Cartan subalgebra of $\mathfrak{g}$ the following elements:

$$H_0 = M_{-1,0}, \quad H_j = -M_{2j-1,2j}, \quad j = 1, \cdots, n+1.$$

Let $\eta, \eta' = \pm 1$. We take the following root vectors:

$$E_{\eta e^j + \eta' e^k} = \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}\right),$$

where $0 \le j < k \le n+1$. This way we obtain a Cartan basis for $\mathfrak{g}$. Therefore, for $\psi_I \in \tilde{\mathscr{H}}_I$, we have

$$\begin{aligned}\tilde\pi(H_0)(\tilde\pi(E_\alpha)(\psi_I)) &= (-I_\mu - 1 + \alpha_0)\tilde\pi(E_\alpha)(\psi_I)\\ &= (-(I-\alpha_0)_\mu - 1)\tilde\pi(E_\alpha)(\psi_I),\end{aligned} \tag{47}$$

where $\alpha_0$ (which can be 0, or $-1$ or 1) is the 0-th component of $\alpha$. It is not hard to see that $\tilde\pi(E_\alpha)(\psi_I)$ is square integrable,[29] so in view of part (2) of Proposition 1, Eq. (47) implies that $\tilde\pi(E_\alpha)(\psi_I) \in \tilde{\mathscr{H}}_{I-\alpha_0}$. (Here $\mathscr{H}_{-1} = 0$.) Therefore, $\tilde\pi(E_\alpha)$ maps any $\tilde{\mathscr{H}}_I$, hence $\tilde{\mathcal{H}}$, into $\tilde{\mathcal{H}}$. By a similar argument, one can show that $\tilde\pi(H_i)$ maps $\tilde{\mathcal{H}}$ into itself. Since $H$'s and $E$'s form a basis for $\mathfrak{g}$, this implies that $\tilde\pi(M_{AB})$ maps $\tilde{\mathcal{H}}$ into itself.

(2) It is equivalent to checking that each $\hat J_{AB} := \frac{1}{\sqrt r} J_{AB}\sqrt r$ is a hermitian operator on $\tilde{\mathcal{H}}$. First of all, it is not hard to see that, when $\mathcal{O} = \pi_\alpha, r, \frac{1}{r}, \sqrt r, \frac{1}{\sqrt r}$, we always have

$$\langle \psi_1, \mathcal{O}\psi_2\rangle = \langle \mathcal{O}\psi_1, \psi_2\rangle \tag{48}$$

for any $\psi_1, \psi_2$ in $\tilde{\mathcal{H}}$. It is equally easy to see that Eq. (48) is always true for any $\psi_1, \psi_2$ in $\tilde{\mathcal{H}}$ when $\mathcal{O}$ is $\hat\Gamma_\alpha = \sqrt r \pi_\alpha \sqrt r$, $\hat X = \sqrt r \pi^2 \sqrt r + \frac{c}{r}$, or $\hat Y = r$. It is then clear from definitions (9) and (10) that Eq. (48) is always true for any $\psi_1, \psi_2$ in $\tilde{\mathcal{H}}$ when $\mathcal{O} = \hat J_{AB}$.

(3) Let us first show that $\tilde\pi(M_{-1,D+1})$, $\tilde\pi(M_{0,D+1})$, and $\tilde\pi(M_{-1,0})$ map each $\tilde\psi_{kl\mathbf{m}}$ into $\tilde{\mathcal{H}}_{l\mathbf{m}}$, so they indeed map $\tilde{\mathcal{H}}_{l\mathbf{m}}$ into $\tilde{\mathcal{H}}_{l\mathbf{m}}$. This is obvious for $\tilde\pi(M_{-1,0})$ because $\tilde\pi(M_{-1,0})(\tilde\psi_{kl\mathbf{m}}) = -\hat\Gamma_{-1}\tilde\psi_{kl\mathbf{m}} = -(k+l_\mu)\tilde\psi_{kl\mathbf{m}}$. Next, we introduce

$$E_\pm = \frac{1}{\sqrt 2}(M_{-1,D+1} \pm i M_{0,D+1}),$$

then one can check from Eq. (30) that $[M_{-1,0}, E_\pm] = \pm E_\pm$. Therefore

$$\tilde\pi(M_{-1,0})(\tilde\pi(E_\pm)(\tilde\psi_{kl\mathbf{m}})) = (-k - l_\mu \pm 1)\tilde\pi(E_\pm)(\tilde\psi_{kl\mathbf{m}}),$$

where $\tilde{\pi}(E_\pm) = \frac{1}{\sqrt{2}}(\hat{T} \pm i\hat{\Gamma}_{D+1})$. It is not hard to see that $\tilde{\pi}(E_\pm)(\tilde{\psi}_{kl\mathbf{m}})$ is square integrable. In view of part (2) of Proposition 1, we conclude that $\tilde{\pi}(E_\pm)(\tilde{\psi}_{kl\mathbf{m}})$ must be proportional to $\tilde{\psi}_{(k\mp 1)l\mathbf{m}}$. (Here, by convention, $\tilde{\psi}_{0l\mathbf{m}} = 0$.) Therefore, operators $\tilde{\pi}(E_\pm)$ map $\tilde{\psi}_{kl\mathbf{m}}$ into $\tilde{\mathcal{H}}_{l\mathbf{m}}$. This proves that $(\tilde{\pi}|_{\mathfrak{sl}(2)}, \tilde{\mathcal{H}}_{l\mathbf{m}})$ is a representation of $\mathfrak{sl}(2)$.

In view of the fact that $\tilde{\pi}(M_{-1,0})(\tilde{\psi}_{1l\mathbf{m}}) = -(l_\mu + 1)\tilde{\psi}_{1l\mathbf{m}} \neq 0$, we conclude that $U(\mathfrak{sl}(2)) \cdot \tilde{\psi}_{1l\mathbf{m}}$ is a *nontrivial* unitary highest weight representation of the noncompact real Lie algebra $\mathfrak{sl}_0(2)$, hence must be the discrete series representation with highest weight $-(l_\mu + 1)$. Since $U(\mathfrak{sl}(2)) \cdot \tilde{\psi}_{1l\mathbf{m}} \subset \tilde{\mathcal{H}}_{l\mathbf{m}}$, and $\dim(\mathscr{H}_I \cap U(\mathfrak{sl}(2)) \cdot \tilde{\psi}_{1l\mathbf{m}}) = \dim(\mathscr{H}_I \cap \tilde{\mathcal{H}}_{l\mathbf{m}})$ for all $I \geq 0$, we conclude that $U(\mathfrak{sl}(2)) \cdot \tilde{\psi}_{1l\mathbf{m}} = \tilde{\mathcal{H}}_{l\mathbf{m}}$. Therefore, $\tilde{\mathcal{H}}_{l\mathbf{m}}$ is a unitary highest weight $\mathfrak{sl}(2)$-module with highest weight $-l_\mu - 1$, which in fact is a unitary highest weight $(\mathfrak{sl}(2), \mathrm{Spin}(2))$-module. Then $\tilde{\mathscr{H}}_{l\mathbf{m}}$ must be the discrete series representation of Spin(2, 1) with highest weight $-l_\mu - 1$. □

To continue the discussion on representations, we prove the following proposition.

*Proposition 3: (1)* $(\tilde{\pi}|_{\mathfrak{s}}, \tilde{\mathscr{H}}_I)$ *is an irreducible unitary representation of* $\mathfrak{s}$*, in fact, it is the highest weight representation with highest weight* $(I + |\mu|, |\mu|, \ldots, |\mu|, \mu)$.

*(2) The unitary action of* $\mathfrak{k}_0$ *on* $\tilde{\mathcal{H}}$ *can be lifted to a unique unitary action of* $K$ *under which*

$$\tilde{\mathcal{H}} = \bigoplus_{l=0}^{\infty} \left( D(-l_\mu - 1) \otimes D^l \right), \tag{49}$$

*where* $D^l$ *is the irreducible module of* Spin$(2n+2)$ *with highest weight* $(l + |\mu|, |\mu|, \cdots, |\mu|, \mu)$ *and* $D(-l_\mu - 1)$ *is the irreducible module of* Spin(2) *with weight* $-l_\mu - 1$.

*(3)* $\tilde{\mathcal{H}}$ *is a unitary* $(\mathfrak{g}, K)$*-module.*

*(4)* $(\tilde{\pi}, \tilde{\mathcal{H}})$ *is irreducible; in fact, it is the unitary highest weight module of* $\mathfrak{g}$ *with highest weight*

$$(-(n + |\mu|), |\mu|, \cdots, |\mu|, \mu).$$

*Proof:* (1) Recall that $\mathfrak{s}$ is the $\mathfrak{so}(2n+2)$ Lie subalgebra of $\mathfrak{g}$ generated by

$$\{H_i, E_{\pm e^j \pm e^k} \mid 1 \leq i \leq n+1, 1 \leq j < k \leq n+1\},$$

and $\mathfrak{s}_0 := \mathfrak{s} \cap \mathfrak{g}_0$ is the compact real form of $\mathfrak{s}$. Since $H_0$ commutes with any element in $\mathfrak{s}$, in view of Remark 2, we conclude that each $\tilde{\mathscr{H}}_I$ is invariant under $\tilde{\pi}(\mathfrak{s})$, i.e., $(\tilde{\pi}|_{\mathfrak{s}}, \tilde{\mathscr{H}}_I)$ is a representation of $\mathfrak{s}$.

Inside $\mathfrak{s}$ there is an $\mathfrak{so}(2n+1)$ Lie subalgebra $\mathfrak{r}$. Note that $H_1$, ..., $H_n$ are the generators of a Cartan subalgebra of $\mathfrak{r}$, and $H_1$, ..., $H_{n+1}$ are the generators of a Cartan subalgebra of $\mathfrak{s}$. Recall from part (4) of Proposition 1,

$$(\tilde{\pi}|_{\mathfrak{r}}, \tilde{\mathscr{H}}_I) = \bigoplus_{l=0}^{I} \tilde{D}_l \tag{50}$$

where $\tilde{D}_l$ is the highest weight $\mathfrak{r}$-module with highest weight $(l + |\mu|, |\mu|, \cdots, |\mu|)$.

By applying the branching rule[30] for $(\mathfrak{s}, \mathfrak{r})$, one finds that there are only two solutions to Eq. (50): $(\tilde{\pi}|_{\mathfrak{s}}, \tilde{\mathscr{H}}_I)$ is the highest weight module of $\mathfrak{s}$ with highest weight equal to either $(I + |\mu|, |\mu|, \cdots, |\mu|, \mu)$ or $(I + |\mu|, |\mu|, \cdots, |\mu|, -\mu)$. Let $\tilde{\psi}_{1I\mathbf{I}} \in \tilde{\mathscr{H}}_I$ be an $\mathfrak{s}$-highest weight vector, which is assumed to have unit norm. Since $\tilde{\pi}(H_{n+1}) = \hat{A}_D$, we have either $\hat{A}_D\tilde{\psi}_{1I\mathbf{I}} = \mu\tilde{\psi}_{1I\mathbf{I}}$ or $\hat{A}_D\tilde{\psi}_{1I\mathbf{I}} = -\mu\tilde{\psi}_{1I\mathbf{I}}$. To determine the sign, we only need to show that $\langle\tilde{\psi}_{1I\mathbf{I}}, A_D\tilde{\psi}_{1I\mathbf{I}}\rangle = \mu$. Note that $A_D = i[\Gamma_D, \Gamma_{D+1}] = i[\Gamma_D, \Gamma_{-1} - r] = i[\Gamma_D, \Gamma_{-1}] - x_D$ and $\tilde{\psi}_{1I\mathbf{I}}$ is an eigenvector of $\hat{\Gamma}_{-1}$, so

$$\begin{aligned} \langle\tilde{\psi}_{1I\mathbf{I}}, \hat{A}_D\tilde{\psi}_{1I\mathbf{I}}\rangle &= -\langle\tilde{\psi}_{1I\mathbf{I}}, x_D\tilde{\psi}_{1I\mathbf{I}}\rangle \\ &= -\int_{\mathbb{R}^D_*} x_D \left|\tilde{\psi}_{1I\mathbf{I}}(r, \Omega)\right|^2 d^D x. \end{aligned} \tag{51}$$

One can show that,[31] up to a multiplicative constant, $\tilde{\psi}_{1I\mathbf{I}}(r, \Omega)$ is equal to

$$r^{I_\mu - n + \frac{1}{2}} e^{-r} \cdot (\sin\theta)^{-(n-1)} (1 - \cos\theta)^{\frac{I_\mu + \mu}{2}} (1 + \cos\theta)^{\frac{I_\mu - \mu}{2}} \cdot Z(\theta_1, \ldots, \theta_{D-3}, \phi).$$

Then

$$
\begin{aligned}
\langle\tilde\psi_{1l\mathbf{I}}, \hat A_D\tilde\psi_{1l\mathbf{I}}\rangle &= -\int_{\mathbb{R}^D_*} x_D\left|\tilde\psi_{1l\mathbf{I}}(r,\Omega)\right|^2 d^Dx\\
&= -\frac{\int_0^\infty r^{2I_\mu+2}e^{-2r}\,dr}{\int_0^\infty r^{2I_\mu+1}e^{-2r}\,dr}\cdot\frac{\int_0^\pi \cos\theta\,(1-\cos\theta)^{I_\mu+\mu}(1+\cos\theta)^{I_\mu-\mu}\sin\theta\,d\theta}{\int_0^\pi (1-\cos\theta)^{I_\mu+\mu}(1+\cos\theta)^{I_\mu-\mu}\sin\theta\,d\theta}\\
yy &= -\frac{\Gamma(2I_\mu+3)}{2\cdot\Gamma(2I_\mu+2)}\cdot\frac{\int_{-1}^1 x\,(1-x)^{I_\mu+\mu}(1+x)^{I_\mu-\mu}\,dx}{\int_{-1}^1 (1-x)^{I_\mu+\mu}(1+x)^{I_\mu-\mu}\,dx}\\
&= -(I_\mu+1)\cdot\left(\frac{\int_{-1}^1 (1-x)^{I_\mu+\mu}(1+x)^{I_\mu+1-\mu}\,dx}{\int_{-1}^1 (1-x)^{I_\mu+\mu}(1+x)^{I_\mu-\mu}\,dx}-1\right)\\
&= -(I_\mu+1)\cdot\left(2\cdot\frac{B(I_\mu+1+\mu, I_\mu+2-\mu)}{B(I_\mu+1+\mu, I_\mu+1-\mu)}-1\right)\\
&= -(I_\mu+1)\cdot\left(2\cdot\frac{\Gamma(I_\mu+2-\mu)\Gamma(2I_\mu+2)}{\Gamma(I_\mu+1-\mu)\Gamma(2I_\mu+3)}-1\right)=\mu.
\end{aligned}
$$

Part (1) is done.

(2) Since $\tilde{\mathscr{H}}_l$ is the space of square integrable solutions of Eq. $\hat\Gamma_{-1}\psi=(l_\mu+1)\psi$ and $\tilde\pi(H_0) = -\hat\Gamma_{-1}$, as a $\mathfrak{k}$-module, $\tilde{\mathscr{H}}_l = D(-l_\mu-1)\otimes D^l$ where $D^l$ is the irreducible module of Spin$(2n+2)$ with highest weight $(l+|\mu|, |\mu|, \cdots, |\mu|, \mu)$ and $D(-l_\mu-1)$ is the irreducible module of Spin(2) with weight $-l_\mu-1$. Since $\mu$ is a half integer, the irreducible unitary action of $\mathfrak{k}_0$ on $\tilde{\mathscr{H}}_l$ can be promoted to a unique irreducible unitary action of $K$. Therefore, $\tilde{\mathcal{H}}$ is a unitary $K$-module and has the following decomposition into isotypic components of $K$:

$$\tilde{\mathcal{H}} = \bigoplus_{l=0}^{\infty}\tilde{\mathscr{H}}_l = \bigoplus_{l=0}^{\infty}\left(D(-l_\mu-1)\otimes D^l\right).$$

(3) From the definition, it is clear that the action of $K$ on $\tilde{\mathcal{H}}$ is compatible with that of $\mathfrak{g}$ on $\tilde{\mathcal{H}}$, and its linearization agrees with the action of $\mathfrak{k}_0$. Part (2) says that $\tilde{\mathcal{H}}$ is $K$-finite. Therefore, $\tilde{\mathcal{H}}$ is a unitary $(\mathfrak{g}, K)$-module.

(4) Let $v\neq 0$ be a vector in $\tilde{\mathscr{H}}_0$ with $\mathfrak{g}$-weight $(-(n+|\mu|), |\mu|, \ldots, |\mu|, \mu)$. Since this weight is the highest among all weights with a nontrivial weight vector in $\tilde{\mathcal{H}}$, $V := U(\mathfrak{g})\cdot v\subset\tilde{\mathcal{H}}$ is the unitary highest weight $\mathfrak{g}$-module with highest weight $(-(n+|\mu|), |\mu|, \ldots, |\mu|, \mu)$. Since $\tilde{\mathscr{H}}_l$ is irreducible under $\mathfrak{s}\subset\mathfrak{g}$, either $\tilde{\mathscr{H}}_l\subset V$ or $\tilde{\mathscr{H}}_l\cap V=0$, so in particular $\tilde{\mathscr{H}}_0\subset V$. We claim that $\tilde{\mathscr{H}}_l\subset V$ for any $l\geq 0$, consequently $V=\tilde{\mathcal{H}}$ and then part (4) is done. To prove the claim, we note that $U(\mathfrak{sl}(2))\cdot v$ must be the discrete series representation of $\mathfrak{sl}(2)$ with highest ($H_0$-) weight $-(n+|\mu|)$ because it is a nontrivial unitary highest weight representation of the noncompact Lie algebra $\mathfrak{sl}_0(2)$. In view of the fact that $\tilde{\mathscr{H}}_l$ is the eigenspace of $\tilde\pi(H_0)$ with eigenvalue $-(l_\mu+1)$, $\tilde{\mathscr{H}}_l\cap(U(\mathfrak{sl}(2))\cdot v)=\mathrm{span}\{\tilde\pi(E_-^l)(v)\}$ must be one-dimensional. Then $\tilde{\mathscr{H}}_l\cap V\neq 0$ because $\dim(\tilde{\mathscr{H}}_l\cap V)\geq\dim(\tilde{\mathscr{H}}_l\cap(U(\mathfrak{sl}(2))\cdot v))=1$. □

### B. Proof of Theorem 1

Viewing the twisting map $\tau$ as an equivalence of representations, we get a representation $\pi$ of $\mathfrak{g}$ equivalent to $\tilde\pi$. Then the two propositions proved in Subsection V A are true if we drop all "tilde" there. Thus $\mathcal{H}$ is the unitary highest weight $(\mathfrak{g}_0, K)$-module with highest weight $(-(n+|\mu|), |\mu|, \ldots, |\mu|, \mu)$. By a standard theorem of Harish-Chandra,[32] we know that $\mathscr{H}$ is the unitary highest weight $G$-module with highest weight $(-(n+|\mu|), |\mu|, \ldots, |\mu|, \mu)$ such that $(\pi, \mathcal{H})$ is the underlying $(\mathfrak{g}_0, K)$-module. One can check that this highest weight module occurs at the first reduction point of the Enright–Howe–Wallach classification diagram.[33] So part (1) is done. Part (2) of Theorem 1 is just a consequence of part (3) of Proposition 2, and part (3) of Theorem 1 is just a consequence of part (2) of Proposition 3.

## APPENDIX: GEOMETRICALLY TRANSPARENT DESCRIPTION

The purpose of this appendix is to give a geometrically transparent description of the unitary highest weight module of Spin$(2, 2n+2)$ with highest weight $(-(n+|\mu|), |\mu|, \cdots, |\mu|, \mu)$.

As usual, we assume $n \geq 1$ is an integer and let $\mathcal{S}^{2\mu}$ be the pullback bundle under the natural retraction $\mathbb{R}^{2n+1}_* \to \mathrm{S}^{2n}$ of the vector bundle $\mathrm{Spin}(2n+1) \times_{\mathrm{Spin}(2n)} \mathbf{s}^{2\mu} \to \mathrm{S}^{2n}$ with the natural Spin$(2n+1)$-invariant connection. Let $d^D x$ be the Lebesgue measure on $\mathbb{R}^{2n+1}$. As is standard in geometry, we use $L^2(\mathcal{S}^{2\mu})$ to denote the Hilbert space of square integrable (with respect to $d^D x$) sections of $\mathcal{S}^{2\mu}$. We have shown that $\tilde{\mathscr{H}}(\mu) = L^2(\mathcal{S}^{2\mu})$, therefore, $\left(\tilde{\pi}, L^2(\mathcal{S}^{2\mu})\right)$ is the unitary highest weight module of Spin$(2, 2n+2)$ with highest weight $(-(n+|\mu|), |\mu|, \cdots, |\mu|, \mu)$. To describe the infinitesimal action of Spin$(2, 2n+2)$ on $C^\infty(\mathcal{S}^{2\mu})$, it suffices to describe how $M_{\alpha,0}$, $M_{D+1,0}$ and $M_{-1,0}$ act as differential operators. It is easy to see that $M_{\alpha,0}$, $M_{D+1,0}$ and $M_{-1,0}$ are equal to $i\sqrt{r}\nabla_\alpha\sqrt{r}$, $\frac{1}{2}\left(\sqrt{r}\Delta_\mu\sqrt{r} + r - \frac{c}{r}\right)$ and $\frac{1}{2}\left(\sqrt{r}\Delta_\mu\sqrt{r} - r - \frac{c}{r}\right)$, respectively. Here $\Delta_\mu$ is the Laplace operator twisted by $\mathcal{S}^{2\mu}$. For example, for $\psi \in C^\infty(\mathcal{S}^{2\mu})$, we have

$$(M_{\alpha,0} \cdot \psi)(r, \Omega) = i\sqrt{r}\nabla_\alpha\left(\sqrt{r}\psi(r, \Omega)\right). \tag{A1}$$


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[13] It is $\sqrt{-1}$ times the local *connection one-form*.
[14] Here we adopt the practice in physics: the Lie algebra generators act as hermitian operators in all unitary representations.
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[30] See, for example, Theorem 3 of page 129 of Ref. 36.

[31] To be more specific, one needs to generalize the work of Ref. 37. Since we are only interested in a sign, we choose to skip the details here.

[32] See, for example, Theorem 7 on page 71 of Ref. 38.

[33] Page 101, Ref. 12. In our case $z = A(\lambda_0) = n + 1$. It is in Case II when $\mu = 0$. For $\mu \neq 0$, it is in Case I for $p = n + 1$ or in Case III depending on the sign of $\mu$. See pages 125–126, Ref. 10. Note that, while there are two reduction points when $\mu = 0$, there is only one reduction point when $\mu \neq 0$.

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