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# Regulatory Capital Modelling for Credit Risk

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A thesis submitted in fulfilment of the requirements  
for the degree of Doctor of Philosophy

Faculty of Science  
University of Sydney

2015





# Abstract

The Basel II internal ratings-based (IRB) approach to capital adequacy for credit risk plays an important role in protecting the Australian banking sector against insolvency. We outline the mathematical foundations of regulatory capital modelling for credit risk, and extend the model specification of the IRB approach to a more general setting than the usual Gaussian case. It rests on the proposition that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles of the portfolio loss distribution. We present a more compact proof of this proposition under weaker assumptions. The IRB approach implements the so-called asymptotic single risk factor (ASRF) model, an asset value factor model of credit risk. The robustness of the model specification of the IRB approach to a relaxation in model assumptions is evaluated on a portfolio that is representative of the credit exposures of the Australian banking sector. We measure the rate of convergence, in terms of number of obligors, of empirical loss distributions to the asymptotic (infinitely fine-grained) portfolio loss distribution; and we evaluate the sensitivity of credit risk capital to dependence structure as modelled by asset correlations and elliptical copulas. A separate time series analysis takes measurements from the ASRF model of the prevailing state of Australia's economy and the level of capitalisation of its banking sector. These readings find general agreement with macroeconomic indicators, financial statistics and external credit ratings. However, given the range of economic conditions, from mild contraction to moderate expansion, experienced in Australia since the implementation of Basel II, we cannot attest to the validity of the model specification of the IRB approach for its intended purpose of solvency assessment. With the implementation of Basel II preceding the time when the effect of the financial crisis of 2007–09 was most acutely felt, our empirical findings offer a fundamental assessment of the impact of the crisis on the Australian banking sector. Access to internal bank data collected by the prudential regulator distinguishes our research from other empirical studies on the IRB approach and recent crisis.

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# Preface

Risk capital models serve management functions including capital allocation, performance attribution, risk pricing, risk identification and monitoring, strategic business planning, and solvency assessment (i.e., capital adequacy). However, model characteristics best suited for different purposes may conflict. For example, models for solvency assessment require precision in the measurement of absolute risk levels under stressed economic conditions, whereas models for capital allocation need only be accurate in the measurement of relative risk under “normal” economic conditions (Basel Committee on Banking Supervision et al. 2010). While economic capital models may serve several management functions, the sole purpose of regulatory capital models is solvency assessment.

Under the Basel II Accord (Basel Committee on Banking Supervision 2006), authorised deposit-taking institutions (ADIs) are required to assess capital adequacy for credit, market and operational risks. ADIs determine regulatory capital for credit risk using either the standardised approach or, subject to approval, the internal rating-based (IRB) approach. The latter is more expensive to administer, but usually produces lower regulatory capital requirements than the former. As a consequence, ADIs using the IRB approach may deploy their capital in pursuit of more (profitable) lending opportunities. This postgraduate research examines the model specification of the IRB approach, the so-called asymptotic single risk factor (ASRF) model.

The Australian Prudential Regulation Authority (APRA) regulates deposit-taking institutions, insurance companies and superannuation funds. Upon implementation of Basel II in 2008, APRA had granted the four largest Australian banks, designated “major” banks, approval to use the IRB approach to capital adequacy for credit risk. Since then, the four major banks have accounted for approximately three-quarters of total assets on the balance sheet of ADIs regulated by APRA. Furthermore, of the regulatory capital reported by the major banks, nearly 90% has been held against unexpected credit losses, and of that more than two-thirds has been assessed using the ASRF model prescribed under the IRB approach. The market dominance of the major banks, when coupled with the concentration of their regulatory capital assessed under the IRB approach, is indicative of the significance of the ASRF model in protecting the Australian banking sector against insolvency.

The financial crisis of 2007–09, also known as the global financial crisis, precipitated the worst global recession since the Great Depression of the 1930s. Prompted by the severity of the

crisis, regulators have since pursued macroprudential tools and policies to measure and manage systemic risk (Financial Stability Board et al. 2011). While difficult to define precisely, systemic risk refers to “the contagion-induced threat to the financial system as a whole, due to the default of one (or more) of its component institutions” (Gleeson et al. 2013). It is usual to model systemic risk in the banking system by subjecting it to exogenous credit shocks, amplifying them as liquidity deteriorates, and propagating them across the banking system through a network of interbank obligations (Eisenberg and Noe 2001; Cifuentes et al. 2005). Models of systemic risk simulate fundamental defaults resulting from exogenous shocks affecting banks’ claims on non-banks, and contagion default due to failure to meet interbank obligations (Drehmann and Tarashev 2011; Gauthier et al. 2012). Clearly credit risk, which has dominated the assessment of minimum capital requirements under the Basel II regime, is integral to modelling systemic risk.

More concretely, the so-called Basel 2.5 package and Basel III Accord introduce reforms to address deficiencies in the Basel II framework exposed by the recent crisis. The Basel 2.5 reforms enhance minimum capital requirements, risk management practices and public disclosures in relation to risks arising from trading activities, securitisation and exposure to off-balance sheet vehicles. The Basel III reforms raise the quality and minimum required level of capital; promote the build up of capital buffers; establish a back-up minimum leverage ratio; improve liquidity and stabilise funding; and assess a regulatory capital surcharge on systemically important financial institutions. These reforms complement, rather than supersede, Basel II. In particular, the ASRF model prescribed under the Basel II IRB approach is unaltered by the introduction of Basel 2.5 and Basel III. It remains as relevant to solvency assessment today as it has been since its implementation in 2008.

The academic contribution of this thesis is both theoretical and empirical. On a theoretical level, we provide a rigorous mathematical exposition of the foundations of the IRB approach. The model specification of the IRB approach, an asset value factor model of credit risk, has its roots in the classical structural approach of Merton (1974). Adapting the single asset model of Merton to a portfolio of credits, Vasicek (2002) derived a function that calculates probabilities of default conditional on a single systematic risk factor explaining dependence across obligors. An expression for conditional expectation of portfolio percentage loss then follows immediately. In assessing regulatory capital charges the IRB approach applies credit value-at-risk (VaR) — an extreme quantile of the portfolio loss distribution that is rarely exceeded — to assign a single numerical value to a random credit loss. Its model specification employs an analytical approximation of credit VaR, which follows from the proposition, due to Gordy (2003), that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles of the portfolio loss distribution. Here, our contribution is a proof that is more compact starting from weaker assumptions. In drawing on the insights of Merton, Vasicek and Gordy to develop the ASRF model, the Basel Committee on Banking Supervision (2005) adopts the standard model of asset values, geometric Brownian motion. More broadly, our theoretical contribution extends the mathematical foundations of the IRB approach to a more general

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setting than the usual Gaussian case.

In effect, the IRB approach implements the one-factor Gaussian copula, popularised by Li (2000), which combines marginal distributions of constituent credits into a multivariate portfolio loss distribution with a chosen default dependence structure. Accordingly, we derive the one-factor Gaussian copula from the single-factor copula model for the general case. It is generally acknowledged that models which assume that financial data follow a Gaussian distribution tend to underestimate tail risk. Addressing this concern, we examine the dependence induced by elliptical copulas, including Gaussian and Student's  $t$  copulas.

The empirical component of this postgraduate research has been supported by APRA. Their support includes access to internal bank data collected by APRA from the institutions that it supervises, which distinguishes our research from other empirical studies on the IRB approach. It forms the basis of our empirical contribution. Constructing a portfolio that is representative of the credit exposures of the major Australian banks, we measure the rate convergence of empirical loss distributions to the asymptotic portfolio loss distribution, and evaluate the robustness of the ASRF model to parameter variations and model misspecification. Then, we render a fundamental assessment of the IRB approach by taking measurements from the ASRF model. Realisations of the single systematic risk factor, interpreted as describing the prevailing state of the Australian economy, are recovered from the ASRF model and compared with macroeconomic indicators. Similarly, estimates of distance-to-default, reflecting the capacity of the Australian banking sector to absorb credit losses, are recovered from the ASRF model and compared with financial statistics and external credit ratings. While our fundamental assessment makes observations about the prevailing state of Australia's economy and the solvency of its banking sector, we do not propose model enhancements, nor draw policy implications. Although, our observations may open a debate on the model specification of the IRB approach or related regulatory policies.

From our empirical analysis emerges a methodology for regulators to monitor the prevailing state of the economy as described by the single systematic risk factor, and the capacity of supervised banks to absorb credit losses as measured by distance-to-default. Measurements from the ASRF model signalling an overheating economy and procyclical movements in capital bases, corroborated by macroeconomic performance indicators including rapidly accelerating credit growth, would prompt supervisors to lean against the prevailing winds by, for example, instructing banks to build up their countercyclical capital buffer introduced under Basel III.

With the implementation of Basel II preceding the time when the effect of the financial crisis of 2007-09 was most acutely felt, we measure the impact of the crisis on the Australian banking sector using internal bank data collected by APRA. We are not the first to attempt to measure the effects of the recent crisis, but we believe that we are the first to do so using regulatory data. Other studies on the financial crisis rely on market data, macroeconomic indicators or published financial statistics. We proceed to compare the market's assessment of the solvency of the Australian banking sector with our fundamental assessment. Accordingly, we estimate distance-to-default analytically using the Merton model, and numerically using the "first passage"

approach. The overreaction of market participants, which is reflected in market prices, explains the divergence of the market's assessment from our fundamental assessment.

The empirical component of this postgraduate research, in effect, applies a single-period, one-factor Gaussian copula sequentially on quarterly time series of internal bank data available since the implementation of Basel II. A natural extension of this thesis that we've explored would model credit defaults, or survival times, in a multi-period framework. Indeed, the path dependent nature of many risks and the requirement to measure portfolio risk over different time horizons leads to a multi-period simulation. It is practical then to simulate all variables, including defaults and survivals, in each time period. In order to consistently represent dependence between multivariate survival times in each period of the simulation, Brigo et al. (2013) model survival times using a continuous-time Markov chain with multivariate exponential distribution (i.e., Marshall-Olkin copula). While the challenge remains to develop an efficient implementation of this dynamic Markovian model for measuring the credit risk of portfolios of large banks, we intend to pursue research on this topic in a postdoctoral capacity.

This thesis is divided into two parts: the theoretical framework of, and empirical findings from the postgraduate research. Part I begins in Chapter 1 with a preliminary discourse that puts the concern of this research in the broader context of risk management topics including risk measures, regulation of ADIs, economic and regulatory capital, and asset value factor models of credit risk.

Chapter 2 derives the theoretical foundations, drawn from the literature, of the ASRF model. The classical structural approach to credit risk modelling, originally developed by Merton (1974), formulates liabilities of a firm as contingent claims on its assets. Adapting the single asset model of Merton to a portfolio of credits, Vasicek (2002) derived a function that transforms unconditional probabilities of default into probabilities of default conditional on a single systematic risk factor. We extend Vasicek's model of conditional independence to a more general setting, one not restricted to Gaussian processes. Gordy (2003) established that conditional on a single systematic risk factor, the portfolio percentage loss converges to its conditional expectation as the portfolio approaches asymptotic granularity — no single credit exposure accounts for more than an arbitrarily small share of total portfolio exposure. Then, assuming conditional independence given a single systematic risk factor, we derive a limiting form of the portfolio loss distribution for the general case. It generalises Vasicek's (2002) formulation of the loss distribution function of an asymptotic, homogeneous portfolio, which models default dependence as a multivariate Gaussian process. The model specification of the IRB approach employs an analytical approximation of credit VaR. It rests on the proposition, due to Gordy (2003), that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles of the portfolio loss distribution. We present a more economical proof of this proposition under weaker assumptions.

In generating a portfolio loss distribution we are, in effect, combining marginal loss distributions of constituent credits into a multivariate distribution capturing default dependence between

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obligors. An approach to modelling default dependence, popularised by Li (2000), uses copula functions. Drawing from Nelsen (2006) and Embrechts, Lindskog, et al. (2003), Chapter 3 introduces the theory of copulas, which combine marginal distributions into a multivariate distribution with a chosen dependence structure. On this basis, we derive the single-factor copula model describing default dependence for the general case. Then, we deal with the special case of the one-factor Gaussian copula, the most commonly applied copula function in credit risk modelling, which is usually cast in the frame of survival time, or time-until-default. Recognising that Gaussian distributions in financial applications tend to underestimate tail risk, we proceed to outline procedures for generating empirical loss distributions described by elliptical copulas, including Gaussian and Student's  $t$  copulas. Moreover, we illustrate the dependence induced by elliptical copulas.

In its implementation of the IRB approach to capital adequacy for credit risk, APRA requires that ADIs set aside provisions for absorbing expected losses, and hold capital against unexpected losses. Assuming that portfolios are infinitely fine-grained so that idiosyncratic risk is fully diversified away, and a single systematic risk factor explains dependence across obligors, Chapter 4 describes an analytical approximation for assessing ratings-based capital charges. While real-world portfolios are not infinitely fine-grained, as a practical matter, credit portfolios of large banks exhibit sufficient granularity to adequately satisfy the asymptotic granularity condition, so it need not pose an impediment to assessing ratings-based capital charges. Again, our contribution extends the model specification of the IRB approach to a more general setting than the usual Gaussian case. We show that ASRF models are portfolio invariant, an important criterion for meeting supervisory needs. Portfolio invariance guarantees that “the capital charge on a given instrument depends only on its characteristics, and not on the characteristics of the portfolio in which it is held” (Gordy 2003).

Upon implementation of Basel II in the first quarter of 2008, APRA had granted the four major Australian banks approval to use the IRB approach to capital adequacy for credit risk. Part II begins in Chapter 5 with a description of data reported to APRA by the major banks on a quarterly basis since the implementation of Basel II. Our fundamental assessment of the model specification of the IRB approach is rendered on the basis of these internal bank data sourced from the statement of financial performance, capital adequacy form and IRB credit risk forms. We contend that these data are representative of the Australian banking sector on the basis of the market dominance of the major banks, and the concentration of their regulatory capital held against unexpected credit losses assessed under the IRB approach. For the purpose of testing the robustness of the ASRF model we proceed to construct a portfolio that is representative of the credit exposures of the major Australian banks.

Chapter 6 evaluates the robustness of the model specification of the IRB approach to a relaxation in model assumptions. We measure the rate of convergence, in terms of number of obligors, of empirical loss distributions to the distribution of conditional expectation of portfolio percentage loss representing an infinitely fine-grained portfolio. In the process we demonstrate that the

proposition, due to Gordy (2003), that underpins the IRB approach holds for a representative credit portfolio that exhibits sufficient granularity. The IRB approach applies the one-factor Gaussian copula, in which default dependence is described by the matrix of pairwise correlations between obligors' asset values. We proceed to evaluate model robustness to parameter variations and model misspecification. In particular, we measure the sensitivity of credit risk capital to dependence structure, as modelled by asset correlations and elliptical copulas. While credit risk capital is quite sensitive to estimates of asset correlation, its sensitivity to the choice of copula function can be much greater.

Chapter 7 renders a fundamental assessment of the model specification of the IRB approach by taking readings of the Australian banking sector since the implementation of Basel II. Firstly, we contextualises the Australian economy over the past decade by comparing its performance with that of the United States and United Kingdom on macroeconomic indicators and financial statistics. Explanations proffered for the recent performance of the Australian economy and its resilience to the financial crisis are discussed in some detail. From the ASRF model prescribed under the IRB approach we recover realisations of the single systematic risk factor describing the prevailing state of Australia's economy, and estimates of distance-to-default reflecting the capacity of its banking sector to absorb credit losses. These readings find general agreement with signals from macroeconomic indicators, financial statistics and external credit ratings. This leads to a favourable assessment of the ASRF model in the context of some of the aforementioned management functions generally served by economic capital models. However, the range of economic conditions, from mild contraction to moderate expansion, experienced in Australia since the first quarter of 2008 translate into observations away from the tail of the portfolio loss distribution. So, we cannot attest to the validity of the ASRF model for the purpose of solvency assessment, and hence regulatory capital modelling. The relatively short quarterly time series available leads to an intuitive assessment of the ASRF model rather than any formal testing of its efficacy. The effects of the financial crisis of 2007–09 were not felt evenly across the globe, and the Australian economy was largely spared. With the implementation of Basel II preceding the time when the effect of the crisis was most acutely felt, our empirical analysis reveals that the crisis imparted a mild stress on the Australian banking sector. Evaluating the model specification of the IRB approach for its intended purpose of solvency assessment would involve taking readings of north Atlantic banking jurisdictions that experienced the full force of the recent crisis. It would supplement the findings of this study. Finally, in a variation on the evaluation of distance-to-default, we employ reverse stress testing to explore stress events that would trigger material supervisory intervention.

In a more or less self-contained chapter, we infer the market's assessment of the capacity of the major Australian banks, in aggregate, to absorb credit losses. Firstly, Chapter 8 imputes the value and volatility of assets from equity prices using the iterative procedure adopted by Vassalou and Xing (2004). Assuming that a firm defaults if the value of its assets falls below a critical threshold, its default point, at a given risk measurement horizon, the analytical Merton model

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assesses solvency in terms of default likelihood indicator (DLI), which is translated into distance-to-default. Then, redefining default as the value of a firm’s assets falling below its default point at any time during the risk measurement horizon, we estimate DLI and distance-to-default using the “first passage” approach. Here, we simulate the time evolution of asset values using Monte Carlo methods, and set the barrier equal to the default point. Obviously, DLI is higher and distance-to-default is narrower under the first passage approach than the corresponding results analytically calculated by the Merton model. The market’s assessment of the solvency of the major banks, in aggregate, is directly comparable with our fundamental assessment recovered by the ASRF model prescribed under the IRB approach. The former, which is much more volatile than the latter, does not comport with signals from macroeconomic indicators, financial statistics or external credit ratings. We contend that this is explained by the overreaction of market participants, which is reflected in market prices. During periods when market participants are gripped by fear (respectively, driven by greed), their perception of risk is heightened (lessened), and markets become undervalued (overpriced).

The electronic submission of this thesis for the degree of Doctor of Philosophy includes:

- L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub> manuscript and supporting files including figures, bibliography and glossary.
- Software developed to undertake the empirical analysis of Part II. Programs are coded in C language on a MacBook Pro (2010) computer. MacPorts, an open source package management system for Mac OS X, is used to install and upgrade ports: GNU C compiler; GNU Make; GNU Scientific Library; and gnuplot.
- Files containing data reported quarterly to APRA by the major banks on their statement of financial performance, capital adequacy form and IRB credit risk forms. These data have been fetched from APRA’s data warehouse and aggregated using SQL database programming. They are input to programs developed for the empirical analysis.
- Data files output by the programs developed for the empirical analysis. Either data contained in these output files are reported in tables presented in the thesis, or the output data files are read by scripts executed by the gnuplot graphing utility.
- Gnuplot scripts that generate figures, or charts, presented in the thesis.
- Statement of financial position, capital adequacy form and IRB credit risk forms, in spreadsheet format, that ADIs submit to APRA as part of their lodgement of statutory returns. Note that the major Australian banks automate their secure electronic data submissions using XBRL (eXtensible Business Reporting Language).





# Acknowledgements

I am especially grateful to Marek Rutkowski, Professor of Financial Mathematics at the University of Sydney, who serves as supervisor of this postgraduate research. While encouraging me to investigate topics that piqued my interest, he offered sage counsel that helped formulate my research topic. His deep expertise in credit risk modelling was invaluable, particularly in developing the theoretical framework that underpins the empirical analysis. Professor Rutkowski always made time to discuss my research, and review and constructively comment on many drafts of working papers that eventually coalesced into this thesis.

The empirical component of this postgraduate research has been conducted with the support of the Australian Prudential Regulation Authority (APRA). It has provided access to professionals with expertise in financial regulation, risk modelling and data management, along with access to internal bank data collected by APRA from the institutions that it supervises. I acknowledge with pleasure the support and encouragement from Charles Littrell, Executive General Manager at APRA, as well as his constructive comments on early drafts of this work. Anthony Coleman and Guy Eastwood of the Credit Risk Analytics team offered valuable input in the design of the empirical analysis and review of the findings. The Statistics team, led by Steve Davies, provided data support for the empirical analysis, and reviewed the thesis for compliance with confidentiality agreements. Finally, I thank the Research team, headed by Bruce Arnold, for its contribution in distilling research topics that are pertinent to prudential regulators.

Financial support for this postgraduate research is gratefully received in the form of an Australian Postgraduate Award, and a scholarship sponsored by Capital Markets Cooperative Research Centre and APRA.



## Part I

# Theoretical Framework



# Preliminaries

The topic of this thesis is regulatory capital modelling for credit risk, and more particularly, the Basel II internal-ratings based (IRB) approach. Some preliminaries introduce topics that provide a broader context for the focus of this postgraduate research. They include: risk measures (Section 1.1), regulation of ADIs (Section 1.2), economic and regulatory capital (Section 1.3), and asset value factor models of credit risk (Section 1.4).

## 1.1 Risk Measures

Generally speaking, a financial risk measure assigns a single numerical value to a random variable representing portfolio profit and loss (P&L). Naturally, relying on a single number to summarise a P&L distribution involves a significant loss of information, so prudent risk management would counsel the use of a range of summary statistics to assess risk. However, in some cases there is little choice but to quantify risk with a single number; for example, the determination of regulatory capital requirements. Before listing desirable properties of a risk measure and examining some common ones, we formally define a risk measure.

**Definition 1.1.** Let  $\Omega$  be the set of possible states of a portfolio of assets at a given risk measurement horizon, and let  $\mathcal{G}$  be the set of random variables (real-valued functions) on  $\Omega$  representing portfolio P&L. Then, risk measure  $\varrho$  is a mapping from  $\mathcal{G}$  to  $\mathbb{R}$ .

In their seminal paper, Artzner et al. (1999) use economic reasoning to define a set of desirable properties of financial risk measures, and refer to measures satisfying these properties as coherent. Where positive, we interpret  $\varrho(X)$  assigned by risk measure  $\varrho$  to random variable  $X \in \mathcal{G}$  as the capital injection required to make the position acceptable to an internal or external risk controller, for example, the prudential regulator. If  $\varrho(X)$  is negative, capital in the amount of  $-\varrho(X)$  may be withdrawn from the position and still remain acceptable to the risk controller. In order to

simplify the presentation, and not inconsistent with the convention adopted in the determination of regulatory capital requirements, we set the return on the reference instrument, or discount rate, equal to zero. *Coherent risk measures* satisfy the following four axioms (Artzner et al. 1999):

**Axiom 1 (Translation invariance)** For all  $X \in \mathcal{G}$  and all  $a \in \mathbb{R}$ ,  $\varrho(X + a) = \varrho(X) + a$ .

**Axiom 2 (Subadditivity)** For all  $X$  and  $Y \in \mathcal{G}$ ,  $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$ .

**Axiom 3 (Positive homogeneity)** For all  $X \in \mathcal{G}$  and  $\lambda \geq 0$ ,  $\varrho(\lambda X) = \lambda \varrho(X)$ .

**Axiom 4 (Monotonicity)** For all  $X$  and  $Y \in \mathcal{G}$  with  $X \leq Y$ ,  $\varrho(Y) \leq \varrho(X)$ .

Translation invariance means that adding (respectively, subtracting) the sure amount  $a$  to an initial position and investing it in the reference instrument reduces (increases) the capital charge by  $a$ . Subadditivity reflects the principle that diversification reduces risk. It implies that merging positions into a portfolio does not create additional risk; conversely, dividing a portfolio into sub-portfolios does not reduce risk. If subadditivity holds, then  $\varrho(nX) \leq n\varrho(X)$  for  $n \in \mathbb{N}$ . Since there is no diversification in this portfolio, it is natural that equality holds, which leads to positive homogeneity. However, positive homogeneity does not penalise the concentration of risk and lack of liquidity that may accompany large positions. Note that a risk measure  $\varrho$  that satisfies subadditivity and positive homogeneity is convex on  $\mathcal{G}$ . Monotonicity is a natural economic property requiring positions that lead to greater losses in every state to incur higher capital charges.

*Remark 1.2.* Throughout this section we consider the general case where possible states of a portfolio of assets at a given risk measurement horizon include profits and losses, which take positive and negative signs, respectively. This is in keeping with the convention adopted by Artzner et al. (1999). In Chapter 2, however, we define a measure of credit risk while adopting the convention that a loss is a positive number. Under this convention, which is common for credit portfolios, the axiom of monotonicity is expressed as:

**Axiom 4 (Monotonicity)** For all  $X$  and  $Y \in \mathcal{G}$  with  $X \leq Y$ ,  $\varrho(X) \leq \varrho(Y)$ .

*Remark 1.3.* The larger class of *convex risk measures* relaxes the axioms of subadditivity and positive homogeneity. The weaker property of convexity is satisfied if for all  $X$  and  $Y \in \mathcal{G}$ :

$$\varrho(\lambda X + (1 - \lambda)Y) \leq \lambda \varrho(X) + (1 - \lambda)\varrho(Y), \quad \lambda \in [0, 1]. \quad (1.1)$$

Risk reduction through diversification provides the economic rationale for the properties of subadditivity and convexity.

The axiomatic characterisation of coherent risk measures based on economic reasoning has attracted strong adherents. Yet, more recent research highlights the shortcomings of coherent risk measures, which should not be neglected. In particular, coherent risk measures have been shown to lack elicibility and robustness, in general.

*Remark 1.4.* Point forecasting methods are typically evaluated and compared by an error measure or scoring function, such as mean squared error or mean absolute error. Gneiting (2011) demonstrates that this common practice can lead to misguided inferences. The author argues that effective point forecasting requires that: (i) the scoring function be disclosed a priori; or (ii) the forecaster receives a directive in the form of statistical functional, such as the mean or quantile of the predictive distribution, and any scoring function applied is consistent with the functional. A functional is *elicitable* if there exists a scoring function that is strictly consistent for it.

*Remark 1.5.* While relaxing the subadditivity property, Kou et al. (2011) propose a set of axioms for a class of risk measures they call natural risk statistics, which includes a subclass characterised by its robustness to model misspecification and small changes in data. A risk measure is said to be *robust* if: (i) it can accommodate a degree of model misspecification and perform well across a set of alternative models; and (ii) it is insensitive to small changes in all observations or large changes in a few observations.

Before discussing a couple of commonly used risk measures in the financial industry, we first define quantiles of a distribution.

**Definition 1.6.** Let  $X$  be a random variable, and let  $\alpha \in (0, 1)$ . Then the  $\alpha$  *quantile* of the distribution of  $X$  is

$$q_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\}. \quad (1.2)$$

*Remark 1.7.* If  $X$  has a continuous and strictly increasing distribution function  $F$ , then the  $\alpha$  quantile of the distribution of  $X$  is given by

$$q_\alpha(X) = F^{-1}(\alpha), \quad (1.3)$$

where  $F^{-1}(\alpha)$ , the inverse distribution function evaluated at  $\alpha$ , is the number  $q_\alpha(X) \in \mathbb{R}$  such that  $F(q_\alpha(X)) = \alpha$ .

Value-at-risk (VaR) is probably the most widely used financial risk measure. It measures the maximum loss which is not exceeded with a typically high probability over a given risk measurement horizon. Its prevalence in the financial industry is partially explained by “its ease of communication as it is ‘well understood by senior management’ as well as quantitative experts” (Basel Committee on Banking Supervision et al. 2010).

**Definition 1.8.** *Value-at-risk* of portfolio P&L random variable  $X$  at the confidence level  $\alpha \in (0, 1)$  over a given risk measurement horizon is the  $\alpha$  quantile of  $-X$ :

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(-X \leq x) \geq \alpha\}. \quad (1.4)$$

VaR is often criticised because at the  $\alpha$  confidence level VaR does not give any information about the severity of losses which occur with probability less than  $1 - \alpha$ . Another common criticism is that it does not satisfy the subadditivity property, in general, and is therefore not a



coherent risk measure. However, VaR is subadditive for elliptical distributions, which we formally define in Appendix 3.A. We denote elliptical distributions by  $X \sim E_n(\mu, \Sigma, \psi)$ , and refer to  $\mu$  as the location vector,  $\Sigma$  as the dispersion (covariance) matrix and  $\psi$  as the characteristic generator of the distribution.

**Theorem 1.9** (Embrechts, McNeil, et al. 2002, Theorem 7.1). *Suppose that  $\mathbf{X} \sim E_n(\mu, \Sigma, \psi)$ , where  $\mathbf{X} = (X_1, \dots, X_n)$  with  $\text{Var}(X_i) < \infty$  for  $i = 1, \dots, n$ . Let*

$$\mathcal{H} = \left\{ Z = \sum_{i=1}^n \lambda_i X_i, \lambda_i \in \mathbb{R} \right\}$$

*be the set of all linear portfolios. Then for any two portfolios  $Z_1, Z_2 \in \mathcal{H}$  and  $0.5 \leq \alpha < 1$ ,*

$$\text{VaR}_\alpha(Z_1 + Z_2) \leq \text{VaR}_\alpha(Z_1) + \text{VaR}_\alpha(Z_2).$$

Expected shortfall (ES), a coherent risk measure, is generally regarded as a superior measure of financial risk. While VaR is prevalent in the financial industry, a number of firms are replacing or supplementing VaR with ES to better measure tail risk (Basel Committee on Banking Supervision et al. 2010). Sometimes referred to as tail conditional expectation or conditional VaR, ES at the  $\alpha$  confidence level is the average of the  $100(1 - \alpha)$  percent worst losses.

**Definition 1.10.** *Expected shortfall* of portfolio P&L random variable  $X$  at the confidence level  $\alpha \in (0, 1)$  over a given risk measurement horizon is given by

$$\text{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(X) \, du. \quad (1.5)$$

ES is not subject to the aforementioned criticisms levelled at VaR. It is subadditive, in general, and accounts for the severity of losses beyond the confidence threshold. On the other hand, ES has been shown to lack elicibility (Remark 1.4) and robustness (Remark 1.5). Gneiting (2011) finds that scoring functions that are consistent for the functional ES do not exist, so ES is not elicitable, despite its popularity in finance. However, the mean and quantiles, and hence VaR, are elicitable risk measures. Kou et al. (2011) show that ES, which is sensitive to modelling assumptions of heaviness of tail distribution and to outliers in the data, lacks the property of robustness. They provide a theoretical basis for using VaR with scenario analysis as a robust external risk measure for prudential regulation.

The Basel Committee on Banking Supervision (BCBS) chose VaR as the risk measure to assess regulatory capital charges under the IRB approach, which models losses on a credit portfolio as a multivariate Gaussian distribution with default dependence described by a matrix of pairwise correlations between obligors' asset values (i.e., an elliptical distribution). Accordingly, the lack of subadditivity of VaR is rendered moot under the model specification of the IRB approach. Moreover, BCBS (2005) sets regulatory capital for credit risk at the 99.9% confidence level in order “to protect against estimation error” in model inputs as well as “other model uncertainties.” The margin for error and uncertainties implicit in setting a very high confidence threshold on

an elliptical loss distribution arguably blunts the criticism that VaR does not account for the severity of losses beyond the confidence threshold. In Chapter 6 we measure the sensitivity of credit risk capital to dependence structure as modelled by a variety of elliptical copulas. We believe that financial stability would be better served by research on modelling the tail of the portfolio loss distribution, rather than building the case for a coherent risk measure, such as ES, to replace VaR. Since the concern of this thesis is the theoretical foundations and empirical analysis of the IRB approach, we conduct our research using VaR.

## 1.2 Regulation of Authorised Deposit-Taking Institutions

The Australian Prudential Regulation Authority (APRA) is the prudential regulator of the financial services industry in Australia. It supervises deposit-taking institutions (i.e., banks, credit unions and building societies), general and life insurance companies, and superannuation funds. This section puts the focus of this postgraduate research into the broader context of the regulation of authorised deposit-taking institutions, as practised in Australia.

APRA has largely adopted the supervisory standards and guidelines developed by BCBS. The 1988 Capital Accord, commonly referred to as Basel I, had the twin objectives of protecting depositors and strengthening financial stability. Over time banks exploited its simplicity by engaging in regulatory arbitrage, undermining the objectives of Basel I. In order to counter the growing problem of regulatory arbitrage and encourage banks to adopt sounder risk management practices, the Basel II Accord promotes greater convergence between regulatory and economic capital. In response to the financial crisis of 2007–09, the Basel 2.5 package and Basel III Accord introduce reforms to address deficiencies in the Basel II framework exposed by the crisis. The reforms complement, rather than supersede, Basel II.

### Basel I

In establishing the 1988 Capital Accord, BCBS (1988) sought to arrest the gradual erosion of capital relative to exposures in the banking system. Faced with declining margins in the preceding years, banks were increasing volume, and hence leverage, to maintain their return on capital. But, with higher leverage came narrower capital bases and a lower capacity to absorb losses. Basel I instituted the Cooke ratio to establish a minimum capital requirement, set at 8% of risk-weighted assets (RWA). The numerator of the Cooke ratio was defined broadly to include shareholders' equity (core or tier 1 capital), and deeply subordinated claims (supplementary or tier 2 capital). The denominator was the sum of a bank's exposures multiplied by risk weights: 0% for OECD government securities, 20% for claims on OECD banks, and 100% for other exposures. Credit equivalent amounts of off-balance sheet assets were also included.

Initially targeted at large, internationally active banks in G10 countries, Basel I was eventually adopted by nearly all countries and extended to all banks. The simplicity of the Cooke ratio, initially considered its strength, became a drawback. The risk weights assigned under Basel I did

not adequately differentiate levels of risk, encouraging banks to engage in regulatory arbitrage — expanding riskier, and thus higher margin, lending while moving less risky, and thus less remunerative, loans off balance sheet through securitisation in order to boost return on regulatory capital. As a consequence of this practice, the exposure-weighted risk on banks’ balance sheet increased without a commensurate increase in capital, undermining the objectives of Basel I. The Cooke ratio had become a widely accepted metric for assessing banks’ performance and soundness. Its adoption for purposes for which it was not originally intended, however, could have misled unsophisticated users (Tieset and Troussard 2005).

### Basel II

A fundamental objective of the Basel II Accord “has been to develop a framework that would further strengthen the soundness and stability of the international banking system while maintaining sufficient consistency that capital adequacy regulation will not be a significant source of competitive inequality among internationally active banks” (Basel Committee on Banking Supervision 2006). Subject to supervisory approval, Basel II permits institutions to use their internal credit ratings and internal models of market and operational risks to assess regulatory capital charges. As expressed by Gordy and Howells (2006), it promotes “better alignment of regulatory capital requirements with ‘economic capital’ demanded by investors and counterparties.” The Basel II framework consists of three mutually reinforcing pillars:

- (1) Minimum capital requirements.
- (2) Supervisory review process.
- (3) Market discipline.

The Basel II framework was implemented in Australia on 1 January 2008 through prudential standards established by APRA.

### Minimum Capital Requirements

In its implementation of Pillar 1 of Basel II, APRA requires ADIs to determine minimum capital requirements for credit risk, operational risk and market risk. For each of these risk classes, ADIs determine regulatory capital using the standardised approach or, subject to APRA’s approval, an advanced internal method. The former is straightforward to administer and produces a relatively conservative estimate of regulatory capital, while the latter is more expensive to administer but usually produces lower regulatory capital requirements.

In relation to capital adequacy for credit risk, ADIs determine regulatory capital using either the standardised approach or IRB approach. The standardised approach (Australian Prudential Regulation Authority 2010a) directly calculates RWA by assigning risk weights to on-balance sheet assets and credit equivalent amounts of off-balance sheet assets. Assigned risk weights are based on credit rating grades, which are broadly aligned with the probability of counterparty

default. Under the IRB approach (Australian Prudential Regulation Authority 2008a) regulatory capital is calculated to cover credit losses at the 99.9% confidence level over a one-year horizon. ADIs that have been granted approval to use the IRB approach generate internal estimates of obligor specific parameters: probability of default, loss given default and exposure at default. The conversion of minimum capital requirements to RWA assumes an 8.0% minimum capital ratio.

ADIs determine regulatory capital for operational risk using either the standardised approach or advanced measurement approach. Under the standardised approach (Australian Prudential Regulation Authority 2008b) an ADI divides its activities into three business lines: retail banking, commercial banking, and all other activities. Regulatory capital for operational risk is calculated as the sum of the 3-year average of assessed capital charges for each of the business lines. Gross outstanding loans and advances for retail and commercial banking, and adjusted gross income are input to this calculation. The advanced measurement approach (Australian Prudential Regulation Authority 2008c) computes regulatory capital for operational risk to cover losses for event types including: internal fraud; external fraud; employment practices and workplace safety; clients products and business practices; damage to physical assets; business disruption; and execution, delivery and process management. An ADI must demonstrate that regulatory capital “meets a soundness standard comparable to a one-year holding period and a 99.9 per cent confidence level.”

ADIs are required to hold capital against market risk arising from trading book positions in fixed income and equity securities, and banking and trading book positions in foreign exchange and commodities. Prudential standards do not distinguish, in principle, between market risk arising from positions in the physical and derivative instruments. Minimum capital requirements for market risk are determined using either the standardised approach or internal model approach (Australian Prudential Regulation Authority 2008d). The standardised approach applies prescriptive capital charges for specific and general market risks. Under the internal model approach regulatory capital is equal to the higher of: (i) the average daily VaR over the preceding sixty trading days multiplied by a scaling factor ( $\geq 3.0$ ) set by APRA; or (ii) the previous day’s VaR. Daily VaR is calculated at the 99% confidence level over a 10-day holding period. An ADI that employs the IRB approach to capital adequacy for credit risk or the advanced measurement approach for operational risk must seek approval from APRA to use the internal model approach for interest rate risk in the banking book (IRRBB) — a component of market risk. Under the internal model approach, ADIs calculate regulatory capital for IRRBB to cover losses arising from repricing, yield curve, basis and optionality risks associated with banking book items at the 99% confidence level over a one-year holding period (Australian Prudential Regulation Authority 2008e).

The conversion of Pillar 1 capital to RWA assumes an 8.0% minimum capital ratio (Australian Prudential Regulation Authority 2008f). An ADI’s RWA under Pillar 1 is equal to the sum of:

- RWA calculated by the standardised or IRB approach to capital adequacy for credit risk; and
- 12.5 ( $= 1.0/0.08$ ) times the sum of regulatory capital charges for operational risk, market risk and IRRBB.

### Supervisory Review Process

APRA implements Pillar 2 of the Basel II Accord through its supervisory review process (Australian Prudential Regulation Authority 2007). ADIs are responsible for developing and maintaining an internal capital adequacy assessment process (ICAAP) proportional to the size, business mix and complexity of their operations. Supervisors review and evaluate ADIs internal capital adequacy assessments and business strategies, and monitor their compliance with regulatory capital requirements. APRA's probability and impact rating system (PAIRS) and supervisory oversight and response system (SOARS) are designed to support the supervisory review process.

As part of the supervisory review process, and subject to a minimum of 8.0% of risk-weighted assets, APRA sets a prudential capital ratio (PCR) for each ADI, which must be maintained at all times. An ADI is required to hold at least half of the capital satisfying its PCR in the form of tier 1 capital, and APRA reserves the right to raise the proportion of tier 1 capital. In setting the PCR for an ADI, APRA takes into account the ADI's exposure to Pillar 2 risks, and qualitative factors such as corporate governance, senior management, and risk management systems and controls. Pillar 2 risks are divided into three categories:

- (a) Pillar 1 inherent risks that are not fully captured by the minimum capital requirement, for example, credit concentration risk.
- (b) Inherent risks that are not covered by Pillar 1 such as liquidity, strategic and reputation risks, as well as the benefits of risk diversification. Given that even during times of serious economic stress, credit risk, operational risk, market risk and other material risks are unlikely to be perfectly correlated, diversification benefits may be recognised in the form of a capital reduction.
- (c) Risks arising from external factors such as macroeconomic risk and systemic risk.

Consistent with their treatment of credit risk under Pillar 1, ADIs are required to determine regulatory capital requirements for securitisation exposures using either the standardised approach or IRB approach (Australian Prudential Regulation Authority 2011). Broadly speaking, risk-weighted assets for securitisation exposures are calculated by applying risk weights, which depend on assigned credit ratings, to on-balance sheet exposures and credit equivalent amounts of off-balance sheet exposures. RWA for securitisation exposures are added to RWA determined under Pillar 1 to arrive at total RWA for an ADI (Australian Prudential Regulation Authority 2008f).

Based on supervisors' qualitative assessments of the inherent risks taken by an ADI, and the effectiveness of its risk management through policies, procedures, systems and controls, PAIRS produces a rating for the ADI which maps to an indicative PCR (Australian Prudential Regulation Authority 2010b). In setting the PCR for an ADI, supervisors exercise their judgement and may adjust the indicative PCR to account for institution specific factors that are not reflected in the PAIRS rating. Finally, to achieve consistency across PCR outcomes, PCRs for ADIs are reviewed collectively by senior management in APRA's frontline supervisory division. Note that an ADI's ICAAP is a crucial input to APRA's supervisory review process, which ultimately determines the PCR assigned to an ADI. The regulatory capital requirement for an ADI then becomes the product of its PCR and total RWA.

### Market Discipline

Disclosure requirements under Pillar 3 encourage transparency and enhance market discipline. Information, qualitative and quantitative, covers capital structure, risk exposures, and risk management procedures for determining and maintaining capital adequacy. The standardised disclosure framework facilitates comparisons across institutions, allowing market participants to reward those that manage their risks prudently.

### Basel 2.5

The so-called Basel 2.5 reform package (Basel Committee on Banking Supervision 2009a) seeks to ensure that minimum capital requirements, risk management practices and public disclosures adequately reflect risks arising from trading activities, securitisation and exposures to off-balance sheet vehicles. The Basel 2.5 reforms came into effect in Australia on 1 January 2012.

### Basel III

The objective of the Basel III Accord (Basel Committee on Banking Supervision 2011) is "to improve the banking sector's ability to absorb shocks arising from financial and economic stress, whatever the source, thus reducing the risk of spillover from the financial sector to the real economy." As with the Basel 2.5 package, the Basel III Accord is intended to address lessons of the financial crisis of 2007–09. The reforms:

- (1) Raise the quality and minimum required level of capital (1 January 2013).
- (2) Promote the build up of capital buffers through a mandatory conservation buffer (1 January 2016) and a discretionary countercyclical buffer.
- (3) Establish a back-up minimum leverage ratio, equal to tier 1 capital divided by average total consolidated assets (1 January 2018).

- (4) Improve liquidity and stabilise funding by introducing a liquidity coverage ratio (1 January 2015) and net stable funding ratio (1 January 2018).
- (5) Assess a regulatory capital surcharge on global systemically important banks and domestic systemically important banks (1 January 2016).

In parentheses are the dates when the reforms came into effect, or are scheduled to come into effect in Australia. Presently, no Australian bank is on the list of global systemically important banks. APRA determines which domestic banks are systemically important on the basis of size, interconnectedness, substitutability and complexity. They are: Commonwealth Bank of Australia, Westpac Banking Corporation, National Australia Bank, and Australia and New Zealand Banking Group. The four largest banks in Australia, domestically they are designated “major” banks. APRA is developing a methodology to ensure that any domestic systemically important bank has higher loss absorbency, commensurate with its systemic importance (Australian Prudential Regulation Authority 2013a).

### 1.3 Economic and Regulatory Capital

Conceptually, economic and regulatory capital reflect the needs of distinct stakeholders, and therefore serve different objectives. In this section we focus on deposit-taking institutions, which we refer to generically as banks. For economic capital the primary stakeholders are the bank’s shareholders, and the objective is to maximise shareholders’ return on invested capital. Economic capital trades off the costs of funding the bank with expensive equity against the benefits of reducing the probability of losing its franchise value. All else being equal, a higher level of shareholders’ equity increases the probability of survival leading to a stronger external credit rating, lower cost of debt financing and access to banking business for which a high quality credit rating is a condition of eligibility. Elizalde and Repullo (2007) define *economic capital* as “the capital that shareholders would choose in the absence of capital regulation.” Banks use economic capital in capital allocation, performance attribution, risk pricing, risk identification and monitoring, strategic business planning, and solvency assessment (capital adequacy).

For regulatory capital the primary stakeholders are the bank’s depositors, and the objective is to minimise the possibility of losses to these depositors. If deposit insurance applies, then the insurer has a major stake as well. In Australia deposit insurance is provided by the government, so the taxpayer has a stake. *Regulatory capital* is defined as the minimum capital required to meet standards of capital adequacy imposed on all institutions by the prudential regulator. Its sole purpose is to absorb losses and protect against insolvency.

As highlighted in Section 1.2, the Basel II Accord sought to counter regulatory arbitrage by encouraging convergence between economic and regulatory capital through the use of internal credit ratings and internal models of market and operational risks to assess regulatory capital charges. However, economic and regulatory capital are not intended to be equivalent. Indeed perfect alignment would not be desirable. The adoption of standardised, advanced techniques by

all institutions for assessing economic and regulatory capital would increase the risk of contagion, undermining financial stability. In the event of a financial crisis each institution, using the same underlying risk models, would interpret and respond to the economic shock in an identical manner (Tieset and Troussard 2005). Important differences between economic and regulatory capital remain, often justified by their distinct objectives:

- Eligible regulatory capital includes shareholders' equity (tier 1 capital) and deeply subordinated claims (tier 2 capital), whereas economic capital comprises tier 1 capital alone (Allen 2006).
- The IRB approach implements an asset value factor model to assess regulatory capital charges for credit risk. It assumes that portfolios are infinitely fine-grained so that idiosyncratic risk is fully diversified away, and a single systematic risk factor explains dependence across obligors. Economic capital assessments may use more sophisticated multi-factor models (e.g., country and industry factors), and include individual exposures to account for credit concentration risk.
- Regulatory and economic capital models measure risk from different bases. The former is determined under book value accounting, while the latter is assessed on the basis of mark-to-market valuation (Giese 2003).
- In practice an institution's management set its level of economic capital to achieve a target external credit rating that supports its business or competitive strategy. Most large, internationally active banks choose a target rating of around AA–, which corresponds to a 99.97% target survival probability, or 0.03% probability of default over a one-year horizon (Tieset and Troussard 2005). The one-year horizon corresponds to the frequency of financial statements, and the management horizon for tapping public debt markets and disposing of assets. The 99.9% target survival probability over a one-year horizon set under the Basel II IRB approach, coupled with the recognition of deeply subordinated claims towards regulatory capital, translates into a target external credit rating of approximately BBB– (Gordy and Howells 2006).
- Credit risk parameters and variables of regulatory and economic capital models are likely to differ, as illustrated in the following examples. The IRB approach to capital adequacy for credit risk takes through-the-cycle (unconditional) probabilities of default as input. By contrast, an economic capital model used to price risk competitively would take point-in-time (conditional) probabilities of default as input. Under the IRB approach asset correlation is a constant or a function of through-the-cycle probability of default depending on the asset class categorisation of the credit exposure. Economic capital models may classify credit exposures differently and produce different estimates of asset correlation (Allen 2006). The model specification of the IRB approach incorporates a maturity adjustment, and APRA



sets a floor on the loss given default (under recessionary conditions) assigned to credit exposures, but neither would necessarily apply in the calculation of economic capital.

- The predominant practice in economic capital modelling follows a compartmentalised, top-down approach that first calculates economic capital for individual risk classes or types (e.g., credit risk, market risk, operational risk, underwriting risk, etc.) separately, and then aggregates capital charges across risk classes in a manner that allows for the recognition of inter-risk diversification benefits (Basel Committee on Banking Supervision 2009b). The variance-covariance approach to inter-risk aggregation is relatively simple and tractable, and widely employed, although its suitability for the purpose of assessing capital adequacy is contested. It uses a matrix of pairwise linear correlations between risk classes to calculate aggregate economic capital from estimates for individual risk classes. Assuming less than perfect pairwise correlations, the variance-covariance approach yields an inter-risk diversification benefit (Giese 2003). Consistent with the predominant practice in economic capital modelling, the Basel II Accord requires banks to first determine regulatory capital for Pillar 1 risks separately, but it does not recognise inter-risk diversification benefits, prescribing the simple addition of regulatory capital charges. Recognition of inter-risk diversification benefits is possible under Pillar 2 through the supervisory review process.
- Economic capital models endeavour to cover all material risks: credit, market, operational, underwriting, liquidity, strategic, reputation and concentration risks. Regulatory capital charges are assessed for credit, market and operational risks under Pillar 1 of the Basel II Accord. Inherent risks not covered by Pillar 1 are addressed under Pillar 2. There is no requirement that specific capital add-ons be modelled for any Pillar 2 risks (Allen 2006). Under its supervisory review process, APRA takes account of a bank's exposure to Pillar 2 risks and other qualitative factors when setting its prudential capital ratio.
- Whereas regulatory capital is assessed for each licensed entity, economic capital is often modelled for a consolidated group, which may include a number of separately licensed entities. At the consolidated group level, aggregate economic capital can reflect concentration, netting and diversification of risks that do not arise at the stand-alone licensed entity level (Allen 2006).

Elizalde and Repullo (2007) employ a numerical procedure to show that economic and regulatory capital do not depend on the same variables. Economic capital depends on intermediation margin and cost of bank capital, while regulatory capital depends on the confidence level set by the prudential regulator. In their empirical study these variables are key to explaining the difference between economic and regulatory capital. Note that both are increasing in probability of default and loss given default. They conclude that the prospect of material supervisory intervention is an effective inducement for banks to hold capital buffers, while policies aimed at increasing market discipline, such as Pillar 3 of the Basel II Accord, have a marginal effect.

Given that the minimum capital requirement under the IRB approach translates into a target external credit rating of approximately BBB−, and most large, internationally active banks target an external credit rating of around AA−, BCBS (2006) contends that “[s]upervisors should expect banks to operate above the minimum regulatory capital ratios and should have the ability to require banks to hold capital in excess of the minimum.” Indeed, this appears to be the experience in Australia since the implementation of Basel II.

## 1.4 Asset Value Factor Models of Credit Risk

The Basel II IRB approach implements an asset value factor model of credit risk. Asset value models posit that default or survival of a firm depends on the value of its assets at a given risk measurement horizon. If the value of its assets falls below a critical threshold, its default point, the firm defaults, otherwise it survives. Asset value models have their roots in Merton’s seminal paper published in 1974. Moody’s KMV, RiskMetrics and most (bank) internal models of credit risk are of the asset value variety. Like the model specification of the IRB approach, they too are generally factor models — a well established, computationally efficient technique for explaining dependence between variables. The classical structural approach of Merton, and the single factor conditional independence model that forms the basis of the IRB approach are explained in detail in Chapter 2. Here we sketch the basic structure of asset value factor models.

Consider a portfolio comprising  $n$  obligors, and assume that the time evolution of obligors’ asset values is modelled as geometric Brownian motion. Then, log-returns on obligors’ asset values are normally distributed with factor model representation

$$r_i = \beta_i \Psi_i + \varepsilon_i \quad (1.6)$$

for  $i = 1, \dots, n$ . In multi-factor models  $\Psi_i$  is the composite factor of obligor  $i$ , a weighted sum of several factors (Bluhm et al. 2010). Notice that (1.6) is a *linear regression* equation, where coefficient  $\beta_i$  is the sensitivity of  $r_i$  to  $\Psi_i$ , and  $\varepsilon_i$  is the residual. It is assumed that residuals  $\varepsilon_i$  are uncorrelated with one another and independent of composite factors  $\Psi_i$ . Accordingly,  $\Psi_i$  and  $\varepsilon_i$  are normally distributed. Coefficient  $\beta_i$  captures the correlation between  $r_i$  and  $\Psi_i$ , that is, log-returns on obligors’ asset values are correlated only through exposure to their composite factors. For this reason  $\Psi_i$  is referred to as the systematic component of  $r_i$ , whereas  $\varepsilon_i$  is termed the specific or idiosyncratic component.

Multi-factor models decompose every composite factor  $\Psi_i$  into  $m$  risk indices  $\psi_k$ :

$$\Psi_i = \sum_{k=1}^m u_{i,k} \psi_k, \quad (1.7)$$

where  $u_{i,k}$  are risk index weights for obligor  $i$ , with  $\sum_{k=1}^m u_{i,k} = 1$  for  $i = 1, \dots, n$ , and  $u_{i,k} \geq 0$  for all  $i$  and  $k$ . Moody’s KMV, for example, decomposes each obligor’s composite factor into country and industry indices (Bluhm et al. 2010). In expanded form (1.6) is analogous to the

multi-factor *arbitrage pricing theory*. For single factor models (1.6) reduces to

$$r_i = \beta_i \Psi + \varepsilon_i \quad (1.8)$$

for  $i = 1, \dots, n$ , where  $\Psi$  is the systematic factor common to all obligors, analogous to the *capital asset pricing model*.

Denote by  $A_i(\tau)$  the value of assets of obligor  $i$  at risk measurement horizon  $\tau > 0$ . Suppose that obligor  $i$  defaults in the period  $[0, \tau]$  if  $A_i(\tau)$  falls below a critical threshold  $B_i$  specific to each obligor. Observing that  $r_i = \log(A_i(\tau)/A_i(0))$ , we give equivalent formulations of the default condition:

$$\begin{aligned} A_i(\tau) &< B_i, \\ \beta_i \Psi_i + \varepsilon_i &< \log \frac{B_i}{A_i(0)}, \\ \varepsilon_i &< \log B_i - \log A_i(0) - \beta_i \Psi_i. \end{aligned} \quad (1.9)$$

The representation of asset value log-returns with respect to underlying systematic and idiosyncratic risk factors is typically expressed in a standardised form. Denote by  $\mathbb{E}[r_i]$  and  $\text{Var}(r_i)$  the expected value and variance, respectively, of log-returns on asset values of obligor  $i$ , and let  $\sigma_i = \sqrt{\text{Var}(r_i)}$ . Then, standardising asset value log-returns, representation (1.6) may be expressed as

$$\tilde{r}_i = \frac{r_i - \mathbb{E}[r_i]}{\sigma_i} = \frac{\beta_i}{\sigma_i} \tilde{\Psi}_i + \tilde{\varepsilon}_i, \quad (1.10)$$

where  $\tilde{r}_i$  and  $\tilde{\Psi}_i$  are normally distributed with mean zero and unit variance,  $\tilde{r}_i \sim \mathcal{N}(0, 1)$  and  $\tilde{\Psi}_i \sim \mathcal{N}(0, 1)$ . It follows that  $\tilde{\varepsilon}_i \sim \mathcal{N}(0, 1 - \beta_i^2/\sigma_i^2)$ . The coefficient of determination,  $R^2$ , of regression equation (1.10) is the proportion of the variability in  $\tilde{r}_i$  explained by its linear relationship with  $\tilde{\Psi}_i$ , equal to  $\beta_i^2/\sigma_i^2$ . The correlation between  $\tilde{r}_i$  and  $\tilde{\Psi}_i$  is equal to  $\sqrt{R^2}$ , or  $\beta_i/\sigma_i$ . Moreover, if dependence is explained by a single factor  $\tilde{\Psi}$  common to all obligors, then the pairwise correlation between obligors' asset values is given by

$$\begin{aligned} \text{Corr}(\tilde{r}_i, \tilde{r}_j) &= \frac{\text{Cov}(\tilde{r}_i, \tilde{r}_j)}{\sqrt{\text{Var}(\tilde{r}_i) \text{Var}(\tilde{r}_j)}} = \frac{\mathbb{E}[\tilde{r}_i \tilde{r}_j] - \mathbb{E}[\tilde{r}_i] \mathbb{E}[\tilde{r}_j]}{\sqrt{\text{Var}(\tilde{r}_i) \text{Var}(\tilde{r}_j)}} \\ &= \mathbb{E}[\tilde{r}_i \tilde{r}_j] = \frac{\beta_i \beta_j}{\sigma_i \sigma_j} \mathbb{E}[\tilde{\Psi}^2] = \frac{\beta_i \beta_j}{\sigma_i \sigma_j}, \end{aligned} \quad (1.11)$$

where  $\text{Cov}(\tilde{r}_i, \tilde{r}_j)$  is the covariance of  $\tilde{r}_i$  and  $\tilde{r}_j$ . Let  $c_i$  be the critical threshold corresponding to  $B_i$  in (1.9) after exchanging  $A_i(\tau)$  for  $\tilde{r}_i$ . Then, the default condition may be formulated as

$$\begin{aligned} \tilde{r}_i &< c_i, \\ \tilde{\varepsilon}_i &< c_i - \frac{\beta_i}{\sigma_i} \tilde{\Psi}_i. \end{aligned} \quad (1.12)$$

In Chapter 2 we deduce a standardised<sup>1</sup> default condition for the single factor conditional independence model that forms the basis of the IRB approach, and proceed to derive a function that transforms unconditional probabilities of default into probabilities of default conditional on a single systematic risk factor.

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<sup>1</sup> Normalised to have mean zero and unit variance.

# Foundations of the Asymptotic Single Risk Factor Model

Under the Basel II IRB approach to capital adequacy for credit risk, regulatory capital charges are assessed to absorb “unexpected” losses, which we define in Chapter 4, on a portfolio of credits over a given risk measurement horizon at a very high confidence level. In this chapter we derive the theoretical foundations of the model specification of the IRB approach, the so-called *asymptotic single risk factor* (ASRF) model. The model is described in Chapter 4, which outlines minimum capital requirements under the IRB approach. Before dealing with the complications that arise with a portfolio of credits, we first consider a single credit, or loan, and introduce some basic terminology and formulae. Define set  $D$ , abstractly, as the event of default. Then, credit loss is a random quantity given by

$$\text{Loss} = \text{EAD} \times \text{LGD} \times \mathbb{1}_D,$$

where:

- *exposure at default* (EAD) is the sum of outstanding (i.e., already drawn) exposures and expected drawdowns on commitments;
- *loss given default* (LGD) is the percentage of exposure that is not recovered in the event of default; and
- indicator function  $\mathbb{1}_D = \begin{cases} 1 & \text{if the event represented by set } D \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$

Assuming that EAD and LGD are deterministic quantities,

$$\text{Expected Loss} = \text{EAD} \times \text{LGD} \times \mathbb{E}[\mathbb{1}_D] = \text{EAD} \times \text{LGD} \times \text{PD},$$

where the expectation of the indicator function is the probability of the associated event, that is, the *probability of default* (PD).

The portfolio loss is simply the sum of losses on constituent credits. If default events in a credit portfolio were independent, then given EAD, LGD and PD estimates for the constituent credits, we could generate independent, uniformly distributed random variables and simulate the portfolio loss distribution. Unfortunately, for the typical credit portfolio default events are not independent, so correlations play a fundamental role in credit risk modelling. An obvious solution to simulating the loss distribution of a credit portfolio is to generate random variables with a correlation matrix that describes default correlations. The usual technique for generating correlated random variables involves the Cholesky decomposition<sup>1</sup> of the correlation matrix. Although Cholesky decomposition is a computationally intensive matrix factorisation for a large portfolio comprising thousands of credits, tuned linear algebra routines perform matrix factorisations rather efficiently. A greater challenge of this approach, however, is estimating default correlations, since defaults do not occur often and the limited data available indicate that correlations change over time. Accordingly, in generating the empirical loss distribution of a credit portfolio, we adopt the conventional approach of modelling asset correlations.<sup>2</sup>

The IRB approach implements an asset value factor model of credit risk. Asset value models posit that default or survival of a firm depends on the value of its assets at (the end of) a given risk measurement horizon. If the value of its assets falls below a critical threshold, its default point, the firm defaults, otherwise it survives. Asset value models have their roots in the classical structural approach of Merton. Factor models are a well established, computationally efficient technique for explaining dependence between variables. Section 2.1 introduces the classical structural approach to credit risk modelling, originally developed by Merton (1974), which formulates liabilities of a firm as contingent claims on its assets. Adapting the single asset model of Merton to a portfolio of credits, Vasicek (2002) derived a function that transforms unconditional PDs into PDs conditional on a single systematic risk factor. This function, derived in Section 2.2, is the kernel of the model specification of the IRB approach. In Section 2.3 we extend Vasicek's model of conditional independence to a more general setting, one not restricted to Gaussian processes. Gordy (2003) established that conditional on a single systematic risk factor, the portfolio percentage loss converges to its conditional expectation as the portfolio approaches asymptotic granularity — no single credit exposure accounts for more than an arbitrarily small share of total portfolio exposure. The proof is given in Section 2.4. Assuming conditional independence given a single systematic risk factor, Section 2.5 derives a limiting form of the portfolio loss distribution for the general case. It generalises Vasicek's (2002) formulation of the loss distribution function

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<sup>1</sup> Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of independent standard Gaussian random variables,  $\mathbf{X} \sim \mathcal{N}(0, I_n)$ . Let  $\mathbf{Y} = (Y_1, \dots, Y_n) = L\mathbf{X}$ , where  $L \in \mathbb{R}^{n \times n}$  is a linear transformation and  $\mathbf{Y} \sim \mathcal{N}(0, \Gamma)$ . Then,

$$\Gamma = \mathbb{E}[\mathbf{Y}\mathbf{Y}^T] = \mathbb{E}[(L\mathbf{X})(L\mathbf{X})^T] = \mathbb{E}[L(\mathbf{X}\mathbf{X}^T)L^T] = L\mathbb{E}[\mathbf{X}\mathbf{X}^T]L^T = LI_nL^T = LL^T,$$

where  $L$  is the Cholesky factor of  $\Gamma$ .

<sup>2</sup> An alternative to structural and reduced form approaches employs a multivariate intensity-based model of correlated defaults.

of an asymptotic, homogeneous portfolio, which models default dependence as a multivariate Gaussian process. The model specification of the IRB approach employs an analytical approximation of credit value-at-risk (VaR), which is defined in Section 2.6. It rests on the proposition, due to Gordy (2003), that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles of the portfolio loss distribution. We present a more compact proof of this proposition starting from weaker assumptions.

## 2.1 Classical Structural Approach

The structural approach to credit risk modelling, originally developed by Merton (1974), formulates liabilities of a firm as contingent claims on its assets.<sup>3</sup> Suppose that a firm with asset value  $A(t)$  at time  $t$  is financed by equity and a zero-coupon bond with face value  $B$  and maturity date  $\tau$ . If the firm cannot meet its contractual obligation to pay investors the principal amount of the bond at maturity, then the bondholders take over the firm. That is, the firm defaults if the value of its assets falls below the principal amount of its debt at maturity, wiping out the firm's equity.

We describe the time evolution of asset values in terms of adapted stochastic processes measurable with respect to a filtration on a probability space. These concepts are formally defined in Appendix 2.C. The standard model of stock prices, and asset values, is geometric Brownian motion. Expressly for the purpose of describing the stochastic differential equation for asset values and its solution, Appendix 2.D defines Brownian motion and an Itô process, and states Itô's formula for Itô processes.

Let  $\mathcal{F}(t), t \geq 0$ , be an associated filtration, and let  $\mu(t)$  and  $\sigma(t)$  be adapted stochastic processes. Define the Itô process

$$X(t) = \int_0^t \left( \mu(s) - \frac{1}{2} \sigma^2(s) \right) ds + \int_0^t \sigma(s) dW(s),$$

which may be expressed in differential form as

$$dX(t) = \left( \mu(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dW(t).$$

Then, consider the integral form of generalised geometric Brownian motion modelling the time-evolution of asset values:

$$A(t) = A(0) \exp \left\{ \int_0^t \left( \mu(s) - \frac{1}{2} \sigma^2(s) \right) ds + \int_0^t \sigma(s) dW(s) \right\},$$

where  $A(0)$  is deterministic and positive. We may express  $A(t)$  as a function of  $X(t)$ , writing  $A(t) = f(X(t)) = A(0) e^{X(t)}$ . Differentiating with respect to  $X(t)$  yields  $f'(X(t)) = A(0) e^{X(t)}$ ,

<sup>3</sup> The structural approach to credit risk modelling relates default to the value of the underlying assets of the obligor, while the reduced form approach prices credit instruments off observable term structures of interest rates for alternative credit qualities.

and  $f''(X(t)) = A(0)e^{X(t)}$ . Next, applying Itô's formula (Remark 2.52) we obtain

$$\begin{aligned}
 dA(t) &= df(X(t)) \\
 &= f'(X(t)) dX(t) + \frac{1}{2}f''(X(t)) d[X, X](t) \\
 &= A(0)e^{X(t)} dX(t) + \frac{1}{2}A(0)e^{X(t)} \sigma^2(t) dt \\
 &= A(t) \left[ (\mu(t) - \frac{1}{2}\sigma^2(t)) dt + \sigma(t) dW(t) \right] + \frac{1}{2}A(t)\sigma^2(t) dt \\
 &= \mu(t)A(t) dt + \sigma(t)A(t) dW(t).
 \end{aligned}$$

This equation describes, in differential form, the time evolution of asset values  $A(t)$  with instantaneous mean rate of return  $\mu(t)$  and volatility  $\sigma(t)$ . In the case of constant  $\mu$  and  $\sigma$ , the solution to the stochastic differential equation is geometric Brownian motion:

$$A(t) = A(0) \exp \left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right\}. \quad (2.1)$$

Taking the natural logarithm of both sides of (2.1) yields

$$\log A(t) = \log A(0) + \mu t - \frac{1}{2}\sigma^2 t + \sigma W(t). \quad (2.2)$$

According to the self-similarity property of Brownian motion,  $W(t) \sim \mathcal{N}(0, t)$  is distributionally equivalent to  $\sqrt{t} W(1)$ , which we write  $\sqrt{t} W$ . Latent random variable  $W$  is standard Gaussian,  $W \sim \mathcal{N}(0, 1)$ . Then, we may write

$$\log A(t) = \log A(0) + \mu t - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t} W. \quad (2.3)$$

Under the postulate that a firm defaults if the value of its assets  $A(\tau)$  falls below the principal amount of its debt  $B$  at maturity  $\tau$ , the unconditional PD is given by

$$\begin{aligned}
 p &= \mathbb{P}(D) = \mathbb{P}(A(\tau) < B) \\
 &= \mathbb{P}(\log A(\tau) < \log B) \\
 &= \mathbb{P}(\log A(0) + \mu\tau - \frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau} W < \log B) \\
 &= \mathbb{P}\left(W < \frac{\log B - \log A(0) - \mu\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \\
 &= \Phi\left(\frac{\log B - \log A(0) - \mu\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right),
 \end{aligned} \quad (2.4)$$

where  $\Phi$  is the standard Gaussian distribution function (Giesecke 2004). Thus, the event of default is more precisely defined by set

$$D = \left\{ W < \frac{\log B - \log A(0) - \mu\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \right\}. \quad (2.5)$$

We proceed to define *distance-to-default* as the negative of the right-hand side of the inequality in (2.5):

$$d = \frac{\log A(0) - \log B + \mu\tau - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}. \quad (2.6)$$

## 2.2 Conditional Independence Model

Adapting the single asset model of Merton to a portfolio of credits, Vasicek (2002) derived a function that transforms unconditional PDs into PDs conditional on a single systematic risk factor. This function is the kernel of the model specification of the IRB approach.

*Remark 2.1.* Vasicek (2002) considered a portfolio consisting of  $n$  loans with equal dollar exposure, and assumed that the probability of default of any one loan is  $p$  and asset correlation between borrowers is  $\rho$ . In developing the model specification of the IRB approach, BCBS (2005) extended Vasicek's model of conditional independence to accommodate obligor specific EAD, LGD, unconditional PD and asset correlation.

Let the sequence of random variables  $\{L_n\}$  be the percentage loss on a credit portfolio comprising  $n \in \mathbb{N}$  obligors over a given risk measurement horizon  $[0, \tau]$ ,  $\tau > 0$ . We make the assumption that the number of credits in the portfolio equals the number of distinct obligors. This can be achieved by aggregating multiple credits of an individual obligor into a single credit by calculating exposure weighted unconditional PDs and LGD rates. Assume that EAD and LGD are deterministic quantities, and denote by  $\delta_i \in \mathbb{R}_+$  and  $\eta_i \in [0, 1]$  the EAD and LGD, respectively, assigned to obligor  $i$ . Also, define set  $D_i$ , abstractly, as the event that obligor  $i$  defaults during the risk measurement horizon. Then, the *portfolio percentage loss* is given by

$$L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{D_i}, \quad (2.7)$$

where  $w_i = \delta_i / \sum_{j=1}^n \delta_j$  is the *exposure weight* of obligor  $i$ , and  $\sum_{i=1}^n w_i = 1$ . Clearly,  $w_i$  depends on  $n$  and could be denoted  $w_i(n)$ , but we adopt the more concise, and more common, notation for exposure weight.

*Remark 2.2.* Define random variable  $\Lambda_k = \delta_k \eta_k \mathbb{1}_{D_k}$ . If  $\{\Lambda_k\}_{k \in \mathbb{N}}$  were a sequence of independent identically distributed random variables, then the portfolio loss distribution would converge, by the central limit theorem (Theorem 2.64), to a Gaussian distribution as the number of credits in the portfolio increases. Firstly, the distributions of  $\Lambda_k$ , for  $k = 1 \dots n$ , are not identical. The dollar loss on a portfolio comprising  $n$  obligors may be interpreted as the partial sum of order  $n$  of the infinite series with terms  $\Lambda_k$  — Poisson trials<sup>4</sup> weighted by their respective EAD and LGD, which may vary from credit to credit. While not identical, the variation in distributions of the sequence of random variables  $\{\Lambda_k\}_{1 \leq k \leq n}$  for the typical credit portfolio is small enough that the portfolio loss distribution would be approximately Gaussian if default events were independent.

Default dependence, however, has a much greater effect on the portfolio loss distribution. Figure 6.1 plots the empirical loss distribution for a representative credit portfolio in which obligors' asset values are correlated, and hence defaults are not independent. Illustrating the

<sup>4</sup> Poisson trials are independent repeated trials of an experiment with two possible outcomes labelled “success” and “failure”. Define random variables  $X_i$  so that  $X_i$  takes the value 1 if the outcome of the experiment is success, and 0 otherwise. The probability function is  $\mathbb{P}(X = 1) = p_i$  and  $\mathbb{P}(X = 0) = 1 - p_i$ ,  $p_i \in (0, 1)$ , and random variables  $X_i$  are all independent.



fundamental role of correlation in credit risk modelling, Figure 6.3 plots the empirical loss distribution for the same representative credit portfolio, except that obligors' asset values are assumed to be uncorrelated, and hence defaults independent. In the latter case the portfolio loss distribution is approximately Gaussian.

It follows from (2.2) that asset values at time  $t$  of obligors constituting a credit portfolio may be expressed in logarithmic form as

$$\log A_i(t) = \log A_i(0) + \mu_i t - \frac{1}{2} \sigma_i^2 t + \sigma_i W_i(t) \quad (2.8)$$

for  $i = 1, \dots, n$ . Dependence across obligors is described by the pairwise correlation between Brownian motions  $W_1(t), \dots, W_n(t)$  modelling the variability in obligors' asset values. In the conditional independence model that emerges, dependence across obligors is induced by the quadratic covariation between Brownian motion  $W_i(t)$ ,  $1 \leq i \leq n$ , and correlated Brownian motion  $Y(t)$  common to all obligors.

Firstly, we decompose Brownian motion  $W_i(t)$  into Itô processes, one with respect to correlated Brownian motion  $Y(t)$ , another which we characterise in Proposition 2.3. Then, assuming constant asset correlations, Proposition 2.5 deduces the pairwise asset correlation between obligors' asset values. Next, applying the self-similarity property of Brownian motion, we reduce  $W_i(t)$  to a standard Gaussian random variable, which is decomposed into systematic and idiosyncratic risk factors. Finally, substituting this representation into the set defining the event of default, we follow Vasicek (2002) in deriving a function that transforms unconditional PDs into PDs conditional on a single systematic risk factor.

**Proposition 2.3.** *Let  $W_i(t)$  and  $Y(t)$  be correlated Brownian motions with quadratic covariation  $d[W_i, Y](t) = \sqrt{\rho_i(t)} dt$ , where  $\rho_i(t) \in (0, 1)$  is an adapted stochastic process. Define stochastic process  $Z_i(t)$ , where  $Z_i(0) = 0$ , such that*

$$W_i(t) = \int_0^t \sqrt{\rho_i(s)} dY(s) + \int_0^t \sqrt{1 - \rho_i(s)} dZ_i(s). \quad (2.9)$$

*Then,  $Y(t)$  and  $Z_i(t)$  are independent Brownian motions.*

The proof uses the one-dimensional Lévy theorem to establish that  $Z_i(t)$  is a Brownian motion, and the two-dimensional Lévy theorem to show that  $Y(t)$  and  $Z_i(t)$  are independent. The one-dimensional Lévy theorem characterises a Brownian motion as a martingale with continuous paths whose quadratic variation is  $[W, W](t) = t$ . Appendix 2.D defines a martingale, enumerates properties of the Itô integral, and states the one- and two-dimensional Lévy theorems.

*Proof of Proposition 2.3.* Equation (2.9) may be written in differential form as

$$dW_i(t) = \sqrt{\rho_i(t)} dY(t) + \sqrt{1 - \rho_i(t)} dZ_i(t). \quad (2.10)$$

Rearranging (2.10),

$$dZ_i(t) = \frac{-\sqrt{\rho_i(t)}}{\sqrt{1 - \rho_i(t)}} dY(t) + \frac{1}{\sqrt{1 - \rho_i(t)}} dW_i(t). \quad (2.11)$$

$Z_i(t)$  is a Brownian motion if it satisfies the conditions of Theorem 2.55:

- (1) By hypothesis,  $Z_i(0) = 0$ .
- (2) Writing (2.11) in integral form yields

$$Z_i(t) = Z_i(0) - \int_0^t \frac{\sqrt{\rho_i(s)}}{\sqrt{1 - \rho_i(s)}} dY(s) + \int_0^t \frac{1}{\sqrt{1 - \rho_i(s)}} dW_i(s), \quad (2.12)$$

where  $Z_i(0) = 0$ . By the continuity property of Theorem 2.54, the Itô integrals on the right-hand side of (2.12) are continuous. It follows that  $Z_i(t)$  has continuous paths.

- (3) By the martingale property of Theorem 2.54, the Itô integrals on the right-hand side of (2.12) are martingales. Accordingly,  $Z_i(t)$  is a martingale.
- (4) The quadratic variation of  $Z_i(t)$  is calculated as

$$\begin{aligned} d[Z_i, Z_i](t) &= \frac{\rho_i(t)}{1 - \rho_i(t)} d[Y, Y](t) + \frac{1}{1 - \rho_i(t)} d[W_i, W_i](t) - \frac{2\sqrt{\rho_i(t)}}{1 - \rho_i(t)} d[W_i, Y](t) \\ &= \frac{\rho_i(t)}{1 - \rho_i(t)} dt + \frac{1}{1 - \rho_i(t)} dt - \frac{2\rho_i(t)}{1 - \rho_i(t)} dt \\ &= dt. \end{aligned}$$

It follows that  $[Z_i, Z_i](t) = t$  for all  $t \geq 0$ .

Thus,  $Z_i(t)$  is a Brownian motion.

Finally, by Theorem 2.56,  $Y(t)$  and  $Z_i(t)$  are independent Brownian motions if their quadratic covariation is zero:

$$\begin{aligned} d[Z_i, Y](t) &= \frac{-\sqrt{\rho_i(t)}}{\sqrt{1 - \rho_i(t)}} d[Y, Y](t) + \frac{1}{\sqrt{1 - \rho_i(t)}} d[W_i, Y](t) \\ &= \frac{-\sqrt{\rho_i(t)}}{\sqrt{1 - \rho_i(t)}} dt + \frac{\sqrt{\rho_i(t)}}{\sqrt{1 - \rho_i(t)}} dt \\ &= 0. \end{aligned}$$

□

In the case of constant  $\rho_i \in (0, 1)$ , (2.9) may be expressed as

$$W_i(t) = \sqrt{\rho_i} Y(t) + \sqrt{1 - \rho_i} Z_i(t), \quad (2.13)$$

where  $Y(t)$  and  $Z_i(t)$  are independent Brownian motions, and  $W_i(t)$  and  $Y(t)$  are correlated Brownian motions with quadratic covariation  $[W_i, Y](t) = \sqrt{\rho_i} t$ . In a credit portfolio comprising  $n$  obligors we suppose that Brownian motions  $Z_1(t), \dots, Z_n(t)$  and  $Y$  are mutually independent, and correlation parameters  $\rho_1, \dots, \rho_n \in (0, 1)$ . Using (2.13) we verify that, consistent with Definition 2.47, Brownian motions  $W_1(t), \dots, W_n(t)$  have expectation

$$\mathbb{E}[W_i(t)] = \sqrt{\rho_i} \mathbb{E}[Y(t)] + \sqrt{1 - \rho_i} \mathbb{E}[Z_i(t)] = 0,$$

and variance

$$\text{Var}(W_i(t)) = \rho_i \text{Var}(Y(t)) + (1 - \rho_i) \text{Var}(Z_i(t)) = \rho_i t + (1 - \rho_i)t = t$$

for  $i = 1, \dots, n$ .

*Remark 2.4.* Equation (2.13), governing correlated Brownian motions, is usually expressed in the form  $W_i(t) = \rho_i Y(t) + \sqrt{1 - \rho_i^2} Z_i(t)$ , where  $\rho_i \in (-1, 1)$ . The formulation we've adopted, which assumes that Brownian motions are positively correlated, is borrowed from Vasicek (2002) and employed in the model specification of the IRB approach. We follow the same convention as the IRB approach for the Gaussian case, because the concern of our research is the IRB approach. In Section 2.3 we extend the conditional independence model developed here to a more general setting, and adopt a formulation of stochastic processes that admits positive and negative correlation.

**Proposition 2.5.** *Let the variability in obligors' asset values be described by (2.13), where  $Y(t)$  and  $Z_i(t)$  are independent Brownian motions, and  $W_i(t)$  and  $Y(t)$  are correlated Brownian motions with quadratic covariation  $[W_i, Y](t) = \sqrt{\rho_i} t$ ,  $\rho_i \in (0, 1)$ . Assume that for the pair  $(W_i(t), W_j(t))$ ,  $i \neq j$ , Brownian motions  $Z_i(t)$  and  $Z_j(t)$  are independent. Then, the pairwise correlation between obligors' assets values  $\text{Corr}(W_i(t), W_j(t)) = \sqrt{\rho_i \rho_j}$ .*

The proof uses Itô's product rule, which is the result of an application of Itô's formula in two-dimensions, where  $X(t)$  and  $Y(t)$  are Itô processes and  $f(t, x, y) = xy$ . Appendix 2.D states Itô's formula in two-dimensions and product rule.

*Proof of Proposition 2.5.* The pair of Brownian motions  $(W_i(t), W_j(t))$ ,  $i \neq j$  may be represented in differential form as

$$dW_i(t) = \sqrt{\rho_i} dY(t) + \sqrt{1 - \rho_i} dZ_i(t),$$

and

$$dW_j(t) = \sqrt{\rho_j} dY(t) + \sqrt{1 - \rho_j} dZ_j(t).$$

Then, applying Itô's product rule (Corollary 2.58) we have

$$\begin{aligned} d(W_i(t)W_j(t)) &= W_j(t) dW_i(t) + W_i(t) dW_j(t) + d[W_i, W_j](t) \\ &= W_j(t) dW_i(t) + W_i(t) dW_j(t) + \sqrt{\rho_i \rho_j} dt. \end{aligned}$$

Integrating and taking expectations of both sides yields

$$\begin{aligned} \mathbb{E}[W_i(t)W_j(t)] &= \mathbb{E} \left[ \int_0^t W_j(s) dW_i(s) \right] + \mathbb{E} \left[ \int_0^t W_i(s) dW_j(s) \right] + \mathbb{E} \left[ \int_0^t \sqrt{\rho_i \rho_j} ds \right] \\ &= \sqrt{\rho_i \rho_j} t. \end{aligned}$$

Note that the first two terms on the right-hand side of the first equality are Itô integrals, which have expectation zero since they are martingales. So, the covariance of  $W_i(t)$  and  $W_j(t)$  is

$$\text{Cov}(W_i(t), W_j(t)) = \mathbb{E}[W_i(t)W_j(t)] - \mathbb{E}[W_i(t)]\mathbb{E}[W_j(t)] = \sqrt{\rho_i \rho_j} t$$

with  $\mathbb{E}[W_i(t)] = \mathbb{E}[W_j(t)] = 0$ . Finally, the correlation between  $W_i(t)$  and  $W_j(t)$  is given by

$$\text{Corr}(W_i(t), W_j(t)) = \frac{\text{Cov}(W_i(t), W_j(t))}{\sqrt{\text{Var}(W_i(t)) \text{Var}(W_j(t))}} = \frac{\sqrt{\rho_i \rho_j} t}{\sqrt{t^2}} = \sqrt{\rho_i \rho_j}. \quad (2.14)$$

□

**Corollary 2.6.** *Let the variability in obligors' asset values be described as in Proposition 2.5. Assume that  $\rho_i = \rho_j = \rho$  for  $i, j \in \mathbb{N}$ . Then obligors' asset values are equicorrelated, and the pairwise correlation between obligors' assets values  $\text{Corr}(W_i(t), W_j(t)) = \rho$ ,  $i \neq j$ .*

*Proof.* Substituting for  $\rho_i$  and  $\rho_j$  in (2.14) yields  $\text{Corr}(W_i(t), W_j(t)) = \sqrt{\rho^2} = \rho$ . □

In our exposition of the structural approach we assume that an obligor defaults if the value of its assets at a given risk measurement horizon falls below its contractual obligations payable. Equation (2.8) expresses the time-evolution of asset values in logarithmic form. By the self-similarity property of Brownian motion,  $W_i(t)$  is distributionally equivalent to  $\sqrt{t} W_i(1)$ , which we write  $\sqrt{t} W_i$ . Hence, the value of assets of obligor  $i$  at risk measurement horizon  $\tau$  may be expressed in logarithmic form as

$$\log A_i(\tau) = \log A_i(0) + \mu_i \tau - \frac{1}{2} \sigma_i^2 \tau + \sigma_i \sqrt{\tau} W_i, \quad (2.15)$$

where latent random variable  $W_i$  is standard Gaussian,  $W_i \sim \mathcal{N}(0, 1)$ . It follows from (2.4) that the *unconditional probability of default* of obligor  $i$  is given by

$$\begin{aligned} p_i &= \mathbb{P}(D_i) = \mathbb{P}(A_i(\tau) < B_i) \\ &= \mathbb{P}(W_i < c_i) \\ &= \Phi(c_i), \end{aligned} \quad (2.16)$$

where

$$c_i = \frac{\log B_i - \log A_i(0) - \mu_i \tau + \frac{1}{2} \sigma_i^2 \tau}{\sigma_i \sqrt{\tau}} = \Phi^{-1}(p_i). \quad (2.17)$$

Then, assuming that unconditional PDs are published as market data, the event that obligor  $i$  defaults during the risk measurement horizon is more precisely defined by set

$$D_i = \{W_i < \Phi^{-1}(p_i)\}. \quad (2.18)$$

Notice that unconditional PD, given by (2.16), is a function of  $\tau$ . In the conditional independence model that follows, we fix  $\tau$ , say a one-year horizon, and conditional PD is a function of  $y$ .

Now, applying the self-similarity property of Brownian motion to (2.13) we may represent latent random variables  $W_1, \dots, W_n$  in (2.15) as

$$W_i = \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i, \quad (2.19)$$

where random variables  $Z_1, \dots, Z_n$  and  $Y$  are standard Gaussian and mutually independent, and  $\rho_1, \dots, \rho_n \in (0, 1)$  are correlation parameters calibrated to market data. Thus,  $W_1, \dots, W_n$  are

conditionally independent given random variable  $Y$ , which is common to all obligors. Systematic risk factor  $Y$  may be interpreted as an underlying risk driver or economic factor, with each realisation describing a scenario of the economy. Random variables  $Z_1, \dots, Z_n$  represent idiosyncratic, or obligor specific, risk. As highlighted in Remark 2.4, we follow the same convention as the IRB approach and adopt representation (2.19) for the Gaussian case.

Conditional independence of obligors' asset values implies that, given  $Y = y$ , observations of the indicator function are independent — Poisson trials with parameter  $p_i(y) = \mathbb{E}[\mathbb{1}_{D_i} | Y = y]$ . Substituting representation (2.19) into set (2.18) representing the event of default, the PD of obligor  $i$  conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ , or *conditional probability of default*, may be expressed as

$$\begin{aligned} p_i(y) &= \mathbb{P}(D_i | Y = y) = \mathbb{P}(W_i < \Phi^{-1}(p_i) | Y = y) \\ &= \mathbb{P}\left(\sqrt{\rho_i} y + \sqrt{1 - \rho_i} Z_i < c_i\right) \\ &= \mathbb{P}\left(Z_i < \frac{c_i - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}}\right), \end{aligned} \quad (2.20)$$

where  $p_i = \mathbb{E}[\mathbb{E}[\mathbb{1}_{D_i} | Y]]$  is the unconditional PD of obligor  $i$ . Equation (2.20) transforms unconditional PDs into PDs conditional on a single systematic risk factor.

Let

$$\zeta_i(y) = \frac{\Phi^{-1}(p_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}} = \Phi^{-1}(p_i(y)) \quad (2.21)$$

for  $i = 1, \dots, n$ . Then, given  $Y = y$ , the portfolio percentage loss is calculated as

$$L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{Z_i < \zeta_i(y)\}}. \quad (2.22)$$

Equation (2.22), which is motivated by the classical structural approach, is widely employed in the simulation of portfolio loss distributions using market data — unconditional PDs, LGD rates and asset correlations published by the likes of Moody's KMV and RiskMetrics.

In the sequel we refer to the model developed in this section as the *Gaussian conditional independence model of a credit portfolio*.

### 2.3 A More General Setting

We proceed to extend the Gaussian conditional independence model to a more general setting. In Section 2.2 we assume that: (i) defaults are conditionally independent given a single systematic risk factor; and (ii) obligors' asset values are modelled by geometric Brownian motion. The latter assumption implies that latent random variables modelling the variability in obligors' asset values, and their component systematic and idiosyncratic risk factors, are Gaussian. By relaxing this assumption, we describe a more general setting for stochastic processes. Firstly, we define self-similar stochastic processes.

**Definition 2.7** (Embrechts, Klüppelberg, et al. 2000, Definition 8.9.1). A stochastic process  $\{X(t), t \in [0, \infty)\}$ , is said to be *self-similar* with index  $H > 0$  ( $H$ -s.s.) if its finite-dimensional distributions satisfy the relation

$$\{X(at)\} \stackrel{D}{=} \{a^H X(t)\}$$

for any  $a > 0$ .

Assume that stochastic processes  $W_1(t), \dots, W_n(t)$  are correlated with stochastic process  $Y(t)$ , and admit representation

$$W_i(t) = \gamma_i Y(t) + \sqrt{1 - \gamma_i^2} Z_i(t).$$

Note that in the general setting developed in this section, stochastic processes  $W_1(t), \dots, W_n(t)$ , and hence  $Z_1(t), \dots, Z_n(t)$  and  $Y(t)$ , are no longer Brownian motions, in general. Suppose, too, that stochastic processes  $W_1(t), \dots, W_n(t)$  are self-similar with index  $b > 0$ . Then, by Definition 2.7 we may write

$$t^b W_i(1) = \gamma_i t^b Y(1) + \sqrt{1 - \gamma_i^2} t^b Z_i(1).$$

Fixing  $t$ , say equal to risk measurement horizon  $\tau$ , and writing  $W_i$  for  $W_i(1)$ , we have

$$W_i = \gamma_i Y + \sqrt{1 - \gamma_i^2} Z_i, \tag{2.23}$$

where  $Z_1, \dots, Z_n$  and  $Y$  are mutually independent random variables, and  $\gamma_1, \dots, \gamma_n \in (-1, 1)$  are correlation parameters calibrated to market data. Again, systematic risk factor  $Y$  is commonly interpreted as the state of the economy, while  $Z_1, \dots, Z_n$  represent idiosyncratic, or obligor specific, risk. Note that here asset values may be positively or negatively correlated. The self-similarity property allows us to standardise the representation of stochastic processes to the unit time interval. For stochastic processes, in general, (2.23) provides a correct representation for any fixed time horizon.

*Remark 2.8.* Since Brownian motion  $W_i(t)$  of Proposition 2.5 is a special case of the general setting in which  $W_i(t)$  is a stochastic process, Proposition 2.5 does not determine pairwise correlation between stochastic processes, in general.

*Remark 2.9.* We may continue to interpret the event that obligor  $i$  defaults during a given risk measurement horizon as the event that the value of its assets at the risk measurement horizon falls below its contractual obligations payable. However, as indicated previously, we no longer imply that the unconditional PD assigned to obligor  $i$ , denoted by  $p_i$ , is derived from fundamental analysis by applying the structural approach. Instead, in the sequel, we assume that  $p_i$ , scaled to a given risk measurement horizon, is a model input determined by the market.

Formalising representation (2.23) of conditionally independent random variables, and recasting set (2.18) representing the event of default for the general case, we define a conditional independence model of a credit portfolio. Unless otherwise stated, results in the sequel apply to the conditional independence model for the general case defined here.

**Definition 2.10.** A conditional independence model of a credit portfolio comprising  $n \in \mathbb{N}$  obligors over a given risk measurement horizon  $[0, \tau]$ ,  $\tau > 0$ , takes the form:

- (1) Let  $\delta_i \in \mathbb{R}$  be the EAD assigned to obligor  $i$ , and  $w_i = \delta_i / \sum_{j=1}^n \delta_j$  its exposure weight. Let  $\eta_i \in [0, 1]$ ,  $\gamma_i \in (-1, 1)$  and  $p_i \in (0, 1)$  be the LGD, asset correlation and unconditional PD, respectively, assigned to obligor  $i$ .
- (2) Suppose that latent random variables  $W_1, \dots, W_n$  are conditionally independent, and admit representation

$$W_i = \gamma_i Y + \sqrt{1 - \gamma_i^2} Z_i, \quad (2.24)$$

where  $Z_1, \dots, Z_n$  and  $Y$  are mutually independent random variables. Systematic risk factor  $Y$  is common to all obligors, while  $Z_1, \dots, Z_n$  represent idiosyncratic, or obligor specific, risk. Denote by  $F_1, \dots, F_n$ ,  $G_1, \dots, G_n$  and  $H$  the continuous and strictly increasing distribution functions of  $W_1, \dots, W_n$ ,  $Z_1, \dots, Z_n$  and  $Y$ , respectively. Clearly,  $F_i$  depends on  $G_i$  and  $H$  for  $i = 1, \dots, n$ .

- (3) The event that obligor  $i$  defaults during the time interval  $[0, \tau]$  is defined by the set

$$D_i = \{W_i < F_i^{-1}(p_i)\} \quad (2.25)$$

for  $i = 1, \dots, n$ .

Portfolio percentage loss  $L_n$  is calculated by (2.7) where, for the general case,  $\mathbb{1}_{D_i}$  is the indicator function of the default event defined by (2.25). Recasting argument (2.21) of the default indicator function and conditional probability function (2.20) for the general case, we deduce a formula for portfolio percentage loss conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ . Thus, given  $Y = y$ , portfolio percentage loss under the conditional independence model of Definition 2.10 is calculated as:

$$L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{Z_i < \zeta_i(y)\}}, \quad (2.26)$$

where

$$\zeta_i(y) = \frac{F_i^{-1}(p_i) - \gamma_i y}{\sqrt{1 - \gamma_i^2}} = G_i^{-1}(p_i(y)), \quad (2.27)$$

and

$$p_i(y) = \mathbb{P}(D_i | Y = y) = G_i \left( \frac{F_i^{-1}(p_i) - \gamma_i y}{\sqrt{1 - \gamma_i^2}} \right) \quad (2.28)$$

for  $i = 1, \dots, n$ .

*Remark 2.11.* In the abstract case where conditional PD, as well as unconditional PD and asset correlation, are known and the state of the economy is sought, we take the inverse of (2.28).

Conditional probability function  $p_i: \mathbb{R} \rightarrow (0, 1)$ , given by (2.28), is continuous and strictly decreasing in  $y$  — conditional PD falls (respectively, rises) as the economy improves (deteriorates). Hence, its inverse  $p_i^{-1}: (0, 1) \rightarrow \mathbb{R}$  is strictly decreasing too. In particular,

$$y = p_i^{-1}(x) = \frac{F_i^{-1}(p_i) - \sqrt{1 - \gamma_i^2} G_i^{-1}(x)}{\gamma_i} \quad (2.29)$$

for all  $x \in (0, 1)$ .

*Remark 2.12.* Let  $y = H^{-1}(1 - \alpha)$ , where  $\alpha \in (0, 1)$ . Then, the PD of obligor  $i$  conditional on  $Y = y$  may be interpreted as the probability of default of obligor  $i$  is no greater than

$$\mathbb{P}(D_i | Y = H^{-1}(1 - \alpha)) = p_i(H^{-1}(1 - \alpha)) = G_i \left( \frac{F_i^{-1}(p_i) - \gamma_i H^{-1}(1 - \alpha)}{\sqrt{1 - \gamma_i^2}} \right) \quad (2.30)$$

in  $(\alpha \times 100)\%$  of economic scenarios.

## 2.4 Conditional Expectation of Portfolio Percentage Loss

The model specification of the IRB approach calculates the expectation of portfolio credit losses conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ . Taking expectations of (2.7) and (2.26), respectively, we define the *expected portfolio percentage loss* as

$$\mathbb{E}[L_n] = \sum_{i=1}^n w_i \eta_i p_i, \quad (2.31)$$

and the *conditional expectation of portfolio percentage loss* as

$$\mathbb{E}[L_n | Y = y] = \sum_{i=1}^n w_i \eta_i p_i(y). \quad (2.32)$$

*Remark 2.13.* Conditional expectation function  $\mathbb{E}[L_n | Y]: \mathbb{R} \rightarrow (0, 1)$ , given by (2.32), is continuous and strictly decreasing in  $y$  — conditional expectation of portfolio percentage loss falls (respectively, rises) as the economy improves (deteriorates).

Key propositions underpinning the model specification of the Basel II IRB approach are derived for an asymptotic portfolio, often described as infinitely fine-grained, in which no single credit exposure accounts for more than an arbitrarily small share of total portfolio exposure. Accordingly, our derivation of the model specification of the IRB approach requires a mathematically more precise definition of asymptotic granularity.

**Definition 2.14.** Let  $\Delta = \sum_{k=1}^{\infty} \delta_k$  be an infinite series whose terms  $\delta_k \in \mathbb{R}_+$  represent EAD assigned to obligors constituting a credit portfolio:

- (1) The partial sums of  $\Delta$  of order  $n$ , are defined for  $n \in \mathbb{N}$  as

$$\Delta_n = \sum_{k=1}^n \delta_k.$$



(2) An *asymptotic portfolio* satisfies

$$\sum_{n=1}^{\infty} \left( \frac{\delta_n}{\Delta_n} \right)^2 < \infty.$$

*Remark 2.15.* An application of Kronecker's lemma (Lemma 2.66) shows that exposure weights of credits constituting a portfolio shrink very rapidly as the number of obligors,  $n$ , tends to infinity:

$$\sum_{k=1}^{\infty} \left( \frac{\delta_k}{\Delta_k} \right)^2 < \infty \implies \frac{1}{\Delta_n^2} \sum_{k=1}^n \delta_k^2 = \sum_{k=1}^n w_k^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (2.33)$$

where  $w_k$  depends on  $n$ .

*Remark 2.16.* Suppose that  $a \leq \delta_k \leq b$  where  $0 < a \leq b < \infty$  for all  $k \in \mathbb{N}$ . Then,

$$\Delta_n = \sum_{k=1}^n \delta_k \geq na \rightarrow \infty \text{ as } n \rightarrow \infty, \quad (2.34)$$

and

$$\sum_{n=1}^{\infty} \left( \frac{\delta_n}{\Delta_n} \right)^2 = \sum_{n=1}^{\infty} \frac{\delta_n^2}{(\sum_{k=1}^n \delta_k)^2} \leq \sum_{n=1}^{\infty} \frac{b^2}{(na)^2} = \frac{b^2}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \quad (2.35)$$

where (2.35) converges by the p-series test (Theorem 2.59). Assuming that EAD assigned to individual obligors is bounded, an entirely uncontroversial claim, total EAD of an asymptotic portfolio diverges, but (2.35) satisfies the definition of an asymptotic portfolio (Bluhm et al. 2010, Example 2.5.3).

*Remark 2.17.* Since LGD is bounded,  $\eta_k \in [0, 1]$  for  $k \in \mathbb{N}$ , we could define an asymptotic portfolio mathematically on the basis of infinite series  $\sum_{k=1}^{\infty} \eta_k \delta_k$ . Adopting a practitioner's rather than theoretician's orientation, we choose to retain the distinction between EAD and LGD throughout our mathematical treatment of credit risk modelling. Also, defining an asymptotic portfolio on the basis of infinite series  $\sum_{k=1}^{\infty} \delta_k$  leads to the more natural interpretation of Remark 2.15, which expresses asymptotic granularity in terms of the convergence of an infinite series whose terms are a function of exposure weight.

Many of the results in this chapter are established under different modes of convergence. Some proofs, too, rely on results established under different modes of convergence. Appendix 2.E provides a digest of definitions and theorems (and lemmas, corollaries, etc.) on the convergence of sequences of random variables and measurable functions. Gordy (2003, Proposition 1) established that, conditional on a single systematic risk factor, the portfolio percentage loss converges, almost surely (Definition 2.61), to its conditional expectation as the portfolio approaches asymptotic granularity.

**Proposition 2.18.** *Assume a conditional independence model of an asymptotic credit portfolio. Then,*

$$\lim_{n \rightarrow \infty} \left( L_n - \sum_{i=1}^n w_i \eta_i p_i(Y) \right) = 0, \quad \mathbb{P}\text{-a.s.} \quad (2.36)$$

*Proof.* It relies on a variant of the strong law of large numbers (Theorem 2.65) based on Kolmogorov's convergence criterion and Kronecker's lemma.

For a credit portfolio comprising  $n$  obligors,  $w_i = \delta_i/\Delta_n$  is the exposure weight of obligor  $i$ . Firstly, observe that given realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ , conditional variances satisfy

$$\sum_{i=1}^{\infty} \text{Var}\left(\frac{\delta_i \eta_i \mathbb{1}_{\{Z_i < \zeta_i(y)\}}}{\Delta_i}\right) = \sum_{i=1}^{\infty} \left(\frac{\delta_i}{\Delta_i}\right)^2 \eta_i^2 p_i(y)(1 - p_i(y)) \leq \sum_{i=1}^{\infty} \left(\frac{\delta_i}{\Delta_i}\right)^2 < \infty,$$

where  $\text{Var}(\mathbb{1}_{\{Z_i < \zeta_i(y)\}}) = p_i(y)(1 - p_i(y))$ . Recall that  $\mathbb{E}[\mathbb{1}_{\{Z_i < \zeta_i(y)\}}] = p_i(y)$ , and the sequence of random variables  $\{\delta_k \eta_k \mathbb{1}_{\{Z_k < \zeta_k(y)\}}/\Delta_k\}$  is independent with respect to  $\mathbb{P}_y$ . Then,

$$\sum_{i=1}^{\infty} \frac{\delta_i \eta_i}{\Delta_i} (\mathbb{1}_{\{Z_i < \zeta_i(y)\}} - p_i(y)) \quad (2.37)$$

converges  $\mathbb{P}_y$ -almost surely by Kolmogorov's convergence criterion (Theorem 2.67). Notice that  $\{\delta_k \eta_k (\mathbb{1}_{\{Z_k < \zeta_k(y)\}} - p_k(y))\}$  constitutes a sequence of random variables, and the sequence of real numbers  $\{\Delta_k\}$  is positive and strictly increasing to infinity. Next, applying Kronecker's lemma (Lemma 2.66) to infinite series (2.37) yields

$$\frac{1}{\Delta_n} \sum_{i=1}^n \delta_i \eta_i (\mathbb{1}_{\{Z_i < \zeta_i(y)\}} - p_i(y)) \xrightarrow{a.s.} 0 \text{ as } n \rightarrow \infty,$$

which implies that

$$\lim_{n \rightarrow \infty} \left( L_n - \sum_{i=1}^n w_i \eta_i p_i(y) \right) = 0,$$

$\mathbb{P}_y$ -almost surely for all  $y \in \mathbb{R}$ . Accordingly, (2.36) holds  $\mathbb{P}$ -almost surely.  $\square$

*Remark 2.19.* The sequence  $\{L_n\}_{n \in \mathbb{N}}$  is bounded, but it is not monotone. Indicator function  $\mathbb{1}_{\{Z_i < \zeta_i(y)\}} \in \{0, 1\}$  and  $\eta_i \in [0, 1]$  for  $i = 1, \dots, n$ , and  $\sum_{i=1}^n w_i = 1$  for all  $n \in \mathbb{N}$ , where  $w_i$  depends on  $n$ . The dependence of exposure weights on the number of obligors is clear when expressed as  $w_i = \delta_i / \sum_{j=1}^n \delta_j$ . Consequently,  $L_n$ , and hence  $\sum_{i=1}^n w_i \eta_i p_i(y)$ , may not converge as  $n \rightarrow \infty$ .

*Remark 2.20.* In an asymptotic portfolio idiosyncratic risk is fully diversified away, so portfolio percentage loss  $L_n$  depends only on systematic risk factor  $Y$ . Appendix 2.B illustrates the vanishing idiosyncratic risk through diversification.

*Remark 2.21.* As a practical matter, credit portfolios of large banks are typically near the asymptotic granularity of Definition 2.14. Thus, given  $Y = y$ , (2.32) provides a statistically accurate estimate of percentage loss on a portfolio containing a large number of credits without concentration in a few names dominating the rest of the portfolio.

## 2.5 Limiting Form of Portfolio Loss Distribution

Risk capital for a credit portfolio is determined from its parametric or empirical loss distribution. Assuming conditional independence given a single systematic risk factor, Vasicek (2002) derived the parametric loss distribution function of an asymptotic, homogeneous credit portfolio. In Section 2.4 we define an asymptotic portfolio, here we define a homogeneous portfolio.

**Definition 2.22.** Assume a conditional independence model of a credit portfolio. A *homogeneous portfolio* comprising  $n$  obligors satisfies:

- (1) Parameters  $\gamma_i = \gamma$  for  $i = 1, \dots, n$  in representation (2.24) of latent random variables  $W_1, \dots, W_n$ .
- (2) Random variables  $Z_1, \dots, Z_n$  are drawn from the same distribution described by the continuous and strictly increasing distribution function  $G$ . Also, denote by  $F$  the common distribution function of  $W_1, \dots, W_n$ .
- (3) Obligor are assigned equal unconditional PD and LGD, that is,  $p_i = p$  and  $\eta_i = \eta$  for  $i = 1, \dots, n$ .

*Remark 2.23.* From the properties of a homogeneous credit portfolio we infer that  $p_i(y) = p(y)$  for  $i = 1, \dots, n$ , and all realisations  $y \in \mathbb{R}$  of systematic risk factor  $Y$ .

*Remark 2.24.* A homogeneous portfolio is often defined as one where constituent credits are assigned equal dollar exposure so that  $w_i = 1/n$  for  $i = 1, \dots, n$ . Key propositions underpinning the Basel II IRB approach do not require the restrictive assumption of equal exposure weights. Rather they rely on the asymptotic granularity of Definition 2.14. Accordingly, Definition 2.22 does not place conditions on the exposure weight of constituent credits.

The following result, derived for the general case, is a corollary of Proposition 2.18.

**Corollary 2.25.** Assume a conditional independence model of an asymptotic, homogeneous credit portfolio. Then,

$$\lim_{n \rightarrow \infty} L_n = \eta p(Y), \quad \mathbb{P}\text{-a.s.} \quad (2.38)$$

Accordingly, the portfolio loss distribution satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l) = 1 - H \left( \frac{F^{-1}(p) - \sqrt{1 - \gamma^2} G^{-1}(l/\eta)}{\gamma} \right) \quad (2.39)$$

for all  $l \in (0, 1)$ .

*Proof.* For the homogeneous case of Definition 2.22, let

$$\zeta(y) = \frac{F^{-1}(p) - \gamma y}{\sqrt{1 - \gamma^2}} = G^{-1}(p(y)).$$

Then, conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ , the portfolio percentage loss is calculated as

$$L_n = \sum_{i=1}^n w_i \eta \mathbb{1}_{\{Z_i < \zeta(y)\}}, \quad (2.40)$$

where  $\mathbb{1}_{\{Z_i < \zeta(y)\}}$  are independent identically distributed Bernoulli random variables with finite mean and variance. By the strong law of large numbers (Theorem 2.65),  $\mathbb{1}_{\{Z_i < \zeta(y)\}}$  converges to its conditional expectation  $p(y)$  as  $n \rightarrow \infty$ ,  $\mathbb{P}_y$ -almost surely for all  $y \in \mathbb{R}$ . Observing that  $\sum_{i=1}^n w_i = 1$  for all  $n \in \mathbb{N}$ , and  $\eta \in [0, 1]$ , (2.38) holds  $\mathbb{P}$ -almost surely.

It follows from (2.38) that given  $Y = y$ ,

$$\lim_{n \rightarrow \infty} L_n = \eta p(y),$$

which is a deterministic quantity. Hence,

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l \mid Y = y) = \mathbb{1}_{\{0 < \eta p(y) \leq l\}} = \mathbb{1}_{\{p^{-1}(l/\eta) \leq y < \infty\}}.$$

Then, integrating over systematic risk factor  $Y$  with density function  $h(y)$ , yields the limiting form of the portfolio loss distribution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l) &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \mathbb{P}(L_n \leq l, Y = y) dy = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \mathbb{P}(L_n \leq l \mid Y = y) h(y) dy \\ &= \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l \mid Y = y) h(y) dy = \int_{-\infty}^{\infty} \mathbb{1}_{\{p^{-1}(l/\eta) \leq y < \infty\}} h(y) dy \\ &= \int_{p^{-1}(l/\eta)}^{\infty} h(y) dy = 1 - \int_{-\infty}^{p^{-1}(l/\eta)} h(y) dy \\ &= 1 - H(p^{-1}(l/\eta)) = 1 - H\left(\frac{F^{-1}(p) - \sqrt{1 - \gamma^2} G^{-1}(l/\eta)}{\gamma}\right). \end{aligned}$$

The dominated convergence theorem (Theorem 2.63) provides conditions under which the limit of integrals of a sequence of functions is the integral of the limiting function. It justifies the third equality, while the last equality follows from (2.29).  $\square$

The following corollary, derived by Vasicek (2002), deals with the special case of an asymptotic, homogeneous portfolio in which default dependence is modelled as a multivariate Gaussian process. We present an alternative proof for the Gaussian case, rather than appealing to Corollary 2.25 and directly substituting into (2.39).

**Corollary 2.26.** *Assume a Gaussian conditional independence model of an asymptotic, homogeneous credit portfolio. Then, the portfolio loss distribution satisfies*

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l) = \Phi\left(\frac{\sqrt{1 - \rho} \Phi^{-1}(l/\eta) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) \quad (2.41)$$

for all  $l \in (0, 1)$ .

*Proof.* In the homogeneous case (2.19) is recast as

$$W_i = \sqrt{\rho} Y + \sqrt{1 - \rho} Z_i.$$

Reformulating (2.20) for the homogeneous case, the PD of any obligor conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$  is given by

$$p(y) = \Phi \left( \frac{\Phi^{-1}(p) - \sqrt{\rho} y}{\sqrt{1 - \rho}} \right).$$

Its inverse is the Gaussian case of the homogeneous version of (2.29). More precisely,

$$y = p^{-1}(x) = \frac{\Phi^{-1}(p) - \sqrt{1 - \rho} \Phi^{-1}(x)}{\sqrt{\rho}} \quad (2.42)$$

for all  $x \in (0, 1)$ . The limiting form of the portfolio loss distribution is deducible:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq l) &= \mathbb{P} \left( \lim_{n \rightarrow \infty} L_n \leq l \right) = \mathbb{P}(\eta p(Y) \leq l) \\ &= \mathbb{P}(Y \geq p^{-1}(l/\eta)) = \mathbb{P}(Y \leq -p^{-1}(l/\eta)) \\ &= \Phi \left( -p^{-1}(l/\eta) \right) = \Phi \left( \frac{\sqrt{1 - \rho} \Phi^{-1}(l/\eta) - \Phi^{-1}(p)}{\sqrt{\rho}} \right). \end{aligned}$$

Since the inverse function  $p_i^{-1}: (0, 1) \rightarrow \mathbb{R}$  is strictly decreasing, the inequality sign in the probability statement is reversed upon its application. The fourth equality is justified by the symmetry of the standard Gaussian density function, and the last equality follows from (2.42).  $\square$

## 2.6 Credit Value-at-Risk

In determining regulatory capital, the Basel II IRB approach applies a risk measure to assign a single numerical value to a random credit loss. The chosen risk measure is VaR (Definition 1.8), one of the most widely used measures in risk management. VaR is an extreme quantile of a loss, or P&L, distribution that is rarely exceeded. Adopting the convention that a loss is a positive number, which differs from that of Definition 1.8, we now define credit VaR.

**Definition 2.27.** *Credit VaR* at the confidence level  $\alpha \in (0, 1)$  over a given risk measurement horizon is the largest portfolio percentage loss  $l$  such that the probability of a loss  $L_n$  exceeding  $l$  is at most  $(1 - \alpha)$ :

$$\text{VaR}_\alpha(L_n) = \inf \{ l \in \mathbb{R} : \mathbb{P}(L_n > l) \leq 1 - \alpha \}. \quad (2.43)$$

In probabilistic terms,  $\text{VaR}_\alpha(L_n)$  is simply the  $\alpha$  quantile (Definition 1.6) of the portfolio loss distribution. Although computationally expensive, Monte Carlo simulation is routinely employed to generate the empirical loss distribution and determine VaR of a credit portfolio. Suppose that we generate the loss distribution of a credit portfolio comprising  $n$  obligors by simulation of (2.26) parameterised by (2.27). Let Monte Carlo simulation perform  $N$  iterations. For each iteration we draw from their respective distributions random variable  $Y$  representing systematic

risk, and random variables  $Z_1, \dots, Z_n$  representing obligor specific risks. Then, conditional on realisation  $y_k \in \mathbb{R}$  of systematic risk factor  $Y$  describing a scenario of the economy, the portfolio percentage loss over the risk measurement horizon is computed as

$$L_{n,k} = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{Z_{i,k} < \zeta_i(y_k)\}} \quad (2.44)$$

for iterations  $k = 1, \dots, N$ . Monte Carlo simulation computes  $N$  portfolio percentage losses constituting the empirical loss distribution described by the function (Bluhm et al. 2010, pp. 30–32):

$$F(l) = \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{0 \leq L_{n,k} \leq l\}}. \quad (2.45)$$

$\text{VaR}_\alpha(L_n)$ , the  $\alpha$  quantile of the empirical loss distribution, is the maximum credit loss at the  $\alpha$  confidence level over a given risk measurement horizon. Expected loss is estimated by calculating the average portfolio percentage loss over  $N$  iterations of the simulation:

$$\mathbb{E}[L_n] = \frac{1}{N} \sum_{k=1}^N L_{n,k}. \quad (2.46)$$

An analytical model of the portfolio loss distribution, on the other hand, facilitates the fast calculation of credit VaR. In the limiting case of an asymptotic, homogeneous credit portfolio,  $\text{VaR}_\alpha(L_n)$  may be determined analytically from distribution function (2.39). However, the assumptions of Corollary 2.25 are too restrictive for real-world credit portfolios. The risk factor model for ratings-based capital charges derived by Gordy (2003) relaxes the homogeneity assumption. His analysis proceeds assuming that:

- (1) Portfolios are infinitely fine-grained so that idiosyncratic risk is fully diversified away.
- (2) A single systematic risk factor explains dependence across obligors.

Under these weaker assumptions, and subject to additional technical conditions, Gordy established that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles of the portfolio loss distribution. The statement and proof of Proposition 5 of Gordy (2003), which leads to an analytical approximation of credit VaR, is relegated to Appendix 2.A. Here, we present a version of this proposition that relaxes the additional technical conditions imposed by Gordy, resulting in a more economical, or parsimonious, proof.

**Proposition 2.28.** *Assume a conditional independence model of a credit portfolio comprising  $n$  obligors. Denote by  $\varphi_n(y)$  the conditional expectation function  $\mathbb{E}[L_n | y]: \mathbb{R} \rightarrow (0, 1)$  given by (2.32), and assume that the sequence  $\{\varphi_n\}_{n \in \mathbb{N}}$  of real-valued functions satisfies:*

(1) For every  $n \in \mathbb{N}$ , function  $\varphi_n$  is strictly monotonic.

(2) For every  $y \in \mathbb{R}$  and  $\varepsilon > 0$  there is a  $\bar{\delta}(\varepsilon) \in \mathbb{R} \setminus \{0\}$  and  $N(\bar{\delta}, \varepsilon) \in \mathbb{N}$  such that  $n > N(\bar{\delta}, \varepsilon)$  implies

$$[\varphi_n(y) - \varepsilon, \varphi_n(y) + \varepsilon] \subset [\varphi_n(y - \bar{\delta}(\varepsilon)), \varphi_n(y + \bar{\delta}(\varepsilon))],$$

where  $\bar{\delta}$  depends on  $\varepsilon$ , and  $N$  depends on  $\bar{\delta}$  and  $\varepsilon$ , in general. While  $\bar{\delta}$  depends on  $\varepsilon$ , and may also depend on  $y$ , we assume that it is independent of  $n$ .

(3) For every  $\xi > 0$  there is an  $\varepsilon > 0$  such that  $0 < |\bar{\delta}(\varepsilon)| < \xi$ , that is,  $\bar{\delta}(\varepsilon)$  tends to zero as  $\varepsilon$  tends to zero.

(4) For every  $y \in \mathbb{R}$  and  $\varepsilon > 0$  there is a  $\underline{\delta}(\varepsilon) \in \mathbb{R} \setminus \{0\}$  and  $N(\underline{\delta}, \varepsilon) \in \mathbb{N}$  such that  $n > N(\underline{\delta}, \varepsilon)$  implies

$$[\varphi_n(y - \underline{\delta}(\varepsilon)), \varphi_n(y + \underline{\delta}(\varepsilon))] \subset [\varphi_n(y) - \varepsilon, \varphi_n(y) + \varepsilon],$$

where  $\underline{\delta}$  is independent of  $n$ .

Then,

$$\lim_{n \rightarrow \infty} (L_n - \varphi_n(y)) = 0, \mathbb{P}\text{-a.s.} \quad \Rightarrow \quad \lim_{n \rightarrow \infty} |q_\alpha(L_n) - \varphi_n(q_{1-\alpha}(Y))| = 0 \quad (2.47)$$

for all  $\alpha \in (0, 1)$ .

*Remark 2.29.* We argue that Conditions (1)–(4) of Proposition 2.28 are quite reasonable assumptions for real-world credit portfolios. Observe that the conditional expectation function  $\varphi_n$ , given by (2.32), is continuous and strictly decreasing in  $y$  by Remark 2.13. Therefore, it satisfies Condition (1), which is also a condition of Lemma 2.31 and Corollary 2.32, both used in the proof of Proposition 2.28.

For Conditions (2)–(4) to hold, the curve of  $\varphi_n$  cannot have horizontal or vertical segments in the neighbourhood of  $q_{1-\alpha}(Y)$ . This is guaranteed by constraints  $\bar{\delta}(\varepsilon) \in \mathbb{R} \setminus \{0\}$ ,  $0 < |\bar{\delta}(\varepsilon)| < \xi$ , and  $\underline{\delta}(\varepsilon) \in \mathbb{R} \setminus \{0\}$ . By inspection of (2.28), Conditions (2)–(4) are satisfied if  $p_i$  and  $\gamma_i^2$  are bounded away from zero and one for  $i = 1, \dots, n$ . Otherwise,  $\varphi_n$  would no longer depend on  $y$ , thus violating Condition (1). As a practical matter, if  $p_i$  were equal to zero, then the capital charge assessed on credit  $i$  would be zero; and if  $p_i$  were equal to one, then the product of EAD and LGD assigned to obligor  $i$  would be charged against profit and loss.

*Remark 2.30.* If on some open interval  $I$  containing  $q_{1-\alpha}(Y)$ ,  $\varphi_n$  were also differentiable on  $I$ , then Conditions (2)–(4) would be satisfied if

$$-\infty < -\Theta \leq \varphi'_n(y) \leq -\theta < 0$$

for all  $y \in I$ , with  $\theta > 0$  and  $\Theta > 0$  independent of  $n$ . Indeed, Proposition 2.34 (Gordy 2003, Proposition 5) assumes that this condition holds on an open interval  $I$  containing  $q_{1-\alpha}(Y)$ .

Recall that  $\text{VaR}_\alpha(L_n) = \mathbf{q}_\alpha(L_n)$ . So, Proposition 2.28 asserts that the  $\alpha$  quantile of the distribution of  $\mathbb{E}[L_n | Y]$ , which is associated with the  $(1-\alpha)$  quantile of the distribution of  $Y$ , may be substituted for the  $\alpha$  quantile of the distribution of  $L_n$  (i.e., credit VaR at the  $\alpha$  confidence level over a given risk measurement horizon). The IRB approach rests on Proposition 2.28. Its proof, presented below, relies on the following lemmas and corollary. Note that in this section,  $F_1, \dots, F_n$  denote a sequence of distribution functions, as distinct from the notation adopted in Section 2.3.

**Lemma 2.31.** *Let  $\{g_n\}_{n \in \mathbb{N}}$  be a sequence of real-valued functions  $g_n: \mathbb{R} \rightarrow \mathbb{R}$  that satisfies Conditions (1)–(3) of Proposition 2.28. Then, for every  $b \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} (a_n - g_n(b)) = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} g_n^{-1}(a_n) = b. \quad (2.48)$$

*Proof.* The sufficient condition for the conclusion in (2.48) states that for every  $\varepsilon > 0$  there is an  $N_0(\varepsilon) \in \mathbb{N}$  such that  $n > N_0(\varepsilon)$  implies

$$|a_n - g_n(b)| \leq \varepsilon,$$

which may be expressed as

$$a_n \in [g_n(b) - \varepsilon, g_n(b) + \varepsilon] \subset [g_n(b - \bar{\delta}(\varepsilon)), g_n(b + \bar{\delta}(\varepsilon))],$$

where the subset relation holds when Condition (2) is satisfied. Observe that since  $g_n$  is strictly monotonic for every  $n \in \mathbb{N}$  by hypothesis,  $g_n$  is one-to-one and  $g_n^{-1}(g_n(b)) = b$ . Also,  $\bar{\delta}(\varepsilon) > 0$  if  $g_n$  is strictly increasing, and  $\bar{\delta}(\varepsilon) < 0$  if  $g_n$  is strictly decreasing. Then, an application of the inverse function  $g_n^{-1}$  yields

$$g_n^{-1}(a_n) \in [b - |\bar{\delta}(\varepsilon)|, b + |\bar{\delta}(\varepsilon)|] \subset [b - \xi, b + \xi],$$

where the subset relation holds when Condition (3) is satisfied. Choosing  $N(\bar{\delta}, \varepsilon) = N_0(\varepsilon)$  such that  $\xi$  is arbitrarily close to zero establishes the necessary condition of the hypothesis in (2.48).  $\square$

**Corollary 2.32.** *Let  $X_n$  and  $Y$  be random variables defined on a common probability space with distribution functions  $F_n$  and  $H$ , respectively. If a sequence  $\{g_n\}_{n \in \mathbb{N}}$  of real-valued functions  $g_n: \mathbb{R} \rightarrow \mathbb{R}$  satisfies Conditions (1)–(3) of Proposition 2.28, then*

$$\lim_{n \rightarrow \infty} (X_n - g_n(y)) = 0, \mathbb{P}\text{-a.s.} \quad \Rightarrow \quad \lim_{n \rightarrow \infty} g_n^{-1}(X_n) = Y. \quad (2.49)$$

Moreover, if  $H$  is continuous, then for every realisation  $y \in \mathbb{R}$  of  $Y$ ,

$$\lim_{n \rightarrow \infty} F_n(g_n(y)) = H(y) \quad (2.50)$$

when functions  $g_n$  are strictly increasing, and

$$\lim_{n \rightarrow \infty} F_n(g_n(y)) = 1 - H(y) \quad (2.51)$$

when functions  $g_n$  are strictly decreasing.



*Proof.* Random variables are real-valued functions on some probability space, so (2.49) is an immediate consequence of Lemma 2.31.

The almost sure convergence of the sufficient condition for the conclusion in (2.49) implies pointwise convergence (Definition 2.60) of the sequence of distribution functions of  $g_n^{-1}(X_n)$  to the distribution function of  $Y$  at every point of continuity of  $H$ . If  $H$  is continuous, then convergence occurs for every realisation  $y \in \mathbb{R}$  of  $Y$ . It follows from the necessary condition of the hypothesis in (2.49) that

$$\mathbb{P}(g_n^{-1}(X_n) \leq y) = \mathbb{P}(X_n \leq g_n(y)) = F_n(g_n(y))$$

converges to  $\mathbb{P}(Y \leq y) = H(y)$  as  $n \rightarrow \infty$  if functions  $g_n$  are strictly increasing, which establishes (2.50). Similarly,

$$\mathbb{P}(g_n^{-1}(X_n) \leq y) = \mathbb{P}(X_n \geq g_n(y)) = 1 - \mathbb{P}(X_n \leq g_n(y)) = 1 - F_n(g_n(y))$$

converges to  $\mathbb{P}(Y \leq y) = H(y)$  as  $n \rightarrow \infty$  if functions  $g_n$  are strictly decreasing, which establishes (2.51).  $\square$

**Lemma 2.33.** *Let  $Y$  be a random variable with continuous and strictly increasing distribution function  $H$ , and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a strictly monotonic function. Then, the  $\alpha$  quantile of the distribution function of  $g(Y)$  is*

$$q_\alpha(g(Y)) = g(H^{-1}(\alpha)) \quad (2.52)$$

*if  $g$  is strictly increasing, and*

$$q_\alpha(g(Y)) = g(H^{-1}(1 - \alpha)) \quad (2.53)$$

*if  $g$  is strictly decreasing.*

*Proof.* By Definition 1.6, the  $\alpha$  quantile of  $g(Y)$  is

$$q_\alpha(g(Y)) = \inf \{g(y) \in \mathbb{R} : \mathbb{P}(g(Y) \leq g(y)) \geq \alpha\}. \quad (2.54)$$

Since  $H$  is continuous and strictly increasing by hypothesis, the  $\alpha$  quantile of the distribution of  $Y$  is given by

$$q_\alpha(Y) = H^{-1}(\alpha), \quad (2.55)$$

where  $H^{-1}(\alpha)$  is the inverse distribution function evaluated at  $\alpha$  (Remark 1.7). Observe that if  $g$  is strictly increasing, then

$$\begin{aligned} \mathbb{P}(g(Y) \leq g(q_\alpha(Y))) &= \mathbb{P}(g(Y) \leq g(H^{-1}(\alpha))) \\ &= \mathbb{P}(Y \leq H^{-1}(\alpha)) \\ &= \alpha, \end{aligned}$$

where the first equality is a consequence of (2.55), and the second equality is the result of an application of inverse function  $g^{-1}$ . Hence, (2.52) follows from (2.54). By a parallel argument, if  $g$  is strictly decreasing, then

$$\begin{aligned}\mathbb{P}(g(Y) \leq g(q_{1-\alpha}(Y))) &= \mathbb{P}(g(Y) \leq g(H^{-1}(1-\alpha))) \\ &= \mathbb{P}(Y \geq H^{-1}(1-\alpha)) \\ &= 1 - \mathbb{P}(Y \leq H^{-1}(1-\alpha)) \\ &= \alpha,\end{aligned}$$

which establishes (2.53).  $\square$

*Proof of Proposition 2.28.* Fix  $\alpha \in (0, 1)$ , set  $\varphi_n(y) = q_\alpha(\varphi_n(Y))$ , and denote by  $F_n$  the distribution function of  $L_n$ . By appealing to results due to Corollary 2.32 and Lemma 2.33 for strictly decreasing functions  $\varphi_n$ , observe that

$$\begin{aligned}\lim_{n \rightarrow \infty} F_n(q_\alpha(\varphi_n(Y))) &= \lim_{n \rightarrow \infty} F_n(\varphi_n(H^{-1}(1-\alpha))) = \lim_{n \rightarrow \infty} F_n(\varphi_n(q_{1-\alpha}(Y))) \\ &= 1 - H(q_{1-\alpha}(Y)) = 1 - H(H^{-1}(1-\alpha)) = \alpha.\end{aligned}\tag{2.56}$$

The first equality follows from (2.53), the second from (2.55), the third from (2.51), and the fourth from (2.55) again. Notice that  $\underline{\delta}(\varepsilon) > 0$  if  $\varphi_n$  is strictly increasing, and  $\underline{\delta}(\varepsilon) < 0$  if  $\varphi_n$  is strictly decreasing. By Remark 2.13,  $\varphi_n(y) = \mathbb{E}[L_n | y]$  is strictly decreasing, thus satisfying Condition (1). Then, on the basis of (2.56) and subject to Condition (4), lower and upper bounds, respectively, on the  $\alpha$  quantile of  $L_n$  are deduced:

$$\lim_{n \rightarrow \infty} F_n(\varphi_n(q_{1-\alpha}(Y) - \underline{\delta}(\varepsilon))) = 1 - H(q_{1-\alpha}(Y) + |\underline{\delta}(\varepsilon)|) < \alpha,$$

and

$$\lim_{n \rightarrow \infty} F_n(\varphi_n(q_{1-\alpha}(Y) + \underline{\delta}(\varepsilon))) = 1 - H(q_{1-\alpha}(Y) - |\underline{\delta}(\varepsilon)|) > \alpha,$$

which may be expressed as

$$q_\alpha(L_n) \in [\varphi_n(q_{1-\alpha}(Y) - \underline{\delta}(\varepsilon)), \varphi_n(q_{1-\alpha}(Y) + \underline{\delta}(\varepsilon))].$$

Finally, by Condition (4), for every  $\varepsilon > 0$  there is a  $\underline{\delta}(\varepsilon) \in \mathbb{R}$  and  $N(\underline{\delta}, \varepsilon) \in \mathbb{N}$  such that  $n > N(\underline{\delta}, \varepsilon)$  implies

$$q_\alpha(L_n) \in [\varphi_n(q_{1-\alpha}(Y)) - \varepsilon, \varphi_n(q_{1-\alpha}(Y)) + \varepsilon],$$

which establishes the necessary condition of the hypothesis in (2.47).  $\square$

## 2.A Proposition due to Gordy (2003, Proposition 5)

The Basel II IRB approach rests on Proposition 5 of Gordy (2003) that quantiles of the distribution of conditional expectation of portfolio percentage loss may be substituted for quantiles

of the portfolio loss distribution. It leads to an analytical approximation of credit VaR. In Section 2.6 we present a version of this proposition that relaxes the technical conditions imposed by Gordy, resulting in a more compact proof. Here, the statement and proof of Proposition 2.34 closely follow that of Gordy (2003).

**Proposition 2.34.** *Consider a credit portfolio comprising  $n$  obligors, and denote by  $L_n$  the portfolio percentage loss. Let  $Y$  be a random variable with continuous and strictly increasing distribution function  $H$ , and denote by  $\varphi_n(Y)$  the conditional expectation of portfolio percentage loss  $\mathbb{E}[L_n | Y]$ . Assume that the following conditions hold:*

- (1)  $\lim_{n \rightarrow \infty} \left( L_n - \sum_{i=1}^n \varphi_n(Y) \right) = 0, \quad \mathbb{P}\text{-a.s.}$
- (2) *There is an open interval  $I$  containing  $H^{-1}(1-\alpha)$ ,  $\alpha \in (0, 1)$ , and  $N_0 \in \mathbb{N}$  such that whenever  $n > N_0$  the conditional expectation of portfolio percentage loss,  $\varphi_n(y) = \mathbb{E}[L_n | Y = y]$ , is strictly decreasing in  $y$  and differentiable on  $I$ .*
- (3) *There is an  $N_0 \in \mathbb{N}$  such that whenever  $n > N_0$ ,*

$$-\infty < -\Theta \leq \varphi'_n(y) \leq -\theta < 0 \quad (2.57)$$

*for all  $y \in I$ , with  $\theta > 0$  and  $\Theta > 0$  independent of  $n$ , and where  $\varphi'_n(y)$  denotes the derivative of  $\mathbb{E}[L_n | Y = y]$  with respect to  $y$ .*

Then,

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_n \leq \varphi_n(H^{-1}(1-\alpha))) = \alpha, \quad (2.58)$$

and

$$\lim_{n \rightarrow \infty} |\text{VaR}_\alpha(L_n) - \varphi_n(H^{-1}(1-\alpha))| = 0. \quad (2.59)$$

*Remark 2.35.* Proposition 2.34 applies more generally than to the conditional independence model of Definition 2.10. However, we are not aware of competing models for which the conditional expectation function  $\varphi_n$  satisfies Condition (1).

*Remark 2.36.* Condition (1) of Proposition 2.34 postulates that portfolio percentage loss converges, almost surely, to its conditional expectation as the portfolio approaches asymptotic granularity. The proof of Proposition 2.34 only requires convergence in probability, but we assume almost sure convergence consistent with Proposition 2.18. For an asymptotic credit portfolio (Definition 2.14), the conditional independence model of Definition 2.10 satisfies Condition (1) by Proposition 2.18.

Conditions (2) and (3) are more technical in nature. Condition (2) postulates that the conditional expectation of portfolio percentage loss rises “smoothly” as the economy deteriorates for a set of states of the economy associated with the tail of the portfolio loss distribution, which is the concern of solvency assessment. In relation to the conditional independence model of Definition 2.10, we quite reasonably assume that the conditional expectation function  $\varphi_n$

satisfies Condition (2). By inspection of (2.28) we observe that Condition (3) holds if  $p_i$  and  $\gamma_i^2$  are bounded away from zero and one for  $i = 1, \dots, n$ .

The proof of Proposition 2.34, presented below, requires a result due to Petrov (1995), which we proceed to state and prove. Note that in Lemma 2.38, variables  $X$  and  $Z$  and functions  $F$  and  $G$  denote arbitrary random variables and distribution functions, and in Proposition 2.34,  $F_n$  and  $G_n$  denote distribution functions distinct from the notation adopted in Section 2.3. Also, the proof of Proposition 2.34 uses the mean value theorem, which we state here for completeness and convenience.

**Theorem 2.37** (Mean value [Wade 2004, Theorem 4.15]). *If function  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  such that*

$$f(b) - f(a) = f'(c)(b - a).$$

**Lemma 2.38.** *Let  $X$  and  $Z$  be random variables defined on a common probability space with distribution functions  $F$  and  $G$ , respectively. For all  $a \in \mathbb{R}$  and  $\varepsilon > 0$ ,*

$$|F(a) - G(a)| \leq \mathbb{P}(|X - Z| > \varepsilon) + \max\{G(a + \varepsilon) - G(a), G(a) - G(a - \varepsilon)\}. \quad (2.60)$$

*Proof.* For all  $a, b \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbb{P}(X < a) &= \mathbb{P}(X < a, Z < a + b) + \mathbb{P}(X < a, Z \geq a + b) \\ &= \mathbb{P}(X < a, Z < a + b) + \mathbb{P}(X < a, a - Z \leq -b) \\ &\leq \mathbb{P}(Z < a + b) + \mathbb{P}(X - Z < -b). \end{aligned}$$

Substituting  $\varepsilon > 0$  for  $b$ , it follows that

$$\mathbb{P}(X < a) \leq \mathbb{P}(Z < a + \varepsilon) + \mathbb{P}(|X - Z| > \varepsilon).$$

Then, subtracting  $\mathbb{P}(Z < a)$  from both sides of the inequality and rearranging yields

$$\begin{aligned} \mathbb{P}(X < a) - \mathbb{P}(Z < a) &\leq \mathbb{P}(|X - Z| > \varepsilon) + \mathbb{P}(Z < a + \varepsilon) - \mathbb{P}(Z < a), \\ F(a) - G(a) &\leq \mathbb{P}(|X - Z| > \varepsilon) + G(a + \varepsilon) - G(a). \end{aligned} \quad (2.61)$$

By a similar argument, we have

$$\begin{aligned} \mathbb{P}(Z < a - b) &= \mathbb{P}(Z < a - b, X < a) + \mathbb{P}(Z < a - b, X \geq a) \\ &= \mathbb{P}(Z < a - b, X < a) + \mathbb{P}(a - Z > b, X \geq a) \\ &\leq \mathbb{P}(X < a) + \mathbb{P}(X - Z > b). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}(Z < a - \varepsilon) &\leq \mathbb{P}(X < a) + \mathbb{P}(|X - Z| > \varepsilon), \\ \mathbb{P}(Z < a) - \mathbb{P}(X < a) &\leq \mathbb{P}(|X - Z| > \varepsilon) + \mathbb{P}(Z < a) - \mathbb{P}(Z < a - \varepsilon), \\ -(F(a) - G(a)) &\leq \mathbb{P}(|X - Z| > \varepsilon) + G(a) - G(a - \varepsilon). \end{aligned} \quad (2.62)$$

Thus, (2.60) follows from (2.61) and (2.62).  $\square$

*Proof of Proposition 2.34.* Denote by  $F_n$  and  $G_n$  the distribution functions of  $L_n$  and  $\varphi_n(Y)$ , respectively. Firstly, by appealing to Lemma 2.38, we show that

$$|F_n(\varphi_n(y)) - G_n(\varphi_n(y))| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for all  $y \in I$ , and deduce (2.58). Set  $X = L_n$ ,  $Z = \varphi_n(Y)$  and  $a = \varphi_n(y)$  in Lemma 2.38. Then, for any  $\varepsilon > 0$ ,

$$\begin{aligned} & |F_n(\varphi_n(y)) - G_n(\varphi_n(y))| \\ & \leq \mathbb{P}(|L_n - \varphi_n(Y)| > \varepsilon) \\ & + \max\{G_n(\varphi_n(y) + \varepsilon) - G_n(\varphi_n(y)), G_n(\varphi_n(y)) - G_n(\varphi_n(y) - \varepsilon)\}. \end{aligned} \quad (2.63)$$

The almost sure convergence asserted by Condition (1) implies convergence in probability (Theorem 2.62). By the definition of convergence in probability (Definition 2.61), for any  $\xi > 0$  and  $\varepsilon > 0$ , choose an  $N_1 \in \mathbb{N}$  (which in general depends on  $\varepsilon$  and  $\xi$ ) such that  $n > N_1$  implies

$$\mathbb{P}(|L_n - \varphi_n(Y)| > \varepsilon) < \frac{\xi}{2} \quad (2.64)$$

for all  $y \in I$ .

The convergence of  $G_n$  in the neighbourhood of  $\varphi_n(H^{-1}(1-\alpha))$  does not immediately follow from the assumption that  $\varphi_n(y)$  is differentiable on  $I$ . By hypothesis,  $\varphi_n(y)$  is differentiable on  $I$  and  $\varphi'_n(y)$  has an upper bound  $-\theta$  on  $I$  whenever  $n > N_0$ . By the mean value theorem, for any  $\varepsilon > 0$  satisfying  $(y - \varepsilon/\theta, y + \varepsilon/\theta) \subset I$ , there is a  $y^* \in (y - \varepsilon/\theta, y)$  such that

$$\varphi_n(y - \varepsilon/\theta) - \varphi_n(y) = \varphi'_n(y^*)(-\varepsilon/\theta) \geq -\theta(-\varepsilon/\theta) = \varepsilon$$

for  $n > N_0$ . Similarly, there is a  $y^* \in (y, y + \varepsilon/\theta)$  such that

$$\varphi_n(y) - \varphi_n(y + \varepsilon/\theta) = \varphi'_n(y^*)(-\varepsilon/\theta) \geq -\theta(-\varepsilon/\theta) = \varepsilon$$

for  $n > N_0$ . By hypothesis,  $\varphi_n(y)$  is strictly decreasing in  $y$  on  $I$  whenever  $n > N_0$ , implying that  $\varphi_n(Y) \leq \varphi_n(y)$  if and only if  $Y \geq y$ . Moreover,  $H$  is continuous and strictly increasing. Accordingly,

$$G_n(\varphi_n(y)) = \mathbb{P}(Y \geq y) = 1 - H(y). \quad (2.65)$$

Hence,

$$G_n(\varphi_n(y) + \varepsilon) \leq G_n(\varphi_n(y - \varepsilon/\theta)) = \mathbb{P}(Y \geq y - \varepsilon/\theta)$$

and

$$G_n(\varphi_n(y) - \varepsilon) \geq G_n(\varphi_n(y + \varepsilon/\theta)) = \mathbb{P}(Y \geq y + \varepsilon/\theta)$$

for  $n > N_0$ . It follows that the continuity of  $H(y)$  on  $I$  implies convergence. That is, for any  $\xi > 0$  and  $\varepsilon > 0$ , there is an  $N_0 \in \mathbb{N}$  (which in general depends on  $\varepsilon$  and  $\xi$ ) such that  $n > N_0$

implies

$$\begin{aligned}
 & \max \{G_n(\varphi_n(y) + \varepsilon) - G_n(\varphi_n(y)), G_n(\varphi_n(y)) - G_n(\varphi_n(y) - \varepsilon)\} \\
 & \leq \max \{\mathbb{P}(Y \geq y - \varepsilon/\theta) - \mathbb{P}(Y \geq y), \mathbb{P}(Y \geq y) - \mathbb{P}(Y \geq y + \varepsilon/\theta)\} \\
 & = \max \{H(y) - H(y - \varepsilon/\theta), H(y + \varepsilon/\theta) - H(y)\} < \frac{\xi}{2}
 \end{aligned} \tag{2.66}$$

for any  $y \in I$ . Combining (2.64) and (2.66) to evaluate (2.63), we claim that for any  $\xi > 0$ , whenever  $n > N = \max\{N_0, N_1\}$ ,

$$|F_n(\varphi_n(y)) - G_n(\varphi_n(y))| = |\mathbb{P}(L_n \leq \varphi_n(y)) - \mathbb{P}(\varphi_n(Y) \leq \varphi_n(y))| < \frac{\xi}{2} + \frac{\xi}{2} = \xi \tag{2.67}$$

for any  $y \in I$ . Substituting for  $G_n(\varphi_n(y))$  in (2.67) from (2.65), and setting  $y = H^{-1}(1-\alpha) \in I$ , (2.67) may be expressed as

$$\lim_{n \rightarrow \infty} |\mathbb{P}(L_n \leq \varphi_n(H^{-1}(1-\alpha))) - \mathbb{P}(Y \geq H^{-1}(1-\alpha))| = 0. \tag{2.68}$$

Now, observing that  $\mathbb{P}(Y \geq H^{-1}(1-\alpha)) = \alpha$  establishes (2.58), and completes the first part of the proof.

It remains to deduce (2.59). Notice that whenever  $n > N$ ,  $(y - \varepsilon/\Theta, y + \varepsilon/\Theta) \subseteq (y - \varepsilon/\theta, y + \varepsilon/\theta)$  and  $\varphi'_n(y)$  has a lower bound  $-\Theta$  on  $I$ . Again, by the mean value theorem, for any  $\varepsilon > 0$  satisfying  $(y - \varepsilon/\Theta, y + \varepsilon/\Theta) \subset I$ , there is a  $y^* \in (y - \varepsilon/\Theta, y)$  such that

$$\varphi_n(y - \varepsilon/\Theta) - \varphi_n(y) = \varphi'_n(y^*)(-\varepsilon/\Theta) \leq -\Theta(-\varepsilon/\Theta) = \varepsilon$$

for  $n > N$ . Similarly, there is a  $y^* \in (y, y + \varepsilon/\Theta)$  such that

$$\varphi_n(y) - \varphi_n(y + \varepsilon/\Theta) = \varphi'_n(y^*)(-\varepsilon/\Theta) \leq -\Theta(-\varepsilon/\Theta) = \varepsilon$$

for  $n > N$ . Setting  $y = H^{-1}(1-\alpha) \in I$ , corresponding to the  $\alpha$  confidence level in Definition 2.27, yields

$$\begin{aligned}
 F_n(\varphi_n(H^{-1}(1-\alpha)) + \varepsilon) & \geq F_n(\varphi_n(H^{-1}(1-\alpha) - \varepsilon/\Theta)) \\
 & = G_n(\varphi_n(H^{-1}(1-\alpha) - \varepsilon/\Theta)) \\
 & = \mathbb{P}(Y \geq H^{-1}(1-\alpha) - \varepsilon/\Theta) \\
 & > \alpha.
 \end{aligned} \tag{2.69}$$

The first equality is a consequence of (2.68), the second of (2.65). It follows from Definition 2.27 and (2.69) that whenever  $n > N$ ,

$$\text{VaR}_\alpha(L_n) < \varphi_n(H^{-1}(1-\alpha)) + \varepsilon. \tag{2.70}$$

By a parallel argument, whenever  $n > N$ ,

$$\begin{aligned}
 F_n(\varphi_n(H^{-1}(1-\alpha)) - \varepsilon) & \leq F_n(\varphi_n(H^{-1}(1-\alpha) + \varepsilon/\Theta)) \\
 & = G_n(\varphi_n(H^{-1}(1-\alpha) + \varepsilon/\Theta)) \\
 & = \mathbb{P}(Y \geq H^{-1}(1-\alpha) + \varepsilon/\Theta) \\
 & < \alpha,
 \end{aligned} \tag{2.71}$$

and

$$\text{VaR}_\alpha(L_n) > \varphi_n(H^{-1}(1-\alpha)) - \varepsilon. \quad (2.72)$$

Together, (2.70) and (2.72) may be expressed as

$$|\text{VaR}_\alpha(L_n) - \varphi_n(H^{-1}(1-\alpha))| < \varepsilon.$$

Setting  $\varepsilon$  arbitrarily close to zero establishes (2.59), and completes the proof.  $\square$

## 2.B Vanishing Idiosyncratic Risk Through Diversification

The Basel II IRB approach rests on Proposition 2.28, which assumes, inter alia, that portfolios are infinitely fine-grained so that idiosyncratic risk is fully diversified away, and a single systematic risk factor explains dependence across obligors. Under these assumptions, idiosyncratic risk vanishes and portfolio percentage loss depends only on systematic risk (Luenberger 1998, pp. 200–201). We demonstrate this proposition using returns on asset values modelled by geometric Brownian motion, and the Gaussian conditional independence model of a credit portfolio that forms the basis of the IRB approach. It is extendable to the general setting adopted in Definition 2.10, in which (2.24) decomposes latent random variables, modelling the variability in obligors' asset values, into systematic and idiosyncratic components.

Recall that the Gaussian conditional independence model of a credit portfolio is motivated by the structural approach to credit risk modelling, which assumes that an obligor defaults if the values of its assets at a given risk measurement horizon falls below its contractual obligations payable. Substituting representation (2.19) for latent random variable  $W_i$  in (2.15), we may express the log-return on the assets of obligor  $i$  over the risk measurement horizon  $[0, \tau]$ ,  $\tau > 0$ , as

$$r_i = \log \left( \frac{A_i(\tau)}{A_i(0)} \right) = \mu_i \tau - \frac{1}{2} \sigma_i^2 \tau + \sigma_i \sqrt{\tau} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right), \quad (2.73)$$

where  $Y$  and  $Z_i$  are independent standard Gaussian random variables, and  $\rho_i \in (0, 1)$ . Then, standardising the log-return on the assets of obligor  $i$ , we have

$$\tilde{r}_i = \frac{r_i - \mathbb{E}[r_i]}{\sqrt{\text{Var}(r_i)}} = \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i, \quad (2.74)$$

where  $\tilde{r}_i$  has mean zero and unit variance.

It follows that for a credit portfolio comprising  $n$  obligors, the standardised log-return is given by

$$\tilde{r} = \sum_{i=1}^n w_i \tilde{r}_i = \sum_{i=1}^n w_i \sqrt{\rho_i} Y + \sum_{i=1}^n w_i \sqrt{1 - \rho_i} Z_i, \quad (2.75)$$

where  $w_i$  is the portfolio weight of obligor  $i$ , and  $\sum_{i=1}^n w_i = 1$ . Observing that random variables  $Z_1, \dots, Z_n$  and  $Y$  are standard Gaussian and mutually independent, it is clear that:

$\text{Var}(Y) = \mathbb{E}[Y^2] = 1$ ;  $\text{Var}(Z_i) = \mathbb{E}[Z_i^2] = 1$  and  $\text{Cov}(Y, Z_i) = \mathbb{E}[YZ_i] = 0$  for  $i = 1, \dots, n$ ; and  $\text{Cov}(Z_i, Z_j) = \mathbb{E}[Z_i Z_j] = 0$  for  $1 \leq i < j \leq n$ . Then, the expected value of (2.75) is

$$\mathbb{E}[\tilde{r}] = \sum_{i=1}^n w_i \mathbb{E}[\tilde{r}_i] = \sum_{i=1}^n w_i \sqrt{\rho_i} \mathbb{E}[Y] + \sum_{i=1}^n w_i \sqrt{1 - \rho_i} \mathbb{E}[Z_i] = 0, \quad (2.76)$$

and its variance is given by

$$\begin{aligned} \text{Var}(\tilde{r}) &= \text{Var}\left(\sum_{i=1}^n w_i \tilde{r}_i\right) \\ &= \text{Var}\left(\sum_{i=1}^n w_i \sqrt{\rho_i} Y\right) + \text{Var}\left(\sum_{i=1}^n w_i \sqrt{1 - \rho_i} Z_i\right) \\ &= \left(\sum_{i=1}^n w_i \sqrt{\rho_i}\right)^2 \text{Var}(Y) + \mathbb{E}\left[\left(\sum_{i=1}^n w_i \sqrt{1 - \rho_i} Z_i\right) \left(\sum_{j=1}^n w_j \sqrt{1 - \rho_j} Z_j\right)\right] \\ &= \left(\sum_{i=1}^n w_i \sqrt{\rho_i}\right)^2 \text{Var}(Y) + \sum_{i=1}^n w_i^2 (1 - \rho_i) \text{Var}(Z_i). \end{aligned} \quad (2.77)$$

Variances in the second equality are additive, since  $Y$  is independent of  $Z_1, \dots, Z_n$ . The last equality uses the facts that  $\text{Cov}(Z_i, Z_j) = \mathbb{E}[Z_i Z_j] = 0$  for  $i \neq j$ , and  $\text{Var}(Z_i) = \mathbb{E}[Z_i^2]$ .

Notice that in deducing (2.74) we recover (2.19). Substituting into (2.18), the event that obligor  $i$  defaults may be redefined as the set

$$D_i = \{\tilde{r}_i < \Phi^{-1}(p_i)\}. \quad (2.78)$$

Then, the portfolio percentage loss is calculated as

$$L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{\tilde{r}_i < \Phi^{-1}(p_i)\}}. \quad (2.79)$$

Having expressed the standardised portfolio log-return,  $\tilde{r}$ , and portfolio percentage loss,  $L_n$ , as functions of random quantities  $\tilde{r}_i \sim \mathcal{N}(0, 1)$ ,  $1 \leq i \leq n$ , we proceed to show that idiosyncratic risk vanishes for infinitely fine-grained portfolios. That is, the variance of standardised log-returns on an asymptotic portfolio is a function of systematic risk factor  $Y$  and independent of idiosyncratic risks, and hence portfolio percentage loss depends only on  $Y$ .

**Proposition 2.39.** *Assume a Gaussian conditional independence model of an asymptotic credit portfolio. Then, assuming that the limits exist, the variance of standardised portfolio log-returns is*

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{r}) = \lim_{n \rightarrow \infty} \text{Var}\left(\sum_{i=1}^n w_i \tilde{r}_i\right) = \text{Var}(Y) \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n w_i \sqrt{\rho_i}\right)^2, \quad (2.80)$$

where  $\tilde{r}_i$  is the return on asset values of obligor  $i$  standardised to mean zero and unit variance, and  $\sum_{i=1}^n w_i = 1$  for all  $n \in \mathbb{N}$ . Thus, portfolio percentage loss,  $L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{\tilde{r}_i < \Phi^{-1}(p_i)\}}$ , depends only on systematic risk factor  $Y$  as  $n \rightarrow \infty$ .



*Proof.* The variance of standardised log-returns on a credit portfolio comprising  $n$  obligors is given by (2.77). Observing that  $w_i \in (0, 1)$ ,  $\rho_i \in (0, 1)$  and  $\text{Var}(Z_i) = 1$  for  $i = 1, \dots, n$ , a comparison test shows that the second term on the right-hand side of (2.77) converges as  $n \rightarrow \infty$ :

$$0 \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i^2 (1 - \rho_i) \text{Var}(Z_i) \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i^2 = 0, \quad (2.81)$$

where Remark 2.15 establishes that  $\sum_{i=1}^n w_i^2 \rightarrow 0$  as  $n \rightarrow \infty$  using Kronecker's lemma (Lemma 2.66). Taking the limit as  $n \rightarrow \infty$  of (2.77) yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Var}(\tilde{r}) &= \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n w_i \sqrt{\rho_i} \right)^2 \text{Var}(Y) + \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i^2 (1 - \rho_i) \text{Var}(Z_i) \\ &= \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n w_i \sqrt{\rho_i} \right)^2 \text{Var}(Y) + 0 \\ &= \text{Var}(Y) \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n w_i \sqrt{\rho_i} \right)^2. \end{aligned}$$

Since the variance of  $\tilde{r}$  for an infinitely fine-grained portfolio is a function of systematic risk factor  $Y$  and independent of idiosyncratic risks, indicator function  $\mathbb{1}_{\{\tilde{r}_i < \Phi^{-1}(p_i)\}}$  depends only on  $Y$ , and so does  $L_n$ .  $\square$

*Remark 2.40.* Observing that under the Gaussian conditional independence model of a credit portfolio, systematic risk factor  $Y \sim \mathcal{N}(0, 1)$ , (2.80) reduces to

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{r}) = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n w_i \sqrt{\rho_i} \right)^2. \quad (2.82)$$

**Corollary 2.41.** *Assume a Gaussian conditional independence model of an asymptotic, homogeneous credit portfolio. Let  $\rho$  be the exposure, or sensitivity, of each obligor to systematic risk factor  $Y$  (i.e., the asset correlation parameter assigned to each obligor), and suppose that equal dollar EAD is assigned to each obligor. Then, the variance of standardised portfolio log-returns is*

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{r}) = \rho. \quad (2.83)$$

*Proof.* Let  $w_i = 1/n$  for  $i = 1, \dots, n$ , and recall that random variables  $Z_1, \dots, Z_n$  and  $Y$  are standard Gaussian and mutually independent. Then, taking the limit as  $n \rightarrow \infty$  of (2.77) yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Var}(\tilde{r}) &= \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \frac{1}{n} \sqrt{\rho} \right)^2 \text{Var}(Y) + \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^2} (1 - \rho) \text{Var}(Z_i) \\ &= \lim_{n \rightarrow \infty} (\sqrt{\rho})^2 \text{Var}(Y) + \lim_{n \rightarrow \infty} \frac{1 - \rho}{n} \\ &= \rho \text{Var}(Y) + 0 \\ &= \rho. \end{aligned}$$

$\square$

## 2.C Probability Spaces

In the lexicon of stochastic calculus, this chapter describes the time-evolution of asset values in terms of adapted stochastic processes measurable with respect to a filtration on a probability space. Here, we formally define these concepts.

**Definition 2.42** (Shreve 2004, Definition 1.1.1). Let *sample space*  $\Omega$  be a non-empty set of all possible outcomes  $\omega$  of an experiment. A  $\sigma$ -algebra, or  $\sigma$ -field,  $\mathcal{F}$  is a collection of subsets of  $\Omega$  satisfying:

- (1) The empty set  $\emptyset$  and the sample space  $\Omega$  belong to  $\mathcal{F}$ .
- (2) Whenever set  $A$  belongs to  $\mathcal{F}$ , its complement  $A^c$  belongs to  $\mathcal{F}$ .
- (3) Whenever a sequence of sets  $A_1, A_2, \dots$  belongs to  $\mathcal{F}$ , their union  $\bigcup_{n=1}^{\infty} A_n$  and intersection  $\bigcap_{n=1}^{\infty} A_n$  also belong to  $\mathcal{F}$ .

**Definition 2.43** (Shreve 2004, Definition 1.1.2). Let  $\mathcal{F}$  be a  $\sigma$ -algebra on sample space  $\Omega$ . A *probability measure*  $\mathbb{P}$  is a function that, to every set  $A \in \mathcal{F}$ , assigns a value in  $[0, 1]$ , called the probability of  $A$  and written  $\mathbb{P}(A)$ . A probability measure  $\mathbb{P}$  on  $(\Omega, \mathcal{F})$  satisfies:

- (1)  $\mathbb{P}(\emptyset) = 0$  and  $\mathbb{P}(\Omega) = 1$ .
- (2) Whenever  $A_1, A_2, \dots$  is a sequence of disjoint sets in  $\mathcal{F}$ , then  $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ .

The triple  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a *probability space*.

**Definition 2.44** (Shreve 2004, Definition 2.1.1). Fix  $\tau > 0$ , and assume that for each  $t \in [0, \tau]$  there is a  $\sigma$ -algebra  $\mathcal{F}(t)$  on sample space  $\Omega$ . Assume further that if  $s \leq t$ , then every set in  $\mathcal{F}(s)$  is also in  $\mathcal{F}(t)$ . We call the collection of  $\sigma$ -algebras  $\mathcal{F}(t)$ ,  $0 \leq t \leq \tau$ , a *filtration*.

**Definition 2.45** (Shreve 2004, Definition 2.1.5). Let  $X$  be a random variable defined on sample space  $\Omega$ , and let  $\mathcal{G}$  be a  $\sigma$ -algebra of the subsets of  $\Omega$ . Denote by  $\sigma(X)$  the  $\sigma$ -algebra generated by  $X$ . If every set in  $\sigma(X)$  is also in  $\mathcal{G}$ , we say that  $X$  is  $\mathcal{G}$ -measurable.

**Definition 2.46** (Shreve 2004, Definition 2.1.6). Let  $\Omega$  be a sample space equipped with a filtration  $\mathcal{F}(t)$ ,  $0 \leq t \leq \tau$ , and let  $X(t)$  be a collection of random variables indexed by  $t \in [0, \tau]$ . We say that this collection of random variables is an adapted stochastic process if, for each  $t$ , the random variable  $X(t)$  is  $\mathcal{F}(t)$ -measurable.

## 2.D Stochastic Calculus

**Definition 2.47** (Shreve 2004, Definition 3.3.1). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. For each  $\omega \in \Omega$ , suppose there is a continuous function  $W(t)$  that satisfies  $W(0) = 0$ , and that depends on  $\omega$ . Then  $W(t), t \geq 0$ , is a *Brownian motion* if for all  $0 = t_0 < t_1 < \dots < t_m$  the increments

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_m) - W(t_{m-1})$$

are independent, and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0,$$

$$\text{Var}(W(t_{i+1}) - W(t_i)) = t_{i+1} - t_i.$$

**Definition 2.48** (Shreve 2004, Definition 4.4.3). Let  $W(t), t \geq 0$ , be a Brownian motion, and let  $\mathcal{F}(t), t \geq 0$ , be an associated filtration. An *Itô process* is a stochastic process of the form

$$X(t) = X(0) + \int_0^t b(s) ds + \int_0^t \sigma(s) dW(s),$$

where  $X(0)$  is deterministic, and  $b(s)$  and  $\sigma(s)$  are adapted stochastic processes.

*Remark 2.49.* In differential notation, an Itô process may be expressed as

$$dX(t) = b(t) dt + \sigma(t) dW(t).$$

*Remark 2.50.* Itô process  $X(t)$  has drift rate  $b(t)$  and variance rate  $a(t) = \sigma^2(t)$ . Adapted stochastic processes  $b(t)$  and  $\sigma(t)$  may, and often do, depend on  $X(t)$  as well.

**Theorem 2.51** (Itô formula [Shreve 2004, Theorem 4.4.6]). *Let  $X(t), t \geq 0$ , be an Itô process as described in Definition 2.48, and let  $f(t, x)$  be a function for which the partial derivatives  $f_t(t, x)$ ,  $f_x(t, x)$  and  $f_{xx}(t, x)$  are defined and continuous. Then, for every  $\tau \geq 0$ ,*

$$\begin{aligned} f(\tau, X(\tau)) &= f(0, X(0)) + \int_0^\tau f_t(t, X(t)) dt + \int_0^\tau f_x(t, X(t)) dX(t) \\ &\quad + \frac{1}{2} \int_0^\tau f_{xx}(t, X(t)) d[X, X](t) \\ &= f(0, X(0)) + \int_0^\tau f_t(t, X(t)) dt + \int_0^\tau f_x(t, X(t)) b(t) dt + \int_0^\tau f_x(t, X(t)) \sigma(t) dW(t) \\ &\quad + \frac{1}{2} \int_0^\tau f_{xx}(t, X(t)) \sigma^2(t) dt. \end{aligned}$$

*Remark 2.52.* Itô's formula may be expressed in differential form as

$$\begin{aligned} df(t, X(t)) &= f_t(t, X(t)) dt + f_x(t, X(t)) dX(t) + \frac{1}{2} f_{xx}(t, X(t)) d[X, X](t) \\ &= f_t(t, X(t)) dt + f_x(t, X(t)) b(t) dt + f_x(t, X(t)) \sigma(t) dW(t) \\ &\quad + \frac{1}{2} f_{xx}(t, X(t)) \sigma^2(t) dt. \end{aligned}$$

**Definition 2.53** (Shreve 2004, Definition 2.3.5). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and fix  $\tau > 0$ . Let  $\mathcal{F}(t)$ ,  $0 \leq t \leq \tau$ , be a filtration of  $\sigma$ -algebras of  $\mathcal{F}$ . Consider an adapted stochastic process  $M(t)$ ,  $0 \leq t \leq \tau$ . If

$$\mathbb{E}[M(t) | \mathcal{F}(s)] = M(s) \text{ for all } 0 \leq s \leq t \leq \tau,$$

we say this process is a *martingale*. It has no tendency to rise or fall.

**Theorem 2.54** (Shreve 2004, Theorem 4.3.1). Fix  $\tau > 0$ , and let  $\sigma(t)$ ,  $0 \leq t \leq \tau$ , be an adapted stochastic process that satisfies  $\mathbb{E}[\int_0^\tau \sigma^2(t) dt] < \infty$ . Then, the Itô integral  $I(t) = \int_0^t \sigma(s) dW(s)$  has the following properties:

- (1) **(Continuity)** As a function of the upper limit of integration  $t$ , the paths of  $I(t)$  are continuous.
- (2) **(Adaptivity)** For every  $t$ ,  $I(t)$  is  $\mathcal{F}(t)$ -measurable.
- (3) **(Linearity)** If  $I(t) = \int_0^t \sigma(s) dW(s)$  and  $J(t) = \int_0^t \phi(s) dW(s)$ , then for every constant  $c$ ,  $cI(t) = \int_0^t c\sigma(s) dW(s)$ ; and  $I(t) \pm J(t) = \int_0^t (\sigma(s) \pm \phi(s)) dW(s)$ .
- (4) **(Martingale)**  $I(t)$  is a martingale.
- (5) **(Itô isometry)**  $E[I^2(t)] = \mathbb{E}[\int_0^t \sigma^2(s) ds]$ .
- (6) **(Quadratic variation)**  $[I, I](t) = \int_0^t \sigma^2(s) ds$ .

**Theorem 2.55** (Lévy, one dimension [Shreve 2004, Theorem 4.6.4]). Let  $M(t)$ ,  $t \geq 0$ , be a martingale relative to a filtration  $\mathcal{F}(t)$ ,  $t \geq 0$ . Assume that  $M(0) = 0$ ,  $M(t)$  has continuous paths, and  $[M, M](t) = t$  for all  $t \geq 0$ . Then,  $M(t)$  is a Brownian motion.

**Theorem 2.56** (Lévy, two dimension [Shreve 2004, Theorem 4.6.5]). Let  $M_1(t)$  and  $M_2(t)$ ,  $t \geq 0$ , be a martingales relative to a filtration  $\mathcal{F}(t)$ ,  $t \geq 0$ . Assume that for  $i = 1, 2$ ,  $M_i(0) = 0$ ,  $M_i(t)$  has continuous paths, and  $[M_i, M_i](t) = t$  for all  $t \geq 0$ . If, in addition,  $[M_1, M_2](t) = 0$  for all  $t \geq 0$ , then  $M_1(t)$  and  $M_2(t)$  are independent Brownian motions.

**Theorem 2.57** (Two-dimensional Itô formula [Shreve 2004, Theorem 4.6.2]). Let  $f(t, x, y)$ ,  $t \geq 0$ , be a function whose partial derivatives  $f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}$  and  $f_{yy}$  are defined and continuous. Let  $X(t)$  and  $Y(t)$  be Itô processes as described in Definition 2.48. The two-dimensional Itô formula in differential form is

$$\begin{aligned} df(t, X(t), Y(t)) &= f_t(t, X(t), Y(t)) dt + f_x(t, X(t), Y(t)) dX(t) + f_y(t, X(t), Y(t)) dY(t) \\ &\quad + \frac{1}{2} f_{xx}(t, X(t), Y(t)) d[X, X](t) + f_{xy}(t, X(t), Y(t)) d[X, Y](t) \\ &\quad + \frac{1}{2} f_{yy}(t, X(t), Y(t)) d[Y, Y](t). \end{aligned}$$

**Corollary 2.58** (Itô product rule [Shreve 2004, Corollary 4.6.3]). *Let  $X(t)$  and  $Y(t)$ ,  $t \geq 0$ , be Itô processes as described in Definition 2.48. Then,*

$$\begin{aligned} d(X(t)Y(t)) &= X(t) dY(t) + Y(t) dX(t) + d[X, Y](t) \\ &= X(t) dY(t) + Y(t) dX(t) + dX(t) dY(t). \end{aligned}$$

## 2.E Convergence of Sequences

Many of the results in this chapter are established under different modes of convergence. Some proofs, too, appeal to results established under different modes of convergence. Here, we provide a digest of definitions and theorems (and lemmas, corollaries, etc.), previously referenced in this chapter, on the convergence of sequences of random variables and measurable functions.

**Theorem 2.59** ( $p$ -series test [Wade 2004, Corollary 6.13]). *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

*converges if and only if  $p > 1$ .*

**Definition 2.60** (Wade 2004, Definition 7.1). Let  $I$  be a nonempty subset of  $\mathbb{R}$ . A sequence of functions  $f_n: I \rightarrow \mathbb{R}$  is said to *converge pointwise* on  $I$  if and only if  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for each  $x \in I$ .

**Definition 2.61** (Grimmett and Stirzaker 2001, Definition 7.2.1). Let  $X, X_1, X_2, \dots$  be random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We say:

- (1)  $X_n$  converges to  $X$  *almost surely*, written  $X_n \xrightarrow{a.s.} X$ , if

$$\{\omega \in \Omega: X_n(\omega) \rightarrow X(\omega) \text{ as } n \rightarrow \infty\}$$

is an event whose probability is 1.

- (2)  $X_n$  converges to  $X$  *in probability*, written  $X_n \xrightarrow{P} X$ , if

$$\mathbb{P}(|X_n - X| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } \varepsilon > 0.$$

- (3)  $X_n$  converges to  $X$  *in distribution*, written  $X_n \xrightarrow{D} X$ , if

$$\mathbb{P}(X_n \leq x) \rightarrow \mathbb{P}(X \leq x) \text{ as } n \rightarrow \infty$$

for all points  $x$  at which the function  $F_X(x) = \mathbb{P}(X \leq x)$  is continuous.

**Theorem 2.62** (Grimmett and Stirzaker 2001, Theorem 7.2.3). *The following implications hold:*

$$(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{P} X) \implies (X_n \xrightarrow{D} X).$$

**Theorem 2.63** (Dominated convergence [Shreve 2004, Theorem 1.4.9]). *Let  $f_1, f_2, \dots$  be a sequence of Borel-measurable functions on  $\mathbb{R}$  converging almost everywhere to a function  $f$ . If there is another non-negative function  $g$  such that  $\int_{-\infty}^{\infty} g(x) dx < \infty$  and  $|f_n| \leq g$  almost everywhere for all  $n$ , then*

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} f(x) dx.$$

**Theorem 2.64** (Central limit [Grimmett and Stirzaker 2001, Theorem 5.10.4]). *Let  $X_1, X_2, \dots$  be a sequence of independent identically distributed random variables with finite mean  $\mu$  and finite non-zero variance  $\sigma^2$ , and let  $S_n = X_1 + X_2 + \dots + X_n$ . Then*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1) \text{ as } n \rightarrow \infty.$$

**Theorem 2.65** (Strong law of large numbers [Grimmett and Stirzaker 2001, Theorem 7.5.1]). *Let  $X_1, X_2, \dots$  be independent identically distributed random variables. Then*

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mu \text{ a.s., as } n \rightarrow \infty,$$

*for some constant  $\mu$ , if and only if  $\mathbb{E}[|X_1|] < \infty$ . In this case  $\mu = \mathbb{E}[X_1]$ .*

**Lemma 2.66** (Kronecker [Gut 2005, Lemma 6.5.1]). *Suppose that  $\{X_n\}_{n \geq 1}$  is a sequence of random variables, and let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that  $0 < a_n \uparrow \infty$  as  $n \rightarrow \infty$ . Then,*

$$\sum_{k=1}^{\infty} \frac{X_k}{a_k} \text{ converges a.s.} \implies \frac{1}{a_n} \sum_{k=1}^n X_k \xrightarrow{\text{a.s.}} 0 \text{ as } n \rightarrow \infty.$$

**Theorem 2.67** (Kolmogorov's convergence criterion [Gut 2005, Theorem 6.5.2]). *Let  $\{X_n\}_{n \geq 1}$  be a sequence of independent random variables. Then,*

$$\sum_{n=1}^{\infty} \text{Var}(X_n) < \infty \implies \sum_{n=1}^{\infty} (X_n - \mathbb{E}[X_n]) \text{ converges a.s.}$$



# Copula Approach to Modelling Default Dependence

In generating a portfolio loss distribution we are, in effect, combining marginal loss distributions of constituent credits into a multivariate distribution capturing default dependence between obligors. An approach to modelling default dependence, popularised by Li (2000), uses copula functions — a statistical technique for combining marginal distributions into a multivariate distribution with a chosen dependence structure.

Drawing from Nelsen (2006) and Embrechts, Lindskog, et al. (2003), Section 3.1 introduces the theory of copulas. On this basis, Section 3.2 derives the single-factor copula model describing default dependence for the general case introduced in Section 2.3. In generating the empirical loss distribution of a credit portfolio by simulation of the conditional independence model of Definition 2.10, we are implicitly applying the single-factor copula model using Monte Carlo methods. Section 3.3 returns to the special case in which defaults are modelled as conditionally independent Gaussian random variables. We derive the one-factor Gaussian copula, the most commonly applied copula function in credit risk modelling. Then, Section 3.4 casts the Gaussian copula in the frame of survival time, or time-until-default. Section 3.5 proceeds to examine tail dependence in portfolio loss distributions described by a variety of elliptical copulas. We outline procedures for generating empirical loss distributions described by Gaussian and Student's  $t$  copulas.

## 3.1 Copula Theory

Our introduction to copulas provides a digest of definitions, theorems and corollaries pertinent to the application of copula functions to modelling default dependence. This material is drawn



primarily from texts by Nelsen (2006) and Embrechts, Lindskog, et al. (2003). Throughout this section, functions  $F, F_1, \dots, F_n$  and  $G$  denote arbitrary distribution functions.

**Definition 3.1.** An  $n$ -dimensional copula (or  $n$ -copula) is a function  $C: [0, 1]^n \rightarrow [0, 1]$  with the following properties:

- (1) For every  $(u_1, \dots, u_n) \in [0, 1]^n$ ,  $C(u_1, \dots, u_n) = 0$  if at least one element  $u_k$  is zero; and  $C(u_1, \dots, u_n) = u_k$  if all elements of  $(u_1, \dots, u_n)$  are equal to 1 except  $u_k$ .
- (2) For any  $n$ -box  $B = [u_1, v_1] \times \dots \times [u_n, v_n] \subseteq [0, 1]^n$ ,

$$V_c(B) = \sum \text{sgn}(w_1, \dots, w_n) C(w_1, \dots, w_n) \geq 0,$$

where the sum is taken over all vertices  $(w_1, \dots, w_n)$  of  $B$ , and

$$\text{sgn}(w_1, \dots, w_n) = \begin{cases} 1 & \text{if } w_k = u_k \text{ for an even number of coordinates } k, \\ -1 & \text{if } w_k = u_k \text{ for an odd number of coordinates } k. \end{cases}$$

The function  $V_c(B)$  is called the  $C$ -volume of  $n$ -box  $B$ . Informally, an  $n$ -copula is a multivariate distribution function on the unit box  $[0, 1]^n$  whose margins are uniformly distributed on the unit interval  $[0, 1]$ . As such, an  $n$ -copula induces a probability measure on  $[0, 1]^n$  via

$$V_c([0, u_1] \times \dots \times [0, u_n]) = C(u_1, \dots, u_n).$$

*Remark 3.2.* For any  $n$ -copula  $C$ ,  $n \geq 3$ , each  $k$ -margin of  $C$  is a  $k$ -copula,  $2 \leq k < n$ .

Sklar's theorem, first published in 1959, is central to the theory of copulas, and is employed in most applications of copulas. In the sequel, let  $\overline{\mathbb{R}}$  denote the extended real number line  $[-\infty, \infty]$ , and  $\text{Dom}G$  and  $\text{Ran}G$  denote the domain and range of function  $G$ , respectively.

**Theorem 3.3** (Sklar). *Let  $F$  be an  $n$ -dimensional distribution function with margins  $F_1, \dots, F_n$ . Then there exists an  $n$ -copula  $C$  such that for all  $(x_1, \dots, x_n) \in \overline{\mathbb{R}}^n$ ,*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \quad (3.1)$$

*If  $F_1, \dots, F_n$  are all continuous, then  $C$  is unique; otherwise,  $C$  is uniquely determined on  $\text{Ran}F_1 \times \dots \times \text{Ran}F_n$ . Conversely, if  $C$  is an  $n$ -copula and  $F_1, \dots, F_n$  are distribution functions, then  $F$  is an  $n$ -dimensional distribution function with margins  $F_1, \dots, F_n$ .*

*Proof.* Refer to Nelsen (2006, Theorem 2.10.9). □

**Definition 3.4.** Let  $G$  be a distribution function. Then the *quasi-inverse* of  $G$  is any function  $G^{(-1)}$  with domain  $[0, 1]$  satisfying:

- (1) If  $t \in \text{Ran}G$ , then  $G^{(-1)}(t)$  is any number  $x \in \overline{\mathbb{R}}$  such that  $G(x) = t$ ; that is, for all  $t \in \text{Ran}G$ ,  $G(G^{(-1)}(t)) = t$ .
- (2) If  $t \notin \text{Ran}G$ , then  $G^{(-1)}(t) = \inf\{x \in \overline{\mathbb{R}}: G(x) \geq t\}$ , using the convention  $\inf \emptyset = -\infty$ .

*Remark 3.5.* If  $G$  is strictly increasing, then  $G$  has a single quasi-inverse, the ordinary inverse denoted  $G^{-1}$ .

**Corollary 3.6.** *Let  $F, F_1, \dots, F_n$  and  $C$  be as in Theorem 3.3, and let  $F_1^{(-1)}, \dots, F_n^{(-1)}$  be quasi-inverses of  $F_1, \dots, F_n$ , respectively. Then for any  $(u_1, \dots, u_n) \in [0, 1]^n$ ,*

$$C(u_1, \dots, u_n) = F(F_1^{(-1)}(u_1), \dots, F_n^{(-1)}(u_n)). \quad (3.2)$$

Consider the copula functions  $M^n, \Pi^n$  and  $W^n$  defined on  $[0, 1]^n$  as:

$$\begin{aligned} M^n(u_1, \dots, u_n) &= \min(u_1, \dots, u_n), \\ \Pi^n(u_1, \dots, u_n) &= u_1 \dots u_n, \\ W^n(u_1, \dots, u_n) &= \max(u_1 + \dots + u_n - n + 1, 0). \end{aligned}$$

$M^n$  and  $W^n$  are the Fréchet-Hoeffding upper and lower bounds, respectively, and  $\Pi^n$  is the product copula. Functions  $M^n$  and  $\Pi^n$  are  $n$ -copulas for  $n \geq 2$ , whereas  $W^n$  is not an  $n$ -copula for  $n > 2$ .

**Theorem 3.7.** *If  $C$  is any  $n$ -copula, then for every  $(u_1, \dots, u_n) \in [0, 1]^n$ ,*

$$W^n(u_1, \dots, u_n) \leq C(u_1, \dots, u_n) \leq M^n(u_1, \dots, u_n). \quad (3.3)$$

*Proof.* For the two-dimensional case, refer to Nelsen (2006, Theorem 2.2.3).  $\square$

Although the Fréchet-Hoeffding lower bound is not an  $n$ -copula for  $n > 2$ , the left-hand inequality in (3.3) is best possible in the following sense.

**Theorem 3.8.** *For any  $(u_1, \dots, u_n) \in [0, 1]^n$  and  $n \geq 3$ , there is an  $n$ -copula  $C$ , which depends on  $(u_1, \dots, u_n)$ , such that*

$$C(u_1, \dots, u_n) = W^n(u_1, \dots, u_n).$$

*Proof.* Refer to Nelsen (2006, Theorem 2.10.13).  $\square$

**Theorem 3.9.** *Let  $(X_1, \dots, X_n)$  be a vector of continuous random variables with  $n$ -copula  $C$ . Then  $X_1, \dots, X_n$  are independent if and only if  $C = \Pi^n$ .*

*Proof.* Follows from Theorem 3.3, and the observation that  $X_1, \dots, X_n$  are independent if and only if  $F(x_1, \dots, x_n) = F_1(x_1) \dots F_n(x_n)$  for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ .  $\square$

Random variables with copula  $M^n$  are often called *comonotonic*.

**Theorem 3.10.** *Let  $(X_1, \dots, X_n)$  be a vector of continuous random variables with  $n$ -copula  $C$ . Then each of the random variables  $X_1, \dots, X_n$  is almost surely a strictly increasing function of any of the others if and only if  $C = M^n$ .*

*Proof.* For the two-dimensional case, refer to Wang and Dhaene (1998).  $\square$

One of the key properties of copulas is that for strictly monotone transformations of continuous random variables, copulas are either invariant or change in predictable and simple ways. Note that a random variable is continuous if its distribution function is continuous. Also, if the distribution function of a random variable  $X$  is continuous, and if  $\varphi$  is a strictly monotone function whose domain contains  $\text{Ran}X$ , then the distribution function of  $\varphi(X)$  is continuous too. We now consider the case of strictly increasing transformations of random variables.

**Theorem 3.11.** *Let  $(X_1, \dots, X_n)$  be a vector of continuous random variables with  $n$ -copula  $C$ . If  $\varphi_1, \dots, \varphi_n$  are strictly increasing on  $\text{Ran}X_1, \dots, \text{Ran}X_n$ , respectively, then  $(\varphi(X_1), \dots, \varphi(X_n))$  has the same copula  $C$ . Thus,  $C$  is invariant under strictly increasing transformations of  $(X_1, \dots, X_n)$ .*

*Proof.* Refer to Embrechts, Lindskog, et al. (2003, Theorem 8.2.6).  $\square$

### 3.2 Single-Factor Copula Model

Consider a credit portfolio comprising  $n$  obligors, and let (2.25) define the event that obligor  $i$  defaults. We may express the unconditional PD of obligor  $i$ , for the general case introduced in Section 2.3, as

$$\mathbb{P}(D_i) = \mathbb{P}(W_i < F_i^{-1}(p_i)), \quad (3.4)$$

and the joint default probability as

$$\mathbb{P}(\mathbb{1}_{D_1} = 1, \dots, \mathbb{1}_{D_n} = 1) = \mathbb{P}(W_1 < F_1^{-1}(p_1), \dots, W_n < F_n^{-1}(p_n)). \quad (3.5)$$

Let  $(u_1, \dots, u_n) = (p_1, \dots, p_n)$  be a vector in  $[0, 1]^n$ , and  $(W_1, \dots, W_n)$  a vector of latent random variables with continuous and strictly increasing distribution functions  $F_1, \dots, F_n$ , respectively. Suppose that  $F$  is an  $n$ -dimensional distribution function with margins  $F_1, \dots, F_n$ . Then, by Theorem 3.3, there is a unique  $n$ -copula  $C$  such that for all  $(w_1, \dots, w_n) \in \overline{\mathbb{R}}^n$ ,

$$F(w_1, \dots, w_n) = C(F_1(w_1), \dots, F_n(w_n)). \quad (3.6)$$

Now, by Corollary 3.6, for any  $(u_1, \dots, u_n) \in [0, 1]^n$ ,

$$\begin{aligned} C(u_1, \dots, u_n) &= F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)) \\ &= \mathbb{P}(W_1 < F_1^{-1}(u_1), \dots, W_n < F_n^{-1}(u_n)). \end{aligned} \quad (3.7)$$

Assuming that defaults are conditionally independent given systematic risk factor  $Y$ , latent random variables  $W_1, \dots, W_n$  may be represented as in (2.24):

$$W_i = \gamma_i Y + \sqrt{1 - \gamma_i^2} Z_i,$$

where  $Z_1, \dots, Z_n$  and  $Y$  are mutually independent random variables with continuous and strictly increasing distribution functions  $G_1, \dots, G_n$  and  $H$ , respectively, and  $\gamma_i \in (-1, 1)$  for  $i = 1, \dots, n$ .

**Lemma 3.12.** *Assume a conditional independence model of a credit portfolio comprising  $n$  obligors. Then, default dependence may be described by the single-factor copula function associated with  $(W_1, \dots, W_n)$ :*

$$C(u_1, \dots, u_n) = \int_{-\infty}^{\infty} \left( \prod_{i=1}^n G_i \left( \frac{F_i^{-1}(u_i) - \gamma_i y}{\sqrt{1 - \gamma_i^2}} \right) \right) dH(y) \quad (3.8)$$

for any  $(u_1, \dots, u_n) \in [0, 1]^n$ .

*Proof.* Integrating conditional marginal distribution functions over systematic risk factor  $Y$ , copula (3.7) may be expressed as

$$\begin{aligned} C(u_1, \dots, u_n) &= \mathbb{P}(W_1 < F_1^{-1}(u_1), \dots, W_n < F_n^{-1}(u_n)) \\ &= \int_{-\infty}^{\infty} \mathbb{P}(W_1 < F_1^{-1}(u_1), \dots, W_n < F_n^{-1}(u_n) | Y = y) dH(y) \\ &= \int_{-\infty}^{\infty} \mathbb{P}(W_1 < F_1^{-1}(u_1) | Y = y) \dots \mathbb{P}(W_n < F_n^{-1}(u_n) | Y = y) dH(y). \end{aligned} \quad (3.9)$$

The integrand in the last equality is the product of obligors' PDs conditional on realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ . Substituting the probability statements in this integrand with the expression for conditional PD in (2.28), and observing that  $u_i = p_i$  for  $i = 1, \dots, n$  establishes (3.8).  $\square$

*Remark 3.13.* Equation (3.8) holds for any distribution function  $H$ . However, other results in this thesis rely on the assumption that  $H$  is continuous and strictly increasing.

The copula application of interest here is the generation of the portfolio loss distribution. We first adapt Lemma 3.12 to measure the probability of  $k$  defaults in a credit portfolio comprising  $n$  obligors. Then, using this result, we deduce an expression for the cumulative probability that the portfolio percentage loss does not exceed a given threshold.

**Proposition 3.14.** *Assume a conditional independence model of a credit portfolio comprising  $n$  obligors, each assigned equal EAD and LGD. That is,  $w_i = 1/n$  and  $\eta_i = \eta$  for  $i = 1, \dots, n$ . Define a permutation  $\pi$  of the set  $\{1, \dots, n\}$  as the one-to-one mapping of the set onto itself. Denote by  $\mathcal{P}$  the set of all such permutations, and  $\pi(j)$  the  $j^{\text{th}}$  element of a permutation  $\pi$ . Then, for  $k = 0, 1, \dots, n$ ,*

$$\begin{aligned} \mathbb{P}\left(L_n = \frac{\eta k}{n}\right) &= \frac{1}{k!(n-k)!} \sum_{\pi \in \mathcal{P}} \int_{-\infty}^{\infty} G_{\pi(1)}(\zeta_{\pi(1)}(y)) \dots G_{\pi(k)}(\zeta_{\pi(k)}(y)) \\ &\quad (1 - G_{\pi(k+1)}(\zeta_{\pi(k+1)}(y))) \dots (1 - G_{\pi(n)}(\zeta_{\pi(n)}(y))) dH(y), \end{aligned} \quad (3.10)$$

where

$$\zeta_{\pi(j)}(y) = \frac{F_{\pi(j)}^{-1}(p_{\pi(j)}) - \gamma_{\pi(j)} y}{\sqrt{1 - \gamma_{\pi(j)}^2}}.$$

*Proof.* There are  $n!$  permutations of the set  $\{1, \dots, n\}$ , so

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1}{k!(n-k)!} \sum_{\pi \in \mathcal{P}} 1.$$

The expression for  $\zeta_{\pi(j)}(y)$  follows from (2.27). Notice that

$$\mathbb{P}(W_{\pi(j)} < F_{\pi(j)}^{-1}(u_{\pi(j)}) | Y = y) = \mathbb{P}(Z_{\pi(j)} < \zeta_{\pi(j)}(y)),$$

where  $u_{\pi(j)} = p_{\pi(j)}$ , and apply (3.9) to measure the probability of  $k$  defaults in a credit portfolio comprising  $n$  obligors. Hence,

$$\begin{aligned} \mathbb{P}\left(L_n = \frac{\eta k}{n}\right) &= \frac{1}{k!(n-k)!} \sum_{\pi \in \mathcal{P}} \mathbb{P}(\mathbb{1}_{D_{\pi(1)}} = 1, \dots, \mathbb{1}_{D_{\pi(k)}} = 1, \mathbb{1}_{D_{\pi(k+1)}} = 0, \dots, \mathbb{1}_{D_{\pi(n)}} = 0) \\ &= \frac{1}{k!(n-k)!} \sum_{\pi \in \mathcal{P}} \int_{-\infty}^{-\infty} \mathbb{P}(Z_{\pi(1)} < \zeta_{\pi(1)}(y)) \dots \mathbb{P}(Z_{\pi(k)} < \zeta_{\pi(k)}(y)) \\ &\quad \mathbb{P}(Z_{\pi(k+1)} \geq \zeta_{\pi(k+1)}(y)) \dots \mathbb{P}(Z_{\pi(n)} \geq \zeta_{\pi(n)}(y)) dH(y). \end{aligned}$$

Observing that  $G_{\pi(j)}$  is the distribution function of  $Z_{\pi(j)}$  establishes (3.10).  $\square$

**Corollary 3.15.** *Under the assumptions of Proposition 3.14, the portfolio loss distribution function is given by*

$$\mathbb{P}(L_n \leq l) = \sum_{k=0}^{\lfloor ln/\eta \rfloor} \mathbb{P}\left(L_n = \frac{\eta k}{n}\right), \quad (3.11)$$

where  $\lfloor ln/\eta \rfloor$  is the largest integer not greater than  $ln/\eta$ .

*Remark 3.16.* Vasicek (1987, 1991) derived the Gaussian equivalent of (3.10) and (3.11), before recoveries, for a homogeneous credit portfolio.

*Remark 3.17.* Monte Carlo simulation computes the portfolio percentage loss for a very large sample of realisations  $y \in \mathbb{R}$  of systematic risk factor  $Y$ . For each realisation  $y$  we use (2.26) to determine the number of defaults  $k$ , and calculate portfolio percentage loss by summing the product of exposure weight and LGD for credits that have defaulted. This simulation procedure is discussed in some detail in Section 6.1. In generating the empirical loss distribution of a credit portfolio by simulation (2.44), we are implicitly applying the single-factor copula model using Monte Carlo methods.

### 3.3 One-Factor Gaussian Copula

Throughout this section, as the title implies, we restrict our attention to the special case in which default dependence is modelled as a multivariate Gaussian process. Consider a credit portfolio comprising  $n$  obligors. Substituting the inverse standard Gaussian distribution function into (3.4) and (3.5), the unconditional PD of obligor  $i$  becomes

$$\mathbb{P}(D_i) = \mathbb{P}(W_i < \Phi^{-1}(p_i)), \quad (3.12)$$

and the joint default probability is given by

$$\mathbb{P}(\mathbb{1}_{D_1} = 1, \dots, \mathbb{1}_{D_n} = 1) = \mathbb{P}(W_1 < \Phi^{-1}(p_1), \dots, W_n < \Phi^{-1}(p_n)). \quad (3.13)$$

Let  $(u_1, \dots, u_n) = (p_1, \dots, p_n)$  be a vector in  $[0, 1]^n$ , and choose a dependence structure described by correlation matrix  $\Gamma$ . Then, the unique Gaussian copula associated with  $(W_1, \dots, W_n)$  is a particular case of (3.7):

$$\begin{aligned} C_\Gamma(u_1, \dots, u_n) &= \Phi_\Gamma(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)) \\ &= \mathbb{P}(W_1 < \Phi^{-1}(u_1), \dots, W_n < \Phi^{-1}(u_n)), \end{aligned} \quad (3.14)$$

for any  $(u_1, \dots, u_n) \in [0, 1]^n$ , where  $\Phi_\Gamma$  is the multivariate standard Gaussian distribution function with correlation matrix  $\Gamma$ .

Now suppose that defaults are conditionally independent given a single systematic risk factor. Then, default dependence is described by correlation matrix:

$$\hat{\Gamma} = \begin{pmatrix} 1 & \sqrt{\rho_1 \rho_2} & \cdots & \sqrt{\rho_1 \rho_n} \\ \sqrt{\rho_1 \rho_2} & 1 & \cdots & \sqrt{\rho_2 \rho_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\rho_1 \rho_n} & \sqrt{\rho_2 \rho_n} & \cdots & 1 \end{pmatrix} \quad (3.15)$$

where  $\sqrt{\rho_i \rho_j} = \text{Corr}(W_i, W_j)$  is the pairwise correlation between obligors' asset values by Proposition 2.5, and  $\sqrt{\rho_i}$  is the exposure of obligor  $i$  to systematic risk factor  $Y$  in (2.19). This special case of the Gaussian copula is the so-called *one-factor Gaussian copula*, the most commonly applied copula function in credit risk modelling (MacKenzie and Spears 2012). The multivariate distribution induced by the one-factor Gaussian copula is an elliptical one. We formally define spherical and elliptical distributions in Appendix 3.A. The following result, a corollary of Lemma 3.12, provides an expression for the one-factor Gaussian copula.

**Corollary 3.18.** *Assume a Gaussian conditional independence model of a credit portfolio comprising  $n$  obligors with pairwise asset correlations defined by matrix (3.15). Then, default dependence may be described by the one-factor Gaussian copula associated with  $(W_1, \dots, W_n)$ :*

$$\begin{aligned} C_{\hat{\Gamma}}(u_1, \dots, u_n) &= \Phi_{\hat{\Gamma}}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)) \\ &= \int_{-\infty}^{\infty} \left( \prod_{i=1}^n \Phi \left( \frac{\Phi^{-1}(u_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}} \right) \right) \phi(y) dy \end{aligned} \quad (3.16)$$

for any  $(u_1, \dots, u_n) \in [0, 1]^n$ .

*Proof.* Substituting the operands of the product operator in (3.8) with the expression for conditional PD in (2.20), and observing that  $d\Phi(y) = \phi(y) dy$  establishes (3.16).  $\square$

*Remark 3.19.* Appealing to Proposition 2.28, the Basel II IRB approach applies the one-factor Gaussian copula to calculate the  $\alpha$  quantile of the distribution of  $\mathbb{E}[L_n | Y]$ , an analytical approximation of the  $\alpha$  quantile of the distribution of  $L_n$ , or  $\text{VaR}_\alpha(L_n)$ .

### 3.4 Copula of Survival Times

It is usual to cast Gaussian copula (3.14) in the frame of survival time, or time-until-default. Assume that marginal distributions of obligors' survival times are known — a typical assumption is that survival times follow an exponential distribution. For a credit portfolio comprising  $n$  obligors, let  $(T_1, \dots, T_n)$  be a vector of survival times with continuous and strictly increasing distribution functions  $F_{T_1}, \dots, F_{T_n}$ . The event that obligor  $i$  defaults during a given risk measurement horizon is defined by (2.25). We now recast this event as set

$$D_i = \{T_i \leq \tau_i\},$$

where the interval  $[0, \tau_i], \tau_i > 0$ , is the risk measurement horizon assigned to obligor  $i$ . Then, the unconditional PD of obligor  $i$ , which is a function of the risk measurement horizon, may be reformulated as

$$p_i = \mathbb{P}(D_i) = \mathbb{P}(T_i \leq \tau_i) = F_{T_i}(\tau_i),$$

and the joint default probability as

$$\mathbb{P}(\mathbf{1}_{D_1} = 1, \dots, \mathbf{1}_{D_n} = 1) = \mathbb{P}(T_1 \leq \tau_1, \dots, T_n \leq \tau_n).$$

Notice that the model accommodates varying risk measurement horizons for different credits. However, the usual practice is to assign the same risk measurement horizon to all constituent credits of the portfolio.

Let  $(u_1, \dots, u_n) = (p_1, \dots, p_n)$  be a vector in  $[0, 1]^n$ . Then, it follows from Corollary 3.6 that default dependence may also be described by a unique  $n$ -copula associated with  $(T_1, \dots, T_n)$ :

$$\begin{aligned} C_T(u_1, \dots, u_n) &= F_T(F_{T_1}^{-1}(u_1), \dots, F_{T_n}^{-1}(u_n)) \\ &= \mathbb{P}(T_1 \leq F_{T_1}^{-1}(u_1), \dots, T_n \leq F_{T_n}^{-1}(u_n)), \end{aligned} \quad (3.17)$$

for any  $(u_1, \dots, u_n) \in [0, 1]^n$ , where  $F_T$  is the multivariate distribution function of the vector of survival times. Since higher realisations of random variables  $T_1, \dots, T_n$  and  $W_1, \dots, W_n$  correspond to wider estimates of distance-to-default, we postulate that there exist strictly increasing functions  $\varphi_1, \dots, \varphi_n$  on  $\text{Ran}W_1, \dots, \text{Ran}W_n$ , respectively, such that

$$(T_1, \dots, T_n) = (\varphi_1(W_1), \dots, \varphi_n(W_n)).$$

Then, by Theorem 3.11,  $(T_1, \dots, T_n)$  and  $(W_1, \dots, W_n)$  share the same copula. That is, (3.7) and (3.17) are equivalent formulations of default dependence:

$$\begin{aligned} \mathbb{P}(W_1 < F_1^{-1}(u_1), \dots, W_n < F_n^{-1}(u_n)) &= \mathbb{P}(T_1 \leq F_{T_1}^{-1}(u_1), \dots, T_n \leq F_{T_n}^{-1}(u_n)), \\ F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)) &= F_T(F_{T_1}^{-1}(u_1), \dots, F_{T_n}^{-1}(u_n)), \\ C(u_1, \dots, u_n) &= C_T(u_1, \dots, u_n). \end{aligned}$$

*Remark 3.20.* Suppose that we choose the one-factor Gaussian copula  $C_{\hat{\Gamma}}$  to model default dependence, and assume that the vector of survival times  $(T_1, \dots, T_n)$  is the image of the vector of latent random variables  $(W_1, \dots, W_n)$  under strictly increasing transformations  $\varphi_1, \dots, \varphi_n$ . Then, observing that

$$\mathbb{P}(D_i) = \mathbb{P}(W_i < \Phi^{-1}(p_i)) = \mathbb{P}(W_i < \Phi^{-1}(F_{T_i}(\tau_i))),$$

we may substitute correlation matrix  $\hat{\Gamma}$  and distribution function  $F_{T_i}(\tau_i)$  into (3.14) to obtain an expression for the joint default probability in terms of survival times:

$$\mathbb{P}(T_1 \leq \tau_1, \dots, T_n \leq \tau_n) = \Phi_{\hat{\Gamma}}(\Phi^{-1}(F_{T_1}(\tau_1)), \dots, \Phi^{-1}(F_{T_n}(\tau_n))). \quad (3.18)$$

Thus, the Gaussian copula defined by (3.18) with asset correlation matrix (3.15) allows us to generate survival times of constituent credits (Li 2000).

*Remark 3.21.* Generation of the single-period, or static, loss distribution of a credit portfolio by simulation of (2.26) parameterised by (2.21) is an application of the one-factor Gaussian copula described by (3.16), which cast in terms of survival times is equivalent to (3.18). Portfolios of large banks contain financial instruments that are influenced by a multitude of underlying risk factors. Internal and external risk controllers, such as the prudential regulator, require risks to be measured over varying holding periods for different risk classes (e.g., one-day or 10-day for market risk, and one-year for credit and operational risks). The path dependent nature of many risks and the requirement to measure portfolio risk over different time horizons leads to a multi-period simulation. It is practical then to simulate all variables, including defaults and survivals, in each time period.

Brigo et al. (2013) demonstrate the pitfalls of iterating marginal survival times with dependence modelled by the Gaussian copula at each time step of a multi-period simulation. This common industry practice may not be consistent with the overall default monitoring of a single-period (through to the terminal horizon) simulation of default times. In order to consistently represent dependence between multivariate survival times in each period of a multi-period simulation, the authors model survival times using a continuous-time Markov chain with multivariate exponential distribution (i.e., Marshall-Olkin copula). This approach borrows from the fatal shock model of reliability theory (Barlow and Proschan 1981; Gupta et al. 2010; Lindskog and McNeil 2001). Survival times are modelled as a matrix differential equation in the transition matrix for the embedded Markov chain. The solution involves the eigendecomposition of the infinitesimal generator of the continuous-time Markov chain. An efficient implementation of this Markovian credit risk model remains a challenge in building a feasible multi-period simulation for measuring risk in massive portfolios.



### 3.5 Elliptical Copulas

In applying the one-factor Gaussian copula to calculate regulatory capital for credit risk, the IRB approach implicitly assumes that a multivariate Gaussian distribution accurately models tail risk of credit portfolios. But, it is generally acknowledged that models which assume that financial data follow a Gaussian distribution tend to underestimate tail risk. For one, Gaussian copulas do not exhibit tail dependence — the tendency for extreme observations (i.e., credit defaults) to occur simultaneously for all random variables. Under a Gaussian copula defaults are said to be *asymptotically independent* in the upper tail (Embrechts, McNeil, et al. 2002). Section 6.3 examines the effect of tail dependence by measuring the sensitivity of credit risk capital to dependence structure as modelled by elliptical copulas, including Gaussian and Student's  $t$  copulas. We abbreviate the latter by  $t$ -copula.

Firstly, we recall some properties of basic probability distributions (Bluhm et al. 2010, pp. 103–105):

- *Chi-square distribution.* Let  $Z_1, \dots, Z_\nu \sim \mathcal{N}(0, 1)$  be independent standard Gaussian random variables. Then, random variable  $X = Z_1^2 + \dots + Z_\nu^2$  is said to be chi-square distributed with  $\nu$  degrees of freedom,  $X \sim \chi^2(\nu)$ .
- *Student's  $t$  distribution.* Let  $Z \sim \mathcal{N}(0, 1)$  and  $V \sim \chi^2(\nu)$ ,  $Z$  independent of  $V$ . Then, random variable  $X = Z\sqrt{\nu/V}$  is said to be  $t$ -distributed with  $\nu$  degrees of freedom,  $X \sim t(\nu)$ . The Student's  $t$  distribution converges to the standard Gaussian distribution as the number of degrees of freedom increases.
- *Multivariate Student's  $t$  distribution.* Given a multivariate vector  $(Z_1, \dots, Z_n) \sim \mathcal{N}(0, \Gamma)$  with correlation matrix  $\Gamma$ , let  $V \sim \chi^2(\nu)$ ,  $V$  independent of  $(Z_1, \dots, Z_n)$ . Then, random vector  $(X_1, \dots, X_n) = \sqrt{\nu/V} (Z_1, \dots, Z_n)$  is said to be  $t$ -distributed with  $\nu$  degrees of freedom,  $(X_1, \dots, X_n) \sim t(\nu, \Gamma)$ . Note that  $(X_1, \dots, X_n)$  inherits its correlation matrix  $\Gamma$  from  $(Z_1, \dots, Z_n)$ . That is,  $\text{Corr}(X_i, X_j) = \text{Corr}(Z_i, Z_j)$  for  $1 \leq i < j \leq n$ .

Now, let  $(X_1, \dots, X_n)$  be a vector of latent random variables modelling default dependence of a portfolio comprising  $n$  obligors. We proceed to illustrate the dependence induced by a variety of elliptical copulas, and outline procedures for randomly generating observations drawn from the resultant multivariate distributions (Bluhm et al. 2010, pp. 106–108):

- *One-factor Gaussian copula.* Observations  $X_1, \dots, X_n$  are randomly generated with

$$X_i = \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i, \quad (3.19)$$

where random variables  $Z_1, \dots, Z_n$  and  $Y$  are standard Gaussian and mutually independent, and correlation parameters  $\rho_1, \dots, \rho_n \in (0, 1)$ . That is, we sample  $(X_1, \dots, X_n)$  from the distribution induced by Gaussian copula (3.14) with correlation matrix (3.15). Note that (3.19) is simply the conditionally independent representation expressed in (2.19).

- *Product copula with Gaussian margins.* By Theorem 3.9 the product copula generates independent, and therefore uncorrelated, standard Gaussian random variables. So, observations  $X_1, \dots, X_n$  are independently drawn from the standard Gaussian distribution. That is, we sample  $(X_1, \dots, X_n)$  from the distribution induced by Gaussian copula (3.14) with correlation matrix  $I_n$ , the  $n$ -by- $n$  identity matrix.
- *$t$ -copula with  $\nu$  degrees of freedom and  $t$ -distributed margins.* Observations  $X_1, \dots, X_n$  are randomly generated with

$$X_i = \sqrt{\frac{\nu}{V}} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right), \quad (3.20)$$

where  $Z_1, \dots, Z_n$  and  $Y \sim \mathcal{N}(0, 1)$ ,  $V \sim \chi^2(\nu)$ , and  $Z_1, \dots, Z_n$ ,  $Y$  and  $V$  are mutually independent. Scaling (2.19) by  $\sqrt{\nu/V}$  transforms standard Gaussian random variables into  $t$ -distributed random variables with  $\nu$  degrees of freedom. Vector  $(X_1, \dots, X_n)$  inherits correlation matrix (3.15).

- *$t$ -copula with  $\nu$  degrees of freedom and Gaussian margins.* The procedure enumerated below samples  $(X_1, \dots, X_n)$  from the resultant multivariate distribution.
  - (1) Generate a vector of conditionally independent  $t$ -distributed random variables with  $\nu$  degrees of freedom using (3.20):

$$\tilde{X}_i = \sqrt{\frac{\nu}{V}} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right),$$

for  $i = 1, \dots, n$ .

- (2) Apply the Student's  $t$  distribution function to coordinates of vector  $(\tilde{X}_1, \dots, \tilde{X}_n)$ :

$$(u_1, \dots, u_n) = \left( \Phi_\nu(\tilde{X}_1), \dots, \Phi_\nu(\tilde{X}_n) \right),$$

where  $\Phi_\nu$  is the Student's  $t$  distribution function with  $\nu$  degrees of freedom, and  $(u_1, \dots, u_n) \in [0, 1]^n$ .

- (3) Apply the inverse standard Gaussian distribution function to coordinates of vector  $(u_1, \dots, u_n)$ :

$$(X_1, \dots, X_n) = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)).$$

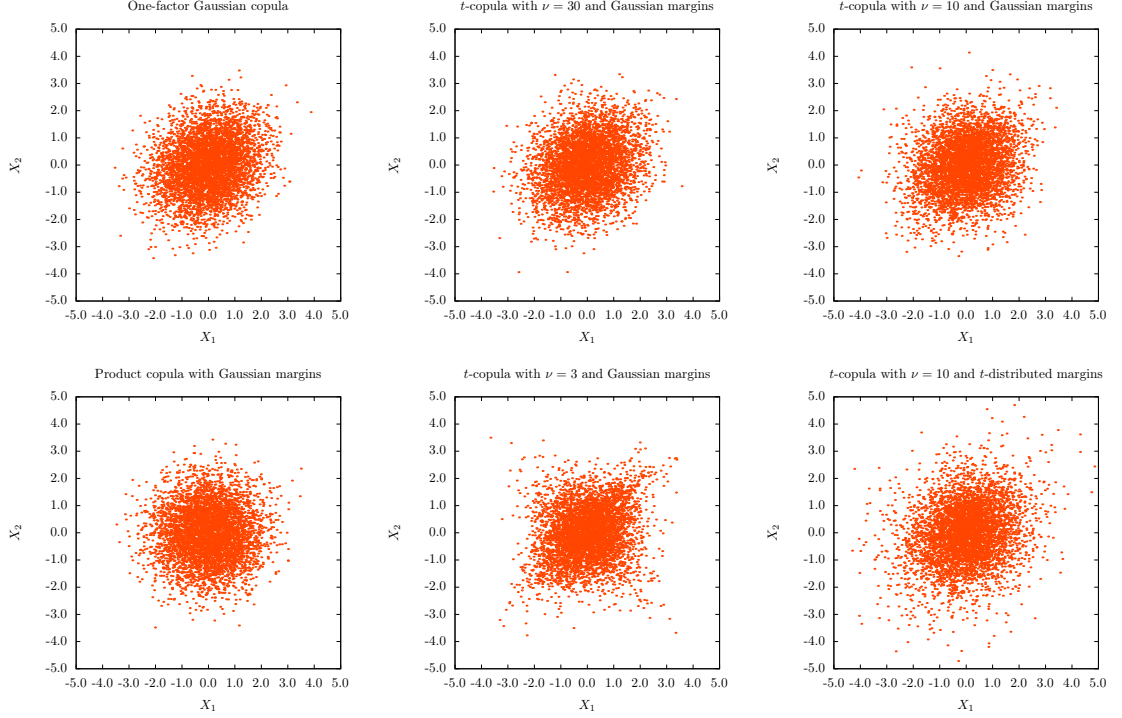
In sum, observations  $X_1, \dots, X_n$  are randomly generated with

$$X_i = \Phi^{-1} \left( \Phi_\nu \left( \sqrt{\frac{\nu}{V}} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right) \right) \right). \quad (3.21)$$

The resultant multivariate distributions are spherical or elliptical, which we formally define in Appendix 3.A. The bivariate scatter plots in Figure 3.1 illustrate the dependence induced by the elliptical copulas described above. For each copula, except the product copula, we set  $\rho_1 = \rho_2 = 0.170$ , the exposure-weighted asset correlation of the representative credit portfolio described in Section 5.2.

### 3. COPULA APPROACH TO MODELLING DEFAULT DEPENDENCE

Figure 3.1: Bivariate scatter plots illustrate the dependence induced by a variety of elliptical copulas. Each point corresponds to an ordered pair  $(X_1, X_2)$ . Except for the product copula where  $X_1$  and  $X_2$  are uncorrelated,  $\rho_1 = \rho_2 = 0.170$ , the exposure-weighted average asset correlation of the representative credit portfolio described in Section 5.2.



*Remark 3.22.* The dependence exhibited by the one-factor Gaussian copula becomes apparent when compared with the product copula, which generates uncorrelated standard Gaussian random variables. In contrast to Gaussian copulas,  $t$ -copulas admit tail dependence with fewer degrees of freedom producing stronger dependence. When the  $t$ -copula is applied to combine Gaussian margins and  $t$ -distributed margins, respectively, the former is more tightly distributed.

Assuming that asset values follow a log-normal distribution, the distribution of  $(X_1, \dots, X_n)$  is determined by the copula function chosen to combine its margins. Section 2.6 outlines the procedure for generating the loss distribution of a credit portfolio comprising  $n$  obligors by simulation of (2.26). In an implementation of the one-factor Gaussian copula, the default indicator function of (2.26) is parameterised by (2.21). Monte Carlo simulation performs  $N$  iterations, (2.44) calculates the portfolio percentage loss for each iteration, and (2.45) describes the empirical loss distribution.

In relation to the  $t$ -copula with Gaussian margins, we continue to assume that unconditional PDs scaled to a given risk measurement horizon are published as market data. Substituting (3.20) into (2.25), we define the event that obligor  $i$  defaults during the risk measurement horizon by

the set

$$D_i = \left\{ \sqrt{\frac{\nu}{V}} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right) < \Phi_\nu^{-1}(p_i) \right\}. \quad (3.22)$$

Then, the PD of obligor  $i$  conditional on  $Y = y$  and  $V = v$  is deducible (Bluhm et al. 2010, pp. 109–111):

$$\begin{aligned} p_i(y, v) &= \mathbb{P}(D_i | Y = y, V = v) \\ &= \mathbb{P} \left( \sqrt{\frac{\nu}{V}} \left( \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \right) < \Phi_\nu^{-1}(p_i) | Y = y, V = v \right) \\ &= \mathbb{P} \left( \sqrt{\frac{\nu}{v}} \left( \sqrt{\rho_i} y + \sqrt{1 - \rho_i} Z_i \right) < \Phi_\nu^{-1}(p_i) \right) \\ &= \mathbb{P} \left( Z_i < \frac{\sqrt{v/\nu} \Phi_\nu^{-1}(p_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}} \right) \\ &= \Phi \left( \frac{\sqrt{v/\nu} \Phi_\nu^{-1}(p_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}} \right). \end{aligned} \quad (3.23)$$

Let

$$\zeta_i(y, v) = \frac{\sqrt{v/\nu} \Phi_\nu^{-1}(p_i) - \sqrt{\rho_i} y}{\sqrt{1 - \rho_i}} \quad (3.24)$$

for  $i = 1, \dots, n$ . Now, given  $Y = y$  and  $V = v$ , the portfolio percentage loss is calculated as

$$L_n = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{Z_i < \zeta_i(y, v)\}}. \quad (3.25)$$

Suppose that we generate the loss distribution of a portfolio comprising  $n$  obligors by simulation of (3.25), an implementation of the  $t$ -copula with Gaussian margins. Let Monte Carlo simulation perform  $N$  iterations. For each iteration we draw from the standard Gaussian distribution random variables  $Z_1, \dots, Z_n$  and  $Y$ , and from the chi-square distribution with  $\nu$  degrees of freedom random variable  $V$ . Then, given  $Y = y_k$  and  $V = v_k$ , the portfolio percentage loss over the risk measurement horizon is computed as

$$L_{n,k} = \sum_{i=1}^n w_i \eta_i \mathbb{1}_{\{Z_{i,k} < \zeta_i(y_k, v_k)\}} \quad (3.26)$$

for iterations  $k = 1, \dots, N$ . Again, (2.45) describes the empirical loss distribution.

### 3.A Spherical and Elliptical Distributions

In Chapters 3 and 6 we examine, theoretically and empirically, tail dependence in portfolio loss distributions described by a variety of elliptical copulas. Firstly, we formally define spherical distributions, distributions of uncorrelated random variables. Then, we define elliptical distributions from the spherical.

**Definition 3.23** (McNeil et al. 2005, Definition 3.18). A random vector  $\mathbf{X} = (X_1, \dots, X_n)'$  has a *spherical distribution* if, for every orthogonal map  $U \in \mathbb{R}^{n \times n}$ ,

$$U\mathbf{X} \stackrel{d}{=} \mathbf{X}.$$

*Remark 3.24.* An orthogonal map  $U \in \mathbb{R}^{n \times n}$  satisfies  $UU' = U'U = I_n$ , the  $n$ -by- $n$  identity matrix. Equivalently,  $U$  preserves lengths, that is, for every  $v \in \mathbb{R}^n$ ,  $\|Uv\| = \|v\|$ . Therefore, spherical random vectors are distributionally invariant under rotations.

**Theorem 3.25** (McNeil et al. 2005, Theorem 3.19). *The following statements are equivalent:*

(1)  $\mathbf{X}$  is spherical.

(2) There exists a function  $\psi$  of a scalar variable such that, for all  $\mathbf{t} \in \mathbb{R}^n$ ,

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[e^{i\mathbf{t}'\mathbf{X}}] = \psi(\mathbf{t}'\mathbf{t}) = \psi(t_1^2 + \dots + t_n^2),$$

where  $\phi_{\mathbf{X}}(\mathbf{t})$  is the characteristic function of  $\mathbf{X}$ .

(3) For every  $\mathbf{t} \in \mathbb{R}^n$ ,

$$\mathbf{t}'\mathbf{X} \stackrel{d}{=} \|\mathbf{t}\|X_1,$$

where  $\|\mathbf{t}\|^2 = \mathbf{t}'\mathbf{t} = t_1^2 + \dots + t_n^2$ .

*Remark 3.26.* Random vector  $\mathbf{X}$  is said to have an  $n$ -dimensional spherical distribution with characteristic generator  $\psi$ , denoted  $\mathbf{X} \sim S_n(\psi)$ .

**Example 3.27.** A random vector  $\mathbf{X}$  drawn from the standard uncorrelated Gaussian distribution  $\mathcal{N}(0, I_n)$  is spherical with characteristic function

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[e^{i\mathbf{t}'\mathbf{X}}] = e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}}.$$

By Theorem 3.25,  $X \sim S_n(\psi)$  with characteristic generator  $\psi(t) = e^{-\frac{1}{2}t}$ .

*Remark 3.28.* Among the spherical distributions,  $\mathcal{N}(0, I_n)$  is the only distribution of independent random variables.

**Definition 3.29** (McNeil et al. 2005, Definition 3.26). A random vector  $\mathbf{X} = (X_1, \dots, X_n)'$  has an *elliptical distribution* if

$$\mathbf{X} \stackrel{d}{=} \mu + A\mathbf{Y},$$

where  $\mathbf{Y} \sim S_k(\psi)$ , and  $A \in \mathbb{R}^{n \times k}$  and  $\mu \in \mathbb{R}^n$  are a matrix and vector of constants, respectively.

*Remark 3.30.* The characteristic function of an elliptical distribution is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[e^{i\mathbf{t}'\mathbf{X}}] = \mathbb{E}[e^{i\mathbf{t}'(\mu + A\mathbf{Y})}] = e^{i\mathbf{t}'\mu} \mathbb{E}[e^{i(A'\mathbf{t})'\mathbf{Y}}] = e^{i\mathbf{t}'\mu} \psi(\mathbf{t}'\Sigma\mathbf{t}),$$

where  $\Sigma = AA'$ . We denote elliptical distributions by  $X \sim E_n(\mu, \Sigma, \psi)$ , and refer to  $\mu$  as the location vector,  $\Sigma$  as the dispersion (covariance) matrix and  $\psi$  as the characteristic generator of the distribution.

# Basel II Capital Requirements Under the Internal Ratings-Based Approach

Under the Basel II Accord (Basel Committee on Banking Supervision 2006), ADIs assess capital adequacy for credit risk using either the standardised approach or, subject to approval, the IRB approach. Our concern is with the theoretical foundations and empirical analysis of the latter approach. In keeping with the Basel II IRB approach to capital adequacy for credit risk, the relevant prudential standard of APRA (2008a) requires that ADIs set aside provisions for absorbing expected losses, and hold capital against unexpected losses.

**Definition 4.1.** *Unexpected loss* on a credit portfolio at the  $\alpha$  confidence level over a given risk measurement horizon is the difference between credit VaR (with the same confidence level and time horizon) and expected loss.

Recall that  $L_n$  denotes the portfolio percentage loss on a credit portfolio comprising  $n$  obligors, and let  $\mathcal{L}_n$  be the dollar loss on the same portfolio. Then,  $\text{VaR}_\alpha(L_n)$  (respectively,  $\text{VaR}_\alpha(\mathcal{L}_n)$ ) denotes the portfolio percentage (dollar) loss at the  $\alpha$  confidence level over a given risk measurement horizon, and  $\mathbb{E}[L_n]$  (respectively,  $\mathbb{E}[\mathcal{L}_n]$ ) the expected portfolio percentage (dollar) loss. We define credit risk capital consistent with the IRB approach of the Basel II Accord and the relevant prudential standard of APRA.

**Definition 4.2.** Let *credit risk capital* be held against unexpected losses. Then,

$$K_\alpha(L_n) = \text{VaR}_\alpha(L_n) - \mathbb{E}[L_n] \quad (4.1)$$

is the capital charge (at the  $\alpha$  confidence level over a given risk measurement horizon) as a percentage of EAD on a credit portfolio comprising  $n$  obligors, and

$$K_\alpha(\mathcal{L}_n) = \text{VaR}_\alpha(\mathcal{L}_n) - \mathbb{E}[\mathcal{L}_n] \quad (4.2)$$

is the capital charge in dollars.

In Section 4.1 we formulate the IRB approach, and extend its model specification to the general setting of Section 2.3. Then, Section 4.2 examines the conformity of the so-called *asymptotic single risk factor* (ASRF) model with the property of portfolio invariance, an important criterion for meeting supervisory needs.

#### 4.1 Model Specification of the Internal Ratings-Based Approach

Appealing to Definition 4.2 and Proposition 2.28, we deduce a function for calculating capital held against unexpected losses on an asymptotic credit portfolio.

**Proposition 4.3.** *Assume a conditional independence model of an asymptotic credit portfolio. Then,*

$$\lim_{n \rightarrow \infty} K_\alpha(L_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i \eta_i G_i \left( \frac{F_i^{-1}(p_i) - \gamma_i H^{-1}(1-\alpha)}{\sqrt{1 - \gamma_i^2}} \right) - \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i \eta_i p_i, \quad (4.3)$$

assuming that the limits on the right-hand side of (4.3) are well defined.

*Proof.* By Definition 4.2, capital held against unexpected losses on an asymptotic credit portfolio may be expressed as

$$\lim_{n \rightarrow \infty} K_\alpha(L_n) = \lim_{n \rightarrow \infty} \text{VaR}_\alpha(L_n) - \lim_{n \rightarrow \infty} \mathbb{E}[L_n], \quad (4.4)$$

assuming that the limits on the right-hand side are well defined. By hypothesis, defaults are modelled as conditionally independent random variables given systematic risk factor  $Y$ . So, adding and subtracting the  $\alpha$  quantile of the distribution of conditional expectation of portfolio percentage loss yields

$$\begin{aligned} \lim_{n \rightarrow \infty} K_\alpha(L_n) &= \lim_{n \rightarrow \infty} (\text{VaR}_\alpha(L_n) - \mathbb{E}[L_n | Y = H^{-1}(1-\alpha)]) \\ &\quad + \lim_{n \rightarrow \infty} \mathbb{E}[L_n | Y = H^{-1}(1-\alpha)] - \lim_{n \rightarrow \infty} \mathbb{E}[L_n] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[L_n | Y = H^{-1}(1-\alpha)] - \lim_{n \rightarrow \infty} \mathbb{E}[L_n], \end{aligned} \quad (4.5)$$

where the limits on the right-hand side of (4.5) are well defined by hypothesis. Recall that in probabilistic terms,  $\text{VaR}_\alpha(L_n)$  is the  $\alpha$  quantile of the portfolio loss distribution,  $q_\alpha(L_n)$ . Hence, the second equality in (4.5) follows from Proposition 2.28, with the  $\alpha$  quantile of the distribution

of  $\mathbb{E}[L_n | Y]$ , in effect, being substituted for  $\text{VaR}_\alpha(L_n)$ . Expanding (4.5) using (2.31) and (2.32) yields

$$\lim_{n \rightarrow \infty} K_\alpha(L_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i \eta_i p_i (H^{-1}(1-\alpha)) - \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i \eta_i p_i. \quad (4.6)$$

Finally, setting  $y = H^{-1}(1-\alpha)$  in (2.28) and substituting into (4.6) establishes (4.3).  $\square$

Proposition 4.3 applies to the abstract case of an infinitely fine-grained credit portfolio for which total EAD diverges to  $\infty$  (Remark 2.16). However, in the real world a credit portfolio contains a finite number of obligors, each assigned a positive and finite EAD. Hence, portfolio EAD is positive and finite, and so is the assessed capital charge, which may be expressed as a percentage of EAD. In Section 4.2 we elaborate on this tension between asymptotic and real-world portfolios in the context of portfolio invariance. Remark 2.21 highlights that while real-world portfolios are not infinitely fine-grained, as a practical matter, credit portfolios of large banks are typically near the asymptotic granularity of Definition 2.14. For the practical application of Proposition 4.3 we redefine credit risk capital for finite portfolios that exhibit “sufficient” granularity.

**Definition 4.4.** Let *credit risk capital* be held against unexpected losses. Then, in practice, the capital charge (at the  $\alpha$  confidence level over a given risk measurement horizon) as a percentage of EAD on a credit portfolio comprising  $n$  obligors that “adequately” satisfies the asymptotic granularity condition of Definition 2.14 may be assessed as

$$\tilde{K}_\alpha(L_n) = \mathbb{E}[L_n | Y = H^{-1}(1-\alpha)] - \mathbb{E}[L_n], \quad (4.7)$$

and the capital charge in dollars may be assessed as

$$\tilde{K}_\alpha(\mathcal{L}_n) = \mathbb{E}[\mathcal{L}_n | Y = H^{-1}(1-\alpha)] - \mathbb{E}[\mathcal{L}_n]. \quad (4.8)$$

Expanding (4.7), the capital charge as a percentage of EAD on a credit portfolio containing a finite number of obligors,  $n$ , that exhibits sufficient granularity is calculated as

$$\begin{aligned} \tilde{K}_\alpha(L_n) &= \sum_{i=1}^n w_i \eta_i p_i (H^{-1}(1-\alpha)) - \sum_{i=1}^n w_i \eta_i p_i \\ &= \sum_{i=1}^n w_i \eta_i G_i \left( \frac{F_i^{-1}(p_i) - \gamma_i H^{-1}(1-\alpha)}{\sqrt{1 - \gamma_i^2}} \right) - \sum_{i=1}^n w_i \eta_i p_i. \end{aligned} \quad (4.9)$$

The ASRF model developed by BCBS describes default dependence as a multivariate Gaussian process. Recasting (4.9) for the Gaussian case, the capital charge (at the  $\alpha$  confidence level over a given risk measurement horizon) on a near asymptotic portfolio comprising  $n$  obligors is calculated as

$$\begin{aligned} \tilde{K}_\alpha(L_n) &= \sum_{i=1}^n w_i \eta_i \Phi \left( \frac{\Phi^{-1}(p_i) - \sqrt{\rho_i} \Phi^{-1}(1-\alpha)}{\sqrt{1 - \rho_i}} \right) - \sum_{i=1}^n w_i \eta_i p_i \\ &= \sum_{i=1}^n w_i \eta_i \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho_i} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_i}} \right) - \sum_{i=1}^n w_i \eta_i p_i. \end{aligned} \quad (4.10)$$



The second equality follows from the symmetry of the standard Gaussian density function. Note that the kernel of ASRF model (4.10) transforms unconditional PDs into PDs conditional on systematic risk factor  $Y$  using (2.20).

Under the Basel II IRB approach regulatory capital is determined at the 99.9% confidence level over a one-year horizon — a 0.1% probability that credit losses will exceed provisions and capital over the subsequent year. In practice, it incorporates a maturity adjustment to account for the greater likelihood of downgrades for longer-term claims, the effects of which are stronger for claims with higher credit ratings. We omit the maturity adjustment from our development of the theoretical framework (Part I), and evaluation of the robustness of the ASRF model (Chapter 6). It is introduced later when taking measurements from the ASRF model of the Australian banking sector (Chapter 7). Thus, regulatory capital for a near asymptotic credit portfolio comprising  $n$  obligors is assessed as

$$\tilde{K}_{99.9\%}(L_n) = \sum_{i=1}^n w_i \eta_i \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho_i} \Phi^{-1}(0.999)}{\sqrt{1 - \rho_i}} \right) - \sum_{i=1}^n w_i \eta_i p_i. \quad (4.11)$$

*Remark 4.5.* In view of Remark 2.12, but with reference to (2.20), the default probability of obligor  $i$  is no greater than

$$\mathbb{P}(D_i | Y = \Phi^{-1}(0.001)) = p_i(-3.090) = \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho_i} \Phi^{-1}(0.999)}{\sqrt{1 - \rho_i}} \right) \quad (4.12)$$

in 99.9% of economic scenarios.

*Remark 4.6.* BCBS (2005) claims that the IRB approach sets regulatory capital for credit risk at a level where losses exceed it, “on average, once in a thousand years.” Qualifying this informal statement of probability, BCBS cautions that the 99.9% confidence level was chosen because tier 2 capital “does not have the loss absorbing capacity of tier 1”, and “to protect against estimation error” in model inputs as well as “other model uncertainties.” With provisions and capital amounting to as little as 2.0–3.0% of EAD under the IRB approach, perhaps the claim of protection against insolvency due to credit losses at the 99.9% confidence level should be interpreted as providing a margin for misspecification of the ASRF model, and not literally protection against a “one in a thousand year” event. The choice of confidence level for the ASRF model may also have been influenced by the desire to produce regulatory capital requirements that are uncontroversial vis-à-vis Basel I. These qualifying remarks warn against the complacency engendered by the high confidence level chosen for the IRB approach.

## 4.2 Portfolio Invariance

An important criterion for meeting supervisory needs in the model specification of the IRB approach is portfolio invariance. The property of portfolio invariance guarantees that “the capital charge on a given instrument depends only on its characteristics, and not on the characteristics of the portfolio in which it is held” (Gordy 2003). Formally, we define this property in terms of the capital charge contribution of a given instrument or asset to any credit portfolio.

**Definition 4.7.** Denote by  $\Pi_k$  the “characteristics” of instruments constituting an arbitrary portfolio comprising  $k$  obligors, and let  $\varphi: \Pi_k \rightarrow \mathbb{R}$  be a real-valued function that assesses capital charges in dollars. Let  $\Pi_n^A$  and  $\Pi_m^B$  describe the characteristics of instruments constituting portfolios  $A$  and  $B$  comprising  $n$  and  $m$  obligors, respectively. Suppose that equal dollar exposures of asset  $a_0$  are added to both portfolios constructing portfolios described by  $\Pi_{n+1}^A$  and  $\Pi_{m+1}^B$ , respectively. We say that  $\varphi$  satisfies *portfolio invariance* if

$$\varphi(\Pi_{n+1}^A) - \varphi(\Pi_n^A) = \varphi(\Pi_{m+1}^B) - \varphi(\Pi_m^B) \quad (4.13)$$

for any  $\Pi_n^A$ ,  $\Pi_m^B$  and  $a_0$ .

Definition 4.7 expresses the capital charge contribution of a given instrument to any portfolio in dollars. Expanding (4.8) of Definition 4.4, the capital charge in dollars on a credit portfolio containing a finite number of obligors,  $n$ , that exhibits sufficient granularity is assessed as

$$\begin{aligned} \tilde{K}_\alpha(\mathcal{L}_n) &= \sum_{i=1}^n \delta_i \eta_i p_i (H^{-1}(1-\alpha)) - \sum_{i=1}^n \delta_i \eta_i p_i \\ &= \tilde{K}_\alpha(L_n) \sum_{i=1}^n \delta_i \\ &= \Delta_n \tilde{K}_\alpha(L_n), \end{aligned} \quad (4.14)$$

where  $\Delta_n = \sum_{i=1}^n \delta_i$  is the portfolio EAD.

*Remark 4.8.* In Definition 4.7  $\varphi$  is defined generically as a real-valued function of the “characteristics” of instruments constituting a given portfolio. Those characteristics may take the form of a vector of variables and parameters describing the instruments constituting the portfolio. Consider a credit portfolio comprising  $n$  obligors. Then, with respect to ASRF model (4.14),  $\Pi_n$  would be represented as vector  $(\delta_1, \dots, \delta_n, \eta_1, \dots, \eta_n, p_1, \dots, p_n, p_1(y), \dots, p_n(y))$  where  $y = H^{-1}(1-\alpha)$ . Note that the conditional PD of obligor  $i$ ,  $p_i(y)$ , is given by (2.28), so  $\Pi_n$  could also be represented as vector  $(\delta_1, \dots, \delta_n, \eta_1, \dots, \eta_n, p_1, \dots, p_n, \gamma_1, \dots, \gamma_n, y)$ .

A corollary of Proposition 4.3 states that the property of portfolio invariance is preserved under ASRF model (4.9), and hence models (4.10) and (4.11). The proof uses (4.14), which converts (4.9) from percentage of EAD to dollars.

**Corollary 4.9.** *ASRF model (4.9) is portfolio invariant.*

*Proof.* Let  $\Pi_n^A$  describe the characteristics of instruments constituting portfolio  $A$  comprising  $n$  obligors that adequately satisfies the asymptotic granularity condition of Definition 2.14. The capital charge in dollars on portfolio  $A$  is assessed using (4.14):

$$\tilde{K}_\alpha(\mathcal{L}_n^A) = \sum_{i=1}^n \delta_i \eta_i p_i (H^{-1}(1-\alpha)) - \sum_{i=1}^n \delta_i \eta_i p_i.$$

Now assign LGD  $\eta_0$ , unconditional PD  $p_0$  and correlation parameter  $\gamma_0$  to asset  $a_0$ , and add exposure  $\delta_0$  of  $a_0$  to portfolio  $A$ . Then, the capital charge on the resultant portfolio described by  $\Pi_{n+1}^A$  is assessed as

$$\tilde{K}_\alpha(\mathcal{L}_{n+1}^A) = (\delta_0 \eta_0 p_0 (H^{-1}(1-\alpha)) - \delta_0 \eta_0 p_0) + \sum_{i=1}^n \delta_i \eta_i p_i (H^{-1}(1-\alpha)) - \sum_{i=1}^n \delta_i \eta_i p_i.$$

Hence, the capital charge contribution of asset  $a_0$  to portfolio  $A$  is

$$\tilde{K}_\alpha(\mathcal{L}_{n+1}^A) - \tilde{K}_\alpha(\mathcal{L}_n^A) = \delta_0 \eta_0 p_0 (H^{-1}(1-\alpha)) - \delta_0 \eta_0 p_0.$$

By a parallel argument the capital charge contribution of exposure  $\delta_0$  of asset  $a_0$  to near asymptotic portfolio  $B$  is given by

$$\begin{aligned} \tilde{K}_\alpha(\mathcal{L}_{m+1}^B) - \tilde{K}_\alpha(\mathcal{L}_m^B) &= \delta_0 \eta_0 p_0 (H^{-1}(1-\alpha)) - \delta_0 \eta_0 p_0 \\ &= \tilde{K}_\alpha(\mathcal{L}_{n+1}^A) - \tilde{K}_\alpha(\mathcal{L}_n^A). \end{aligned}$$

Thus, ASRF model (4.9), which converted to dollars is expressed as (4.14), is portfolio invariant by Definition 4.7.  $\square$

We conclude that the capital charge contribution of asset  $a_0$  to a credit portfolio, as assessed by ASRF model (4.9), is the same for any portfolio. Or in the parlance of Gordy (2003), the capital charge on asset  $a_0$  depends only on its characteristics, and not on the characteristics of the portfolio in which it is held. The proof of Corollary 4.9, which refers specifically to (4.9), is almost trivial. Bearing in mind that (4.9) follows from Definition 4.4, we infer that portfolio invariance is satisfied if:

- (1) Portfolios are sufficiently fine-grained to adequately satisfy the asymptotic granularity condition of Definition 2.14, and diversify away idiosyncratic risk.
- (2) A single systematic risk factor explains dependence across obligors, that is, defaults are conditionally independent given a single systematic risk factor.

Condition (1) recognises that prudential regulators require that capital charges be assessed on credit portfolios, which in the real world are not infinitely fine-grained. In Section 6.1 we examine the rate of convergence, in terms of number of obligors, of empirical loss distributions to the asymptotic portfolio loss distribution (i.e., the loss distribution of an infinitely fine-grained portfolio). It quantifies the imprecise notion of a credit portfolio “adequately” satisfying the asymptotic granularity condition, or exhibiting “sufficient” granularity. Our findings support the anecdotal evidence of Remark 2.21 that credit portfolios of large banks are typically near the asymptotic granularity of Definition 2.14. So, ASRF model (4.9) is a pragmatic solution for assessing capital charges on finite, real-world credit portfolios. Furthermore, Basel II admits a “granularity adjustment”, which assesses a capital charge for the contribution of name concentrations (i.e., undiversified idiosyncratic risk) to portfolio risk through the supervisory review

process (Pillar 2). Therefore, moderate departures from asymptotic granularity need not pose an impediment to assessing ratings-based capital charges.

*Remark 4.10.* In the context of the model specification of the IRB approach, the granularity adjustment estimates the difference between  $\text{VaR}_{99.9\%}(L_n)$  and  $\mathbb{E}[L_n | Y = \Phi^{-1}(0.001)]$ . The first order granularity adjustment derived by Gordy and Lütkebohmert (2007) is an extension of the one-factor CreditRisk+ model. Classified as an actuarial model, CreditRisk+ is essentially a Poisson mixture model with a factor (sector) structure. The approach of Gordy and Lütkebohmert (2007) is methodologically similar to that of Emmer and Tasche (2005). The latter developed a function for calculating a granularity adjustment based on the one-factor RiskMetrics model in default mode. As with Moody's KMV and the ASRF model, RiskMetrics is an extension of the asset value model of Merton (1973).

*Remark 4.11.* Proposition 4.3, which appeals to Proposition 2.28, assumes that credit portfolios satisfy the asymptotic granularity condition of Definition 2.14, rather than the imprecisely formed Condition (1). Could we instead show that ASRF model (4.3), the limiting case of model (4.9), is portfolio invariant? Suppose that portfolios  $A$  and  $B$  of Definition 4.7 were infinitely fine-grained. Then, in view of Remark 2.16, portfolio EAD diverges to  $\infty$  for  $A$  and  $B$ , and asymptotic granularity is preserved upon adding positive and finite exposures  $\delta_0$  of asset  $a_0$  to both portfolios. Now, divide the exposure of each portfolio to asset  $a_0$  into  $k$  exposures  $\delta_0/k$  with the same instrument characteristics (i.e., LGD, unconditional PD and asset correlation). Then, as  $k \rightarrow \infty$ , (4.3) assesses capital charges on finite exposure  $\delta_0$  to asset  $a_0$ . The problem with this neat mathematical construct that attempts to accommodate the finiteness of real-world portfolios is that (4.3) assumes that defaults are conditionally independent given a single systematic risk factor. But, given a realisation  $y \in \mathbb{R}$  of systematic risk factor  $Y$ , defaults on infinitely divisible exposures to asset  $a_0$  are clearly not independent, thus violating Condition (2). Unfortunately, in order to assess capital charges on finite, real-world credit portfolios we must settle for a pragmatic solution, ASRF model (4.9).

Condition (2) above, on the other hand, is more stringent; if relaxed, the property of portfolio invariance no longer holds, in general. The following examples illustrate the rigidity of the latter condition.

**Example 4.12.** Assume that dependence across obligors is explained by a global risk factor  $Y$ , describing the general level of economic activity, with continuous and strictly increasing distribution function  $H$ . That is, defaults are modelled as conditionally independent random variables given global risk factor  $Y$ . Moreover, assume that the conditional expectation of portfolio percentage loss  $\mathbb{E}[L_n | Y]$  is strictly decreasing in  $y$ . Then, by Proposition 2.28,  $\mathbb{E}[L_n | Y = H^{-1}(1-\alpha)]$  provides an analytical approximation of  $\text{VaR}_\alpha(L_n)$  for an asymptotic portfolio. Now, suppose that for a sub-portfolio dependence across obligors is also influenced by a local risk factor  $X$ , say, the price of some commodity. Then, conditional on a realisation  $y$  of global risk factor  $Y$ , obligors constituting the sub-portfolio would no longer be independent,

violating the assumptions of Proposition 2.18. That is, the additivity of capital charges on credits constituting the sub-portfolio would fail to hold. So long as credit exposures in the sub-portfolio affected by the local risk factor  $X$ , in aggregate, account for an arbitrarily small share of total portfolio exposure, the dependence of obligors on  $X$  can be ignored.

**Example 4.13.** A multi-factor model may offer greater explanatory power for dependence across obligors. Rather than a single systematic risk factor describing the general level of economic activity, a multi-factor model may include independent variables such as industrial production, retail sales, employment, inflation, interest rates, commodities prices, and consumer and business confidence. Without loss of generality, suppose that dependence across obligors is explained by the index of industrial production  $X$  and the short rate  $Y$ , which we assume are not perfectly comonotonic. Denote by  $H_{X,Y}(x,y)$  the joint distribution function of risk factors  $X$  and  $Y$ . Consider two portfolios with widely different characteristics: portfolio  $A$  with high exposure to  $X$  and low exposure to  $Y$ ; and portfolio  $B$  with low exposure to  $X$  and high exposure to  $Y$ . Credit VaR at the  $\alpha$  confidence level over a given risk measurement horizon for the respective portfolios would be associated with widely different realisations of pair  $(x,y)$  drawn from  $H_{X,Y}$ . Furthermore, for a given increment of exposure, we expect that an obligor with high exposure to  $X$  and low exposure to  $Y$  would incur a higher capital charge when added to portfolio  $A$  than when added to portfolio  $B$  — its addition to portfolio  $B$  would provide a greater diversification benefit. Tasche (2005) illustrates this diversification benefit, as measured by a multi-factor model, with a numerical example. Therefore, in the case of a multi-factor model of default dependence the capital charge on an instrument depends not only on its characteristics, but also on the characteristics of the portfolio in which it is held. That is, the property of portfolio invariance is not preserved.

*Remark 4.14.* By Theorem 3.10 random variables  $X$  and  $Y$  with distribution functions  $H_X$  and  $H_Y$ , respectively, are comonotonic if and only if  $X$  is almost surely a strictly increasing function of  $Y$ . In the case of continuous distribution functions  $H_X$  and  $H_Y$ ,  $X = \psi(Y)$  almost surely, with  $\psi = H_X^{-1} \circ H_Y$  increasing (Embrechts, McNeil, et al. 2002, Definition 7.5). Comonotonic risk factors may, in effect, be reduced to a single risk factor explaining dependence across obligors.

## Part II

# Empirical Findings



## Empirical Data

Under its implementation of Basel II, APRA requires ADIs to assess capital adequacy for credit, market and operational risks. ADIs determine regulatory capital for credit risk using either the standardised approach or, subject to approval, the IRB approach. The former applies prescribed risk weights to credit exposures based on asset class and credit rating grade to arrive at an estimate of RWA. Then, the minimum capital requirement is simply 8% of RWA. The standardised approach, which is an extension of Basel I, is straightforward to administer and produces a relatively conservative estimate of regulatory capital. The IRB approach, which implements the ASRF model described in Section 4.1, is a more sophisticated method requiring more input data estimated at higher precision. Its greater complexity makes it more expensive to administer, but usually produces lower regulatory capital requirements than the standardised approach. As a consequence, ADIs using the IRB approach may deploy their capital in pursuit of more (profitable) lending opportunities.

Upon implementation of Basel II in the first quarter of 2008, APRA had granted the four largest Australian banks, designated “major” banks, approval to use the IRB approach to capital adequacy for credit risk. They include: Commonwealth Bank of Australia (CBA), Westpac Banking Corporation (WBC), National Australia Bank (NAB), and Australia and New Zealand Banking Group (ANZ). WBC acquired St. George Bank (SGB) on 1 December 2008, and CBA acquired Bank of Western Australia (BWA) on 19 December 2008. Putting them in a global context, all four major Australian banks have been ranked in the top 20 banks in the world by market capitalisation, and top 50 by assets during 2013. Since the implementation of Basel II, the major banks have accounted for, on average, 74.6% of total assets on the balance sheet of ADIs regulated by APRA. Furthermore, of the regulatory capital reported by the major banks, on average, 88.0% has been assessed for credit risk, 6.9% for operational risk and 4.4% for market



risk, with other capital charges applied by APRA accounting for the remaining 0.7%.<sup>1,2</sup>

As discussed in Section 1.2, the Basel 2.5 package and Basel III Accord introduce reforms to address deficiencies in the Basel II framework exposed by the financial crisis of 2007–09. The Basel 2.5 reforms enhance minimum capital requirements, risk management practices and public disclosures in relation to risks arising from trading activities, securitisation and exposure to off-balance sheet vehicles. The Basel III reforms raise the quality and minimum required level of capital; promote the build up of capital buffers; establish a back-up minimum leverage ratio; improve liquidity and stabilise funding; and assess a regulatory capital surcharge on systemically important financial institutions. These reforms supplement Basel II. In particular, the model specification of the IRB approach under Basel II, described in Section 4.1, is unaltered by the implementation of Basel 2.5 and Basel III. The empirical analysis conducted in Part II examines the ASRF model prescribed under the IRB approach, and hence remains relevant today.

The primary contribution of our empirical analysis is to render a fundamental assessment of the model specification of the IRB approach. Section 5.1 describes the data reported to APRA by the major banks on a quarterly basis since the implementation of Basel II. Using these data we recover from the ASRF model measurements of the prevailing state of Australia’s economy and the capacity of its banking sector to absorb credit losses. In Section 5.2 we construct a portfolio that is representative of the credit exposures of the major Australian banks. Characteristics of the representative credit portfolio are used to measure the rate of convergence to the asymptotic portfolio loss distribution, and evaluate model robustness to parameter variations and model misspecification.

## 5.1 Capital Adequacy Reporting

Our fundamental assessment of the model specification of the IRB approach is rendered on the basis of internal bank data that is input to the ASRF model, which assesses regulatory capital charges for credit risk. The internal bank data is sourced from statutory returns lodged by ADIs with APRA. Chapter 7 uses these data to recover from ASRF model (4.10) realisations of the single systematic risk factor describing the prevailing state of Australia’s economy, and estimates of distance-to-default reflecting the capacity of its banking sector to absorb credit losses.

ADIs lodge their statutory returns with APRA using a secure electronic data submission system (Australian Prudential Regulation Authority 2014). A return is a collection of related forms covering the same reporting period. A form is a dataset containing information on a specific topic (e.g., capital, risk class/sub-class, financial statement, etc.). For reference purposes, forms in spreadsheet format and instructions are available at [www.apra.gov.au](http://www.apra.gov.au). Given the volume of data processed in preparing statutory returns, the major Australian banks automate

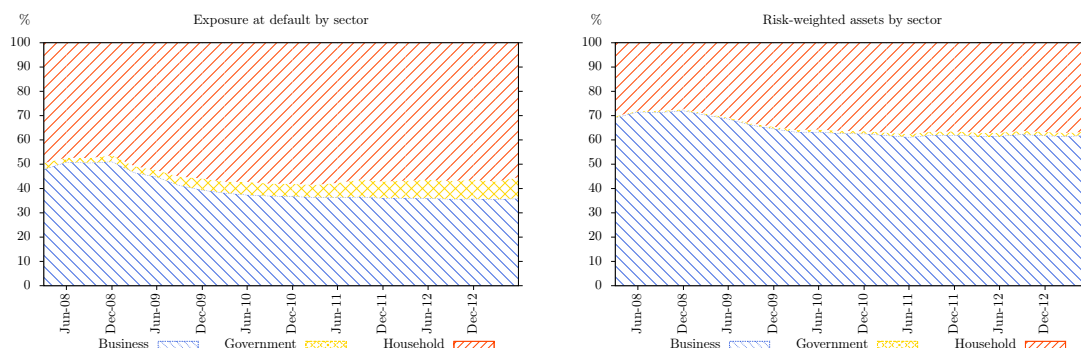
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<sup>1</sup> Average of figures for quarters ending 31 March 2008 through 30 June 2013 published by APRA (2013b) in its quarterly issue of ADI performance statistics.

<sup>2</sup> Data are reported for the major Australian banks, in aggregate, so as not to violate confidentiality agreements.

Figure 5.1: EAD and RWA, respectively, of IRB credit exposures of the major Australian banks are decomposed by sector (business, government and household).

Source: Australian Prudential Regulation Authority.



their electronic data submissions using XBRL (eXtensible Business Reporting Language) — an XML-based language for preparing, publishing, extracting and exchanging business and financial information. Once forms are validated and returns submitted, data are stored in APRA’s data warehouse. We use SQL programming to fetch and aggregate data reported to APRA by the major banks on a quarterly basis from 31 March 2008 through 30 June 2013.<sup>3</sup> Specifically, data reported on the statement of financial performance, capital adequacy form, and IRB credit risk forms.<sup>4</sup>

Banking book exposures are reported on credit risk forms<sup>5</sup> by IRB asset class: corporate (non-financial), small- and medium-sized enterprises (SME), bank, sovereign, residential mortgages, retail qualified revolving, and other retail.<sup>6</sup> For presentation purposes we merge IRB asset classes, and report credit exposures as business, government or household. Henceforth, we refer to these banking book exposures as IRB credit exposures. Figure 5.1 decomposes EAD and RWA, respectively, of IRB credit exposures into business, government and household sectors. Since the implementation of Basel II, RWA for credit risk reported by the major Australian banks has been divided, on average, 71.9/28.1 between IRB credit exposures and other banking book exposures. The market dominance of the major banks, coupled with the concentration of their regulatory capital held against unexpected losses on IRB credit exposures, convey the significance of the ASRF model in protecting the Australian banking sector against insolvency.

<sup>3</sup> NAB did not adopt the IRB approach to capital adequacy for credit risk until the second quarter of 2008. Therefore, in measuring the effect of the financial crisis, we omit NAB from our major banks’ aggregate for the quarter ending 31 March 2008.

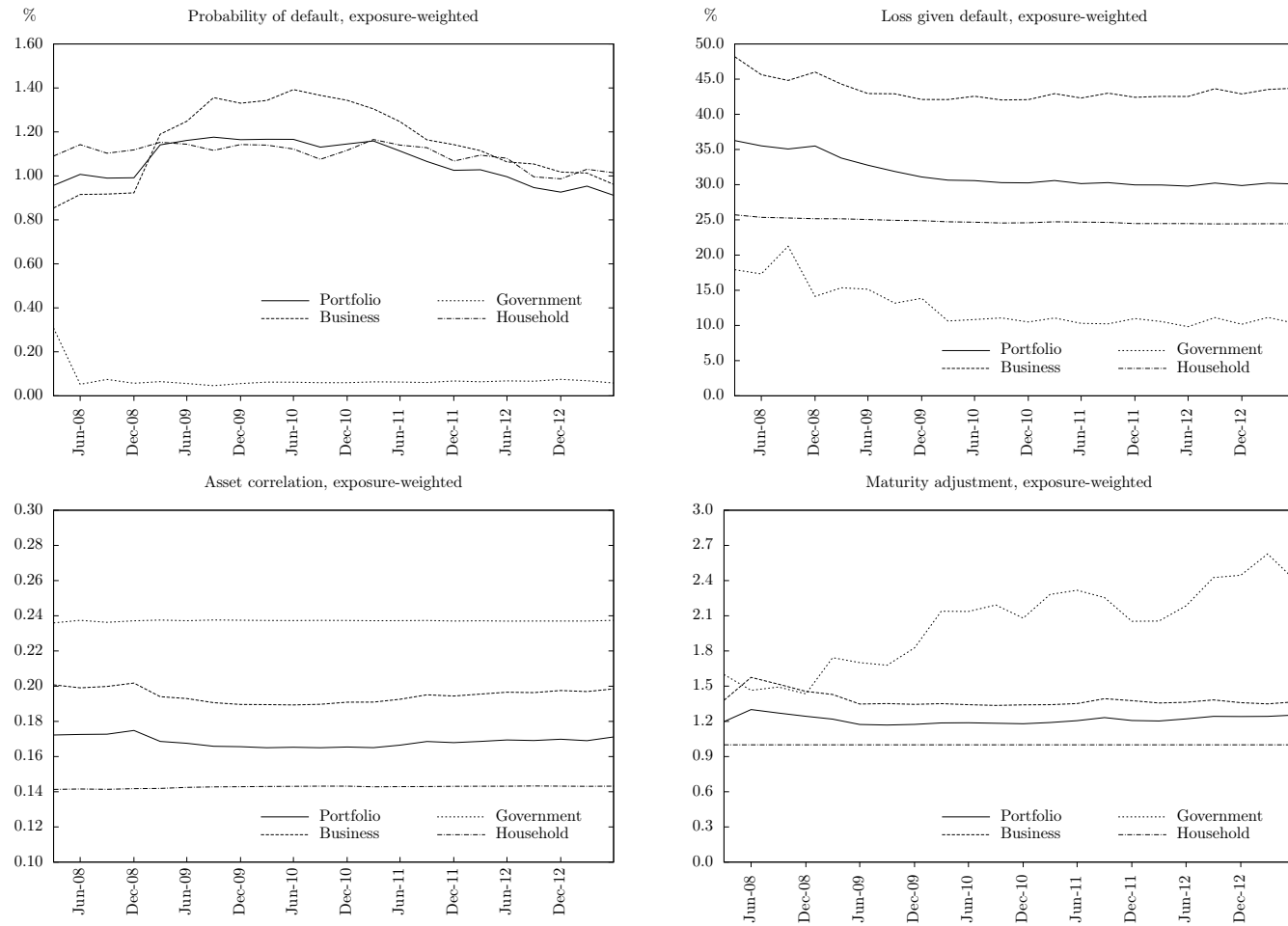
<sup>4</sup> For a number of quarters after the acquisitions of BWA and SGB by CBA and WBC, respectively, BWA and SGB reported credit risk using the IRB approach in parallel with the consolidated reporting of CBA and WBC, which determined RWA for credit risk in the banking books of BWA and SGB using the standardised approach. In our analysis we have included the parallel IRB credit risk forms submitted by BWA and SGB, and deducted the corresponding RWA, as determined by the standardised approach, from the consolidated RWA reported by CBA and WBC, respectively.

<sup>5</sup> Reporting forms and instructions for ADIs available at [www.apra.gov.au](http://www.apra.gov.au): ARF\_113\_1A, ARF\_113\_1B, ARF\_113\_1C, ARF\_113\_1D, ARF\_113\_3A, ARF\_113\_3B, ARF\_113\_3C, and ARF\_113\_3D.

<sup>6</sup> APRA (2008a) provides a full definition of IRB asset classes.

Figure 5.2: Risk characteristics of credit portfolios (IRB credit exposures) of the major Australian banks, rebalanced quarterly, by sector (business, government and household), as well as for the whole portfolio. Credit risk variables and parameters include: exposure-weighted unconditional PD, economic-downturn LGD, asset correlation and maturity adjustment.

Sources: Australian Prudential Regulation Authority.



Accordingly, we contend that, while focussing exclusively on IRB credit exposures of the major banks, our empirical analysis draws a representative sample of the credit risk assumed by the Australian banking sector.

Under the IRB approach, ADIs assign their on- and off-balance sheet credit exposures to internally-defined obligor grades reflecting PD bands, and LGD bands. Between 2008 and 2013 RWA of IRB credit exposures held in the banking book of the major Australian banks has been divided, on average, 75.4/24.6 between on-balance sheet assets and off-balance sheet exposures. EAD, RWA, expected loss, and exposure weighted LGD, unconditional PD, maturity and firm size are reported for each obligor grade. We assign IRB credit exposures reported by the major banks to standardised PD bands (i.e., consistent across the major banks), and calculate risk parameters characterising each of these standardised obligor grades. The ASRF model incorporates a maturity adjustment, which is a function of maturity and unconditional PD, for business and government credit exposures. Asset correlation is constant for residential mortgages and retail qualified revolving credit exposures; a function of unconditional PD for corporate, bank, sovereign and other retail credit exposures; and a function of firm size and unconditional PD for SME credit exposures (Basel Committee on Banking Supervision 2005). We construct a commingled credit portfolio by pooling obligor grades across IRB asset classes. Figure 5.2 plots exposure-weighted PD, LGD, asset correlation and maturity adjustment for commingled credit portfolios, rebalance quarterly, by sector (business, government and household), as well as for the whole portfolio.

Prudential standards published by APRA instruct ADIs to assign one-year unconditional (through-the-cycle) PDs and economic-downturn (stressed) LGD rates to IRB credit exposures. If the major banks were reporting unconditional PDs, one would expect that temporal variations in reported PDs would be explained exclusively by changes in portfolio composition. In reality though, the prevailing state of the economy likely affects reported PDs to a degree. That is, reported PDs are the result of perturbations of through-the-cycle estimates by point-in-time influences. Figure 5.2 shows the variation in reported exposure-weighted PD for the whole portfolio ranging from 0.91% to 1.18%. Assuming that portfolio composition was unchanged between 2008 and 2013, and using an exposure-weighted asset correlation of 16.8%,<sup>7</sup> the corresponding range of realisations of the single systematic risk factor describing the state of the economy is less than one-quarter of one standard deviation. This is a fraction of the range of realisations of the single systematic risk factor recovered from the ASRF model in Section 7.2. Accordingly, we don't believe that our conclusions are materially effected by point-in-time influences on reported PDs.

During an economic downturn losses on defaults are likely to be higher than under "normal" economic conditions. In developing the IRB approach to capital adequacy for credit risk, BCBS considered implementing a function that translates average, or through-the-cycle, LGD rates into

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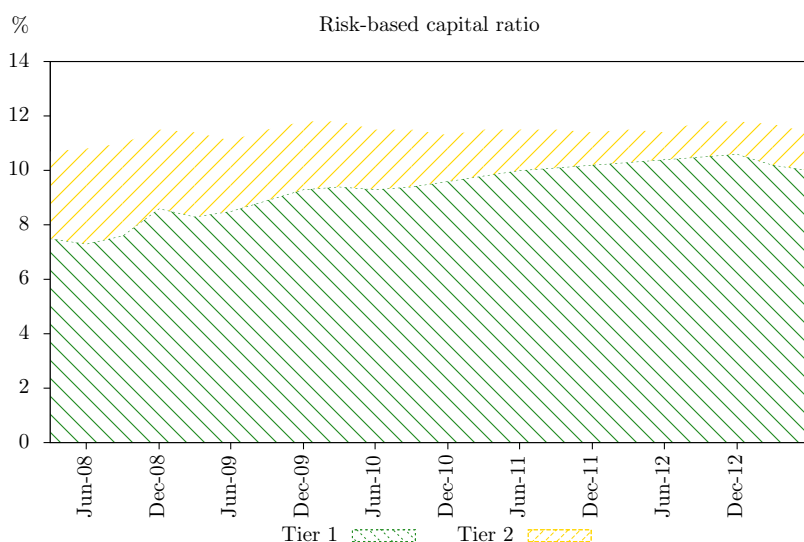
<sup>7</sup> The exposure-weighted asset correlation of the IRB credit exposures of the major Australian banks averaged 0.168 for quarters ending 31 March 2008 through 30 June 2013.

downturn LGD rates, similar to (2.20) which transforms unconditional PDs into conditional PDs. In principle, this function could depend on several factors including the state of the economy, magnitude of the average LGD, and asset class of the exposure. Given the evolving nature of LGD modelling and practices for estimating LGD at the time, BCBS elected to instruct ADIs to provide estimates of downturn LGD rates (Basel Committee on Banking Supervision 2005). There is no inconsistency here when assessing regulatory capital charges, since capital is held against unexpected losses arising under recessionary conditions. However, using downturn LGD rates to recover realisations of the single systematic risk factor from the ASRF model yields a prevailing state of the economy that is more expansionary than would be recovered using point-in-time LGD rates, assuming that the economy was not experiencing a (severe) downturn. The empirical analysis of Section 7.2 addresses this complication by recovering realisations of the single systematic risk factor describing the prevailing state of the Australian economy using downturn and through-the-cycle LGD rates. Note that distance-to-default, which measures the level of capitalisation, reflects the capacity to absorb credit losses under stressed economic conditions. So, it is appropriate to use downturn LGD rates when translating realisations of the single systematic risk factor recovered from the ASRF model into estimates of distance-to-default.

In a criticism of the “quasi” state dependence of the model specification of the IRB approach, Jarrow (2007) argues that the concern that making regulatory capital state dependent would be procyclical and exacerbate the business cycle is overstated and perhaps unwarranted. It is known that PD and LGD are state dependent (e.g., dependent on the state of the economy).

Figure 5.3: Risk-based capital ratio (capital / RWA) of the major Australian banks decomposed into tier 1 and tier 2 capital.

Source: Australian Prudential Regulation Authority.



Recasting (2.30) for the Gaussian case, Remark 2.12 infers that the PD of obligor  $i$  conditional on  $Y = \Phi^{-1}(1-\alpha)$  may be interpreted as the probability of default of obligor  $i$  is no greater than  $p_i(\Phi^{-1}(1-\alpha))$  in  $(\alpha \times 100)\%$  of economic scenarios. It follows from Remark 4.5 that PD is state dependent in so far as  $\alpha = 0.999$ . Similarly, LGD is state dependent in so far as BCBS instructs regulated institutions to provide estimates of economic-downturn LGD rates. Regulatory capital, though, serves the sole purpose of solvency assessment, so the 99.9% confidence level and downturn LGD rates appear to be prudent choices for reducing the risk of bank failures.

RWA, capital and provisions are reported on the capital adequacy form.<sup>8</sup> RWA are reported by risk class, and within the credit risk class, for IRB asset classes and the standardised approach. These data are aggregated across capital adequacy forms submitted by the major Australian banks. Note that the minimum capital requirement is simply 8% of RWA. Then, subject to a minimum 8% of RWA, APRA sets a prudential capital ratio for each ADI, and an ADI typically holds a capital buffer above its prudential capital requirement (Australian Prudential Regulation Authority 2007). Figure 5.3 decomposes the aggregate capital base of the major banks, measured as a percentage of RWA, into tier 1 and tier 2 capital.

Finally, we take credit losses as charges for bad and doubtful debts reported on the statement of financial performance,<sup>9</sup> or income statement. Credit losses are aggregated across income statements reported to APRA by the major banks.

## 5.2 A Representative Credit Portfolio

The robustness of the model specification of the IRB approach to a relaxation in model assumptions is evaluated on a portfolio that is representative of the IRB credit exposures of the major Australian banks as at 31 December 2012. That is, exposure weight of each PD band within an IRB asset class is equal to the corresponding EAD reported by major banks as a percentage of IRB credit exposures. Credits assigned the same obligor grade share the same risk characteristics: LGD, unconditional PD and asset correlation. For presentation purposes Table 5.1 categorises credit exposures as business, government or household, and groups them into credit rating grades on the basis of obligors' unconditional PD.<sup>10</sup> It reports EAD, and exposure weighted LGD, unconditional PD and asset correlation for each credit rating grade by sector. Chapter 6 uses characteristics of this representative credit portfolio to measure the rate of convergence, in terms of number of obligors, to the asymptotic portfolio loss distribution, and evaluate the sensitivity of credit risk capital to dependence structure as modelled by asset correlations and elliptical copulas.

<sup>8</sup> Reporting form ARF\_110\_0\_2 and instructions for ADIs available at [www.apra.gov.au](http://www.apra.gov.au).

<sup>9</sup> Reporting form ARF\_330\_0\_C and instructions for ADIs available at [www.apra.gov.au](http://www.apra.gov.au).

<sup>10</sup> We use the mapping of S&P credit rating grades to KMV expected default frequency values derived by Lopez (2002).

Table 5.1: Characteristics (EAD and exposure-weighted LGD, unconditional PD and asset correlation) of a portfolio that is representative of the IRB credit exposures of the major Australian banks as at 31 December 2012 are reported for each credit rating grade by sector (business, government and household) and for the whole portfolio.

		Credit rating grade							
		Total	AAA	AA	A	BBB	BB	B	C <sup>a</sup>
Business	EAD, $\delta_i$ (\$)	3,552	7	549	1,001	874	728	327	66
	LGD, $\eta_i$	0.429	0.219	0.519	0.474	0.418	0.356	0.339	0.412
	Unconditional PD, $p_i$ (%)	1.02	0.02	0.03	0.11	0.41	1.24	3.08	18.56
	Asset correlation, $\rho_i$	0.198	0.239	0.238	0.231	0.206	0.159	0.112	0.091
Government	Exposure at default, $\delta_i$ (\$)	785	538	203	18	10	13	3	
	Loss given default, $\eta_i$	0.102	0.088	0.085	0.223	0.496	0.387	0.408	
	Unconditional PD, $p_i$ (%)	0.07	0.01	0.03	0.09	0.40	1.74	3.00	
	Asset correlation, $\rho_i$	0.237	0.239	0.238	0.235	0.218	0.171	0.149	
Household	Exposure at default, $\delta_i$ (\$)	5,663			2,581	1,725	919	291	147
	Loss given default, $\eta_i$	0.244			0.233	0.225	0.264	0.357	0.324
	Unconditional PD, $p_i$ (%)	0.99			0.08	0.39	1.10	3.68	17.80
	Asset correlation, $\rho_i$	0.143			0.144	0.147	0.143	0.123	0.127
Portfolio	Exposure at default, $\delta_i$ (\$)	10,000	545	752	3,600	2,609	1,660	621	213
	Loss given default, $\eta_i$	0.299	0.090	0.402	0.300	0.291	0.306	0.348	0.350
	Unconditional PD, $p_i$ (%)	0.93	0.01	0.03	0.09	0.40	1.16	3.36	18.03
	Asset correlation, $\rho_i$	0.170	0.239	0.238	0.169	0.167	0.150	0.117	0.116

<sup>a</sup>CCC, CC and C credit ratings.

# Robustness of the Asymptotic Single Risk Factor Model

Regulatory capital models, which serve the sole purpose of solvency assessment, require precision in the measurement of absolute risk levels under stressed economic conditions. In this chapter we evaluate the robustness of the ASRF model prescribed under the Basel II IRB approach to a relaxation of model assumptions using a portfolio that is representative of the IRB credit exposures of the major Australian banks. Section 6.1 measures the rate of convergence, in terms of number of obligors, of empirical loss distributions to the asymptotic portfolio loss distribution (i.e., the distribution of conditional expectation of portfolio percentage loss representing an infinitely fine-grained portfolio). In the process we demonstrate that Proposition 2.28 holds for a credit portfolio that exhibits sufficient granularity. The IRB approach applies the one-factor Gaussian copula with default dependence described by the matrix of pairwise correlations between obligors' asset values. We proceed to evaluate model robustness to parameter variations and model misspecification. Section 6.2 measures the sensitivity of credit risk capital to dependence structure as described by asset correlations. Section 6.3 examines the effect of tail dependence by measuring the sensitivity of credit risk capital to dependence structure, as modelled by a variety of elliptical copulas.

## 6.1 Rate of Convergence to the Asymptotic Distribution

Before measuring the rate of convergence to the asymptotic portfolio loss distribution, we demonstrate that Proposition 2.28, on which the IRB approach is premised, holds for a representative credit portfolio within a static (i.e., single-period) framework. Table 5.1 describes the representative credit portfolio. In order to construct a portfolio that exhibits sufficient granularity



we impose the constraint that no credit accounts for more than one basis point exposure. This exercise demonstrates that the  $\alpha$  quantile of the distribution of  $\mathbb{E}[L_n | Y]$ , which is associated with the  $(1 - \alpha)$  quantile of the distribution of  $Y$ , may be substituted for the  $\alpha$  quantile of the distribution of  $L_n$ . Firstly, capital held against unexpected losses at the 99.9% confidence level over a one-year horizon,  $\tilde{K}_{99.9\%}(L_n)$ , is calculated analytically by ASRF model (4.11). The expectation of portfolio percentage loss over a one-year horizon conditional on a state of the economy that is worse than at most 99.9% of economic scenarios,  $\mathbb{E}[L_n | Y = \Phi^{-1}(0.001)]$ , is readily calculated by the first term on the right-hand side of (4.11). Expected loss,  $\mathbb{E}[L_n]$ , is given by the second term of the right-hand side of (4.11). And  $\tilde{K}_{99.9\%}(L_n)$  is the difference between  $\mathbb{E}[L_n | Y = \Phi^{-1}(0.001)]$  and  $\mathbb{E}[L_n]$ .

Next,  $K_{99.9\%}(L_n)$  is determined by computationally intensive simulation (2.44). Credit VaR at the 99.9% confidence level over a one-year horizon,  $\text{VaR}_{99.9\%}(L_n)$ , is determined from the empirical loss distribution of the representative credit portfolio, which is generated by simulation of (2.26) parameterised by (2.21).<sup>1</sup> Monte Carlo simulation performs 1,000,000 iterations to generate the empirical loss distribution (Figure 6.1). Equation (2.44) calculates the portfolio percentage loss for each iteration, and credit VaR is the 99.9% quantile of the empirical loss

<sup>1</sup> Gaussian random variables are generated using GNU Scientific Library routine `gsl_ran_gaussian`. It implements the Box-Muller algorithm, which make two calls to the MT19937 generator of Makoto Matsumoto and Takuji Nishimura.

Figure 6.1: The empirical loss distribution of the representative credit portfolio described in Table 5.1 is generated by Monte Carlo simulation. The portfolio is constructed such that no credit accounts for more than one basis point exposure, and credit losses are reported as a percentage of EAD.

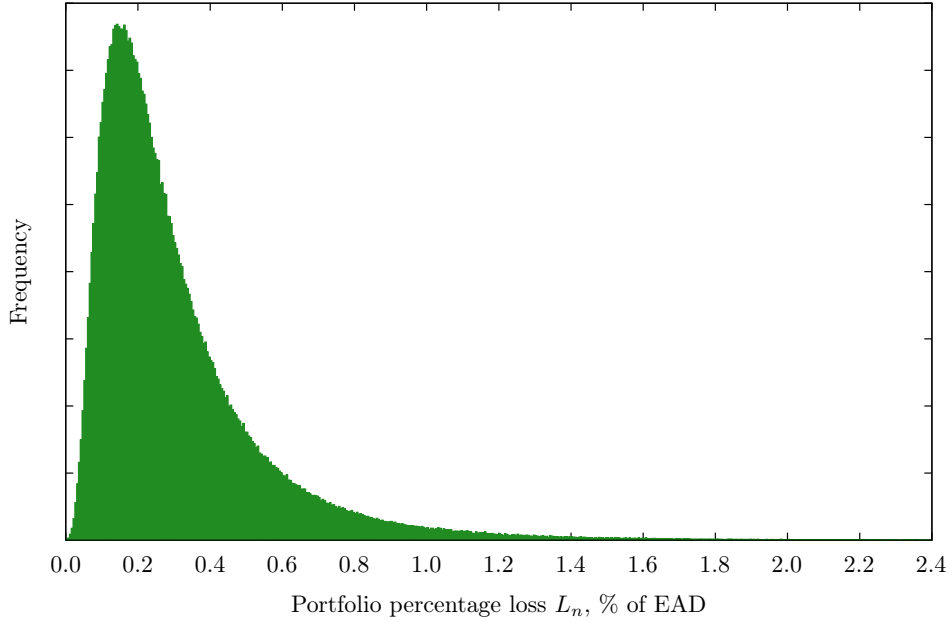


Table 6.1: Capital charges, at the 99.9% confidence level over a one-year horizon, assessed on the representative credit portfolio described in Table 5.1. The portfolio is constructed such that no credit accounts for more than one basis point exposure. Credit VaR at the 99.9% confidence level is computed numerically by Monte Carlo simulation, and its analytical approximation is calculated by the ASRF model.

	% of EAD		
	ASRF	Simulation	
$\mathbb{E}[L_n   Y = \Phi^{-1}(0.001)]$	2.18	2.19	$\text{VaR}_{99.9\%}(L_n)$
$\mathbb{E}[L_n]$	0.31	0.31	$\mathbb{E}[L_n]$
$\tilde{K}_{99.9\%}(L_n)$	1.87	1.88	$K_{99.9\%}(L_n)$

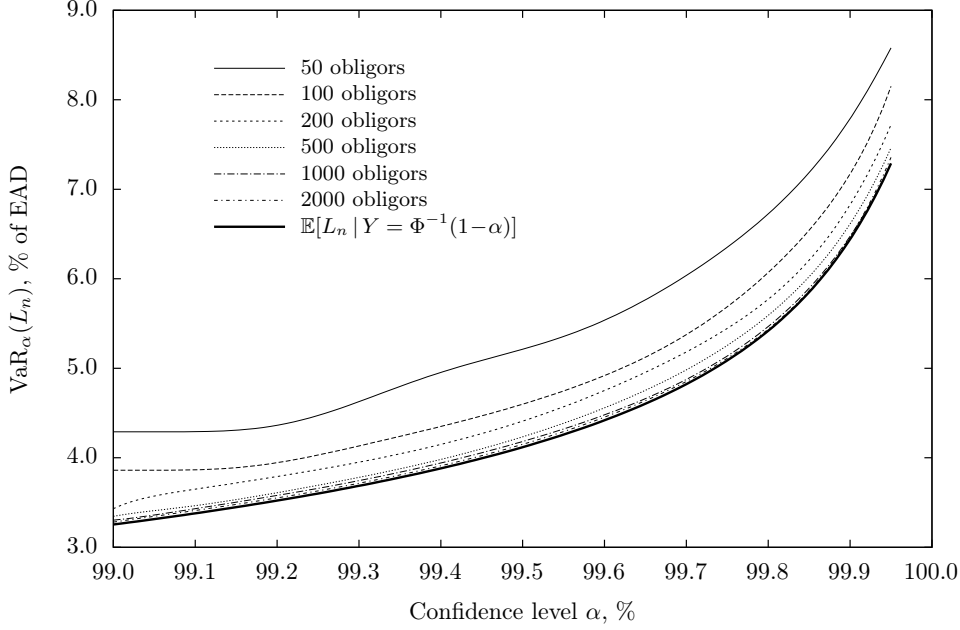
distribution described by (2.45). Expected loss,  $\mathbb{E}[L_n]$ , is given by (2.46). And  $K_{99.9\%}(L_n)$  is the difference between  $\text{VaR}_{99.9\%}(L_n)$  and  $\mathbb{E}[L_n]$ .

It remains to compare the estimates produced by the analytical and simulation models. Given that this representative portfolio contains a large number of credits without concentration in a few names dominating the rest of the portfolio, we anticipate that as the number of simulation iterations increases, its empirical loss distribution will converge to the distribution of conditional expectation of portfolio percentage loss. Confirming our intuition, Table 6.1 reports that estimates from the ASRF model and Monte Carlo simulation are within one basis point of one another. Recall that ASRF model (4.11) describes defaults as conditionally independent Gaussian random variables, and the Monte Carlo simulation of (2.26), parameterised by (2.21), draws random variables representing the single systematic risk factor and obligor specific risks from a Gaussian distribution. So, it would be surprising if this convergence did not occur. Indeed, for a number of iterations large enough and a portfolio exhibiting sufficient granularity, the simulation does little more than demonstrate that the expectation of indicator function  $\mathbf{1}_{\{Z_i < \zeta_i(y)\}}$  is conditional PD  $p_i(y)$ . Our findings provide empirical support for Proposition 2.28.

Evidently, the representative credit portfolio described in Table 5.1, with no credit accounting for more than one basis point exposure, adequately satisfies the asymptotic granularity condition of Definition 2.14, an assumption of Proposition 2.28. But how large need the number of credits, or obligors, constituting a portfolio be for  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$  to produce a statistically accurate estimate of  $\text{VaR}_\alpha(L_n)$ ? We proceed to address this question by constructing portfolios comprising obligors ranging in number from 50 to 2000. For each portfolio we assign its constituent obligors equal dollar EAD, and exposure weighted LGD, unconditional PD and asset correlation as calculated for exposures to the business sector of the representative credit portfolio described in Table 5.1. We choose to conduct this exercise on portfolios representative of IRB credit exposures of the major banks to the business sector, because corporate loans are typically “lumpier” than residential mortgages.

We generate empirical loss distributions, one for each of the constituted portfolios containing a finite number of obligors, by simulation of (2.26) parameterised by (2.21). Monte Carlo simulation performs 1,000,000 iterations to generate an empirical loss distribution for each of

Figure 6.2: Credit VaR at the  $\alpha$  confidence level,  $99.0\% \leq \alpha < 100.0\%$ , for portfolios comprising obligors ranging in number from 50 to 2000. The portfolios are representative of IRB credit exposures of the major Australian banks to the business sector reported in Table 5.1. The curves illustrate the rate of convergence in the tail of empirical loss distributions,  $\text{VaR}_\alpha(L_n)$ , to the distribution of conditional expectation of portfolio percentage loss,  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$ , representing the loss distribution of an infinitely fine-grained portfolio.



the constituted portfolios. The distribution of conditional expectation of portfolio percentage loss represents the loss distribution of an infinitely fine-grained portfolio. Solvency assessment is concerned with the tail of the portfolio loss distribution, so we examine the confidence interval bounded by  $99.0\% \leq \alpha < 100.0\%$ .  $\text{VaR}_\alpha(L_n)$  for each of the constituted portfolios is the  $\alpha$  quantile of the empirical loss distributions described by (2.45).  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$  is given by the first term on the right-hand side of ASRF model (4.10). Figure 6.2 plots the tail of each of the empirical loss distributions along with the tail of the distribution of conditional expectation of portfolio percentage loss, illustrating the rate of convergence in terms of number of obligors. By inspection we argue that a statistically accurate estimate of  $\text{VaR}_\alpha(L_n)$  is given by  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$  for portfolios comprising 1,000 obligors or more, without concentration in a few names dominating the rest of the portfolio.

In a numerical exercise Gordy and Lütkebohmert (2007) applied their granularity adjustment to German bank portfolios, which they classify as large (more than 4,000 credit exposures), medium (between 1,000 and 4,000 exposures), small and very small. By comparison with Figure 6.2, they estimated a granularity adjustment of 12–14 basis points for large portfolios and 14–36 basis points for medium portfolios. In view of Remark 2.21, we argue that the commingled

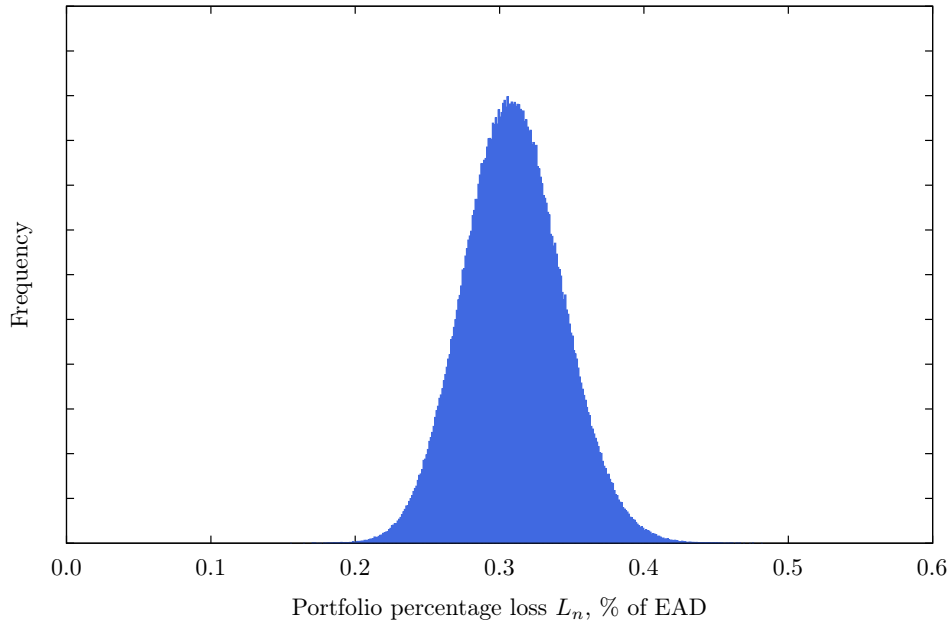
credit portfolios constructed by pooling IRB credit exposures of the major Australian banks, as outlined in Section 5.1, for the purpose of undertaking the empirical analysis of Chapter 7 would approach the asymptotic granularity of Definition 2.14.

## 6.2 Sensitivity to Dependence As Described by Asset Correlations

The IRB approach, in effect, applies the one-factor Gaussian copula of Corollary 3.18 with matrix (3.15) describing pairwise correlations between obligors' asset values. Choosing the one-factor Gaussian copula to model default dependence, we evaluate the sensitivity of credit risk capital to dependence structure as described by asset correlations.

To illustrate the fundamental role of correlation in credit risk modelling, Figure 6.3 plots the empirical loss distribution of the representative credit portfolio described in Table 5.1, except that obligors' asset values are assumed to be uncorrelated. That is, Figure 6.3 assumes that default events are independent. Vis-à-vis the empirical loss distribution plotted in Figure 6.1, in which the exposure-weighted asset correlation is 0.170,  $\mathbb{E}[L_n]$  is unchanged at 0.31% of EAD, but  $\text{VaR}_{99.9\%}(L_n)$  falls from 2.19% to 0.42% of EAD. So, the effect of asset correlation on the representative credit portfolio described in Table 5.1 is to increase credit VaR, at the 99.9% confidence level over a one-year horizon, more than five times the credit VaR had default events

Figure 6.3: The empirical loss distribution, generated by Monte Carlo simulation, of the representative credit portfolio described in Table 5.1, except that default events are assumed to be independent. The portfolio is constructed such that no credit accounts for more than one basis point exposure, and credit losses are reported as a percentage of EAD.

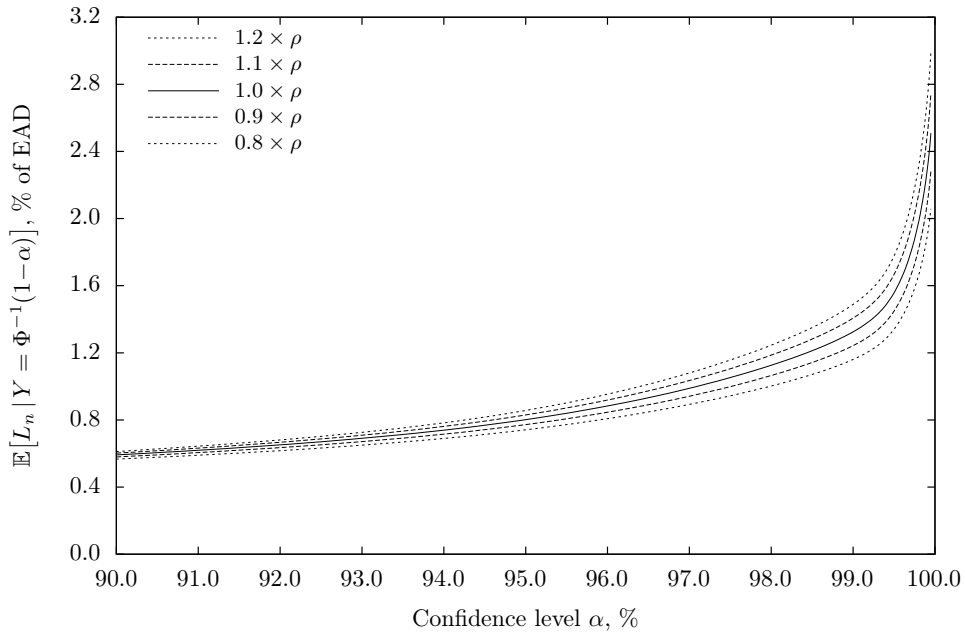


been independent.

Recall from Section 5.1 that the IRB approach models asset correlation as a constant for residential mortgages and retail qualified revolving credit exposures; a function of unconditional PD for corporate, bank, sovereign and other retail credit exposures; and a function of firm size and unconditional PD for SME credit exposures. Parameters of the asset correlation functions prescribed under the IRB approach were derived from an analysis of times series collected by G10 supervisors. Even if the dependence described by parameters derived from this time series analysis continues to hold, the actual asset correlation will lie in some distribution around the estimate given by the model specification of the IRB approach. So, some measure of the sensitivity of credit risk capital to the error in asset correlation estimates would be informative.

Figure 6.4 plots the sensitivity of  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$ , and hence credit risk capital, to dependence structure as described by asset correlations. Here, we adopt the credit risk capital interpretation of Definition 4.4, which applies to portfolios that adequately satisfy the asymptotic granularity condition of Definition 2.14.  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$ , given by the first term on the right-hand side of (4.10), is calculated for the representative credit portfolio described in Table 5.1, and then recalculated after adjusting the asset correlation assigned to constituent obligors by  $\pm 10\%$  and  $\pm 20\%$ . Since solvency assessment is concerned with the tail of the portfolio loss distribution, we examine the confidence interval bounded by  $90.0\% \leq \alpha < 100.0\%$ . Conditional expectation of portfolio percentage loss becomes more sensitive to asset correlation as

Figure 6.4:  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$ ,  $90.0\% \leq \alpha < 100.0\%$ , for the representative credit portfolio described in Table 5.1. The sensitivity of  $\mathbb{E}[L_n | Y = \Phi^{-1}(1-\alpha)]$  to default dependence structure is measured by adjusting the asset correlation assigned to constituent obligors by  $\pm 10\%$  and  $\pm 20\%$ .



one moves further into the tail of the portfolio loss distribution. At the 99.9% confidence level the relative error in asset correlation estimates affects the calculation of credit risk capital by a similar magnitude. Naturally, this heuristic for the sensitivity of credit risk capital to asset correlation only applies deep in the tail of the loss distribution of near asymptotic portfolios with characteristics (viz., unconditional PDs and asset correlations) not very different from those described in Table 5.1.

As noted above, BCBS has calibrated the level of correlation parameter  $\rho_i$  in (4.10) by IRB asset class. Our test of the robustness of the model specification of the IRB approach to variations in the correlation parameter takes asset correlations assigned under the IRB approach and adjusts them  $\pm 10\%$  and  $\pm 20\%$ . We make no attempt to recalibrate the asset correlation parameter. Using time series of pooled credit loss observations Fitch Ratings (2008) empirically estimated asset correlations for each IRB asset class. They find that across asset classes, correlation parameters assigned under the IRB approach are conservative relative to their empirically derived estimates. Fitch Ratings contend that this conservatism provides a margin for error during periods of economic distress when correlations tend to spike. Moreover, it accommodates observed geographic variations in correlation parameters.

### 6.3 Sensitivity to Dependence As Modelled by Elliptical Copulas

While it's generally acknowledged that models which assume that financial data follow a Gaussian distribution tend to underestimate tail risk, the IRB approach to solvency assessment does apply the one-factor Gaussian copula. As discussed in Section 3.5, Gaussian copulas do not exhibit tail dependence — the tendency for extreme observations (i.e., credit defaults) to occur simultaneously for all random variables. We examine the effect of tail dependence by measuring the sensitivity of credit risk capital to dependence structure as modelled by elliptical copulas, including Gaussian and Student's  $t$  copulas. The latter, which we abbreviate by  $t$ -copula, admits tail dependence with fewer degrees of freedom producing stronger dependence (Figure 3.1).

In order to evaluate the sensitivity of credit risk capital to default dependence structure, we generate empirical loss distributions of the representative credit portfolio described in Table 5.1 using elliptical copulas: one-factor Gaussian copula, and  $t$ -copulas with 30, 10 and 3 degrees of freedom and Gaussian margins. Again, the portfolio is constructed such that no credit accounts for more than one basis point exposure. Assuming that asset values follow a log-normal distribution, distributional differences are attributed to the copula function modelling default dependence. The one-factor Gaussian copula is implemented by simulation of (2.26) parameterised by (2.21), and generates the empirical loss distribution plotted in Figure 6.1.  $t$ -copulas with Gaussian margins are implemented by simulation of (3.25) parameterised by (3.24).<sup>2</sup> Monte Carlo simulation performs 1,000,000 iterations.

<sup>2</sup> Normally and chi-square distributed random variables are generated using GNU Scientific Library routines `gsl_ran_gaussian` and `gsl_ran_chisq`, respectively.

Figure 6.5:  $\text{VaR}_\alpha(L_n)$ ,  $90.0\% \leq \alpha < 100.0\%$ , for the representative credit portfolio described in Table 5.1. The portfolio is constructed such that no credit accounts for more than one basis point exposure. The sensitivity of  $\text{VaR}_\alpha(L_n)$  to default dependence structure is measured by generating empirical loss distributions using elliptical copulas and assuming that asset values follow a log-normal distribution.

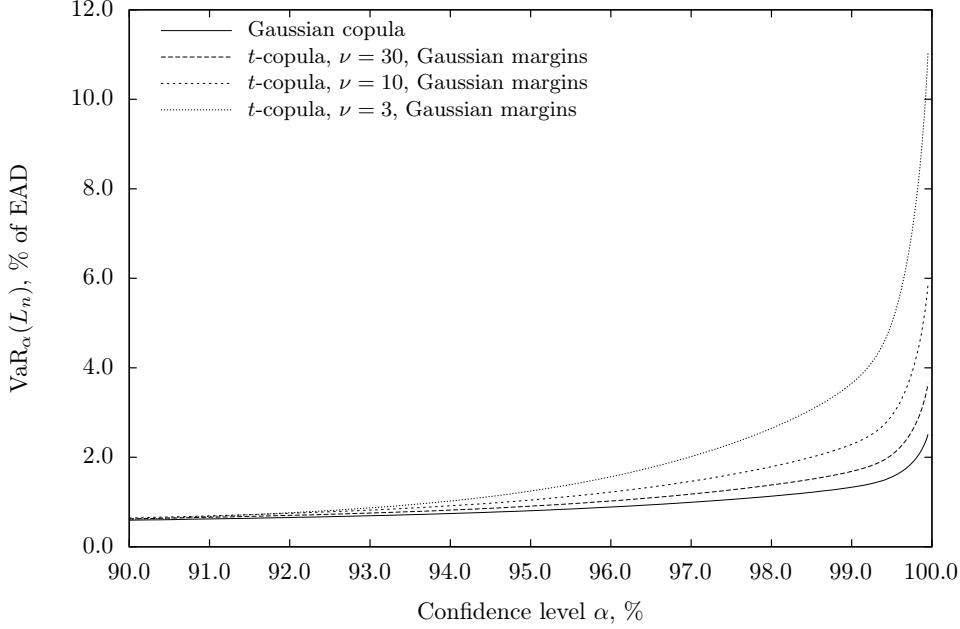


Figure 6.5 plots the sensitivity of credit VaR, and hence credit risk capital, to default dependence structure as modelled by the one-factor Gaussian copula and  $t$ -copulas with Gaussian margins. As in Section 6.2, we examine the confidence interval bounded by  $90.0\% \leq \alpha < 100.0\%$ . At the 90.0% confidence level there is little difference in estimates of credit VaR computed by the respective elliptical copulas. However, as one moves further into the tail of the empirical loss distribution estimates of credit VaR diverge at an accelerating rate. At the 99.9% confidence level credit VaR computed by the  $t$ -copula with  $\nu = 10$  is more than double that computed by the Gaussian copula; and the  $t$ -copula with  $\nu = 3$  estimates credit VaR to be more than four times the estimate produced by the Gaussian copula. Unfortunately, the calibration of degrees of freedom is not easy and to some extent subjective, which may explain why the one-factor Gaussian copula prevails in practice. The sensitivity of credit risk capital to the choice of elliptical copula can be much greater than its sensitivity to asset correlation.

We reach a similar conclusion to that of Frey et al. (2001), who examine model misspecification. They simulated empirical loss distributions for a range of correlation parameters and obligor default probabilities assuming that default dependence is modelled by multivariate Gaussian and Student's  $t$  distributions. As noted in Section 3.5, a multivariate  $t$  distribution admits tail dependence with fewer degrees of freedom producing stronger dependence, and it inherits the correlation matrix of the multivariate Gaussian distribution that it generalises. Assuming

that default dependence is modelled by a multivariate Gaussian distribution, credit losses at high quantiles of the empirical loss distribution would be seriously understated if the true model of dependence were a multivariate  $t$  distribution. Frey et al. (2001) argue that asset correlations are not enough to describe default dependence, and “[a]n assumption of multivariate normality may not model the potential extreme risk in the portfolio.” However, they also recommend the use of historical default data to estimate true default correlations and calibrate credit risk models.

Hamerle and Röscher (2005) demonstrate the robustness of the Gaussian copula, which underlies the IRB approach, to model misspecification. They simulated empirical loss distributions assuming that default dependence is modelled by the Student’s  $t$  copula, and then empirically estimated correlation parameters using the misspecified Gaussian copula. While parameter estimates for asset correlation exhibit large biases, quantiles of empirical loss distributions subsequently generated by the misspecified Gaussian copula with the overestimated asset correlations no longer seriously underestimate true credit losses. However, the economic damage wrought by the financial crisis of 2007–09 in the north Atlantic suggests that parameter estimation errors did not conveniently offset model uncertainties during the recent crisis.





# Measurements from the Asymptotic Single Risk Factor Model of the Australian Banking Sector

Our empirical analysis renders a fundamental assessment of the model specification of the Basel II IRB approach on the basis of evidence from Australia. We take measurements from the ASRF model, described in Section 4.1, of the Australian banking sector. Accordingly, we make observations about the prevailing state of Australia's economy and the solvency of its banking sector, which inform our fundamental assessment. But, we do not propose model enhancements, nor do we draw policy implications from these observations. Although, our observations may open a debate on the model specification of the IRB approach or related regulatory policies.

The ASRF model prescribed under the IRB approach is an asset value factor model. Other asset value models include: Moody's KMV, RiskMetrics, and most internal bank models. They too are generally factor models. Note that multi-factor asset value models typically express asset values as a function of a composite factor obtained by the superposition of underlying independent risk indices (Bluhm et al. 2010). We believe that the empirical findings presented in this chapter are applicable to the broader class of asset value factor models of credit risk.

The financial crisis of 2007–09, also known as the global financial crisis, precipitated the worst global recession since the Great Depression of the 1930s. With the implementation of Basel II preceding the time when the effect of the crisis was most acutely felt, our empirical analysis produces a fundamental evaluation of the impact of the crisis on the Australian banking sector using internal bank data collected by APRA. We are not the first to attempt to measure the effects of the recent crisis, but we believe that we are the first to do so using regulatory data. Other studies on the financial crisis rely on market data, macroeconomic indicators or published

financial statistics.<sup>1</sup> In arguing the resilience of the Australian economy to the crisis, McDonald and Morling (2011), and Brown and Davis (2010) describe its performance primarily in terms of macroeconomic indicators and financial statistics. Our measurements from the ASRF model of the Australian banking sector corroborate their observations. Allen and Powell (2012), on the other hand, rely on market data and reach a markedly different conclusion regarding the condition of the Australian banking sector during the financial crisis. We submit that their results are biased by plummeting market prices and spiking volatility reflecting the overreaction of market participants gripped by fear at the depth of the crisis.

There are many more international studies on the causes, effects and implications of the financial crisis. For instance, Jagannathan et al. (2013) argue that conditions for the crisis were created by the transformation of global labour markets through technical innovation, capital flows induced by current account imbalances, and weaknesses in financial regulation and governance. Cecchetti et al. (2011) examine whether the macroeconomic performance of countries during the recent crisis was due to pre-crisis policy decisions. They conclude that national economies that exhibited a better capitalised banking sector, low loan-to-deposit ratios, a current account surplus, high level of foreign exchange reserves, and low levels and growth of private sector credit-to-GDP were less vulnerable. Introducing the financial crisis as a shock to the supply of credit, Laeven and Valencia (2013) study the effect of public policy interventions on firm growth in 50 countries. They find that bank recapitalisations and discretionary fiscal policy supported the growth of firms that were more dependent on external financing. Flannery et al. (2013) analyse the market response during the crisis to the information asymmetry of banks. They observe that banks' microstructure properties (specifically, stock bid-ask spreads and price impacts) were significantly higher than those for non-banks during the crisis, which is consistent with a sharp increase in the relative opacity of banks. Suh (2012) proposes a factor-augmented correlated default model to measure the systemic risk of large US financial institutions using stock market data. With time-varying asset correlations rising during periods of financial distress, he estimates that systemic risk reached historically high levels during the recent crisis. We reiterate that access to internal bank data collected by the prudential regulator distinguishes our research from other studies on the financial crisis.

From our assessment of the model specification of the IRB approach emerges a methodology for regulators to monitor the prevailing state of the economy as described by the single systematic risk factor, and the capacity of supervised banks to absorb credit losses as measured by distance-to-default. While confidentiality agreements preclude us from publishing results for individual banks, regulators would be at liberty to conduct their analysis on an individual bank basis. Measurements from the ASRF model signalling an overheating economy and procyclical movements in capital bases, corroborated by macroeconomic performance indicators, would prompt supervisory intervention. For example, banks could be instructed to build up their countercyclical capital buffer introduced under Basel III in order to rein in rapidly accelerating credit growth.

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<sup>1</sup> Measures of financial position (stocks) and performance (flows), national and company level.

Furthermore, a longer history of the time series of the prevailing state of the economy recovered from the ASRF model could serve to validate and calibrate macroeconomic-based models for estimating conditional (point-in-time) PDs. Macroeconomic-based models for estimating point-in-time PDs usually express the single systematic risk factor as a function of macroeconomic variables and a random economic shock (Chan-Lau 2006). This time series could also inform the development of economic scenarios for stress testing, a standard tool of prudential supervision.

This chapter begins in Section 7.1 by comparing the performance of the Australian economy over the past decade with that of the United States and United Kingdom on macroeconomic indicators and financial statistics. Both the latter economies experienced the full force of the financial crisis of 2007–09. Then, applying the ASRF model, with appropriate substitutions, to internal bank data collected by APRA since the implementation of Basel II, our empirical analysis generates quarterly time series of: (i) the single systematic risk factor describing the prevailing state of Australia’s economy; and (ii) distance-to-default measuring the capacity of its banking sector to absorb credit losses. The former is plotted in Section 7.2, while the latter is reported in Section 7.3. Comparing these time series with macroeconomic indicators, financial statistics and external credit ratings we render a fundamental assessment of the model specification of the IRB approach. The relatively short quarterly time series available leads to an intuitive assessment of the ASRF model rather than any formal testing of its efficacy. Since the depths of the financial crisis were reached after the implementation of Basel II, this time series analysis measures the impact of the crisis on the Australian banking sector. Finally, in a variation on the evaluation of distance-to-default, Section 7.4 employs reverse stress testing to explore stress events that would trigger material supervisory intervention.

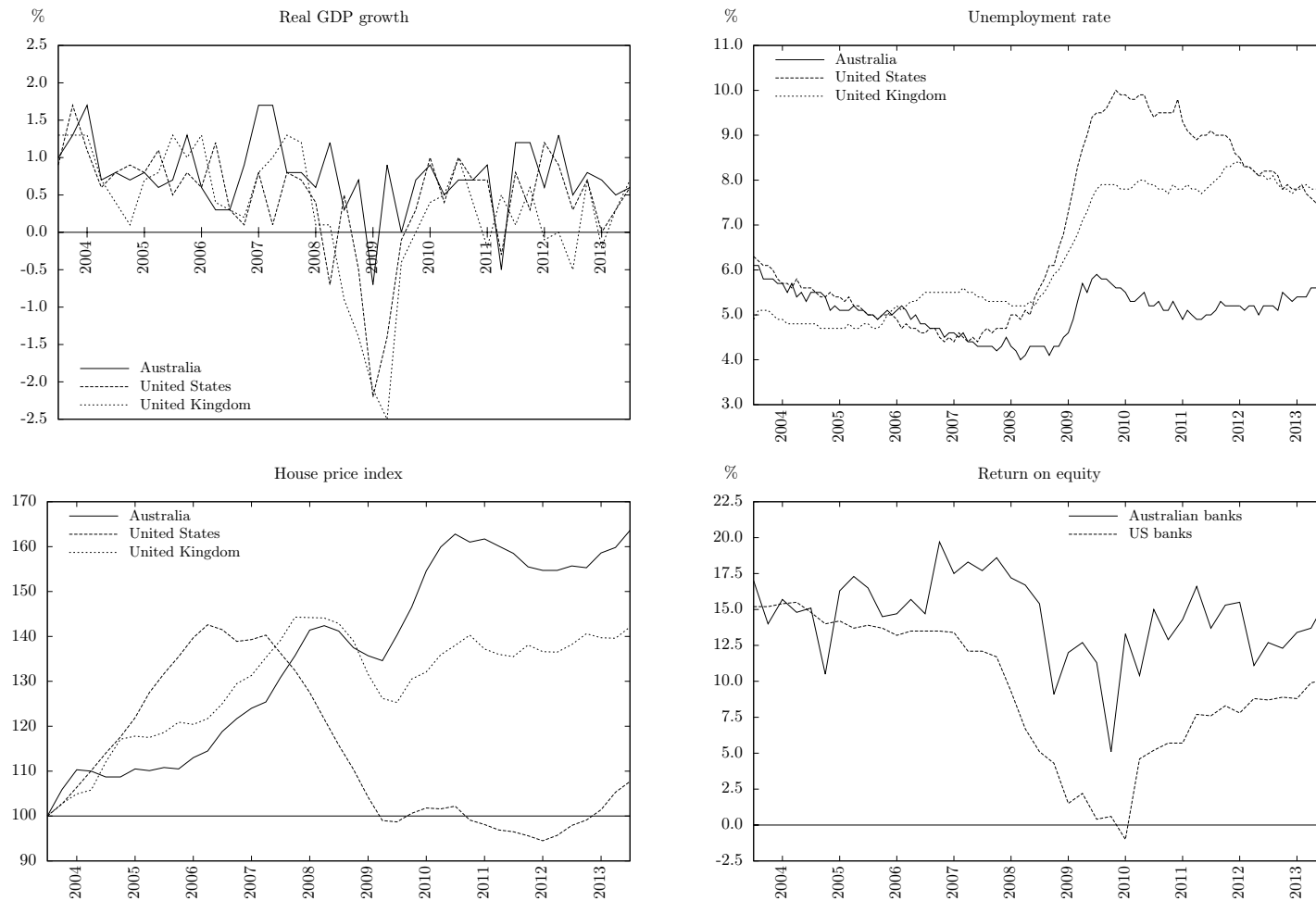
## 7.1 Recent Performance of the Australian Economy

In conducting our fundamental evaluation of the model specification of the Basel II IRB approach, we use internal bank data collected by APRA to solve for realisations of the single systematic risk factor describing the prevailing state of the Australian economy. These readings of the Australian banking sector are then compared with signals from macroeconomic indicators and financial statistics to render our assessment. Here, we describe the performance of the Australian economy over the past decade, a period which includes the financial crisis of 2007–09. While the crisis precipitated the worst global recession since the Great Depression, its effects were not evenly felt across the globe. In order to contextualise the Australian economy we compare its performance with that of the United States and United Kingdom, economies that experienced the full force of the recent crisis. The resilience of the Australian economy to the crisis is evident from a review of macroeconomic indicators and financial statistics that portend or reflect credit stresses: real GDP growth, unemployment rate, house prices, and return on equity of the banking sector.

A number of explanations for the resilience of the Australian economy have been proffered:

Figure 7.1: Resilience of the Australian economy to the financial crisis of 2007–09. Its performance is compared with that of the United States and United Kingdom on macroeconomic indicators and financial statistics: real GDP growth, unemployment rate, house price index, and return on equity of the banking sector.

Sources: Australian Bureau of Statistics; Australian Prudential Regulation Authority; US Bureau of Economic Analysis; US Bureau of Labor Statistics; US Federal Reserve Economic Data; Standard & Poor's; UK Office of National Statistics.



- Since the mid 2000s Australia has benefited from a favourable movement in terms of trade driven by the strong global demand for commodities, much of it coming from Asia. An appreciating foreign exchange rate over the same period has been a key factor in the relatively smooth adjustment of the economy to the increase in terms of trade (Bishop et al. 2013). Inflation has been consistent with the target set by the Reserve Bank of Australia (RBA), unemployment has remained relatively low, and economic growth has mostly been around trend — the economy registered a mild contraction in the fourth quarter of 2008, but did not experience a recession. These economic fundamentals have shielded Australia from the financial crisis.
- Residential mortgages, typically floating rate with full recourse to the borrower, have accounted for 50–60% of loans on the balance sheet of Australian banks over the past decade. With the vast bulk of residential mortgages originated by the banks and held to maturity, the risks associated with the banks’ residential mortgage portfolios are comparatively small. Other notable structural factors of the Australian financial sector which serve to constrain excessive risk taking include: the government’s “four pillars” policy precluding mergers between the four major banks, which dominate the Australian financial sector; and the ability of the Australian banks to pay fully-franked dividends to shareholders (Lewis 2013). The attention to risk quantification and management by the major banks, in order to achieve Basel II IRB approval in 2008, arguably helped discourage excessive risk-taking too (Brown and Davis 2010).
- APRA, which is responsible for the regulation of deposit-taking institutions, insurance companies and superannuation funds, distinguishes prudential supervision in Australia by its active, risk-based approach (Lewis 2013). BCBS (2012) calls for greater focus on risk-based supervision in its core principles for effective banking supervision: “[t]his risk-based process targets supervisory resources where they can be utilised to the best effect, focussing on outcomes as well as processes, moving beyond passive assessment of compliance rules.” There were significant failures of non-bank financial companies during the financial crisis with investors, rather than taxpayers, bearing the losses. The list of failed companies includes: Absolute Capital, Allco Finance Group, Babcock and Brown, Opes Prime, RAMS Home Loans and Storm Financial. Australia’s banking sector, on the other hand, experienced no failures. Indeed, while banks’ profitability declined during the crisis, it remained quite healthy. Also, the major Australian banks maintained their AA credit rating through the crisis (Brown and Davis 2010; D’Aloisio 2010).
- In October 2008 the Australian government introduced two schemes, announced with a three-year duration, to guarantee bank funding. Under the Financial Claims Scheme, all deposits up to \$1 million with locally-incorporated ADIs were automatically guaranteed by the government with no fee payable. Under the Funding Guarantee Scheme, the government provided a guarantee, for a fee, on deposits over \$1 million and wholesale funding

with maturity out to five years. The RBA believes that these schemes achieved their goal of maintaining public confidence in the Australian banking sector (Senate Standing Committees on Economics 2011).

- At the onset of the financial crisis, in October 2007, the RBA sought to restore liquidity to dysfunctional credit markets by expanding the range of securities it would accept as collateral for repurchase agreements to include residential mortgage-backed securities and asset-backed commercial paper (Debelle 2007). Then, as the crisis spread to the real economy, the RBA slashed its target cash rate from 7.25% in August 2008 to 3.0% in April 2009. Importantly, much of the monetary policy easing was passed through to borrowers. With most household and business loans in Australia being variable, lower interest rates translated into higher disposable incomes. Falling interest rates coupled with rising incomes improved housing affordability somewhat, and helped avert a sharp correction in the housing market, which had boomed over the previous decade (McDonald and Morling 2011).
- Between October 2008 and February 2009 the Australian government announced substantial fiscal stimulus packages: \$10.4 billion Economic Security Strategy; \$15.2 billion Council of Australian Governments funding package; \$4.7 billion Nation Building package; and \$42 billion Nation Building and Jobs Plan. The Treasury estimates that, absent the fiscal stimulus, GDP growth would have been negative for three consecutive quarters (McDonald and Morling 2011).
- Finally, the “lucky” country was probably not without a dose of good fortune.

The charts in Figure 7.1 clearly indicate that the Australian economy was largely cushioned from the adverse effects of the recent crisis. Putting the mild contraction experienced by the Australian economy during the financial crisis of 2007–09 in an historical context, we compare it with Australia’s most recent recession in 1990. During the financial crisis real GDP growth in Australia fell 0.7% in the fourth quarter of 2008, and unemployment peaked at 5.9% in June 2009. By comparison, during the 1990 recession the Australian economy contracted 1.7% and unemployment rose to 10.8%. A number of financial institutions failed including the State Bank of Victoria, the State Bank of South Australia, the largest credit union (Teachers Credit Union of Western Australia), the second largest building society (Pyramid Building Society), and several merchant banks. Moreover, two of the four major Australian banks incurred heavy losses and had to be recapitalised. The 1990 recession is considered similar in magnitude to those of 1974 and 1961, but not as severe as the recession of 1982 when the economy contracted 3.7% and unemployment rose to 10.5% (Macfarlane 2006).

## 7.2 Prevailing State of the Australian Economy

Taking measurements from the ASRF model of the Australian banking sector requires internal bank data collected by APRA under Basel II. Data reported on IRB credit risk forms by the

major banks are available since the first quarter of 2008 on a quarterly basis. Recall that IRB credit risk forms assign credit exposures to an obligor grade identifying a PD band within an IRB asset class. Pooling obligor grades, reported as at the end of a given quarter, across IRB asset classes constitutes a commingled credit portfolio in which each obligor grade is represented as a single credit. The quarterly observations between 31 March 2008 and 30 June 2013 establish a time series of commingled credit portfolios.

Recall that  $\delta_i(t)$ ,  $\eta_i(t)$ ,  $p_i(t)$  and  $\rho_i(t)$  denote the EAD, LGD, unconditional PD and asset correlation assigned to credit  $i$  as at the end of quarter  $t$ . Now, for the purpose of taking readings of the Australian banking sector we introduce the maturity adjustment previously omitted from our exposition in Section 4.1 of the ASRF model developed by BCBS (2005). Denote by  $\nu_i(t)$  the maturity adjustment assigned to credit  $i$  as at the end of quarter  $t$ . Prudential standards published by APRA instruct ADIs to assign one-year unconditional (through-the-cycle) PDs to IRB credit exposures. Accordingly,  $\delta_i(t)$ ,  $\eta_i(t)$ ,  $p_i(t)$ ,  $\rho_i(t)$  and  $\nu_i(t)$  reported as at the end of quarter  $t$  are input to the ASRF model, which makes projections of losses on IRB credit exposures over the subsequent four-quarter interval  $[t+1, t+4]$ . As noted in Section 5.1, these parameters and variables are reported on IRB credit risk forms submitted to APRA by the majors banks on a quarterly basis.

The model specification of the IRB approach assesses regulatory capital charges by calculating the expectation of credit losses conditional on a realisation of the single systematic risk factor. Substituting losses incurred on IRB credit exposures into the ASRF model, we recover realisations of the single systematic risk factor describing states of the Australian economy experienced since the implementation of Basel II. Then, applying the standard Gaussian distribution function yields the quantile of the distribution of economic scenarios.

For any one-year horizon over which the ASRF model makes projections of credit losses, we choose to associate the model inputs with the realisation of the single systematic risk factor as at the midpoint of the corresponding risk measurement horizon. So, measuring time in quarterly increments, realisation  $y(t)$  of systematic risk factor  $Y$  describing the state of the economy as at the end of quarter  $t$  is associated with credit losses projected over the time interval  $[t-1, t+2]$  using credit risk parameters and variables reported as at the end of quarter  $t-2$ .

Suppose that for each quarter in the time series, there is a total of  $r$  banking book exposures subject to credit risk, of which  $n$  IRB credit exposures are held in the aforementioned commingled portfolio, where  $n \leq r$ . Denote by  $\mathcal{R}_n(t)$ ,  $\mathcal{R}_r(t)$  and  $\mathcal{R}(t)$  the RWA of IRB credit exposures, RWA for credit risk and total RWA, respectively, as at the end of quarter  $t$ . Let  $\mathcal{L}_r(s)$  be the total credit losses incurred during quarter  $s$ , which is reported on the statement of financial performance. We do not observe  $\mathcal{L}_n(s)$ , losses incurred on IRB credit exposures during quarter  $s$ . Therefore, we choose to allocate total credit losses incurred during the four-quarter interval  $[s-1, s+2]$  between IRB credit exposures and other banking book exposures in proportion to RWA as at the end of the quarter  $t-2$ . So, losses on IRB credit exposures incurred during quarters  $s-1$ ,  $s$ ,  $s+1$  and



$s+2$  is given by

$$\frac{\mathcal{R}_n(t-2)}{\mathcal{R}_r(t-2)} (\mathcal{L}_r(s-1) + \mathcal{L}_r(s) + \mathcal{L}_r(s+1) + \mathcal{L}_r(s+2)). \quad (7.1)$$

In taking measurements from the ASRF model of the Australian banking sector, we assume that there is no delay in the recognition of bad debts. (Later, we allow for delays in the recognition of bad debts.) Setting  $s = t$ , projected credit losses over any one-year horizon are compared with credit losses incurred during the same one-year interval. Then, applying the formula for conditional expectation of portfolio credit losses:

$$\begin{aligned} & \frac{\mathcal{R}_n(t-2)}{\mathcal{R}_r(t-2)} (\mathcal{L}_r(t-1) + \mathcal{L}_r(t) + \mathcal{L}_r(t+1) + \mathcal{L}_r(t+2)) \\ &= \sum_{i=1}^n \delta_i(t-2) \eta_i(t-2) \nu_i(t-2) \Phi \left( \frac{\Phi^{-1}(p_i(t-2)) - \sqrt{\rho_i(t-2)} y(t)}{\sqrt{1 - \rho_i(t-2)}} \right), \end{aligned} \quad (7.2)$$

we solve for realisation  $y(t)$  of systematic risk factor  $Y$ .<sup>2</sup> Repeating this static analysis for each quarter  $t$  from 30 September 2008 through 31 December 2012 generates a time series of systematic risk factor  $Y$ . The ASRF model assumes that the conditional expectation of portfolio credit losses rises as the economy deteriorates — a strictly decreasing function of the single systematic risk factor. It follows from Lemma 2.33 that the  $\alpha$  quantile of the distribution of  $\mathbb{E}[L_n | Y]$  is associated with the  $1 - \alpha$  quantile of the distribution of  $Y$ . Hence, confidence level  $\alpha(t)$  corresponding to realisation  $y(t)$  of systematic risk factor  $Y$  is given by

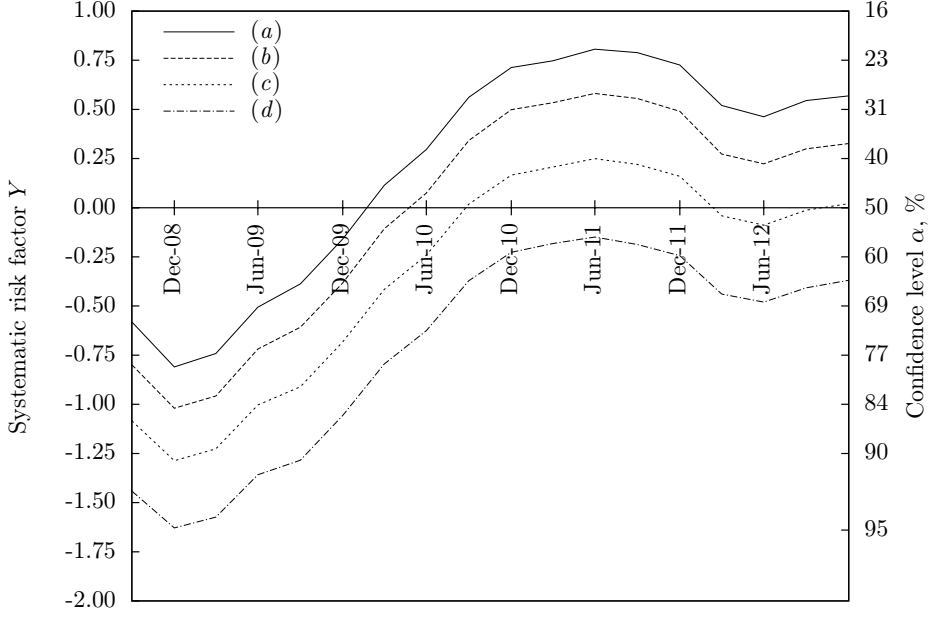
$$\alpha(t) = 1 - \Phi(y(t)). \quad (7.3)$$

As discussed in Section 5.1, prudential standards published by APRA instruct ADIs to assign economic downturn (stressed) LGD rates to IRB credit exposures. Denote by  $\underline{\eta}_i(t)$  the downturn LGD assigned to credit  $i$  as at the end of quarter  $t$ . Setting  $\eta_i(t) = \underline{\eta}_i(t)$  for  $i = 1, \dots, n$  in (7.2) and solving for realisation  $y(t)$  of systematic risk factor  $Y$  yields a prevailing state of the economy that is more expansionary than would be recovered using point-in-time LGD rates, assuming that the economy was not experiencing a (severe) downturn. In solving for realisations of the single systematic risk factor describing the prevailing state of the economy, we also examine the effect of adjustments to downturn LGD rates. Recognising that banks may not be able to produce internal estimates of downturn LGD, the Office of the Comptroller of the Currency et al. (2007) proposed a “linear supervisory mapping function” for translating through-the-cycle LGD rates into downturn LGD rates:  $\underline{\eta}_i(t) = 0.08 + 0.92\bar{\eta}_i(t)$ , where  $\bar{\eta}_i(t)$  is the through-the-cycle LGD assigned to credit  $i$  at the end of quarter  $t$ . The adjustment varies between zero and eight percent, decreasing linearly from low to high through-the-cycle LGD rates. Using historical LGD rates from their database, Moody’s derived an adjustment method to meet the downturn LGD requirement imposed under the IRB approach. Their adjustment to through-the-cycle LGD

<sup>2</sup> We use the GNU Scientific Library routine `gsl_root_fsolver_falsepos`, which implements the *false position algorithm*, to solve for  $y(t)$ . A root bracketing algorithm based on linear interpolation, its convergence is linear, but it is usually faster than the *bisection method*.

Figure 7.2: Realisations of the single systematic risk factor describing the prevailing state of the Australian economy, and the effect of adjustments to downturn LGD. Confidence level  $\alpha$  is the probability of the state of the economy being better than the economic scenario described by realisation  $y$ . For credits  $i = 1, \dots, n$  constituting the commingled portfolio at the end of quarter  $t$ :

- (a)  $\eta_i(t) = \underline{\eta}_i(t)$ , where  $\underline{\eta}_i(t)$  is downturn LGD.
- (b)  $\eta_i(t) = \bar{\eta}_i(t)$ , where  $\bar{\eta}_i(t)$  is through-the-cycle LGD and  $\underline{\eta}_i(t) = 0.08 + 0.92\bar{\eta}_i(t)$ .
- (c)  $\eta_i(t) = \frac{1}{3}\underline{\eta}_i(t)$ .
- (d)  $\eta_i(t) = \frac{1}{2}\underline{\eta}_i(t)$ .



rates depends on the seniority of the obligations, with the magnitude also varying from zero to eight percent (Emery 2007). We demonstrate the effect of adjustments to downturn LGD rates by solving (7.2) using four definitions of LGD:  $\eta_i(t) = \underline{\eta}_i(t)$ ,  $\eta_i(t) = \bar{\eta}_i(t)$ ,  $\eta_i(t) = \frac{1}{3}\underline{\eta}_i(t)$ , and  $\eta_i(t) = \frac{1}{2}\underline{\eta}_i(t)$  for  $i = 1, \dots, n$ .

Figure 7.2 plots realisations of the single systematic risk factor describing the prevailing state of the economy. On the basis of downturn LGD rates, as reported by the major banks, the economic shock imparted by the financial crisis of 2007–09 propagated through the Australian banking system inflicting credit losses incurred, on average, once every five years. That is, at the depths of the crisis, the prevailing state of the Australian economy as recovered from (7.2) is summarised by realisation  $y = -0.81$  on 31 December 2008. Appealing to Proposition 2.28,  $y = -0.81$  translates into the 79.1% quantile of the portfolio loss distribution — credit losses incurred were no greater than expected in 79.1% of economic scenarios, which we characterise informally as (approximately) a one-in-five-year event. We argue that the single systematic risk factor is a germane measure of the impact of the financial crisis, because the IRB approach chooses realisation  $y = -3.090$  of systematic risk factor  $Y$  to assess regulatory capital charges that are

expected to absorb credit losses in 99.9% of economic scenarios. Bear in mind, though, that measurements from the ASRF model describing the prevailing state of the Australian economy capture policy responses of government departments and agencies (i.e., The Treasury, RBA and APRA) designed to mitigate the recent crisis. Naturally, it is not possible to isolate these policy responses in order to ascribe a value to their mitigating effects.

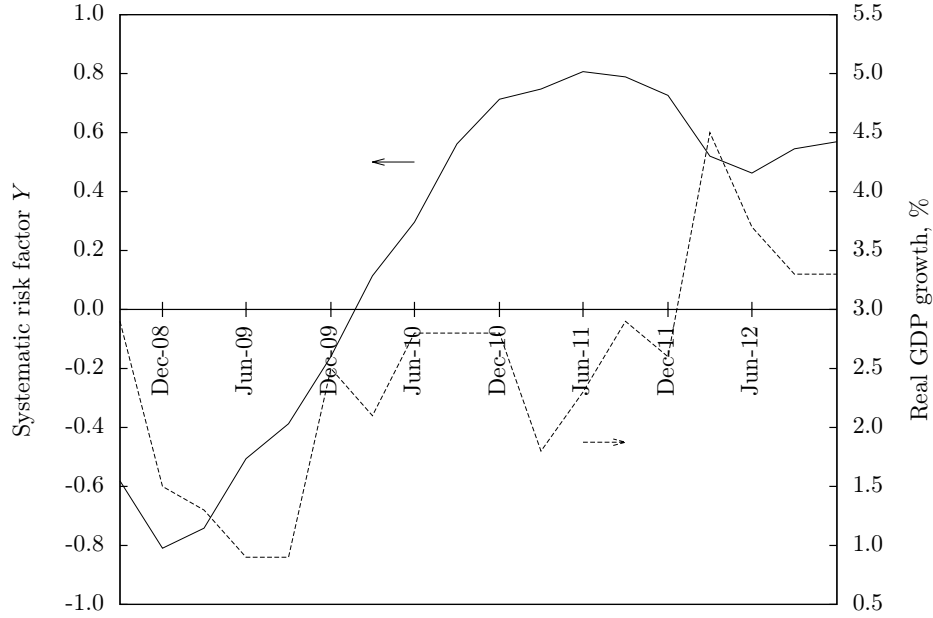
Translating downturn LGD rates into through-the-cycle LGD rates using the linear supervisory mapping function proposed by Office of the Comptroller of the Currency et al. (2007), realisations of the single systematic risk factor recovered from (7.2) indicate that the recent crisis inflicted credit losses on the Australian banking system incurred, on average, once every seven years. Given that the Australian economy contracted during the financial crisis, and assuming that through-the-cycle LGD rates obtained from downturn LGD rates using the linear supervisory mapping function are good estimates, we would argue that the shock experienced by the Australian economy at the depths of the crisis was between a one-in-five-year and one-in-seven-year event. This seems consistent with the perception that Australian banks weathered the recent crisis quite well, reporting good profitability and high capital ratios, and maintaining strong credit ratings.

For reference, Figure 7.2 also plots realisations of the single systematic risk factor recovered from (7.2) using LGD estimates that are one-third less than, and half of downturn LGD rates. Under these assumptions for point-in-time LGD at the depths of the crisis, the shock experience by the Australian economy was a one-in-ten-year and one-in-twenty-year event, respectively. In the analysis that follows we use the time series of the single systematic risk factor recovered from (7.2) using downturn LGD rates, as reported by the major Australian banks. We would reach broadly the same conclusions using point-in-time LGD rates.

Real GDP growth, plotted for the Australian economy in Figure 7.1, is conventionally reported as seasonally adjusted, quarter-over-quarter. Quarterly realisations of the single systematic risk factor are recovered by equating credit losses projected over a one-year horizon with credit losses incurred during the same one-year interval. Accordingly, we restate quarterly observations of real GDP growth for the Australian economy as year-over-year. Figure 7.3 plots time series of the single systematic risk factor describing the prevailing state of the Australian economy and real GDP growth, year-over-year. The time series are quite strongly correlated (+0.60), suggesting that the single systematic risk factor serves as a reasonable proxy for the relative state of the economy. Indeed the correlation between the time series rises to +0.72 when realisations of the single systematic risk factor are lagged by one or two quarters. Arguably, realisation  $y(t)$  of systematic risk factor  $Y$  leads real GDP growth, but realisation  $y(t)$  cannot be computed until credit losses  $\mathcal{L}_r(t+2)$  are recognised in profit and loss for quarter  $t+2$ . Overall, observations of the prevailing state of the Australian economy recovered from (7.2), and plotted in Figures 7.2 and 7.3, agree rather well with the macroeconomic indicators and financial statistics plotted in Figure 7.1.

As noted above, we do not observe losses on IRB credit exposures. Hence, we choose to allo-

Figure 7.3: Realisations  $y(t)$  of systematic risk factor  $Y$  describing the prevailing state of the Australian economy and real GDP growth, y-o-y. The time series exhibit a moderately strong correlation (+0.60). Source: Australian Bureau of Statistics.



cate total credit losses incurred between IRB credit exposures and other banking book exposures in proportion to RWA. Figure 7.4 illustrates that even if we were to attribute credit losses entirely to IRB credit exposures and use downturn LGD rates in (7.2), the financial crisis of 2007–09 would have inflicted credit losses on the Australian banking system not exceeding those incurred, on average, once every eight years — a lower bound on the severity of the crisis in Australia.

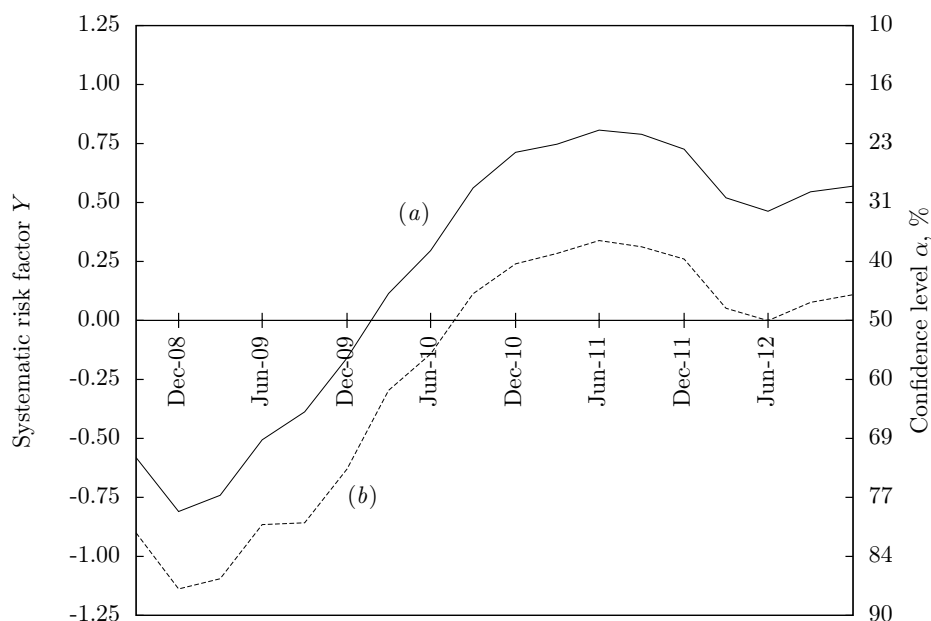
The model specification of the IRB approach was developed for the purpose of solvency assessment, or capital adequacy. A credit risk model of capital adequacy requires precision in the measurement of absolute risk levels under stressed economic conditions associated with the tail of the portfolio loss distribution (Basel Committee on Banking Supervision et al. 2010). Recognising that realisations of the single systematic risk factor describing states of the Australian economy experienced since the implementation of Basel II correspond to observations away from the tail of its distribution, we cannot attest to the validity of the ASRF model for regulatory capital modelling. However, we argue that our findings support a favourable assessment of the ASRF model, and asset value factor models of credit risk in general, for the purposes of capital allocation, performance attribution and risk monitoring. These management functions, generally served by economic capital models, need only be accurate in the measurement of relative risk under “normal” economic conditions.

An evaluation of the ASRF model for the purpose of solvency assessment, would involve

Figure 7.4: Impact of the financial crisis of 2007–09 on the credit portfolios of the major Australian banks. Realisation  $y$  of systematic risk factor  $Y$  describes the prevailing state of the economy. Confidence level  $\alpha$  is the probability of the state of the economy being better than the economic scenario described by realisation  $y$ .

(a) Credit losses are allocated between IRB credit exposures and other banking book exposures in proportion to RWA.

(b) Credit losses are allocated entirely to IRB credit exposures providing a lower bound on  $Y$ .

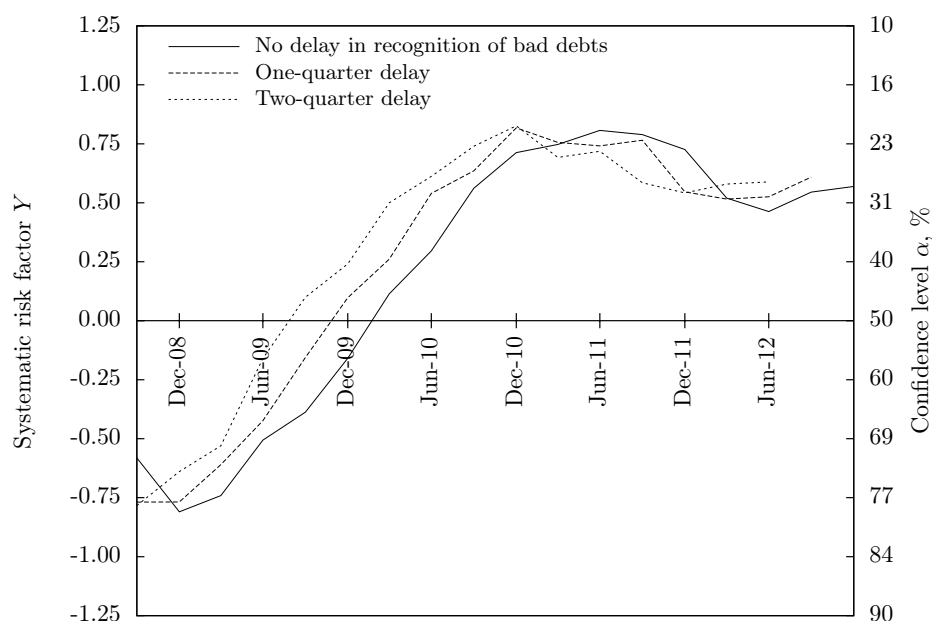


taking readings of north Atlantic banking jurisdictions that experienced the full force of the financial crisis of 2007–09. It is unlikely to be as favourable as the qualified evaluation presented here. The UK banking sector, which has operated under the Basel II framework since 2008, experienced losses during the crisis on a scale that led to the failure or nationalisation of large banks including Bradford & Bingley, HBOS, Lloyds Banking Group, Northern Rock and Royal Bank of Scotland. Moreover, the Basel 2.5 and Basel III reform packages have been developed by BCBS to address deficiencies of the Basel II framework exposed by the recent crisis.

It could be argued that delays in the recognition of bad debts warrant introducing a lag in the association of projected credit losses with credit losses incurred. Figure 7.5 plots the time series of the prevailing state of the economy (i.e., realisations of the single systematic risk factor) assuming no delay in the recognition of bad debts ( $s = t$ ), along with time series assuming delays of one quarter ( $s = t + 1$ ) and two quarters ( $s = t + 2$ ). Introducing lags to account for possible delays in the recognition of bad debts does not materially alter our measurement of the impact of the financial crisis on the credit portfolios of the major Australian banks at the depths of the crisis — it remains, informally speaking, a one-in-five-year event (using downturn LGD rates).

This empirical analysis begins with the initial submission of capital adequacy and IRB credit

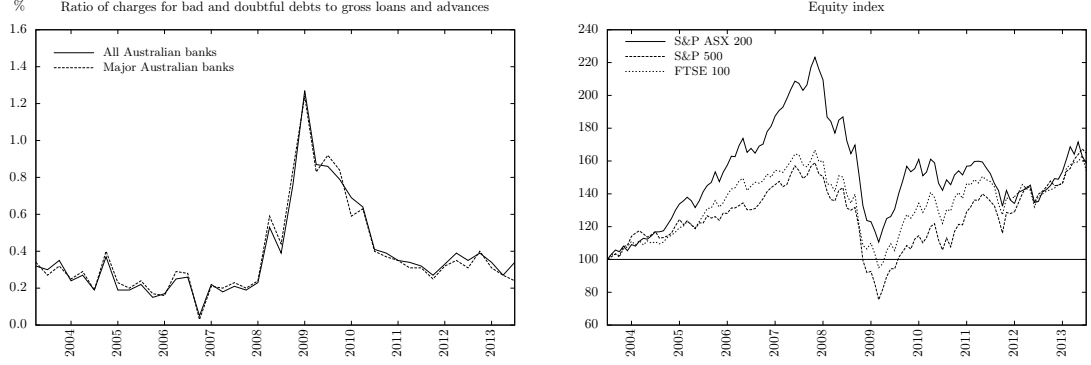
Figure 7.5: Impact of the financial crisis of 2007–09 on the credit portfolios of the major Australian banks. Sensitivity of the single systematic risk factor describing the prevailing state of the economy to delays in the recognition of bad debts. Confidence level  $\alpha$  is the probability of the state of the economy being better than the economic scenario described by realisation  $y$  of systematic risk factor  $Y$ .



risk forms as at 31 March 2008 and credit losses recognised in profit and loss between 1 April 2008 and 31 March 2009, and solves for the single systematic risk factor describing the state of the economy as at 30 September 2008. Our measurement of the impact of the financial crisis on the Australian banking sector is necessarily predicated on the assumption that the effect of the crisis was most acutely felt after 31 March 2008. Figure 7.6 indicates that credit losses incurred by the major Australian banks peaked in the fourth quarter of 2008 and remained elevated through 2009, and global equity indices plumbed their lows during the first quarter of 2009 — S&P ASX 200 index fell 54% between November 2007 and March 2009.

Our measurement of the impact of the financial crisis on the Australian banking sector is buttressed by the macroeconomic stress test administered by APRA on the major Australian banks in 2012. The three-year economic scenario developed for the stress test by APRA in conjunction with the RBA and Reserve Bank of New Zealand, was designed to be comparable with the actual experience of the United States and United Kingdom during the recent crisis. In particular, the macroeconomic stress test envisaged: real GDP contracting 5%; unemployment rising to 12%; house prices falling 35%; and commercial property prices falling 40%. None of the major Australian banks would have failed under this severe but plausible economic scenario, nor would any of the major banks have breached the 4% minimum tier 1 capital requirement of the Basel II Accord (Laker 2012).

Figure 7.6: Dating the inception of the financial crisis of 2007–09. Credit losses incurred by Australian banks peaked, and global equity indices plunged their lows, after the implementation of Basel II. Sources: Australian Prudential Regulation Authority; Bloomberg.



### 7.3 Australian Banks' Capacity to Absorb Credit Losses

Distance-to-default, which measures the level of capitalisation, reflects the capacity to absorb credit losses. Substituting provisions set aside for absorbing expected losses and capital held against unexpected losses on IRB credit exposures into the ASRF model, and translating realisations of the single systematic risk factor into distance-to-default, we measure the level of capitalisation of the major Australian banks, in aggregate,<sup>3</sup> since the implementation of Basel II.

Suppose that a bank sets aside provisions and holds capital that are sufficient to absorb credit losses at the  $\alpha$  confidence level. Denote by  $\mathcal{Q}_r(t)$  provisions set aside for absorbing expected credit losses as at the end of quarter  $t$ , and  $\mathcal{K}(t)$  the capital base as at the end of quarter  $t$ , both reported on the capital adequacy form. Recall that APRA sets a prudential capital ratio, subject to a minimum 8% of RWA, for each ADI, and an ADI typically holds a capital buffer above its prudential capital requirement. Since we observe neither provisions set aside for absorbing expected losses nor capital held against unexpected losses on IRB credit exposures, we choose an allocation procedure. Provisions are allocated between IRB credit exposures and other banking book exposures in proportion to expected losses over the subsequent four quarters as projected at the end of quarter  $t$ :

$$\mathcal{Q}_n(t) = \frac{\mathbb{E}[\mathcal{L}_n(t+1) + \mathcal{L}_n(t+2) + \mathcal{L}_n(t+3) + \mathcal{L}_n(t+4)]}{\mathbb{E}[\mathcal{L}_r(t+1) + \mathcal{L}_r(t+2) + \mathcal{L}_r(t+3) + \mathcal{L}_r(t+4)]} \mathcal{Q}_r(t). \quad (7.4)$$

Capital is allocated between IRB credit exposures and other risk exposures (i.e., non-IRB credit exposures, and operational, market and securitisation risks) in proportion to RWA as at the end of quarter  $t$ :

$$\mathcal{K}_n(t) = \frac{\mathcal{R}_n(t)}{\mathcal{R}(t)} \mathcal{K}(t). \quad (7.5)$$

<sup>3</sup> Results are reported for the major Australian banks, in aggregate, so as not to violate confidentiality agreements.

Hence, provisions set aside for absorbing expected losses and capital held against unexpected losses on IRB credit exposures as at the end of quarter  $t$  is equal to  $\mathcal{Q}_n(t) + \mathcal{K}_n(t)$ .

We estimate distance-to-default by solving for realisation  $\tilde{y}(t)$  of systematic risk factor  $Y$  that equates the sum of (7.4) and (7.5) to the conditional expectation of portfolio credit losses:<sup>4</sup>

$$\mathcal{Q}_n(t) + \mathcal{K}_n(t) = \sum_{i=1}^n \delta_i(t) \eta_i(t) \nu_i(t) \Phi \left( \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_i(t)} \tilde{y}(t)}{\sqrt{1 - \rho_i(t)}} \right). \quad (7.6)$$

Recall that APRA's prudential standards instruct ADIs to assign unconditional PDs and downturn LGD rates to IRB credit exposures. Since distance-to-default reflects the capacity to absorb credit losses, usually under stressed economic conditions, it is appropriate to use downturn LGD rates to estimate distance-to-default. Hence,  $\eta_i(t) = \underline{\eta}_i(t)$  for  $i = 1, \dots, n$  and for all  $t$  in (7.6). Note that the sum of provisions set aside for absorbing expected credit losses and capital held against unexpected credit losses in (7.6), replaces the sum of expected losses and regulatory capital in the ASRF model described in Section 4.1. Denote by  $\tilde{d}(t)$  the distance-to-default as at the end of quarter  $t$ . Then,

$$\tilde{d}(t) = -\tilde{y}(t), \quad (7.7)$$

which translates into confidence level

$$\tilde{\alpha}(t) = 1 - \Phi(-\tilde{d}(t)) = \Phi(\tilde{d}(t)). \quad (7.8)$$

It follows from (7.5) that the risk-based capital ratio as at the end of quarter  $t$ , plotted in Figure 5.3, is given by

$$\kappa(t) = \frac{\mathcal{K}(t)}{\mathcal{R}(t)} = \frac{\mathcal{K}_n(t)}{\mathcal{R}_n(t)}. \quad (7.9)$$

Table 7.1 reports quarterly risk-based capital ratio of the major banks, in aggregate, along with implied distance-to-default. Since 2008 they have maintained a capital base that is consistent with targeting a credit rating between A and AA (i.e., a target confidence level between 99.96% and 99.99%). Note that, under the prudential standards of APRA (2008a), if provisions set aside are insufficient to absorb expected credit losses, the shortfall is deducted from capital. The general agreement of our estimates of distance-to-default with credit ratings issued by external rating agencies seemingly lends further support to a favourable assessment of the model specification of the IRB approach. However, distance-to-default is a measure of capital adequacy, and Section 7.2 argues that we cannot attest to the suitability of the ASRF model for the purpose of solvency assessment, because realisations of the single systematic risk factor experienced in Australia since the implementation of Basel II correspond to observations away from the tail of its distribution. Therefore, we report estimates of distance-to-default with the caveat that (7.6) models default dependence in the tail of the portfolio loss distribution as a multivariate Gaussian process. It is generally acknowledged that models which assume that financial data follow a

<sup>4</sup> We use the GNU Scientific Library routine `gsl_root_fsolver_falsepos`, which implements the *false position algorithm*, to solve for  $\tilde{y}(t)$ .



Table 7.1: As reported risk-based capital ratio  $\kappa(t)$ , and implied distance-to-default  $\tilde{d}(t)$  of the major Australian banks, in aggregate. The reverse stress test uncovers the weakest economic shock  $\hat{y}(t)$  that would result in a breach of the capital ratio floor  $\underline{\kappa}$ . Realisation  $y(t)$  describes the prevailing state of the economy.

$t$	As reported			Reverse stress test				Prevailing state of the economy	
	$\kappa(t)$ , %	$\tilde{d}(t)$	$\tilde{\alpha}(t)$ , %	$\underline{\kappa} = 4.0\%$		$\underline{\kappa} = 8.0\%$		$y(t)$	$\alpha(t)$ , %
				$\hat{y}(t)$	$\hat{\alpha}(t)$ , %	$\hat{y}(t)$	$\hat{\alpha}(t)$ , %		
31-Mar-2008	10.67	3.504	99.977	-2.915	99.822	-1.90	97.10		
30-Jun-2008	10.73	3.413	99.968	-2.845	99.778	-1.87	96.94		
30-Sep-2008	10.98	3.508	99.978	-2.946	99.839	-2.01	97.78	-0.58	71.9
31-Dec-2008	11.51	3.588	99.983	-3.044	99.883	-2.17	98.49	-0.81	79.1
31-Mar-2009	11.38	3.621	99.985	-3.044	99.883	-2.12	98.29	-0.74	77.1
30-Jun-2009	11.10	3.620	99.985	-3.018	99.873	-2.04	97.94	-0.51	69.4
30-Sep-2009	11.52	3.682	99.988	-3.095	99.902	-2.17	98.51	-0.39	65.1
31-Dec-2009	11.85	3.732	99.991	-3.160	99.921	-2.28	98.88	-0.16	56.3
31-Mar-2010	11.82	3.717	99.990	-3.145	99.917	-2.26	98.82	0.11	45.5
30-Jun-2010	11.51	3.677	99.988	-3.090	99.900	-2.17	98.48	0.30	38.4
30-Sep-2010	11.50	3.680	99.988	-3.092	99.901	-2.16	98.48	0.56	28.7
31-Dec-2010	11.27	3.653	99.987	-3.057	99.888	-2.10	98.20	0.71	23.8
31-Mar-2011	11.49	3.682	99.989	-3.098	99.902	-2.17	98.49	0.75	22.7
30-Jun-2011	11.42	3.660	99.987	-3.076	99.895	-2.14	98.38	0.81	21.0
30-Sep-2011	11.48	3.647	99.987	-3.083	99.897	-2.18	98.54	0.79	21.5
31-Dec-2011	11.42	3.642	99.986	-3.077	99.895	-2.17	98.51	0.73	23.4
31-Mar-2012	11.48	3.637	99.986	-3.077	99.896	-2.18	98.55	0.52	30.2
30-Jun-2012	11.44	3.633	99.986	-3.075	99.895	-2.18	98.54	0.46	32.2
30-Sep-2012	11.72	3.659	99.987	-3.121	99.910	-2.28	98.86	0.54	29.3
31-Dec-2012	11.75	3.663	99.988	-3.126	99.911	-2.29	98.89	0.57	28.5
31-Mar-2013	11.69	3.698	99.989	-3.165	99.923	-2.34	99.04		
30-Jun-2013	11.42	3.657	99.987	-3.118	99.909	-2.27	98.83		

Gaussian distribution tend to underestimate tail risk. If we were to model default dependence by heavier-tailed distributions, we would recover narrower estimates of distance-to-default. Also, credit losses incurred at the depths of the financial crisis would correspond to a quantile of the portfolio loss distribution further from the tail, and thus associated with a less contractionary, or more expansionary, state of the economy.

In contrast to our evaluation of the capacity of the major banks to absorb credit losses during the financial crisis, Allen and Powell (2012) argue that Australian banks did not fare much better than their global counterparts. They compared the distance-to-default of the Australian banking sector with that of the banking sectors in the United States, Europe and Canada. In their implementation of the KMV/Merton structural methodology, distance-to-default is a function of implied market value and volatility of assets imputed from equity prices. Allen and Powell observed that the distance-to-default for Australian banks narrowed sharply from a peak of 11.31 in 2005 to 0.60 in 2008. Note that a distance-to-default of 0.60 translates into roughly a one-in-four chance of bank failure, but in reality there were none. An implicit assumption of their analysis is that security markets consistently price risk fairly. But it's not uncommon for markets to exhibit bouts of manic-depressive behaviour. During periods when market participants are gripped by fear (respectively, driven by greed), their perception of risk is heightened (lessened), and markets become undervalued (overpriced). We submit that their results are biased by plummeting equity prices and spiking volatility reflecting the overreaction of market participants gripped by fear at the depths of the crisis — the S&P ASX 200 Banks index fell 58% between November 2007 and January 2009. With access to internal bank data collected by APRA, we produce a fundamental evaluation of the effect of the recent crisis and solvency of Australian banks that differs markedly from the stock market's assessment.

## 7.4 Reverse Stress Testing

Stress testing is an important risk management tool promoted by supervisors through the Basel II framework. In its principles for sound stress testing, BCBS (2009c) recommends that supervisors make regular and comprehensive assessments of banks' stress testing programs, and encourages supervisors to conduct stress tests based on common scenarios for banks in their jurisdiction.

A key principle of APRA's supervisory review is that regulated institutions are responsible for developing and maintaining an internal capital adequacy assessment process (ICAAP) proportional to the size, business mix and complexity of their operations. ICAAP is an integrated and documented approach to risk and capital management that assesses the risk appetite of an institution and establishes the commensurate level and quality of capital. As part of ICAAP, institutions are expected to conduct their own and supervisory-led stress tests. APRA believes that stress testing goes beyond assessing capital adequacy to informing risk appetite, setting risk limits, identifying vulnerabilities and developing mitigating actions. The results of stress tests are used by supervisors "to anchor expectations for the level of capital that an institution should

hold in normal times to provide a sufficient buffer to withstand a challenging environment.” The results also serve “to inform our [supervisors] risk assessment of institutions as part of the development of supervisory action plans” (Laker 2012).

Usually, a stress test begins with the development of an economic scenario that is the manifestation of some stress event or economic shock. It is then translated into conditional (point-in-time) PDs assigned to obligors constituting a bank’s credit portfolio. Finally, an analytical or simulation model estimates portfolio credit losses subject to the stress event, which are charged against provisions and capital to produce an assessment of the bank’s solvency. A variation on this methodology is reverse stress testing — a technique in which “losses that would render an institution unviable or subject to material regulatory intervention are identified and attention then focussed on the types of scenarios that would generate these losses” (Laker 2010).

Clearly, realisation  $\tilde{y}(t) = -\tilde{d}(t)$ , recovered from (7.6), describes an economic scenario that would render the major banks, in aggregate, insolvent at the end of quarter  $t$  if the associated credit losses were incurred and recognised instantaneously. We extend this rudimentary form of reverse stress testing to uncover economic scenarios that would cause the major banks to breach some designated capital ratio floor  $\underline{\kappa}$ , and consequently trigger material supervisory intervention. Suppose that the expectation of portfolio credit losses conditional on stress event  $\hat{y}(t)$  is instantaneously incurred and recognised in profit and loss at the end of quarter  $t$ . Then, the capital ratio floor would be breached at the end of quarter  $t$  if credit losses exceeded

$$\mathcal{Q}_n(t) + \mathcal{K}_n(t) - \underline{\kappa}\mathcal{R}_n(t). \quad (7.10)$$

Economic scenarios worse than that described by realisation  $\hat{y}(t)$  of systematic risk factor  $Y$  satisfying (7.11) would result in a breach of the capital ratio floor at the end of quarter  $t$ :<sup>5</sup>

$$\mathcal{Q}_n(t) + \mathcal{K}_n(t) - \underline{\kappa}\mathcal{R}_n(t) = \sum_{i=1}^n \delta_i(t) \eta_i(t) \nu_i(t) \Phi \left( \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_i(t)} \hat{y}(t)}{\sqrt{1 - \rho_i(t)}} \right). \quad (7.11)$$

Then, the probability of the state of the economy being better than the economic scenario described by realisation  $\hat{y}(t)$  is a most

$$\hat{\alpha}(t) = 1 - \Phi(\hat{y}(t)). \quad (7.12)$$

Finally, macroeconomic-based models would translate realisations of the single systematic risk factor into observations of macroeconomic indicators (e.g., real GDP growth, unemployment rate, house prices, etc.). This final translation is beyond the scope of this paper.

Our rudimentary stress testing methodology has its limitations: it is static; focusses exclusively on credit risk, assuming away market, liquidity and operational risks; fails to consider diversification benefits or compounding effects arising from the interaction between risk classes; excludes net interest income earned during the period in which credit losses are incurred; and

<sup>5</sup> We use the GNU Scientific Library routine `gsl_root_fsolver_falsepos`, which implements the *false position algorithm*, to solve for  $\hat{y}(t)$ .

does not consider the possibility of raising fresh capital. Yet it puts the severity of the financial crisis, as experienced by the Australian banking sector, in perspective by comparing it with stress events that would trigger material supervisory intervention.

Table 7.1 reports the weakest economic shock  $\hat{y}(t)$  imparted at the end of quarter  $t$  that would result in a breach of capital ratio floors set at 4.0% and 8.0% if credit losses were instantaneously incurred and recognised in profit and loss. Section 7.2 made reference to the macroeconomic stress test administered by APRA on the major banks in 2012, describing the severe and plausible economic scenario under which none of the major banks would have breached the 4% minimum tier 1 capital requirement. Table 7.1 indicates that with  $\underline{\kappa} = 4.0\%$ , approximately 99.9% of economic scenarios, presumably including the severe but plausible one developed for the macroeconomic stress test administered by APRA, would not have resulted in the major banks, in aggregate, breaching the capital ratio floor in 2012. Even with  $\underline{\kappa} = 8.0\%$ , fewer than 1.5% of economic scenarios would have resulted a breach of the capital ratio floor in 2012. During the recent crisis the capacity of the major banks, in aggregate, to absorb credit losses appears to have been “stretched” to the point where 0.22% (respectively, 3.06%) of economic scenarios would have resulted in a breach of the 4.0% (8.0%) capital ratio floor during the quarter ending 30 June 2008. For comparison, Table 7.1 also reports the prevailing state of the economy described by quarterly realisations  $y(t)$  recovered from (7.2) using downturn LGD rates, and plotted in Figures 7.2 and 7.3.



# Market Assessment of the Solvency of the Australian Banking Sector

The theoretical framework of Part I underpins the primary contribution of our empirical analysis, which renders a fundamental assessment of the model specification of the Basel II IRB approach. In Chapter 7 we argue that measurements from the ASRF model of the prevailing state of Australia's economy and the level of capitalisation of its banking sector find general agreement with macroeconomic indicators, financial statistics and external credit ratings. Moreover, we submit that the market's assessment of the solvency of the Australian banking sector during the financial crisis of 2007–09 is biased by plummeting prices and spiking volatility reflecting the overreaction of market participants gripped by fear at the depths of the crisis. Using market data, Allen and Powell (2012) find that Australian banks did not fair much better than their global counterparts during the recent crisis. In this more or less self-contained chapter, we broadly follow their methodology to infer the market's assessment of the capacity of the major Australian banks, in aggregate, to absorb credit losses. The results are then directly comparable with our fundamental assessment.

The structural approach to credit risk modelling (Merton 1974) formulates liabilities of a firm as contingent claims on its assets. Contingent claims analysis is a generalisation of the option pricing theory developed by Black and Scholes (1973) and Merton (1973). When applied to credit risk, it is commonly referred to as the “Merton model.” Section 8.3 applies this analytical model to infer the market's assessment of the solvency of the major Australian banks, in aggregate. Assuming that a firm defaults if the value of its assets falls below its default point at the risk measurement horizon, solvency is assessed in terms of default likelihood indicator (DLI), which is translated into distance-to-default. We borrow the term DLI from Vassalou and Xing (2004) to distinguish this measure of default risk from conditional PD given by the function derived by

Vasicek (2002). The Merton model uses the market value of equity, which presumably contains forward-looking information, to estimate default risk. Market models are commonly regarded as superior to accounting models, which use information derived from financial statements, and are therefore inherently backward looking. Section 8.1 describes the market data, along with the financial statement data, used to infer the market’s assessment. The Merton model takes into account the volatility of implied asset values in its estimate of default risk. Implied asset values are imputed from equity prices using the iterative procedure adopted by Vassalou and Xing (2004) and described in Section 8.2. Assuming that a firm defaults if the value of its assets falls below its default point at any time during the risk measurement horizon, Section 8.4 estimates DLI and distance-to-default of the major banks, in aggregate, using the “first passage” approach. The stochastic time evolution of asset values is simulated using Monte Carlo methods, and the barrier is set to the default point. Finally, the market’s assessment of the solvency of the major Australian banks, as inferred by the Merton model and first passage approach, is compared with our fundamental assessment recovered from the model specification of the IRB approach.

## 8.1 Market Data

Chapters 6 and 7, respectively, use internal bank data collected by the prudential regulator to evaluate the robustness of the ASRF model, and render an assessment on the IRB approach. In the latter case, our fundamental assessment recovers from the ASRF model the prevailing state of Australia’s economy and the capacity of its banking sector to absorb credit losses. By contrast, this chapter applies the Merton model and first passage approach using market data to infer the market’s assessment of the capacity of Australia’s banking sector to absorb credit losses. As with Chapters 6 and 7, the empirical analysis presented in this chapter is conducted on the major Australian banks.

The Merton model expresses the equity value, or market capitalisation, of a firm as a European call option on the value of its assets with the strike price equal to the value of its liabilities, or contractual obligations payable. Closing stock price, shares outstanding and market capitalisation of the major Australian banks are downloaded from Bloomberg for each trading day from 31 December 2004 through 31 December 2013. A daily time series of equity value is generated by summing the market capitalisation of the major banks. Also, we construct a capitalisation-weighted index of the major banks. Using daily closing stock prices and shares outstanding, we calculate daily closing prices for this index in accordance with the index methodology described in S&P Dow Jones Indices (2014).

The strike price of the European call option for the Merton model is set to the book value of contractual obligations payable. The major Australian banks report their liabilities on the statement of financial position, or balance sheet.<sup>1</sup> Aggregate liabilities of the major banks are published by APRA (2013b) in its quarterly issue of ADI performance statistics. We extract

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<sup>1</sup> Reporting form ARF\_322\_0 and instructions for ADIs available at [www.apra.gov.au](http://www.apra.gov.au).

time series of aggregate total liabilities, decomposed into current liabilities and long-term debt, from the quarter ending 31 December 2004 through the quarter ending 31 December 2013.

The market value of assets is not observed. So, using the numerical procedure described in Section 8.2, which solves the Black-Scholes formula for a European call option, implied asset values are imputed from equity prices. As well as equity value and strike price, the risk-free interest rate is input to this numerical procedure. We proxy the risk-free interest rate with the Australian Government 2-year bond rate. The Reserve Bank of Australia publishes tables of historical yields on Australian Government bonds. We download Australian Government 2-year bond rates for each trading day from 31 December 2004 through 31 December 2013.

Using these time series data, Section 8.3 infers the market's assessment of the solvency of the major Australian banks, in aggregate, since the beginning of 2006, which predates the financial crisis of 2007–09. From 2006 the major banks have accounted for between 88% and 98% of the weight of banks<sup>2</sup> in the ASX All Ordinaries index, which includes an eight percent jump on the acquisition of St. George Bank by Westpac Banking Corporation in the fourth quarter of 2008. Again, we argue that as a consequence of the market dominance of the major banks, the empirical findings presented in this chapter are reflective of the Australian banking sector.

## 8.2 Implied Asset Values

The Merton model expresses the equity market capitalisation of a firm as a European call option on the market value of its assets with strike price equal to the face value of a zero-coupon bond representing the firm's contractual obligations payable, and expiration coinciding with the maturity of the bond (Giesecke 2004). If the firm cannot meet its contractual obligation to pay investors the principal amount of the bond at maturity, the bondholders take over the firm. That is, the firm defaults if the value of its assets falls below the principal amount of its debt at maturity, wiping out shareholders' equity.

The Merton model assumes that Itô process  $A(t)$  describing the time evolution of asset values is expressed in differential form as

$$dA(t) = \mu_A A(t) dt + \sigma_A A(t) dW(t), \quad (8.1)$$

where  $A(t)$  has constant drift rate  $\mu_A$  and volatility  $\sigma_A$ , and  $W(t) \sim \mathcal{N}(0, t)$  is a Brownian motion. The solution to stochastic differential equation (8.1) is geometric Brownian motion:

$$A(t) = A(0) \exp \left\{ \left( \mu_A - \frac{1}{2} \sigma_A^2 \right) t + \sigma_A W(t) \right\}. \quad (8.2)$$

Let  $B(t)$  be the face value of a zero-coupon bond, representing the firm's contractual obligations payable, with maturity  $t + \tau$ ,  $\tau > 0$ , which coincides with the time to expiration for the European call option. Denote by  $r$  the constant risk-free interest rate. Then, the equity value  $E(t)$ , or

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<sup>2</sup>Classified as *Banks* under FTSE ICB (Industry Classification Benchmark) sectors.



market capitalisation, of a firm at time  $t$  satisfies

$$E(t) = A(t)\Phi(d_1(t)) - e^{-r\tau} B(t)\Phi(d_2(t)), \quad (8.3)$$

where

$$\begin{aligned} d_1(t) &= \frac{\log(A(t)/B(t)) + (r + \frac{1}{2}\sigma_A^2)\tau}{\sigma_A(t)\sqrt{\tau}}, \\ d_2(t) &= d_1(t) - \sigma_A\sqrt{\tau}, \end{aligned}$$

and  $\Phi$  is the standard Gaussian distribution function.

The market value of assets is not observed, so  $A(t)$  and  $\sigma_A$  in (8.3) are unknowns. The standard model of equity values, or stock prices, is geometric Brownian motion. Analogously to (8.1), the time evolution of equity values is expressed in differential form as

$$dE(t) = \mu_E E(t) dt + \sigma_E E(t) dW(t), \quad (8.4)$$

where  $E(t)$  has constant drift rate  $\mu_E$  and volatility  $\sigma_E$ . Expressing  $E(t)$  as a function of  $A(t)$ , we may write  $E(t) = f(A(t))$ . Applying Itô's formula (Remark 2.52) yields

$$\begin{aligned} dE(t) &= f'(A(t)) dA(t) + \frac{1}{2} f''(A(t)) d[A, A](t) \\ &= \frac{\partial E(t)}{\partial A(t)} dA(t) + \frac{1}{2} \frac{\partial^2 E(t)}{\partial A(t)^2} d[A, A](t) \\ &= \left( \frac{\partial E(t)}{\partial A(t)} \mu_A A(t) + \frac{1}{2} \frac{\partial^2 E(t)}{\partial A(t)^2} \sigma_A^2 A^2(t) \right) dt + \frac{\partial E(t)}{\partial A(t)} \sigma_A A(t) dW(t), \end{aligned} \quad (8.5)$$

where  $dA(t)$  is given by (8.1), and  $d[A, A](t) = \sigma_A^2 A^2(t) dt$ . Equating coefficients of  $dW(t)$  in (8.4) and (8.5), we obtain an expression that relates asset volatility to equity volatility:

$$\begin{aligned} \sigma_E &= \left( \frac{A(t)}{E(t)} \right) \frac{\partial E(t)}{\partial A(t)} \sigma_A \\ &= \left( \frac{A(t)}{E(t)} \right) \Phi(d_1(t)) \sigma_A. \end{aligned} \quad (8.6)$$

It can be shown that the partial derivative of  $E(t)$  with respect to  $A(t)$ , or the delta of a European call option, is equal to  $\Phi(d_1(t))$  by differentiating (8.3) with respect to  $A(t)$  — a tedious calculation using the chain rule.

In the preceding theoretical outline we introduce continuous-time models of asset and equity values indexed by  $t$ . For consistency, we proceed to express  $A(t)$ ,  $E(t)$  and  $B(t)$  in (8.3) as functions of time, which serves our empirical analysis of monthly data from 2006 to 2013. But we judiciously refrained from explicitly expressing  $r$ ,  $\mu_A$  and  $\sigma_A$  in (8.3) as functions of time, since the Merton model assumes that these parameters are constant over the option's term  $\tau$ . In the remainder of this section we generate monthly time series of market value, drift rate and volatility of assets by applying an iterative procedure that solves (8.3) using daily historical market data. Section 8.3 applies the Merton model to the monthly time series data to analytically calculate

DLI. Section 8.4 simulates the continuous-time evolution of asset values, parameterised by the monthly time series data, to compute DLI using the first passage approach. Accordingly, in the sequel  $r$ ,  $\mu_A$  and  $\sigma_A$  are indexed by time when dealing with daily or monthly time series data, while assuming that for each execution of (8.8), (8.12) or (8.14) these parameters are constant over the option's term, or time to maturity of the zero-coupon bond,  $\tau$ .

As discussed in Section 8.1, daily equity market capitalisation and quarterly contractual obligations payable are available for the major Australian banks.  $E(t)$  represents the sum of market capitalisation of the major banks at the close of trading day  $t$ . For each trading day  $t$ ,  $B(t)$  is the face value of a zero-coupon bond with maturity  $t + \tau$  representing the aggregate contractual obligations payable of the major banks reported on their quarterly statement of financial position as at the end of the previous fiscal quarter. In other words,  $B(t)$  is the default point of the major banks, in aggregate, on trading day  $t$ . Time to maturity  $\tau$  of the zero-coupon bond, which coincides with the risk measurement horizon, is set to one year. We estimate DLI for three definitions of default point based on book values of: (i) total liabilities; (ii) current liabilities plus one-half of long-term debt; and (iii) current liabilities. In assessing the effect of default risk on equity returns, Vassalou and Xing (2004) set  $B(t)$  to current liabilities plus one-half of long-term debt. They argue that it is important to include long-term debt, which matures beyond the risk measurement horizon, because a firm needs to service its long-term debt whose interest payments are current liabilities, and a firm's long-term debt affects its ability to rollover its current liabilities. Their choice of setting the default point to current liabilities plus one-half of long-term debt is consistent with KMV, who argue that it adequately captures the financing constraints of firms. Bharath and Shumway (2008) compare the forecasting performance of the Merton model with a "naïve" alternative, where they too set the default point to current liabilities plus one-half of long-term debt; as do Allen and Powell (2012) in evaluating the default risk of Australian banks during the financial crisis of 2007–09.

With the risk measurement horizon, which coincides with the time to maturity of the zero-coupon bond and time to expiration of the European call option, set to one year, our implementation of (8.3) calls for the one-year risk-free interest rate. The RBA publishes time series of the monthly cash rate and daily yield on the Australian Government 2-year bond. The objective of this chapter is not to obtain a precise measure of default risk, but to compare the market's assessment of the solvency of the major Australian banks with our fundamental assessment from Section 7.3. Choosing the cash rate, yield on the Australian Government 2-year bond or some interpolated yield does not alter our conclusions. We set the risk-free interest rate,  $r(t)$ , to the yield on the Australian Government 2-year bond as at the close of trading day  $t$ .

The numerical procedure (Vassalou and Xing 2004) for estimating monthly time series of the value of assets  $A(s)$  and asset volatility  $\sigma_A(s)$  proceeds as follows. Firstly, adopting the convention of 252 trading days per year ( $m = 252$ ), we use closing equity values of the major Australian banks, in aggregate, from the current month-end trading day  $s$  and preceding  $m$  trading days to estimate equity volatility  $\sigma_E(s)$  — annualised standard deviation of the daily

continuously compounded rate of change in equity values. The initial guess for asset volatility<sup>3</sup> is then set to

$$\sigma_A(s) = \frac{E(s)}{E(s) + A(s)} \sigma_E(s). \quad (8.7)$$

Next, for each trading day  $t = \{s-m, \dots, s-1, s\}$  during the past year, we solve for  $A(t)$  given  $E(t)$ ,  $B(t)$ ,  $r(t)$ ,  $\tau$  and our initial guess for  $\sigma_A(s)$ .<sup>4</sup>

$$E(t) = A(t)\Phi(d_1(t)) - e^{-r(t)\tau} B(t)\Phi(d_2(t)), \quad (8.8)$$

where

$$d_1(t) = \frac{\log(A(t)/B(t)) + (r(t) + \frac{1}{2}\sigma_A^2(s))\tau}{\sigma_A(s)\sqrt{\tau}},$$

$$d_2(t) = d_1(t) - \sigma_A(s)\sqrt{\tau}.$$

Then, we calculate the annualised standard deviation of the daily continuously compounded rate of change in asset values,  $\{\log(A(s-m-1)/A(s-m)), \dots, \log(A(s)/A(s-1))\}$ , which becomes the value of  $\sigma_A(s)$  used for the next iteration. This step is iterated until estimates of  $\sigma_A(s)$  from consecutive iterations converge.<sup>5</sup> The drift rate of asset values,  $\mu_A(s) = \log(A(s)/A(s-m))$ , is calculated from daily asset values recovered from (8.8) in the (last) iteration that achieves

<sup>3</sup> Setting the initial guess for asset volatility using the daily time series of either equity value (equity market capitalisation) or price of our index of the major Australian banks works equally well.

<sup>4</sup> We use the GNU Scientific Library routine `gsl_root_fdfsolver_newton`, which implements Newton's method to solve for  $A(t)$  in function (8.8). Newton's method uses the derivative of function (8.8),  $\frac{\partial E(t)}{\partial A(t)} = \Phi(d_1(t))$ , to find each iterate. It exhibits quadratic convergence locally.

<sup>5</sup> Convergence is achieved when the absolute difference in values of  $\sigma_A(s)$  from consecutive iterations is less than  $1\text{E}-4$ . It usually only takes a few iterations for  $\sigma_A(s)$  to converge.

Figure 8.1: Book and market value of assets of the major Australian banks. Implied (market) value of assets are imputed from equity prices with contractual obligations payable represented by a zero-coupon bond with face value equal to: (a) total liabilities; (b) current liabilities plus one-half of long-term debt; and (c) current liabilities. Financial leverage ratio is measured on book value (total assets / shareholders' equity) and market value (implied value of assets / equity market capitalisation) bases.

Sources: Australian Prudential Regulation Authority; Bloomberg.

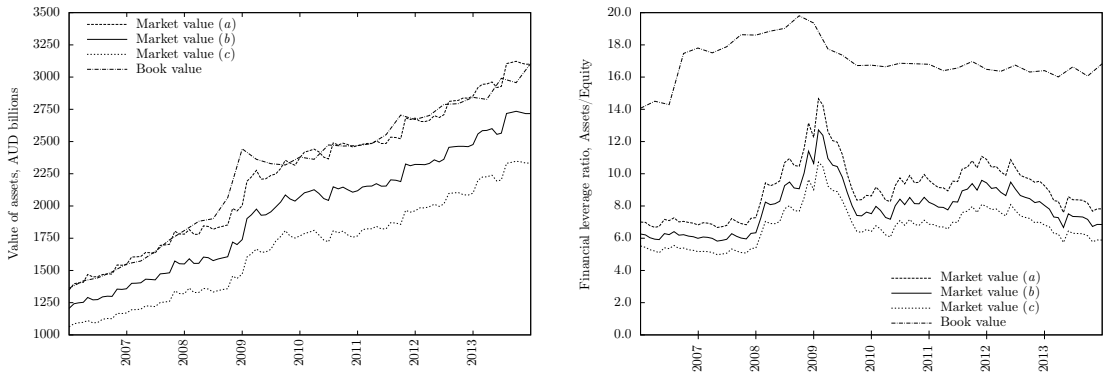
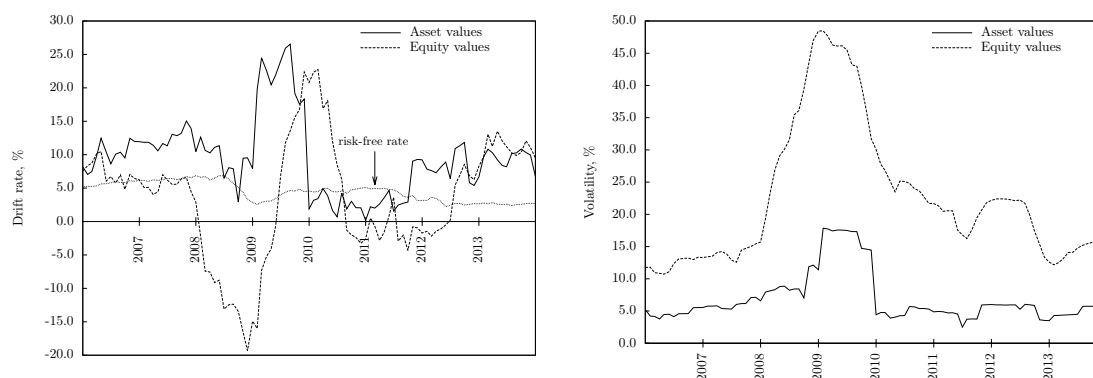


Figure 8.2: Drift rate (trailing 12-month continuously compounded rate of change) and volatility (annualised standard deviation of the daily continuously compounded rate of change for the last 12 months) of the market value of assets and equity of the major Australian banks. Implied (market) value of assets are imputed from equity prices with contractual obligations payable represented by a zero-coupon bond with face value equal to current liabilities plus one-half of long-term debt. The value of equity is the sum of market capitalisation of the major banks. The risk-free interest rate is the yield on the Australian Government 2-year bond.

Sources: Australian Prudential Regulation Authority; Bloomberg, Reserve Bank of Australia.



convergence, and the month-end asset value  $A(s)$  is also from the last iteration. This procedure is repeated for each month-end trading day from December 2005 through December 2013, generating monthly time series of  $A(s)$ ,  $\sigma_A(s)$  and  $\mu_A(s)$ .

With reference to Figures 8.1 and 8.2, we make some observations from the monthly time series between 2006 and 2013 generated by the procedure outlined above.

- The jump in the book value of assets in the fourth quarter of 2009 is the result of acquisitions: Westpac Banking Corporation acquired St. George Bank, and Commonwealth Bank of Australia acquired Bank of Western Australia.
- With the financial leverage ratio (total assets / shareholders' equity) averaging 17.0 on a book value basis, market and book value of assets are broadly in line with each other for the case where the default point is set to total liabilities.
- The market value of equity, or equity market capitalisation, has averaged 1.95 times shareholders' equity. Consequently, the financial leverage ratio on a market value basis (implied value of assets / equity market capitalisation) has averaged 9.0.
- Drift rate and volatility of the implied value of assets varies little between definitions of default point. We choose to plot drift rate and volatility for the case where the default point is set to current liabilities plus one-half of long-term debt.
- Drift rate (trailing 12-month continuously compounded rate of change) of the implied value of assets remains positive (1.8%–26.5%) as major banks expanded their balance sheet through the financial crisis, while the drift rate of the market value of equity, driven by

stock prices, ranges from -19.3% to 22.7%. Both time series are volatile relative to the risk-free interest rate.

- As implied by expression (8.6) relating asset volatility to equity volatility, and the naïve estimate of asset volatility given by (8.7) (Bharath and Shumway 2008), asset volatility averages around one-third equity volatility on a market value basis.

### 8.3 Default Likelihood Indicator

In assessing the effect of default risk on equity returns, Vassalou and Xing (2004) deduce an analytical expression for default probability from the assumptions of the Merton model, which underlie asset value models of credit risk, in general. They contend that default probabilities calculated in this manner are not true default probabilities in the manner of default probabilities estimated by KMV from an empirical distribution of defaults. Accordingly, they choose not to call their measure default probability, but rather default likelihood indicator (DLI). Conceding their argument, we adopt the term DLI, with the added clarity brought by distinguishing this measure of default risk from conditional PD given by (2.20) and (2.28).

Recall that by the self-similarity property of Brownian motion,  $W(t)$  is distributionally equivalent to  $\sqrt{t}W(1)$ , which we write  $\sqrt{t}W$ . Adapting (8.2) for the risk measurement horizon  $[s, s + \tau]$ , taking natural logarithms, and applying the self-similarity property of Brownian motion yields

$$\log A(s + \tau) = \log A(s) + \mu_A \tau - \frac{1}{2} \sigma_A^2 \tau + \sigma_A \sqrt{\tau} W, \quad (8.9)$$

where  $W \sim \mathcal{N}(0, 1)$ .

The Merton model assumes that a firm defaults if the market value of its assets falls below a critical threshold, its default point, at a given risk measurement horizon. Denote by  $\lambda(s)$  the *default likelihood indicator* (DLI) at time  $s$  over the risk measurement horizon  $[s, s + \tau]$ . Then,

$$\begin{aligned} \lambda(s) &= \mathbb{P}(A(s + \tau) < B(s)) \\ &= \mathbb{P}(\log A(s + \tau) < \log B(s)) \\ &= \mathbb{P}(\log A(s) + \mu_A \tau - \frac{1}{2} \sigma_A^2 \tau + \sigma_A \sqrt{\tau} W < \log B(s)) \\ &= \mathbb{P}\left(W < \frac{\log B(s) - \log A(s) - \mu_A \tau + \frac{1}{2} \sigma_A^2 \tau}{\sigma_A \sqrt{\tau}}\right) \\ &= \Phi\left(\frac{\log B(s) - \log A(s) - \mu_A \tau + \frac{1}{2} \sigma_A^2 \tau}{\sigma_A \sqrt{\tau}}\right), \end{aligned} \quad (8.10)$$

where the firm's default point  $B(s)$  corresponds to the face value of a zero-coupon bond with maturity  $s + \tau$  representing the firm's contractual obligations payable. Denote by  $d(s)$  the *distance-to-default* at time  $s$ . Then, taking the negative of the inverse standard distribution function translates DLI into distance-to-default:

$$\begin{aligned} d(s) &= -\Phi^{-1}(\lambda(s)) \\ &= \frac{\log A(s) - \log B(s) + \mu_A \tau - \frac{1}{2} \sigma_A^2 \tau}{\sigma_A \sqrt{\tau}}. \end{aligned} \quad (8.11)$$

DLI is estimated for three definitions of default point  $B(s)$  based on book values of: (i) total liabilities; (ii) current liabilities plus one-half of long-term debt; and (iii) current liabilities. While the Merton model assumes that the market value of a firm's assets  $A(t)$  follows a geometric Brownian motion with constant drift rate  $\mu_A$  and volatility  $\sigma_A$  over the risk measurement horizon  $[s, s + \tau]$ , the drift rate and volatility change every month for which DLI is estimated by (8.10) using monthly time series data generated in Section 8.2. Accordingly, for each month-end trading day  $s$  from December 2005 through December 2013, DLI and distance-to-default are calculated as

$$\lambda(s) = \Phi \left( \frac{\log B(s) - \log A(s) - \mu_A(s)\tau + \frac{1}{2}\sigma_A^2(s)\tau}{\sigma_A(s)\sqrt{\tau}} \right), \quad (8.12)$$

and

$$d(s) = -\Phi^{-1}(\lambda(s)),$$

respectively, where  $\tau$  is set to one year.

Figure 8.3 plots the monthly time series of DLI as calculated by (8.12), and its distance-to-default translation, for our three definitions of default point. While the market value of assets is a function of default point (Figure 8.1), distance-to-default tends to narrow as the default point increases from current liabilities to total liabilities. DLI (respectively, distance-to-default) is a volatile time series ranging from less than one-in-a-billion (6.034) in March 2006 to 19.3% (0.866) in November 2008 for the case where the default point is set to current liabilities plus one-half of

Figure 8.3: Default likelihood indicator and distance-to-default of the major Australian banks implied by the Merton model. The default point is set to: (a) total liabilities; (b) current liabilities plus one-half of long-term debt; and (c) current liabilities.

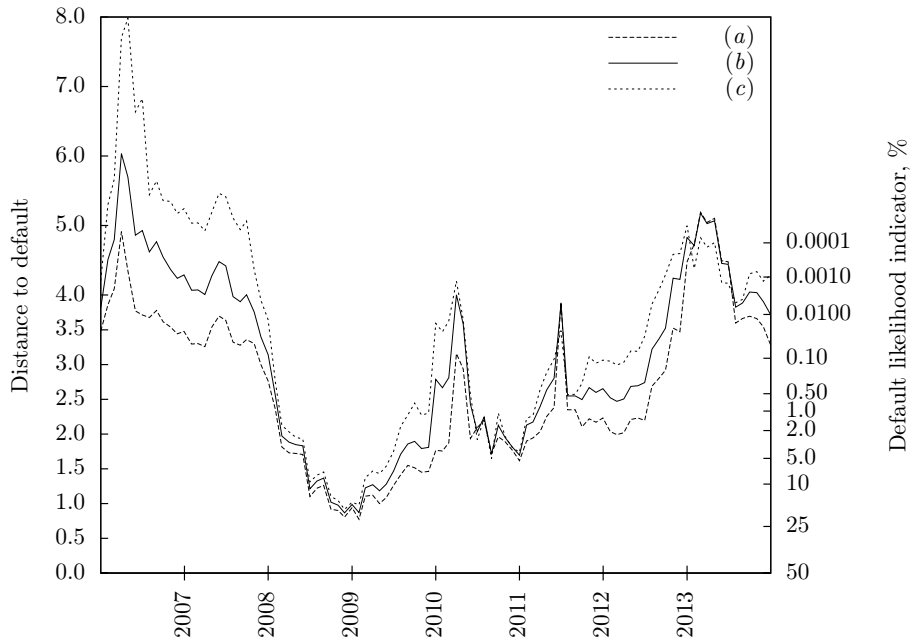
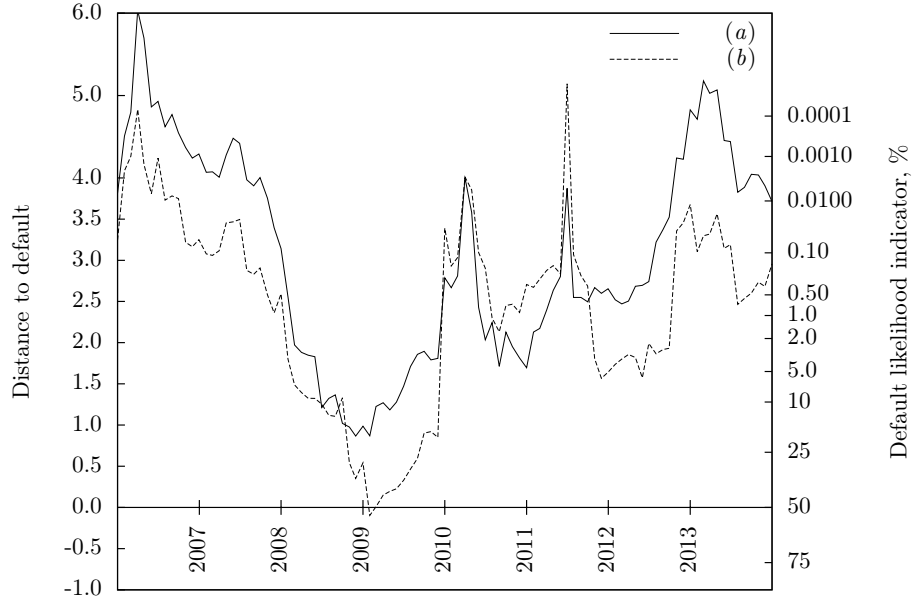


Figure 8.4: Default likelihood indicator and distance-to-default of the major Australian banks implied by the Merton model. Default point is equal to current liabilities plus one-half of long-term debt. Drift rate of asset values is set to: (a) trailing 12-month continuously compounded rate of change in asset values; and (b) yield on the Australian Government 2-year bond.



long-term debt. The volatility of DLI is explained by the variability in drift rate and volatility of implied asset values (Figure 8.2).

Notice that the drift rate of assets,  $\mu_A$ , does not appear in (8.3), an example of the risk-neutral valuation principle of option pricing. However, DLI and distance-to-default depend on the future value of assets, and hence  $\mu_A$  appears in (8.10) and (8.11). In Figure 8.3 the expected annual drift rate  $\mu_A(s)$  in (8.12) is set to the trailing 12-month continuously compounded rate of change in asset values, a volatile time series. Figure 8.4 compares DLI calculated by (8.12) with that calculated by substituting the comparatively steady risk-free interest rate  $r(s)$  for  $\mu_A(s)$  in (8.12). For much of the time from 2006 to 2013, the trailing 12-month continuously compounded rate of change in asset values exceeded the yield on the Australian Government 2-year bond (Figure 8.2) as the major Australian banks, in aggregate, expanded their balance sheet. During periods when  $\mu_A(s)$  exceeds  $r(s)$ , DLI calculated by (8.12) is further from default than that calculated by substituting  $r(s)$  for  $\mu_A(s)$  in (8.12). Both time series remain volatile on account of the variability in asset volatility  $\sigma_A(s)$  (Figure 8.2).

## 8.4 First Passage Approach

Assuming that a firm defaults if the market value of its assets  $A(s+\tau)$  falls below its default point  $B(s)$ , at a given risk measurement horizon  $s + \tau$ , analytical equations (8.10) and (8.11) calculate DLI and distance-to-default, respectively. Under these assumptions, (8.3) models equity market capitalisation of a firm as a European call option on the market value of its asset with strike price equal to its default point and term  $\tau$ . The default point of the firm corresponds to the face value a zero-coupon bond  $B(s)$  with maturity  $s + \tau$  representing its contractual obligations payable.

Redefining default as the market value of a firm's assets  $A(t)$  falling below a barrier equal to  $B(s)$  at any time during the interval  $[s, s + \tau]$ , Giesecke (2004) deduces an analytical expression for the default probability of the firm.<sup>6</sup> The default probability, or DLI, under this “first passage” approach is obviously higher than the corresponding DLI under the classical structural approach of Section 8.3. Under the first passage approach the equity market capitalisation of a firm is modelled as a European down-and-out call option on the market value of its asset with strike price equal to the barrier. The barrier is set to the face value of a zero-coupon bond representing the firm's contractual obligations payable, and option expiration coincides with the maturity of the bond. Giesecke (2004) remarks that the down-and-out call option is worth no more than the corresponding vanilla call option.

In this section we implement the first passage approach using Monte Carlo simulation. From Section 8.2 we have time series of  $A(s)$ ,  $\mu_A(s)$  and  $\sigma_A(s)$  for month-end trading days from December 2005 through December 2013. For each month-end trading day  $s$  we simulate asset values  $A(t)$  as at the close of each of the  $m$  trading days during the risk measurement horizon  $[s, s + \tau]$ , where  $\tau$  is set to one year. If at any time during the interval  $[s, s + \tau]$ ,  $A(t)$  falls below the barrier  $B(s)$  set to the face value of a zero-coupon bond with maturity  $s + \tau$  representing the firm's contractual obligations payable, the firm defaults. Itô process (8.1) describing the time evolution of asset values may be expressed as a stochastic differential equation in  $\log A(t)$ :

$$d \log A(t) = \frac{dA(t)}{A(t)} = \mu_A dt + \sigma_A dW(t). \quad (8.13)$$

Discretising (8.13) for the  $k^{\text{th}}$  iteration of the Monte Carlo simulation parameterised by the monthly time series data generated in Section 8.2, we obtain the difference equation

$$\log A_k(t+1) = \log A_k(t) + \mu_A(s)\Delta t + \sigma_A(s)\sqrt{\Delta t}\epsilon_k(t-s), \quad (8.14)$$

<sup>6</sup> Under the first passage approach with the barrier  $B(s)$  set to the face value of a zero-coupon bond with maturity  $s + \tau$  representing the firm's contractual obligations payable, the default probability of the firm is given by

$$\tilde{p}(s) = \Phi\left(\frac{\log B(s) - \log A(s) - \mu_A\tau + \frac{1}{2}\sigma_A^2\tau}{\sigma_A\sqrt{\tau}}\right) + \left(\frac{B(s)}{A(s)}\right)^{\frac{2\mu_A - \sigma_A^2}{\sigma_A^2}} \Phi\left(\frac{\log B(s) - \log A(s) + \mu_A\tau - \frac{1}{2}\sigma_A^2\tau}{\sigma_A\sqrt{\tau}}\right).$$



where  $t = \{s, s+1, \dots, s+m-1\}$ ,  $\Delta t = \tau/m$ , and  $\epsilon_k(t-s)$  is the  $(t-s)^{\text{th}}$  random variable drawn from the standard Gaussian distribution<sup>7</sup> for the  $k^{\text{th}}$  iteration of the simulation (Luenberger 1998, pp. 308–312).

For each month-end trading day  $s$ , we simulate  $N = 10,000,000$  asset value paths. Then, for each asset value path we check whether  $A(t) < B(s)$  as at the close of any of the  $m$  trading days during the risk measurement horizon  $[s, s + \tau]$ , and increment the count  $n$  of asset value paths on which default occurs accordingly. DLI by the first passage approach is calculated as

$$\check{\lambda}(s) = \frac{n}{N}, \quad (8.15)$$

and

$$\check{d}(s) = -\Phi^{-1}(\check{\lambda}(s)) \quad (8.16)$$

translates DLI into distance-to-default.

<sup>7</sup> Gaussian random variables are generated using GNU Scientific Library routine `gsl_ran_gaussian`. It implements the Box-Muller algorithm, which make two calls to the MT19937 generator of Makoto Matsumoto and Takuji Nishimura.

Figure 8.5: Market and fundamental assessments of the solvency of the major Australian banks, as measured by default likelihood indicator and distance-to-default. The market's assessment imputes asset values from equity prices, and represents contractual obligations payable by a zero-coupon bond with face value equal to current liabilities plus one-half of long-term debt. Our fundamental assessment uses internal bank data collected by the prudential regulator.

- (a) Market assessment is calculated analytically using the Merton model.
- (b) Market assessment is computed by Monte Carlo simulation using the first passage approach.
- (c) Fundamental assessment is recovered from the ASRF model prescribed under the IRB approach.

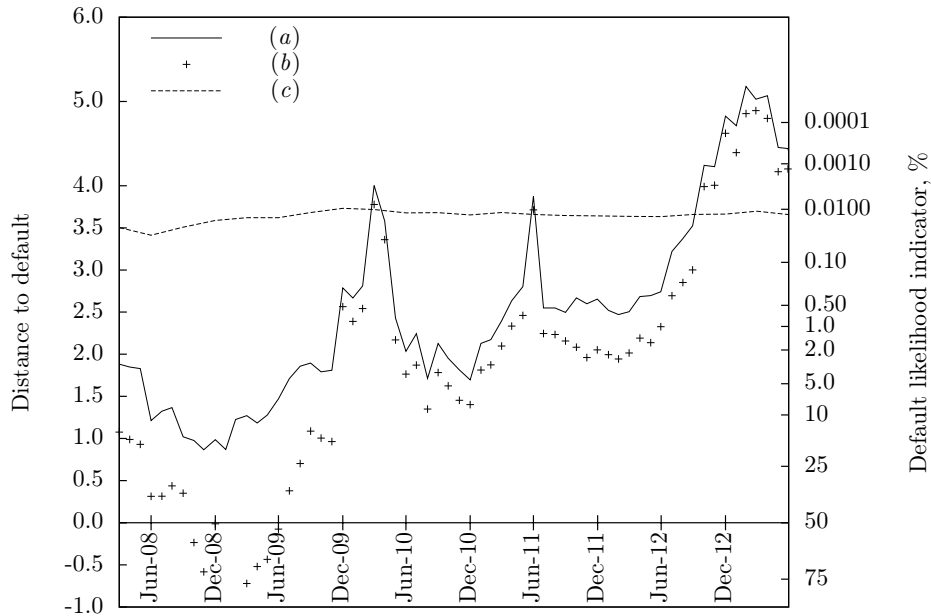


Figure 8.5 plots DLI of the major Australian banks, in aggregate, computed by Monte Carlo simulation using the first passage approach. As highlighted above, distance-to-default simulated by the first passage approach plots below (i.e., is narrower than) that calculated by the Merton model under the classical structural approach. Figure 8.5 also compares the market's assessment of the solvency of the major banks with our fundamental assessment recovered from the ASRF model prescribed under the IRB approach. In the latter case, distance-to-default, reported in Table 7.1, is estimated from internal bank data collected by the prudential regulator. The market's assessment, which is much more volatile than our fundamental assessment, does not comport with the state of Australia's economy as described by macroeconomic indicators, nor the condition of its banking sector as reflected by financial statistics and external credit ratings. We reiterate that markets do not consistently price risk fairly, but rather reflect the overreaction of market participants. During periods when market participants are gripped by fear (respectively, driven by greed), their perception of risk is heightened (lessened), and markets become undervalued (overpriced).



# Bibliography

- Allen, Bob. 2006. "Internal affairs." *Risk* 19 (6): 45–49.
- Allen, David E., and Robert Powell. 2012. "The fluctuating default risk of Australian banks." *Australian Journal of Management* 37 (2): 297–325.
- Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath. 1999. "Coherent measures of risk." *Mathematical Finance* 9 (3): 203–228.
- Australian Prudential Regulation Authority. 2007. *Implementation of Basel II capital framework: Supervisory review process*. Information Paper. December. <http://apra.gov.au/adi/pages/basel-ii-implementation-in-australia.aspx>.
- . 2008a. *Capital adequacy: Internal ratings-based approach to credit risk*. ADI Prudential Standard 113. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2008b. *Capital adequacy: Standardised approach to operational risk*. ADI Prudential Standard 114. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2008c. *Capital adequacy: Advanced measurement approaches to operational risk*. ADI Prudential Standard 115. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2008d. *Capital adequacy: Market risk*. ADI Prudential Standard 116. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2008e. *Capital adequacy: Interest rate risk in the banking book (advanced ADIs)*. ADI Prudential Standard 117. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2008f. *Capital adequacy*. ADI Prudential Standard 110. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.

- Australian Prudential Regulation Authority. 2010a. *Capital adequacy: Standardised approach to credit risk*. ADI Prudential Standard 112. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2010b. *Probability and impact rating system*. Information Paper. November. [http://apra.gov.au/crossindustry/documents/pairs\\_nov\\_2010.pdf](http://apra.gov.au/crossindustry/documents/pairs_nov_2010.pdf).
- . 2011. *Securitisation*. ADI Prudential Standard 120. <http://apra.gov.au/adi/PrudentialFramework/Pages/prudential-standards-and-guidance-notes-for-adis.aspx>.
- . 2013a. *Domestic systemically important banks in Australia*. Information Paper. December. <http://www.apra.gov.au/adi/Publications/Pages/other-information-for-adis.aspx>.
- . 2013b. *Quarterly authorised deposit-taking institution performance*. Statistics. <http://apra.gov.au/adi/Publications/Pages/adi-quarterly-performance-statistics.aspx>.
- . 2014. *D2A guide for APRA regulated institutions*. Help Guide. June. <http://www.apra.gov.au/CrossIndustry/Pages/D2A.aspx>.
- Barlow, Richard E., and Frank Proschan. 1981. *Statistical Theory of Reliability and Life Testing: Probability Models*. Silver Spring, MD: To Begin With.
- Basel Committee on Banking Supervision. 1988. *International convergence of capital measurement and capital standards*. Bank for International Settlements, July. <http://www.bis.org/publ/bcbs04a.htm>.
- . 2005. *An explanatory note on the Basel II IRB risk weight functions*. Bank for International Settlements, July. <http://www.bis.org/bcbs/irbriskweight.htm>.
- . 2006. *Basel II: International convergence of capital measurement and capital standards. A revised framework, comprehensive version*. Bank for International Settlements, June. <http://www.bis.org/publ/bcbs128.htm>.
- . 2009a. *Enhancements to the Basel II framework*. Bank for International Settlements, July. <http://www.bis.org/publ/bcbs157.htm>.
- . 2009b. *Findings on the interaction of market and credit risk*. Working Paper 16. Bank for International Settlements, May. [http://www.bis.org/publ/bcbs\\_wp16.htm](http://www.bis.org/publ/bcbs_wp16.htm).
- . 2009c. *Principles for sound stress testing practices and supervision*. Bank for International Settlements, May. <http://www.bis.org/publ/bcbs155.htm>.

- 
- . 2011. *Basel III: A global regulatory framework for more resilient banks and banking systems*. Revised version. Bank for International Settlements, June. <http://www.bis.org/publ/bcbs189.htm>.
- . 2012. *Core principles for effective banking supervision*. Bank for International Settlements, September. <http://www.bis.org/publ/bcbs230.htm>.
- Basel Committee on Banking Supervision, International Organisation of Securities Commissions, and International Association of Insurance Supervisors. 2010. *Developments in modelling risk aggregation*. Joint Forum. Bank for International Settlements, October. <http://www.bis.org/publ/joint25.htm>.
- Bharath, Sreedhar T., and Tyler Shumway. 2008. “Forecasting default with the Merton distance to default model.” *The Review of Financial Studies* 21 (3): 1339–1369.
- Bishop, James, Christopher Kent, Michael Plumb, and Vanessa Rayner. 2013. “The resources boom and the Australian economy: A sectoral analysis.” *Bulletin* 34 (March): 39–49. Reserve Bank of Australia.
- Black, Fischer, and Myron Scholes. 1973. “The pricing of options and corporate liabilities.” *Journal of Political Economy* 81 (3): 637–654.
- Bluhm, Christian, Ludger Overbeck, and Christoph Wagner. 2010. *Introduction to Credit Risk Modeling*. 2nd ed. Boca Raton, FL: Chapman & Hall / CRC.
- Brigo, Damiano, Jan-Frederik Mai, and Matthias Scherer. 2013. “Consistent iterated simulation of multi-variate default times: A Markovian indicators characterization.” arXiv: [1306.0887 \[q-fin.RM\]](https://arxiv.org/abs/1306.0887).
- Brown, Christine, and Kevin Davis. 2010. “Australia’s experience in the global financial crisis.” Chap. 66 in *Lessons from the Financial Crisis: Causes, Consequences, and Our Economic Future*, edited by Robert W. Kolb, 537–544. Hoboken, NJ: John Wiley & Sons, Inc.
- Cecchetti, Stephen G., Michael R. King, and James Yetman. 2011. *Weathering the financial crisis: Good policy or good luck?* Working Paper 351. Bank for International Settlements, August. <http://www.bis.org/publ/work351.htm>.
- Chan-Lau, Jorge A. 2006. *Fundamental-based estimation of default probabilities: A survey*. Working Paper 06/149. International Monetary Fund, June. <http://www.imf.org/external/pubs/ft/wp/2006/wp06149.pdf>.
- Cifuentes, Rodrigo, Gianluigi Ferrucci, and Hyun Song Shin. 2005. “Liquidity risk and contagion.” *Journal of the European Economic Association* 3:556–566.

- D'Aloisio, Tony. 2010. *Responding to the global financial crisis: The ASIC story*. Charman's Boardroom Series. Trans-Tasman Business Circle, Sydney, November 30. Australian Securities & Investments Commission.
- Debelle, Guy. 2007. *Open market operations and domestic securities*. Annual Conference. Australian Securitisation Forum, Sydney, November 29. Reserve Bank of Australia, <http://www.rba.gov.au/speeches/2007/index.html>.
- Drehmann, Mathias, and Nikola Tarashev. 2011. *Measuring the systemic importance of interconnected banks*. Working Paper 342. Bank for International Settlements, March. <http://www.bis.org/publ/work342.htm>.
- Eisenberg, Larry, and Thomas H. Noe. 2001. "Systemic risk in financial systems." *Management Science* 47 (2): 236–249.
- Elizalde, Abel, and Rafael Repullo. 2007. "Economic and regulatory capital in banking: What is the difference?" *International Journal of Central Banking* 3 (3): 87–117.
- Embrechts, Paul, Claudia Klüppelberg, and Thomas Mikosch. 2000. *Modelling Extremal Events for Insurance and Finance*. Berlin: Springer.
- Embrechts, Paul, Filip Lindskog, and Alexander McNeil. 2003. "Modelling dependence with copulas and applications to risk management." Chap. 8 in *Handbook of Heavy Tailed Distributions in Finance*, edited by Svetlozar T. Rachev, 329–383. Amsterdam: Elsevier/North-Holland.
- Embrechts, Paul, Alexander J. McNeil, and Daniel Straumann. 2002. "Correlation and dependence in risk management: Properties and pitfalls." Chap. 7 in *Risk Management: Value at Risk and Beyond*, edited by M.A.H. Dempster, 176–223. Cambridge: Cambridge University Press.
- Emery, Kenneth. 2007. "Adjusting Moody's LGD assessments to meet Basel II downturn requirements." Moody's Corporation, November. <http://www.moodys.com>.
- Emmer, Susanne, and Dirk Tasche. 2005. "Calculating credit risk capital charges with the one-factor model." *Journal of Risk* 7 (2): 85–101.
- Financial Stability Board, International Monetary Fund, and Bank for International Settlements. 2011. *Macroprudential policy tools and frameworks: Update to the G20 finance ministers and central bank governors*, February. [http://www.financialstabilityboard.org/list/fsb\\_publications/index.htm](http://www.financialstabilityboard.org/list/fsb_publications/index.htm).
- Fitch Ratings. 2008. "Basel II correlation values: An empirical analysis of EL, UL and the IRB model." May. <http://www.fitchratings.com>.
- Flannery, Mark J., Simon H. Kwan, and Mahendrarajah Nimalendran. 2013. "The 2007–2009 financial crisis and bank opacity." *Journal of Financial Intermediation* 22 (1): 55–84.

- Frey, Rüdiger, Alexander J. McNeil, and Mark A. Nyfeler. 2001. "Copulas and credit models." *RISK Magazine* (October): 111–114.
- Gauthier, Céline, Alfred Lehar, and Moez Souissi. 2012. "Macroprudential capital requirements and systemic risk." *Journal of Financial Intermediation* 21 (4): 594–618.
- Giese, Guido. 2003. "Economic capital versus regulatory capital — a market benchmark." *Risk* 16 (5): S17–S20.
- Giesecke, Kay. 2004. "Credit risk modelling and valuations: An introduction." Chap. 16 in *Credit Risk: Models and Management*, 2nd ed., edited by David Shimko, 487–526. London: Risk Books.
- Gleeson, James P., T.R. Hurd, Sergey Melnik, and Adam Hackett. 2013. "Systemic risk in banking networks without Monte Carlo simulation." Chap. 2 in *Advances in Network Analysis and its Applications: Models and Management*, edited by Evangelos Kranakis, 27–55. Berlin: Springer-Verlag.
- Gneiting, Tilmann. 2011. "Making and evaluating point forecasts." *Journal of the American Statistical Association* 106 (494): 746–762.
- Gordy, Michael B. 2003. "A risk-factor model foundation for ratings-based bank capital rules." *Journal of Financial Intermediation* 12 (3): 199–232.
- Gordy, Michael B., and Bradley Howells. 2006. "Procyclicality in Basel II: Can we treat the disease without killing the patient?" *Journal of Financial Intermediation* 15 (3): 395–417.
- Gordy, Michael B., and Eva Lütkebohmert. 2007. *Granularity adjustment for Basel II*. Discussion Paper 01/2007. Deutsche Bundesbank.
- Grimmett, Geoffrey R., and David R. Stirzaker. 2001. *Probability and Random Processes*. 3rd ed. New York: Oxford University Press.
- Gupta, Arjun K., Wei-Bin Zeng, and Yanhong Wu. 2010. *Probability and Statistical Models: Foundations for Problems in Reliability and Financial Mathematics*. New York: Springer.
- Gut, Allan. 2005. *Probability: A Graduate Course*. New York: Springer.
- Hamerle, Alfred, and Daniel Röscher. 2005. "Misspecified copulas in credit risk models: How good is Gaussian?" *Journal of Risk* 8 (1): 41–58.
- Jagannathan, Ravi, Mudit Kapoor, and Ernst Schaumburg. 2013. "Causes of the great recession of 2007–2009: The financial crisis was the symptom not the disease!" *Journal of Financial Intermediation* 22 (1): 4–29.
- Jarrow, Robert A. 2007. "A critique of revised Basel II." *Journal of Financial Services Research* 32 (1): 1–16.



- Kou, Steven, Xianhua Peng, and Christopher C. Heyde. 2011. "Robust external risk measures." In *Wiley Encyclopedia of Operations Research and Management Science*, edited by James J. Cochran. John Wiley & Sons, Inc.
- Laeven, Luc, and Fabián Valencia. 2013. "The real effects of financial sector interventions during crises." *Journal of Money, Credit and Banking* 45 (1): 147–177.
- Laker, John F. 2010. *The Australian banking system under stress?* Australian Business Economists, Sydney, June 9. Australian Prudential Regulation Authority, <http://apra.gov.au/Speeches/Pages/the-australian-banking-system-under-stress.aspx>.
- . 2012. *The Australian banking system under stress, again?* Randstad Leaders Lecture. Australian Banking & Finance, Brisbane, November 8. Australian Prudential Regulation Authority, <http://apra.gov.au/Speeches/Pages/The-Australian-banking-system-under-stress-again.aspx>.
- Lewis, David. 2013. *Risk-based supervision: How can we do better? An Australian supervisory perspective.* Program on Supervisory Experiences in Implementing Global Banking Reforms. Toronto Centre, Toronto, June 19. Australian Prudential Regulation Authority, <http://apra.gov.au/Speeches/Pages/Risk-based-supervision-How-can-we-do-better.aspx>.
- Li, David X. 2000. "On default correlation: A copula function approach." *Journal of Fixed Income* 9 (4): 43–54.
- Lindskog, Filip, and Alexander J. McNeil. 2001. "Common Poisson shock models: Applications to insurance and credit risk modelling." *ASTIN Bulletin* 33 (1): 209–238.
- Lopez, Jose A. 2002. *The empirical relationship between average asset correlation, firm probability of default and asset size.* Working Paper 2002–05. Federal Reserve Bank of San Francisco, June. <http://www.frbsf.org/economic-research/files/wp02-05bk.pdf>.
- Luenberger, David G. 1998. *Investment Science.* New York: Oxford University Press.
- Macfarlane, Ian. 2006. *The search for stability: The recession of 1990 and its legacy.* Boyer Lectures (Lecture 4). Australian Broadcasting Corporation, December 3. <http://www.abc.net.au/radionational/programs/boyerlectures>.
- MacKenzie, Donald, and Taylor Spears. 2012. *The formula that killed Wall Street: The Gaussian copula and the material cultures of modelling.* Working Paper. University of Edinburgh, June. [http://www.sps.ed.ac.uk/\\_\\_data/assets/pdf\\_file/0003/84243/Gaussian14.pdf](http://www.sps.ed.ac.uk/__data/assets/pdf_file/0003/84243/Gaussian14.pdf).
- McDonald, Tony, and Steve Morling. 2011. "The Australian economy and the global downturn: Reasons for resilience." *Economic Roundup* 2:1–31. The Treasury, Australian Government.

- McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts. 2005. *Quantitative Risk Management*. Princeton: Princeton University Press.
- Merton, Robert C. 1973. "Theory of rational option pricing." *The Bell Journal of Economics and Management Science* 4 (1): 141–183.
- . 1974. "On the pricing of corporate debt: The risk structure of interest rates." *Journal of Finance* 29:449–470.
- Nelsen, Roger B. 2006. *An Introduction to Copulas*. 2nd ed. New York: Springer.
- Office of the Comptroller of the Currency, Board of Governors of the Federal Reserve System, Federal Deposit Insurance Corporation, and Office of Thrift Supervision. 2007. *Risk-based capital standards: Advanced capital adequacy framework*. Notice of Proposed Rulemaking. November. [http://www.federalreserve.gov/generalinfo/basel2/FinalRule\\_BaselIII](http://www.federalreserve.gov/generalinfo/basel2/FinalRule_BaselIII).
- Petrov, Valentin V. 1995. *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*. New York: Oxford University Press.
- Senate Standing Committees on Economics. 2011. "Government guarantees and support." Chap. 12 in *Competition Within the Australian Banking Sector*, 227–241. Parliament of Australia, May 6.
- Shreve, Steven E. 2004. *Stochastic Calculus for Finance II: Continuous-Time Models*. New York: Springer.
- S&P Dow Jones Indices. 2014. "Index mathematics methodology." McGraw Hill Financial, November. <http://www.spindices.com>.
- Suh, Sangwon. 2012. "Measuring systemic risk: A factor-augmented correlated default approach." *Journal of Financial Intermediation* 21 (2): 341–358.
- Tasche, Dirk. 2005. "Risk contributions in a multi-factor framework." In *Concentration Risk in Credit Portfolios: A Workshop on Applied Banking Research*. Frankfurt. Deutsche Bundesbank, Basel Committee on Banking Supervision and Journal of Credit Risk, November. <https://www.bis.org/bcbs/events/crcp05tasche.pdf>.
- Tiesset, Muriel, and Philippe Troussard. 2005. *Regulatory capital and economic capital*. Financial Stability Review 7. Banque de France, November. <https://www.banque-france.fr/en/publications/financial-stability-review.html>.
- Vasicek, Oldrich. 1987. "Probability of loss on loan portfolio." KMV Corporation, February. <http://www.moodysanalytics.com>.
- . 1991. "Limiting loan loss probability distribution." KMV Corporation, August. <http://www.moodysanalytics.com>.
- . 2002. "Loan portfolio value." *RISK Magazine* (December): 160–162.

- Vassalou, Maria, and Yuhang Xing. 2004. "Default risk in equity returns." *The Journal of Finance* 59 (2): 831–868.
- Wade, William R. 2004. *An Introduction to Analysis*. 3rd ed. Upper Saddle River, NJ: Pearson Prentice Hall.
- Wang, Shaun, and Jan Dhaene. 1998. "Comonotonicity, correlation order and premium principles." *Insurance: Mathematics and Economics* 22 (3): 235–242.

# Notation Index

$\omega$	Outcome of an experiment	49
$\Omega$	Sample space of all possible outcomes of an experiment	49
$\mathcal{F}$	Collection of subsets of the sample space called a $\sigma$ -algebra, or $\sigma$ -field	49
$\mathbb{P}(A)$	Probability of set $A \in \mathcal{F}$	49
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space	49
$\mathcal{F}(t)$	Collection of $\sigma$ -algebras called a filtration	49
$\mathcal{N}(0, 1)$	Standard normal distribution with mean zero and unit variance	21
$\Phi$	Standard normal distribution function	21
$t(\nu)$	Student's $t$ distribution with $\nu$ degrees of freedom	62
$\Phi_\nu$	Student's $t$ distribution function with $\nu$ degrees of freedom	63
$\chi^2(\nu)$	Chi-square distribution with $\nu$ degrees of freedom	62
$r_i$	Log-return on asset values of obligor $i$	15, 46
$\tilde{r}_i$	Standardised log-return on asset values of obligor $i$	16, 46
$\tilde{r}$	Portfolio standardised log-return	46
$\mathbb{E}[\tilde{r}]$	Expected value of $\tilde{r}$	47
$\text{Var}(\tilde{r})$	Variance of $\tilde{r}$	47
$\tau$	Risk measurement horizon, bond maturity, or option term	22, 113
$A(t)$	Implied value of assets of a firm at time $t$	20, 113
$\mu(t)$	Instantaneous mean rate of return at time $t$	21
$\sigma(t)$	Instantaneous volatility of returns at time $t$	21
$W(t)$	Brownian motion	19, 113
$W$	Standard Gaussian random variable	21
$B$	Face value of zero-coupon bond, representing the debt of a firm, with maturity $\tau$	21
$B(t)$	Face value of zero-coupon bond, representing contractual obligations payable of a firm, with maturity $t + \tau$	113

$E(t)$	Market value of equity of a firm at time $t$	113
$\mu_A$	Drift rate of asset values	113
$\mu_E$	Drift rate of equity values	114
$\sigma_A$	Volatility of asset values	113
$\sigma_E$	Volatility of equity values	114
$D$	Set representing the event of default	21
$d$	Distance-to-default	21
$\delta_i$	Exposure at default assigned to obligor $i$	22, 30
$\Delta_n$	Exposure at default of a credit portfolio comprising $n$ obligors	31, 71
$w_i$	Exposure (portfolio) weight of obligor $i$	22, 30
$\eta_i$	Loss given default assigned to obligor $i$	22, 30
$D_i$	Event that obligor $i$ defaults	27, 30
$\mathbf{1}_{D_i}$	Indicator function of the event represented by set $D_i$	22
$L_n$	Random variable representing loss as a percentage of exposure at default on a credit portfolio comprising $n$ obligors over a given risk measurement horizon	28, 30
$L_{n,k}$	Portfolio percentage loss computed for the $k^{\text{th}}$ iteration of a simulation	37, 65
$\mathcal{L}_n$	Random variable representing dollar loss on a credit portfolio comprising $n$ obligors over a given risk measurement horizon	67
$W_i(t)$	Brownian motion modelling the variability in asset values of obligor $i$	25
$Y(t)$	Brownian motion correlated with $W_i(t)$ and common to all obligors	25
$Z_i(t)$	Brownian motion that is a component of $W_i(t)$ , but independent of $Y(t)$	25
$\mathbb{E}[W_i(t)]$	Expectation of $W_i(t)$	25
$\text{Var}(W_i(t))$	Variance of $W_i(t)$	25
$[W_i, Y](t)$	Quadratic covariation of $W_i(t)$ and $Y(t)$	25
$\text{Cov}(W_i(t), W_j(t))$	Covariance of $W_i(t)$ and $W_j(t)$ , $i \neq j$	26
$\text{Corr}(W_i(t), W_j(t))$	Correlation between $W_i(t)$ and $W_j(t)$ (i.e., correlation between asset values of obligors $i$ and $j$ ), $i \neq j$	26
$W_i$	Latent random variable modelling the variability in asset values of obligor $i$	27, 30
$Z_i$	Random variable representing idiosyncratic risk factor specific to obligor $i$	27, 30

$Y$	Random variable representing the single (systematic) risk factor common to all obligors	27, 30
$y$	Realisation of systematic risk factor $Y$	28
$F_i$	Distribution function of $W_i$ in the general setting	30
$G_i$	Distribution function of $Z_i$ in the general setting	30
$H$	Distribution function of $Y$ in the general setting	30
$\rho_i$	Asset correlation assigned to obligor $i$ for the Gaussian case	25
$\gamma_i$	Asset correlation assigned to obligor $i$ in the general setting	30
$p_i$	Unconditional (through-the-cycle) probability of default of obligor $i$ , $\mathbb{P}(D_i)$	27, 30
$p_i(y)$	Conditional (point-in-time) probability of default of obligor $i$ , $\mathbb{P}(D_i   Y = y)$	28, 30
$\zeta_i(y)$	Argument of the indicator function of default of obligor $i$	28, 30
$\mathbb{E}[L_n]$	Expected portfolio percentage loss	31, 67
$\mathbb{E}[L_n   Y = y]$	Conditional expectation of portfolio percentage loss given $Y = y$	31
$X$	Random variable representing portfolio profit and loss (P&L)	3
$\alpha$	Confidence level	5, 36
$\varrho(X)$	Value assigned by risk measure $\varrho$ to $X$	3
$q_\alpha(X)$	$\alpha$ quantile of the distribution of $X$	5
$q_\alpha(L_n)$	$\alpha$ quantile of the portfolio loss distribution expressed as a percentage of exposure at default	39
$\text{VaR}_\alpha(X)$	Value-at-risk of $X$ at the $\alpha$ confidence level over a given risk measurement horizon	5
$\text{VaR}_\alpha(L_n)$	Credit value-at-risk of portfolio percentage loss at the $\alpha$ confidence level over a given risk measurement horizon	36, 67
$\text{ES}_\alpha(X)$	Expected shortfall of $X$ at the $\alpha$ confidence level over a given risk measurement horizon	6
$C(u_1, \dots, u_n)$	$n$ -dimensional copula	55
$M^n(u_1, \dots, u_n)$	Fréchet-Hoeffding upper bound	55
$W^n(u_1, \dots, u_n)$	Fréchet-Hoeffding lower bound	55
$\Pi^n(u_1, \dots, u_n)$	Product copula	55
$T_i$	Random variable representing survival time of obligor $i$	60
$\hat{\Gamma}$	Correlation matrix of the one-factor Gaussian copula	59
$C_{\hat{\Gamma}}(u_1, \dots, u_n)$	One-factor Gaussian copula	59

$K_\alpha(L_n)$	Capital charge (at the $\alpha$ confidence level over a given risk measurement horizon) assessed on a credit portfolio comprising $n$ obligors	67
$\tilde{K}_\alpha(L_n)$	Capital charge (at the $\alpha$ confidence level over a given risk measurement horizon) assessed on a credit portfolio comprising $n$ obligors that exhibits sufficient granularity	69
$\delta_i(t)$	Exposure at default assigned to credit $i$ at the end of quarter $t$	98
$\eta_i(t)$	Loss given default assigned to credit $i$ at the end of quarter $t$	98
$\underline{\eta}_i(t)$	Economic downturn (stressed) loss given default assigned to credit $i$ at the end of quarter $t$	100
$\bar{\eta}_i(t)$	Through-the-cycle loss given default assigned to credit $i$ at the end of quarter $t$	100
$\rho_i(t)$	Asset correlation assigned to credit $i$ at the end of quarter $t$	98
$p_i(t)$	Unconditional probability of default of credit $i$ at the end of quarter $t$	98
$\nu_i(t)$	Maturity adjustment assigned to credit $i$ at the end of quarter $t$	98
$\mathcal{R}_n(t)$	Risk-weighted assets of IRB credit exposures as at the end of quarter $t$	99
$\mathcal{R}_r(t)$	Risk-weighted assets for credit risk as at the end of quarter $t$	99
$\mathcal{R}(t)$	Total risk-weighted assets as at the end of quarter $t$	99
$\mathcal{L}_r(s)$	Credit losses incurred during quarter $s$	99
$\mathcal{Q}_r(t)$	Provisions set aside for absorbing expected credit losses as at the end of quarter $t$	106
$\mathcal{K}(t)$	Capital base as at the end of quarter $t$	106
$\kappa(t)$	Risk-based capital ratio as at the end of quarter $t$	107
$\underline{\kappa}$	Capital ratio floor	109
$y(t)$	Realisation of systematic risk factor $Y$ as at the end of quarter $t$	100
$\tilde{d}(t)$	Distance-to-default as at the end of quarter $t$	107
$r(t)$	Risk-free interest rate as at the close of trading day $t$	115
$\lambda(s)$	Default likelihood indicator at time $s$ over the risk measurement horizon $[s, s + \tau]$	118
$d(s)$	Distance-to-default at time $s$	118